2η Σειρά Ασκήσεων: Σήματα και Συστήματα

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Άσκηση 1:

a) $h_d[n] = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} j\frac{\omega}{T_s}\cdot \exp(j\omega n)\cdot d\omega \overset{n\neq 0}{=}$

$$= \frac{1}{2\pi\cdot T_s\cdot n}\int_{-\pi}^{\pi}(j\omega n)\cdot \exp(j\omega n)\cdot d\omega$$

Θέτω $j\omega n = x \Rightarrow \omega = \frac{x}{jn}$ και $d\omega = \frac{dx}{jn}$

$\omega = \pi \Rightarrow x = \pi j n$

$\omega = -\pi \Rightarrow x = -\pi j n$

$$h_d[n] = \frac{1}{2\pi T_s\cdot n}\int_{-\pi j n}^{\pi j n} x\cdot e^x\cdot\frac{dx}{jn} = \frac{1}{2\pi\cdot T_s\cdot n^2 j}\int_{-\pi j n}^{\pi j n} x\cdot e^x\, dx = \frac{1}{2\pi T_s n^2 j}\cdot I_1$$

$$\int x\cdot e^x\, dx = x\cdot e^x - \int e^x\, dx = x\cdot e^x - e^x = (x-1)\cdot e^x + c$$

$$I_1 = (j\pi n - 1)\cdot e^{j\pi n} + (j\pi n + 1)\cdot e^{-j\pi n} = j\pi n\,(e^{j\pi n} + e^{-j\pi n}) - (e^{j\pi n} - e^{-j\pi n}) =$$
$$= 2\cdot j\pi n\cdot \cos(\pi n) - \cancel{2j\cdot \sin(\pi n)}^{0} = 2j\pi n\cdot(-1)^n \quad , n\neq 0.$$

Για $n=0$: $h_d[0] = \frac{j}{2\pi T_s}\cdot\int_{-\pi}^{\pi}\omega\cdot d\omega = \frac{j}{2\pi T_s}\cdot\frac{\omega^2}{2}\Big|_{-\pi}^{\pi} \Rightarrow$

$h_d[0] = \frac{j}{2\pi T_s}\left(\cancel{\frac{\pi^2}{2} - \frac{\pi^2}{2}}^{0}\right) = 0$ και $h_d[n] = \frac{2\pi j n(-1)^n}{2\pi T_s n^2 j} = \frac{(-1)^n}{T_s n}$, $n\neq 0$

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Άρα $h_d[n] = \begin{cases} 0 & , n=0 \\ \dfrac{(-1)^n}{n \cdot T_s} & , n \neq 0 \end{cases}$

β). $y_d[n] = h_d[n] * x_d[n] \Rightarrow$

$$\boxed{y_d[n] = \frac{(-1)^n}{n \cdot T_s} * x_d[n]}$$

Επίσης στο πεδίο της συχνότητας, σύμφωνα με τη λειτουργία του συστήματος μας θα ισχύει:

$$Y_c(\omega) = H_r(\omega) \cdot Y_s(\omega) = H_r(\omega) \cdot Y(\omega T_s) = H_r(\omega) \cdot X(\omega T_s) \cdot H_d(\omega T_s) =$$
$$= H_r(\omega) \cdot X_s(\omega) \cdot H_d(\omega T_s) = X_c(\omega) \cdot H_d(\omega T_s) =$$
$$= X_c(\omega) \cdot H_c(\omega).$$

όπου $X_s(\omega) = X(\omega T_s)$, $Y_s(\omega) = Y(\omega T_s)$, $H_r(\omega)$ απόκριση συχνότητας του φίλτρου και $H_c(\omega)$ απόκριση συχνότητας του συστήματος.

Άρα στο πεδίο του χρόνου έχουμε:

$y_c(t) = x_c(t) * h_c(t)$ (1).

$$H_c(\omega) = \begin{cases} H_d(\omega T_s) & , |\omega| \leq \pi/T_s \\ 0 & , |\omega| > \pi/T_s \end{cases} \Rightarrow$$

$$H_c(\omega) = \begin{cases} j\omega & , |\omega| \leq \pi/T_s \\ 0 & , |\omega| > \pi/T_s \end{cases}$$

και $h_c(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} H_c(\omega) \cdot e^{j\omega t} d\omega = \dfrac{1}{2\pi} \displaystyle\int_{-\pi/T_s}^{\pi/T_s} j\omega \cdot e^{j\omega t} \cdot d\omega$ (t≠0)

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$$= \frac{1}{2\pi t} \cdot \int_{-\pi/T_s}^{\pi/T_s} (j\omega t) \cdot e^{j\omega t} \cdot d\omega$$

Θέτω $j\omega t = x \Rightarrow \omega = \frac{x}{jt} \Rightarrow d\omega = \frac{dx}{jt}$

$\omega = -\frac{\pi}{T_s} \Rightarrow x = \frac{-\pi j t}{T_s}$

$\omega = \frac{\pi}{T_s} \Rightarrow x = \frac{\pi j t}{T_s}$

$$h_c(t) = \frac{1}{2\pi t} \cdot \int_{-\frac{\pi j t}{T_s}}^{\frac{\pi j t}{T_s}} x \cdot e^{x} \cdot \frac{dx}{jt} = \frac{1}{2\pi j t^2} \int_{-\frac{\pi j t}{T_s}}^{\frac{\pi j t}{T_s}} x \cdot e^{x} dx$$

$$\overset{\Sigma 1}{=} \frac{1}{2\pi j t^2} \left[ \left( \frac{j\pi t}{T_s} - 1 \right) \cdot e^{\frac{j\pi t}{T_s}} + \left( \frac{j\pi t}{T_s} + 1 \right) \cdot e^{-\frac{j\pi t}{T_s}} \right] =$$

$$= \frac{1}{2\pi j t^2} \left[ \frac{j\pi t}{T_s} \cdot \left( e^{\frac{j\pi t}{T_s}} + e^{-\frac{j\pi t}{T_s}} \right) - \left( e^{\frac{j\pi t}{T_s}} - e^{-\frac{j\pi t}{T_s}} \right) \right] =$$

$$= \frac{j\pi t}{\cancel{2}\pi j t^2 \cdot T_s} \cdot \cancel{2} \cos\left( \frac{\pi t}{T_s} \right) - \frac{1}{\cancel{2}\pi j t^2} \cdot \cancel{2} j \sin\left( \frac{\pi t}{T_s} \right) =$$

$$= \frac{1}{t \cdot T_s} \cdot \cos\left( \frac{\pi t}{T_s} \right) - \frac{1}{\pi t^2} \cdot \sin\left( \frac{\pi t}{T_s} \right), \quad t \neq 0$$

Για $t = 0$: $h_c(0) = \frac{1}{2\pi} \int_{-\pi/T_s}^{\pi/T_s} j\omega \, d\omega = \frac{j}{2\pi} \left. \frac{\omega^2}{2} \right|_{-\pi/T_s}^{\pi/T_s} = \frac{j}{2\pi} \left( \cancel{\frac{(\frac{\pi}{T_s})^2}{2} - \frac{(\frac{\pi}{T_s})^2}{2}} \right) = 0$

$h_c(0) = 0$.

Δηλαδή (1) $\Rightarrow y_c(t) = \begin{cases} x_c(t) * \frac{1}{t \cdot T_s} \cdot \cos\left(\frac{\pi t}{T_s}\right) - \frac{1}{\pi t^2} \cdot \sin\left(\frac{\pi t}{T_s}\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$

Το ολικό σύστημα είναι ένα ΓΧΑ συνεχούς χρόνου με κέντρο ένα ΓΧΑ διακριτού χρόνου με απόκριση $H_d - H_d$. Το δεύτερο εκτελεί την ψηφιακή επεξεργασία του σήματος $x[n]$ που δέχεται ως είσοδο του. Το ολικό σύστημα είναι ένα σύστημα ψηφιακής επεξεργασίας αναλογικών σημείων που δέχεται το σήμα $x_c(t)$ και παράγει το $y_c(t)$

γ). $x_c(t) = \frac{\sin(\frac{\pi t}{T})}{\pi t}$ και ψάχνω $y_d[n]$:

$Y_d(\Omega) = H_d(\Omega) \cdot X_d(\Omega)$.

$x_d[n] = x_c(nT_s) = \frac{\sin\left(\frac{\pi n \cdot T_s}{T}\right)}{\pi n T_s} = \frac{1}{T_s} \cdot \frac{\sin\left(\frac{\pi T_s \cdot n}{T}\right)}{\pi n}$

Από DTFT: $\left\{\frac{\sin(wn)}{\pi n}\right\} \xleftrightarrow{DTFT} \left\{\begin{array}{l} X(\Omega) = \begin{cases} 1, & 0 \le |\Omega| \le w \\ 0, & w < |\Omega| \le \pi \end{cases} \\ X(\Omega) = X(\Omega + 2\pi) \end{array}\right\}$

για $0 < w < \pi$.

Όπου $w = \frac{\pi \cdot T_s}{T} < \frac{\pi}{2} < \pi$ λόγω προϋπόθεσης δειγματοληψίας

$(T_s < \frac{T}{2})$. Άρα $X_d(\Omega) = \begin{cases} \frac{1}{T_s}, & 0 \le |\Omega| \le \frac{\pi \cdot T_s}{T} \\ 0, & \frac{\pi \cdot T_s}{T} < |\Omega| \le \pi. \end{cases}$

Άρα. $Y_d(\Omega) = \begin{cases} j\Omega / T_s^2, & 0 \le |\Omega| \le \frac{\pi \cdot T_s}{T} \\ 0, & \frac{\pi \cdot T_s}{T} < |\Omega| \le \pi. \end{cases}$

Επομένως $y_d[n] = \frac{1}{2\pi} \int^{\pi} Y_d(\Omega) \cdot e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\frac{\pi T_s}{T}}^{\frac{\pi T_s}{T}} \frac{j\Omega}{T_s^2} \cdot e^{j\Omega n} \cdot d\Omega \overset{n \neq 0}{=}$

$= \frac{1}{2\pi T_s^2 n} \int_{-\frac{\pi T_s}{T}}^{\frac{\pi T_s}{T}} (j\Omega n) \cdot e^{j\Omega n} d\Omega \overset{j\Omega n = x}{=} \frac{1}{2\pi T_s^2 n} \int_{-\frac{jn\pi T_s}{T}}^{\frac{jn\pi T_s}{T}} x \cdot e^x \frac{dx}{jn} \Rightarrow$

$y_d[n] = \frac{1}{2\pi T_s^2 n^2 j} \left[\left(\frac{jn\pi T_s}{T} - 1\right) \cdot e^{\frac{jn\pi T_s}{T}} + \left(\frac{j\pi n T_s}{T} + 1\right) \cdot e^{-\frac{jn\pi T_s}{T}}\right] =$

$$= \frac{1}{2\pi T_s^2 n^2 j}\left[\frac{j\pi n T_s}{T}\left(e^{\frac{j\pi n T_s}{T}}+e^{-\frac{j\pi n T_s}{T}}\right)-\left(e^{\frac{j\pi n T_s}{T}}-e^{-\frac{j\pi n T_s}{T}}\right)\right] =$$

$$= \frac{\cancel{j\pi} n T_s \cancel{2}}{\cancel{2\pi} T_s^2 n^2 \cancel{j} T}\cos\left(\frac{\pi n T_s}{T}\right) - \frac{1 \cdot \cancel{2j}}{\cancel{2}\pi T_s^2 n^2 \cancel{j}}\cdot \sin\left(\frac{\pi n T_s}{T}\right) =$$

$$= \frac{1}{T_s n \cdot T}\cdot \cos\left(\frac{\pi n T_s}{T}\right) - \frac{1}{\pi T_s^2 n^2}\cdot \sin\left(\frac{\pi n T_s}{T}\right), \quad n \neq 0$$

Για $n=0$: $y_d[0] = \frac{1}{2\pi}\int_{-\frac{\pi T_s}{T}}^{\frac{\pi T_s}{T}} \frac{j\theta}{T_s^2}\, d\theta = \frac{j}{2\pi T_s^2}\cdot \int_{-\frac{\pi T_s}{T}}^{\frac{\pi T_s}{T}} \theta\, d\theta = \frac{j}{2\pi T_s^2}\left.\frac{\theta^2}{2}\right|_{-\frac{\pi T_s}{T}}^{\frac{\pi T_s}{T}} =$

$$= \frac{j}{2\pi T_s^2}\left[\frac{1}{2}\left(\frac{\pi T_s}{T}\right)^2 - \frac{1}{2}\cdot\left(\frac{\pi T_s}{T}\right)^2\right] = 0$$

- $$y_d[n] = \begin{cases} \frac{1}{T_s \cdot n \cdot T}\cdot \cos\left(\frac{\pi n T_s}{T}\right) - \frac{1}{T_s^2 n^2 \pi}\cdot \sin\left(\frac{\pi n T_s}{T}\right), & n \neq 0 \\ 0, & n = 0 \end{cases}$$

δ) $Y_c(\omega) = X_c(\omega) \cdot H_c(\omega)$.

$$\frac{\sin(Wt)}{\pi t} \overset{F}{\longleftrightarrow} X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Άρα με $W = \frac{\pi}{T}$ : $X_c(\omega) = \begin{cases} 1, & |\omega| < \frac{\pi}{T} \\ 0, & |\omega| > \frac{\pi}{T} \end{cases}$

και επομένως: $Y_c(\omega) = \begin{cases} j\omega, & |\omega| < \frac{\pi}{T} \\ 0, & |\omega| > \frac{\pi}{T} \end{cases}$, αφού $\frac{\pi}{T_s} > \frac{\pi}{T}$

$$y_c(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y_c(\omega)\cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} j\omega \cdot e^{j\omega t}\, d\omega \overset{t \neq 0}{=}$$

⑥

$$= \frac{1}{2\pi t} \cdot \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} (j\omega t) e^{j\omega t} d\omega \overset{j\omega t = x}{=} \frac{1}{2\pi t} \cdot \int_{-\frac{\pi j t}{T}}^{\frac{\pi j t}{T}} x \cdot e^{x} \cdot \frac{dx}{jt} \overset{I_1}{=}$$

$$= \frac{1}{2\pi \cdot t^2 j} \left[ \left(\frac{\pi j t}{T} - 1\right) \cdot e^{\frac{\pi j t}{T}} + \left(\frac{\pi j t}{T} + 1\right) \cdot e^{-\frac{\pi j t}{T}} \right] =$$

$$= \frac{1}{2\pi t^2 j} \left[ \frac{\pi j t}{T} \left( e^{\frac{\pi j t}{T}} + e^{-\frac{\pi j t}{T}} \right) - \left( e^{\frac{\pi j t}{T}} - e^{-\frac{\pi j t}{T}} \right) \right] =$$

$$= \frac{\cancel{\pi j t} \cdot \cancel{2}}{\cancel{2\pi} t^{\cancel{2}} \cdot \cancel{j} \cdot T} \cdot \cos\left(\frac{\pi t}{T}\right) - \frac{1 \cdot 2j}{2\pi t^2 \cdot j} \sin\left(\frac{\pi t}{T}\right) =$$

$$= \frac{1}{t \cdot T} \cdot \cos\left(\frac{\pi t}{T}\right) - \frac{1}{\pi t^2} \cdot \sin\left(\frac{\pi t}{T}\right), \ t \neq 0$$

Για $t=0$: $y_c(0) = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} j\omega \, d\omega = \frac{j}{2\pi} \left. \frac{\omega^2}{2} \right|_{-\frac{\pi}{T}}^{\frac{\pi}{T}} = \frac{j}{2\pi} \left( \cancel{\frac{(\frac{\pi}{T})^2}{2} - \frac{(\frac{\pi}{T})^2}{2}} \right) = 0$

Άρα $y_c(t) = \begin{cases} \frac{1}{t \cdot T} \cos\left(\frac{\pi t}{T}\right) - \frac{1}{\pi t^2} \cdot \sin\left(\frac{\pi t}{T}\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$

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Άσκηση 2:

α) $y[n] = y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

Χαρακτηριστική εξίσωση: $z^2 - z + 1/2 = 0 \Rightarrow z_1 = \frac{1+j}{2}, z_2 = \frac{1-j}{2}$

Τότε: $y_{zi}[n] = \sum_{k=1}^{2} c_k (z_k)^n = c_1 z_1^n + c_2 z_2^n$

$y[-1] = \frac{3}{4} \Rightarrow c_1 z_1^{-1} + c_2 z_2^{-1} = 3/4$

$y[-2] = \frac{1}{4} \Rightarrow c_1 \cdot z_1^{-2} + c_2 \cdot z_2^{-2} = 1/4$

επίλυση συστήματος: $c_1 = \frac{5}{16} + j\frac{1}{16}$, $c_2 = \frac{5}{16} - j\frac{1}{16}$

$y_{zi}[n] = \left(\frac{5}{16} + j\frac{1}{16}\right)\left(\frac{1+j}{2}\right)^n + \left(\frac{5}{16} - j\frac{1}{16}\right)\left(\frac{1-j}{2}\right)^n \Rightarrow$

$$\boxed{y_{zi}[n] = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \frac{5}{8} \cos\left(\frac{\pi n}{5}\right) - \left(\frac{1}{\sqrt{2}}\right)^n \cdot \frac{1}{8} \sin\left(\frac{\pi n}{5}\right)}$$

Επίσης: $y_{zs}[n] = K \cdot b^n \overset{b=\frac{1}{2}}{=} K\left(\frac{1}{2}\right)^n$

Από εξίσωση διαφορών και με το ανώτερο παίρνω:

$K\left(\frac{1}{2}\right)^n = K\left(\frac{1}{2}\right)^{n-1} - \frac{1}{2}K\left(\frac{1}{2}\right)^{n-2} + \frac{1}{2}\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} \Rightarrow$

$K\left(\frac{1}{2}\right)^2 = K\left(\frac{1}{2}\right) - \frac{1}{2}K + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{2}\cdot\frac{1}{2} \Rightarrow \boxed{K = \frac{3}{2}}$

Άρα $\boxed{y_{zs} = \frac{3}{2}\left(\frac{1}{2}\right)^n \cdot u(n)}$

$y = y_{zi} + y_{zs} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \frac{5}{8}\cos\left(\frac{n\pi}{5}\right) - \left(\frac{1}{\sqrt{2}}\right)^n \cdot \frac{1}{8}\sin\left(\frac{n\pi}{5}\right) + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n]$

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β) $y(n) = y(n-1) - \frac{1}{2}y(n-2) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$ $\Rightarrow$

$y(n) = -\sum_{i=1}^{p} a_i\, y(n-i) + \sum_{i=0}^{q} b_i\, x(n-i)$

$a_1\, y(n-1) = -1\, y(n-1) \Rightarrow a_1 = -1$

$a_2\, y(n-2) = \frac{1}{2}y(n-2) \Rightarrow a_2 = \frac{1}{2}$

$b_0\, x(n) = \frac{1}{2}x(n) \Rightarrow b_0 = \frac{1}{2}$

$b_1\, x(n-1) = \frac{1}{2}x(n-1) \Rightarrow b_1 = \frac{1}{2}$

$A(z) = 1 + \sum_{i=1}^{p} a_i \cdot z^{-i} = 1 + (-1)\cdot z^{-1} + \frac{1}{2}\cdot z^{-2} = 1 - z^{-1} + \frac{1}{2}\cdot z^{-2}$

$B(z) = \sum_{i=0}^{q} b_i \cdot z^{-i} = \frac{1}{2} + \frac{1}{2}\cdot z^{-1}$

$Y_{zs}(z) = \dfrac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \cdot X(z)$, $X(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$

$Y_{zs}(z) = \dfrac{\frac{1}{2} + \frac{1}{2}z^{-1}}{(1 - z^{-1} + \frac{1}{2}\cdot z^{-2})(1 - \frac{1}{2}z^{-1})} = \dfrac{\frac{1}{2}z^3 + \frac{1}{2}z^2}{(z^2 - z + \frac{1}{2})(z - \frac{1}{2})} =$

$= \dfrac{z^2(\frac{1}{2}z + \frac{1}{2})}{(z^2 - z + \frac{1}{2})(z - \frac{1}{2})} \Rightarrow$

$\dfrac{Y_{zs}(z)}{z} = \dfrac{3}{2z - 1} - \dfrac{z - 3/2}{z^2 - z + 1/2} \Rightarrow$

$Y_{zs}(z) = \dfrac{3z}{2z - 1} - \dfrac{z(z - 3/2)}{z^2 - z + 1/2}$

$\dfrac{3z}{2z - 1} = \dfrac{3}{2}\cdot\dfrac{z}{z - \frac{1}{2}} = \dfrac{3}{2}\cdot\dfrac{1}{1 - \frac{1}{2}z^{-1}} \overset{Z}{\longleftrightarrow} \frac{3}{2}\left(\frac{1}{2}\right)^n u(n)$

τα: $\dfrac{z(z - \frac{3}{2})}{z^2 - z + \frac{1}{2}} = \dfrac{1 - \frac{3}{2}\cdot z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \dfrac{1 - \frac{1}{2}\cdot z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} - 2\cdot\dfrac{\frac{1}{2}\cdot z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \overset{Z}{\longleftrightarrow}$

$\left(\frac{1}{\sqrt{2}}\right)^2 \cos\left(\frac{\pi}{4}n\right)u(n) - 2\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{4}n\right)\cdot u(n).$

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άρα $y_{zs}(n) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^n u(n) + 2\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{3}n\right)u(n) - \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{3}n\right)\cdot u(n).$

και

$$Y_{zi}(z) = -\frac{1}{A(z)}\sum_{i=1}^{p} a_i \sum_{k=1}^{i} y(-k)\cdot z^{-i+k} =$$

$$= -\frac{1}{A(z)}\sum_{i=1}^{3} a_i \sum_{k=1}^{i} z^{-i+k} =$$

$$= -\frac{3}{8}\cdot\frac{z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{5}{8}\cdot\frac{1}{1-z^{-1}+\frac{1}{2}z^{-2}} =$$

$$= -\frac{3}{4}\,\frac{\frac{1}{2}\cdot z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{5}{8}\,\frac{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}} \Rightarrow$$

$$y_{zi}(n) = \frac{5}{8}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi n}{5}\right) - \frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi n}{5}\right)$$

Άρα $Y(z) = Y_{zi}(z) + Y_{zs}(z) \Rightarrow$

$$Y(z) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^n u(n) + 2\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{3}n\right)u(n) - \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{3}n\right)u(n) + \frac{5}{8}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{5}n\right) - \frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi n}{5}\right).$$

γ) DTFT: $Y(\Omega) = e^{-j\Omega}Y(\Omega) - \frac{1}{2}e^{-2j\Omega}Y(\Omega) + \frac{1}{2}X(\Omega) + \frac{1}{2}e^{-j\Omega}X(\Omega) \Rightarrow$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{2}\cdot\frac{1+e^{-j\Omega}}{1-e^{-j\Omega}+\frac{1}{2}e^{-2j\Omega}}$$

δ). $H(\Omega) = \frac{1}{2}\cdot\frac{e^{j\Omega}(e^{j\Omega}+1)}{e^{2j\Omega}-e^{j\Omega}+\frac{1}{2}}$

και $h(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{j\Omega(n+1)}(e^{j\Omega}+1)}{e^{2j\Omega}-e^{j\Omega}+\frac{1}{2}}\cdot d\Omega = \ldots$

Άσκηση 3:

α) $x[n] = \begin{cases} \sum_{k=-n}^{n} a^{|k|} &, n \geq 0 \\ 0 &, n < 0 \end{cases}$ με $|a| < 1$

• $n \geq 0$: $x[n] = \sum_{k=-n}^{-1} a^{|k|} + 1 + \sum_{k=1}^{n} a^{|k|} = 2\sum_{k=1}^{n} a^{k} + 1 =$

$= 2 \cdot \frac{a^{n+1} - a}{a-1} + 1 = \frac{2a^{n+1} - 2a + a - 1}{a-1} \Rightarrow$

$x[n] = \frac{2a^{n+1} - a - 1}{a-1} = \frac{2a \cdot a^{n}}{a-1} - \frac{a+1}{a-1}$

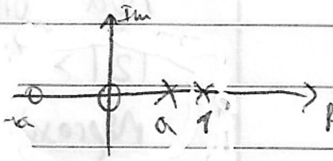
Άρα $x[n] = \frac{2a}{a-1} \cdot a^{n} \cdot u[n] - \frac{a+1}{a-1} \cdot u[n]$

Άρα $X(z) = \frac{2a}{a-1} \cdot \frac{1}{1 - a z^{-1}} - \frac{a+1}{a-1} \cdot \frac{z}{z-1} =$

$= \frac{2a}{a-1} \cdot \frac{z}{z-a} - \frac{a+1}{a-1} \cdot \frac{z}{z-1} \Rightarrow$

$X(z) = \frac{z}{\cancel{a-1}} \cdot \frac{\cancel{(a-1)}(z+a)}{(z-a)(z-1)} = \frac{z(z+a)}{(z-a)(z-1)}$, $|z| > 1$.

Πόλοι: $p_1 = a$, $p_2 = 1$.
Μηδενικά: $z_1 = 0$, $z_2 = -a$.




Σύγκλιση για $|z| > 1$, άρα ο DTFT δεν υπάρχει αφού ο μοναδιαίος κύκλος είναι εκτός περιοχής σύγκλισης.

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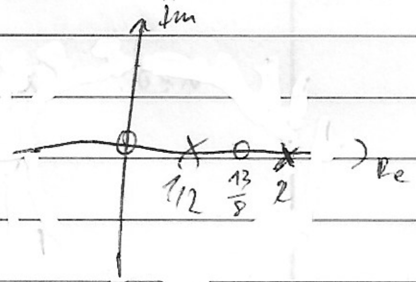
β) $x[n] = 2^n \cdot u[n] + 3\left(\frac{1}{2}\right)^n \cdot u[n]$

$$X(z) = \frac{z}{z-2} + 3 \cdot \frac{z}{z-\frac{1}{2}}, \quad |z| > 2 \text{ και } |z| > \frac{1}{2} \Rightarrow |z| > 2.$$

$$X(z) = \frac{z\left(4z - \frac{13}{2}\right)}{(z-2)(z-1/2)}$$




Πόλοι: $p1 = 1/2$, $p2 = 2$.

Μηδενικά: $z1 = 0$, $z2 = \frac{13}{8}$.

Σύγκλιση για $|z| > 2$, άρα ο DTFT δεν υπάρχει, αφού ο μοναδιαίος κύκλος είναι εκτός της περιοχής συγκλίσεως.

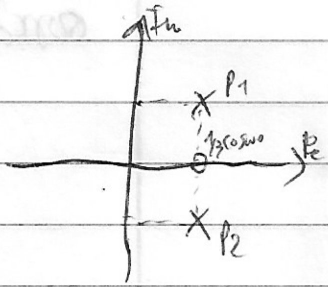
γ). $x[n] = \left(\frac{1}{3}\right)^n \cdot \cos(n w_0) \cdot u[n]$

$$X(z) = \frac{1 - \frac{1}{3} \cdot \cos(w_0) \cdot z^{-1}}{1 - \frac{2}{3} \cdot \cos(w_0) \cdot z^{-1} + \frac{1}{9} \cdot z^{-2}}, \quad |z| > \frac{1}{3}.$$

Άρα υπάρχει ο DTFT, αφού έχουμε σύγκλιση για $|z| > 1/3$ και ο μοναδιαίος κύκλος ανήκει στην περιοχή σύγκλισης.

$$X(z) = \frac{z^2 - \frac{1}{3}\cos(w_0) \cdot z}{z^2 - \frac{2}{3}\cos(w_0) \cdot z + \frac{1}{9}} = \frac{9z^2 - 3\cos(w_0) \cdot z}{9z^2 - 6\cos(w_0) \cdot z + 1} =$$

$$= \frac{3z(3z - \cos(w_0))}{[3z - (\cos w_0 + j\sin w_0)][3z - (\cos w_0 - j\sin w_0)]}$$




Πόλοι: $p1 = \frac{1}{3}\cos w_0 + \frac{1}{3} j \sin w_0$, $p2 = \frac{1}{3}\cos w_0 - \frac{1}{3} j\sin w_0$

Μηδενικά: $z1 = 0$, $z2 = \frac{1}{3} \cdot \cos(w_0)$

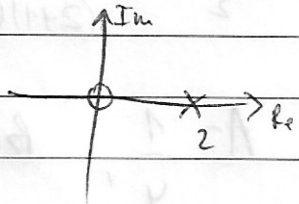
στ) $x[n] = -(2)^n \cdot u[-n-1]$.

$$X(z) = \frac{1}{1-2\cdot z^{-1}} = \frac{z}{z-2}, \quad |z| < 2.$$

Άρα υπάρχει ο DTFT, αφού έχουμε σύγκλιση για $|z|<2$ και ο μοναδιαίος κύκλος ανήκει στην περιοχή σύγκλισης.

Πόλοι: $p_1 = 2$

Μηδενικά: $z_1 = 0$.




ε) $x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right] \cdot u[n-10] = \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{n-10} \cdot u[n-10] + \left(\frac{3}{4}\right)^{10}\left(\frac{3}{4}\right)^{n-10} u[n-10]$.

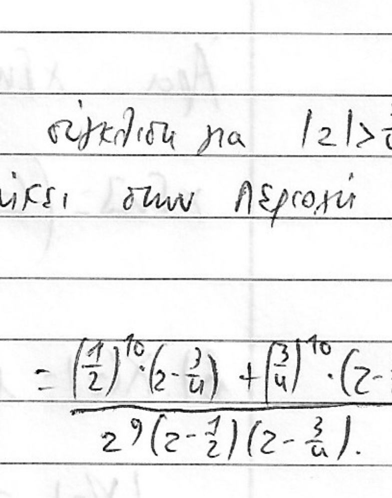
$$X(z) = \left(\frac{1}{2}\right)^{10} \cdot \frac{z\cdot z^{-10}}{z-\frac{1}{2}} + \left(\frac{3}{4}\right)^{10} \cdot \frac{z\cdot z^{-10}}{z-\frac{3}{4}}, \quad |z| > \frac{1}{2} \text{ και } |z| > \frac{3}{4} \Rightarrow |z| > \frac{3}{4}$$

Άρα υπάρχει ο DTFT, αφού έχουμε σύγκλιση για $|z| > \frac{3}{4}$ και ο μοναδιαίος κύκλος ανήκει στην περιοχή σύγκλισης.

$$X(z) = \left(\frac{1}{2}\right)^{10} \cdot \frac{1}{z^9\left(z-\frac{1}{2}\right)} + \left(\frac{3}{4}\right)^{10} \cdot \frac{1}{z^9\left(z-\frac{3}{4}\right)} = \frac{\left(\frac{1}{2}\right)^{10}\left(z-\frac{3}{4}\right) + \left(\frac{3}{4}\right)^{10}\cdot\left(z-\frac{1}{2}\right)}{z^9\left(z-\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}.$$

Πόλοι: $p_1 = 0$ (πολλαπλότητας 9), $p_2 = \frac{1}{2}$, $p_3 = \frac{3}{4}$

Μηδενικά: $\dfrac{\frac{3}{8}\left[\left(\frac{1}{2}\right)^9 + \left(\frac{3}{4}\right)^9\right]}{\left(\frac{1}{2}\right)^{10} + \left(\frac{3}{4}\right)^{10}} > \frac{3}{8}$ αλλά $< \frac{1}{2}$



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Άσκηση 4:

α). $X(z) = \dfrac{1}{(1-z^{-1})(1-z^{-2})}$ , $|z|>1$.

$X(z) = \dfrac{z^3}{(z-1)(z^2-1)} = \dfrac{z^3}{(z+1)(z-1)^2}$

$\dfrac{X(z)}{z} = \dfrac{z^2}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{(z-1)^2} + \dfrac{\Gamma}{z-1}$

$A = \dfrac{1}{4}$, $B = \dfrac{1}{2}$, $\Gamma = \dfrac{3}{4}$.

$X(z) = \dfrac{1}{4} \cdot \dfrac{z}{z+1} + \dfrac{1}{2} \cdot \dfrac{z}{(z-1)^2} + \dfrac{3}{4} \cdot \dfrac{z}{z-1}$ , $|z|>1$.

Άρα $x[n] = \dfrac{1}{4} \cdot (-1)^n \cdot u[n] + \dfrac{1}{2} \cdot n \cdot u[n] + \dfrac{3}{4} \cdot u[n] \Rightarrow$

$x[n] = \left(\dfrac{(-1)^n}{4} + \dfrac{n}{2} + \dfrac{3}{4}\right) \cdot u[n]$ $\left(|z|>1, |z|>1, |z|>|-1|=1\right)^{\checkmark}$

β). $X(z) = \log\left(1 - \frac{1}{2}z^{-1}\right)$ , $|z|>1/2$.

$\dfrac{dX(z)}{dz} = \dfrac{1}{1-\frac{1}{2}z^{-1}} \cdot \dfrac{1}{2} \cdot z^{-2} = \dfrac{1}{2} \cdot \dfrac{z^{-2}}{1-\frac{1}{2}z^{-1}} \Rightarrow$

$\dfrac{dX(z)}{dz} = \dfrac{1}{2} \cdot \dfrac{1}{z^2 - \frac{1}{2}z} = \dfrac{1}{2} \cdot \dfrac{1}{z(z-\frac{1}{2})} \Rightarrow$

$z \cdot \dfrac{dX(z)}{dz} = \dfrac{1}{2} \cdot \dfrac{1}{z-\frac{1}{2}} \Rightarrow -z \cdot \dfrac{dX(z)}{dz} = -\dfrac{1}{2} \cdot \dfrac{1}{z-\frac{1}{2}}$

Άρα $n x[n] = -\frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} \cdot u[n-1] \Rightarrow$

$$x[n] = -\frac{1}{2n} \cdot \left(\frac{1}{2}\right)^{n-1} \cdot u[n-1] \qquad (|z| > 1/2) \checkmark$$

γ) $X(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{4} z^{-2}}, \quad |z| > 1/2.$

$$X(z) = \frac{z^2 - z}{z^2 - \frac{1}{4}} \Rightarrow \frac{X(z)}{z} = \frac{z-1}{(z-\frac{1}{2})(z+\frac{1}{2})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{2}}$$

$A = -\frac{1}{2}, \quad B = \frac{3}{2}$

$$\frac{X(z)}{z} = \frac{3}{2} \cdot \frac{1}{z+\frac{1}{2}} - \frac{1}{2} \cdot \frac{1}{z-\frac{1}{2}} \Rightarrow X(z) = \frac{3}{2} \cdot \frac{z}{z+\frac{1}{2}} - \frac{1}{2} \cdot \frac{z}{z-\frac{1}{2}}$$

Άρα $x[n] = \left[\frac{3}{2} \cdot \left(-\frac{1}{2}\right)^n - \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n\right] \cdot u[n]$. $\quad (|z| > |-\frac{1}{2}| = 1/2, \; |z| > 1/2) \checkmark$

δ). $X(z) = \dfrac{1}{1 + 3z^{-1} + 2z^{-2}}, \quad |z| > 2.$

$$X(z) = \frac{z^2}{z^2 + 3z + 2} \Rightarrow \frac{X(z)}{z} = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$A = -1, \quad B = 2$

$$\frac{X(z)}{z} = \frac{2}{z+2} - \frac{1}{z+1} \Rightarrow X(z) = 2 \cdot \frac{z}{z+2} - \frac{z}{z+1}$$

Άρα $x[n] = [2 \cdot (-2)^n - (-1)^n] \cdot u[n]$. $\quad (|z| > |-2| = 2, \; |z| > |-1| = 1) \checkmark$

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ε) $X(z) = \dfrac{3 - \frac{5}{6}\cdot z^{-1}}{(1-\frac{1}{4}\cdot z^{-1})(1-\frac{1}{3}\cdot z^{-1})}$, $|z| > 1/3$

$X(z) = \dfrac{3z^2 - \frac{5}{6}\cdot z}{(z-\frac{1}{4})(z-\frac{1}{3})} \Rightarrow \dfrac{X(z)}{z} = \dfrac{3z - \frac{5}{6}}{(z-\frac{1}{4})(z-\frac{1}{3})} = \dfrac{A}{z-\frac{1}{4}} + \dfrac{B}{z-\frac{1}{3}}$

$A = 1$, $B = 2$

$X(z) = \dfrac{z}{z-\frac{1}{4}} + 2\cdot\dfrac{z}{z-\frac{1}{3}}$

Άρα $x[n] = \left[\left(\frac{1}{4}\right)^n + 2\left(\frac{1}{3}\right)^n\right]\cdot u[n]$ $\left(|z| > \frac{1}{3} > \frac{1}{4},\ |z| > \frac{1}{3}\right)$ ✓

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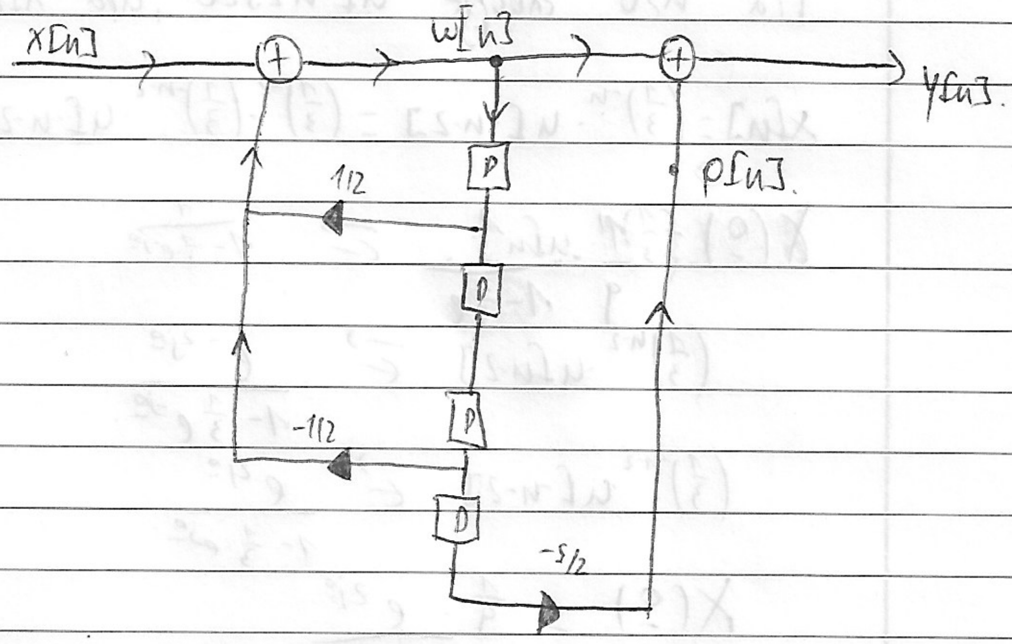
## Άσκηση 5:

α) Κάθε D ισούται με $z^{-1}$ καθυστέρηση.




$$w(z) = X(z) + \frac{1}{2}\cdot z^{-1}\cdot W(z) - \frac{1}{2}\cdot z^{-3}\cdot w(z) \Rightarrow$$

$$w(z)\cdot\left[1-\frac{1}{2}\cdot z^{-1}+\frac{1}{2}\cdot z^{-3}\right] = X(z) \Rightarrow w(z) = \frac{X(z)}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-3}}$$

$$p(z) = -\frac{5}{2}\cdot z^{-4}\cdot W(z) \Rightarrow p(z) = \frac{-\frac{5}{2}z^{-4}\cdot X(z)}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-3}}$$

$$Y(z) = W(z) + p(z) = \frac{X(z)}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-3}} - \frac{\frac{5}{2}z^{-4}\cdot X(z)}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-3}} \Rightarrow$$

$$Y(z) = X(z)\cdot\frac{1-\frac{5}{2}\cdot z^{-4}}{1-\frac{1}{2}\cdot z^{-1}+\frac{1}{2}\cdot z^{-3}}$$

β) $Y(z) - Y(z)\cdot\frac{1}{2}\cdot z^{-1} + Y(z)\cdot\frac{1}{2}\cdot z^{-3} = X(z) - X(z)\cdot\frac{5}{2}\cdot z^{-4}$ $\left(\xrightarrow{z^{-1}}\right)$

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{2}y[n-3] = x[n] - \frac{5}{2}\cdot x[n-4].$$

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Άσκηση 6:

α) $x[n] = \left(\frac{1}{3}\right)^{|n|} \cdot u[-n-2]$.

Για $n>0$ έχουμε $u[-n-2]=0$, άρα $x[n]=0$.

$$x[n] = \left(\frac{1}{3}\right)^{-n} \cdot u[-n-2] = \left(\frac{1}{3}\right)^{2} \cdot \left(\frac{1}{3}\right)^{-n-2} \cdot u[-n-2] = \frac{1}{9} \cdot \left(\frac{1}{3}\right)^{-n-2} \cdot u[-n-2].$$

$$\left(\frac{1}{3}\right)^{n} \cdot u[n] \rightleftarrows \frac{1}{1-\frac{1}{3} e^{-j\Omega}}$$

$$\left(\frac{1}{3}\right)^{n-2} \cdot u[n-2] \rightleftarrows \frac{e^{-2j\Omega}}{1-\frac{1}{3} e^{-j\Omega}}.$$

$$\left(\frac{1}{3}\right)^{-n-2} u[-n-2] \rightleftarrows \frac{e^{2j\Omega}}{1-\frac{1}{3} \cdot e^{j\Omega}}$$

$$X(\Omega) = \frac{1}{9} \cdot \frac{e^{2j\Omega}}{1-\frac{1}{3} \cdot e^{j\Omega}}.$$

β). $x[n] = (n+1)\left(\frac{1}{7}\right)^{n} \cdot u[n]$.

$$(n+1) \cdot a^n u[n] \rightleftarrows \frac{1}{(1-ae^{-j\Omega})^2} \quad , |a|<1.$$

$$X(\Omega) = \frac{1}{\left(1-\frac{1}{7} \cdot e^{-j\Omega}\right)^2} \quad , \left|\frac{1}{7}\right|<1.$$

γ). $X(\Omega) = \begin{cases} -j &, 0 < \Omega \leq \pi \\ j &, -\pi \leq \Omega < 0 \end{cases}$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{0} j e^{j\Omega n} \, d\Omega - \frac{1}{2\pi} \int_{0}^{\pi} j \cdot e^{j\Omega n} \, d\Omega \overset{n \neq 0}{=}$$

$$= \frac{1}{2\pi} \cdot j \frac{1}{jn} \cdot e^{j\Omega n} \Big|_{-\pi}^{0} - \frac{1}{2\pi} \cdot j \cdot \frac{1}{jn} \cdot e^{j\Omega n} \Big|_{0}^{\pi} = ?$$

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$$x[n] = \frac{1}{2\pi n}\left(1 - e^{-j\pi n}\right) - \frac{1}{2\pi n}\left(e^{j\pi n} - 1\right) = \frac{1}{2\pi n}\left(1 - e^{-j\pi n} - e^{j\pi n} + 1\right) =$$

$$= \frac{2}{2\pi n} - \frac{1}{2\pi n}\cdot\left(e^{j\pi n} + e^{-j\pi n}\right) = \frac{1}{\pi n} - \frac{1}{\cancel{2}\pi n}\cdot\cancel{2}\cos(n\pi) =$$

$$= \frac{1}{\pi n} - \frac{1}{\pi n}\cdot(-1)^n = \frac{1-(-1)^n}{\pi n} = \frac{1+(-1)^{n+1}}{\pi n}, \quad n \neq 0.$$

Για $n=0$ : $x[0] = \frac{1}{2\pi}\left(\int_{-\pi}^{0} j\,d\theta - \int_{0}^{\pi} j\,d\theta\right) = \frac{j}{2\pi}\left(\int_{-\pi}^{0} d\theta - \int_{0}^{\pi} d\theta\right) =$

$$= \frac{j}{2\pi}\left(0 + \cancel{\pi} - \cancel{\pi} + 0\right) = 0$$

Άρα $x[n] = \begin{cases} 0, & n=0 \\ \frac{1+(-1)^{n+1}}{\pi n}, & n \neq 0 \end{cases}$

δ). $X(\theta) = \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\theta - \frac{\pi k}{2}\right)$

$$\sum_{k=\langle N\rangle} a_k \cdot e^{j\frac{2\pi k n}{N}} \rightleftarrows 2\pi \cdot \sum_{k=-\infty}^{\infty} a_k \delta\left(\theta - \frac{2\pi k}{N}\right).$$

$a_k = a_{k+N}$

Άρα $\frac{1}{2} = \frac{2}{N} \Rightarrow N=4$ και $a_{k+N} = (-1)^{k+4} = (-1)^k = a_k$.

Άρα $x[n] = \sum_{k=\langle 4\rangle} (-1)^k \cdot e^{\frac{j\pi k n}{2}}$

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## Άσκηση 7:

α) $x[n] = \cos(n\Omega_0)$, $0 \le n \le N-1$.

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} \cos(n\Omega_0) \cdot e^{-j\frac{2\pi k n}{N}} =$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{N-1} \left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) \cdot e^{-j\frac{2\pi k n}{N}} =$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} \left[e^{jn(\Omega_0 - \frac{2\pi k}{N})} + e^{-jn(\Omega_0 + \frac{2\pi k}{N})}\right]$$

- Αν $\Omega_0 = \frac{2\pi k_0}{N}$ τότε:

$$X[k] = \frac{1}{2}\left[\sum_{n=0}^{N-1} e^{-\frac{2\pi j n(k-k_0)}{N}} + \sum_{n=0}^{N-1} e^{-\frac{2\pi j n}{N}(k+k_0)}\right] =$$

$$= \frac{1}{2}\left[N \cdot \delta(k-k_0) + N \cdot \delta(k+k_0)\right],$$

διότι $\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k-k_0)} = \begin{cases} 0, & \forall k \neq k_0 \\ N, & k = k_0 \end{cases} = N\delta[k-k_0]$

Άρα $X[k] = \begin{cases} 0, & \text{αλλού} \\ \frac{N}{2}, & k = k_0 \text{ ή } k = N-k_0 \end{cases}$

διότι $X[k] = \frac{N}{2}(\delta(k-k_0) + \delta(k-N+k_0))$ αφού $X[k]$ περιοδικό με περίοδο $N$.

- Για $\Omega \neq \frac{2\pi k_0}{N}$

$$X[k] = \frac{1}{2} \cdot \sum_{n=0}^{N-1} \left(e^{j\Omega n} + e^{-j\Omega n}\right) \cdot e^{-j\frac{2\pi k n}{N}} =$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N} - \Omega_0\right) \cdot n} + \frac{1}{2} \cdot \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N} + \Omega_0\right) n} =$$

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$$= \frac{1}{2} \cdot \sum_{n=0}^{N-1} \left(e^{-j\left(\frac{2\pi k}{N} - \Omega_0\right)}\right)^n + \frac{1}{2} \cdot \sum_{n=0}^{N-1} \left(e^{-j\left(\frac{2\pi k}{N} + \Omega_0\right)}\right)^n =$$

$$= \frac{1}{2} \cdot \frac{e^{-j\left(\frac{2\pi k}{N} - \Omega_0\right) \cdot N} - 1}{e^{-j\left(\frac{2\pi k}{N} - \Omega_0\right)} - 1} + \frac{1}{2} \cdot \frac{e^{-j\left(\frac{2\pi k}{N} + \Omega_0\right) \cdot N} - 1}{e^{-j\left(\frac{2\pi k}{N} + \Omega_0\right)} - 1} =$$

$$= \left[\frac{e^{-j\left(\frac{2\pi k}{N} - \Omega_0\right) \cdot \frac{N-1}{2}}}{2} \cdot \frac{\sin\left(\pi k - \frac{N\Omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} - \frac{\Omega_0}{2}\right)}\right] + \left[\frac{e^{-j\left(\frac{2\pi k}{N} + \Omega_0\right) \frac{N-1}{2}}}{2} \cdot \frac{\sin\left(\pi k + \frac{N\Omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\Omega_0}{2}\right)}\right]$$

Ισχύει επίσης, ότι:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} \overset{N}{\Rightarrow} X(\Omega) = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\Omega n}$$

Άρα ο DFT έχει μόνο συγκεκριμένα δείγματα του DTFT του X. Επίσης, αν $\Omega_0 \neq \frac{2\pi}{N}$ γενικά $X[k] \neq 0$, ενώ για $\Omega_0 = \frac{2\pi}{N}$ έχουμε και μηδενικές τιμές εκτός από $k = k_0$ ή $k = N - k_0$, όπου $X(k) = \frac{N}{2}$.

6).a). $x(n) = 4 + \cos^2\left(\frac{2\pi n}{N}\right) = 4 + \frac{1}{2}\left(1 + \cos\left(\frac{4\pi n}{N}\right)\right) \Rightarrow$
$x(n) = \frac{9}{2} + \frac{1}{2}\cos\left(\frac{4\pi n}{N}\right)$ άρα $\Omega_0 = \frac{4\pi}{N}$ με $m_0 = 2$.

Άρα $X[m] = \sum_{n=0}^{N-1} \left(\frac{9}{2} + \frac{1}{2}\cos(\Omega_0 n)\right) \cdot e^{-j\frac{2\pi m n}{N}} \Rightarrow$

$$X[m] = \frac{9}{2} + \frac{9}{2} \cdot \frac{e^{-j\frac{2\pi m}{N}} - e^{-j\frac{2\pi m}{N}(N-1)} - 1}{e^{-j\frac{2\pi m}{N}} - 1} + \frac{N}{2}\left(\delta[k-2] + \delta[k+2]\right)$$

6). $x(n) = a^n$, $0 \leq n < N \Rightarrow$

$$X(n) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi m n}{N}} = \sum_{n=0}^{N-1} a^n \cdot e^{-j\frac{2\pi m n}{N}} =$$

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$$= \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi m}{N}}\right)^n = 1 + \sum_{n=1}^{N-1} \left(a e^{-j\frac{n 2 m}{N}}\right)^n =$$

$$= 1 + a e^{-j\frac{2\pi m}{N}} \cdot \frac{a e^{-j\frac{2\pi m}{N}(N-1)} - 1}{a e^{-j\frac{2\pi m}{N}} - 1} \Rightarrow$$

$$X[m] = \frac{1 + a^2 e^{-j2\pi m} - a e^{-j\frac{2\pi m}{N}}}{a e^{-j\frac{2\pi m}{N}} - 1}$$

γ) $x[n] = u[n] - u[n-n_0]$, $0 < n_0 < N$.

$$x[n] = \begin{cases} 0, & n<0,\ n \geq n_0 \\ 1, & 0 \leq n < n_0 \end{cases}$$

Άρα: $$X[m] = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi m n}{N}} = \sum_{n=0}^{n_0-1} e^{-j\frac{2\pi m n}{N}} \Rightarrow$$

$$X[m] = \sum_{n=0}^{n_0-1} e^{-j\frac{2\pi m n}{N}} = 1 + \sum_{n=1}^{n_0-1} \left(e^{-j\frac{2\pi m}{N}}\right)^n \Rightarrow$$

$$X[m] = 1 + \frac{e^{-j\frac{2\pi m}{N} \cdot n_0} - e^{-j\frac{2\pi m}{N}}}{e^{-j\frac{2\pi m}{N}} - 1}$$

Τέλος