Linear Algebra Notes

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Online Resources

Readings

- http://chortle.ccsu.edu/vectorlessons/vectorindex.html
- http://mathforum.org/mathimages/index.php/Math_for_Computer_Graphics_and_Computer_Vision
- http://cs229.stanford.edu/section/cs229-linalg.pdf
- http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/, Computer graphics
- http://joshua.smcvt.edu/linearalgebra/

Videos

- http://www.khanacademy.org/math/linear-algebra

Matrices

- A matrix is a set of scalars organized into rows and columns.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrix addition, subtraction, multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiplication is *not* commutative! $MN \neq NM$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \dots \\ \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & \dots \\ \dots & \dots \end{bmatrix}$$

Mathematical Vectors

$$\begin{aligned} \vec{v} &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= [a \; b \; c]^T \\ &= (a, b, c) \end{aligned}$$

- A vector is an N row by 1 column matrix.
- We will use mathematical vectors to represent both *points* and *vectors* in space.

Matrices as transforms

- Multiplication of an N-vector by an $N \times N$ matrix *on the left* changes it into another N-vector.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Matrix inverses

- Identity matrix:
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- $AI = IA = A$
- Some matrices have an inverse: $AA^{-1} = A^{-1}A = I$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

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Determinant of a matrix

$$
\begin{aligned}
A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
\det(A) &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\
&= ad - bc \\
A^{-1} &= \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{\det(A)}
\end{aligned}
$$

Determinant of a matrix

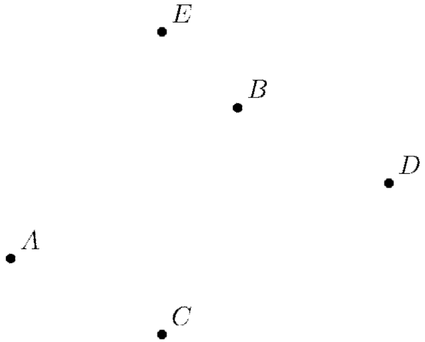
$$
\begin{aligned}
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= (aei - afh) - (bdi - bfg) + (cdh - ceg) \\
&= aei + bfg + cdh - afh - bdi - ceg
\end{aligned}
$$

Inverse of a matrix

$$\left[\begin{array}{cccccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

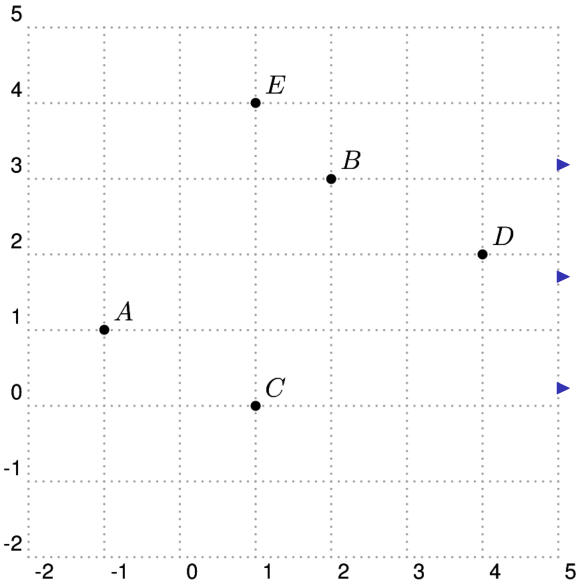
- Stick the identity on the right.
- Add multiples of one row to another until the identity is on the left.
- The inverse is now on the right

Points in Space



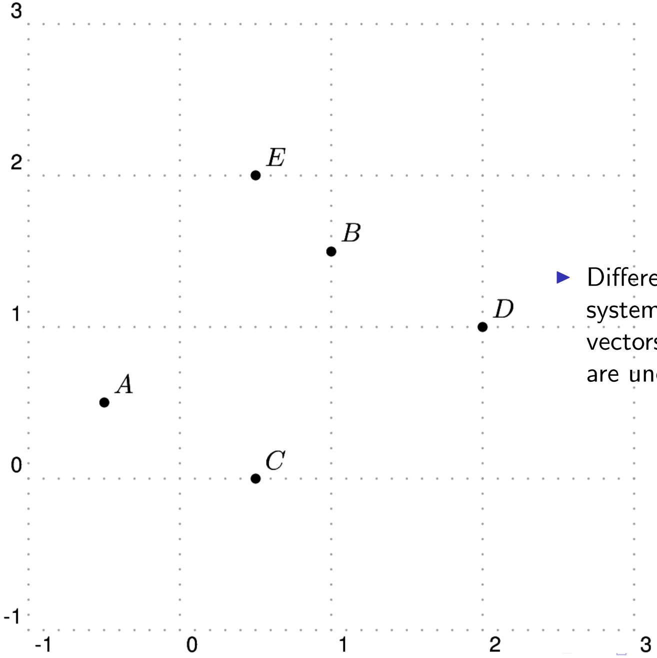
- Points exist in space without a coordinate system.
- But with only labels it's difficult to compute with them.

Points in a Coordinate System



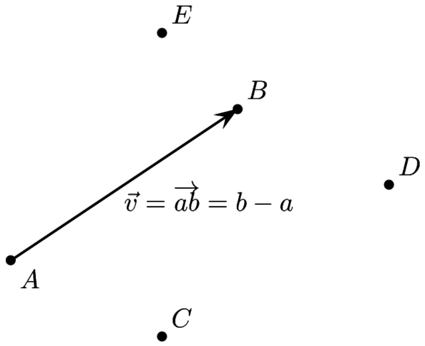
- A coordinate system gives positions to points.
- Relates *points* to *tuples of numbers*, or *mathematical vectors*.
- However, points are *not* vectors!

Different Coordinates



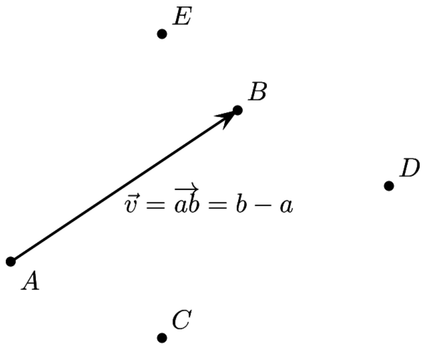
- Different coordinate systems give different vectors, but the *points* are unchanged.

Physical vectors are differences between points.



- Physical vectors are *not* mathematical vectors.
- But given a coordinate system, you can represent the points as mathematical vectors, and then subtract.
- But these mathematical vectors are not the same thing!
- Different coordinate systems will give you different mathematical vectors for the *same* physical vector.

Points and vectors are not the same thing!



- A point is a position in space.
- A vector has a magnitude and a direction.
- The vector from a to b is the point difference: $\vec{v} = \overrightarrow{ab} = b - a$
- You can add two vectors.
- You *cannot* add two points!
- You can add points and vectors: $b = a + \vec{v} = a + (b - a)$

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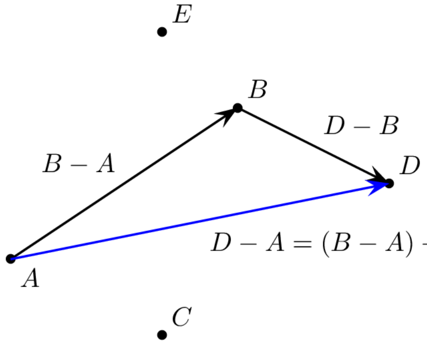
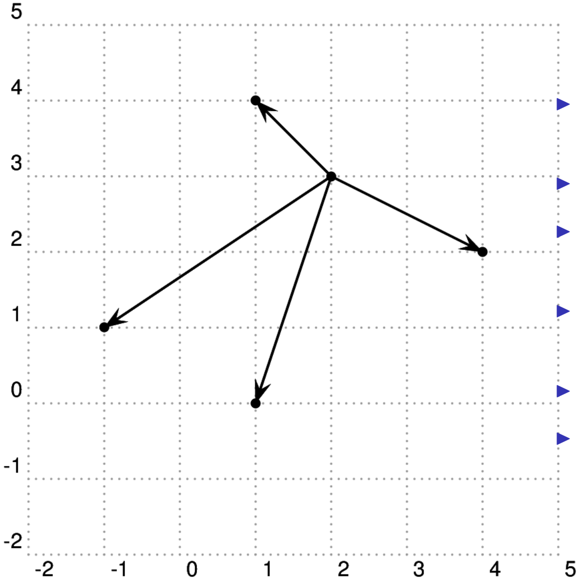
Vector Addition



$D - A = (B - A) + (D - B)$

- vector + vector = vector
- point + vector = point
- point - point = vector
- point + point = *nonsense*

Coordinates give mathematical vectors for physical vectors.



- Subtract the components.
- $(1, 4) - (2, 3) = (-1, 1)$
- $(-1, 1) - (2, 3) = (-1, -2)$
- $(1, 0) - (2, 3) = (-1, -3)$
- $(4, 2) - (2, 3) = (2, -1)$
- Note: we subtract two *points* to get a *vector*.

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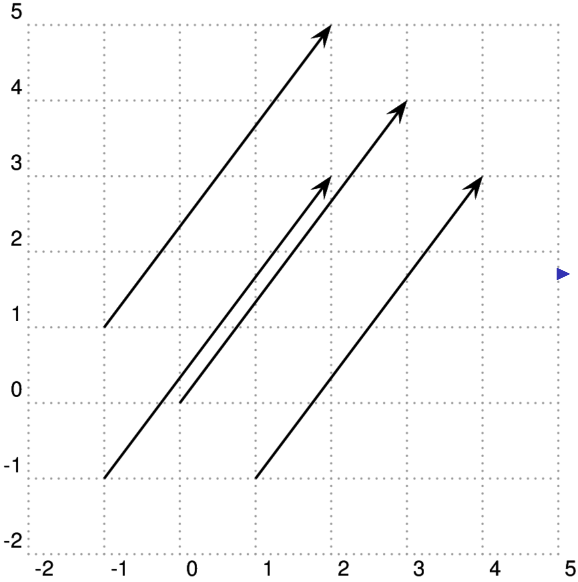
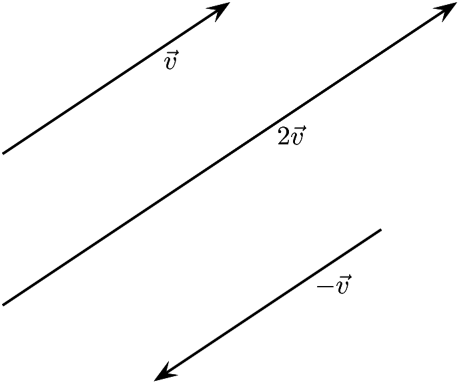
Vectors do not have positions



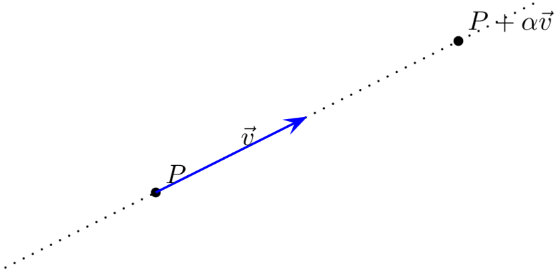
- Each of these vectors is the *same* vector.

Vectors can be multiplied by scalars



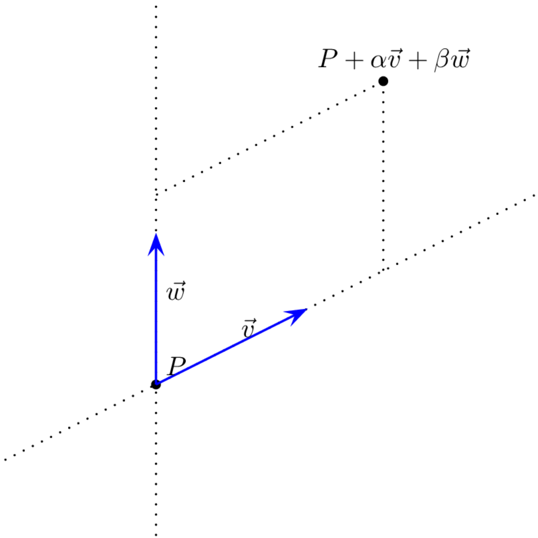
- Multiplication is repeated addition.

Lines



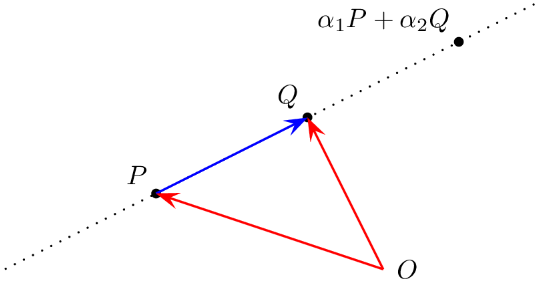
- The line through P in the direction v is the set of all points $P + \alpha v$ for some $\alpha \in \mathbb{R}$

Planes (in 3 dimensions)



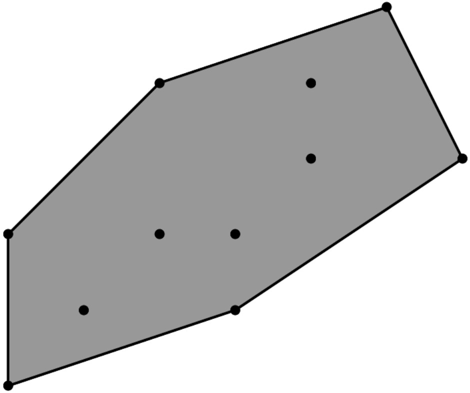
- The plane through P spanned by v and w is the set of all points $P + \alpha v + \beta w$ for some $\alpha, \beta \in \mathbb{R}$

Affine sums



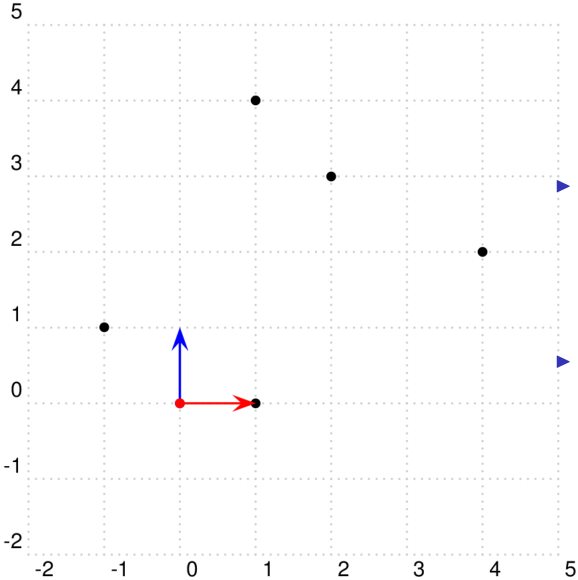
- $P + \alpha(Q - P)$
  $= (1 - \alpha)P + \alpha Q$
  $= \alpha_1 P + \alpha_2 Q$
- $\alpha_1 + \alpha_2 = 1$
- Think of each point as the vector from some arbitrary point:
  $P \equiv P - O$
  $Q \equiv Q - O$
- If $0 \leq \alpha_i$ then the point is between P and Q.

Convex hull



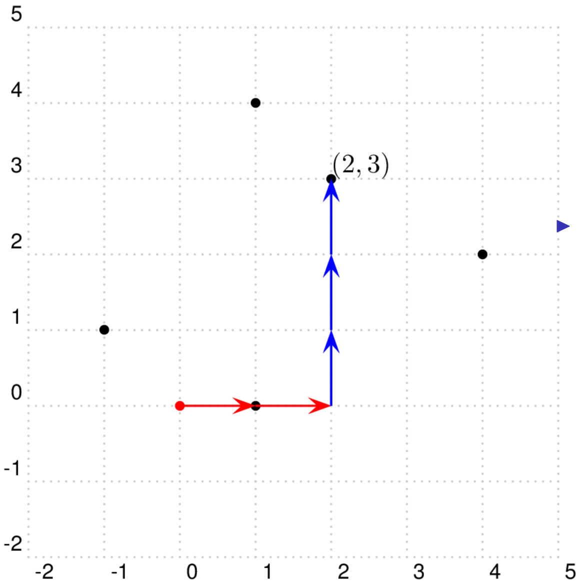
- $P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$
- $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$
- $0 \leq \alpha_i$

Frames



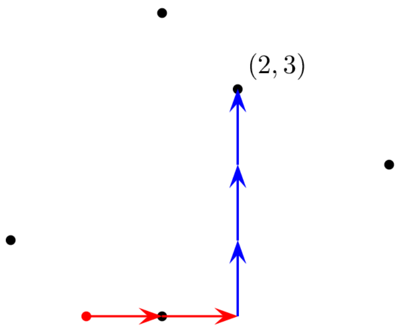
- A coordinate system can be thought of as a single point, the *origin*, and a set of *basis vectors*.
- Such a set is called a *frame*.

Frames



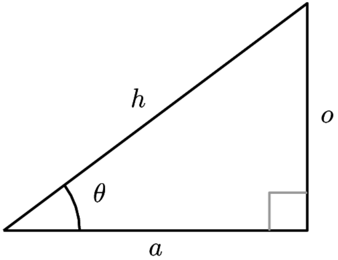
- The coordinates of a point are how many copies of the basis vectors you have to add to the origin.

Frames



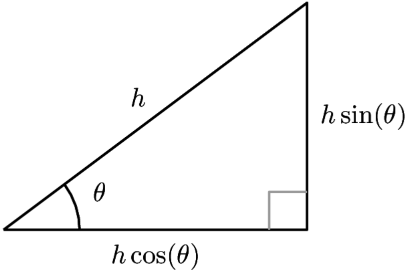
- Note that a frame gives sense to coordinates without anything other than points and vectors.
- A coordinate system is nothing more than an origin and a set of basis vectors, a *frame*.
- An *orthonormal* frame is one in which all the vectors are of unit length and perpendicular to each other.

Trigonometry



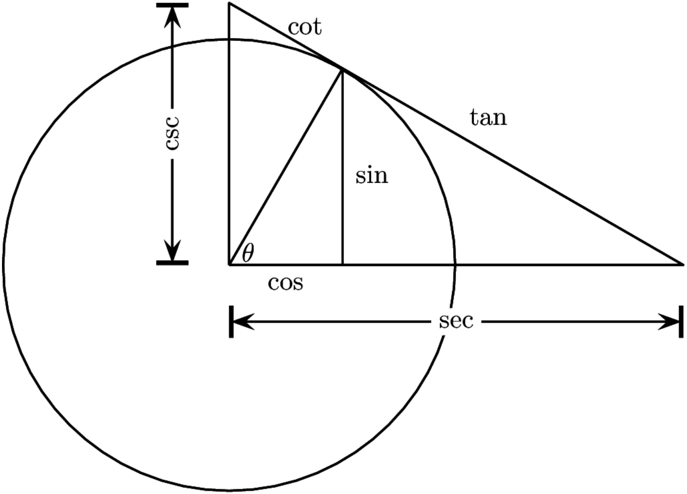
- $\sin(\theta) = o/h$
- $\cos(\theta) = a/h$
- $\tan(\theta) = o/a$

Trigonometry

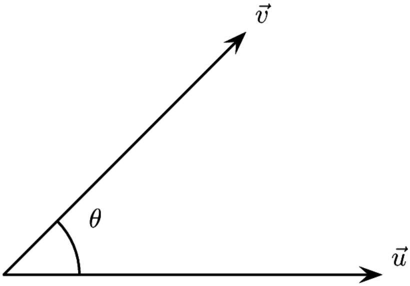


- $\tan(\theta) = \sin(\theta)/\cos(\theta)$

Trigonometry

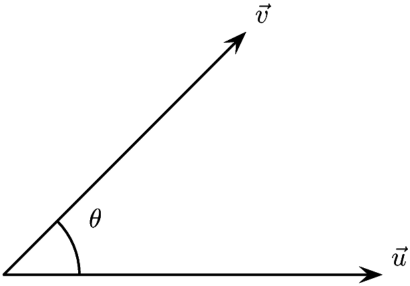


Dot product (Inner product)



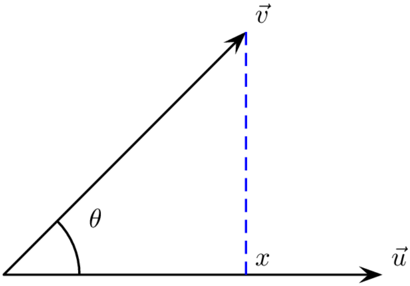
- $u \cdot v = \cos(\theta)|u||v|$

Dot product (Inner product)



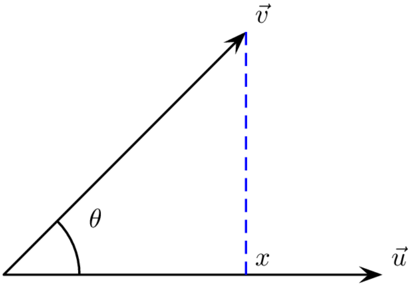
- $u \cdot v = \cos(\theta)|u||v|$
- $|u| = \sqrt{u \cdot u}$

Projection of one vector on another



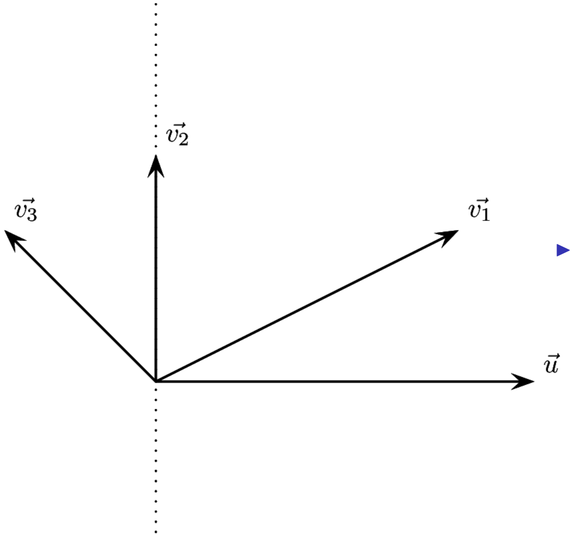
- What is x?

Projection of one vector on another



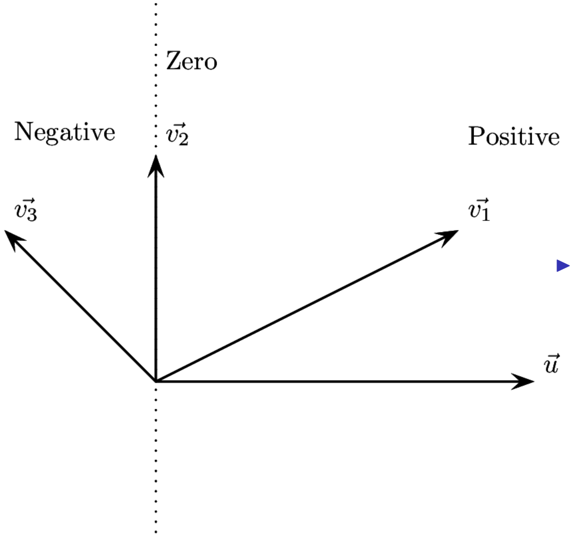
- $x = \cos(\theta)|v|$
- $x = u \cdot v / |u|$

Same direction, opposite direction



- What is the sign of $u \cdot v_i$?

Same direction, opposite direction



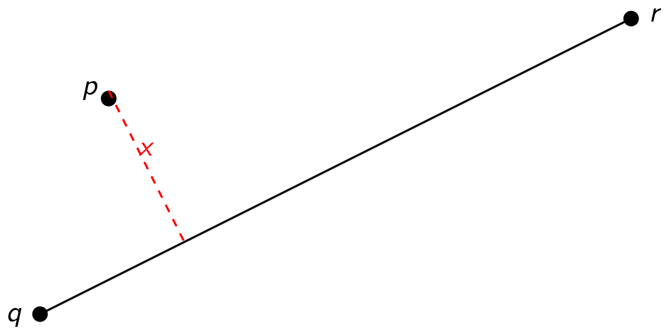
- Sign of $u \cdot v_i$

AMAZING theorem about the dot product.

- In any coordinate system whatsoever:

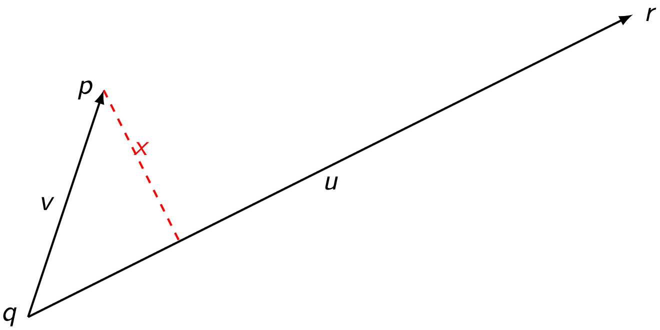
$$
\begin{aligned}
u \cdot v &= (u_x, u_y, u_z) \cdot (v_x, v_y, v_z) \\
&= u_x v_x + u_y v_y + u_z v_z \\
&= [u_x \; u_y \; u_z] \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \\
&= u^T v
\end{aligned}
$$

Example use of the dot product



- An object at p is approaching a wall determined by points q and r.
- How far away is the wall?

How far away is the wall?

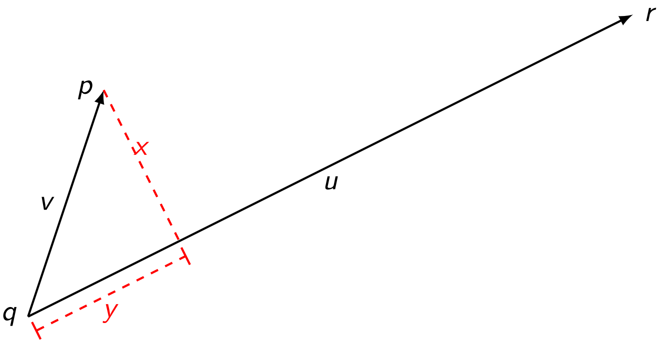


- Let's get some vectors:

$$\begin{aligned} v &= p - q \\ u &= r - q \end{aligned}$$

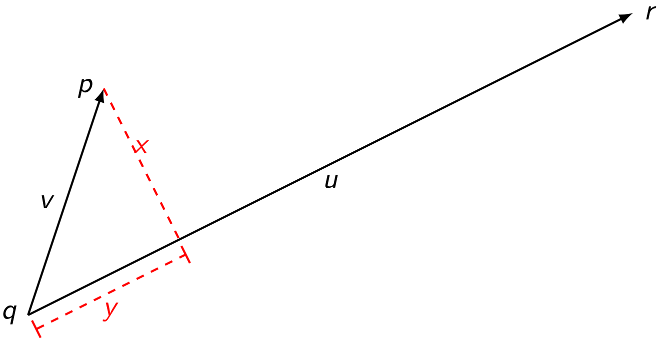
- Now what?

How far away is the wall?



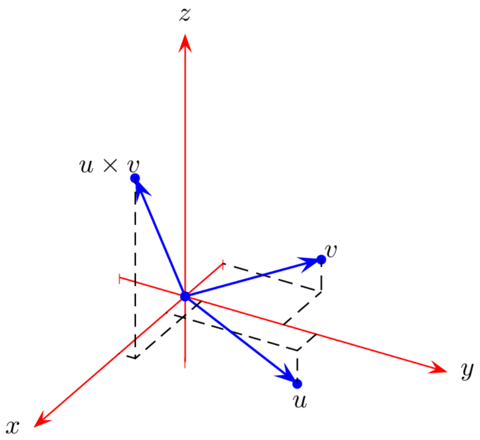
- Can we find y?
- Will that give us x?

How far away is the wall?



$$
\begin{aligned}
y &= \frac{u}{|u|} \cdot v \\
x &= \left| p - y \frac{u}{|u|} \right| \\
&= \sqrt{|v|^2 - y^2}
\end{aligned}
$$

Cross product (vector product)



- A vector at right angles to u and v.
- $u \times v = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$
- Mnemonic:

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- $|u \times v| = |u||v| \sin(\theta)$