Geometric Data Structures: Range Queries

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Indian Institute of Technology Bombay

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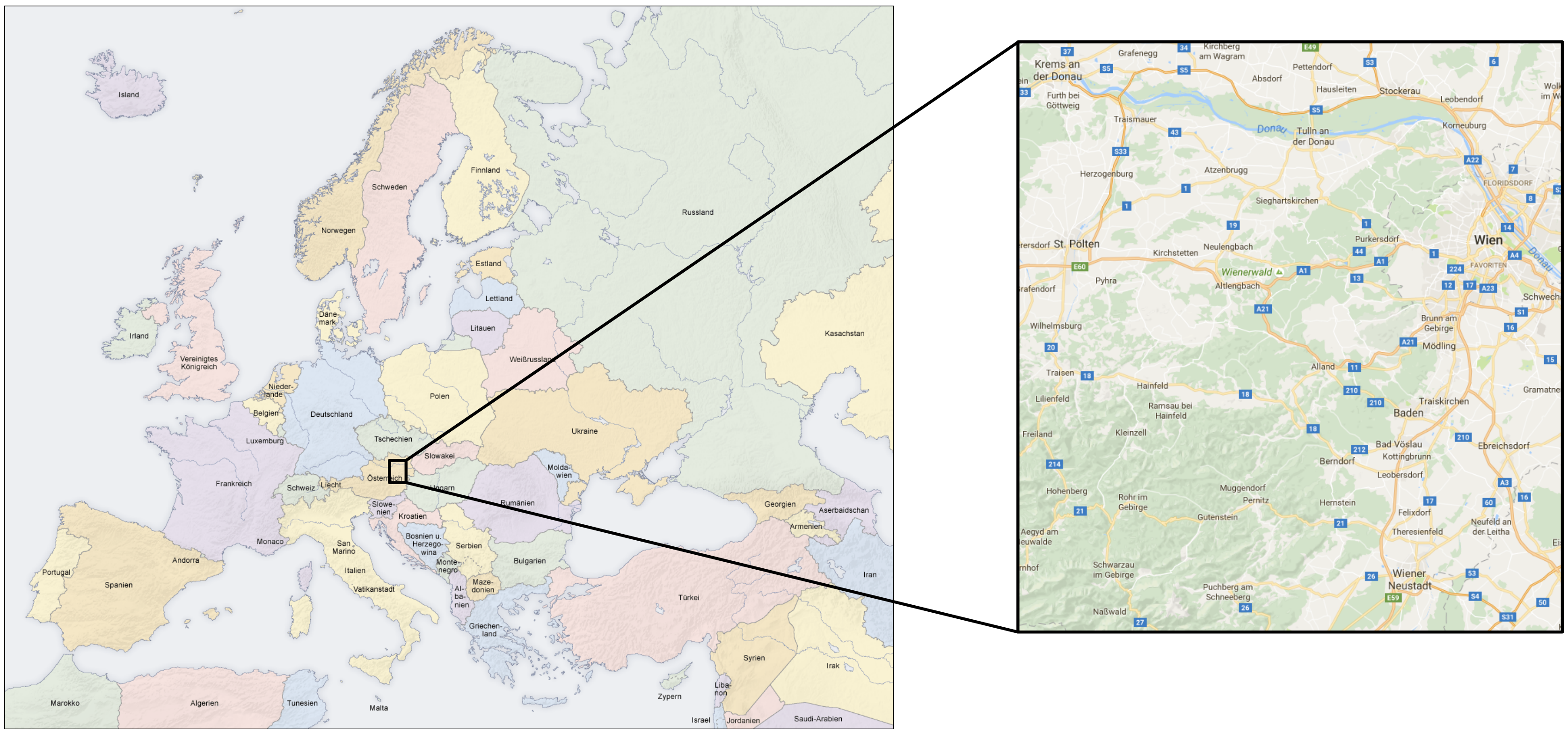

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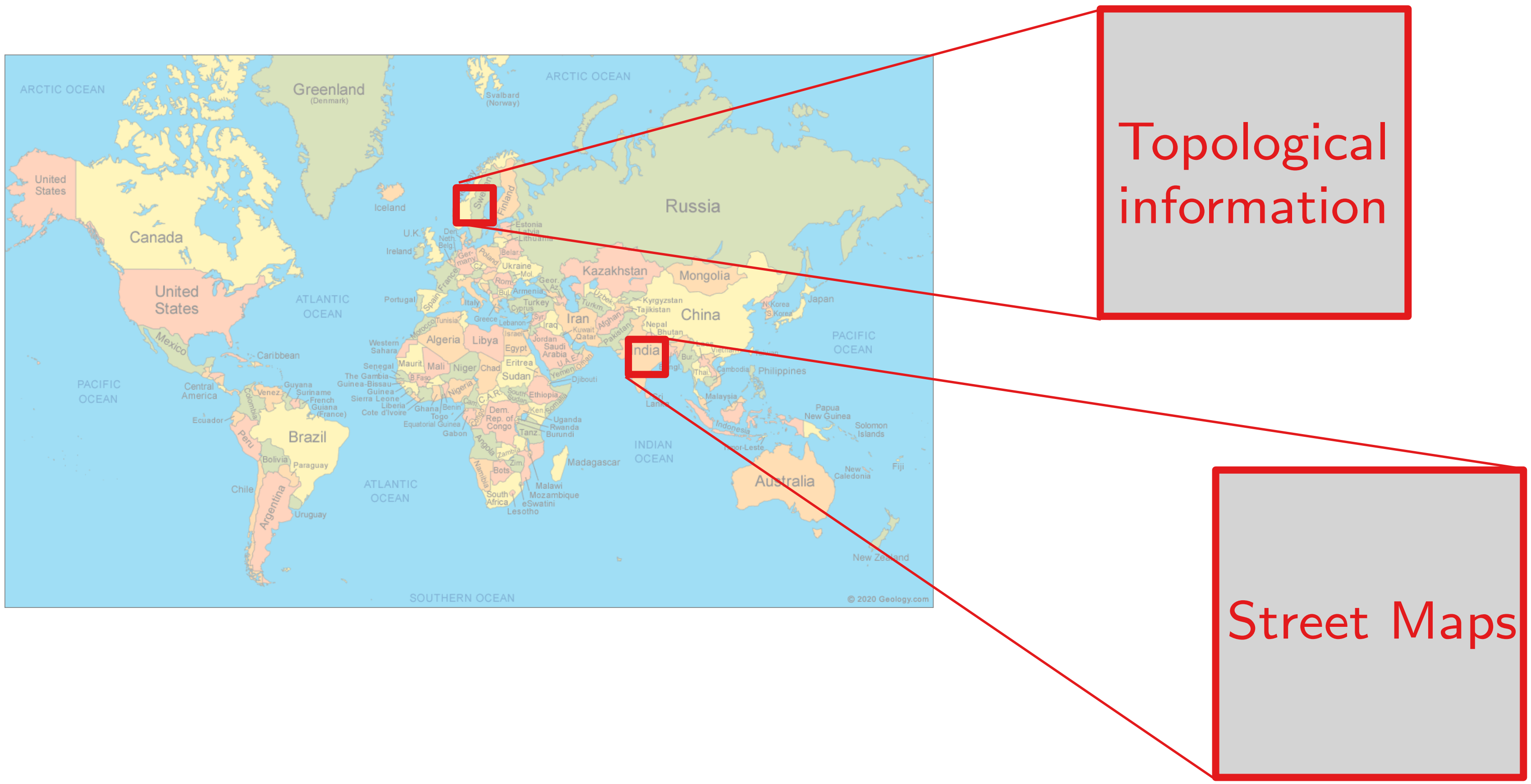

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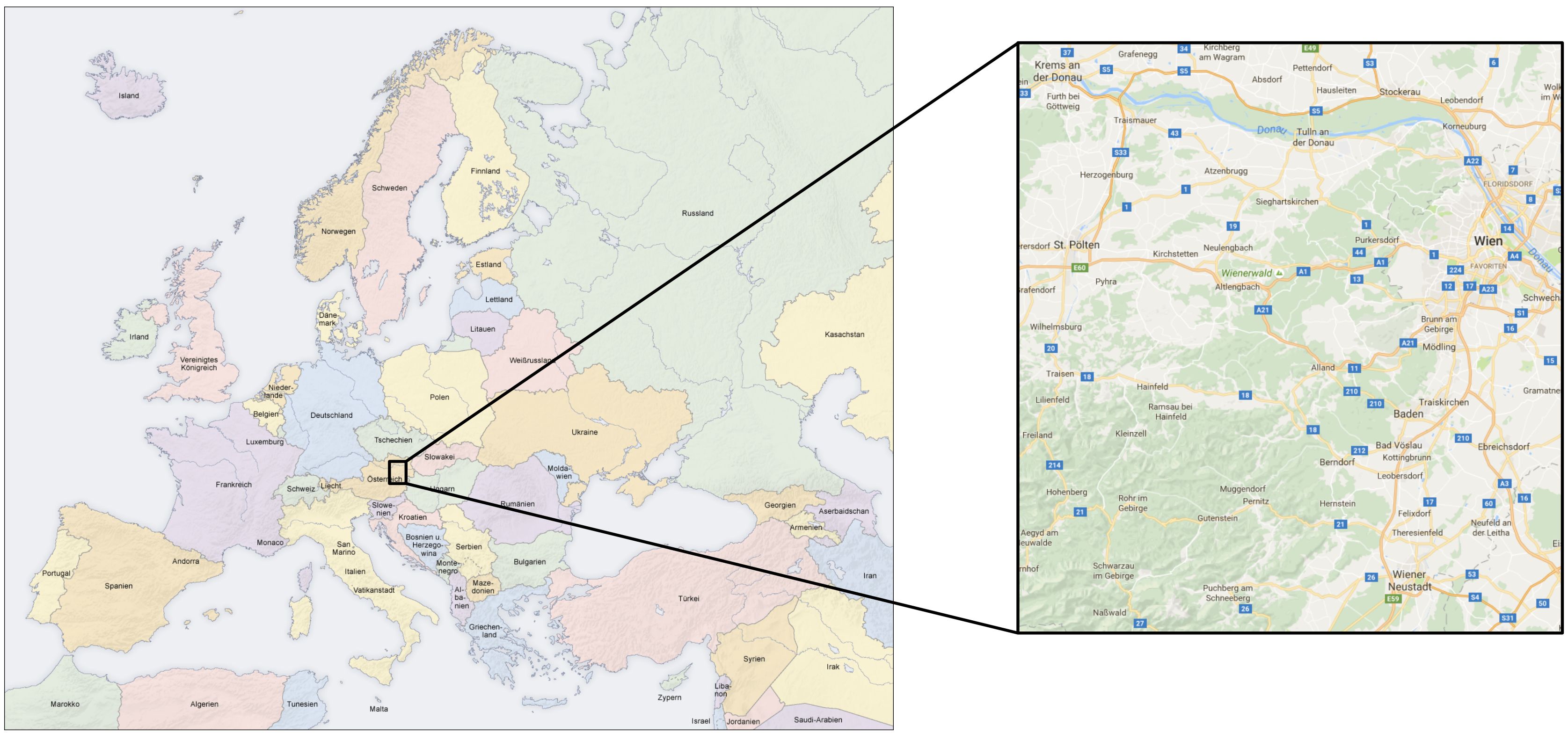

Evaluating each map feature is unrealistic.

We want a fast data structure for answering range queries

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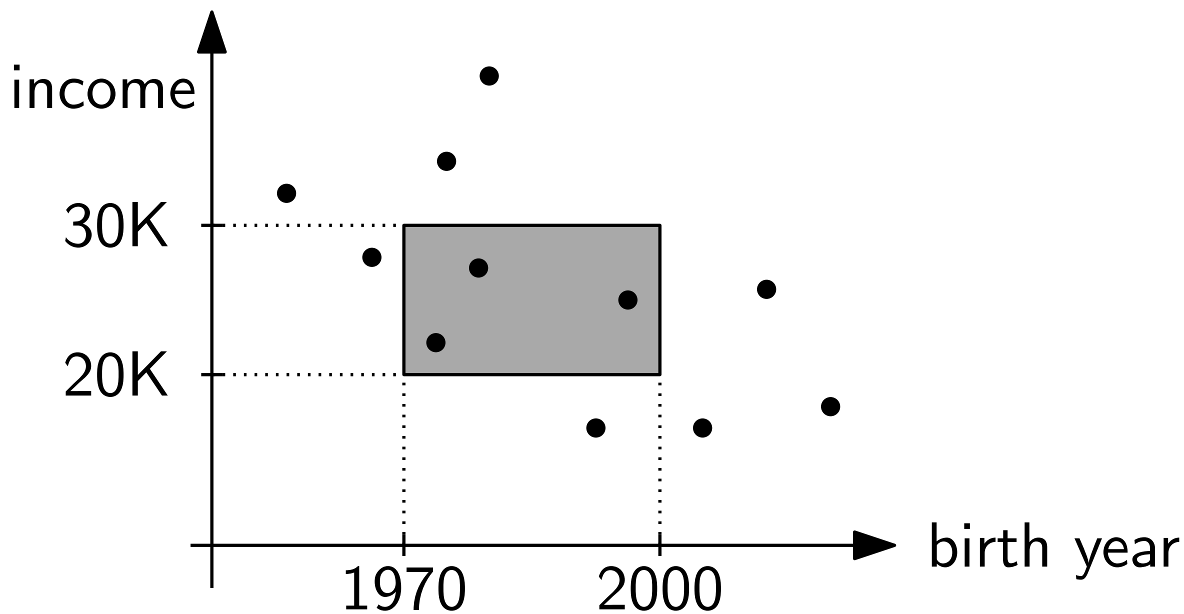

Geometric Interpretation:
Entries are points: (birth year, income level) and the query is an axis-parallel rectangle

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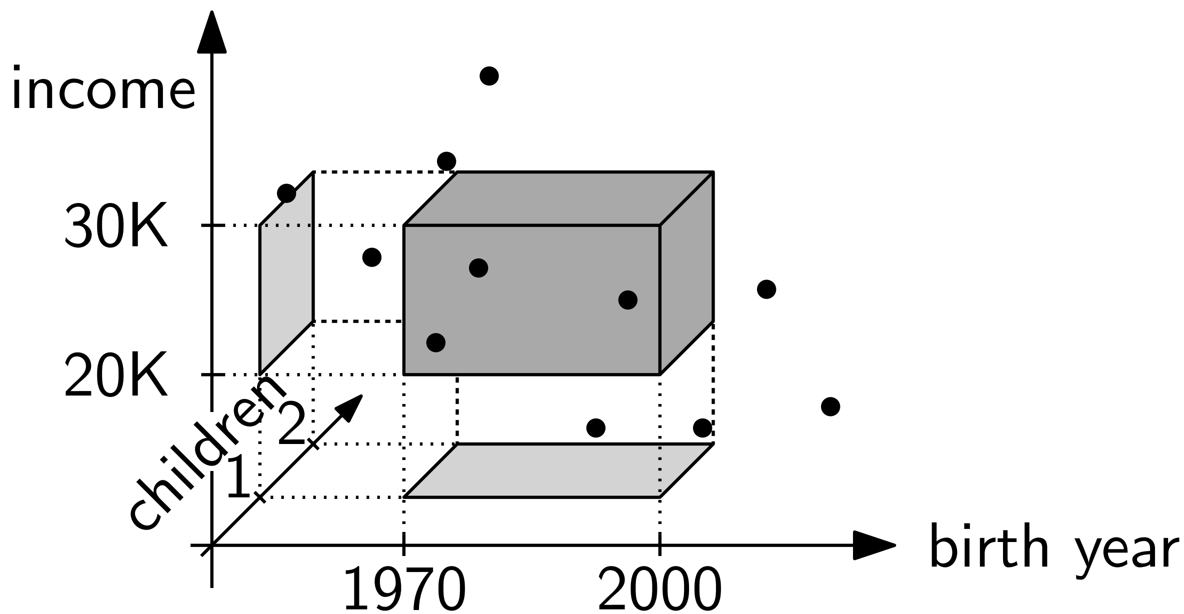

- This problem can easily be generalized to d dimensions.

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Given: n points in $\mathbb{R}^d$

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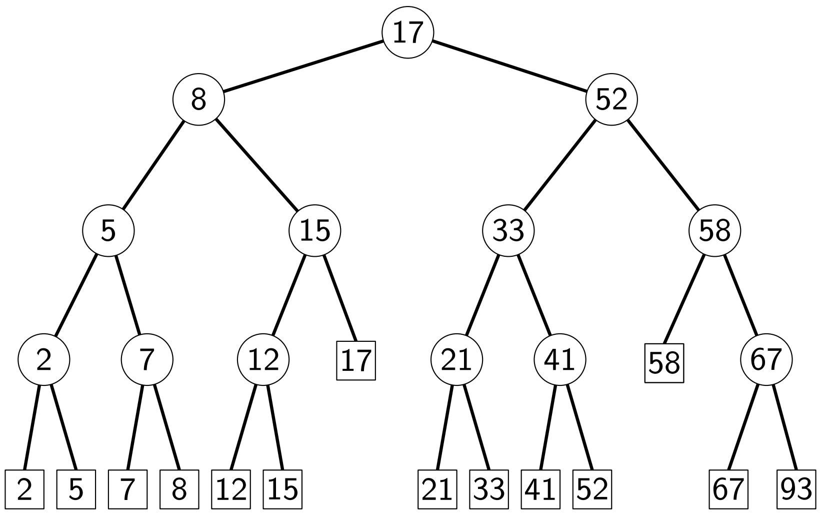

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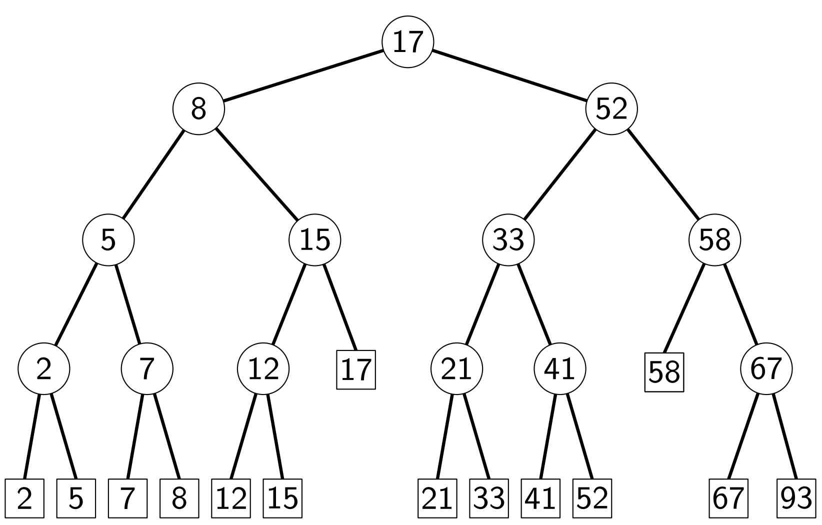
largest element in the left subtree



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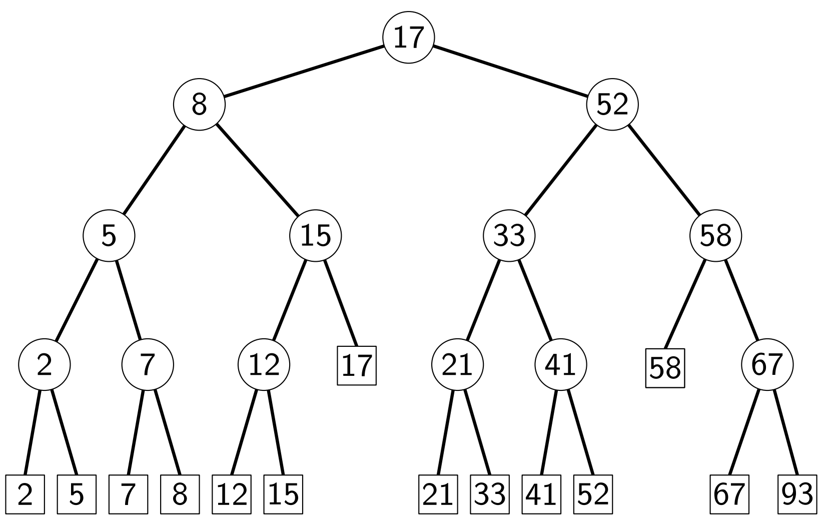

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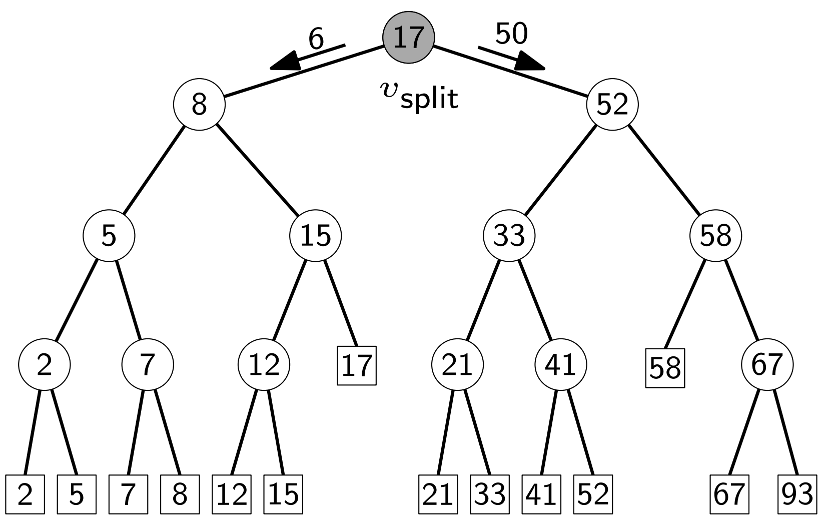

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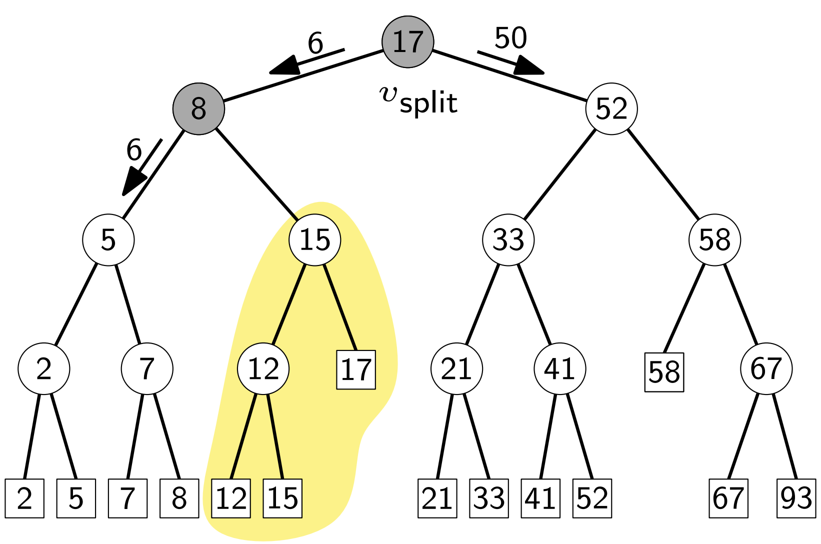

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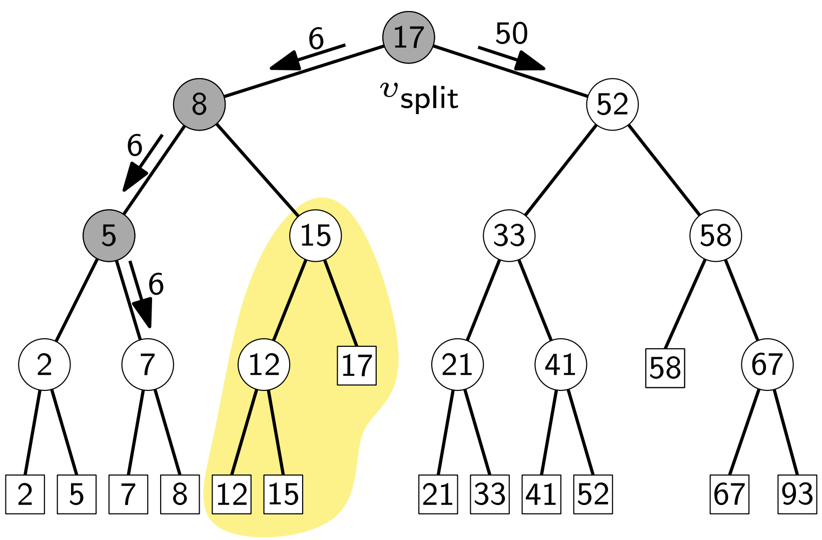

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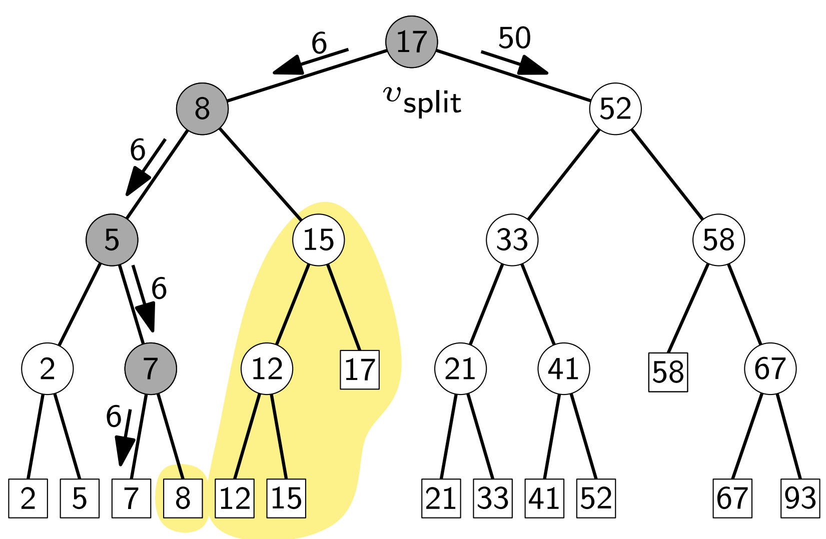

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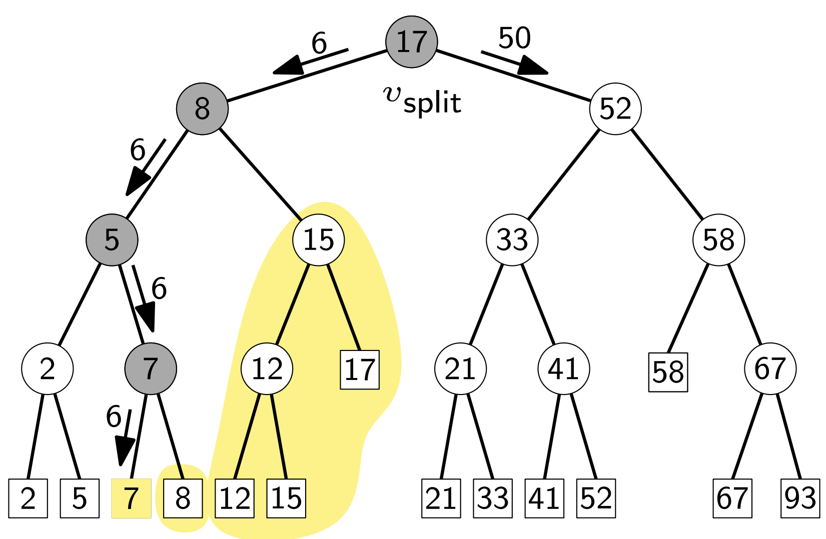

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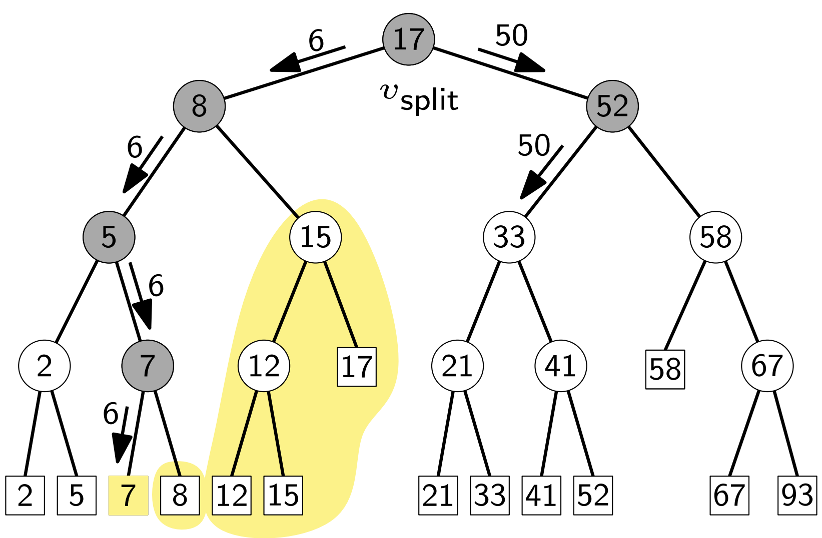

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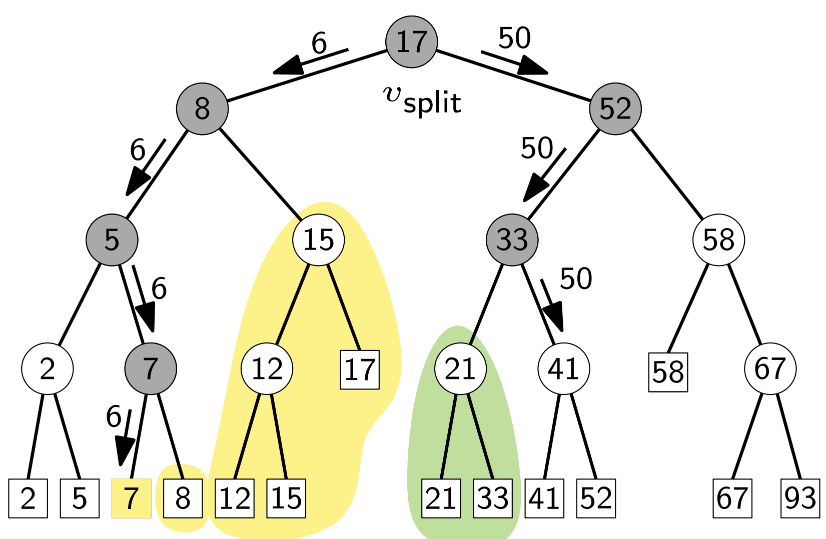

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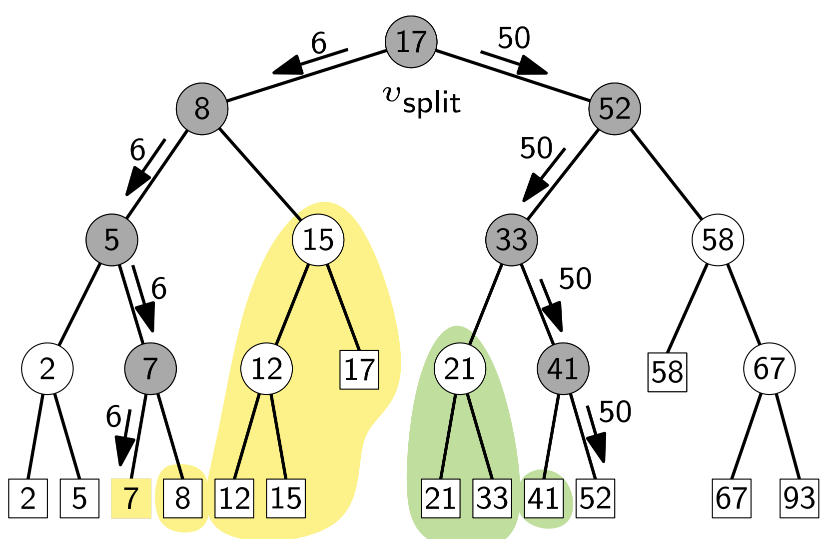

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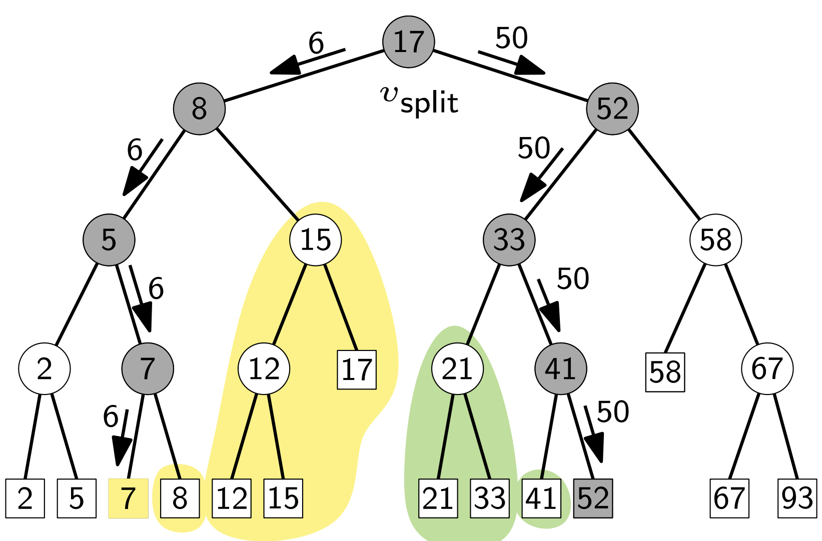

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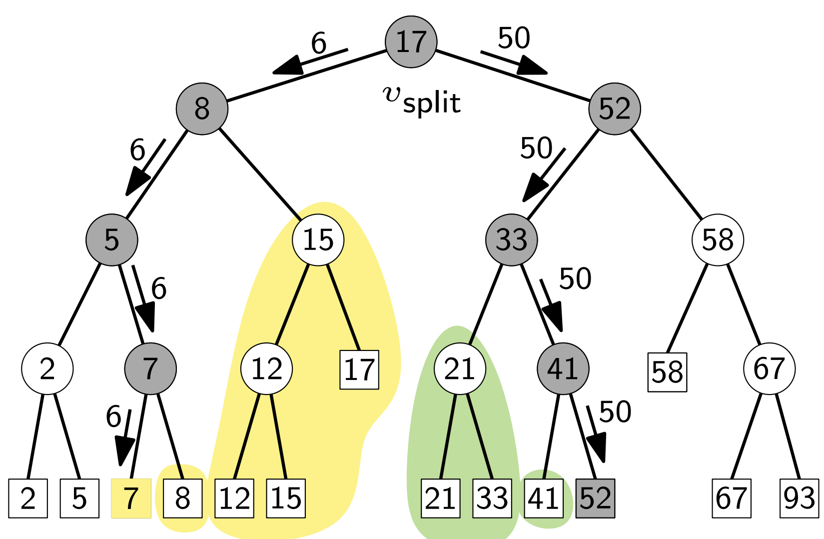

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Answer:

Points in the leaves between the search paths, i.e., $\{7,8,12,15,17,21,33,41\}$

1dRangeQuery

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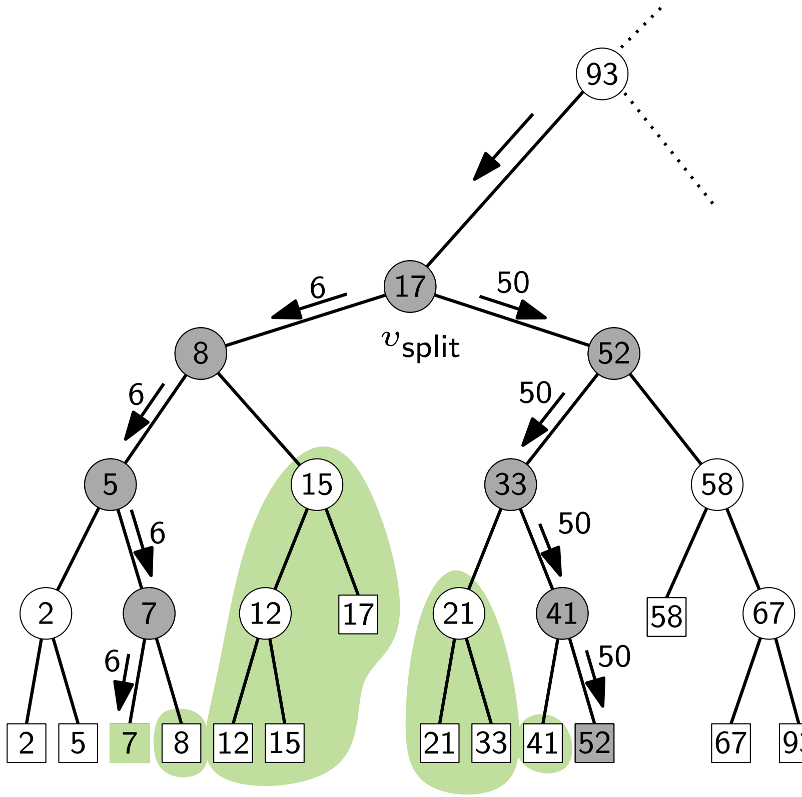

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while v not a leaf and $(x' \le x_v$ or $x > x_v)$ **do**

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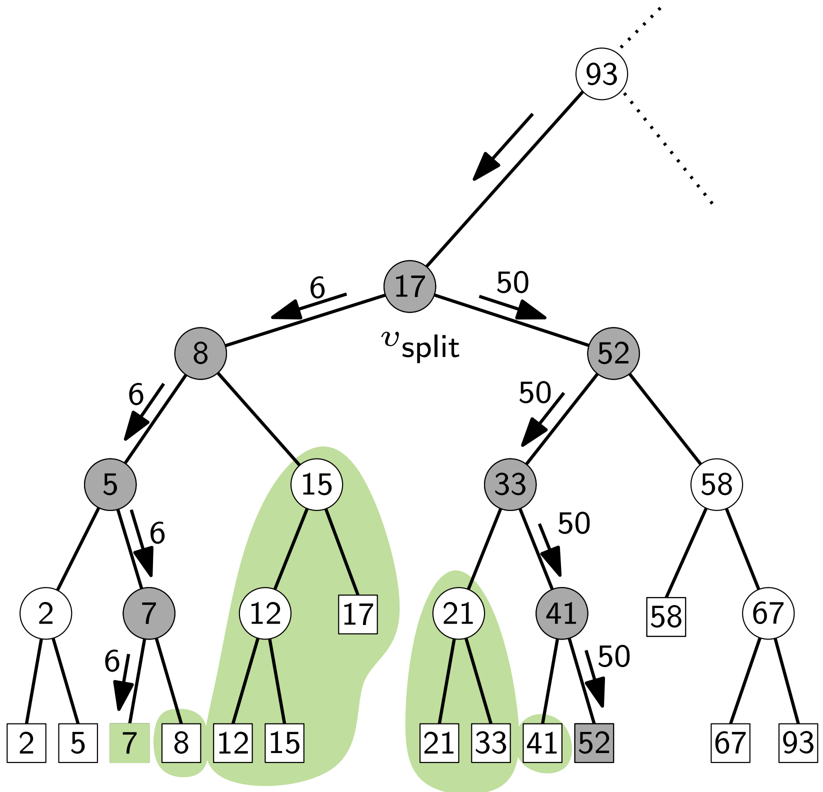

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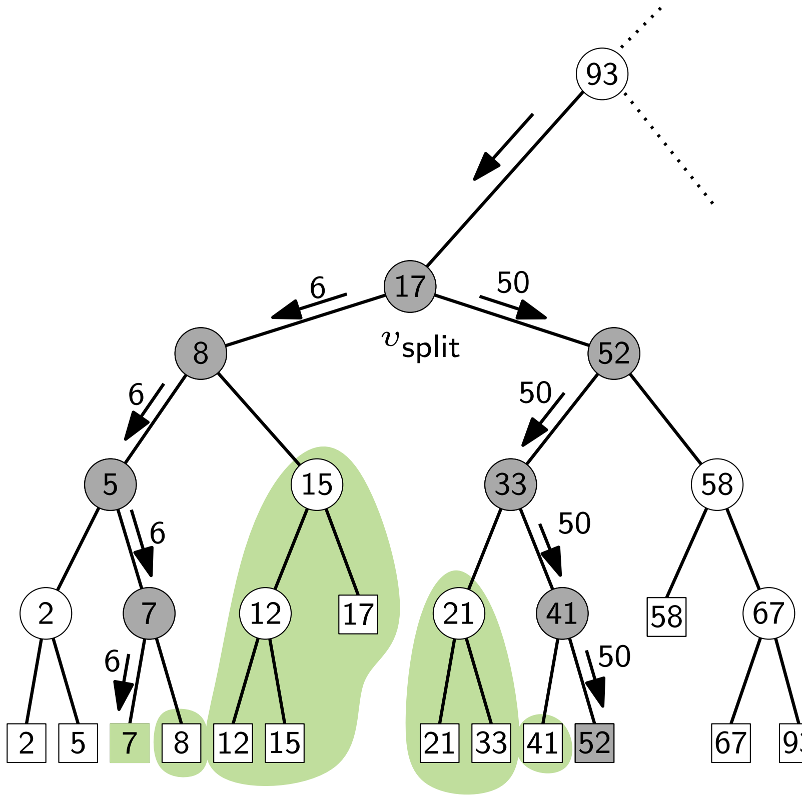

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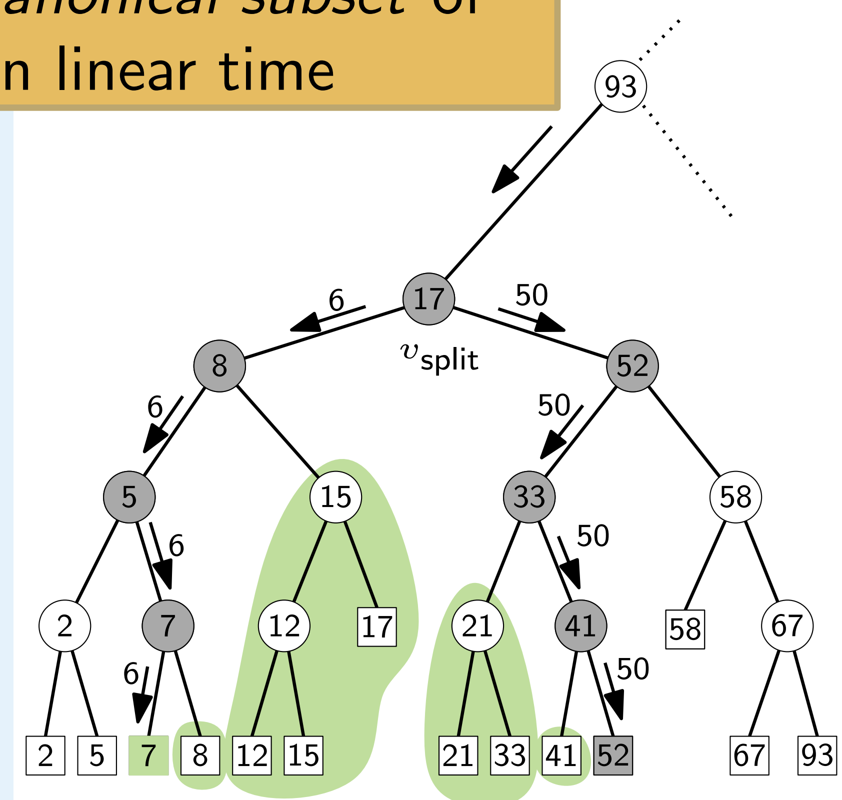
Can find *canonical subset* of all leaves in linear time



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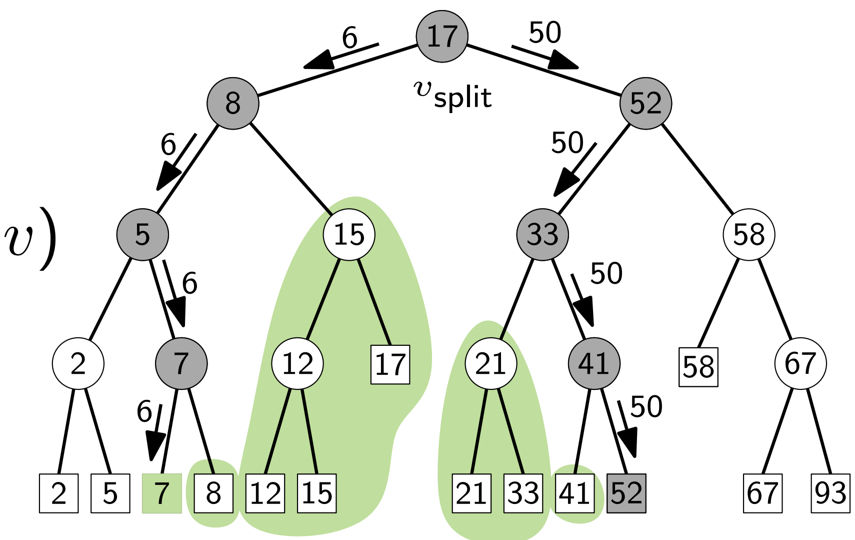

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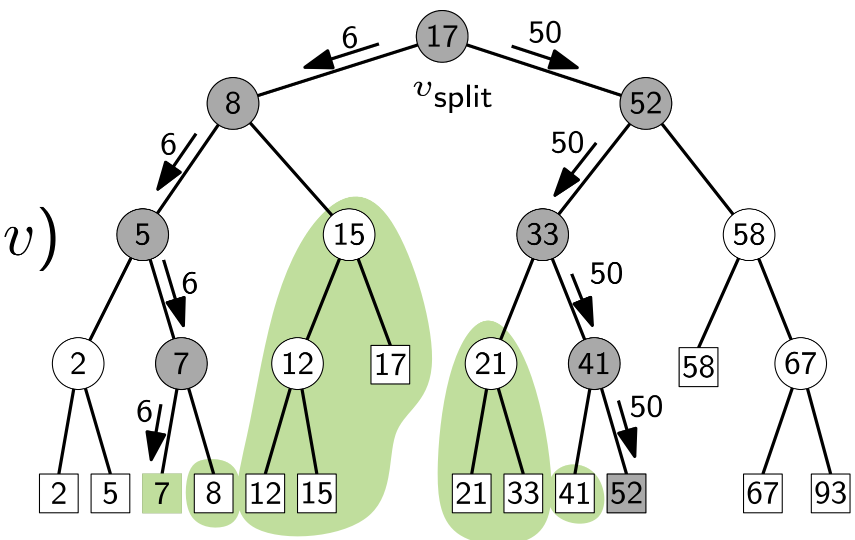

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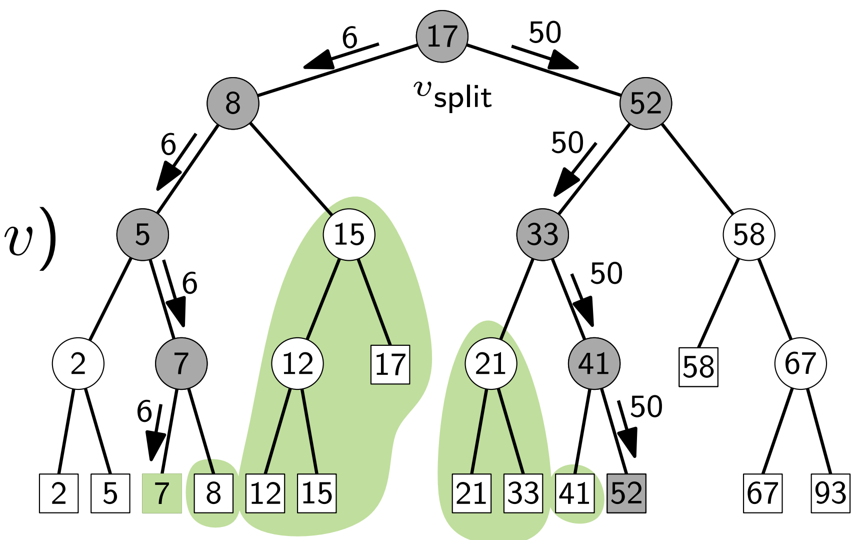

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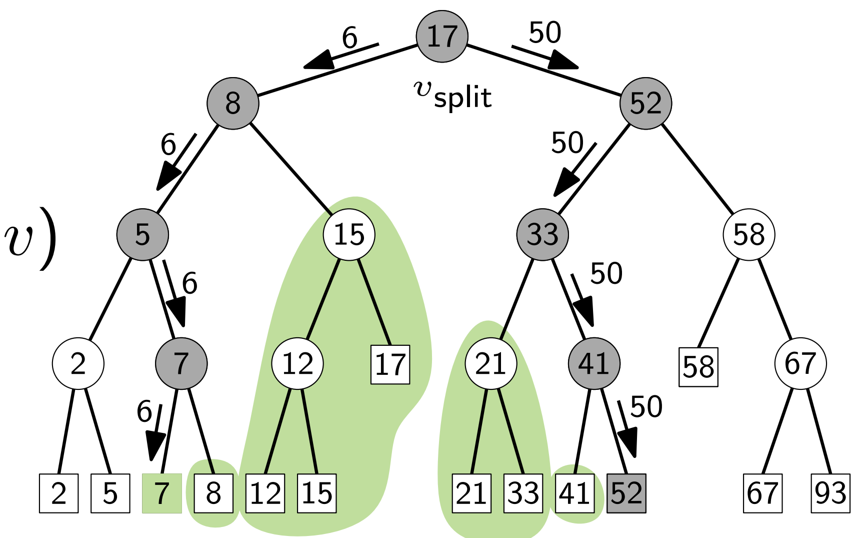

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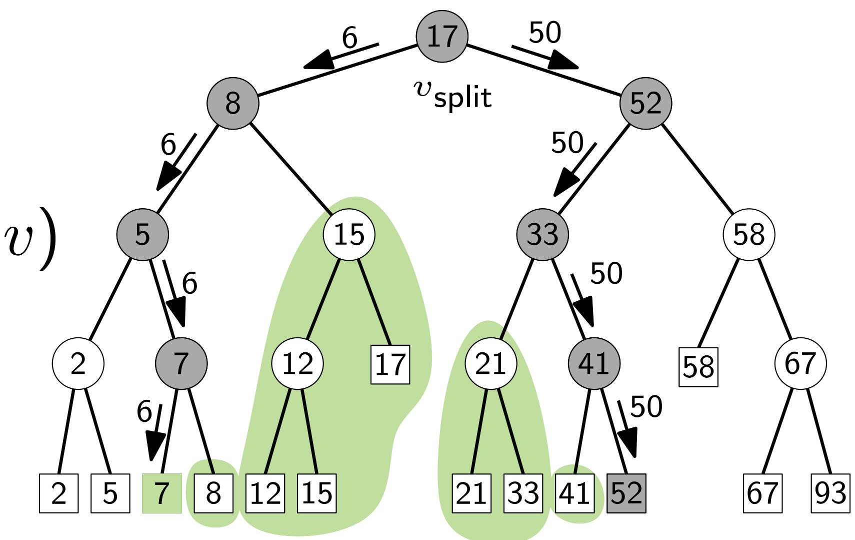

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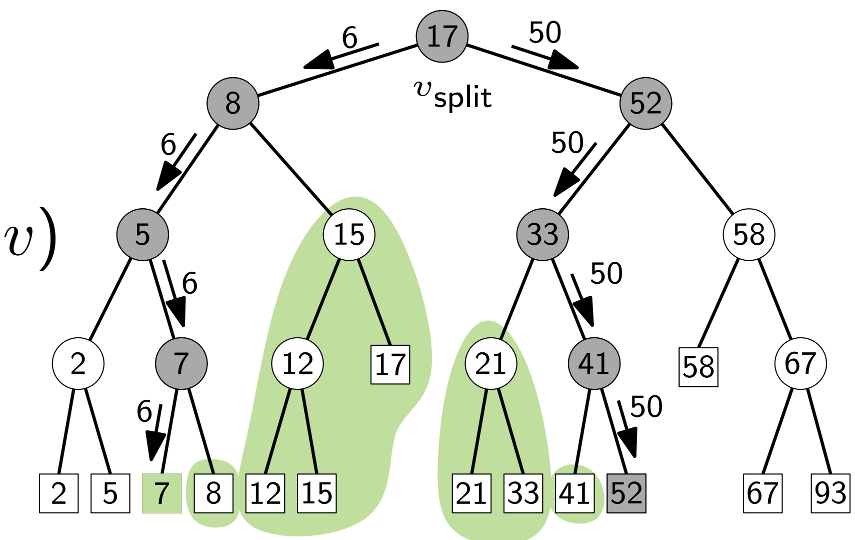
Proof sketch -
- two search paths of length $O(\log n)$.
- reporting subtree takes $O(k_v)$ time, where k_v is the # of leaves in the subtree of v.
- $\sum k_v \implies O(\log n + \sum k_v) = O(k \log n)$.



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Goal: A data structure to efficiently answer range queries of the form $R = [x, x'] \times [y, y']$

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- *one* search tree, alternate search for x and y coordinates
  $\rightarrow$ kd-**Tree**
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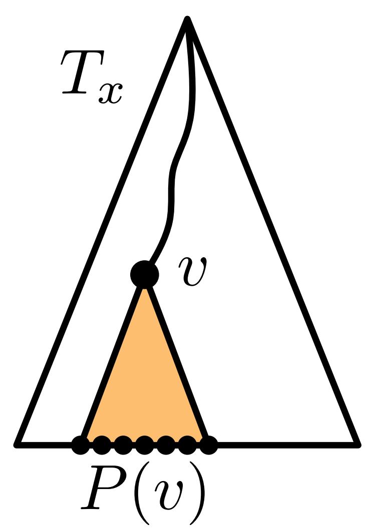

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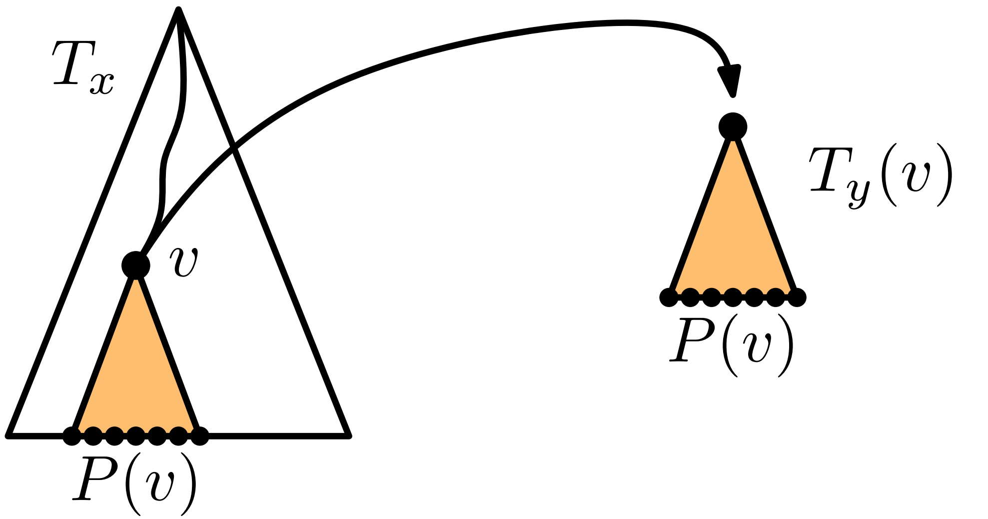
- a 1d search tree T_x on x-coordinates
- in each node v of T_x a 1d search tree $T_y(v)$ to store the canonical subset $P(v)$ based on y-coordinates



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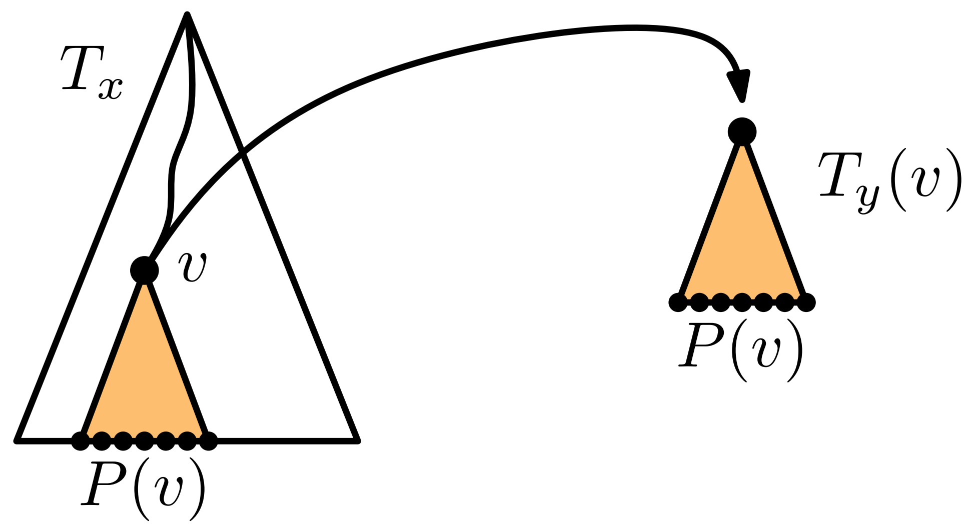

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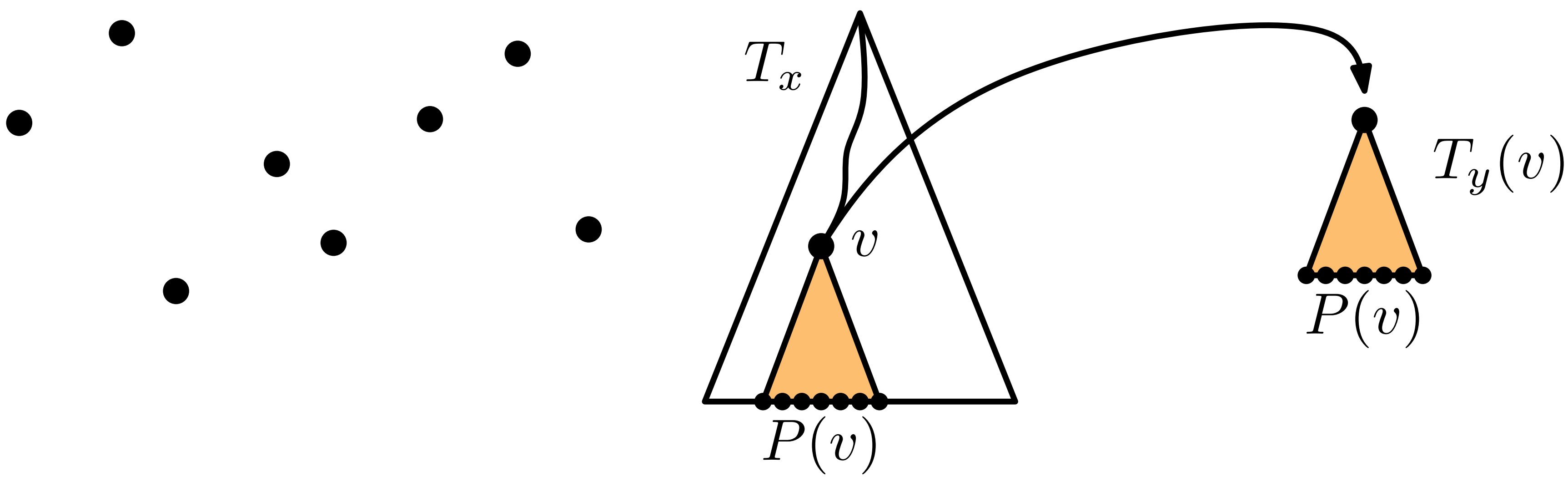

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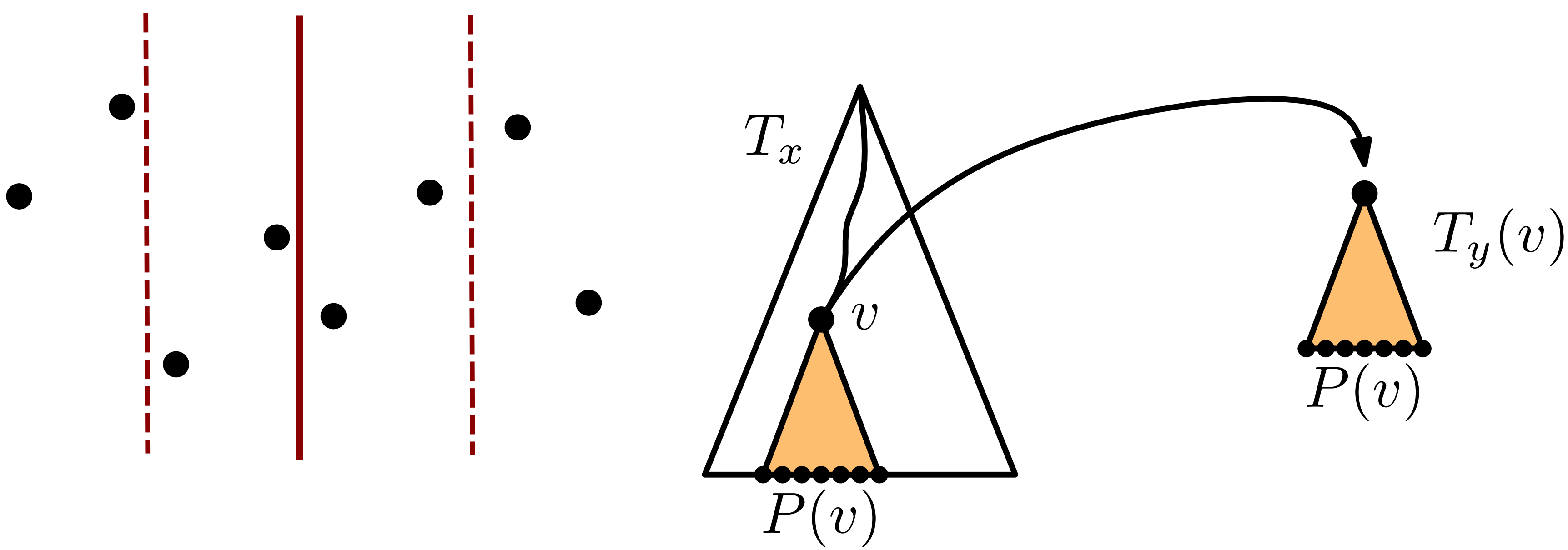

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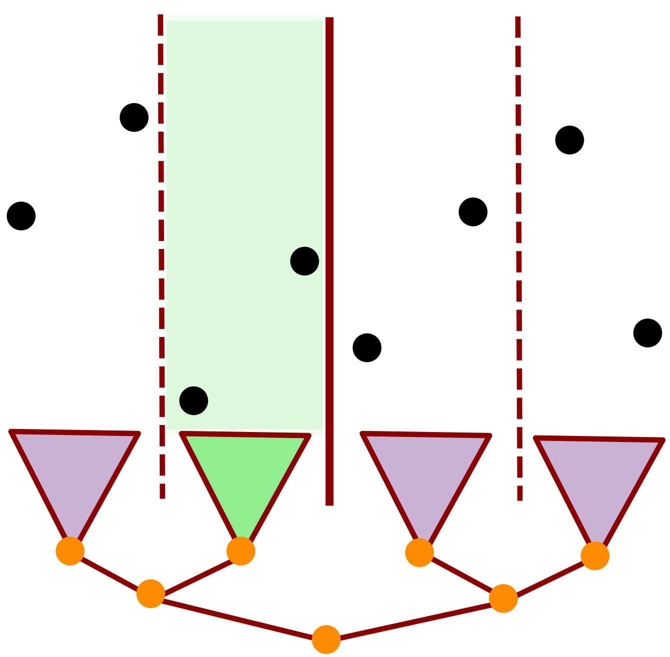

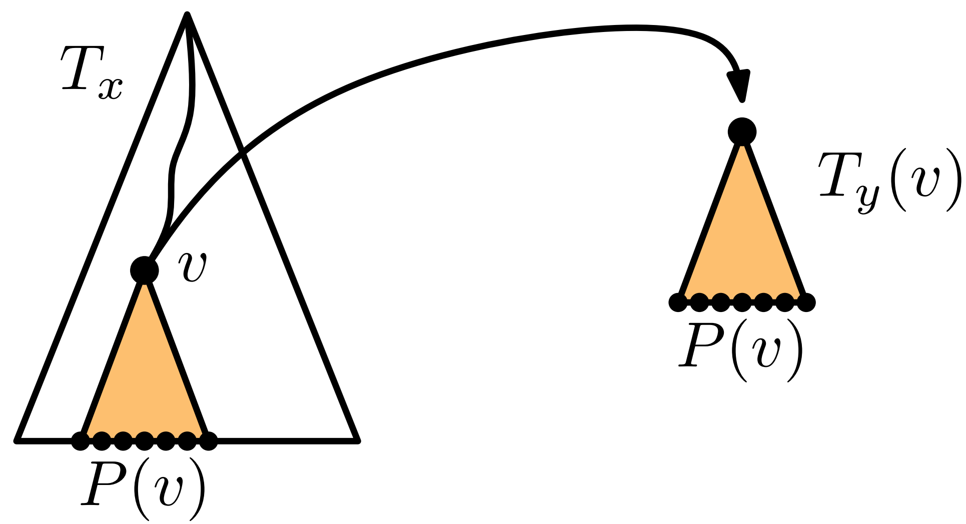

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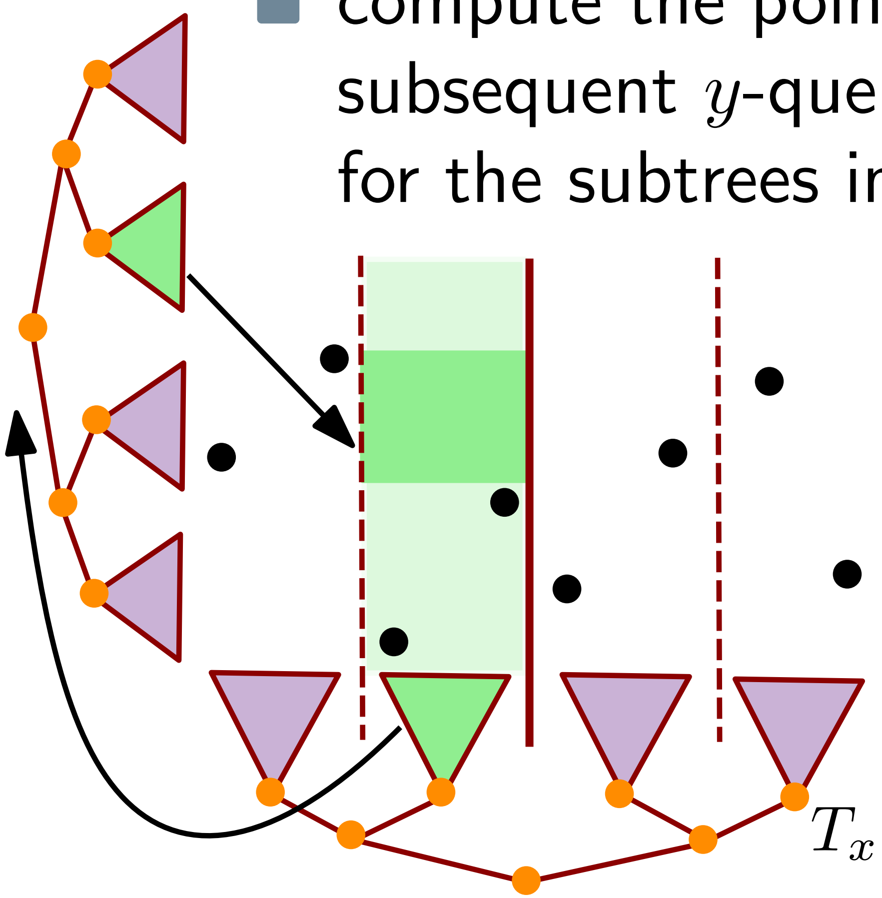

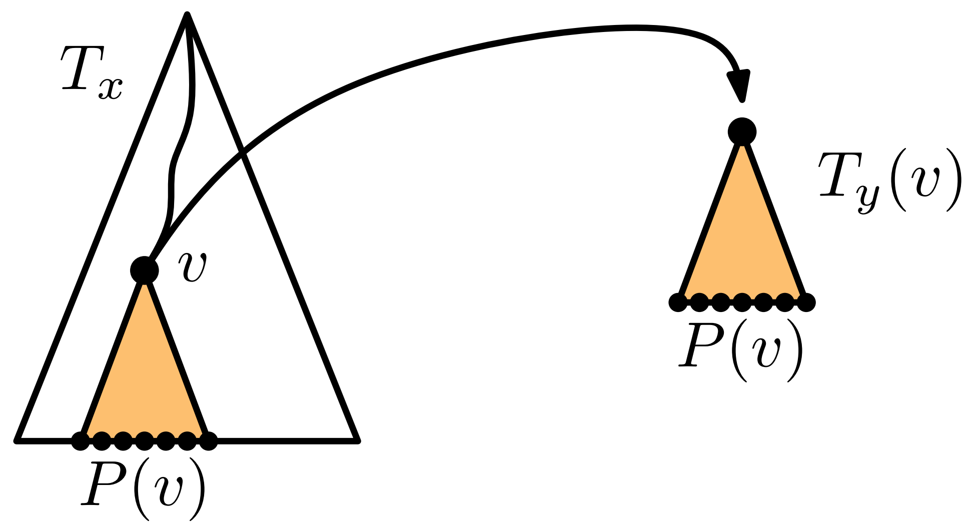

Range Trees: Construction

BuildRangeTree(P)

if $|P| = 1$ **then**

  create leaf v for the point in P

else

  split P at x_{median} into $P_1 = \{p \in P \mid p_x \leq x_{\text{median}}\}$ and $P_2 = P \setminus P_1$

  $v_{\text{left}} \leftarrow$ BuildRangeTree(P_1)

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Task: How much space and runtime does BuildRangeTree use?

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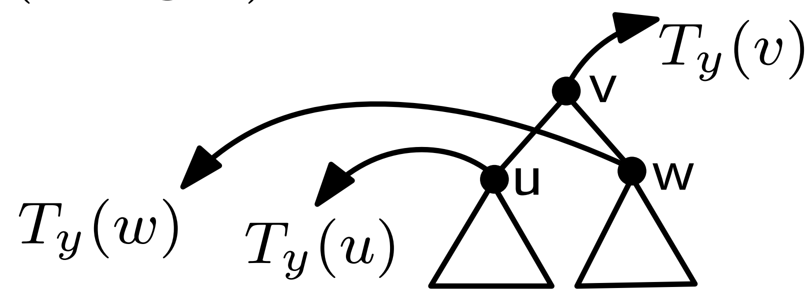

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Space -

- Tree T_x uses $O(n)$ space.
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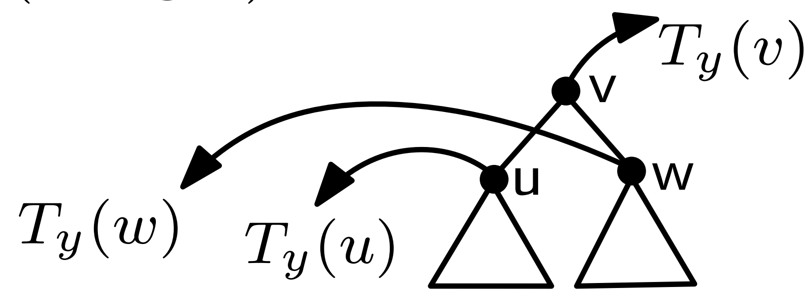

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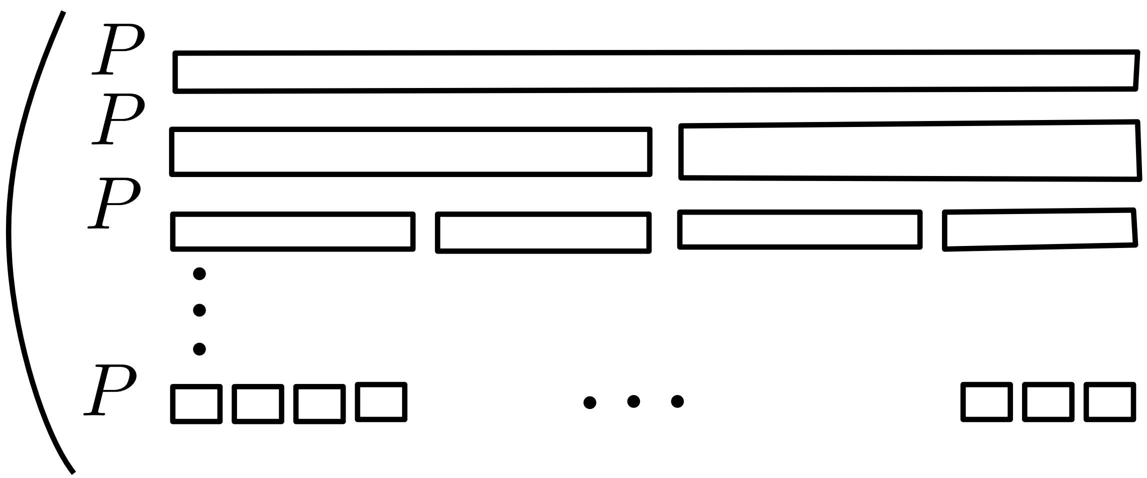

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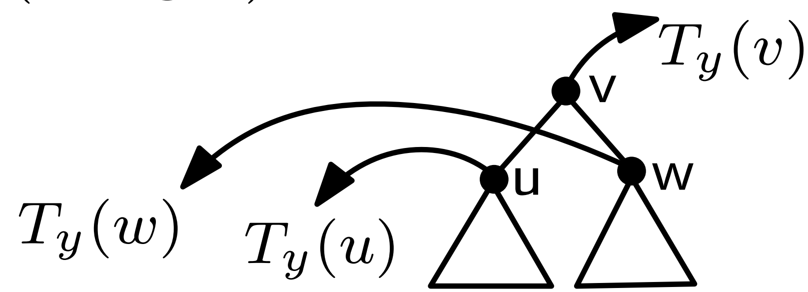

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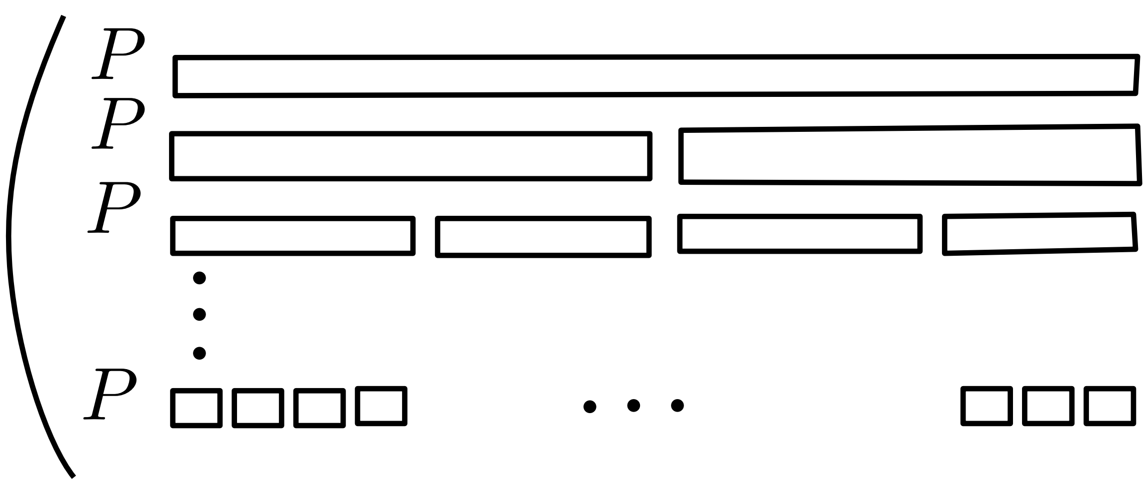

Note: we have $O(n)$ secondary trees, but, total space is $O(n \log n)$.

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Time -

- Constructing T_x takes $O(n \log n)$ time as before.
- For the T_y' pre-sort P by y-coordinates once in $O(n \log n)$ time.
- From this sorted list build $T_y(v)$ bottom-up in $O(n)$ time for the root v of T_x.
- Filter the sorted list into sorted lists of subsets P_1 and P_2 and construct $T_y(u)$, $T_y(w)$ again in $O(n)$ time per level for $u = lc(v)$, $w = rc(v)$; then recurse.
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Range Queries in a Range Tree

Reminder:

1dRangeQuery(T, x, x')

$v_{\text{split}} \leftarrow$ FindSplitNode(T, x, x')
if v_{split} is leaf **then** check v_{split}
else
    $v \leftarrow$ lc(v_{split})
    while v not leaf **do**
        if $x \leq x_v$ **then**
            ReportSubtree(rc(v))
            $v \leftarrow$ lc(v)
        else $v \leftarrow$ rc(v)
    report v
    `//` `analogous for` x' `and rc(`v_{split}`)`

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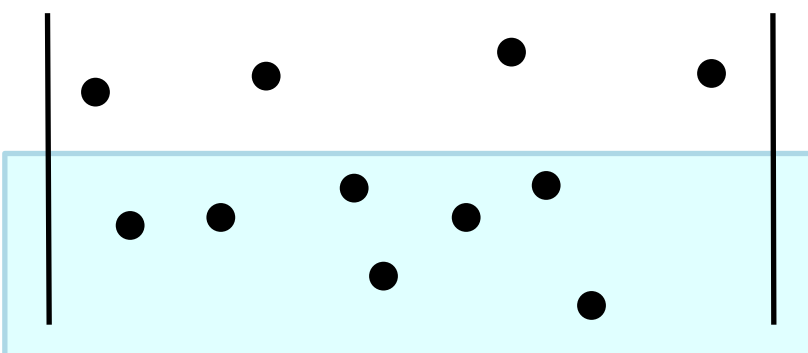

Cannonical subset in x-dim.

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- each visited node v in T_x: $O(1) + 1D$-RangeQuery $(O(k_v + \log n))$ ... recall $k_v = |p_v|$.
- $\sum O(k_v + \log n) = \underbrace{\sum O(k_v)}_{O(k)} + \sum O(\log n) = O(k) + O(\log^2 n)$.

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Observation: Range queries in a Range Tree perform $O(\log n)$ 1d range queries, each taking $O(\log n + k_v)$ time. The query interval $[y, y']$ is always the same!

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Can we do better than two binary searches?

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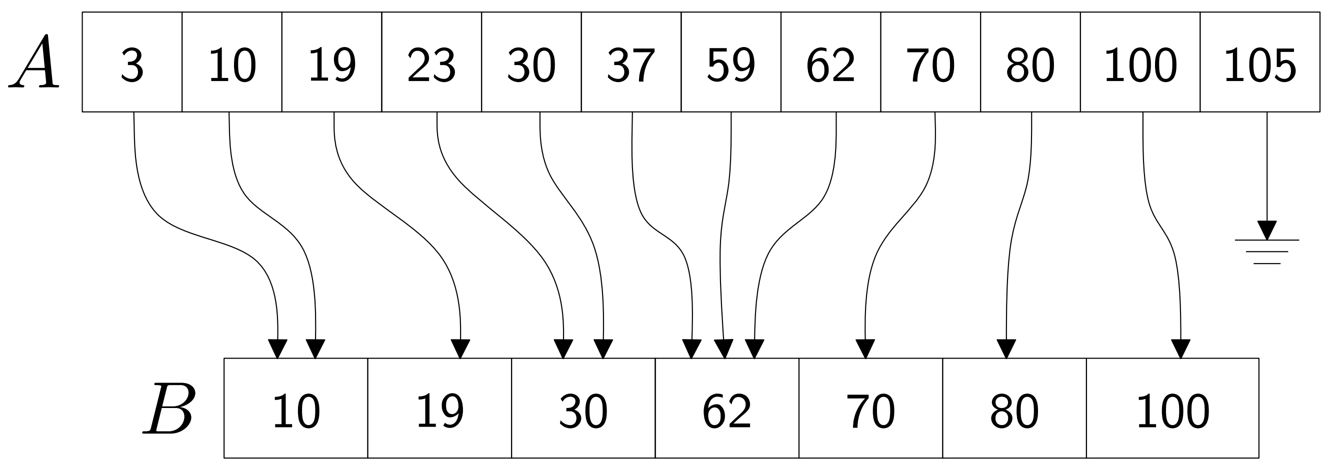

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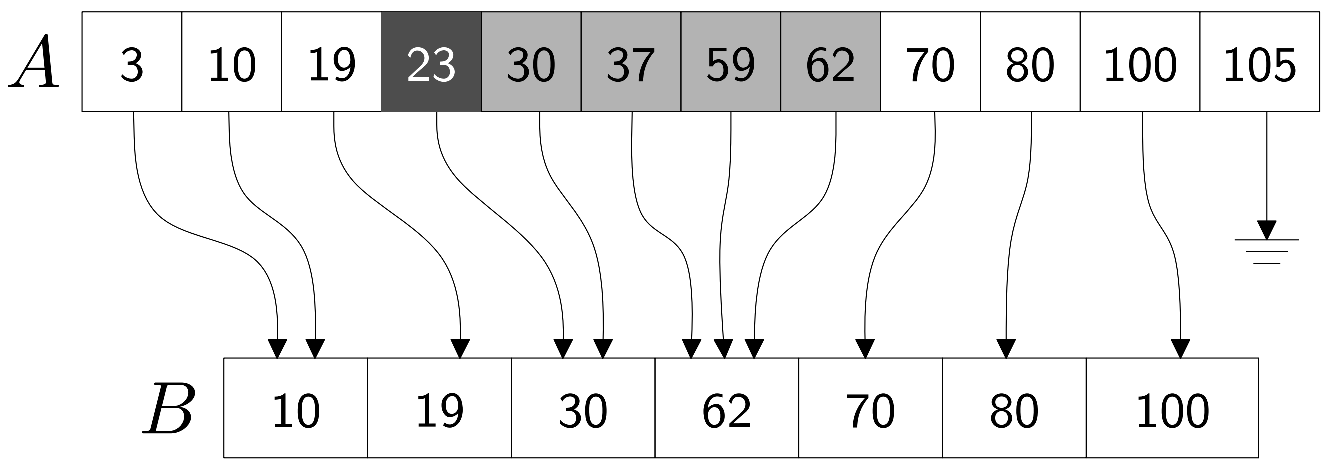

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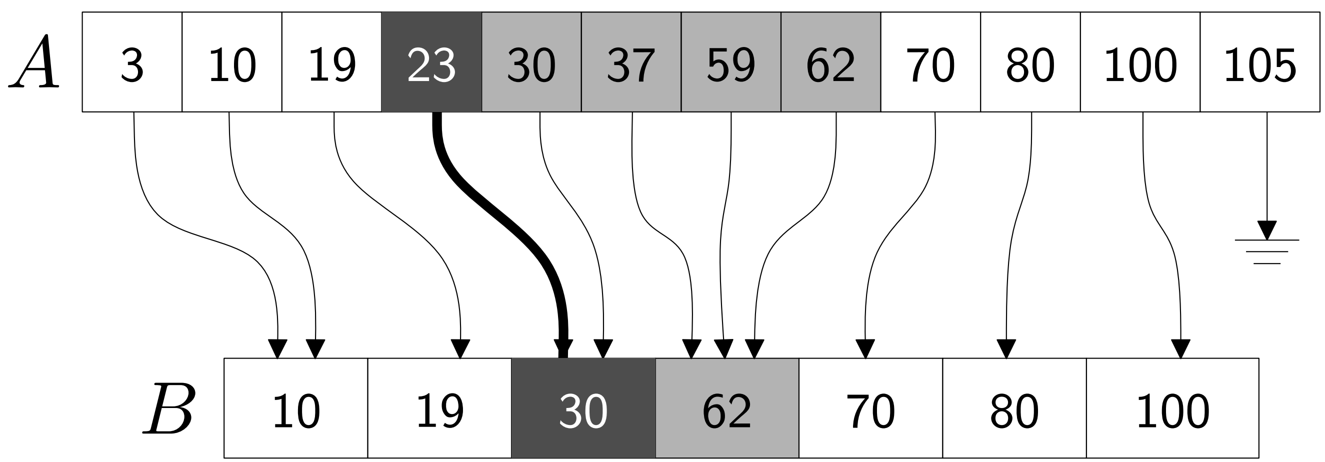

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Pointer yields starting point for second search in $O(1)$ time

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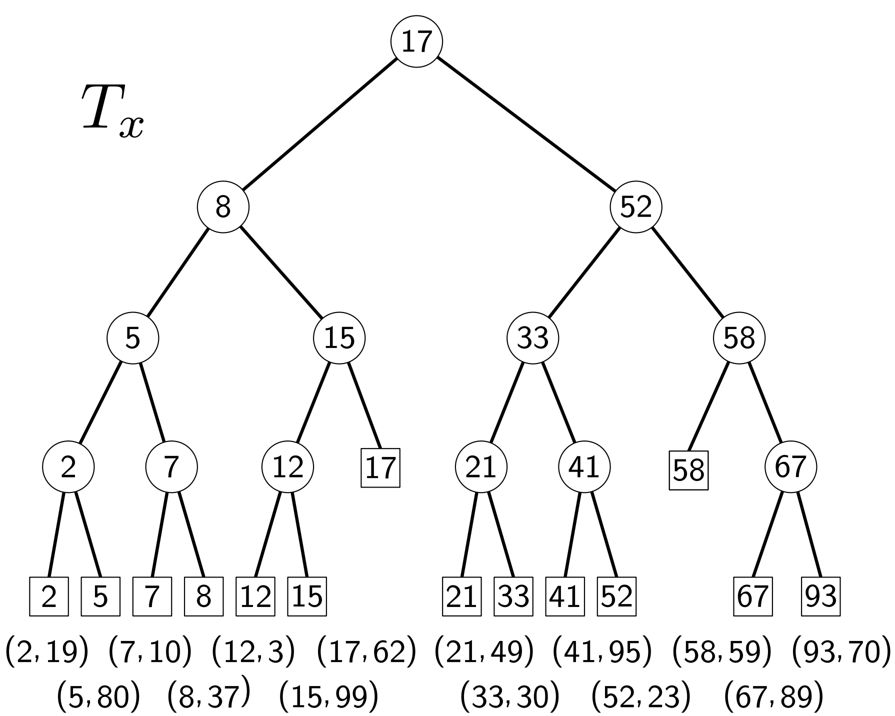

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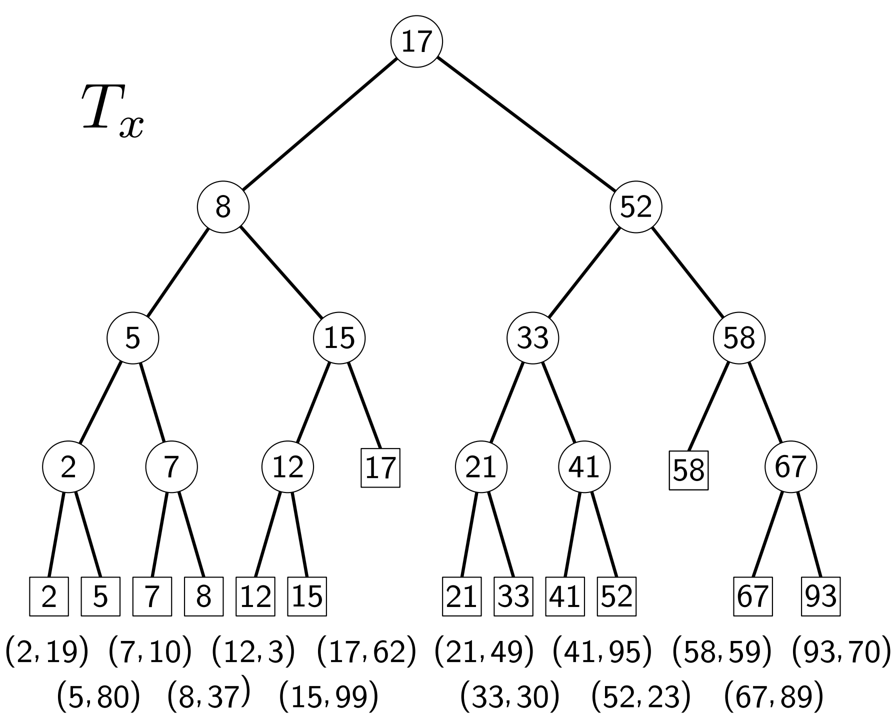

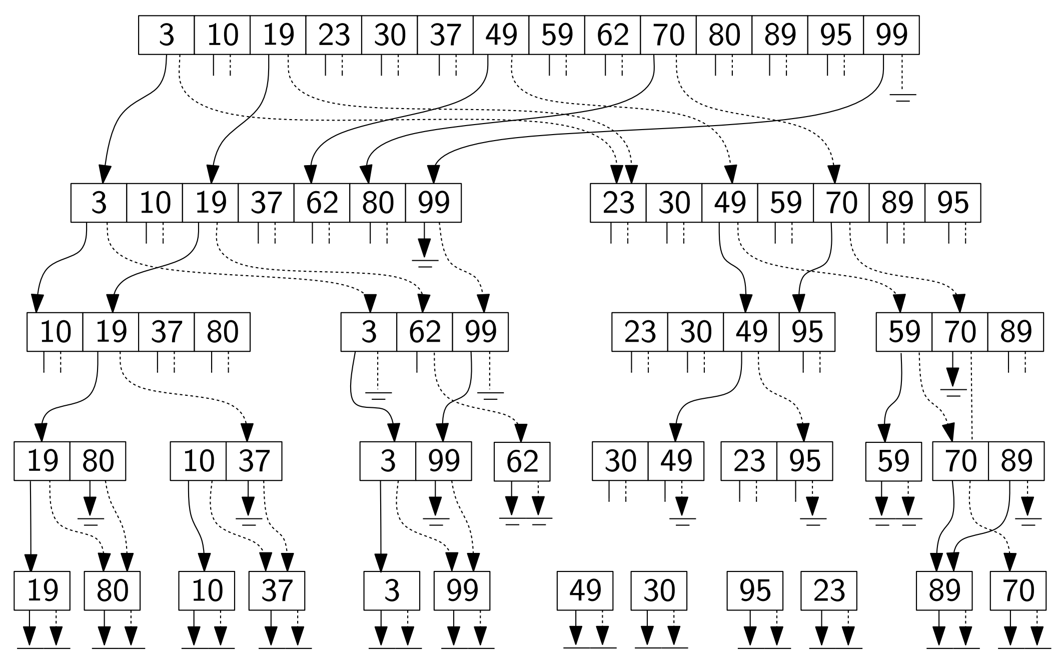

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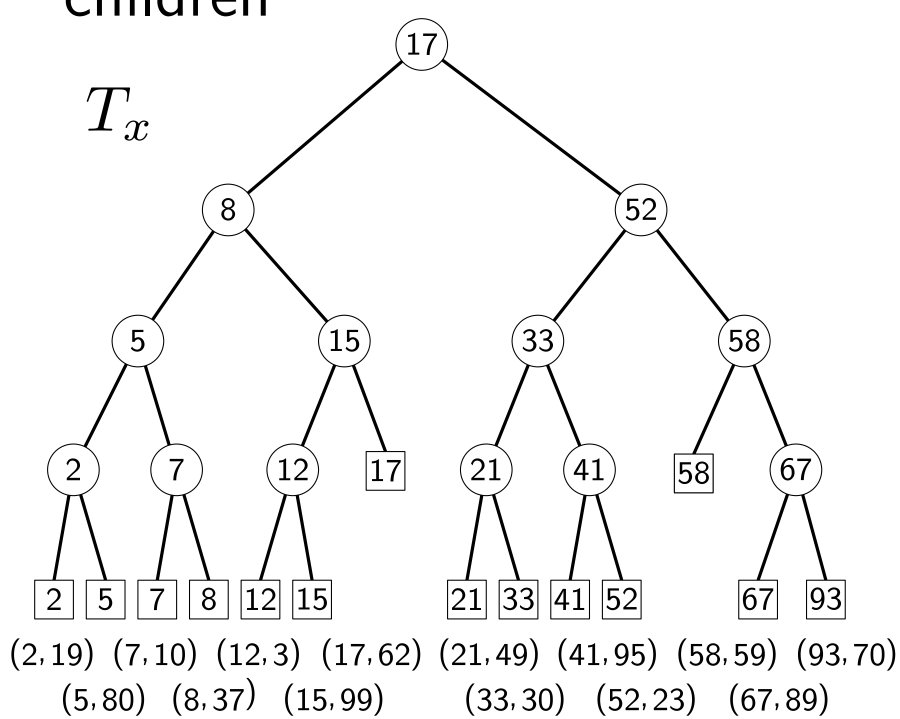

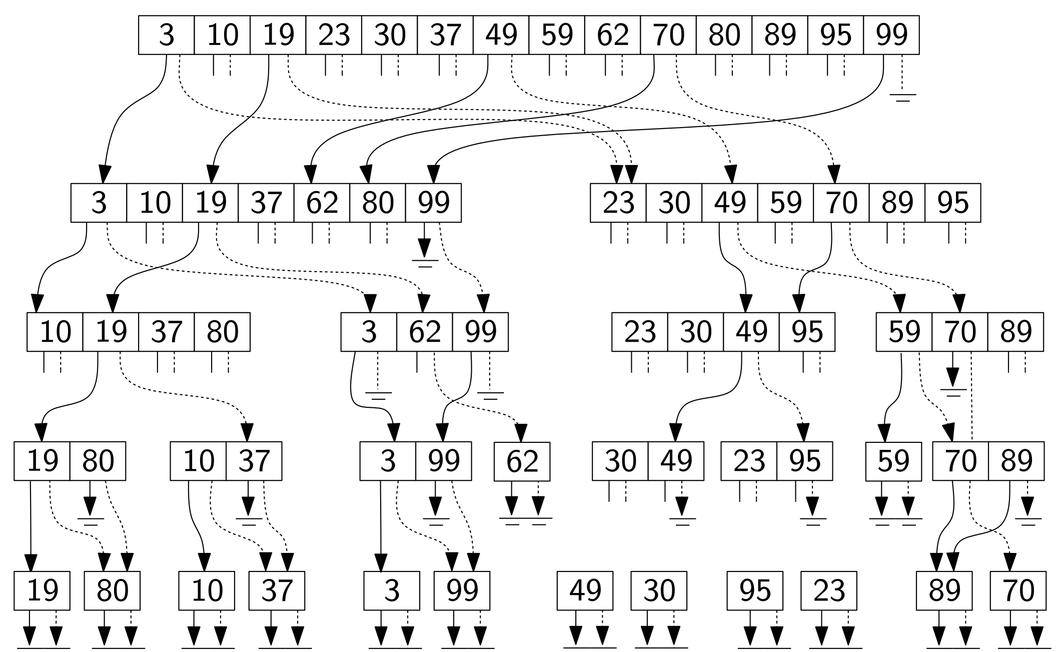

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Theorem 2: A Layered Range Tree on n points in $\mathbb{R}^2$ can be constructed in $O(n \log n)$ time and space. Range queries take $O(\log n + k)$ time, where k is the number of reported points.

Arbitrary Point Sets

So far: points in general position, that is no two points have the same x- or y-coordinate

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$$\downarrow$$

$$\hat{R} = [(x|-\infty), (x'|+\infty)] \times [(y|-\infty), (y'|+\infty)]$$

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So far: points in general position, that is no two points have the same x- or y-coordinate

Idea: Use pairs of numbers $(a|b)$ with lexicographic order (algorithm only assumes totally ordered set)

$$p = (p_x, p_y) \longrightarrow \hat{p} = \big((p_x|p_y),\ (p_y|p_x)\big)$$

unique coordinates

Rectangle $R = [x, x'] \times [y, y']$

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Then: $p \in R \ \Leftrightarrow\ \hat{p} \in \hat{R}$

Summary

Given: n points in $\mathbb{R}^d$

Output: data structure that efficiently answers queries of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$

$\rightarrow$ We have seen two alternatives

	kd-Tree	Range Tree
Preprocessing		
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Fractional Cascading

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Yes, range searching for other objects (e.g. polygons) can be reduced to higher-dimensional range searching for points (see exercise).
Queries for objects that are only partially contained in the query range will be covered in the next lecture.

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- Removing points

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2) **Augmented dynamic range trees** [Mehlhorn, Näher '90]
support updates in $O(\log n \log \log n)$ time and queries in $O(\log n \log \log n + k)$ time