Linear Algebra Notes

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Fall 2011

Online Resources

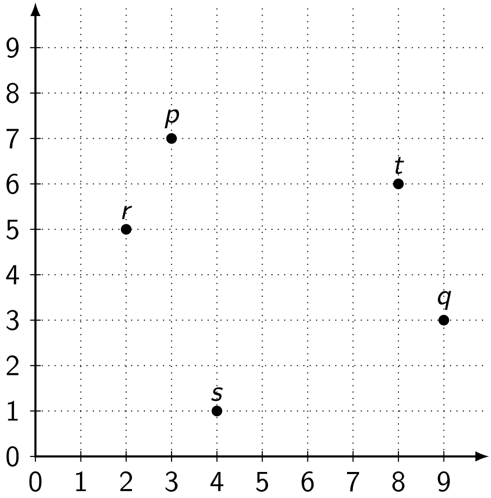
Readings

- http://chortle.ccsu.edu/vectorlessons/vectorindex.html
- http://mathforum.org/mathimages/index.php/Math_for_Computer_Graphics_and_Computer_Vision
- http://cs229.stanford.edu/section/cs229-linalg.pdf
- http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/, Computer graphics
- http://joshua.smcvt.edu/linearalgebra/

Videos

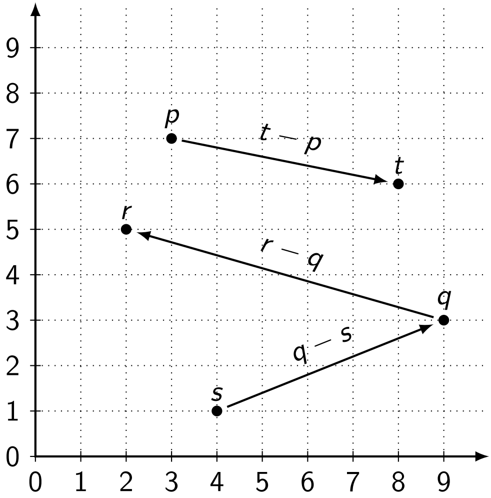
- http://www.khanacademy.org/math/linear-algebra

Points in space



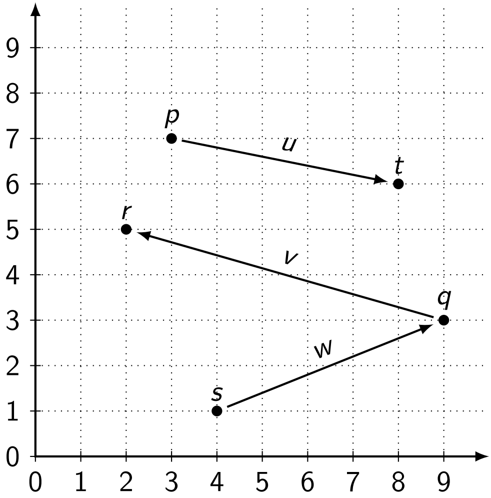
- Points exist in space without a coordinate system.
- We introduce coordinates to make it easier to compute with them.
- But the coordinates are arbitrary, so long as they're consistent.

Vectors



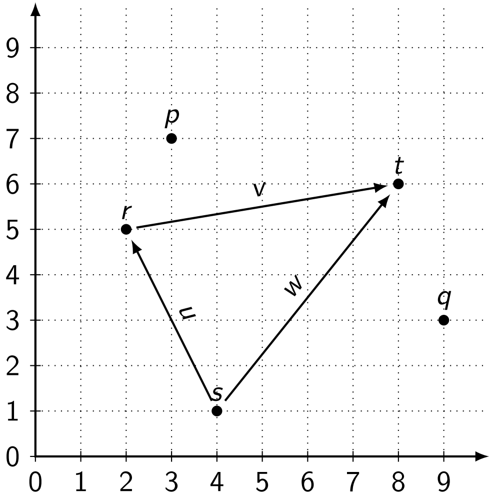
- Subtracting two points results in a vector.
- Both points and vectors in n dimensions are represented by n real numbers.

Points *vs.* Vectors



- A point is a position.
- A vector is a magnitude and a direction.

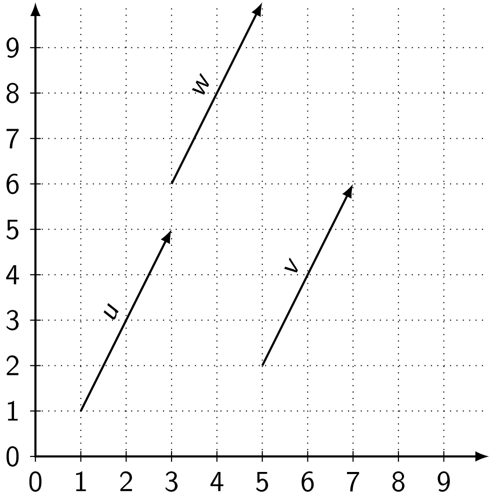
Vector Addition



$$
\begin{aligned}
u &= r - s \\
v &= t - r \\
w &= t - s \\
w &= u + v \\
r &= s + u \\
t &= r + v \\
t &= s + w
\end{aligned}
$$

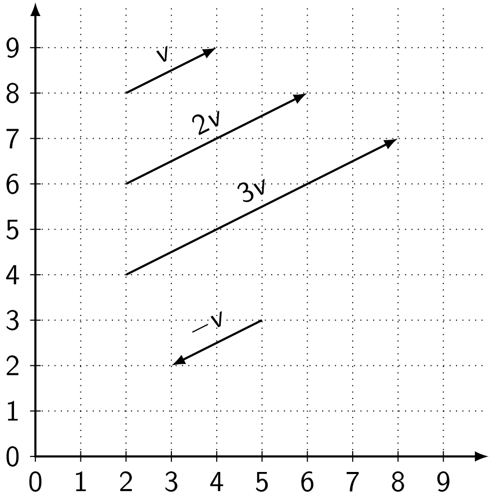
You can't add points!

Vectors do not have positions



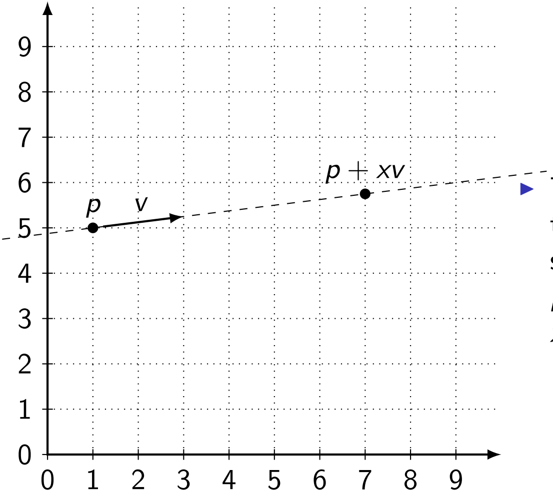
- These are all the same vector
- $u = v = w$

Vectors multiplied by scalars



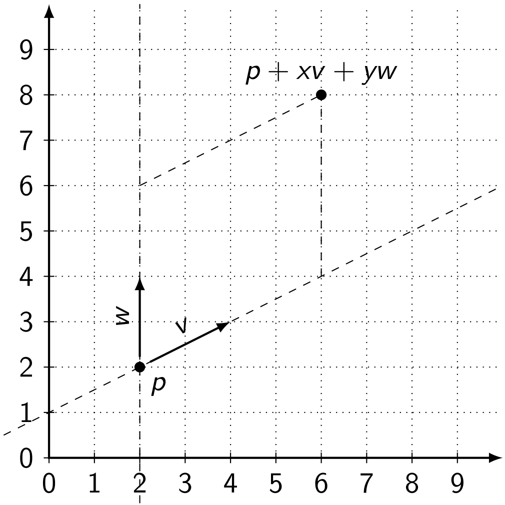
- Multiplication is repeated addition.

Lines defined by point and vector



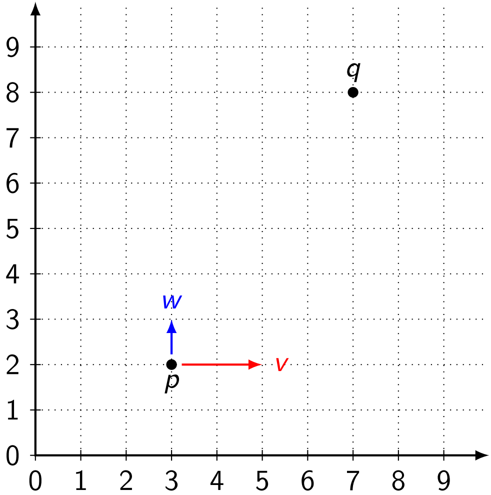
- The line through p in the direction v is the set of all points $p + xv$ for some $x \in \mathbb{R}$

Planes defined by point and two vectors



- The plane through p aligned with v and w is the set of all points $p + xv + yw$ for some $x, y \in \mathbb{R}$

Frames



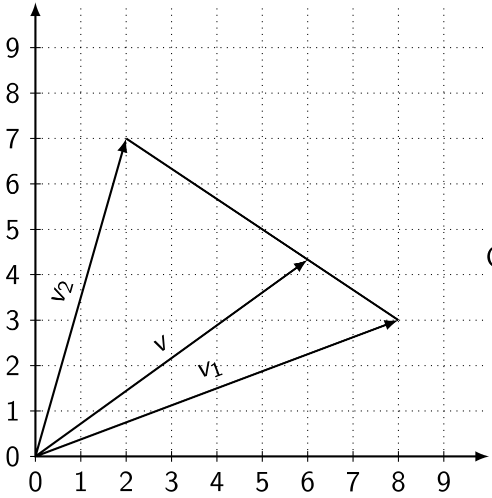
- A tuple consisting of a point (the origin) and n vectors is a *frame*.

$$f = \langle p, v, w \rangle$$

- A frame gives coordinates to points.
- $q = p + 2v + 6w$
- $q = (2, 6)_f$

An *orthonormal* frame is one in which all the vectors are unit length and perpendicular to each other.

Affine sums of vectors



$$0 \leq a \leq 1$$

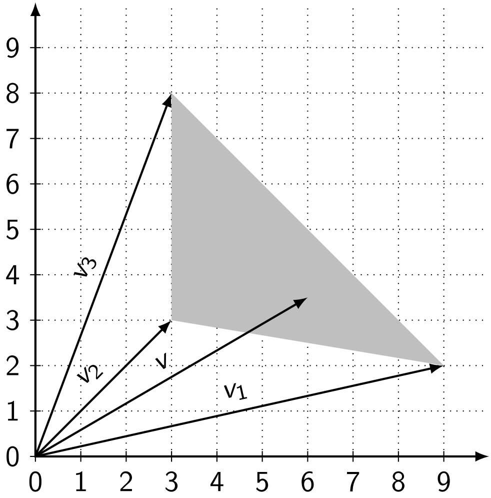
$$\begin{aligned} v &= (1-a)v_1 + av_2 \\ &= v_1 + a(v_2 - v_1) \end{aligned}$$

Or:

$$0 \leq a_i \leq 1$$

$$\begin{aligned} 1 &= a_1 + a_2 \\ v &= a_1 v_1 + a_2 v_2 \end{aligned}$$

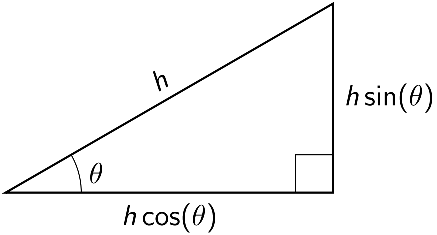
Affine sums of vectors



$$0 \leq a_i \leq 1$$

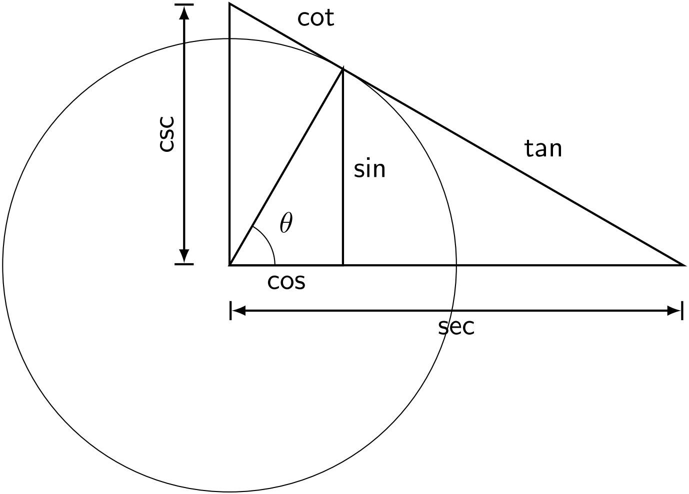
$$\begin{aligned} 1 &= a_1 + a_2 + a_3 \\ v &= a_1 v_1 + a_2 v_2 + a_3 v_3 \end{aligned}$$

Trigonometry

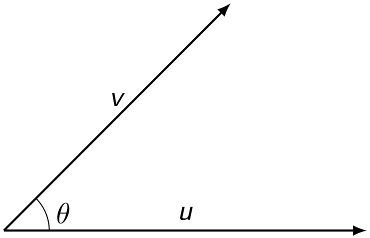


$$
\begin{aligned}
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{1}{\cot(\theta)} \\
\sin(\theta) &= \frac{1}{\csc(\theta)} \\
\cos(\theta) &= \frac{1}{\sec(\theta)}
\end{aligned}
$$

More Trigonometry



Dot product



In 2D

$$
\begin{aligned}
u \cdot v &= \cos(\theta)|u||v| \\
&= u_x v_x + u_y v_y
\end{aligned}
$$

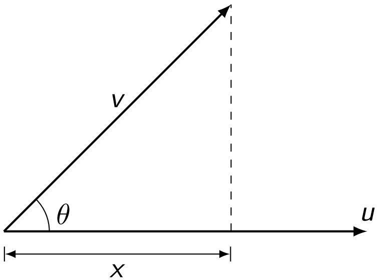
In 3D

$$
\begin{aligned}
u \cdot v &= \cos(\theta)|u||v| \\
&= u_x v_x + u_y v_y + u_z v_z
\end{aligned}
$$

Note:

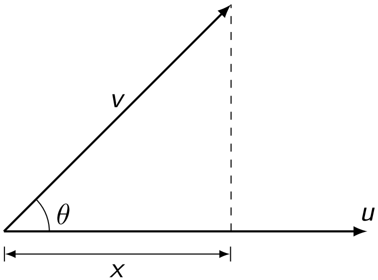
$$
\begin{aligned}
u \cdot u &= \cos(\theta)|u||u| \\
&= u_x u_x + u_y u_y + u_z u_z \\
&= |u|^2
\end{aligned}
$$

Projection of one vector on another



- What is x?

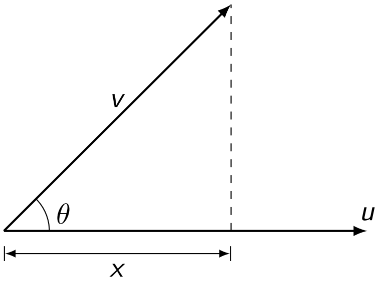
Projection of one vector on another



- What is x?

$$x = \cos(\theta)|v|$$

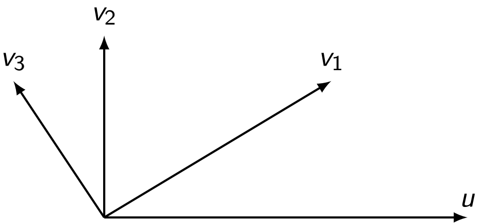
Projection of one vector on another



- What is x?

$$
\begin{aligned}
x &= \cos(\theta)|v| \\
v \cdot u &= \cos(\theta)|u||v| \\
\cos(\theta) &= \frac{u \cdot v}{|u||v|} \\
x &= \frac{u \cdot v}{|u|}
\end{aligned}
$$

Same direction, opposite direction



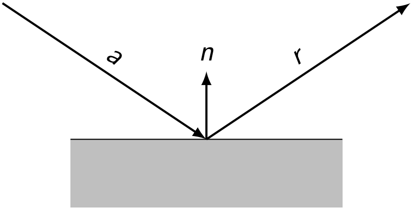
- What is the sign of $u \cdot v_i$?

AMAZING theorem about the dot product.

- In any coordinate system whatsoever:

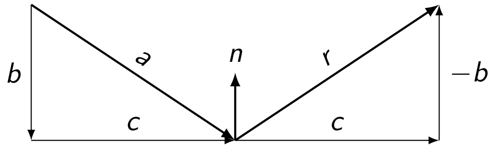
$$
\begin{aligned}
u \cdot v &= (u_x, u_y, u_z) \cdot (v_x, v_y, v_z) \\
&= u_x v_x + u_y v_y + u_z v_z \\
&= [u_x \; u_y \; u_z] \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \\
&= u^T v
\end{aligned}
$$

Example use of dot product: reflected ray



- How do we reflect ray a about normal n to get r?

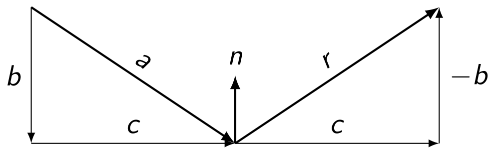
Reflected ray



$$
\begin{aligned}
a &= b + c \\
r &= -b + c
\end{aligned}
$$

How do we find b and c?

Reflected ray



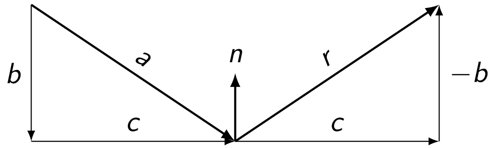
$$\begin{aligned} a &= b + c \\ r &= -b + c \end{aligned}$$

How do we find b and c?
Assume $|n| = 1$

$$\begin{aligned} b &= (a \cdot n)n \\ c &= a - b \end{aligned}$$

What if $|n| \neq 1$?

Reflected ray



$$\begin{aligned} a &= b + c \\ r &= -b + c \end{aligned}$$

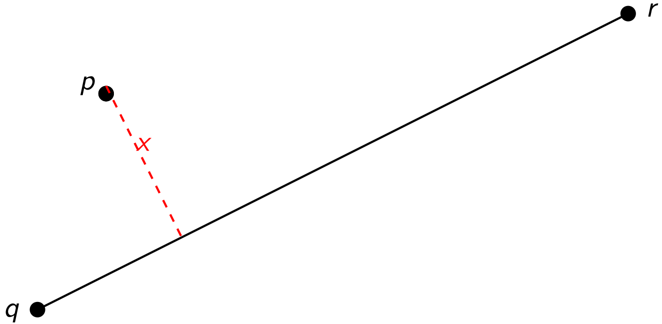
How do we find b and c?
Assume $|n| = 1$

$$\begin{aligned} b &= (a \cdot n)n \\ c &= a - b \end{aligned}$$

What if $|n| \neq 1$?

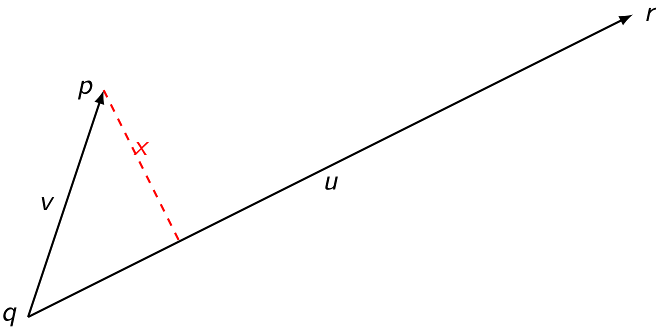
$$b = \frac{(a \cdot n)}{(n \cdot n)} n$$

Example use of the dot product



- An object at p is approaching a wall determined by points q and r.
- How far away is the wall?

How far away is the wall?

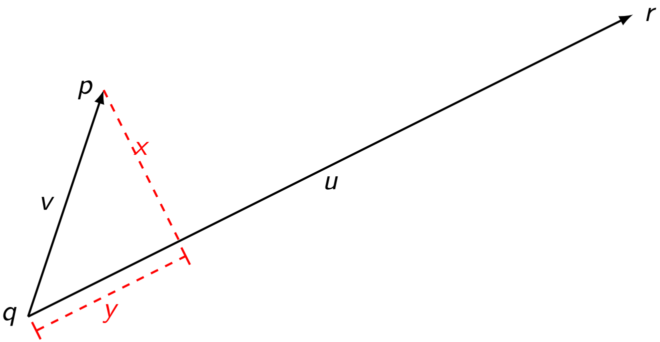


- Let's get some vectors:

$$\begin{aligned} v &= p - q \\ u &= r - q \end{aligned}$$

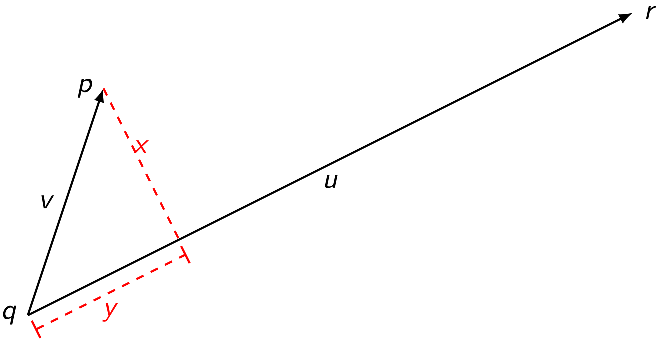
- Now what?

How far away is the wall?



- Can we find y?
- Will that give us x?

How far away is the wall?



$$
\begin{aligned}
y &= \frac{u}{|u|} \cdot v \\
x &= \left| p - y \frac{u}{|u|} \right| \\
&= \sqrt{|v|^2 - y^2}
\end{aligned}
$$