geographical analysis

Geographical Analysis (2020) **52**, 480–493

Special Issue

Overview of Contributions in *Geographical Analysis*: Waldo Tobler

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The academic contributions of Waldo Tobler are noteworthy and significant, spanning essentially all disciplines that involve the study of geographic phenomena. While much attention has been given to his observations of the first law of geography, there is much more substance to his larger body of research. It is especially fitting that this commemorative special issue is appearing in Geographical Analysis as Tobler published extensively in the journal, beginning in the first volume in 1969 up to volume 42 in 2010, making important contributions to quantitative theoretical geography. His research helped to build and sustain the journal, laying the foundation for what is the premier quantitative geography outlet today. This article reviews his publication activity in Geographical Analysis.

Introduction

The academic career of Waldo Tobler (1930–2018) spanned an important time for quantitative geography, heralding in computing, and introducing the need for sound methods to support analysis, planning, management, and policy using this new technology. While his influence was extensive in geography, cartography, and spatial analysis, it extended beyond to impact all disciplines that involve the study of geographic phenomena. *Geographical Analysis* published volume 1 in 1969. Although it was founded by the Department of Geography at Ohio State University, Tobler supported the journal in a number of ways. This is a bit surprising given that he was a faculty member at the University of Michigan at the time. The modern-day perception is that the two universities (and states for that matter) have a competitive disdain for each other, though it is no doubt exaggerated through press and media coverage associated with collegiate athletics. The point, however, is that he contributed regularly to the journal, publishing Tobler (1969) in volume 1 and Yoo, Kyriakidis, and Tobler (2010) in volume 42 and many in-between, as will be detailed below.

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Submitted: January 17 2020; Revised version accepted: August 20 2020

doi: 10.1111/gean.12257

The journal has grown in prominence in a number of different ways, now representing the premier outlet for quantitative theoretical geography. Tobler's contributions to quantitative geography are many, and his research has appeared in a range of journals, including:

- *Economic Geography*
- *Journal of the American Statistical Association*
- *Journal of Environmental Psychology*
- *Annals of the Association of American Geographers*
- *Cartography and Geographic Information Science* (since 1999, *Cartography and Geographic Information Systems* from 1990 to 1998, and *The American Cartographer* from 1974 to 1989)
- *Geographical Review* (no longer published)
- *Geographical Analysis*
- *Urban Geography*
- *Papers of the Regional Science Association* (now *Papers in Regional Science*)
- *International Journal of Population Geography*
- *Area*
- *Nature*
- *Journal of Regional Science*
- *Environment and Planning A*
- *American Journal of Ophthalmology*
- *The Cartographic Journal*
- *Computers, Environment and Urban Systems*
- *Proceedings of the National Academy of Sciences*
- *International Journal of Geographical Information Science*
- *Geographical and Environmental Modelling* (no longer published)
- *Professional Geographer*
- others as well

Tobler (1970) in *Economic Geography* has received considerable attention regarding the observations of the first law of geography, some 7,327 citations to date.[1] In particular, the notion that given two observations, A and B, it is generally the case that when such observations have a geographic context then $P(A \cap B) \neq P(A)\,P(B)$, where $P()$ is the probability of an event. That is, observed attribute values are likely not independent in a statistical sense. Tobler's first law of geography, therefore, is "everything is related to everything else, but near things are more related than distant things" (Tobler 1970). It is significant because this reflects the essence of positive spatial autocorrelation. Of course, his body of research extended well beyond this single publication, as reflected in the many journals noted above. *Geographical Analysis* is particularly significant because it is the outlet in which he most frequently published over the course of his career, 14 publications in total.

The goal of this article is intentionally different, like Tobler himself. There have been a number of obituary and reflection pieces already, including Clarke (2018), Dorling and Hennig (2018), and Unwin (2019). Further, many of the articles in this special issue review and/or address topics for which Tobler made particularly important contributions. This article instead focuses on individual publications by Tobler appearing in *Geographical Analysis*, highlighting their utility and significance from a technical perspective. Six of the 14 publications are actually research notes: Barton and Tobler (1971), Tobler (1973, 1975), Tobler and Lau (1978), Tobler and Kennedy (1983), and Tobler and Chen (1986), with 306 total citations (18, 161, 96, 10, 17, and 4, respectively). We are opting to say little more about them since they are short research notes generally lacking technical details. The eight remaining articles have been organized into different categories based on general topics and ordered by their total number of citations to date. The next section discusses bidimensional regression, followed by scale variance, geographical filters, flow, and spatial interpolation. The review ends with concluding comments.

Bidimensional regression

Tobler (1994) introduces bidimensional regression and is the most cited publication that he authored appearing in *Geographical Analysis*, with some 230 citations to date. A related article is Tobler (1978) with 93 citations, limited in technical details. The publication by Tobler (1994) has an interesting history as it was a 1977 discussion article that was accompanied by a software program implementing the work. Bidimensional regression was introduced as a statistical approach to measure the resemblance between two or more configurations of points and is an extension of classic (unidimensional) regression in which both the dependent and independent variables are two-dimensional. Proposed by Tobler (1994) are four different bidimensional regression models. An important emphasis of the article is the need for such an approach in order to compare data that are two-dimensional in nature, especially geographic information. In particular, there is an interest in assessing the similarity of the pattern. Bidimensional regression effectively seeks to measure the magnitude of the transformation of one layer to the other layer, independent of the particular coordinate system.

Tobler (1994) summarizes the goal of regression analysis as relating an independent variable, Z, to a dependent variable, W. He views this as a mapping $Z \rightarrow \widehat{W}$ such that the fitted $\widehat{W}$ is as close as possible to the observed W. In bidimensional regression, both Z and W are two-dimensional. That is, Z is referenced as (x, y) and W is referenced as (u, v), where x and y are a coordinate pair in the original data layer and u and v are a coordinate pair in the mapped (or transformed) layer. Extensions to multivariate cases are also possible.

Bidimensional regression can be formalized as follows:

$$u = f(x, y) \tag{1a}$$

$$v = g(x, y) \tag{1b}$$

where $f()$ and $g()$ are real functions representing the best mapping (or transformation) possible. The challenge, therefore, is the identification and specification of the best functions $f()$ and $g()$.

Proposed by Tobler (1994) is a family of four regression models as estimates of the functions $f()$ and $g()$, including three linear models (Euclidean, affine, and predictive transformations) and a curvilinear method. The geometric interpretation of linearity in this context denotes the preservation of lines, for example, a straight line in the independent variable layer would be expected to be a straight line in the dependent variable layer.

The Euclidean transformation is:

$$u = \alpha_1 + \beta_{11} x - \beta_{12} y \tag{2a}$$

$$v = \alpha_2 + \beta_{12}x + \beta_{11}y \tag{2b}$$

This then leaves four parameters, α_1, α_2, β_{11}, and β_{12}, to be estimated. The Euclidean model is a rigid transformation in which the independent variable coordinates, x and y, are scaled, rotated, and translated by the same values so that the transformation gives the same shape.

The affine transformation is:

$$u = \alpha_1 + \beta_{11}x + \beta_{12}y \tag{3a}$$

$$v = \alpha_2 + \beta_{21}x + \beta_{22}y \tag{3b}$$

The affine transformation contains six parameters, two more (β_{21} and β_{22}) than the Euclidean. This enables the independent variable coordinates, x and y, to be scaled independently. Thus, the dependent variable layer will not necessarily maintain shape compared to the independent layer. For example, if the data in the independent variable layer formed a circle, this may become an ellipse in the dependent variable layer.

The projective transformation is:

$$u = \left(\beta_{11}x + \beta_{12}y + \beta_{13}\right) / \left(\beta_{31}x + \beta_{32}y + \beta_{33}\right) \tag{4a}$$

$$v = \left(\beta_{21}x + \beta_{22}y + \beta_{23}\right) / \left(\beta_{31}x + \beta_{32}y + \beta_{33}\right) \tag{4b}$$

The difference, in this case, is the scaling reflected in the denominator, requiring nine parameters to be estimated. The projective transformation allows the size, shape, and orientation of the independent variable layer to be modified in order to obtain the best fit correspondence with the dependent variable layer.

Non-linear transformations are possible too. Tobler (1994) reviews a variety of curvilinear approximation possibilities, including global and local models, for doing this. The method adopted by Tobler (1994) is based on linear interpolation. Points on a regular lattice are used to fit a surface that minimizes the sum of the squares of the partial derivatives.

A major challenge, whether the approach is linear or non-linear, is estimating model parameters. Tobler (1994) relied on the least squares method to derive parameters. The non-linear methods can in fact be viewed as a conjunction of infinitely many local linear transformations. Thus, it is possible to approximate a non-linear model with the affine transformation locally and then solve for best parameter values using least squares. Interestingly, there is no explicit specification of errors in the detailed models, but the mention of minimizing the sum of the squared differences between predictions and observed values does imply the recognition of errors. Nakaya (1997) assumed statistical properties of the error term and derived estimates by maximum likelihood as well as undertook statistical inferences on model estimates and comparisons. Another noteworthy extension is the weighted bidimensional regression work of Schmid, Marx, and Samal (2011). They weigh landmarks to account for different levels of variability and measurement accuracy. Their model estimates were derived by a weighted least squares approach.

As an important technique to compute the similarity between two layers of points with choices of linear and non-linear models, bidimensional regression can be broadly applied to compare two-dimensional patterns in various contexts. Of course, it is not limited to geographic applications, as suggested by Tobler (1994). In addition, there is an R package called "BiDimRegression" that can be readily accessed for bidimensional analysis. This package includes options for Euclidean, affine, and projective transformations (Carbon 2013).

Scale variance

Moellering and Tobler (1972) emphasize the significance of scale variance, with 224 citations to date this is his second most highly cited publication appearing in *Geographical Analysis*. The focus of the article is on the effects of spatial scale, effectively suggesting that researchers should assess and evaluate this if the analysis can be varied with respect to scale or spatial unit definition. The implication of course is that if there are scale effects, care would be necessary for the interpretation and significance of any analysis, planning/management or decision making that relies on data that can vary (or be varied) in terms of its spatial representation.

Proposed by Moellering and Tobler (1972) is an approach based on spectral analysis. Ultimately, however, the method is related to the analysis of variance (ANOVA), and is based on the spatial scale, where it is assumed that the data is or can be hierarchically organized. As a representative example, consider U.S. Census data that is organized in terms of the following major categories:

Nation ⇒ States ⇒ Counties ⇒ Tracts ⇒ Block Groups ⇒ Blocks

Blocks are properly contained within a specific Block Group. Block Groups are properly contained within a Tract. Tracts are properly contained within a County. Counties are properly contained within a State. Finally, States are contained within the Nation. This is a typical hierarchical structure for organizing information, in this case corresponding to the spatial nested relationship of administrative reporting units. Further, these reporting units provide attribute information on individuals residing within. This is the essence of the context considered.

Moellering and Tobler (1972) illustrate an example of linking three levels. Of course, this may be generalized, and the reported analysis did, indeed, consider more levels. The following notation is defined:

i index of spatial units at level 1 (entire set I); j = index of spatial units at level 2 (entire set J_i for each i); k = index of spatial units at level 3 (entire set K_{ij} for each i, j pair); and, X_{ijk} = observed the attribute value of unit k (level 3) contained in unit j (level 2) contained in unit i (level 1).

The highest level (3 in this example) corresponds to the most disaggregate resolution, with the lowest level (1 in this case) reflecting more coarse spatial representation. The statistical model reflecting the intended relationships is the following:

$$X_{ijk} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} \tag{5}$$

where, μ = overall mean; α_i = effect of spatial unit i (level 1); β_{ij} = effect of spatial unit j (level 2) contained in unit i (level 1); and, γ_{ijk} = effect of spatial unit k (level 3) contained in unit j (level 2) contained in unit i (level 1).

This means that they are interested in explaining a hierarchical relationship in an observed attribute value that could be reported at a number of different geographical scales. Technically speaking, since Moellering and Tobler (1972) place the work in the context of analysis of variance (ANOVA), the following assumptions hold:

$$\alpha_i \sim N\left(0, \sigma_1^2\right).$$

$$\beta_{ij} \sim N\left(0, \sigma_2^2\right).$$

$$\gamma_{ijk} \sim N\left(0, \sigma_3^2\right).$$

Given the irregular hierarchy associated with this three-level spatial structure example, Moellering and Tobler (1972) propose an ANOVA-like approach for essentially testing the hypothesis that the variances are the same for each level:

$$H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$$

They derive the following decomposition relationship between the total sum of squares and the sum of squares for each spatial scale (level):

$$SS_T = SS_1 + SS_2 + SS_3 \tag{6}$$

In matrix notation this may be stated as:

$$\left(X_{ijk} - \overline{X}_{...}\right)^2 = \left(\overline{X}_{i..} - \overline{X}_{...}\right)^2 + \left(\overline{X}_{ij.} - \overline{X}_{i..}\right)^2 + \left(X_{ijk} - \overline{X}_{ij.}\right)^2 \tag{7}$$

where

$$\overline{X}_{...} = \sum_{i \in I} \sum_{j \in J_i} \sum_{k \in K_{ij}} X_{ijk} / \sum_{i \in I} \sum_{j \in J_i} \left|K_{ij}\right| \tag{8}$$

This is a classic statistical approach for denoting summation across a particular index, with $\overline{X}_{...}$ corresponding to the overall mean and other means similarly defined (e.g., $\overline{X}_{i..}$, $\overline{X}_{ij.}$). Note that $||$ is an operator indicating the number of elements in the set. Thus, $\left|K_{ij}\right|$ is the number of spatial units in the set K_{ij} defining a level 3 area. The denominator then is simply the total number of individual spatial units at the most disaggregate spatial scale. Other means are similarly defined, with $\overline{X}_{i..}$ corresponding to the level 1 mean and $\overline{X}_{ij.}$ the level 2 mean. Accordingly, the corresponding variables introduced in equation (5) are defined as $\mu = \overline{X}_{...}$, $\alpha_i = \overline{X}_{i..} - \overline{X}_{...}$, $\beta_{ij} = \overline{X}_{ij.} - \overline{X}_{i..}$ and $\gamma_{ijk} = X_{ijk} - \overline{X}_{ij.}$

The test statistic is simply the classic ANOVA approach of assessing whether the associated decomposed component is too different. That is, the question is whether variance associated with any spatial scale (level) is excessive. This is performed by examining the mean sum of squares for each level:

$$MSS_1 = \left(\overline{X}_{i..} - \overline{X}_{...}\right)^2 / (|I| - 1) \tag{9a}$$

$$MSS_2 = \left(\overline{X}_{ij.} - \overline{X}_{i..}\right)^2 / \sum_i \left(|J_i| - 1\right) \tag{9b}$$

$$MSS_3 = \left(X_{ijk} - \overline{X}_{ij.}\right)^2 / \sum_i \sum_{j \in J_i} \left(\left|K_{ij}\right| - 1\right) \tag{9c}$$

If the comparison of the mean sum of squares, (9a)–(9c), indicates any proportion of the total sum of squares SS_T is significantly different, then the hypothesis of similar variances across spatial scales would be rejected.

Interestingly, there is no error term and therefore no assumed statistical distribution in this classic decomposition case. What Moellering and Tobler (1972) actually did in order to identify the effects of the scale was to compare the magnitude and scale of variance components numerically and graphically rather than conduct a formal statistical test. Nevertheless, this approach is effectively equivalent to nested (hierarchical) ANOVA when an appropriate statistical distribution is imposed (see Snedecor and Cochran 1989; Collins and Woodcock 2000), accessible in R as well as most commercial statistical software.

Geographical filters

Published in the first issue of *Geographical Analysis* was the article by Tobler (1969), with 179 citations to date. The notion of geographical filtering is motivated by Tobler (1969) based on the idea that it is “clearly impossible” to display reality in all of its complexity, prompting the need for the identification and analysis of only the most critical components contributing to the spatial heterogeneity of phenomena amidst the noise, error, and trivial signals in which they are embedded. Filters, therefore, reflect the importance of weighting and functional specification that explains observed variability. As befitting a publication in an inaugural volume of a journal, Tobler (1969) provides a survey of different functions and methods organized into two broad categories: (1) identifying geographic trends and (2) creating or recreating a geographic distribution. Further, he also highlights the importance of inverse filters, or application of the procedure in reverse, which allows the restoration of data that has been aggregated or generalized to a coarser resolution, effectively returning it to its initial form.

Tobler (1969) suggests decomposing a geographic distribution into its various explanatory component trends. A general mathematical form for this is:

$$Z(x, y) = f_1(x, y) + f_2(x, y) + \ldots + f_{|L|}(x, y) + \varepsilon(x, y) \tag{10}$$

where (x, y) is the geographical coordinate of a location, $Z(x, y)$ is an observed measure at (x, y), $f_l(x, y)$ is interpreted as trend component l ($l \in L$, where L is the set of trend components), and $\varepsilon(x, y)$ is a residual. Thus, indicated in (10) is that any surface, a geographical observation on a lattice, can be described by $|L|$ functions and a residual function ε. Each function $f_l(x, y)$ represents a trend component, and the $|L|$ functions combine to describe the observed heterogeneous distribution. Tobler (1969) indicates that such a form is amenable to least squares methods for estimation, making trend analysis an extension of multiple regression. He details four specific mathematical models applicable to different geographical phenomena.

The second form of geographical filtering suggested by Tobler (1969), the geographic spread function, borrows from diffusion models in physics amenable to modeling geographic processes like population or migration change. Further, it is observed that if we assume that geographic events happen at different scales, then filtering by scale can separate out these processes. To introduce this concept, he asserts that a Fourier series can be used to approximate some geographic event, where modification of the coefficients through a transfer function acts more generally as a "filter." In other domains this may be applied and referred to as "smoothing" or "blurring," but in a geographic context can be used to help separate a spatial process at two scales or two points in time. Tobler (1969) presents numerous examples to demonstrate the utility of a geographic filter of this character.

The interest in geographic filtering is explained by its broad applicability in both analyzing trends in geographic data and in generating one geographic data layer from another of a similar nature. Any analysis involving a specific spatial unit can benefit from the use of a geographic filter, as it has the potential to identify the loss or gain of information when aggregating or disaggregating geographic units by mathematically recovering the process of that loss or gain. The importance of geographic trends is subsequently summarized by Tobler (2000). In the United States, the data collected or displayed at a state level can detect features of 650 km in size, while the data at a county scale can detect features of 80 km in size. This discrepancy means that when the data are aggregated from the county level to the state level, as an example, spatial detail is lost. Geographical filtering, therefore, provides a basis for consistent aggregation and disaggregation. Many other examples and associated approaches can be noted as well, including the expansion method (Casetti 1972), local decomposition (Anselin 1995), spatial filtering (Getis and Griffith 2002), factorial kriging (Goovaerts, Jacquez, and Greiling 2005), multiscale geographically weighted regression (Yu et al. 2020), etc.

Flow

The focus of Tobler (1981), with 132 citations, is flow, or rather the movement of individuals. This is a basic and fundamental concept in geography, reflecting many different types of spatial interaction. Proposed is a model capable of describing an observed flow (or movement or relocation) process, both for a discrete network and in continuous space. This approach assumes that interaction between two locations is proportional to the difference in attractiveness ("attractivities") and inversely proportional to geographical separation. The attractivities term is used to effectively denote that attractiveness could be related to the number of activities available in a particular area, the specifics of which depend on study context.

Given change over a time interval, the following notation is defined as:

i (and j) = index of places; d_{ij} = measure of geographical separation of places i and j; a_i = attractiveness of place i; m_{ij} = magnitude of movement from place i to place j; and, n_{ij} = net flow of movement from place i to place j.

One, therefore, has observed interaction between places i and j, but the intent of Tobler (1981) is to explain this as a function of attributes, a_i, and distance, d_{ij}. Uncertainty arises because the activities are effectively unknown. A spatial interaction model is hypothesized and detailed to explain this:

$$\hat{n}_{ij} = \kappa \left(a_j - a_i\right) / d_{ij} \tag{11}$$

where the coefficient κ is a scaling constant and $\hat{n}_{ij}$ is the estimation of observed movement n_{ij} (with $n_{ij} = m_{ij} - m_{ji}$). However, a_i are not known per se, and the intent of the model is to identify what these attractiveness values are given empirically observed movement between places over some time period.

Solution to identify a_i is shown through matrix algebra:

$$A = D^{-1}\Delta \tag{12}$$

where

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} \tag{13}$$

$$\Delta = \begin{bmatrix} \sum_i m_{i1} - \sum_j m_{1j} \\ \sum_i m_{i2} - \sum_j m_{2j} \\ \sum_i m_{i3} - \sum_j m_{3j} \\ \vdots \end{bmatrix} \tag{14}$$

$$D = \begin{bmatrix} \left(\sum_{i(i \neq 1)} \frac{1}{d_{i1}}\right) & -\frac{1}{d_{12}} & -\frac{1}{d_{13}} & \cdots \\ -\frac{1}{d_{21}} & \left(\sum_{i(i \neq 2)} \frac{1}{d_{i2}}\right) & -\frac{1}{d_{23}} & \\ -\frac{1}{d_{31}} & -\frac{1}{d_{32}} & \left(\sum_{i(i \neq 1)} \frac{1}{d_{i3}}\right) & \\ \vdots & & & \ddots \end{bmatrix} \tag{15}$$

Column vectors A and Δ are of dimension $|I| \times 1$, assuming I is the set of places, with A being the unknown attractiveness values for each place and Δ the net flow change by place. The matrix D is related to inverse distances, and is $|I| \times |I|$ in dimension. Condition (14) arises through the evaluation of total new arrivals in a place. The solution of the simultaneous system of linear equations is possible in order to determine the column of unknowns, A.

Tobler (1981) also discusses a model based on linear programming, with the following objective:

$$Minimize \sum_i \sum_j \hat{n}_{ij} d_{ij} \tag{16}$$

No constraints are given, but it is suggested that this is a transportation problem. Non-linear alternatives are also detailed as well as an extension to address continuous space application.

Spatial interaction reflected in movements over time continues to be of great significance. The Tobler (1981) publication had a profound influence on spatial interaction studies and provided a mathematical basis to study spatial interaction across scale according to the perceived attractiveness of a place. These place-based differences can reveal characteristics of changing regional industries, burgeoning economic development, and other financial sinks and flows. Brunsdon and Charlton (2006) position the work of Tobler (1981) in the context of local trend statistics, providing illustrated examples of their relative significance. More recent work on movement analytics by Dodge (2020) attempts to establish a theoretical framework within which movement and motion can be considered, effectively including many of the themes that Tobler (1981) discussed. A bridge between theory and applied data-processing is reflected in Flow mapper software (http://www.csiss.org/clearinghouse/FlowMapper/) made accessible by Tobler, and still garnering interest and use. Tobler's interest in movement geography was not singular in nature but proved to be a consistent theme across various analytical developments.

Spatial interpolation

An important thread of articles was associated with spatial interpolation, including Tobler (1979a) with 67 citations, Tobler and Kennedy (1985) with 43 citations and Yoo, Kyriakidis, and Tobler (2010) with 30 citations. Spatial interpolation is associated with intelligent estimation, where the value of an attribute at a location or within an area is sought given limited information. In some instances, one may have a sample set of observed or measured values at certain locations. This is the classic point interpolation problem: estimate the attribute value at a particular location given a set of sample observations. In other instances, one has observed attribute information for areal units, and from this, the goal is also to estimate the value of the attribute at a particular location. Theoretically, then some function $f(x, y)$ would define a surface describing the spatial distribution of the attribute z. In practice, however, it is usually difficult or impossible to perfectly describe this distribution, making the mathematical specification of any function challenging. This is the essence of the geographical filter that Tobler (1969) discusses. Spatial interpolation, therefore, represents approximate ways to estimate an attribute under different conditions.

Tobler (1979a) assumed a regular lattice of points for which the corresponding attribute value is to be estimated given a set of sampled (known) attribute observations. The intent is to devise an approach for estimating values that minimize the backward interpolation error in estimates. That is, use the sample of observed values to estimate the regular lattice values, then use the estimates to evaluate at sample observations. If this backward interpolation does not reproduce observed values, then this indicates an error. The goal, therefore, is to devise an approach that results in the least amount of error.

To accomplish this, a general case extension of bilinear weighted interpolation is detailed by Tobler (1979a) for a regular grid using row i and column j references for estimation between rows i and $i+1$ and columns j and $j+1$. A model formulation is the following:

$$f(x,y)=\frac{1}{(x_{i+1}-x_i)\,(y_{j+1}-y_j)}$$
$$\left[(x_{i+1}-x)\,(y_{j+1}-y)\,z_{ij}+(x_{i+1}-x)\,(y-y_j)\,z_{ij+1}+(x-x_i)\,(y_{j+1}-y)\,z_{i+1j}+(x-x_i)\,(y-y_j)\,z_{i+1j+1}\right] \tag{17}$$

where z_{ij} is the attribute value at lattice point corresponding to i (row) and j (column), x_i is the x-coordinate value of row i, and y_j is the y-coordinate value of column j. Note that lattice size is accounted for as $x_{i+1}-x_i$ in the x direction and $y_{j+1}-y_j$ in the y direction. This defines a plane, indicating that the continuous surface is approximated by a collection of lattice planes between neighboring row-column pairs.

Of course, the attribute value z_{ij} is interpolated based upon an observed point sample set to begin with. The issue of interest, therefore, is tuning the interpolated estimates so that they are distance-aware of actual observed values.

Assuming that backward interpolation has a non-zero root mean square error, the right-hand side of equation (20) can be rewritten as:

$$\frac{1}{(x_{i+1}-x_i)\,(y_{j+1}-y_j)}$$
$$\left[(x_{i+1}-x)\,(y_{j+1}-y)\,(z_{ij}+\delta_{ij})+(x_{i+1}-x)\,(y-y_j)\,(z_{ij+1}+\delta_{ij+1})+(x-x_i)\,(y_{j+1}-y)\,(z_{i+1j}+\delta_{i+1j})+(x-x_i)\,(y-y_j)\,(z_{i+1j+1}+\delta_{i+1j+1})\right] \tag{18}$$

The δ_{ij} values can, therefore, be considered the tuning process. Of note is that Tobler (1979a) views this as an optimization problem, attempting to ensure estimates are closer to observed values when they are nearby.

Tobler and Kennedy (1985) seek to smooth an estimated surface through an averaging, similar in many ways to that of Tobler (1979a). Rather than focus on interpolation with a spatial reference, suppose that we have spatial units with known neighbors. Consider the following notation definitions:

i (and j) = index of spatial units; and, Ω_i = set of units that are a spatial neighbor of unit i.

This means that an attribute estimate for any spatial unit could be defined as:

$$\hat{z}_i=\sum_{j\in\Omega_i} w_{ij} z_j \tag{19}$$

where z_j is the known attribute value of unit j and w_{ij} is a normalized weight reflecting the influence of the attribute associated with unit j on unit i. Accordingly, $\hat{z}_i$ is the associated estimate if only neighboring unit attributes are considered. Tobler and Kennedy (1985) discuss that (19) is an instance of Laplace's equation. Further, the weights w_{ij} are based on the normalized inverse distance of units i and j. Finally, neighbors are defined based on adjacency, where units i and j share a common edge.

A slightly different spatial interpolation problem is reported in the article by Yoo, Kyriakidis, and Tobler (2010), applying a Laplacian equation to construct a variogram. This is equivalent to the de wijsian model in geostatistics. The pycnophylatic property of a smooth surface is reviewed by Tobler (1979b), exploring the smoothing method in discrete collection units with the Laplacian approach in relation to the pycnophylatic property. Yoo, Kyriakidis, and Tobler (2010) also consider spatial units that are area-based objects. In general, the attribute for all spatial units is known. What is sought is a surface $f(x, y)$ that can give an estimate at a particular location (x, y), but also that pycnophylactic properties are maintained. That is, the use of the surface should effectively reproduce the known attribute values of individual units. Mathematically this may be stated for unit i as:

$$\iint_{o_i} f(x, y)\, \partial x \partial y \approx z_i \tag{20}$$

where o_i is the spatial object representation of unit i. Thus, equation (20) reflects integration in order to identify the total attribute distributed over the surface of unit i. Of course, this would need to hold true for all spatial units.

Spatial interpolation is a particularly active area of ongoing research in both geography and geostatistics and is an important facet of basic GIS functionality. Overviews are given by Cromley, Hanink, and Bentley (2012) and Leyk, Nagle, and Buttenfield (2013), highlighting the basic utility as well as extension potential. Currently, major proprietary and open-source GIS packages provide access to a range of interpolation techniques (see Mitas and Mitasova 2005).

Conclusions

The stream of 14 articles published in *Geographical Analysis* by Waldo Tobler was significant, both in numbers as well as overall impact. This work has garnered some 1,304 citations to date. Embodied in his research was the importance of spatial proximity and in particular the first law of geography. Beyond this connection, there are a number of underlying themes that resonate through his publications having to do with frame independence, transformations/projections, and analytical cartography.

The reviewed work establishes a foundation for his subsequent work characterizing frame independence. Tobler (1989) suggests that references in the literature to the modifiable areal unit problem are effectively mischaracterizing the real issue. If a quantitative approach or model produces different results when applied to data that can be represented in different ways, either by a change in scale or a change in unit definition, then it is frame-dependent. If results are frame-dependent, then the approach or model is of limited utility. Therefore, Tobler (1989) advocates the development and search for approaches, methods, models, etc. that are frame independent. This is noteworthy because it has spawned the development of subsequent methods designed to be frame independent, or at least less frame-dependent than previous approaches. Examples include the work of Murray and Weintraub (2002), Wong (2003), Murray (2018), among others.

Many of the publications by Tobler appearing in *Geographical Analysis* reflect his expertise and interest in transformations and map projections, foundational elements of modern geographic information systems (GIS). Certainly, the bidimensional regression approach represents a transformation and/or projection. Similarly, many of his developed methods and approaches have an analytical cartography flavor, such as the work on geographical filtering, spatial interaction, and interpolation. Part of this is associated with the explanation of the distribution of an attribute(s) but also with improving visual aesthetics, such as the development and use of smoothing functions and geographical filters for more appealing flow and contour mapping.

The career of Waldo Tobler was a remarkable one, which is reflected in his election to the National Academy of Sciences along with other notable recognitions. He published extensively in *Geographical Analysis*, beginning in the first volume in 1969 and continuing through volume 42 in 2010. His work reflects important contributions to quantitative theoretical geography specifically, but also has and will continue to have broad impacts on all disciplines engaged in spatial analysis.

Note

1 All citation counts reported in this review were obtained using Google Scholar on 1/13/20.

References

Anselin, L. (1995). "Local Indicators of Spatial Association—LISA." *Geographical Analysis* 27(2), 93–115.

Barton, B., and W. Tobler. (1971). "A Spectral Analysis of Innovation Diffusion." *Geographical Analysis* 3(2), 182–6.

Brunsdon, C., and M. Charlton. (2006). "Local Trend Statistics for Directional Data—A Moving Window Approach." *Computers, Environment and Urban Systems* 30(2), 130–42.

Carbon, C. C. (2013). "BiDimRegression: Bidimensional Regression Modeling Using R." *Journal of Statistical Software* 52(1), 1–11.

Casetti, E. (1972). "Generating Models by the Expansion Method: Applications to Geographical Research." *Geographical Analysis* 4(1), 81–91.

Clarke, K. C. (2018). "Waldo R. Tobler (1930–2018)." *Cartography and Geographic Information Science* 45(4), 287–8.

Collins, J. B., and C. E. Woodcock. (2000). "Combining Geostatistical Methods and Hierarchical Scene Models for Analysis of Multiscale Variation in Spatial Data." *Geographical Analysis* 32(1), 50–63.

Cromley, R. G., D. M. Hanink, and G. C. Bentley. (2012). "A Quantile Regression Approach to Areal Interpolation." *Annals of the Association of American Geographers* 102(4), 763–77.

Dodge, S. (2020). "A Data Science Framework for Movement." *Geographical Analysis*. https://doi.org/10.1111/gean.12212.

Dorling, D., and B. Hennig. (2018). "Waldo Tobler: Remembering a Genius." *The Cartographic Journal* 55(3), 303–4.

Getis, A., and D. A. Griffith. (2002). "Comparative Spatial Filtering in Regression Analysis." *Geographical Analysis* 34(2), 130–40.

Goovaerts, P., G. M. Jacquez, and D. Greiling. (2005). "Exploring Scale-Dependent Correlations Between Cancer Mortality Rates Using Factorial Kriging and Population-Weighted Semivariograms." *Geographical Analysis* 37(2), 152–82.

Leyk, S., N. N. Nagle, and B. P. Buttenfield. (2013). "Maximum Entropy Dasymetric Modeling for Demographic Small Area Estimation." *Geographical Analysis* 45(3), 285–306.

Mitas, L., and H. Mitasova. (2005). "Spatial Interpolation." In Geographical Information Systems: Principles, Techniques, Management and Applications, 2nd ed., 481–92, edited by P. A. Longley, M. F. Goodchild, D. J. Maguire and D. W. Rhind. New York: Wiley.

Moellering, H., and W. Tobler. (1972). "Geographical Variances." *Geographical Analysis* 4(1), 34–50.

Murray, A. T. (2018). "Evolving Location Analytics for Service Coverage Modeling." *Geographical Analysis* 50(3), 207–22.

Murray, A. T., and A. Weintraub. (2002). "Scale and Unit Specification Influences in Harvest Scheduling with Maximum Area Restrictions." *Forest Science* 48(4), 779–89.

Nakaya, T. (1997). "Statistical Inferences in Bidimensional Regression Models." *Geographical Analysis* 29(2), 169–86.

Schmid, K. K., D. B. Marx, and A. Samal. (2011). "Weighted Bidimensional Regression." *Geographical Analysis* 43(1), 1–13.

Snedecor, G. W., and W. G. Cochran. (1989). Statistical Methods. 8th ed. Ames: Iowa State University Press.

Tobler, W. R. (1969). "Geographical Filters and their Inverses." *Geographical Analysis* 1(3), 234–53.

Tobler, W. R. (1970). "A Computer Movie Simulating Urban Growth in the Detroit Region." *Economic Geography* 46(sup1), 234–40.

Tobler, W. R. (1973). "Choropleth Maps Without Class Intervals." *Geographical Analysis* 5(3), 262–5.

Tobler, W. (1975). "A Preliminary Analysis of the Spread of the Depression." *Geographical Analysis* 7(4), 427–34.

Tobler, W. R. (1978). "Comparison of Plane Forms." *Geographical Analysis* 10(2), 154–62.

Tobler, W. R. (1979a). "Lattice Tuning." *Geographical Analysis* 11(1), 36–44.

Tobler, W. R. (1979b). "Smooth Pycnophylactic Interpolation for Geographical Regions." *Journal of the American Statistical Association* 74(367), 519–30.

Tobler, W. R. (1981). "A Model of Geographical Movement." *Geographical Analysis* 13(1), 1–20.

Tobler, W. R. (1989). "Frame Independent Spatial Analysis." In Accuracy of Spatial Databases, 115–22, edited by M. Goodchild and S. Gopal Bristol: Taylor and Francis.

Tobler, W. R. (1994). "Bidimensional Regression." *Geographical Analysis* 26(3), 187–212.

Tobler, W. (2000). "The Development of Analytical Cartography: A Personal Note." *Cartography and Geographic Information Science* 27(3), 189–94.

Tobler, W., and Z. T. Chen. (1986). "A Quadtree for Global Information Storage." *Geographical Analysis* 18(4), 360–71.

Tobler, W. R., and S. Kennedy. (1983). "Geographic Interpolation." *Geographical Analysis* 15(2), 151–6.

Tobler, W. R., and S. Kennedy. (1985). "Smooth Multidimensional Interpolation." *Geographical Analysis* 17(3), 251–7.

Tobler, W., and J. Lau. (1978). "Isopleth Mapping Using Histosplines." *Geographical Analysis* 10(3), 273–9.

Unwin, D. (2019). "Waldo Tobler: An Appreciation of the Contributions Made to Geographic Information Science." *Transactions in GIS* 23(2), 387–93.

Wong, D. W. (2003). "Spatial Decomposition of Segregation Indices: A Framework Toward Measuring Segregation at Multiple Levels." *Geographical Analysis* 35(3), 179–94.

Yoo, E. H., P. C. Kyriakidis, and W. Tobler. (2010). "Reconstructing Population Density Surfaces from Areal Data: A Comparison of Tobler's Pycnophylactic Interpolation Method and Area-to-Point Kriging." *Geographical Analysis* 42(1), 78–98.

Yu, H., A. S. Fotheringham, Z. Li, T. Oshan, W. Kang, and L. J. Wolf. (2020). "Inference in Multiscale Geographically Weighted Regression." *Geographical Analysis* 52(1), 87–106.