1. (a) The RREF form of matrix A is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) from (a) we know that $\begin{cases} x_1 - x_3 = 0 \\ x_2 = 0 \end{cases}$, so solution is $x = x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The dimension of it is 1 and $N(A) \in R^3$

(c) The Column space of A: $x_1\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$, and x_3 is a free variable, we can set it as 0. Therefore, the column space of A is $x_1\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. Dimension of $R(A)$ is 2 and $R(A) \in R^4$.

(d) The rank of A equal $\dim R(A) = 2$. In matrix A, $n = 3$ $\dim R(A) < n$ and $m > n$. Thus, it is possible to have a solution but the solution is not unique.

2. (a) $(A^TA)^T = A^TA$, according to principle of transpose of product of matrices. That's $(AB)^T = B^TA^T$, so in this problem $(A^TA)^T = A^T(A^T)^T = A^TA$. Since $(A^TA)^T = A^TA$, A^TA is symmetric.

(b) $y^T y = \begin{bmatrix} y_1, y_2, \ldots, y_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \sum_{i=1}^{m} y_i^2$

(c) From (a) we know that $(AB)^T = B^T A^T$, $x^T(A^T A)x = x^T A^T A x$

$= (Ax)^T Ax$

$= y^T y$

So $y = Ax$

(d) Since $y = Ax$, $\underbrace{y^T y = \sum_{i=1}^{m} y_i^2 \geq 0}$, $y^T y = x^T(A^T A)x = \sum_{i=1}^{m} y_i^2 \geq 0$

because $y_i^2 \geq 0$, sum of them should be ≥ 0.

(e) i: $\dim R(A) = \text{rank}(A) = n$, $\dim N(A) = n - \text{rank}(A) = 0$

ii: Since $\text{rank}(A) = n$, $Ax = 0$ only has anunique solution. We know that when x equals to zero vector, $Ax = 0$. The solution is unique, so there is no nonzero vector x to make $Ax = 0$.

iii: If we let $y = Ax$ and $y^T y = (Ax)^T Ax$, then $y^T y = \sum_{i=1}^{m} (y_i)^2 = \sum_{i=1}^{m} (Ax)_i^2$. From (ii) we know that if $x \neq 0$, $Ax \neq 0$

So $y \neq 0$, then $y^T y \neq 0$ and thus $\sum_{i=1}^{m} (Ax)_i^2 \neq 0$

iv: If $x \neq 0$, $\sum_{i=1}^{m} (Ax)_i^2 \neq 0$, thus $\sum_{i=1}^{n} y_i^2 \neq 0 \Rightarrow y^T y \neq 0 \Rightarrow x^T(A^T A)x \neq 0$

From (d), we know that $x^T(A^T A)x \geq 0$, so $x^T(A^T A)x > 0$ if $x \neq 0$.