Ch3: Generalized inverse, individual activity

Hang Chen

1.(a) $G = USV^T$

$G^T = (USV)^T = VS^TU^T$

So $G^TG = VS^TU^T \cdot USV^T$

Since $U^TU = \vec{e}_i$

Then $G^TG = VS^TSV^T$

(b) According to Theorem A.7 in the textbook, If A is a real symmetric maxtrix, then $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$, Q is real orthogonal matrix, Λ is a real diagonal matrix

G^TG is a real symmetric maxtrix.

Then from (a) $G^TG = US^TSU^T$

If $m>n$ $S^TS = \begin{bmatrix} S_1^2 & & \\ & \ddots & \\ & & S_n^2 \end{bmatrix}$

If $m<n$ $S^TS = \begin{bmatrix} S_1^2 & & \\ & \ddots & \\ & & S_m^2 \end{bmatrix}$

And we know that V is real orthogonal matrix. S^TS is diagnal matrix, thus the eigenvalues of G^TG are square values of G.

(c) We know that $G^+ = V_pS_p^{-1}U_p^T$ from

$$m = (G^TG)^{-1}G^Td$$
$$= (V_pS_p^2V_p^T)^{-1}G^Td$$
$$= V_pS_p^{-2}V_p^TV_pS_p^TU_p^Td$$
$$= \underbrace{V_pS_p^{-1}U_p^T}_{G^+}d$$

so $m^+ = V_pS_p^{-1}U_p^Td$

$$= \underbrace{V_pS_pU_p^T}_{G^T}U_pS_p^{-2}U_p^Td$$
$$= G^T(U_pS_p^{-2}U_p^T)d$$
$$= (GG^T)^{-1}d \Rightarrow \text{equation from chapter 2}$$

2. $G^+ = V_p S_p^{-1} U_p^T$

$G = U_p S_p V_p^T$

a. $GG^+G = U_p S_p V_p^T \cdot (V_p S_p^{-1} U_p^T) U_p S_p V_p^T$

$= U_p S_p S_p^{-1} S_p V_p^T$

$= U_p S_p V_p^T = G$

b. $G^+GG^+ = (U_p S_p^{-1} U_p^T)(U_p S_p V_p^T)(V_p S_p^{-1} U_p^T)$

$= U_p S_p^{-1} S_p S_p^{-1} U_p^T$

$= U_p S_p^{-1} U_p^T = G^+$

$\cancel{c.\ (GG^+)^T = (U_p S_p^{-1} U_p^T)^T [(U_p S_p V_p^T)(V_p S_p^{-1} U_p^T)]^T}$

$\cancel{= (U_p S_p^{-1} U_p^T)^T (U_p S_p U_p^T)^T}$

$\cancel{= (U_p (S_p^{-1})^T U_p^T)(U_p^T S_p^T)} U_p$

c. If we prove GG^+ is symmetric then $(GG^+)^T = GG^+$

$GG^+ = U_p S_p V_p^T \cdot V_p S_p^{-1} U_p^T = U_p S_p S_p^{-1} U_p^T = (U_p S_p S_p^{-1} U_p^T)^T$

So it is symmetric

d. Same as c: $G^+G = V_p S_p^{-1} U_p^T \cdot U_p S_p V_p^T = V_p S_p^{-1} S_p V_p^T = (V_p S_p^{-1} S_p V_p^T)^T$

So G^+G is symmetric

Question 3

(a)

m =

-3.68867239266073e-07

-8.69669914110091e-06

1.39889971370030e-06

3.03300858899103e-07

-6.70220057259940e-06

2.73223304703363e-06

9.73223304703364e-06

2.06556638036697e-06

3.53553390593274e-06

(b)

m =

-7.02200572599404e-07

-6.96446609406727e-06

0

-1.42893218813453e-06

-6.36886723926607e-06

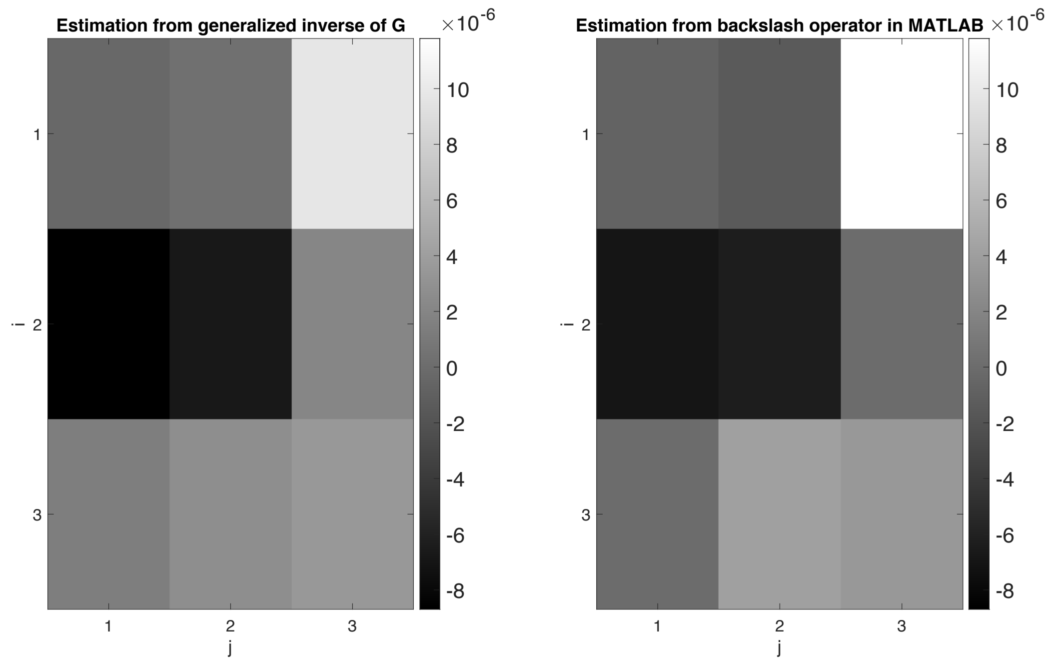
4.13113276073393e-06

1.17977994274006e-05

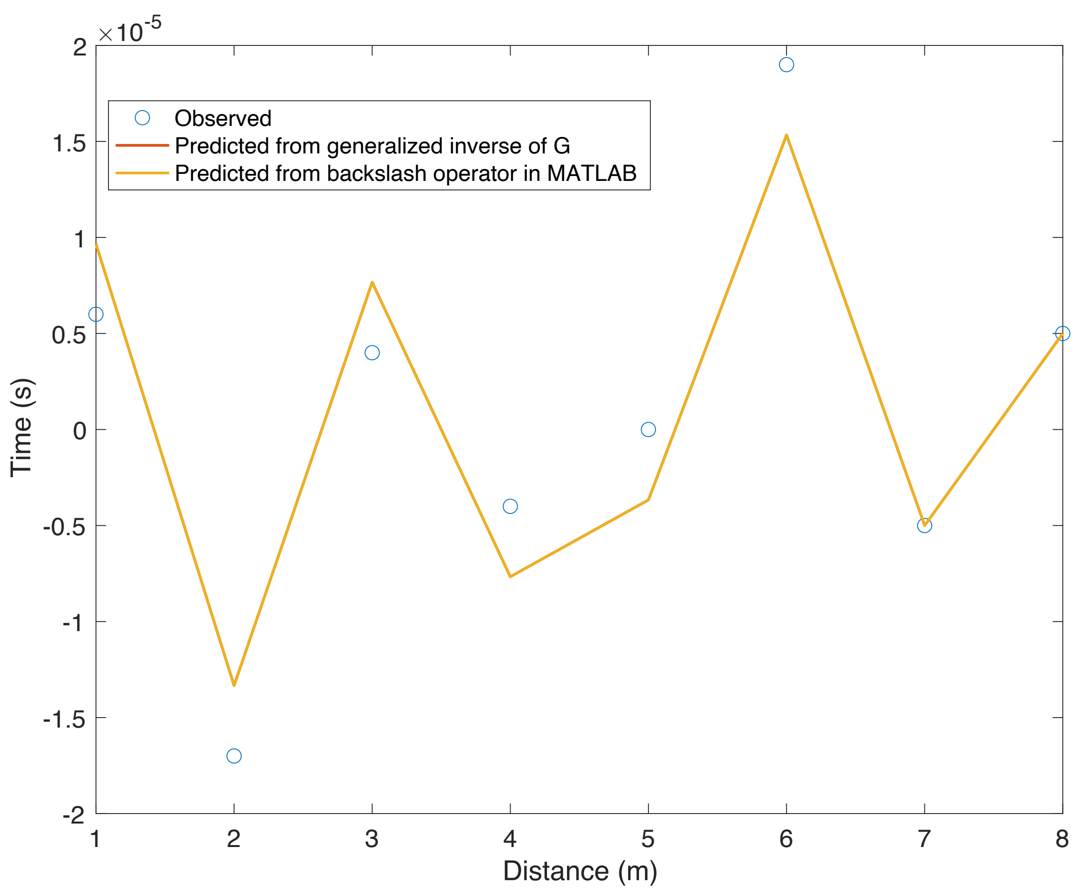
0

3.53553390593274e-06

The comparison of model parameters and predicted data are shown in next page. We can find that the model parameters are slightly different from two methods but the predicted data is exactly the same.

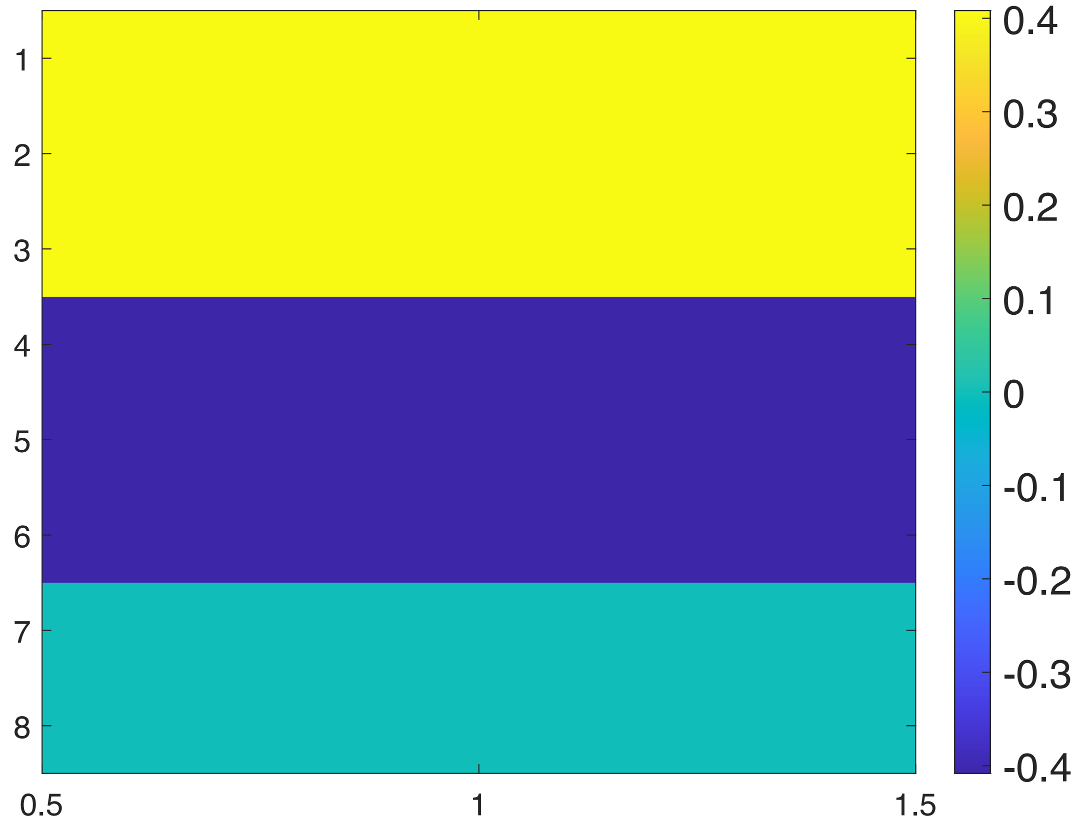


Estimation from generalized inverse of G
Estimation from backslash operator in MATLAB
×10^-6
i
j

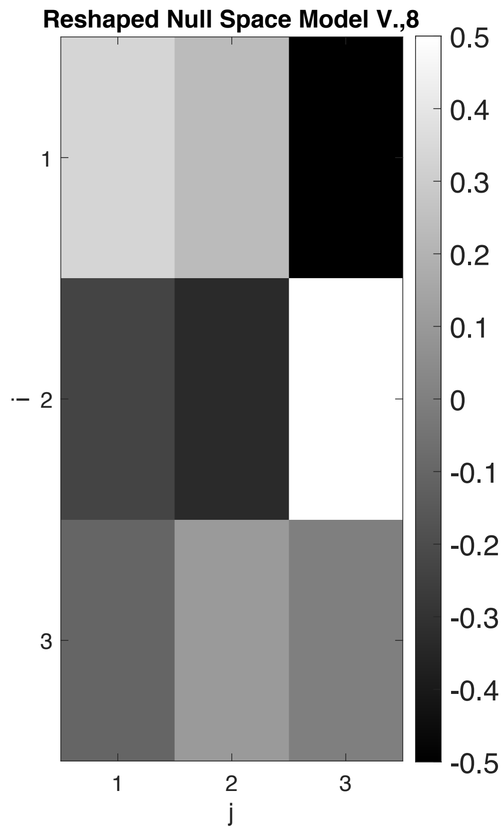


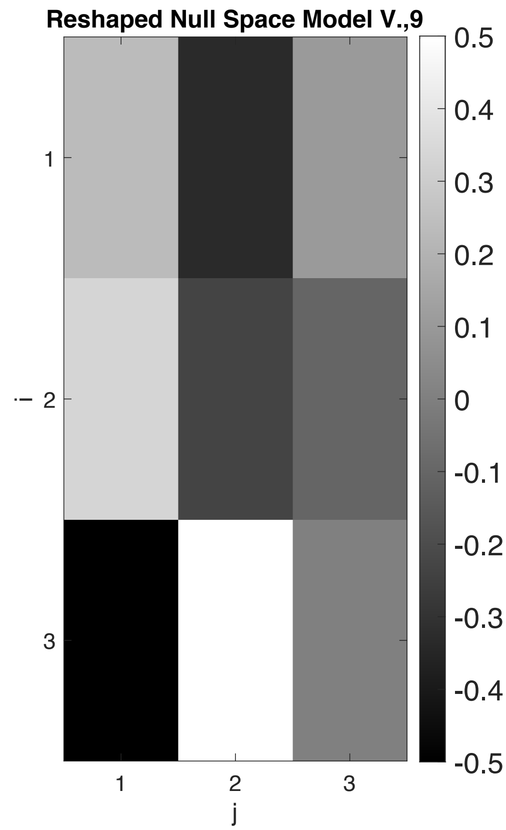
×10^-5
Observed
Predicted from generalized inverse of G
Predicted from backslash operator in MATLAB
Time (s)
Distance (m)

(b) The dimension of the data null space is the 8 – 7 (rank of the G) = 1 but with 8 elements i.e., 8*1. The vector is shown as



(c) The model null space is 9 - 7 (rank of the G) = 2 with 9 elements i.e., 9*2. They are shown as





(d) Yes, it is possible, because G is not full rank and inverse solution is not unique. Like the solution to this question in (a)

(e) No, it is impossible. G is unique so if we give a specific m then will be specific Gm.

Codes

```
%

clear
clc
%% question 3 (a)

% Construct system matrix for the ray path models
s2=sqrt(2);
G = [1,0,0,1,0,0,1,0,0;
     0,1,0,0,1,0,0,1,0;
     0,0,1,0,0,1,0,0,1;
     1,1,1,0,0,0,0,0,0;
     0,0,0,1,1,1,0,0,0;
     0,0,0,0,0,0,1,1,1;
     s2,0,0,0,s2,0,0,0,s2;
     0,0,0,0,0,0,0,0,s2];

t = [6e-06,-1.7e-05,4e-06,-4e-06,0,1.9e-05,-5e-06,5e-06].';

% Find and display system rank
p=rank(G);

% for question i
[U,S,V] = svd(G);
Up = U(:,1:p);
temp = diag(S);
Sp = diag(temp(1:p));
Vp = V(:,1:p);
m = Vp*Sp^-1*Up'*t;

% for question ii
m2 = G\t;

figure
subplot(1,2,1)
colormap('gray')
imagesc(reshape(m,3,3))
caxis([min([m;m2]),max([m;m2])]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3,4,5,6,7,8,9]);
set(gca,'ytick',[1,2,3,4,5,6,7,8,9]);
xlabel('j')
ylabel('i')
title('Estimation from generalized inverse of G')
set(gca,'fontsize',14)
subplot(1,2,2)
colormap('gray')
imagesc(reshape(m2,3,3))
caxis([min([m;m2]),max([m;m2])]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3,4,5,6,7,8,9]);
set(gca,'ytick',[1,2,3,4,5,6,7,8,9]);
xlabel('j')
```

```
ylabel('i')
title('Estimation from backslash operator in MATLAB')
set(gca,'fontsize',14)


figure
plot(t,'o','MarkerSize',8)
hold on
plot(G*m,'LineWidth',1.5)
hold on
plot(G*m2,'LineWidth',1.5)
xlabel('Distance (m)')
legend('Observed','Predicted from generalized inverse of G', ...
    'Predicted from backslash operator in MATLAB')
ylabel('Time (s)')
set(gca,'fontsize',14)

%% Question 3 (b)

% Display data space null vector
disp('Data null space vector')
U(:,p+1)
figure
imagesc(U(:,p+1))
set(gca,'ytick',[1,2,3,4,5,6,7,8]);
set(colorbar,'Fontsize',18);
set(gca,'fontsize',14)

%% Question (c)

% Display null space vectors reshaped to match tomography example geometry
disp('Model null space vectors reshaped into matrices')
m01=reshape(V(:,p+1),3,3)'
m02=reshape(V(:,p+2),3,3)'

figure
subplot(1,2,1)
colormap('gray')
imagesc(m01)
caxis([-0.5 0.5]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
set(gca,'fontsize',14)
title('Reshaped Null Space Model V.,8');
subplot(1,2,2)
colormap('gray')
imagesc(m02)
caxis([-0.5 0.5]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
```

```
ylabel('i')
title('Reshaped Null Space Model V.,9');
set(gca,'fontsize',14)
```