Equations 1D

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1 Mass conservation

$$\begin{cases} \dfrac{\partial H}{\partial t} = 0 \\ \\ \dfrac{\partial \theta}{\partial t} = \dfrac{\partial}{\partial z}\left[K\left(\dfrac{\partial \psi}{\partial z} + 1\right)\right] \end{cases} \tag{1}$$

The conservation equation for the surface water may be discretized as:

$$H_i(\psi_i^{n+1}) - \Delta t\left[0 - K_{i-\frac{1}{2}}^n \frac{\psi_i^{n+1} - \psi_{i-1}^{n+1}}{\Delta z}\right] = d_i^n \tag{2}$$

where

$$d_i^n = H_i(\psi_i^n) + \Delta t[J^n - K_{i-\frac{1}{2}}^n] \tag{3}$$

and the function $H(\psi)$ is defined as

$$H(\psi) = \begin{cases} \psi & \text{if } \psi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Richards' equation is discretized as:

$$\theta_i^{n+1} - \Delta t\left[K_{i+\frac{1}{2}}^n \frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta z} - K_{i-\frac{1}{2}}^n \frac{\psi_i^{n+1} - \psi_{i-1}^{n+1}}{\Delta z}\right] = b_i^n \tag{5}$$

where

$$b_i^n = \theta_i^n + \Delta t\left[K_{i+\frac{1}{2}}^n - K_{i-\frac{1}{2}}^n\right] \tag{6}$$

$$\theta_i^n = \theta(\psi_i^n)\Delta z_i \tag{7}$$

Eq.(2) and Eq.(5) can be written in a metrix form:

$$V(\psi) + \mathbf{T}\psi = b \tag{8}$$

2 Conservation equation

Within the soil the energy conservation equation reads as:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left(\rho_w c_w J T - \lambda \frac{\partial T}{\partial z}\right) = 0 \tag{9}$$

where $J = -K\frac{\partial}{\partial z}(\psi+z)$, and the internal energy per unit volume $u = \rho_w c_w \theta(\psi)(T - T_{ref}) + (1-\theta_s)\rho_s c_s (T - T_{ref})$. In the above equation expliciting the time derivative one obtains

$$c_T \frac{\partial T}{\partial t} + \rho_w c_w J \frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left(\lambda(\theta)\frac{\partial T}{\partial z}\right) \tag{10}$$

By making use of up-wind scheme for convection part and centered difference for diffusion part we have:

$$\begin{aligned} c_{T_i}^{n+1} T_i^{n+1} = c_{T_i}^{n} T_i^{n} - \rho_w c_w \frac{\Delta t}{\Delta z_i}[\frac{1}{2}(J_{i+\frac{1}{2}}^{n+1})(T_{i+1}^{n+1} + T_i^{n+1}) - \frac{1}{2}|J_{i+\frac{1}{2}}^{n+1}|(T_{i+1}^{n+1} - T_i^{n+1}) \\ - \frac{1}{2}(J_{i-\frac{1}{2}}^{n+1})(T_i^{n+1} + T_{i-1}^{n+1}) + \frac{1}{2}|J_{i-\frac{1}{2}}^{n+1}|(T_i^{n+1} - T_{i-1}^{n+1})] \\ + \frac{\Delta t}{\Delta z_i}\left[\lambda_{i+\frac{1}{2}}^n \frac{T_{i+1}^{n+1} - T_i^{n+1}}{\Delta z_i} - \lambda_{i-\frac{1}{2}}^n \frac{T_i^{n+1} - T_{i-1}^{n+1}}{\Delta z_i}\right] \end{aligned} \tag{11}$$

The above scheme may be rewritten in a more concise form as

$$c_{T_i}^{n+1}\Delta z_i T_i^{n+1} = c_{T_i}^{n}\Delta z_i T_i^{n} - \Delta t\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right] \tag{12}$$

$$C_{T_i}^{n+1} T_i^{n+1} = C_{T_i}^{n} T_i^{n} - \Delta t\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right] \tag{13}$$

where

$$\begin{aligned} f_{i+\frac{1}{2}} = \rho_w c_w \left(\frac{1}{2}(J_{i+\frac{1}{2}}^{n+1})(T_{i+1}^{n+1} + T_i^{n+1}) - \frac{1}{2}\left| J_{i+\frac{1}{2}}^{n+1} \right|(T_{i+1}^{n+1} - T_i^{n+1})\right) \\ - \lambda_{i+\frac{1}{2}}^n \frac{T_{i+1}^{n+1} - T_i^{n+1}}{\Delta z_i} \end{aligned} \tag{14}$$

Remark:

- $C_{T_i}^n = c_{T_i}^n \Delta z_i = \rho_s c_s (1-\theta_s)\Delta z_i + \rho_w c_w \theta(\psi_i^n)\Delta z_i = \rho_s c_s(1-\theta_s)\Delta z_i + \rho_w c_w \theta_i^n$
- velocities to be used is that computed solving mass equation: this guarantees the max-min property.

The approach used to couple surface and subsurface flow can be used here to extend the energy equation to the surface. At the surface we have

$$\frac{\partial \rho_w c_w T}{\partial t} + \frac{\partial}{\partial z}\left(\rho_w c_w J T - \lambda\frac{\partial T}{\partial z}\right) + \rho_w c_w E_T = 0 \tag{15}$$

where E_T is the water representing the evapotranspiration. (To discretize the equation this term has to be the same computed through mass conservation equation)

$$\rho_w c_w \frac{\partial T}{\partial t} + \frac{\partial}{\partial z}\left(\rho_w c_w J\, T - \lambda \frac{\partial T}{\partial z}\right) + \rho_w c_w E_T = 0 \tag{16}$$

Integrating over the control volume:

$$H(\psi)\rho_w c_w \frac{\partial T}{\partial t} + \int \rho_w c_w J\, T - \lambda \frac{\partial T}{\partial z}\ dS + \rho_w c_w E_T = 0 \tag{17}$$

Discretizing the above equation and applying boundary conditions

$$\begin{aligned} H_i^{n+1}\rho_w c_w T_i^{n+1} &= H_i^n \rho_w c_w T_i^n - \rho_w c_w \Delta t [Rain^{n+1} T_a^{n+1} \\ &\quad - \frac{1}{2}(J_{i-\frac{1}{2}}^{n+1})(T_i^{n+1} + T_{i-1}^{n+1}) + \frac{1}{2}|J_{i-\frac{1}{2}}^{n+1}|(T_i^{n+1} - T_{i-1}^{n+1})] \\ +\Delta t &\left[R_{sw}^{n+1} + \epsilon_i \epsilon_a \sigma (T_a^{n+1})^4 - \epsilon_i \sigma (T_i^n)^4 + \rho_a c_p u_a \frac{T_a^{n+1} - T_i^{n+1}}{r_a} - \lambda_{i-\frac{1}{2}}^n \frac{T_i^{n+1} - T_{i-1}^{n+1}}{\Delta z_i}\right] \\ &\quad + \rho_w c_w E_T \Delta t \end{aligned} \tag{18}$$

If there is no ponding water the above equation reduces to the energy balance of the surface in which the the storage of energy in the layer can be neglected, otherwise the thermal inertia of water is considered. To be consistent with the physics

$$\epsilon_i = \begin{cases} \epsilon_{water} & \text{if } \psi > 0 \\ \epsilon_{soilsurface} & \text{otherwise} \end{cases} \tag{19}$$

Coupling in this way mass and energy equation it is possible to compute a better approximation of soil temperature. The upwelling long wave radiation is evaluated at time level n instead of $n+1$ since T is raised to the power of 4. However it is a better approximation than perform the surface energy budget using for surface temperature the air one.

References