Topology Exercise Sheet 1

October 26, 2021

Excercise 1

1.

Let $X := \{0,\ 1\}$. For a topology $\mathcal{I}$ it holds that $\mathcal{I} \subset \mathcal{P}(X) = \{\emptyset,\ \{0\},\ \{1\},\ \{0,\ 1\}\}$. Since $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = X$, $\{\emptyset,\ X\} \subset \mathcal{I}$ leaving only three possiblities.

- $\mathcal{I} = \{X,\ \emptyset\}$ is a topology because
  - $X \cup \emptyset = X \in \mathcal{I}$
  - $X \cap \emptyset = \emptyset \in \mathcal{I}$.
- $\mathcal{I} = \{X,\ \{0\},\ \emptyset\}$ is a topology because
  - $X \cup \emptyset = X \in \mathcal{I}$
  - $X \cup \{0\} = X \in \mathcal{I}$
  - $\{0\} \cup \emptyset = \{0\} \in \mathcal{I}$
  - $X \cap \emptyset = \emptyset \in \mathcal{I}$
  - $X \cap \{0\} = \{0\} \in \mathcal{I}$
  - $\{0\} \cap \emptyset = \emptyset \in \mathcal{I}$.
- $\mathcal{I} = \{X,\ \{1\},\ \emptyset\}$ is a topology because it is the second case with 0 replaced by 1.
- $\mathcal{I} = \{X,\ \{1\},\ \{0\},\ \emptyset\}$ is the discrete topology.

2.

If a topology $\mathcal{I}$ is metrizable, there exists a metric d with $d(0,1) =: b \neq 0$, because otherwise $d(0,1) = 0 \implies 0 = 1$ which is a contradiction. Then $B_{\frac{b}{2}}(0) = \{0\}$, $B_{\frac{b}{2}}(1) = \{1\}$, $B_{2b}(0) = B_{2b}(1) = \{0,\ 1\}$ are open sets in the topology $\mathcal{I}$. Then

- $\mathcal{I} = \{X,\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(0) = \{0\} \notin \mathcal{I}$
- $\mathcal{I} = \{X,\ \{0\},\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(1) = \{1\} \notin \mathcal{I}$

- $\mathcal{I} = \{X,\ \{1\},\ \emptyset\}$ is not metrizable because $B_{\frac{b}{2}}(0) = \{0\} \notin \mathcal{I}$
- $\mathcal{I} = \{X, \{1\},\ \{0\},\ \emptyset\}$ is metrizable because for every $U \in \mathcal{I}$ and every $x \in U$ there exists the ball $B := B_{\frac{b}{2}}(x) = \{x\}$ satisfying $B \subset U$. Thus if U is in $\mathcal{I}$, it is also in the induced topology $\mathcal{I}_d$. And if $U \in \mathcal{I}_d$, $U \in \mathcal{I} = \mathcal{P}(X)$.

Excercise 2

Let $U_i = \mathbb{V}(F_i)^c$ be open sets with $F_i \subset \mathbb{C}[z_1,\ \ldots,\ z_n]$ for $i \in \mathcal{J}$. Then

$$\bigcup_{i\in\mathcal{J}} U_i = \bigcup_{i\in\mathcal{J}} \mathbb{V}(F_i)^c = \left(\bigcap_{i\in\mathcal{J}} \mathbb{V}(F_i)\right)^c = \left(\mathbb{V}\left(\bigcup_{i\in\mathcal{J}} F_i\right)\right)^c$$

because

$$\begin{aligned} x \in \bigcap_{i\in\mathcal{J}} \mathbb{V}(F_i) &\Leftrightarrow f(x) = 0\ \forall f \in F_i\ \forall i \in \mathcal{J} \\ &\Leftrightarrow f(x) = 0\ \forall f \in \bigcup_{i\in\mathcal{J}} F_i. \end{aligned}$$

Thus $U_i = \mathbb{V}(F_i)^c$ is open.
Let $U_i = \mathbb{V}(F_i)^c$ be open sets with $F_i \subset \mathbb{C}[z_1,\ \ldots,\ z_n]$ for $i \in \mathcal{J} := \{1,\ \ldots,\ M\}$. Let

$$G := \{\Pi_{i=1}^M p_i \mid p_i \in F_i\} \subset \mathbb{C}[z_1,\ \ldots,\ z_n].$$

Then

$$\bigcap_{i=1}^M U_i = \bigcap_{i=1}^M \mathbb{V}(F_i)^c = \left(\bigcup_{i=1}^M \mathbb{V}(F_i)\right)^c$$

and

$$\begin{aligned} x \in \bigcup_{i=1}^M \mathbb{V}(F_i) &\Leftrightarrow \exists\ i \in \mathcal{J} :\ x \in \mathbb{V}(F_i) \\ &\Leftrightarrow \exists\ i \in \mathcal{J} :\ \forall\ f \in F_i :\ f(x) = 0 \\ &\Leftrightarrow \forall\ p = p_1 \cdots p_i \cdots p_m \in G :\ p(x) = 0 \end{aligned}$$

where the last equivalence holds because if $p_i(x) = 0$ for all $p_i \in F_i$, then $p(x) = 0$ for all $p = p_1 \cdots p_i \cdots p_m \in G$; and if for every $i \in \mathcal{J}$ there exists $f_i \in F_i$ with $f_i(x) \neq 0$ then $p(x) = f_1(x) \cdots f_M(x) \neq 0$. Thus $x \in \bigcup_{i=1}^M \mathbb{V}(F_i) \Leftrightarrow \forall\ p \in G :\ p(x) = 0 \Leftrightarrow x \in \mathbb{V}(G)$. Then

$$\bigcap_{i=1}^M U_i = \left(\bigcup_{i=1}^M \mathbb{V}(F_i)\right)^c = (\mathbb{V}(G))^c$$

is an open set. Hence the Zarski topology is a topology.

2.

Let $U = \mathbb{V}(F)^c$. It is to show that there exists $B \subset \mathcal{B}$ with $U = \bigcup B$. Let $B := \{\mathbb{V}(f) \mid f \in F\}$. Then

$$\begin{aligned} x \in \bigcup B &\Leftrightarrow \exists\, f \in F: \; x \in \mathbb{V}(f)^c \\ &\Leftrightarrow \exists\, f \in F: f(x) \neq 0 \\ &\Leftrightarrow x \notin \mathbb{V}(F) \Leftrightarrow x \in \mathbb{V}(F)^c = U \end{aligned}$$

and hence $U = \bigcup B$.

Excercise 3

1.

Let $\mathcal{I} \subset \mathcal{P}(V)$ be the topology induced by $\|.\|$ and $\mathcal{I}'$ the topology induced by $\|.\|'$. Let $U \in \mathcal{I}$ be open and $x \in U$. Because $\{B_\epsilon^{\|\cdot\|}(y) \mid y \in V, \epsilon > 0\}$ is a basis of $\mathcal{I}$, there exists $\epsilon > 0$ with $B_\epsilon^{\|\cdot\|}(x) \subset U$. Then $B_{A\epsilon}^{\|\cdot\|'}(x) \subset B_\epsilon^{\|\cdot\|}(x)$, because for all $y \in B_{A\epsilon}^{\|\cdot\|'}(x)$

$$\|y - x\| \leq \frac{1}{A}\|y - x\|' \; < \frac{1}{A}(A\epsilon) = \epsilon.$$

Hence for all $x \in U$ exists $\delta = A\epsilon > 0$ such that $B_\delta^{\|\cdot\|'}(x) = B_{A\epsilon}^{\|\cdot\|'}(x) \subset B_\epsilon^{\|\cdot\|}(x) \subset U$ and thus $U \in \mathcal{I}'$.
If $U \in \mathcal{I}'$ and $x \in U$, since $\{B_\epsilon^{\|\cdot\|'}(y) \mid y \in V, \epsilon > 0\}$ is a basis of I', there exists $\epsilon > 0$ with $B_\epsilon^{\|\cdot\|'}(x) \subset U$. Then $B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x) \subset B_\epsilon^{\|\cdot\|'}(x)$, because for all $y \in B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x)$

$$\|y - x\|' \leq B\|y - x\| < B\frac{\epsilon}{B} = \epsilon.$$

Hence for all $x \in U$ exists $\delta = \frac{\epsilon}{B} > 0$ such that $B_\delta^{\|\cdot\|}(x) = B_{\frac{\epsilon}{B}}^{\|\cdot\|}(x) \subset B_\epsilon^{\|\cdot\|'}(x) \subset U$ and thus $U \in \mathcal{I}$.

2.

It is known from Analysis I, that $\|x\|_\infty := \max_{i \in \{1,\, \ldots,\, n\}} |x_i|$ defines a norm on $\mathbb{R}^n$. Let $1 \leq p < \infty$, then for every $x \in V$ there exists $m \in \{1, \ldots, n\}$ with $\|x\|_\infty = |x_m|$ and

$$\begin{aligned} \|x\|_\infty = |x_m| = (|x_m|^p)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \|x\|_p \\ &\leq (n|x_m|^p)^{\frac{1}{p}} = n^{\frac{1}{p}}|x_m| = n^{\frac{1}{p}}\|x\|_\infty \\ &\Rightarrow \|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\|x\|_\infty. \end{aligned}$$

From this already follows that all p-norms are equivalent because if $1 \leq p, q < \infty$, then

$$\begin{aligned} & \|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty \leq n^{\frac{1}{p}} \|x\|_q \leq n^{\frac{1}{p}+\frac{1}{q}} \|x\|_\infty \leq n^{\frac{1}{p}+\frac{1}{q}} \|x\|_q. \\ \implies & \|x\|_p \leq n^{\frac{1}{p}} \|x\|_q \leq n^{\frac{1}{p}+\frac{1}{q}} \|x\|_q \\ \implies & n^{\frac{-1}{p}} \|x\|_p \leq \|x\|_q \leq n^{\frac{1}{q}} \|x\|_q. \end{aligned}$$

3.

Set $f_n(x) := x^n$, then $f_n \in C^1([0,\, 1], \mathbb{R})$ for all $n \in \mathbb{N}$. Then

$$\|f_n\|_0 = \max_{x \in [0,\, 1]} |f_n(x)| = \max_{x \in [0,\, 1]} x^n = 1$$

for all $n \in \mathbb{N}$ and

$$\|f_n\|_1 = \|f_n\|_0 + \|f_n'\|_0 = 1 + \max_{x \in [0,\, 1]} |nx^{n-1}| = 1 + n \max_{x \in [0,\, 1]} |x^{n-1}| = 1 + n.$$

for all $n \in \mathbb{N}$. If $\|.\|_0$ and $\|.\|_1$ were equivalent, there would exist $A > 0$ with

$$\|f_n\|_1 \leq A\|f_n\|_0.$$

for all $n \in \mathbb{N}$. Then

$$\infty = \lim_{n\to\infty} 1 + n = \lim_{n\to\infty} \|f_n\|_1 \leq \lim_{n\to\infty} A\|f_n\|_0 = \lim_{n\to\infty} A = A.$$

This is a contradiction and hence the two given norms are not equivalent.

Excercise 4

1.

Let $\mathcal{B} \subset \mathcal{T}$. It is to show that $\bigcup \mathcal{B} \in \mathcal{T}$.

- If $\mathcal{B} = \emptyset$, then $\bigcup \mathcal{B} = \emptyset \in \mathcal{T}$.
- Next consider the case that $\mathcal{B} \neq \emptyset$ and $\mathbb{R} \notin \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then there exists a set $B \neq \emptyset$ such that $\mathcal{B} = \{(-\infty,\, x) \mid x \in B\}$. Set

  $$S := \sup_{x \in B} x \in \mathbb{R} \cup \{\infty\}.$$

  Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = S$. Then

  $$y \in \bigcup \mathcal{B} \Leftrightarrow y \in \bigcup_{x \in B} (-\infty,\, x) \Leftrightarrow y < x \text{ for a } x \in B \Leftrightarrow y \in (-\infty,\, S),$$

  because if $y < x$ for a $x \in \mathcal{B}$, then $y < \sup_{x\in B} = S$. And if $y < \sup_{x \in B} = S$, then $y < x$ for a $x \in \mathcal{B}$. Then $\bigcup \mathcal{B} = (-\infty,\, S) \in \mathcal{T}$.

- Next consider the case that $\mathbb{R} \in \mathcal{B}$. Then $\bigcup \mathcal{B} = \mathbb{R} \in \mathcal{T}$.
- Next consider the case that $\emptyset \in \mathcal{B}$ and $\mathbb{R} \notin \mathcal{B}$. Then $\bigcup \mathcal{B} = \bigcup (\mathcal{B} \setminus \{\emptyset\}) \in \mathcal{T}$ according to cases 1 or 2.

Let $\mathcal{B} \subset \mathcal{T}$ be finite. It is to show that $\bigcap \mathcal{B} \in \mathcal{T}$.

- If $\mathcal{B} = \emptyset$, then $\bigcap \mathcal{B} = \mathbb{R}$.
- Next consider the case that $\mathcal{B} \neq \emptyset$ and $\mathbb{R} \notin \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then there exists a set $B \neq \emptyset$ such that $\mathcal{B} = \{(-\infty, x) \mid x \in B\}$. Set

  $$S := \inf_{x \in B} x \in \mathbb{B}$$

  because $\mathcal{B}$ is finite. Then

  $$y \in \bigcap \mathcal{B} \Leftrightarrow y \in \bigcap_{x \in B} (-\infty, x) \Leftrightarrow y < x \text{ for all } x \in B \Leftrightarrow y \in (-\infty, S),$$

  because if $y < x$ for all $x \in \mathcal{B}$, then $y < S \in \mathcal{B}$. And if $y < S$, then $y < S \leq x$ for all $x \in \mathcal{B}$. Then $\bigcap \mathcal{B} = (-\infty, S) \in \mathcal{T}$.
- Next consider the case that $\emptyset \in \mathcal{B}$. Then $\bigcap \mathcal{B} = \emptyset \in \mathcal{T}$.
- Next consider the case that $\mathbb{R} \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Then $\bigcap \mathcal{B} = \bigcap (\mathcal{B} \setminus \{\mathbb{R}\}) \in \mathcal{T}$ according to cases 1 or 2.

2.

$$\begin{aligned}
\overline{(0,\, 1)} &= \bigcap \{C \supset (0,\, 1) \mid C^c = (-\infty,\, x),\, x \in \mathbb{R}\} \\
&= \bigcap \{C \supset (0,\, 1) \mid C = [x,\, \infty),\, x \in \mathbb{R}\} \\
&= \bigcap \{C \supset (0,\, 1) \mid C = [x,\, \infty),\, x \leq 0\} \\
&= \bigcap \{[x, \infty) \mid x \leq 0\} \\
&= [0, \infty)
\end{aligned}$$

$$\begin{aligned}
[0,\, 1]^\circ &= \bigcup \{A \subset [0,\, 1] \mid A = (-\infty,\, x),\, x \in \mathbb{R}\} \\
&= \emptyset
\end{aligned}$$

because for all $(-\infty,\, x) \in \mathcal{T}$ $\min\{x,\, 0\} - 1 \in (\infty, x) \setminus [0,\, 1]$.

Excercise 5

1.

not open:

If X is open, there exists ϵ such that $B_\epsilon(-1) \subset X$ because $-1 \in X$ and because $\{B_\delta(y) \mid \delta > 0,\, y \in \mathbb{R}\}$ is a Basis of the euclidean Topology on $\mathbb{R}$. Let $x_n := -1 - \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$ for every $n \in \mathbb{N}$. Then $x_n \notin \mathbb{R} \setminus X$. Since $\lim_{n\to\infty} x_n = -1$, there exists an $n \in \mathbb{N}$ such that $x_n \in B_\epsilon(-1) \subset X$. This is a contradiction to $x_n \notin X$. Hence X is not open.

not closed:

If X is closed, X^c is open and there exists $\epsilon > 0$ such that $B_\epsilon(\frac{5}{2}) \subset X^c$. Let $x_n := \frac{5}{2} - \frac{1}{n+100} \in (4 - \frac{1}{2}, 4 + \frac{1}{2}) \subset X$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = \frac{5}{2}$. Then there exists $n \in \mathbb{N}$ such that $x_n \in B_\epsilon(\frac{5}{2}) \subset X^c$. This is a contradiction to $x_n \notin X^c$. Hence X is not closed.

2.

Claim: $X^\circ = \bigcup_{n\in\mathbb{N}} (2n - \frac{1}{n},\, 2n + \frac{1}{n}) := U$**.**

Proof:
Show that $X \supset U$:
For all $n \in \mathbb{N}$ $(2n - \frac{1}{n},\, 2n + \frac{1}{n})$ is open. Then U is open as a union of open sets.
Since $U \subset X$, $U \subset X^\circ$.
Show that $X^\circ \subset U$:
Assume there exists $x \in X^\circ \setminus U \subset X \setminus U$. Then $x \in \mathbb{Q} \cap ((-\infty,\, 1] \cup [3,\, \pi))$ and there exists $\frac{1}{2} > \epsilon > 0$ such that $B_\epsilon(x) \subset X^\circ \subset X$ because X° is open.

- If $x \in \mathbb{Q} \cap [3,\, \pi)$, then for $y := x + \lfloor 2\epsilon \rfloor (\pi - x) \in \mathbb{R} \setminus \mathbb{Q}$

  $$|y - x| = |x + \lfloor 2\epsilon \rfloor (\pi - x) - x| = \lfloor 2\epsilon \rfloor (\pi - x) < \lfloor 2\epsilon \rfloor \frac{1}{2} < \frac{2\epsilon}{2} = \epsilon.$$

  Then $y \in B_\epsilon(x) \subset X$ but $y \notin X$ because

  $$\begin{aligned} &(x - \pi) < 0 \text{ and } (1 - 2\epsilon) > 0 \\ \Rightarrow\ &(x - \pi)(1 - 2\epsilon) < 0 \\ \Rightarrow\ &(1 - 2\epsilon)x + (2\epsilon - 1)\pi < 0 \\ \Rightarrow\ &x + 2\epsilon(\pi - x) - \pi < 0 \\ \Rightarrow\ &y < \pi \\ \Rightarrow\ &y \in [3,\, \pi) \setminus \mathbb{Q}. \end{aligned}$$

  This is a contradiction and hence

- $x \in \mathbb{Q} \cap (-\infty, 1]$. Then for $y := x - \frac{\lfloor \epsilon \rfloor}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$

  $$|y - x| = \left| x - \frac{\lfloor \epsilon \rfloor}{\sqrt{2}} - x \right| < \left| \frac{\epsilon}{\sqrt{2}} \right| < \epsilon.$$

  Then $y \in B_\epsilon(x) \subset X$ but $y \notin X$ because $y < x \leq 1 \;\Rightarrow\; y \in (-\infty, 1) \setminus \mathbb{Q}$. This is a contradiction and hence $X^\circ \subset U$.

Claim: Set of all cluster points S is $\bigcup_{n \in \mathbb{N},\, n \geq 1} [2n - \frac{1}{n}, 2n + \frac{1}{n}] \cup \mathbb{R}_{\leq \pi} =: V$

Proof:
It is

$$\overline{X} = \bigcup_{n \in \mathbb{N},\, n \geq 1} \overline{(2n - \frac{1}{n}, 2n + \frac{1}{n})} \cup \overline{\{x \in \mathbb{Q} \mid x < \pi\}}.$$

Where $\overline{(2n - \frac{1}{n}, 2n + \frac{1}{n})} = [2n - \frac{1}{n}, 2n + \frac{1}{n}]$ for every $n \in \mathbb{N}$ because $[2n - \frac{1}{n}, 2n + \frac{1}{n}]$ is closed (because $[2n - \frac{1}{n},\, 2n + \frac{1}{n}]^c = (-\infty,\, 2n - \frac{1}{n}) \cup (2n + \frac{1}{n},\, \infty)$ is open as a union of open sets) and if $(2n - \frac{1}{n}, 2n + \frac{1}{n}) \subset U \subsetneq [2n - \frac{1}{n}, 2n + \frac{1}{n}]$ for a closed U, then $2n - \frac{1}{n} \notin U$ or $2n + \frac{1}{n} \notin U$.

- If $2n + \frac{1}{n} \notin U$ then there exists $\frac{1}{n} > \epsilon > 0$ such that $B_\epsilon(2n + \frac{1}{n}) \subset U^c$ because U^c is open. Then for $x_m := 2n + \frac{1}{n} - \frac{1}{m} \in (2n - \frac{1}{n},\, 2n + \frac{1}{n}) \subset U$ $\lim_{m \to \infty} x_m = 2n + \frac{1}{n}$. Then there exists $m \in \mathbb{N}$ such that $x_m \in (B_\epsilon(2n + \frac{1}{n}) \cap U) \subset (U^c \cap U)$. This is a contradiction.
- If $2n - \frac{1}{n} \notin U$ then there exists $\frac{1}{n} > \epsilon > 0$ such that $B_\epsilon(2n - \frac{1}{n}) \subset U^c$ because U^c is open. Then for $x_m := 2n - \frac{1}{n} + \frac{1}{m} \in (2n - \frac{1}{n},\, 2n + \frac{1}{n}) \subset U$ $\lim_{m \to \infty} x_m = 2n - \frac{1}{n}$. Then there exists $m \in \mathbb{N}$ such that $x_m \in (B_\epsilon(2n - \frac{1}{n}) \cap U) \subset (U^c \cap U)$. This is a contradiction.

Then $\overline{(2n - \frac{1}{n},\, 2n + \frac{1}{n})} = \bigcap \{U \supset (2n - \frac{1}{n},\, 2n + \frac{1}{n}) \mid U \text{ closed}\} = [2n - \frac{1}{n},\, 2n + \frac{1}{n}]$.

And $\overline{\{x \in \mathbb{Q} \mid x < \pi\}} = \{x \in \mathbb{R} \mid x \leq \pi\}$ because $\{x \in \mathbb{R} \mid x \leq \pi\}$ is closed because $(\pi,\, \infty)$ is open. And if $\{x \in \mathbb{Q} \mid x < \pi\} \subset U \subsetneq \{x \in \mathbb{R} \mid x \leq \pi\}$ for a closed U, then there exists $x \in \{x \in \mathbb{R} \mid x \leq \pi\} \setminus U$ and $\epsilon > 0$ such that $(x - \epsilon,\, x + \epsilon) \subset U^c \subset \{x \in \mathbb{Q} \mid x < \pi\}^c$ because U^c is open.
Furthermore $(x - \epsilon,\; x - \frac{\epsilon}{2}) \subset (-\infty,\; \pi)$ because $x - \frac{\epsilon}{2} < x \leq \pi$. According to Analysis I there exists a $q \in \mathbb{Q}$ in every open interval, hence there exists $q \in \mathbb{Q}$ with $q \in (x - \epsilon,\, x - \frac{\epsilon}{2})$. Then q satisfies

- $q \in (x - \epsilon,\; x - \frac{\epsilon}{2}) \subset (x - \epsilon,\; x + \epsilon) \subset \{x \in \mathbb{Q} \mid x < \pi\}^c$, impying that either $q \in \mathbb{R} \setminus \mathbb{Q}$ or $q > \pi$.
- And $q \in (x - \epsilon,\; x - \frac{\epsilon}{2}) \implies q < \pi$ and $q \in \mathbb{Q}$.

This is a contradiction and hence such U does not exist. Thus

$$\overline{\{x \in \mathbb{Q} \mid x < \pi\}} = \bigcap \{U \supset \{x \in \mathbb{Q} \mid x < \pi\} | U \text{ closed }\} = \{x \in \mathbb{R} \mid x \leq \pi\}.$$

Thus $\overline{X} = \bigcup_{n\in\mathbb{N},\, n\geq 1}[2n - \frac{1}{n}, 2n + \frac{1}{n}] \cup \{x \in \mathbb{R} \mid x \leq \pi\} = V$.
In order to proof $V = \overline{X} = S$, it is sufficient to show, that $X \subset S$ because due to $\overline{X} = X \cup S$, $\overline{X} = S$ follows immediately.
Let $x \in X$, and $B_\epsilon(x)$ be given, show that $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$

- If $x \in (2n-\frac{1}{n},\ 2n+\frac{1}{n})$ for $n \in \mathbb{N}$, there exists $y \in (\max\{2n-\frac{1}{n},\ x-\epsilon\}, x) \subset ((2n-\frac{1}{n},\ 2n+\frac{1}{n}) \cap (x-\epsilon,\ x+\epsilon) \setminus \{x\}) \subset (X \cap (x-\epsilon,\ x+\epsilon) \setminus \{x\})$. Thus $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$.
- if $x \in \mathbb{Q}_{<\pi}$, then $x - \frac{\epsilon}{2} < x \leq \pi$ and $(x-\epsilon,\ x-\frac{\epsilon}{2}) \cap \mathbb{Q}_{<\pi} \setminus \{x\} = (x-\epsilon,\ x-\frac{\epsilon}{2}) \cap \mathbb{Q} \neq \emptyset$ because every interval contains a rational number according to Analysis I. Thus $X \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$.

Hence $V = \overline{X} = S$.

Determine set of all boundary points B

It is $B = \overline{X^c} \cap \overline{X}$ and

$$\begin{aligned}
\overline{X^c} &= (X^\circ)^c = \mathbb{R} \setminus \bigcup_{i=1}^{\infty}(2n - \frac{1}{n},\ 2n + \frac{1}{n}) \\
\overline{X} &= \bigcup_{i=1}^{\infty}[2n - \frac{1}{n},\ 2n + \frac{1}{n}] \cup \mathbb{R}_{\leq\pi}.
\end{aligned}$$

Hence

$$\begin{aligned}
\overline{X^c} \cap \overline{X} &= \mathbb{R} \setminus \bigcup_{i=1}^{\infty}(2n - \frac{1}{n},\ 2n + \frac{1}{n}) \cap \left(\bigcup_{i=1}^{\infty}[2n - \frac{1}{n},\ 2n + \frac{1}{n}] \cup \mathbb{R}_{\leq\pi}\right) \\
&= \mathbb{R} \cap \bigcap_{i=1}^{\infty}(2n - \frac{1}{n},\ 2n + \frac{1}{n})^c \cap \left(\bigcup_{i=1}^{\infty}[2n - \frac{1}{n},\ 2n + \frac{1}{n}] \cup \mathbb{R}_{\leq\pi}\right) \\
&= \left(\bigcap_{i=1}^{\infty}(2n - \frac{1}{n},\ 2n + \frac{1}{n})^c \cap \bigcup_{i=1}^{\infty}[2n - \frac{1}{n},\ 2n + \frac{1}{n}]\right) \cup \left(\bigcap_{i=1}^{\infty}(2n - \frac{1}{n},\ 2n + \frac{1}{n})^c \cap \mathbb{R}_{\leq\pi}\right) \\
&= \left(\bigcup_{i=1}^{\infty}[2n - \frac{1}{n},\ 2n + \frac{1}{n}] \setminus (2n - \frac{1}{n},\ 2n + \frac{1}{n})\right) \cup (\mathbb{R}_{\leq\pi} \setminus (1,\ 3)) \\
&= \bigcup_{i=1}^{\infty}\{2n - \frac{1}{n}, 2n + \frac{1}{n}\} \cup (-\infty,\ 1) \cup (3,\ \pi] \\
&= \bigcup_{i=2}^{\infty}\{2n - \frac{1}{n}, 2n + \frac{1}{n}\} \cup (-\infty,\ 1] \cup [3,\ \pi]
\end{aligned}$$