Topology Exercise Sheet 1

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NAMEN, MATRIKELNUMMER

Excercise 1

1.

Let $X := \{0, 1\}$. For a topology $\mathcal{I}$ it holds that $\mathcal{I} \subset \mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Since $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = X$, $\{\emptyset, X\} \subset \mathcal{I}$ leaving only three possiblities.

- $\mathcal{I} = \{X, \emptyset\}$ is a topology because
    - $X \cup \emptyset = X \in \mathcal{I}$
    - $X \cap \emptyset = \emptyset \in \mathcal{I}$.

- $\mathcal{I} = \{X, \{0\}, \emptyset\}$ is a topology because
    - $X \cup \emptyset = X \in \mathcal{I}$
    - $X \cup \{0\} = X \in \mathcal{I}$
    - $\{0\} \cup \emptyset = \{0\} \in \mathcal{I}$
    - $X \cap \emptyset = \emptyset \in \mathcal{I}$
    - $X \cap \{0\} = \{0\} \in \mathcal{I}$
    - $\{0\} \cap \emptyset = \emptyset \in \mathcal{I}$.

- $\mathcal{I} = \{X, \{1\}, \emptyset\}$ is a topology because it is the second case with 0 replaced by 1.

- $\mathcal{I} = \{X, \{1\}, \{0\}, \emptyset\}$ is the discrete topology.

2.

If a topology $\mathcal{I}$ is metrizable, there exists a metric d with $d(0, 1) =: b \neq 0$, because otherwise $d(0, 1) = 0 \implies 0 = 1$ which is a contradiction. Then $B_{\frac{b}{2}}(0) = \{0\}$, $B_{\frac{b}{2}}(1) = \{1\}$, $B_{2b}(0) = B_{2b}(1) = \{0, 1\}$ are open sets in the topology $\mathcal{I}$. Then

- $\mathcal{I} = \{X,\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(0) = \{0\} \notin \mathcal{I}$

- $\mathcal{I} = \{X,\ \{0\},\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(1) = \{1\} \notin \mathcal{I}$

- $\mathcal{I} = \{X,\ \{1\},\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(0) = \{0\} \notin \mathcal{I}$

- $\mathcal{I} = \{X, \{1\},\ \{0\},\ \emptyset\}$ is metrizable because for every $U \in \mathcal{I}$ and every $x \in U$ there exists the ball $B := B_{\frac{b}{2}}(x) = \{x\}$ satisfying $B \subset U$. Thus if U is in $\mathcal{I}$, it is also in the induced topology $\mathcal{I}_d$. And if $U \in \mathcal{I}_d$, $U \in \mathcal{I} = \mathcal{P}(X)$.

Excercise 2

Let $U_i = \mathbb{V}(F_i)^c$ be open sets with $F_i \subset \mathbb{C}\,[z_1,\ \ldots,\ z_n]$ for $i \in \mathcal{J}$. Then

$$\bigcup_{i \in \mathcal{J}} U_i = \bigcup_{i \in \mathcal{J}} \mathbb{V}(F_i)^c = \left(\bigcap_{i \in \mathcal{J}} \mathbb{V}(F_i) \right)^c = \left(\mathbb{V}\left(\bigcup_{i \in \mathcal{J}} F_i \right) \right)^c$$

because

$$x \in \bigcap_{i \in \mathcal{J}} \mathbb{V}(F_i) \Leftrightarrow f(x) = 0 \ \forall f \in F_i \ \forall i \in \mathcal{J}$$

$$\Leftrightarrow f(x) = 0 \ \forall f \in \bigcup_{i \in \mathcal{J}} F_i.$$

Thus $\bigcup_{i \in \mathcal{J}} U_i = \mathbb{V}(F_i)^c$ is open.
Let $U_i = \mathbb{V}(F_i)^c$ be open sets with $F_i \subset \mathbb{C}\,[z_1,\ \ldots,\ z_n]$ for $i \in \mathcal{J} := \{1,\ \ldots,\ M\}$.
Let

$$G := \{\Pi_{i=1}^M p_i \mid p_i \in F_i\} \subset \mathbb{C}\,[z_1,\ \ldots,\ z_n].$$

Then

$$\bigcap_{i=1}^M U_i = \bigcap_{i=1}^M \mathbb{V}(F_i)^c = \left(\bigcup_{i=1}^M \mathbb{V}(F_i) \right)^c$$

and

$$x \in \bigcup_{i=1}^M \mathbb{V}(F_i) \Leftrightarrow \exists\, i \in \mathcal{J} :\ x \in \mathbb{V}(F_i)$$

$$\Leftrightarrow \exists\, i \in \mathcal{J} :\ \forall\, f \in F_i :\ f(x) = 0$$
$$\Leftrightarrow \forall\, p = p_1 \cdots p_i \cdots p_m \in G :\ p(x) = 0$$

where the last equivalence holds because if $p_i(x) = 0$ for all $p_i \in F_i$, then $p(x) = 0$ for all $p = p_1 \cdots p_i \cdots p_m \in G$; and if for every $i \in \mathcal{J}$ there exists

$f_i \in F_i$ with $f_i(x) \neq 0$ then $p(x) = f_1(x) \cdots f_M(x) \neq 0$. Thus $x \in \bigcup_{i=1}^{M} \mathbb{V}(F_i) \Leftrightarrow$ $\forall\, p \in G : \; p(x) = 0 \Leftrightarrow x \in \mathbb{V}(G)$. Then

$$\bigcap_{i=1}^{M} U_i = \left(\bigcup_{i=1}^{M} \mathbb{V}(F_i) \right)^c = (\mathbb{V}(G))^c$$

is an open set. Hence the Zariski topology is a topology.

2.

Let $U = \mathbb{V}(F)^c$. It is to show that there exists $B \subset \mathcal{B}$ with $U = \bigcup B$. Let $B := \{ \mathbb{V}(f) \mid f \in F \}$. Then

$$\begin{aligned}
x \in \bigcup B &\Leftrightarrow \exists\, f \in F : \; x \in \mathbb{V}(f)^c \\
&\Leftrightarrow \exists\, f \in F : f(x) \neq 0 \\
&\Leftrightarrow x \notin \mathbb{V}(F) \Leftrightarrow x \in \mathbb{V}(F)^c = U
\end{aligned}$$

and hence $U = \bigcup B$.

Excercise 3

1.

Let $\mathcal{I} \subset \mathcal{P}(V)$ be the topology induced by $\|.\|$ and $\mathcal{I}'$ the topology induced by $\|.\|'$. Let $U \in \mathcal{I}$ be open and $x \in U$. Because $\{ B_\epsilon^{\|\cdot\|}(y) \mid y \in V, \, \epsilon > 0 \}$ is a basis of $\mathcal{I}$, there exists $\epsilon > 0$ with $B_\epsilon^{\|\cdot\|}(x) \subset U$. Then $B_{A\epsilon}^{\|\cdot\|'}(x) \subset B_\epsilon^{\|\cdot\|}(x)$, because for all $y \in B_{A\epsilon}^{\|\cdot\|'}(x)$

$$\|y - x\| \leq \frac{1}{A}\|y - x\|' \; < \frac{1}{A}(A\epsilon) = \epsilon.$$

Hence for all $x \in U$ exists $\delta = A\epsilon > 0$ such that $B_\delta^{\|\cdot\|'}(x) = B_{A\epsilon}^{\|\cdot\|'}(x) \subset B_\epsilon^{\|\cdot\|}(x) \subset U$ and thus $U \in \mathcal{I}'$.

If $U \in \mathcal{I}'$ and $x \in U$, since $\{ B_\epsilon^{\|\cdot\|'}(y) \mid y \in V, \, \epsilon > 0 \}$ is a basis of I', there exists $\epsilon > 0$ with $B_\epsilon^{\|\cdot\|'}(x) \subset U$. Then $B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x) \subset B_\epsilon^{\|\cdot\|'}(x)$, because for all $y \in B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x)$

$$\|y - x\|' \leq B\|y - x\| < B\frac{\epsilon}{B} = \epsilon.$$

Hence for all $x \in U$ exists $\delta = \frac{\epsilon}{B} > 0$ such that $B_\delta^{\|\cdot\|}(x) = B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x) \subset B_\epsilon^{\|\cdot\|'}(x) \subset U$ and thus $U \in \mathcal{I}$.

2.

It is known from Analysis I, that $\|x\|_\infty := \max_{i \in \{1,\,\ldots,\,n\}} |x_i|$ defines a norm on $\mathbb{R}^n$. Let $1 \leq p < \infty$, then for every $x \in V$ there exists $m \in \{1,\,\ldots,\,n\}$ with $\|x\|_\infty = |x_m|$ and

$$\|x\|_\infty = |x_m| = (|x_m|^p)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} = \|x\|_p$$

$$\leq (n|x_m|^p)^{\frac{1}{p}} = n^{\frac{1}{p}} |x_m| = n^{\frac{1}{p}} \|x\|_\infty$$

$$\Rightarrow \|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty.$$

From this already follows that all p-norms are equivalent because if $1 \leq p, q < \infty$, then

$$\|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty \leq n^{\frac{1}{p}} \|x\|_q \leq n^{\frac{1}{p} + \frac{1}{q}} \|x\|_\infty \leq n^{\frac{1}{p} + \frac{1}{q}} \|x\|_q.$$

$$\implies \|x\|_p \leq n^{\frac{1}{p}} \|x\|_q \leq n^{\frac{1}{p} + \frac{1}{q}} \|x\|_q$$

$$\implies n^{\frac{-1}{p}} \|x\|_p \leq \|x\|_q \leq n^{\frac{1}{q}} \|x\|_q.$$

3.

Set $f_n(x) := x^n$, then $f_n \in C^1([0, 1], \mathbb{R})$ for all $n \in \mathbb{N}$. Then

$$\|f_n\|_0 = \max_{x \in [0,\,1]} |f_n(x)| = \max_{x \in [0,\,1]} x^n = 1$$

for all $n \in \mathbb{N}$ and

$$\|f_n\|_1 = \|f_n\|_0 + \|f_n'\|_0 = 1 + \max_{x \in [0,\,1]} |nx^{n-1}| = 1 + n \max_{x \in [0,\,1]} |x^{n-1}| = 1 + n.$$

for all $n \in \mathbb{N}$. If $\|.\|_0$ and $\|.\|_1$ were equivalent, there would exist $A > 0$ with

$$\|f_n\|_1 \leq A\|f_n\|_0.$$

for all $n \in \mathbb{N}$. Then

$$\infty = \lim_{n \to \infty} 1 + n = \lim_{n \to \infty} \|f_n\|_1 \leq \lim_{n \to \infty} A\|f_n\|_0 = \lim_{n \to \infty} A = A.$$

This is a contradiction and hence the two given norms are not equivalent.

Excercise 4

1.

Let $\mathcal{B} \subset \mathcal{T}$. It is to show that $\bigcup \mathcal{B} \in \mathcal{T}$.

- If $\mathcal{B} = \emptyset$, then $\bigcup \mathcal{B} = \emptyset \in \mathcal{T}$.

- Next consider the case that $\mathcal{B} \neq \emptyset$ and $\mathbb{R} \notin \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then there exists a set $B \neq \emptyset$ such that $\mathcal{B} = \{(-\infty, x) \mid x \in B\}$. Set

$$S := \sup_{x \in B} x \in \mathbb{R} \cup \{\infty\}.$$

  Then there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$ such that $\lim_{n \to \infty} x_n = S$. Then

$$y \in \bigcup \mathcal{B} \Leftrightarrow y \in \bigcup_{x \in B} (-\infty, x) \Leftrightarrow y < x \text{ for a } x \in B \Leftrightarrow y \in (-\infty, S),$$

  because if $y < x$ for a $x \in \mathcal{B}$, then $y < \sup_{x \in B} = S$. And if $y < \sup_{x \in B} = S$, then $y < x$ for a $x \in \mathcal{B}$. Then $\bigcup \mathcal{B} = (-\infty, S) \in \mathcal{T}$.

- Next consider the case that $\mathbb{R} \in \mathcal{B}$. Then $\bigcup \mathcal{B} = \mathbb{R} \in \mathcal{T}$.

- Next consider the case that $\emptyset \in \mathcal{B}$ and $\mathbb{R} \notin \mathcal{B}$. Then $\bigcup \mathcal{B} = \bigcup (\mathcal{B} \setminus \{\emptyset\}) \in \mathcal{T}$ according to cases 1 or 2.

Let $\mathcal{B} \subset \mathcal{T}$ be finite. It is to show that $\bigcap \mathcal{B} \in \mathcal{T}$.

- If $\mathcal{B} = \emptyset$, then $\bigcap \mathcal{B} = \mathbb{R}$.

- Next consider the case that $\mathcal{B} \neq \emptyset$ and $\mathbb{R} \notin \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then there exists a set $B \neq \emptyset$ such that $\mathcal{B} = \{(-\infty, x) \mid x \in B\}$. Set

$$S := \inf_{x \in B} x \in \mathbb{B}$$

  because $\mathcal{B}$ is finite. Then

$$y \in \bigcap \mathcal{B} \Leftrightarrow y \in \bigcap_{x \in B} (-\infty, x) \Leftrightarrow y < x \text{ for all } x \in B \Leftrightarrow y \in (-\infty, S),$$

  because if $y < x$ for all $x \in \mathcal{B}$, then $y < S \in \mathcal{B}$. And if $y < S$, then $y < S \leq x$ for all $x \in \mathcal{B}$. Then $\bigcap \mathcal{B} = (-\infty, S) \in \mathcal{T}$.

- Next consider the case that $\emptyset \in \mathcal{B}$. Then $\bigcap \mathcal{B} = \emptyset \in \mathcal{T}$.

- Next consider the case that $\mathbb{R} \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then $\bigcap \mathcal{B} = \bigcap (\mathcal{B} \setminus \{\mathbb{R}\}) \in \mathcal{T}$ according to cases 1 or 2.

2.

$$\begin{aligned}
\overline{(0, 1)} &= \bigcap \{C \supset (0, 1) \mid C^c = (-\infty, x), \, x \in \mathbb{R}\} \\
&= \bigcap \{C \supset (0, 1) \mid C = [x, \infty), \, x \in \mathbb{R}\} \\
&= \bigcap \{C \supset (0, 1) \mid C = [x, \infty), \, x \leq 0\} \\
&= \bigcap \{[x, \infty) \mid x \leq 0\} \\
&= [0, \infty)
\end{aligned}$$

$$[0, 1]^\circ = \bigcup \{A \subset [0, 1] \mid A = (-\infty, x), \, x \in \mathbb{R}\}$$
$$= \emptyset$$

because for all $(-\infty, x) \in \mathcal{T}$ $\min\{x, 0\} - 1 \in (\infty, x) \setminus [0, 1]$.

Excercise 5

1.

not open:

If X is open, there exists ϵ such that $B_\epsilon(-1) \subset X$ because $-1 \in X$ and because $\{B_\delta(y) \mid \delta > 0, \, y \in \mathbb{R}\}$ is a Basis of the euclidean Topology on $\mathbb{R}$. Let $x_n := -1 - \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$ for every $n \in \mathbb{N}$. Then $x_n \notin \mathbb{R} \setminus X$. Since $\lim_{n \to \infty} x_n = -1$, there exists an $n \in \mathbb{N}$ such that $x_n \in B_\epsilon(-1) \subset X$. This is a contradiction to $x_n \notin X$. Hence X is not open.

not closed:

If X is closed, X^c is open and there exists $\epsilon > 0$ such that $B_\epsilon(\frac{5}{2}) \subset X^c$. Let $x_n := \frac{5}{2} - \frac{1}{n+100} \in (4 - \frac{1}{2}, 4 + \frac{1}{2}) \subset X$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} x_n = \frac{5}{2}$. Then there exists $n \in \mathbb{N}$ such that $x_n \in B_\epsilon(\frac{5}{2}) \subset X^c$. This is a contradiction to $x_n \notin X^c$. Hence X is not closed.

2.

Claim: $X^\circ = \bigcup_{n \in \mathbb{N}}(2n - \frac{1}{n}, \, 2n + \frac{1}{n}) := U$.

Proof:
Show that $X \supset U$:
For all $n \in \mathbb{N}$ $(2n - \frac{1}{n}, \, 2n + \frac{1}{n})$ is open. Then U is open as a union of open sets.
Since $U \subset X$, $U \subset X^\circ$.
Show that $X^\circ \subset U$:
Assume there exists $x \in X^\circ \setminus U \subset X \setminus U$. Then $x \in \mathbb{Q} \cap ((-\infty, 1] \cup [3, \pi))$ and there exists $\frac{1}{2} > \epsilon > 0$ such that $B_\epsilon(x) \subset X^\circ \subset X$ because X° is open.

- If $x \in \mathbb{Q} \cap [3, \pi)$, then for $y := x + \lfloor 2\epsilon \rfloor (\pi - x) \in \mathbb{R} \setminus \mathbb{Q}$

$$|y - x| = |x + \lfloor 2\epsilon \rfloor (\pi - x) - x| = \lfloor 2\epsilon \rfloor (\pi - x) < \lfloor 2\epsilon \rfloor \frac{1}{2} < \frac{2\epsilon}{2} = \epsilon.$$

Then $y \in B_\epsilon(x) \subset X$ but $y \notin X$ because

$$(x - \pi) < 0 \text{ and } (1 - 2\epsilon) > 0$$
$$\Rightarrow (x - \pi)(1 - 2\epsilon) < 0$$
$$\Rightarrow (1 - 2\epsilon)x + (2\epsilon - 1)\pi < 0$$
$$\Rightarrow x + 2\epsilon(\pi - x) - \pi < 0$$
$$\Rightarrow y < \pi$$
$$\Rightarrow y \in [3, \pi) \setminus \mathbb{Q}.$$

This is a contradiction and hence

- $x \in \mathbb{Q} \cap (-\infty, 1]$. Then for $y := x - \frac{\lfloor \epsilon \rfloor}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$

$$|y - x| = \left| x - \frac{\lfloor \epsilon \rfloor}{\sqrt{2}} - x \right| < \left| \frac{\epsilon}{\sqrt{2}} \right| < \epsilon.$$

Then $y \in B_\epsilon(x) \subset X$ but $y \notin X$ because $y < x \le 1 \Rightarrow y \in (-\infty, 1) \setminus \mathbb{Q}$.
This is a contradiction and hence $X^\circ \subset U$.

Claim: Set of all cluster points S is $\bigcup_{n \in \mathbb{N},\, n \ge 1}[2n - \frac{1}{n}, 2n + \frac{1}{n}] \cup \mathbb{R}_{\le \pi} =: V$

Proof:
It is

$$\overline{X} = \bigcup_{n \in \mathbb{N},\, n \ge 1} \overline{(2n - \frac{1}{n}, 2n + \frac{1}{n})} \cup \overline{\{x \in \mathbb{Q} \mid x < \pi\}}.$$

Where $\overline{(2n - \frac{1}{n}, 2n + \frac{1}{n})} = [2n - \frac{1}{n}, 2n + \frac{1}{n}]$ for every $n \in \mathbb{N}$ because $[2n - \frac{1}{n}, 2n + \frac{1}{n}]$ is closed (because $[2n - \frac{1}{n}, 2n + \frac{1}{n}]^c = (-\infty, 2n - \frac{1}{n}) \cup (2n + \frac{1}{n}, \infty)$ is open as a union of open sets) and if $(2n - \frac{1}{n}, 2n + \frac{1}{n}) \subset U \subsetneq [2n - \frac{1}{n}, 2n + \frac{1}{n}]$ for a closed U, then $2n - \frac{1}{n} \notin U$ or $2n + \frac{1}{n} \notin U$.

- If $2n + \frac{1}{n} \notin U$ then there exists $\frac{1}{n} > \epsilon > 0$ such that $B_\epsilon(2n + \frac{1}{n}) \subset U^c$ because U^c is open. Then for $x_m := 2n + \frac{1}{n} - \frac{1}{m} \in (2n - \frac{1}{n}, 2n + \frac{1}{n}) \subset U$ $\lim_{m \to \infty} x_m = 2n + \frac{1}{n}$. Then there exists $m \in \mathbb{N}$ such that $x_m \in (B_\epsilon(2n + \frac{1}{n}) \cap U) \subset (U^c \cap U)$. This is a contradiction.

- If $2n - \frac{1}{n} \notin U$ then there exists $\frac{1}{n} > \epsilon > 0$ such that $B_\epsilon(2n - \frac{1}{n}) \subset U^c$ because U^c is open. Then for $x_m := 2n - \frac{1}{n} + \frac{1}{m} \in (2n - \frac{1}{n}, 2n + \frac{1}{n}) \subset U$ $\lim_{m \to \infty} x_m = 2n - \frac{1}{n}$. Then there exists $m \in \mathbb{N}$ such that $x_m \in (B_\epsilon(2n - \frac{1}{n}) \cap U) \subset (U^c \cap U)$. This is a contradiction.

Then $\overline{(2n - \frac{1}{n}, 2n + \frac{1}{n})} = \bigcap \{U \supset (2n - \frac{1}{n}, 2n + \frac{1}{n}) \mid U \text{ closed}\} = [2n - \frac{1}{n}, 2n + \frac{1}{n}]$.

And $\overline{\{x \in \mathbb{Q} \mid x < \pi\}} = \{x \in \mathbb{R} \mid x \le \pi\}$ because $\{x \in \mathbb{R} \mid x \le \pi\}$ is closed because (π, ∞) is open. And if $\{x \in \mathbb{Q} \mid x < \pi\} \subset U \subsetneq \{x \in \mathbb{R} \mid x \le \pi\}$ for

a closed U, then there exists $x \in \{x \in \mathbb{R} \mid x \leq \pi\} \setminus U$ and $\epsilon > 0$ such that $(x - \epsilon,\, x + \epsilon) \subset U^c \subset \{x \in \mathbb{Q} \mid x < \pi\}^c$ because U^c is open.

Furthermore $(x - \epsilon,\, x - \frac{\epsilon}{2}) \subset (-\infty,\, \pi)$ because $x - \frac{\epsilon}{2} < x \leq \pi$. According to Analysis I there exists a $q \in \mathbb{Q}$ in every open interval, hence there exists $q \in \mathbb{Q}$ with $q \in (x - \epsilon,\, x - \frac{\epsilon}{2})$. Then q satisfies

- $q \in (x - \epsilon,\, x - \frac{\epsilon}{2}) \subset (x - \epsilon,\, x + \epsilon) \subset \{x \in \mathbb{Q} \mid x < \pi\}^c$, impying that either $q \in \mathbb{R} \setminus \mathbb{Q}$ or $q > \pi$.

- And $q \in (x - \epsilon,\, x - \frac{\epsilon}{2}) \implies q < \pi$ and $q \in \mathbb{Q}$.

This is a contradiction and hence such U does not exist. Thus

$$\overline{\{x \in \mathbb{Q} \mid x < \pi\}} = \bigcap \{U \supset \{x \in \mathbb{Q} \mid x < \pi\} | U \text{ closed }\} = \{x \in \mathbb{R} \mid x \leq \pi\}.$$

Thus $\overline{X} = \bigcup_{n \in \mathbb{N},\, n \geq 1}[2n - \frac{1}{n}, 2n + \frac{1}{n}] \cup \{x \in \mathbb{R} \mid x \leq \pi\} = V$.

In order to proof $V = \overline{X} = S$, it is sufficient to show, that $X \subset S$ because due to $\overline{X} = X \cup S$, $\overline{X} = S$ follows immediately.

Let $x \in X$, and $B_\epsilon(x)$ be given, show that $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$

- If $x \in (2n - \frac{1}{n},\, 2n + \frac{1}{n})$ for $n \in \mathbb{N}$, there exists $y \in (\max\{2n - \frac{1}{n},\, x - \epsilon\}, x) \subset ((2n - \frac{1}{n},\, 2n + \frac{1}{n}) \cap (x - \epsilon,\, x + \epsilon) \setminus \{x\}) \subset (X \cap (x - \epsilon,\, x + \epsilon) \setminus \{x\})$. Thus $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$.

- if $x \in \mathbb{Q}_{<\pi}$, then $x - \frac{\epsilon}{2} < x \leq \pi$ and $(x - \epsilon,\, x - \frac{\epsilon}{2}) \cap \mathbb{Q}_{<\pi} \setminus \{x\} = (x - \epsilon,\, x - \frac{\epsilon}{2}) \cap \mathbb{Q} \neq \emptyset$ because every interval contains a rational number according to Analysis I. Thus $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$.

Hence $V = \overline{X} = S$.

Determine set of all boundary points B

It is $B = \overline{X^c} \cap \overline{X}$ and

$$\overline{X^c} = (X^\circ)^c = \mathbb{R} \setminus \bigcup_{i=1}^{\infty}\left(2n - \frac{1}{n},\, 2n + \frac{1}{n}\right)$$

$$\overline{X} = \bigcup_{i=1}^{\infty}\left[2n - \frac{1}{n},\, 2n + \frac{1}{n}\right] \cup \mathbb{R}_{\leq \pi}.$$

Hence

$$
\begin{aligned}
\overline{X^c} \cap \overline{X} \;=\;& \mathbb{R} \setminus \bigcup_{i=1}^{\infty}(2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}) \cap \left(\bigcup_{i=1}^{\infty}[2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}] \cup \mathbb{R}_{\le \pi} \right) \\
=\;& \mathbb{R} \cap \bigcap_{i=1}^{\infty}(2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n})^c \cap \left(\bigcup_{i=1}^{\infty}[2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}] \cup \mathbb{R}_{\le \pi} \right) \\
=\;& \left(\bigcap_{i=1}^{\infty}(2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n})^c \cap \bigcup_{i=1}^{\infty}[2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}] \right) \cup \left(\bigcap_{i=1}^{\infty}(2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n})^c \cap \mathbb{R}_{\le \pi} \right) \\
=\;& \left(\bigcup_{i=1}^{\infty}[2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}] \setminus (2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}) \right) \cup \left(\mathbb{R}_{\le \pi} \setminus (1,\, 3) \right) \\
=\;& \bigcup_{i=1}^{\infty}\{2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}\} \cup (-\infty,\; 1) \cup (3,\; \pi] \\
=\;& \bigcup_{i=2}^{\infty}\{2n - \tfrac{1}{n},\; 2n + \tfrac{1}{n}\} \cup (-\infty,\; 1] \cup [3,\; \pi]
\end{aligned}
$$