Geodesic Convolutional Neural Networks

Outline

- Introduction & Motivation
- Background
  i. Manifolds and Metrics
  ii. Point Clouds & Meshes
  iii. Laplace Beltrami Operator & Heat Diffusion (optional)
  iv. Spectral Shape Descriptors (optional)
- Methods
- Results
- Demonstration

Introduction & Motivation

Aim to generalize the idea of a convolutional “filter” to process patches of mesh objects instead of patches of images.

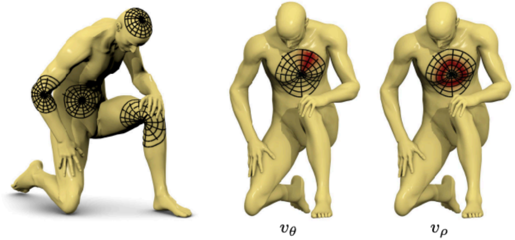


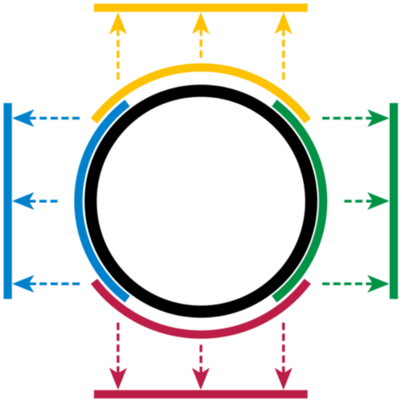
Figure 1: Geodesic patches on a shape

Manifolds

Each point has a neighborhood which is homeomorphic to an open subset of Euclidean space.

Can cover the manifold with **charts** that accomplish this mapping, and together they form an **atlas**

Example:



Riemannian Metric

A Riemannian metric assigns to each point p in our manifold, a positive definite inner product $g_p : T_pM \times T_pM \to \mathbb{R}$

The metric g is smooth (infinitely differentiable).

Distance on a Riemannian Manifold

The length of a differentiable curve $L(\gamma)$ on a Riemannian manifold with metric g can be given by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma(t)}, \dot{\gamma(t)})}dt \tag{1}$$

Consequently, the distance $d(p, q)$ on a Riemannian manifold is $L(\gamma^*)$, where γ^* is the infimum of all differentiable curves which satisfy $\gamma(a) = p$, $\gamma(b) = q$

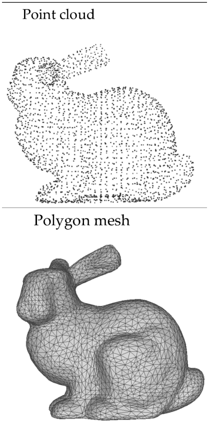
Finding the geodesic distance on a mesh can be done numerically using any Boundary Value Problem solver (fast marching algorithm, etc)

Formal Definition of a 3D Shape

1. Connected, smooth compact two-dimensional manifold X
2. Locally, each point x is homeomorphic to a 2-D Euclidean space (tangent plane, T_xX)

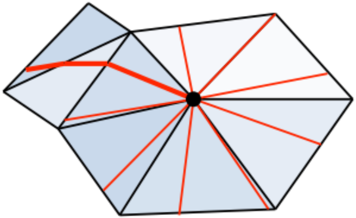
Discretization of a 3D Shape from Point Clouds

Given a realized point cloud $\{x_1, x_2, \ldots x_N\} \in X$, we can define a **triangular mesh** (V, E, F)



Discretization of a 3D Shape from Point Clouds (contd.)

1. Each *interior* edge $ij \in E$ is only shared by 2 triangular faces $ikj, jhi \in F$ while *boundary* edges only have 1 associated triangular face
2. Vertices are located at $\{x_1, x_2, \dots x_N\}$
3. A function $f : X \to \mathbb{R}$ is sampled on V and can be defined by $\mathbf{f} = (f(x_1), f(x_2), \dots f(x_N))^T$, a N dimensional vector
4. The set of vertices directly connected to i is called the *1-ring* of i



Laplace Beltrami Operator

Generalization of the Laplacian to non-Euclidean space

1. Intrinsic (dependent only on the Riemannian metric)
2. Isometric (invariant to distance preserving deformations of a manifold)
3. Yields an eigen-decomposotion with real non-negative eigenvalues λ_i, and an orthonormal basis of eigenfunctions $\phi_i(x)$.

Laplace Beltrami Operator (on a mesh!)

Since we can't work in the function space (we only have points on the manifold), we work on the discretizated version defined by

$$L = A^{-1}W$$

where L is a $N \times N$ matrix.

- We can define its eigenvalues and orthonormal basis with the traditional matrix eigen-decomposition of L.

The main takeaway is that you can construct a mesh, and define an operator on it.

Spectral Shape Descriptors

Most take the form of

$$f(x) = \sum_{k \geq 1} \tau(\lambda_k)\phi_k^2(x) \approx \sum_{k}^{K} \tau(\lambda_k)\phi_k^2(x) \tag{2}$$

where $\tau(\cdot)$ is some transfer function, and λ_k and $\phi_k(\cdot)$ are the respective eigenvalues and eigenvectors of the LBO.

1. Heat Kernel Signature
   a) $\tau_t(\lambda) = e^{-\lambda t}$
   b) Poor localization
2. Wave Kernel Signature
   a) $\tau_\nu(\lambda) = e^{\frac{log\nu - log\lambda}{2\sigma^2}}$
   b) Poor globalization
3. Optimal Spectral Descriptors
   a) $\tau_q(\lambda) = \sum_{m=1}^{M} a_{qm}\beta_m(\lambda)$
   b) Have to learn the spline parameters

Geodesic Convolution

1. Defining a patch operator
2. Defining a convolution

Defining a Patch Operator

Let $B_{\rho_0}(x)$ be a geodesic ball of size ρ_0.

$\Omega(x) : B_{\rho_0}(x) \to [0, \rho_0] \times [0, 2\pi]$

Patch operator interpolates a function f in local coordinates

$$(Df(x))(\rho, \theta) = (f \circ \Omega^{-1}(x))(\rho, \theta) \tag{3}$$

$$(Df(x))(\rho, \theta) = \int_X v_{\rho,\theta}(x, y) f(y) dy \tag{4}$$

$$v_{\rho,\theta}(x, y) = v_\rho(x, y) v_\theta(x, y) \tag{5}$$

1. $v_\rho(x, y) \approx$ geodesic distance between x, y
2. $v_\theta(x, y) \approx$ geodesic distance between the point y and the geodesic generated at x, in the direction θ

Defining a Patch Operator (Discrete's Version)

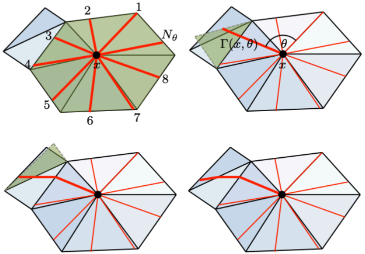


Figure 2: Construction of local geodesic polar coordinates on a triangular mesh. Shown clock-wise: division of 1-ring of vertex x_i into N_θ equi-angular bins; propagation of a ray (bold line) by unfolding the respective triangles (marked in green).

Defining A Convolution

$$(f \star a)(x) = \sum_{\rho,\theta} a(\theta + \Delta\theta, \rho)(Df(x))(\rho, \theta) \tag{6}$$

where $a(\cdot, \cdot)$ is a filter.

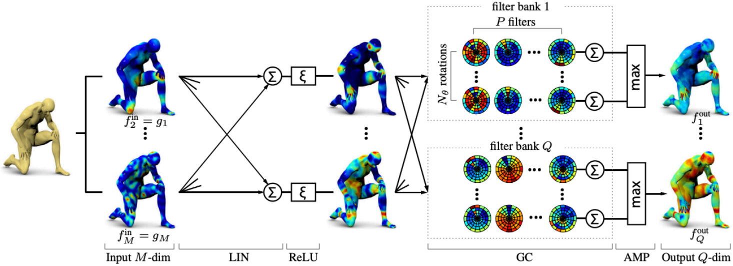
Effectively, we are projecting x onto local angular coordinates, and performing a convolution on those coordinates.

Convolutional Layers

1. Linear Layer (standard)
2. Geodesic Convolution (GC)
   - $\sum_{p=1}^{P}(f_p \star a_{\Delta\theta,qp})(x)$
   - is computed for all N_θ (similar to other GCNN paper)
3. Angular Max Pooling (AMP)
   - $\max_{\Delta\theta} f^{in}_{\Delta\theta,p}(x)$
   - follows GC layer
4. Fourier Transform Magnitude
   - $f^{out}_p(\rho, w) = |\sum_\theta e^{-iw\theta}(Df(x))(\rho,\theta)|$
   - removes rotational ambiguity
5. Covariance (COV)
   - $f^{out} = \int_X (f^{in}(x) - \mu)(^{in}(x) - \mu)^T dx$
   - produces a global descriptor

The spectral shape descriptors can be recovered from some specific parametrization of the above.

Example Architecture



Results

Three tasks:

1. Invariant descriptors
   - produces a local descriptor of x
2. Shape Correspondence
   - vertext labeling problem
3. Shape Retrieval
   - discriminate between classes of shapes

Invariant Descriptors

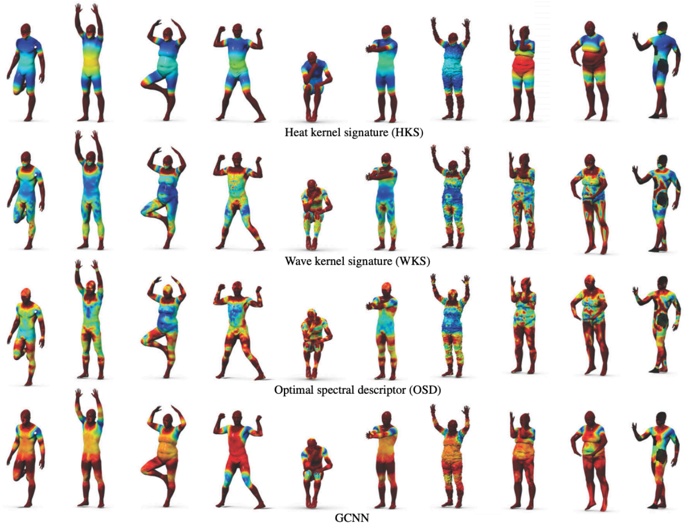


Figure 4: Normalized Euclidean distance between the descriptor at a reference point on the shoulder (white sphere) and the descriptors computed at the rest of the points for different transformations (shown left-to-right: near isometric deformations, non-isometric deformations, topological noise, geometric noise, uniform/non-uniform subsampling, missing parts). Cold and hot colors represent small and large distances, respectively; distances are saturated at the median value. Ideal descriptors would produce a distance map with a sharp minimum at the corresponding point and no spurious local minima at other locations.

Shape Correspondence

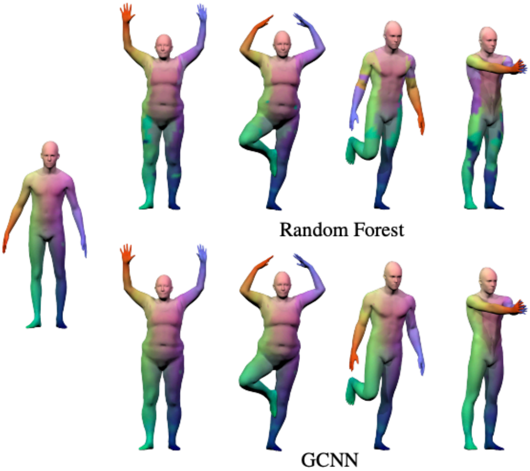


Figure 7: Example of correspondence obtained with GCNN (bottom) and random forest (top). Similar colors encode corresponding points.

Shape Retrieval

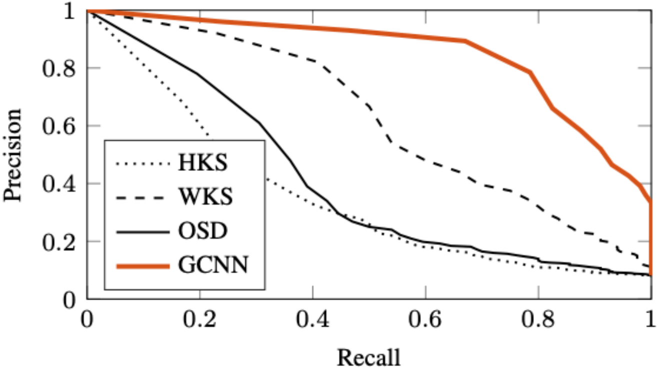


Figure 8: Performance (in terms of Precision-Recall) of shape retrieval on the FAUST dataset using different descriptors. Higher curve corresponds to better performance.

Code Demo

Some notes:

1. Code lives at https://github.com/andreasMazur/geoconv
2. The `environment.yml` file contains the necessary python dependencies from (1)
3. Use the following

```
conda env create -f environment.yml
conda activate ece594n_geodesic_convolutional_networks
git clone https://github.com/andreasMazur/geoconv.git
cd ./geoconv
pip install .
```