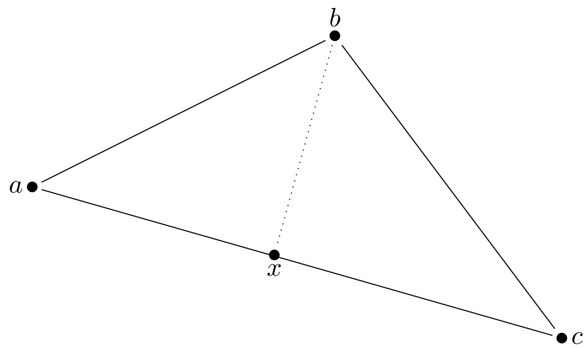
Solutions CSCI 480, Fall 2015, Math Homework # 2

1. In the picture below, a, b, and c are arbitrary points, and the dotted line from b to x is perpendicular to the line from a to c. Give formulas to find each of the distances from a, b, and c to x as a function of the points a, b, and c. Use point subtraction and dot products. Each formula should stand on its own and not depend on the other formulas.



Distance a to x: Normal vector from a pointing at c:

$$\frac{c-a}{|c-a|}$$

Distance from a to x is given by the projection of $b-a$ onto $c-a$:

$$(b-a)\cdot\frac{c-a}{|c-a|}$$

Distance b to x: Use Pythagoras formula with the distances between a and b and a and x:

$$\sqrt{|a-b|^2-\left((b-a)\cdot\frac{c-a}{|c-a|}\right)^2}$$

Distance c to x: Similar to distance a to x:

$$(b-c)\cdot\frac{a-c}{|a-c|}$$

2. For each of the following implicitly defined quadric surfaces, find fomulas for the coefficients for the quadratic equation, $at^2+bt+c=0$ to determine the value of t where a ray defined by $p+tv$ intersects the surface.

   (a) Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$

$$\begin{aligned}
\frac{(p_x+tv_x)^2}{a^2}+\frac{(p_y+tv_y)^2}{b^2}-(p_z+tv_z) &= \left(\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}\right)t^2+\left(\frac{2v_x}{a^2}+\frac{2v_y}{b^2}-v_z\right)t+\left(\frac{p_x^2}{a^2}+\frac{p_y^2}{b^2}-p_z\right)\\
a &= \left(\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}\right)t^2\\
b &= \left(\frac{2v_x}{a^2}+\frac{2v_y}{b^2}-v_z\right)\\
c &= \left(\frac{p_x^2}{a^2}+\frac{p_y^2}{b^2}-p_z\right)
\end{aligned}$$

(b) Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z \; = 0$$

$$\begin{aligned} a &= \left(\frac{v_x^2}{a^2} - \frac{v_y^2}{b^2}\right) t^2 \\ b &= \left(\frac{2v_x}{a^2} - \frac{2v_y}{b^2} - v_z\right) \\ c &= \left(\frac{p_x^2}{a^2} - \frac{p_y^2}{b^2} - p_z\right) \end{aligned}$$

(c) Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} \; = 1$$

$$\begin{aligned} a &= \left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2}\right) t^2 \\ b &= \left(\frac{2v_x}{a^2} + \frac{2v_y}{b^2} - \frac{2v_z}{c^2}\right) \\ c &= \left(\frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2}\right) - 1 \end{aligned}$$

(d) Elliptic hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\begin{aligned} a &= \left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_z^2}{c^2}\right) t^2 \\ b &= \left(\frac{2v_x}{a^2} + \frac{2v_y}{b^2} - \frac{2v_z}{c^2}\right) \\ c &= \left(\frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} - \frac{p_z^2}{c^2}\right) + 1 \end{aligned}$$

3. Find a formula for a vector normal to each of the following surfaces, given a point (x, y, z) on the surface.

(a) Elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$$

$$\nabla f(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, -1 \right\rangle$$

(b) Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z \; = 0$$

$$\nabla f(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{-2y}{b^2}, -1 \right\rangle$$

(c) Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

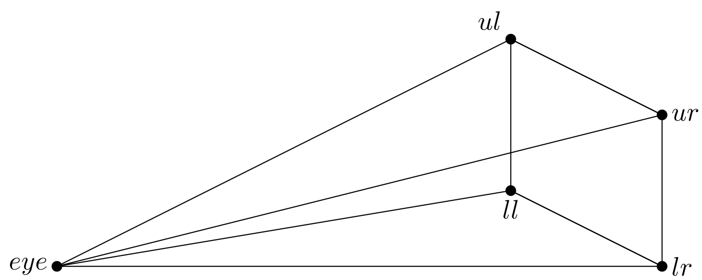
$$\nabla f(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{-2z}{c^2} \right\rangle$$

(d) Elliptic hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\nabla f(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{-2z}{c^2} \right\rangle$$

4. Suppose we specify a camera by five points: the eye point and the four corners of the image plane (upper left, upper right, lower left, lower right), as in the figure below (the left side of the image plane is deeper into the picture than the right side).



Given a position in the image plane defined by x and y, each scaled to $[0, 1]$, and with the origin of the image plane understood as the lower left corner, write an expression giving the vector for a ray from the eye to that point in the image plane. You do not need to normalize the vector.

$$(ll + x(lr - ll) + y(ul - ll)) - eye$$

5. Given a camera specified as in the lecture notes, with an eye point, normalized right, up, and foward vectors, and scalars depth, width and height, $\langle p, r, u, f, d, w, h \rangle$, write expressions for each of the five points in the camera representation from the previous problem.

(a)

$$eye = eye$$

(b)

$$ul = eye + d(f) - (w/2)r + (h/2)u$$

(c)

$$ur = eye + d(f) + (w/2)r + (h/2)u$$

(d)

$$ll = eye + d(f) - (w/2)r - (h/2)u$$

(e)

$$lr = eye + d(f) + (w/2)r - (h/2)u$$