# Golden Fields: A Case for the Heptagon

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One of the best-kept secrets in plane geometry is the family of ratios of diagonal to side in the regular polygons. So much attention has been given to one member of this family, the *golden ratio* $\phi$ in the pentagonal case, that the others live in undeserved obscurity. But the wealth of material that pours from the pentagon—proportional sections, recursive sequences, and quasiperiodic systems—can be matched wonder-for-wonder by any other polygon. We will look at some general properties of regular polygons and, in particular, at the case of the heptagon.

## I. The Diagonal Product Formula

What are the lengths of the diagonals of a regular polygon with unit side length? As FIGURE 1 shows, the triangle has no diagonals, the square and pentagon each have one kind of diagonal, and the hexagon and heptagon have two. Some diagonal lengths can be discovered by using the Pythagorean theorem or a cosine or sine of a special angle, but to obtain closed forms for $\phi$ in the pentagon and for $\rho$ and $\sigma$ in the heptagon we need more. The starred pentagons in FIGURE 2 highlight similar triangles that imply the proportion and the quadratic:

$$\frac{a}{b} = \frac{a+b}{a} = \frac{\phi}{1} \Rightarrow \frac{a}{b} = \frac{a}{a} + \frac{b}{a} \Rightarrow \phi = 1 + \frac{1}{\phi} \Rightarrow \phi^2 - \phi - 1 = 0,$$

which has the positive root $\phi = (1+\sqrt{5})/2 \approx 1.618$.

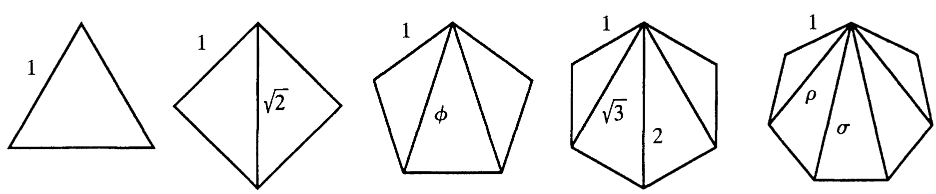



**FIGURE 1**
Diagonals of regular polygons

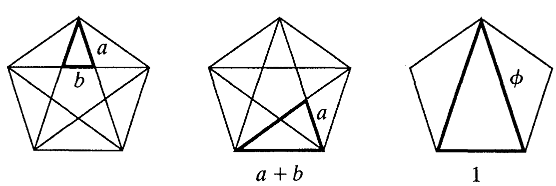



**FIGURE 2**
Derivation of $\phi$

The level of difficulty jumps when we analyze the heptagon. Similar triangles reveal many identities, such as $\rho\sigma = \rho + \sigma$ and $\rho^2 = 1 + \sigma$. These two, solved simultaneously, yield $\rho^3 - \rho^2 - 2\rho + 1 = 0$, and any closed form of the solution involves complex radicals. Rho and sigma are both cubic numbers (roots of irreducible cubic polynomials), and their closed forms are far less useful than the trigonometric forms:

$$\rho = 2\cos(\pi/7) \approx 1.80194, \qquad \sigma = 4\cos^2(\pi/7) - 1 \approx 2.24698.$$

The lengths themselves are less interesting than identities like $\rho\sigma = \rho + \sigma$. Look again at FIGURE 1. The diagonals are arranged in a fan shape to illustrate a pattern: at least up to the hexagon, products of two diagonal lengths are also sums. For the hexagon, we have:

$$\sqrt{3}\cdot 2 = \sqrt{3} + \sqrt{3}, \quad \sqrt{3}\cdot\sqrt{3} = 1 + 2, \quad \text{and} \quad 2\cdot 2 = 1 + 2 + 1.$$

The pattern suggests that the product of two diagonals is a sum of a sequence of diagonals (in the fan, every other one) centered on the longer of the two. This is true also for the heptagon and, indeed, for all regular polygons. We can state it explicitly as follows.

PROPOSITION: DIAGONAL PRODUCT FORMULA. *Consider a regular $n$-gon* (FIGURE 3) *and let $d_0$ be the length of a side, $d_k$ the length of a $k$th internal diagonal, with $k \le \lfloor (n-2)/2 \rfloor$, and $r_k = d_k/d_0$. Then*

$$\begin{aligned}
&r_0 r_k = r_k,\\
&r_1 r_k = r_{k-1} + r_{k+1}, \quad 1 \le k,\\
&r_2 r_k = r_{k-2} + r_k + r_{k+2}, \quad 2 \le k,\\
&r_3 r_k = r_{k-3} + r_{k-1} + r_{k+1} + r_{k+3}, \quad 3 \le k,\\
&\dots
\end{aligned}$$

$$r_h r_k = \sum_{i=0}^{h} r_{k-h+2i} \quad \text{and} \quad d_h d_k = d_0 \sum_{i=0}^{h} d_{k-h+2i}, \quad h \le k. \tag{1}$$

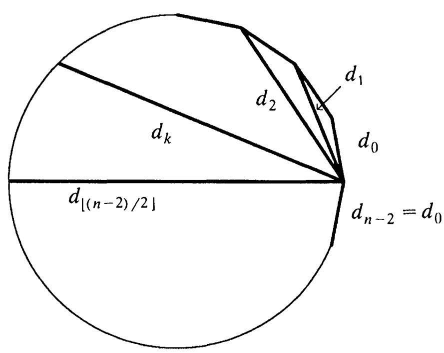



**FIGURE 3**
Regular $n$-gon

This can be proved using the $n$-gon whose vertices are the $n$th roots of unity in the complex plane. Here is a sketch. Let $\omega = e^{2\pi i/n}$, and express any ratio of diagonal to side as $r_k = |\omega^{k+1} - 1|/|\omega - 1|$. Substitute these ratios in (1) and write out the sum,

$$\frac{|\omega^{h+1} - 1||\omega^{k+1} - 1|}{|\omega - 1|} = |\omega^{k-h+1} - 1| + |\omega^{k-h+3} - 1| + \cdots + |\omega^{k+h+1} - 1|, \tag{2}$$

and manipulate the left side as follows:

$$\begin{aligned}
&|\omega^h + \omega^{h-1} + \omega^{h-2} + \cdots + 1||\omega^{k+1} - 1| \\
&\quad = |\omega^{k+h+1} - \omega^h + \omega^{k+h} - \omega^{h-1} + \omega^{k+h-1} - \omega^{h-2} + \cdots + \omega^{k+1} - 1| \\
&\quad = |\omega^{k+h+1} - 1 + \omega^{k+h} - \omega + \omega^{k+h-1} - \omega^2 + \cdots + \omega^{k+1} - \omega^h|.
\end{aligned}$$

The last expression gives a clue for a substitution in the sum on the right side of (2). After substituting appropriately named diagonals, the equation is seen to be an application of the triangle inequality to parallel segments, thus implying equality. Reversing the steps completes the proof.

The diagonal product formula (DPF)(1) allows us to work in the extension field $\mathbb{Q}(r_1)$, wherein we may express products and quotients of diagonals (with $d_0 = 1$) as linear combinations of diagonals. For the pentagon and heptagon the DPF yields the familiar golden ratio identities, $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$, and the surprising identities:

$$\begin{array}{lll}
\rho^2 = 1 + \sigma & \sigma/\rho = \sigma - 1 & 1/\sigma = \sigma - \rho \\
\rho\sigma = \rho + \sigma & \rho/\sigma = \rho - 1 & 1/\rho + 1/\sigma = 1 \\
\sigma^2 = 1 + \rho + \sigma & & 1/\rho = 1 + \rho - \sigma
\end{array} \tag{3}$$

(The quotients are obtained by simple manipulations of the products.)

The polynomials of which diagonal ratios are roots constitute a rich subject for study. The golden polynomial, $x^2 - x - 1 = 0$, has the two roots $\phi$ and $-1/\phi$. Closer inspection of the cubics for $\rho$ and $\sigma$ shows that

$$\begin{array}{ll}
x^3 - x^2 - 2x + 1 = 0 & \text{has roots } \rho, 1/\sigma, -\sigma/\rho; \\
x^3 - 2x^2 - x + 1 = 0 & \text{has roots } \sigma, 1/\rho, -\rho/\sigma.
\end{array} \tag{4}$$

In absolute value all six heptagonal ratios occur as roots in the two cubics. (I know of no general pattern of roots in such polynomials.)

Using a chain of substitutions in the DPF, one can derive for the regular $n$-gon a general polynomial ((4) is one case) that has $r_1 = 2\cos \pi/n$ as a root. For odd $n$ this polynomial is

$$\begin{aligned}
&\binom{k}{0}x^k - \binom{k-1}{1}x^{k-2} + \binom{k-2}{2}x^{k-4} - \cdots \\
&\quad = \binom{k-1}{0}x^{k-1} - \binom{k-2}{1}x^{k-3} + \binom{k-3}{2}x^{k-5} - \cdots, \text{ where } k = \frac{n-1}{2}.
\end{aligned} \tag{5}$$

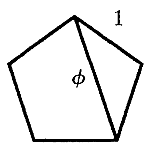
If (5) is written $\boldsymbol{P}_k(x) = 0$, then we have the recurrence $\boldsymbol{P}_{k+1}(x) = x\boldsymbol{P}_k(x) - \boldsymbol{P}_{k-1}(x)$, and $\boldsymbol{P}_k$ can be expressed in terms of derivatives of the Chebychev polynomials. $\boldsymbol{P}_k$ is irreducible over $\mathbb{Q}$ if and only if $n$ is prime. For the 11-gon, for example, $r_1$ is a quintic number. For the 15-gon, $\boldsymbol{P}_k$ has degree seven, with an irreducible quartic factor that has $r_1$ as a root. Moreover, when $n$ is prime, the unit side and the set of diagonals form a basis for the field $\mathbb{Q}(r_1)$.

The property of a field that all products and quotients are expressible as linear combinations of basis elements is the feature essential to our study of the behavior of these special numbers. For this reason (and for others to come) I will call the set $\mathbb{Q}(\phi) = \{a\phi + b\colon a, b \in \mathbb{Q}\}$ the *first golden field*. The *second golden field*, $\mathbb{Q}(\rho) = \{a\sigma + b\rho + c\colon a, b, c \in \mathbb{Q}\}$, we will study in more detail.

## II. A Family of Golden Proportions

Since $\rho$ and $\sigma$ are cubic numbers, the heptagon is not classically constructible, which may explain the ancients' silence on the matter. The Greek geometers' method of investigation was construction. This and their limited understanding of irrational numbers would inhibit their analysis of figures like the heptagon. Archimedes **[1]** at least constructed the heptagon with a *marked* straightedge, and may have discovered more. The derivation of $\phi$ and its properties by similar triangles (FIGURE 2) has been known since ancient times, and one would think that the Greeks would have applied the same reasoning to other figures despite their inconstructibility. (Perhaps they did. Dijksterhuis **[1]** cites evidence of a lost Archimedean manuscript entitled *On the Heptagon in a Circle.*)

Besides the DPF there is another remarkable pattern that the ancients were able to find but which apparently escaped their notice. The Greeks defined $\phi$ in terms of a section (cut) of a segment (see FIGURE 4a): *The whole is to the larger part as the larger is to the smaller part.* (Sections such as this began a design tradition, continuing today, in which a harmonious arrangement of elements is defined as one that realizes some special ratio, such as $\phi$, and repeats it in proportion. One of the diagonals of the octagon, $\theta = 1 + \sqrt{2}$, is known to architects as the Sacred Cut. See **[4]** and **[7]**.) Numerically, though, the golden *proportion* (realized in the *section*) is the unique solution to the problem of forming a non-trivial proportion, which requires four entries, using only two quantities. When stated as $(a+b)/a = a/b$, the unique solution is $a/b = \phi/1$.




$$\frac{AC}{BC} = \frac{BC}{AB} = \phi \quad \text{or} \quad \frac{\text{whole}}{\text{larger}} = \frac{\text{larger}}{\text{smaller}}$$

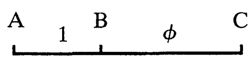
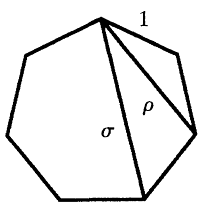



**FIGURE 4a**
Optimal (golden) bisection and proportion

To express $\rho$ and $\sigma$ in terms of a section (FIGURE 4b), draw a segment of length $1 + \sigma + \rho$ $(= \sigma^2)$. Now the three quantities 1, $\rho$, $\sigma$, in all their combinations as the




$$\begin{aligned} &AB : CD : BC \\ &= CD : AC : BD \\ &= BC : BD : AD \end{aligned} \quad \text{or} \quad \begin{aligned} &1 : \rho : \sigma \\ &= \rho : 1+\sigma : \rho+\sigma \\ &= \sigma : \rho+\sigma : 1+\rho+\sigma \end{aligned}$$


**FIGURE 4b**
Optimal trisection and proportion

six internal segments, fill the 18 entries in these three triple proportions:

$$\frac{AD}{BC}=\frac{BD}{CD}=\frac{BC}{AB}=\sigma \quad \text{or} \quad \frac{\text{whole}}{\text{large}}=\frac{\text{large + medium}}{\text{medium}}=\frac{\text{large}}{\text{small}};$$

$$\frac{BD}{BC}=\frac{AC}{CD}=\frac{CD}{AB}=\rho \quad \text{or} \quad \frac{\text{large + medium}}{\text{large}}=\frac{\text{large + small}}{\text{medium}}=\frac{\text{medium}}{\text{small}};$$

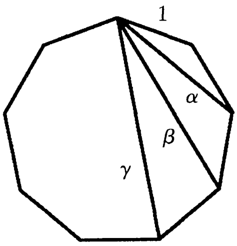
$$\frac{AD}{BD}=\frac{BD}{AC}=\frac{BC}{CD}=\frac{\sigma}{\rho} \quad \text{or} \quad \frac{\text{whole}}{\text{large + medium}}=\frac{\text{large + medium}}{\text{large + small}}=\frac{\text{large}}{\text{medium}}.$$

The enneagon's four diagonals 1, $\alpha$, $\beta$, $\gamma$ (FIGURE 4c) fill 48 entries in six quadruple proportions.

$$\alpha=\frac{DE}{AB}=\frac{AC}{DE}=\frac{CE}{BC}=\frac{BD}{CD} \qquad \frac{\beta}{\alpha}=\frac{BC}{DE}=\frac{CE}{AC}=\frac{AD}{CE}=\frac{BE}{BD}$$

$$\beta=\frac{BC}{AB}=\frac{CE}{DE}=\frac{AD}{BC}=\frac{BE}{CD} \qquad \frac{\gamma}{\alpha}=\frac{CD}{DE}=\frac{BD}{AC}=\frac{BE}{CE}=\frac{AE}{BD}$$

$$\gamma=\frac{CD}{AB}=\frac{BD}{DE}=\frac{BE}{BC}=\frac{AE}{CD} \qquad \frac{\gamma}{\beta}=\frac{CD}{BC}=\frac{BD}{CE}=\frac{BE}{AD}=\frac{AE}{BE}$$




$$\begin{array}{l} AB:DE:BC:CD \\ =DE:AC:CE:BD \\ =BC:CE:AD:BE \\ =CD:BD:BE:AE \end{array} \quad \text{or} \quad \begin{array}{l} 1:\alpha:\beta:\gamma \\ =\alpha:1+\beta:\alpha+\gamma:\beta+\gamma \\ =\beta:\alpha+\gamma:1+\beta+\gamma:\alpha+\beta+\gamma \\ =\gamma:\beta+\gamma:\alpha+\beta+\gamma:1+\alpha+\beta+\gamma \end{array}$$


**FIGURE 4c**
Optimal quadrisection and proportion

Like the golden section, these are also unique solutions, but to precisely what general problem? To answer this question, first rewrite these proportions as in FIGURE 4, as $k \times k$ proportions, consisting of $k$ equal $k$-part ratios. It is not difficult to show that unity and the diagonals of any regular $(2k+1)$-gon, $1, r_1, r_2, r_3, \ldots, r_{k-1}$, can be arranged end-to-end in such a way that the composite $k$-part segment illustrates a $k \times k$ proportion using all $\binom{k+1}{2}$ internal segments—a sort of optimal harvest of proportions. The heptagon's $3 \times 3$ and the enneagon's $4 \times 4$ proportions are shown in FIGURE 4. In fact, for each odd $n$-gon, only one arrangement (and its reverse) will accomplish this. Only the sequence $1, r_2, r_4, r_6, \ldots, r_{n-5}=r_3, r_{n-3}=r_1$ can accommodate all linear combinations given by the DPF (because of its sums of every-other diagonal), so this is the only sequence that can represent all products as internal segments. But suppose we ask whether any other sets of real numbers can simultaneously fit a $k \times k$ proportion and a sectioned segment. This would define the general problem and the question of uniqueness of solutions.

Let $x_1 = 1 < x_2 < x_3 < \cdots < x_k$, and observe that

$$\begin{aligned} & 1 : x_2 : x_3 : \cdots : x_k \\ = \; & x_2 : x_2^2 : x_2 x_3 : \cdots : x_2 x_k \\ = \; & x_3 : x_2 x_3 : x_3^2 : \cdots : x_3 x_k \\ = \; & \cdots \\ = \; & x_k : x_2 x_k : x_3 x_k : \cdots : x_k^2. \end{aligned}$$

(Each $k$-part ratio is equal to the first, since it is the first multiplied through by some $x_i$.) Now, in order to model a $k$-sectioned segment, we require that each product in this $k \times k$ proportion be equal to a linear combination of the $x_i$ such that every combination occurs *consecutively* in some fixed permutation of the $x_i$. This is the general problem we are looking for. For $k = 2$,

$$\begin{aligned} & 1 : x \\ & = x : 1 + x \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \phi = (1 + \sqrt{5})/2, \end{aligned}$$

but for $k = 3$ there are three permutations depending on which quantity is in the middle of the segment, and so there are three cases. For $1 < x < y$, place each in turn in the middle.

| order $1, x, y \Rightarrow$ | order $x, 1, y \Rightarrow$ | order $1, y, x \Rightarrow$ |
|---|---|---|
| $1 : x \quad : y$ | $1 : x \quad : y$ | $1 : x \quad : y$ |
| $= x : 1 + x : x + y$ | $= x : 1 + x : 1 + y$ | $= x : 1 + y : x + y$ |
| $= y : x + y : 1 + x + y.$ | $= y : 1 + y : 1 + x + y.$ | $= y : x + y : 1 + x + y.$ |

The first two cases above have no real solutions, and the third (realized in FIGURE 4b) uniquely determines that $x = \rho$ and $y = \sigma$, establishing the unique optimal trisection. Similarly, the case $k = 4$ has twelve permutations yielding sixteen cases; fifteen have no real solutions, and the sixteenth uniquely determines the diagonals of the enneagon! The obvious question is whether *all* odd $n$-gon proportions are unique. I do not know.

(When $k = 4$, twelve permutations yield sixteen cases because of ambiguous inequalities. For instance, if $1 < x < y < z$, then is $1 + z$ less or greater than $x + y$? Both cases must be checked. In general, $k$ induces $k!/2$ segment permutations and $k^{k-2}$ possible proportion cases. So settling the uniqueness of the 11-gon's diagonals for an optimal pentasection involved checking 125 cases, and the pentasection is the largest that has been verified.)

## III. Sequences

In number theory, terms of the well-known Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$ occur as coefficients of linear combinations expressing powers of $\phi$.

$$\begin{aligned} & \phi = 1\phi + 0; \quad \phi^2 = 1\phi + 1; \quad \phi^3 = 2\phi + 1; \quad \phi^4 = 3\phi + 2; \quad \phi^5 = 5\phi + 3; \\ & \phi^6 = 8\phi + 5; \quad \phi^7 = 13\phi + 8 \ldots \end{aligned} \tag{6}$$

The recurrence relation, $\phi^k = a_k\phi + b_k \Rightarrow \phi^{k+1} = (a_k + b_k)\phi + a_k$, yields new coefficients from old. Also, successive ratios of coefficients tend in the limit to $\phi$: $a_k/b_k = a_k/a_{k-1} \to \phi$.

Taking powers of $\sigma$ yields the following:

$$\begin{aligned} \sigma &= 1\sigma + 0 + 0 \\ \sigma^2 &= 1\sigma + 1\rho + 1 \\ \sigma^3 &= 3\sigma + 2\rho + 1 \\ \sigma^4 &= 6\sigma + 5\rho + 3 \\ \sigma^5 &= 14\sigma + 11\rho + 6 \\ \sigma^6 &= 31\sigma + 25\rho + 14 \\ \sigma^7 &= 70\sigma + 56\rho + 31\ldots \end{aligned} \tag{7}$$

The recurrence relation is $\sigma^k = a_k\sigma + b_k\rho + c_k \Rightarrow \sigma^{k+1} = (a_k + b_k + c_k)\sigma + (a_k + b_k)\rho + a_k$. In the limit $a_k : b_k : c_k \to \sigma : \rho : 1$, and $x_k/x_{k-1} \to \sigma$ for $x = a$, $b$, or $c$. (The sequences $1, 1, 3, 6, 14, \ldots$ and $0, 1, 2, 5, 11, \ldots$ have been included in **[5]**.)

Another interesting sequence and recurrence holds for powers of $\rho$, and one may also consider linear expansions in terms of the bases $\{1, \rho, \rho^2\}$ or $\{1, \sigma, \sigma^2\}$ instead of $\{1, \rho, \sigma\}$. Note how the recurrence relation in (7) naturally generalizes that in (6). This recurrence pattern can be extended to all odd $n$-gons for powers of the longest diagonal. (Terauchi **[6]** calls the coefficients in (7) "higher-order Fibonacci sequences." The name has been applied to many things and is now in use by crystal-diffraction physicists. Physicists also employ the terms silver mean, bronze mean, copper mean, etc., but these, with one exception, are unrelated to the polygonal ratios.)

When the transformation $\langle a, b\rangle \to \langle a + b, a\rangle$ is applied to *any* non-negative numbers $a$ and $b$ (initially above $a = 1$, $b = 0$) and iterated, the ratio $a : b$ approaches $\phi : 1$. Similarly, the transformation $\langle a, b, c\rangle \to \langle a + b + c, a + b, a\rangle$ can be applied to any non-negative numbers (above $a = 1$, $b = c = 0$), and the ratio $a : b : c$ must approach $\sigma : \rho : 1$! For its utter simplicity, this may be the most striking manifestation of these algebraic numbers. Extending the recurrence pattern, the transform $\langle a, b, c, d\rangle \to \langle a + b + c + d, a + b + c, a + b, a\rangle$ drives numbers toward the ratios of the enneagon.

Considering the prevalence of the golden proportion and the Fibonacci sequence in the growth of natural forms, one wonders whether these other ratios and sequences occur in nature. Flower symmetries, for example, occur mainly in 4-, 5-, 6-, 7-, and 8-fold types, and the algebraic numbers associated with all but one of these types—the heptagon—have been widely studied in a great variety of natural contexts. Has the heptagon been overlooked? Can $\rho$ and $\sigma$ be found in plain view?

## IV. Quasiperiodics

Numbers of the golden family generate some very deep dissection puzzles, packing problems, and geometric novelties—too many to discuss here. The most interesting and current, though, are quasiperiodic (beware that the term *quasiperiodic* has no universally accepted definition yet).

The first analyzed example of quasiperiodicity was Fibonacci's own model of rabbit populations. In one generation, every baby becomes an adult and every adult reproduces once. Equivalently, we may rule that the quantities 1 and $\phi$ be multiplied by $\phi$. We convey this rule symbolically by $1 \to \phi$, $\phi \to 1\phi$.

When iterated, this rule generates an infinite sequence on two letters. From the initial word 1 we have:

$$\begin{aligned}1 &\to \phi \to 1\phi \to \phi 1\phi \to 1\phi\phi 1\phi \to \phi 1\phi 1\phi\phi 1\phi \to 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi \\ &\to \phi 1\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi \\ &\to 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi \to \cdots\end{aligned} \tag{8}$$

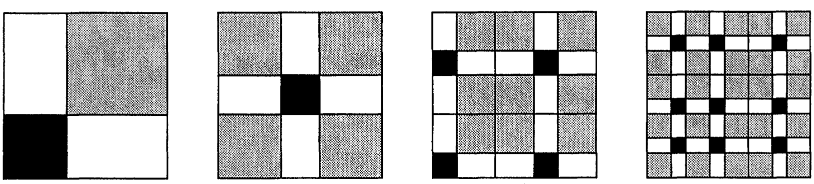
In the limiting infinite sequence the characters $\phi$ and 1 occur in the ratio $\phi : 1$, and their frequencies in any iteration are consecutive Fibonacci numbers. The sequence is not periodic since it cannot be generated by translations of a fundamental domain (the ratio of characters would then be rational), but it is quasiperiodic (QP) in the sense that it exhibits local-isomorphism and local-symmetry properties to be described shortly.

Using the identities (3) given by the DPF we can generate several QP sequences on the characters 1, $\rho$, $\sigma$. Here is one. Begin by applying the following rule, amounting to multiplication by $\sigma$:

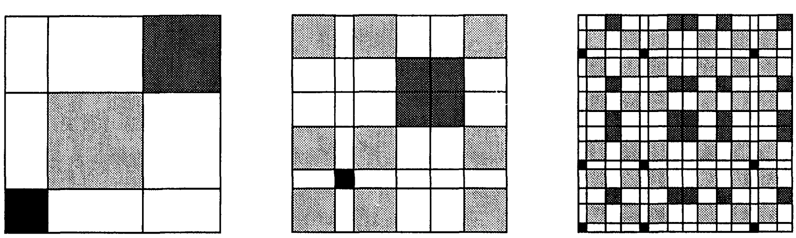
$$\begin{aligned}&1 \to \sigma, \quad \rho \to \rho\sigma, \quad \sigma \to 1\sigma\rho. \quad \text{Then} \\ &1 \to \sigma \to 1\sigma\rho \to \sigma 1\sigma\rho\rho\sigma \to 1\sigma\rho\sigma 1\sigma\rho\rho\sigma\rho\sigma 1\sigma\rho \\ &\to \sigma 1\sigma\rho\rho\sigma 1\sigma\rho\sigma 1\sigma\rho\rho\sigma\rho\sigma 1\sigma\rho\rho\sigma 1\sigma\rho\sigma 1\sigma\rho\rho\sigma \to \cdots\end{aligned} \tag{9}$$

The characters will occur in the expected ratios, and their frequencies in an iteration we have already seen as coefficients in (7). The order of characters in the last rule ($\sigma \to 1\sigma\rho$) was chosen so that the characters 1 and $\rho$ will never be adjacent, in order to conform to the section order shown in FIGURE 4. This makes it possible to find a linear combination for any power of $\rho$ or $\sigma$ as a contiguous subsequence infinitely often in the sequence. Another QP sequence based on multiplication by $\rho$ has this same property, known as local isomorphism, and so of course does the $\phi$-sequence (8) in powers of $\phi$.

Crossing a QP sequence with itself on the Euclidean plane is a very simple way to produce an aperiodic tiling. FIGURE 5 shows the results of applying either the



1 $\phi$



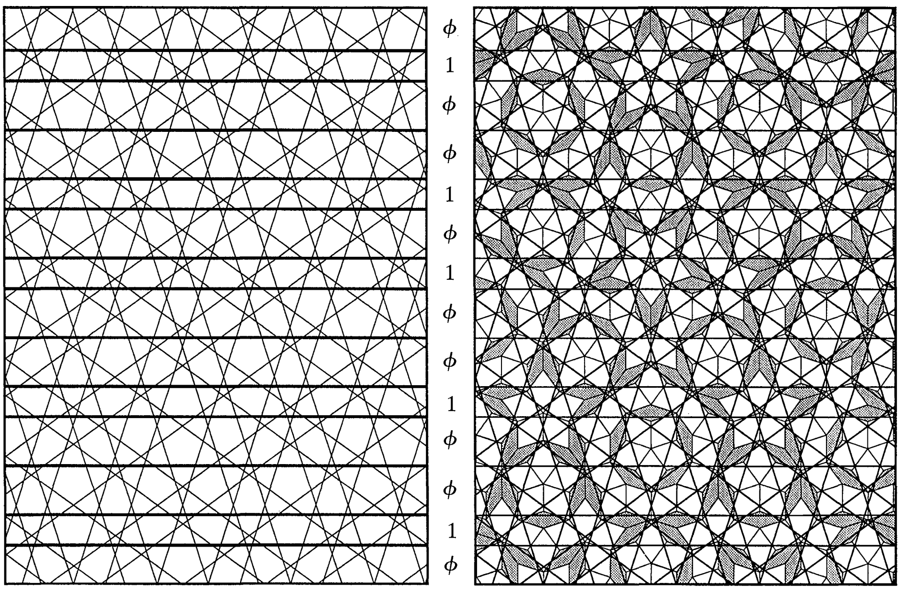
1 $\sigma$ $\rho$

**FIGURE 5**

$\phi$- and $\sigma$-aperiodic tilings

$\phi$-transform (8) or the $\sigma$-transform (9) to the sides of a square. Successive subdivision and magnification covers the plane with tiles aperiodically (though the tilings have one reflective symmetry across the diagonal).

In a very different way the $\phi$-sequence can be used to generate Penrose QP tilings. As in FIGURE 6a, form an infinite set of parallel lines, spaced according to the $\phi$-sequence. Five sets of these lines, oriented at multiples of $2\pi/5$ and arranged appropriately, form an Ammann 5-grid **[3]**, to which the Penrose rhombus tiling is a sort of dual (FIGURE 6b). If the tiles are decorated with segments from the grid (actually, all varieties of Penrose tiles are mere decorations on the Ammann grid), and tiles are assembled so that segments of the grid form straight lines, then a plane-covering QP tiling is guaranteed.




a b

**FIGURE 6**
Ammann grid and Penrose tiling

Most interesting is the way that the local isomorphism property of the $\phi$-sequence is manifest in the tiling as *local symmetry*: arbitrarily large regions of pentagonal ($D_5$) symmetry occur infinitely often in the tiling. The same process, sequence to grid to tiling, applied to the Sacred Cut, $\theta = 1 + \sqrt{2}$, produces a beautiful tiling that has local octagonal ($D_8$) symmetry. This local symmetry property has become the motivation for a search for QP tilings with other dihedral symmetries, and so has become one of the defining elements of quasiperiodicity.

Now if the reader is expecting a heptagonal revelation, there is only a puzzle. The problem of deciding whether the plane can be covered with copies of a given set of tiles is in general unsolvable; it is akin to the so-called "algebraic word problem." What we have just seen is a more fruitful way of exploring tiling: Establish the desired properties by means of a grid, then make tiles to order, that is, dissect the grid into tiles so that the pattern of grid lines is constant on each type of tile. Historically, the grid method has revealed many properties of these tilings. But though there are several $\rho$, $\sigma$-QP sequences analogous to the $\phi$-sequence, remarkably, the expected

7-grid has not been found. (The claim was made in **[3]**, but no explanation has appeared.) Analyses to date suggest that there must exist tilings with $D_7$ local symmetry, but that centers of such symmetry are very far apart, and that if there is a rhombic heptiling analogous to the Penrose sort, then each type of rhombus will appear with more than one type of decoration or matching rule, and the number of types of decorations is unknown. (For an interesting alternative, see Franco's tiling **[2]**. This is a radial tiling; it has one center of global rotational symmetry.)

The difficulty of the grid method in the heptagonal case prompts two observations. First, all of the richly symmetric QP grids found so far—with $D_5$, $D_8$, $D_{12}$, and $D_{17}$ local dihedral symmetry—are based on *constructible* polygons. Do we face the same barrier that stopped the Greek geometers, but for different reasons? Second, the regular pentagon is prohibited from periodic systems of dimension less than four, but can occur in QP systems in $\mathbb{R}^2$ and $\mathbb{R}^3$. The regular heptagon cannot occur in periodic systems of dimension less than six. At what level does the heptagon—using a grid and a finite number of tile shapes—fully participate in the quasiperiodic arena?

**Acknowledgment.** My study of these patterns began many years ago when I told Abraham Hillman of the University of New Mexico that I wanted to find the Seventh Roots of Unity. "You may find them," he replied, "but they won't be of any use to you." I found them and he was right. (At the time I didn't know that he never speaks hastily.) But the path I took showed me many useful and beautiful things.

## REFERENCES

1. E. J. Dijksterhuis, *Archimedes*, Ejnar Munksgaard, Copenhagen, 1956.
2. B. J. O. Franco, Third-order Fibonacci sequence associated to a heptagonal quasiperiodic tiling of the plane. *Physics Letters* A 178 (1993) 119–122.
3. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman and Co., New York, NY, 1987.
4. Robert Lawlor, *Sacred Geometry*, Thames & Hudson, London, 1992.
5. Neil J. Sloane and Simon Plouffe, *The Encyclopedia of Integer Sequences*. Academic Press, New York, NY, 1995.
6. H. Terauchi, et al., Synchrotron X-ray diffraction study of third-order Fibonacci lattices. *J. Phys. Soc. Japan* 59 (1990) 405.
7. Kim Williams, The sacred cut revisited: the pavement of the Baptistery of San Giovanni, Florence. *The Mathematical Intelligencer* 16:2 (1994) 18.