

Parametric Surfaces

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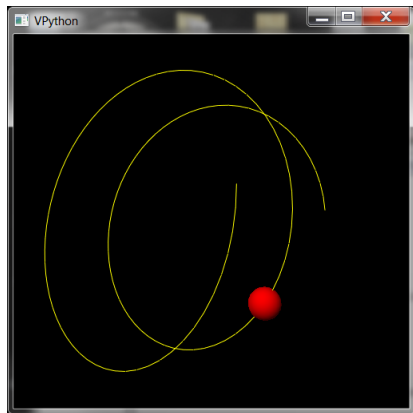
November 3, 2015

Online Resources

Readings

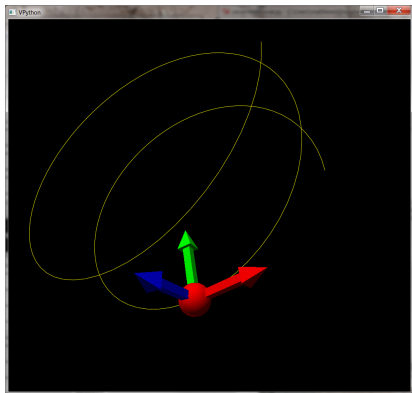
- ▶ <https://www.math.duke.edu//education/ccp/materials/mvcalc/parasurfs/para1.html>
- ▶ http://en.wikipedia.org/wiki/Parametric_surface
- ▶ <http://www.math.uri.edu/~bkaskosz/flashmo/tools/parsur/>
- ▶ <http://people.cs.clemson.edu/~dhouse/courses/405/notes/implicit-parametric.pdf>
- ▶ <http://msenux.redwoods.edu/Math4Textbook/Plotting/ParametricSurfaces.pdf>

Parametric Curves



$$f(t) = (10 \cos(t), 10 \sin(t), t)$$

Parametric Curves: Tangent, Normal, Binormal



$$f(t) = (10 \cos(t), 10 \sin(t), t)$$

$$\frac{df}{dt} = (-10 \sin(t), 10 \cos(t), 1)$$

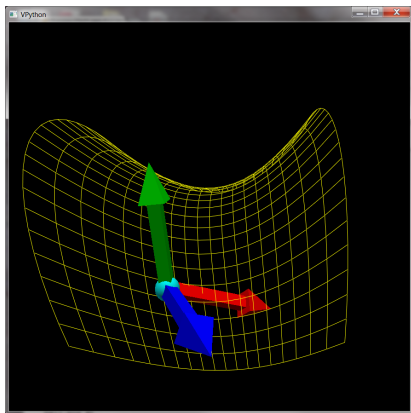
$$\frac{d^2f}{dt^2} = (-10 \cos(t), -10 \sin(t), 0)$$

$$\frac{d^3f}{dt^3} = (10 \sin(t), -10 \cos(t), 0)$$

$$\frac{df}{dt} \times \frac{d^2f}{dt^2} \propto (\sin(t), \cos(t), 10)$$

$$\frac{df}{dt} \cdot \frac{d^2f}{dt^2} = \frac{d^2f}{dt} \cdot \frac{d^3f}{dt^3} = \frac{df}{dt} \cdot \frac{d^3f}{dt^3} = 0$$

Parametric Surfaces



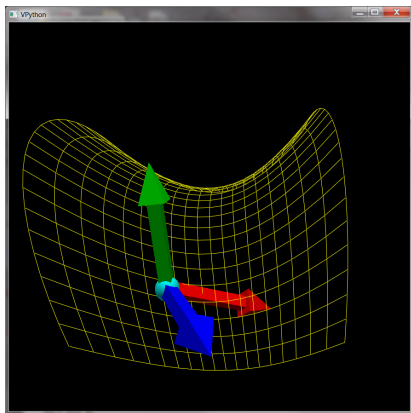
Explicit:

$$f(s, t) = (s, t, s^2 - t^2)$$

Implicit:

$$F(x, y, z) = 0$$

Parametric Surfaces



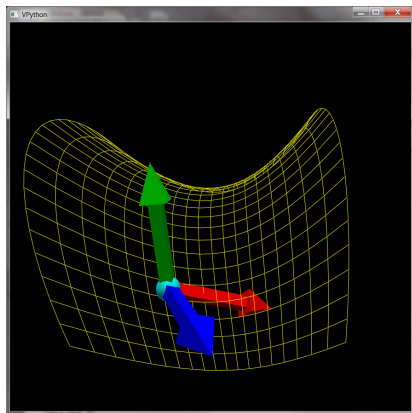
Explicit:

$$f(s, t) = (s, t, s^2 - t^2)$$

Implicit:

$$\begin{aligned} F(x, y, z) &= 0 \\ x^2 - y^2 &= z \end{aligned}$$

Parametric Surfaces



Explicit:

$$f(s, t) = (s, t, s^2 - t^2)$$

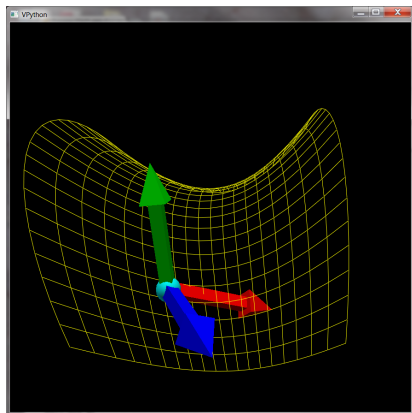
Implicit:

$$F(x, y, z) = 0$$

$$x^2 - y^2 = z$$

$$x^2 - y^2 - z = 0$$

Parametric Surfaces: Normal, Tangent, Binormal



$$f(s, t) = (s, t, s^2 - t^2)$$

$$\frac{\partial f}{\partial s} = (1, 0, 2s)$$

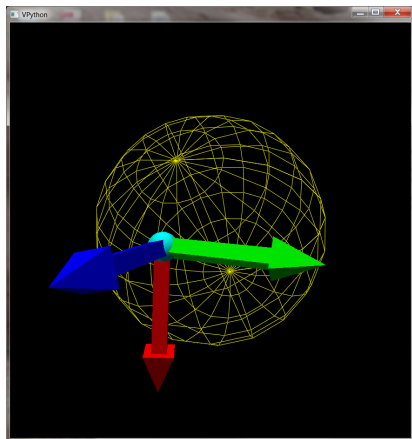
$$\frac{\partial f}{\partial t} = (0, 1, -2t)$$

$$\frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} = (-2s, 2t, 1)$$

$$F(x, y, z) = x^2 - y^2 - z$$

$$\nabla F(x, y, z) = (2x, -2y, -1)$$

Parametric Surfaces: Sphere



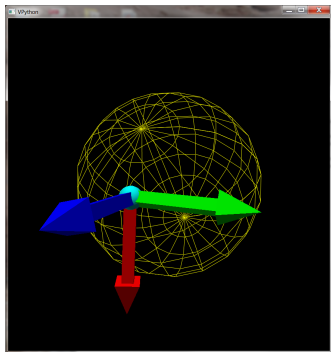
Explicit:

$$f(s, t) = \begin{pmatrix} \cos(t) \cos(s), \\ \cos(t) \sin(s), \\ \sin(t) \end{pmatrix}$$

Implicit:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + y^2 + z^2 - 1 &= 0 \end{aligned}$$

Parametric Surfaces: Normal, Tangent, Binormal



$$\begin{aligned} F(x, y, z) &= x^2 + y^2 + z^2 - 1 \\ \nabla F(x, y, z) &= (2x, 2y, 2z) \end{aligned}$$

$$\begin{aligned} f(s, t) &= (\cos(t) \cos(s), \cos(t) \sin(s), \sin(t)) \\ \frac{\partial f}{\partial s} &= (-\cos(t) \sin(s), \cos(t) \cos(s), 0) \\ \frac{\partial f}{\partial t} &= (-\sin(t) \cos(s), -\sin(t) \sin(s), \cos(t)) \end{aligned}$$

Approximations to Tangent, Normal, Binormal

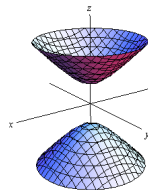
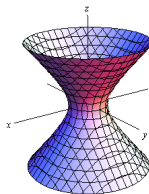
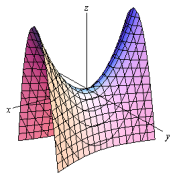
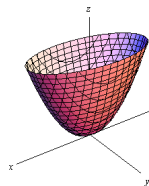
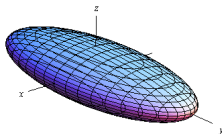
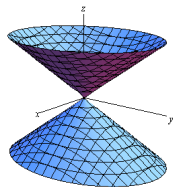
$$\frac{\partial f}{\partial s} \approx \frac{f(s + \epsilon, t) - f(s, t)}{|f(s + \epsilon, t) - f(s, t)|}$$

$$\frac{\partial f}{\partial t} \approx \frac{f(s, t + \epsilon) - f(s, t)}{|f(s, t + \epsilon) - f(s, t)|}$$

$$\text{Normal} = \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial g}$$

Quadrics

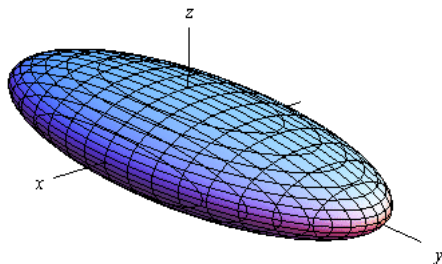
<http://tutorial.math.lamar.edu/Classes/CalcIII/QuadricSurfaces.aspx>



Ellipsoid

$$x^2 + y^2 + z^2 = 1$$

$$(\cos(v) \cos(u), \cos(v) \sin(u), \sin(v))$$
$$v \in [-\pi/2, \pi/2], u \in [-\pi, \pi]$$

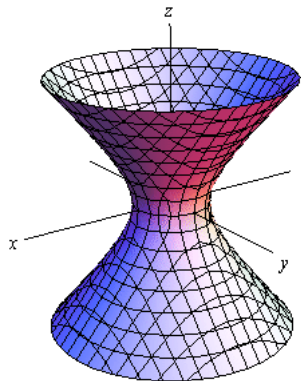


Equations from *Computer Graphics Using Open GL, 2E*

Hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

$$(\sec(v) \cos(u), \sec(v) \sin(u), \tan(v))$$
$$v \in [-\pi/2, \pi/2], u \in [-\pi, \pi]$$

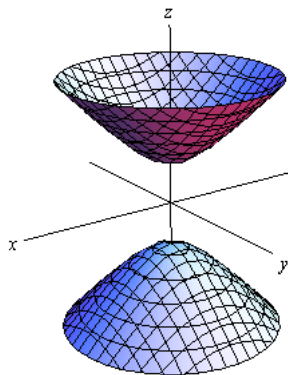


Hyperboloid of two sheets

$$x^2 - y^2 - z^2 = 1$$

$$(\sec(v) \cos(u), \sec(v) \tan(u), \tan(v))$$
$$v \in [-\pi/2, \pi/2], [\pi/2, 3\pi/2], u \in [-\pi, \pi]$$

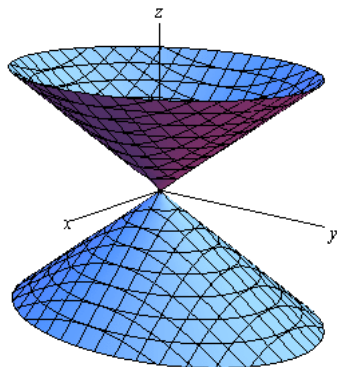
?



Elliptic cone

$$x^2 + y^2 - z^2$$

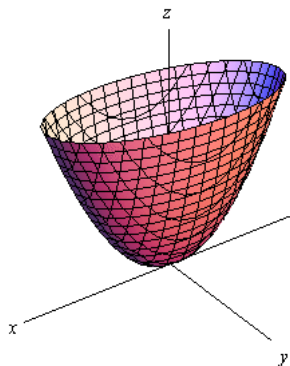
$$(v \cos(u), v \sin(u), v)$$
$$v \in [-\infty, +\infty], u \in [-\pi, \pi]$$



Elliptic paraboloid

$$x^2 + y^2 - z$$

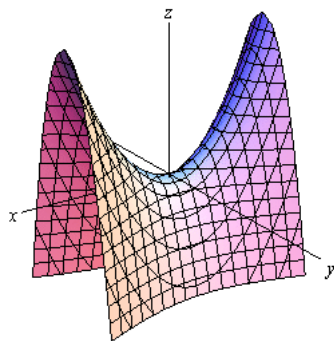
$$(v \cos(u), v \sin(u), v^2)$$
$$v \in [0, +\infty], u \in [-\pi, \pi]$$



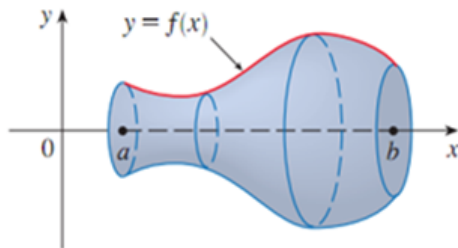
Hyperbolic paraboloid

$$-x^2 + y^2 - z$$

$$(v \tan(u), v \sec(u), v^2)$$
$$v \in [0, +\infty], u \in [-\pi, \pi]$$



Surfaces of Revolution



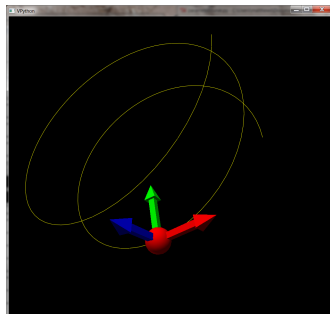
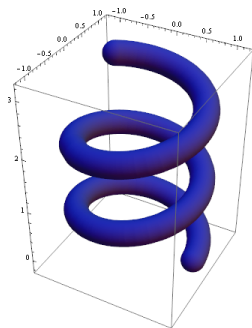
(a) Surface of revolution



If $y = f(x)$ is determined by a series of points $(x_0, y_0), \dots, (x_n, y_n)$, then let $s \in [a, \dots, b]$ and $t \in [0, 2\pi]$, and the parametric surface is

$$\begin{aligned} g(s, t) &= (s, f(s) \cos(t), f(s) \sin(t)) \\ \frac{\partial g}{\partial s} &= (1, f'(s) \cos(t), f'(s) \sin(t)) \\ \frac{\partial g}{\partial t} &= (s, -f(s) \sin(t), f(s) \cos(t)) \end{aligned}$$

Tubes along 3D curves



- ▶ Assume the curve is parameterized by $f(s)$, $s \in [a, b]$
- ▶ Find the tangent, normal, and binormal to the curve: t, n, b
- ▶ The parametric surface, parameterized by $s \in [a, b]$, $t \in [0, 2\pi]$:

$$f(s) + \cos(t)n + \sin(t)b$$

- ▶ Could you extrude an arbitrary shape instead of a circle?