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Dirac Equation Path Integral: Interpreting the Grassmann Variables (*) (**).

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Summary. — A functional integral for a particle obeying the Dirac equation is presented. In earlier work (reviewed here) we showed that 1) such a particle could be described as a massless particle randomly flipping direction and helicity at a complex rate i/m and 2) its between-flips propagation could be written as a sum over paths for a Grassmann variable valued stochastic process. We here extend the earlier work by providing a geometrical interpretation of the Grassmann variables as forms on $SU(2)$. With this interpretation we clarify the supersymmetric correspondence relating products of Grassmann variables to spatial coordinates.

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1. – Introduction.

Is the path integral more than another tool for calculation? For anyone who has thought there was physics lurking behind the mathematics—and readers of Feynman's original paper (1) could hardly think otherwise—the absence of a

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(**) To speed up publication, the authors have agreed not to receive proofs which have been supervised by the Scientific Committee.
(1) R. P. FEYNMANN: *Rev. Mod. Phys.*, **20**, 367 (1948).

simple path integral for the Dirac equation has been a continuing source of discomfort and, in some cases, challenge. You cannot, after all, have a fundamental theory that ignores both spin and relativity. True, there are path integral representations of quantum fields, but the formal nature of these objects, especially for fermion fields, brings home the realization that one has leap frogged over the Dirac equation and gone on to hard problems before solving the easier ones. It may also be said that the fluctuating fortunes of quantum field theory—this year in favour, not so long ago disdained—suggest that the Dirac equation, based as it is on little beyond the Poincaré group (2), is more robust than the field theories that build upon it.

In this paper we will describe our own formulation of a path integral for the Dirac equation. The new material we will present is a geometric interpretation of the Grassmann variables that we used previously in this connection. In addition we review work of the second giant of path integration, Mark Kac, with whom we had the privilege of collaborating (3) a few years ago. The conference at which this paper is being presented commemorates the fortieth anniversary of Feynman's original work and sadly has also turned out to be a forum for his eulogies. We hope in the present paper to mention some small portion of the oeuvre of Mark Kac, and, sadly too, we dedicate this article to his memory.

In sect. 2 and 3 we review the material on which the new work is based. In sect. 2 we cover early efforts by Feynman, rescued from the geniza by Schweber (4), but first publicized in the book by Feynman and Hibbs (5). We give a modified derivation of the extension of these ideas by Kac, Jacobson and ourselves (3,6). In sect. 3 is a précis of the Grassmann variable stochastic process that we developed (7) to fill in the gaps (in a literal sense) of ref. (3). In sect. 4 we present the beginnings of a geometrical interpretation of the Dirac equation path integral and finally in sect. 5 there is a general discussion. The reader should also be aware that the search for a simple path integral for the Dirac equation has occupied many investigators and several alternative approaches exist (8).

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(2) This was developed by E. P. Wigner and V. Bargmann and is concisely expressed in the introductory sections of S. WEINBERG: *Phys. Rev. B*, **133**, 1318 (1964).

(3) B. GAVEAU, T. JACOBSON, M. KAC and L. S. SCHULMAN: *Phys. Rev. Lett.*, **53**, 419 (1984).

(4) Box 13, Folder 3 of the Caltech Feynman archives, to be precise. See S. SCHWEBER, *Rev. Mod. Phys.*, **58**, 449 (1986). Note especially Schweber's fig. 8.

(5) R. P. FEYNMAN and A. R. HIBBS: *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, N.Y., 1965). See also G. V. RIAZANOV: *JETP*, **6**, 1107 (1958); [*J. Exptl. Theoret. Phys.* (*USSR*), **33**, 1437 (1957)].

(6) T. JACOBSON and L. S. SCHULMAN: *J. Phys. A*, **17**, 375 (1984).

(7) B. GAVEAU and L. S. SCHULMAN: *Phys. Rev. D*, **36**, 1135 (1987).

(8) T. ICHINOSE and H. TAMURA: *J. Math. Phys.* (*N.Y.*), **25**, 1810 (1984); *Prog. Theor. Phys. Suppl.*, **92**, 144 (1987); B. GAVEAU: *J. Funct. Anal.*, **58**, 310 (1984); T. JACOBSON: *J. Phys. A*, **17**, 2433 (1984); A. O. BARUT and N. ZANGHI: *Phys. Rev. Lett.*, **52**, 2009 (1984);

As mentioned above, we here present an interpretation of the Grassmann variable functional integral for the Dirac equation. The appearance of Grassmann variables in the Dirac equation path integral now seems to us pretty much unavoidable. Such a judgment is a matter of taste, but once one gets used to manipulating these strange objects their place in the equations acquires a certain naturalness. What they do not acquire is the ability to extend to the relativistic case what Feynman advertised in the title of his paper (1), a *space-time* approach to quantum mechanics. It is this last issue that we here begin to address. What is the origin of Grassmann variables after all? They entered mathematics as geometric objects, differential forms on manifolds, and, as we shall see below, giving this interpretation to the Grassmann variables that appeared in our earlier work (7) allows us to take the first steps toward giving geometrical meaning to our Dirac equation path integral. In this context too we are able to find a natural meaning for the fundamental rule of supersymmetry, namely that an even product of Grassmann variables generates a spatial translation.

2. – Checkerboards and Poisson processes: the role of mass.

In both one and three space dimensions the Dirac equation has the form

$$i\hbar\frac{\partial\psi}{\partial t}=i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=mc^2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}-i\hbar\begin{pmatrix}Q&0\\0&-Q\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=H\psi\,. \tag{2.1}$$

For one dimension ψ_1 and ψ_2 are functions and

$$Q=c\partial/\partial x-ieA\,. \tag{2.2}$$

In three dimensions ψ_1 and ψ_2 are themselves 2-component spinors and

$$Q=\boldsymbol{\sigma}\cdot(c\partial/\partial\boldsymbol{x}-ie\boldsymbol{A})\,. \tag{2.3}$$

There are two conceptually distinct terms in H: the mass term and the propagation associated with Q. The triviality of Q in one dimension (for $A=0$) leads to the checkerboard path integral construction of Feynman (4,5) but, since this has been published many times, we will adopt a more general mathematical viewpoint that allows simultaneous treatment of one and three dimensions.

We recall that it was the achievement of Kac to recognize in Feynman's path integral an analytic continuation of the Wiener integral for Brownian motion. He

A. O. Barut and I. H. Duru: *Phys. Rev. Lett.*, **53**, 2355 (1984); F. Ravndal: *Phys. Rev. D*, **21**, 2823 (1980); H. B. Nielsen and D. Rohrlich: *Nucl. Phys. B*, **299**, 471 (1988); G. J. Papadopoulos and J. T. Devreese: *Phys. Rev. D*, **13**, 2227 (1976).

turned this observation into a powerful tool for the unification of probability and potential theory. In fact, one of the uses to which he put this tool was the study of the Telegrapher's equation which as we will soon see is closely related to (2.1).

Anticipating an analytic continuation we introduce the variable

$$a = mc^2/i\hbar \tag{2.4}$$

and use Pauli matrices to simplify the writing of (2.1) which now takes the form

$$\frac{\partial \psi}{\partial t} = a\sigma_x \psi - Q\sigma_z \psi . \tag{2.5}$$

Introduce the multicomponent function

$$\phi = \exp[-at]\psi$$

which satisfies

$$\frac{\partial \phi}{\partial t} = a(\sigma_x - 1)\phi - Q\sigma_z \phi . \tag{2.6}$$

If we assume that a is a positive number instead of a pure imaginary quantity, then this equation allows the following probabilistic interpretation: Suppose a system exists in one of two states, labeled «1» or «2». Within each of these states it is further described by a point in a Hilbert space $\mathcal{H}$. While in state 1 its time evolution is generated by the operator $-Q$ and while in state 2 it evolves by $+Q$. Thus we describe the state of the system by the pair (ϕ_1, ϕ_2), $\phi_i \in \mathcal{H}$, $i = 1, 2$ (this ϕ will turn out to be the same as that in (2.6)). Now suppose that it is possible for the system to spontaneously change from being in state 1 to state 2, and vice versa; let the rate of these spontaneous occurrences be a. Then at a time $t + \Delta t$ the vector in state 1 will be

$$\phi_1(t+\Delta t) = (1 - a\,\Delta t)\exp[-Q\,\Delta t]\,\phi_1(t) + a\,\Delta t \exp[Q\,\Delta t]\,\phi_2(t) \tag{2.7a}$$

and in state 2

$$\phi_2(t+\Delta t) = (1 - a\,\Delta t)\exp[Q\,\Delta t]\,\phi_2(t) + a\,\Delta t \exp[-Q\,\Delta t]\,\phi_1(t) . \tag{2.7b}$$

Writing $\phi_i(t+\Delta t) = \phi_i(t) + \Delta t\,(\partial\phi_i/\partial t)$, expanding (2.7) to order Δt, noting the cancellation of the zeroth-order terms and dividing by Δt leads precisely to eq. (2.6).

The «spontaneous» changes in the evolution law are recognized as a Poisson process and the above framework can include many situations, for example the propagation of waves in a random medium. From his studies of the Telegrapher's equation, Kac developed a «sum over paths» representation for the propagator [9,10] and this is how we will solve (2.6). However, our derivation will be different from that of Kac and is presented because it provides a unification of approach with the nonrelativistic path integral (it is similar in spirit to the use of the Trotter product formula) and because it makes clear how by a slight modification one gets Feynman's original (1-dim) Dirac equation prescription. Returning to eq. (2.6) we write its propagator in the usual way

$$\left\{\begin{aligned} &K(t) \equiv \exp[t(a(\sigma_x - 1) - Q\sigma_z)] = \lim_{N\to\infty} K_N(t)\,,\\ &K_N(t) \equiv \{\exp[\varepsilon a(\sigma_x - 1)]\exp[-\varepsilon Q\sigma_z]\}^N \end{aligned}\right. \tag{2.8}$$

with $\varepsilon = t/N$. Recall that (2.8) is derived by neglecting $O(\varepsilon^2)$ terms (that multiply $[\sigma_x, Q\sigma_z]$) because for $N \to \infty$ these disappear. The same feature is now used to replace $\exp[\varepsilon a(\sigma_x - 1)]$ by $1 + \varepsilon a(\sigma_x - 1)$ and then to replace $-\varepsilon a + \exp[-\varepsilon Q\sigma_z]$ by $(1 - \varepsilon a)\exp[-\varepsilon Q\sigma_z]$. Thus

$$\begin{aligned} K_N(t) &= \{(1-\varepsilon a)\exp[-\varepsilon Q\sigma_z] + \varepsilon a\sigma_x\}^N = \\ &= \sum_{j=0}^{N} (\varepsilon a)^j (1-\varepsilon a)^{N-j} \sum_{\{n_l\}} \exp[-n_{j+1}\varepsilon Q\sigma_z]\,\sigma_x \ldots \sigma_x \exp[-n_1\varepsilon Q\sigma_z]\,, \end{aligned} \tag{2.9}$$

where the second sum in (2.9) is over sequences $(n_1, \ldots, n_{j+1})$ such that $n_1 + \ldots + n_{j+1} = N - j$. (For each j there are $\binom{N}{j}$ terms, but (2.9) is nevertheless equivalent to the usual checkerboard expansion [5,6] with its Bessel functions and products of *two* combinatorial coefficients. This is because, when K is given spatial arguments, one only takes terms in (2.9) for which $\frac{1}{N}\sum(-1)^l n_l \sim$ (spatial displacement).)

The key observation about the sum in (2.9) is that the values of j that give the important contributions to the sum do *not* go to infinity as $N \to \infty$. This is why we will get a Poisson process for which the number of reversals is a physical quantity. We make our assertion precise in terms of the series $\sum (1/N)^j \binom{N}{j}$, which has the same structure as (2.9). Let

$$A_l^N = \sum_{j=l}^{N} \left(\frac{1}{N}\right)^j \binom{N}{j}.$$

([9]) M. KAC: *Rocky Mount. J. Math.*, 4, 497 (1974), based on 1956 notes from the Magnolia Petroleum Company colloquium lectures.
([10]) R. HERSH: *Rocky Mount. J. Math.*, 4, 443 (1974).

Then we assert that for any δ there exists a l such that $A_l^N < \delta$ for all N. This uniformity in N will allow us to take $N \to \infty$ in (2.9). The proof of the assertion is obvious if one writes

$$A_l^N = \sum_{j=l}^{N} \frac{1}{j!} \frac{N(N-1)\dots(N-j+1)}{N^j} \leqslant \sum_{j=l}^{N} \frac{1}{j!}.$$

The $N \to \infty$ limit of (2.9) is

$$K(t) = E[\exp[-T_\nu Q\sigma_x]\sigma_x \dots \sigma_x \exp[-T_2 Q\sigma_z]\sigma_x \exp[-T_1 Q\sigma_z]], \tag{2.10}$$

where the expectation «E» is over sample paths of a Poisson process with rate a. The T_k are the random times between flips. The number of reversals, ν, appearing within the expectation is a dummy random variable (as are T_1, T_2, etc.) and may be even or odd. To see how all this relates to the Feynman checkerboard path integral we note that

$$\sigma_x \exp[-TQ\sigma_z]\sigma_x = \exp[TQ\sigma_z]. \tag{2.11}$$

(This is a property of the Pauli matrices and from a $SU(2)$ perspective says that a 180° rotation about the x-axis sends z into $-z$.) Equation (2.11) allows (2.10) to be rewritten as

$$K(t) = E[\sigma_x^\nu \exp[(-1)^\nu T_\nu Q\sigma_z] \dots \exp[-T_3 Q\sigma_z]\exp[T_2 Q\sigma_z]\exp[-T_1 Q\sigma_z]]. \tag{2.12}$$

Equation (2.12) is a compact statement of the checkerboard path integral as well as of our own formulation of the three-dimensional Dirac particle in terms of alternate propagation as a left or right handed massless particle [3]. In the one-dimensional case Q is a translation generator plus a field and when the translations are commuted past the fields in the ordered product of (2.12) one gets the usual checkerboard sum with paths weighted by $\exp[ie\int A\, dx/\hbar c]$ (with the integral evaluated along the particular path), as is known. The advantage of expression (2.12) over previous formulations is that it does not require a listing of cases $\phi_j \to \phi_k$, $k = j$ and $k \neq j$, with a sorting out of suitable matrix elements and paths.

A yet more compact way of writing the propagator can be achieved by defining the random variables of the Poisson process directly. Let $N(t)$ be the number of events of the process («flips» or «reversals» in our case) up to and including time t. $N(0) = 0$. Then

$$\mathrm{Prob}\,(N(t+\Delta t) - N(t) = 1) = a\,\Delta t + O((\Delta t)^2) \tag{2.13}$$

and (as a consequence), if T_k and T_{k+1} are the times of successive flips,

$$\operatorname{Prob}(T_{k+1} - T_k \geqslant \tau) = \exp[-a\tau]. \tag{2.14}$$

In (2.12) we are interested in the parity of $N(t)$ so it is natural to define

$$\lambda(s) = (-1)^{N(s)}. \tag{2.15}$$

It follows that

$$K(t) = E\left[\sigma_x^{N(t)} \exp\left[-\int_0^t \lambda(s)\, Q\sigma_z\, \mathrm{d}s\right]\right] \tag{2.16}$$

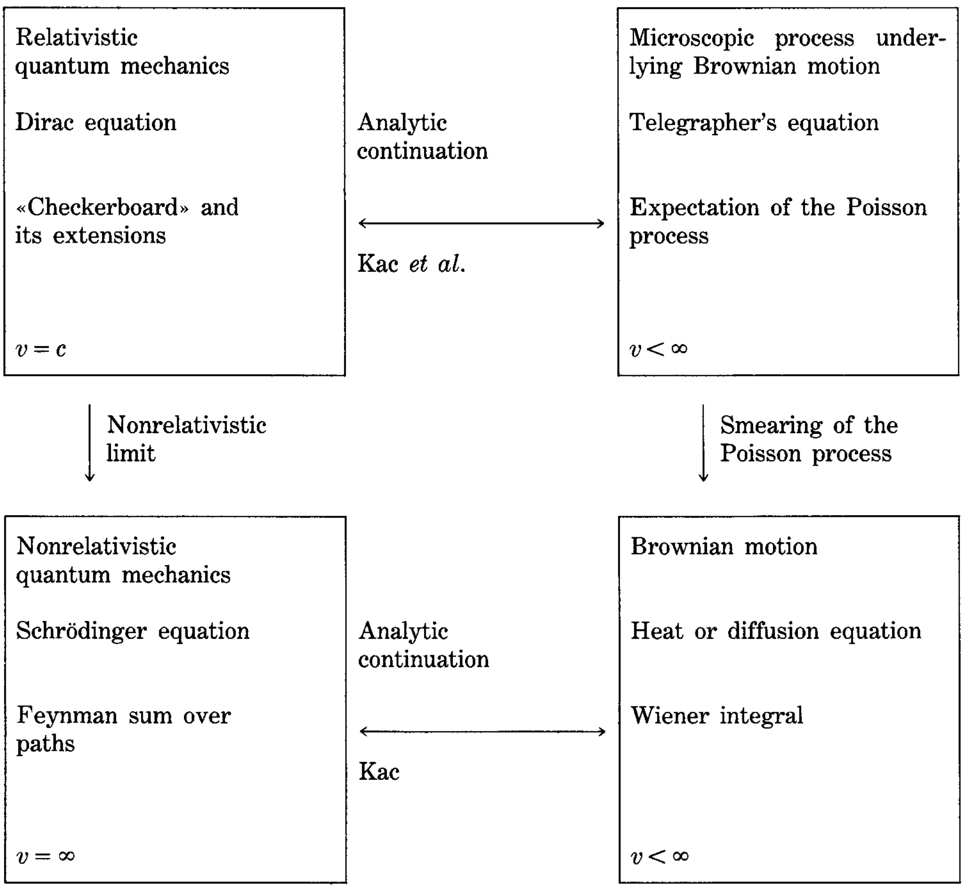

Fig. 1. – Interrelations of physical systems.

with time ordered product (*i.e.* product integral) understood, in case Q has explicit time dependence.

For the purposes of our Poisson process discussion we have pretended that the transition rate a was real. To connect to the Dirac equation, however, we see from (2.4) that a should be pure imaginary. In this way we have used a stochastic process and by analytic continuation found way to derive the appropriate complex sum over paths for the quantum particle. This is exactly what Kac [11] did for Feynman's nonrelativistic path integral 37 years ago, making use of the Wiener process. And now for the relativistic case he and his collaborators made the analogous identification for the Poisson process.

By uncomplicated mathematics, the smearing or coarse graining of the Poisson process yields a Wiener process and this provides a neat way to take the nonrelativistic limit of the Dirac equation. We have discussed these matters elsewhere [6,3,12] but would like to take this opportunity to publish a diagram (fig. 1) that illustrates the various processes and limits.

3. – Grassmann valued process for the «between flips» propagation.

In this section we summarize the results of ref. [7], adapted to the presentation of the last section.

The goal is to represent the operator $\exp[-tQ\hat{\sigma}_z]$ for the three-dimensional case ($Q = \boldsymbol{\sigma}\cdot(c\partial/\partial\boldsymbol{x} - ie\boldsymbol{A})$) as an integral over a kind of stochastic process. (We have placed a circumflex over the σ_z to emphasize that the «$\hat{\sigma}_z$» in the exponent and the «$\boldsymbol{\sigma}$» in Q act on different spaces and therefore commute. Note that we did not this distinction earlier in the paper since the Pauli matrix within Q was not explicit.) The essential problem is to evaluate $\exp[\pm tQ]$ since the $\hat{\sigma}_z$ in $\exp[-t\hat{\sigma}_z Q]$ is diagonal. This means we are integrating the Weyl equation, namely the equation for a left or right handed massless particle. This equation is obtained by setting $m = 0$ in eq. (2.1) or $a = 0$ in eq. (2.6) and confining attention to the upper or lower subspace. We introduce Grassmann variables in order to separate $\boldsymbol{\sigma}$ and $\partial/\partial\boldsymbol{x}$ in Q. Let $\theta_\alpha(s)$ be a set of Grassmann variables with $\alpha \in \{1, 2, 3\}$ and $s \in \mathfrak{R}$, satisfying

$$\theta_\alpha(s)\,\theta_\beta(s') + \theta_\beta(s')\,\theta_\alpha(s) = 0 \qquad \text{for all } \alpha, \beta, s, s'. \tag{3.1}$$

([11]) M. KAC: *On some connections between probability theory and differential and integral equations*, in *Proceedings of the Second Berkely Symposium on Math. Stat. Prob.*, edited by J. NEYMAN (Berkely, 1951), p. 189.

([12]) L. S. SCHULMAN: *Introduction to the Path Integral, Lectures at the Adriatico Research Conference on Path Integration, Trieste, 1987*, in *Path Summation: Achievements and Goals*, edited by S. O. LUNDQVIST, A. RANFAGNI, V. SA-YAKANIT and L. S. SCHULMAN (World Scientific, Singapore, 1988).

For the Graßmann variables there is defined a Berezin integral

$$\int 1\,\mathrm{d}\theta_\alpha(s)=0\,,\qquad \int\theta_\beta(s)\,\mathrm{d}\theta_\alpha(s)=\delta_{\alpha\beta}\,. \tag{3.2}$$

If M is an antisymmetric 3×3 matrix and $\boldsymbol{\sigma}$ the Pauli spin matrices we have the following identity

$$\int\exp[i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}]\exp[\boldsymbol{\theta}M\boldsymbol{\theta}]\,\mathrm{d}\boldsymbol{\theta}=1+i\sum_{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma}\sigma_\alpha M_{\beta\gamma}\,, \tag{3.3}$$

where $\boldsymbol{\theta}M\boldsymbol{\theta}=\sum_{\alpha,\beta}\theta_\alpha M_{\alpha\beta}\theta_\beta$, $\mathrm{d}\boldsymbol{\theta}=\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3$, the s label is suppressed and $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. From a 3-vector $\boldsymbol{\eta}$ we build an antisymmetric matrix by setting $M_{\beta\gamma}=\frac{1}{2}\sum\eta_\alpha\varepsilon_{\alpha\beta\gamma}$. Equation (3.3) can now be written

$$\begin{aligned}\frac{\exp[i\boldsymbol{\eta}\cdot\boldsymbol{\sigma}]}{\sqrt{1-\eta^2}}&=1+i\boldsymbol{\sigma}\cdot\boldsymbol{\eta}=\int\exp[i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}]\exp[\boldsymbol{\theta}M\boldsymbol{\theta}]\,\mathrm{d}\boldsymbol{\theta}=\\&=\int\exp[i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}]\exp\left[\frac{1}{2}\boldsymbol{\eta}\cdot\boldsymbol{\theta}\times\boldsymbol{\theta}\right]\mathrm{d}\boldsymbol{\theta}.\end{aligned} \tag{3.4}$$

Equation (3.4) is the fundamental identity behind our method. On the left (in two forms) is a product $\boldsymbol{\eta}\cdot\boldsymbol{\sigma}$; on the right the factors are split at the price of doing an integral. It is entirely analogous to the way in which uncompleting the square or the Gaussian trick introduces path integration in the nonrelativistic situation.

We do the usual $O(1/N)$ expansion for $\exp[-tQ]$ ($t\to -t$ will give the formulae for $\exp[+tQ]$)

$$\exp[-tQ]=\exp[-t\boldsymbol{\sigma}\cdot(\partial-i\boldsymbol{a})]=\lim_{N\to\infty}[1-\varepsilon\boldsymbol{\sigma}\cdot(\partial-i\boldsymbol{a})]^N \tag{3.5}$$

with $\varepsilon=t/N$, $c=1$, $\boldsymbol{a}=e\boldsymbol{A}$ and a chronological product taken if necessary. Individual terms in the product in (3.5) are labeled by $k=1,\dots,N$ and we let $t_k=k\varepsilon$. By repeated application of (3.4) we have

$$\begin{aligned}&\exp[-tQ]=\\&=\lim_{N\to\infty}\int\left(\prod_{k=1}^N\mathrm{d}\boldsymbol{\theta}\right)\prod_{k=1}^N\exp[i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}]\exp\left[\frac{i}{2}\varepsilon\boldsymbol{a}\cdot\boldsymbol{\theta}\times\boldsymbol{\theta}\right]\exp\left[-\frac{1}{2}\varepsilon\partial\cdot\boldsymbol{\theta}\times\boldsymbol{\theta}\right],\end{aligned} \tag{3.6}$$

where each $\boldsymbol{\theta}$ is understood to mean $\boldsymbol{\theta}(t_k)$ and we have used the fact that even terms in θ commute with everything. This last observation as well as the way in which $\boldsymbol{\theta}\times\boldsymbol{\theta}$ finds itself matched with the translation generator ∂ leads to the natural definition

$$\delta\xi = \frac{1}{2}\varepsilon\boldsymbol{\theta}\times\boldsymbol{\theta} \tag{3.7}$$

(so that, *e.g.*, $\delta\xi_1 = \varepsilon\theta_2\theta_3$). We will see that $\delta\xi$ acts like a differential position. Recalling the index k implicit in (3.7) we further define

$$\xi(t_k) = \sum_{j=1}^{k} \delta\xi(t_j)\,. \tag{3.8}$$

One now performs several manipulations on (3.6). All the $\delta\xi$'s are collected on the right but we retain $\prod \exp[i\boldsymbol{\sigma}\cdot\theta(t_k)]$ as a chronological product. Next we bring all translation operators to the right by means of the formula $\exp[u\partial]f(v) = f(u+v)\exp[u\partial]$. These «translations» of course are in steps of $\delta\xi$, a position variable derived from our Grassmann variables. The interpretation of the $\delta\xi$ as a translation variable in ordinary space is exactly the content of the rule of supersymmetry. See ref. [13].

In discrete notation our formula reads

$$\exp[-tQ] = \lim_{N\to\infty}\int\left(\prod_{k=1}^{N} \mathrm{d}\boldsymbol{\theta}(t_k)\right)\left(\prod_{k=1}^{N}\exp[i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}(t_k)]\right)\cdot$$
$$\cdot\exp\left[+i\sum_{k=1}^{N}\delta\xi(t_k)\cdot\boldsymbol{a}(\cdot-\xi(t_N)+\xi(t_k))\right]\exp[-\xi(t_N)\cdot\partial]\,,$$

t_N is t and the dot in the argument of $\boldsymbol{a}$ is a dummy variable corresponding to the argument of the function on which $\exp[-tQ]$ may act. Therefore in continuum notation we have for the Weyl equation propagator

$$\exp[-tQ] = \tag{3.9}$$
$$= \int \mathscr{D}\boldsymbol{\theta}(s)\exp\left[i\boldsymbol{\sigma}\cdot\int_0^t\boldsymbol{\theta}(s)\right]\exp\left[ie\int \boldsymbol{A}(\cdot-\xi(t)+\xi(s))\,\mathrm{d}\xi(s)\right]\exp[-\xi(t)\cdot\partial]\,,$$

where the term $\exp[i\boldsymbol{\sigma}\cdot\int\boldsymbol{\theta}(s)]$ is understood to mean a chronological product. For $\exp[+tQ]$ one lets $t\to-t$, which is exactly the same as letting $\delta\xi\to-\delta\xi$. (The term $\exp[i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}]$ has no $\varepsilon = t/N$ factor in it and is unchanged in this

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[13] F. BEREZIN: *Analysis and algebra with anticommuting variables* (Nauka, Moscow, 1982).

transformation.) This makes it useful to define

$$\delta\zeta(s) = \lambda(s^+)\,\delta\xi(s)\,, \tag{3.10}$$

where $\lambda(s^+)$ is the limit from above of the random variable $\lambda(s)$ (defined in (2.15)) and the need for the s^+ can be seen from the discrete form. The quantity

$$\zeta(t) = \int_0^t \mathrm{d}\zeta(s) \tag{3.11}$$

thus becomes the net «distance» covered if one interprets $\varepsilon\boldsymbol{\theta}\times\boldsymbol{\theta}$ as a distance and the Poisson process as a direction flip, the latter in keeping with the one-dimensional checkerboard picture.

The principle term within the expectation for the propagator in formula (2.16) is $\exp\left[-\int_0^t \lambda(s)\,Q\hat{\sigma}_z\,\mathrm{d}s\right]$. This does not mix the upper and lower subspaces and we can clearly write, say, the upper component as

$$\exp\left[-\int_0^t \lambda(s)\,Q\,\mathrm{d}s\right] = \int \mathscr{D}\boldsymbol{\theta}(s)\exp\left[i\boldsymbol{\sigma}\cdot\int_0^t \boldsymbol{\theta}(s)\right]\cdot$$
$$\cdot\exp\left[ie\int_0^t \boldsymbol{A}(\cdot-\zeta(t)+\zeta(s))\,\mathrm{d}\zeta(s)\right]\exp[-\zeta(t)\cdot\partial]\,, \tag{3.12}$$

so that the entire effect of $\lambda(s)$ is expressed through the change $\xi\to\zeta$. This allows the complete propagator to be written as

$$K(t) = E\left\{\hat{\sigma}_x^{N(t)}\int \mathscr{D}\boldsymbol{\theta}(s)\exp\left[i\boldsymbol{\sigma}\cdot\int_0^t \boldsymbol{\theta}(s)\right]\cdot\right.$$
$$\left.\cdot\exp\left[i\hat{\sigma}_z e\int_0^t \boldsymbol{A}(\cdot-\hat{\sigma}_z(\zeta(t)-\zeta(s)))\,\mathrm{d}\zeta(s)\right]\exp[-\hat{\sigma}_z\zeta(t)\cdot\partial]\right\}. \tag{3.13}$$

The $\hat{\sigma}_z$ in the argument of $\boldsymbol{A}$ is given meaning, as usual, in the Taylor series for $\boldsymbol{A}$. This meaning was already implicit in our use of Grassmann valued arguments for $\boldsymbol{A}$. Again note that the «$\boldsymbol{\sigma}$» multiplying $\boldsymbol{\theta}(s)$ acts on the Hilbert space $\mathscr{H}$ of the two component spinors; it is the same $\boldsymbol{\sigma}$ that was part of Q in eq. (2.3). The other Pauli matrices in (3.13) (the explicit $\hat{\sigma}_x$ and $\hat{\sigma}_z$'s) act on the two-dimensional space that doubles $\mathscr{H}$ and makes the Dirac particle a 4-spinor. For the sake of explicitness we rewrite (3.13) without the $\hat{\sigma}_x$ and $\hat{\sigma}_z$'s, but keep the $\boldsymbol{\sigma}$ belonging to Q. This makes $K(t)$ a 2×2 matrix. We attack labels λ, λ' taking values $+$ (upper component) and $-$(lower component). Then

$$(3.14)\qquad K_{\lambda\lambda'}(t)=E\left\{\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}^{N(t)}\right)_{\lambda\lambda'}\int \mathscr{D}\boldsymbol{\theta}(s)\exp\left[i\boldsymbol{\sigma}\cdot\int_0^t\boldsymbol{\theta}(s)\right]\cdot\right.$$
$$\left.\cdot\exp\left[ie\int_0^t \boldsymbol{A}(\cdot-\lambda'\zeta(t)+\lambda'\zeta(s))\cdot\lambda'\,\mathrm{d}\zeta(s)\right]\exp[-\lambda'\zeta(t)\cdot\partial]\right\}$$

(no summation over λ').

Expressions (3.13) and (3.14) represent our final form for the propagator. To the extent that one can attach meaning to the Grassmann variable stochastic process, the particle is described as wandering about in the Grassmann space with an associated motion in physical coordinate space given by $\varepsilon\lambda(t)\frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\theta}$. If there were no flips, it would move at light velocity, but the changing $\lambda(t)$ means that these lightlike steps have some cancellation and a lower net velocity is observed.

We have written the propagator with a translation operator on the right. This form is more common in the mathematical literature than in the physical but, as we showed in ref. (7), it is fully equivalent to the expressions one uses in physics. Further discussion of this point can be found in ref. (12).

4. – Representation of the abstract Grassmann variables.

4·1. *Introduction.* – Historically Grassmann variables have their origins in the algebra of differential forms. We will show that realizing the Grassmann variables of the last section as differential forms on the group manifold of $SU(2)$ allows the formal (Berezin) integration to be interpreted as ordinary integration on $SU(2)$. Moreover, it will be seen that it is $SU(2)$ that is singled out here (not just forms on any manifold) in the sense that it has the correct transformation properties for the supersymmetric analysis. In a sense the correspondence is *too* good. This is because an attempt to associate the Grassmann variables that appear in other contemporary physical applications with forms would not be expected to have all the natural relationships that make our example work.

We first establish basic facts and notation for the association we will make.

4·2. *Invariant differential forms in* $SU(2)$. – For any Lie group G, the differential form of degree 1, $g^{-1}\mathrm{d}g$, has its value in the Lie algebra of G. Thus for $G=SU(n)$ the elements of G satisfy $gg^\dagger=1$, where † is Hermitian adjoint. Taking the differential of this equation we get

$$(\mathrm{d}g)g^\dagger+g(\mathrm{d}g)^\dagger=0$$

so that $g^{-1}\mathrm{d}g = -(\mathrm{d}g^{\dagger})(g^{\dagger})^{-1}$ and $g^{-1}\mathrm{d}g$ is a 1 form with values in the anti-Hermitian matrices.

Now we concentrate on $SU(2)$. A basis of the Lie algebra of $SU(2)$ is given by $i\sigma_1$, $i\sigma_2$ and $i\sigma_3$ where σ_1, σ_2 and σ_3 are the usual Pauli matrices. We expand $g^{-1}\mathrm{d}g$ in this basis and write

$$g^{-1}\mathrm{d}g = \sum_{j=1}^{3} (i\sigma_j)\,\omega_j\,, \tag{4.1}$$

where now ω_j are differential 1-forms with real values (scalar valued forms). By the definition of exterior product, they anticommute:

$$\omega_j \wedge \omega_k + \omega_k \wedge \omega_j = 0\,. \tag{4.2}$$

4·3. *The volume element on* $SU(2)$. – Let us compute $(g^{-1}\mathrm{d}g)\wedge \wedge (g^{-1}\mathrm{d}g)\wedge(g^{-1}\mathrm{d}g)$ where we consider $g^{-1}\mathrm{d}g$ to be a 1-form with values in the anti-Hermitian matrices and the wedge, $\wedge$, is the exterior product of 1-forms. Using (4.1) we see that

$$g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g = 6(i\sigma_1)(i\sigma_2)(i\sigma_3)\,\omega_1\wedge\omega_2\wedge\omega_3\,,$$

where we have used (4.2) and the anticommutation relations of the Pauli matrices. But $\sigma_1\sigma_2\sigma_3 = iI$, so that

$$g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g = 6I(\omega_1\wedge\omega_2\wedge\omega_3)\,. \tag{4.3}$$

Now $g^{-1}\mathrm{d}g$ is a left invariant form; that is for g_0 any constant matrix in $SU(2)$

$$(g_0 g)^{-1}\mathrm{d}(g_0 g) = g^{-1}g_0^{-1}g_0\,\mathrm{d}g = g^{-1}\mathrm{d}g\,.$$

It follows that $(g^{-1}\mathrm{d}g)^{\wedge^3}$ is also left invariant, so that $\omega_1\wedge\omega_2\wedge\omega_3$ is the volume element of $SU(2)$. Let us call C_0 the volume integral of $SU(2)$, that is the pure number given by

$$C_0 = \int_{SU(2)} \omega_1\wedge\omega_2\wedge\omega_3\,. \tag{4.4}$$

Remark. A direct computation gives

$$g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g = -2[i\sigma_3\,\omega_1\wedge\omega_2 - i\sigma_2\,\omega_1\wedge\omega_3 + i\sigma_1\,\omega_2\wedge\omega_3] = -\,\varepsilon_{\alpha\beta\gamma}\,i\sigma_\alpha\,\omega_\beta\wedge\omega_\gamma\,.$$

4·4. *Interpretation of a fundamental identity.* – In sect. 3 we employed a fundamental identity, eq. (3.3):

$$\int \exp[i\boldsymbol{\theta\sigma}]\exp(\boldsymbol{\theta}M\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = I + i\sum \varepsilon_{\alpha\beta\gamma}\,\sigma_\alpha M_{\beta\gamma} \tag{4.5}$$

for $(M_{\beta\gamma}) \equiv M$ an antisymmetric matrix. Here $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ are Grassmann variables and $\int d\boldsymbol{\theta} = \int d\theta_1 \, d\theta_2 \, d\theta_3$ is the Berezin integral which is an abstractly defined object. We want to reinterpret this identity in a classical way. Let us start with the usual 1-forms $\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3)$ introduced above and consider the quantity

$$\exp\left[i \sum_{j=1}^{3} \omega_j \sigma_j\right] \exp[\boldsymbol{\omega} \wedge M\boldsymbol{\omega}], \tag{4.6}$$

where the product in the second exponential is the usual exterior product so that

$$\boldsymbol{\omega} \wedge M\boldsymbol{\omega} = \sum_{\alpha} \omega_\alpha \wedge \left(\sum_{\beta} M_{\alpha\beta} \omega_\beta\right)$$

(this is not zero because M is antisymmetric). If we expand (4.6) in a series, using the exterior product, we shall be led to the same computation that appears for the case of abstract θ_α because all we used was the Grassmann property of the θ_α's.

Thus, if we expand (4.6), we get a linear combination of differential forms with matrix values, of degree 0, 1, 2, 3. Now we can integrate *in the usual sense* (no longer in the Berezin sense) this linear combination of forms on the three-dimensional manifold $SU(2)$. Only the component of degree 3 survives (because by definition the integral of a q-form in a n-manifold is 0 if $q \neq n$). Thus

$$\int_{SU(2)} \exp\left[i \sum_{j=1}^{3} \omega_j \sigma_j\right] \exp[\boldsymbol{\omega} \wedge M\boldsymbol{\omega}]$$

is exactly

$$\int_{SU(2)} \left[\exp\left[i \sum_{j=1}^{3} \boldsymbol{\omega}_j \sigma_j\right] \exp[\boldsymbol{\omega} \wedge M\boldsymbol{\omega}]\right]_{[3]},$$

where the bracketed subscript 3 means component of degree 3 in the expansion. But this is then

$$\int_{SU(2)} (I + i \sum \varepsilon_{\alpha\beta\gamma} \sigma_\alpha M_{\beta\gamma}) \, \omega_1 \wedge \omega_2 \wedge \omega_3 .$$

Now $I + i \sum \varepsilon_{\alpha\beta\gamma} \sigma_\alpha M_{\beta\gamma}$ is a constant matrix-valued function on $SU(2)$, so it can be taken out of the integral and we find by (4.4)

$$C_0(I + i \sum \varepsilon_{\alpha\beta\gamma} \sigma_\alpha M_{\beta\gamma}) .$$

Thus, the fundamental identity (3.3) *becomes*

$$\frac{1}{C_0}\int_{SU(2)} \exp\left[i\sum_{j=1}^{3}\omega_j\sigma_j\right]\exp\left[\boldsymbol{\omega}\wedge M\boldsymbol{\omega}\right] = I + i\sum \varepsilon_{\alpha\beta\gamma}\sigma_\alpha M_{\beta\gamma}\,. \tag{4.7}$$

This is no longer a Berezin integral; it is the usual integral in $SU(2)$. Moreover, there is no need to introduce an extra category of abstract objects; everything is well behaved with the usual exterior differential and integral calculus.

4·5. *Interpretation of the Grassmann stochastic process.* – As above we discretize the time $0, \varepsilon, 2\varepsilon, \dots, n\varepsilon = t$, and we introduce, for each k, $1 \leqslant k \leqslant n$, an exemplar of $SU(2)$ denoted $SU(2)_k$ with its invariant forms denoted $\omega_\alpha(k\varepsilon)$, $\alpha = 1, 2, 3$. In other words, we consider the product manifold M_n:

$$M_n = SU(2)\times SU(2)\times \dots \times SU(2) \qquad (n \text{ times})\,.$$

Then the objects $\omega_\alpha(k\varepsilon)$ are just the components of the form $g^{-1}\mathrm{d}g$ in the group M_n in the canonical basis of its Lie algebra. They anticommute naturally; for any α, β, k, k'

$$\omega_\alpha(k\varepsilon)\wedge\omega_\beta(k'\varepsilon) + \omega_\beta(k'\varepsilon)\wedge\omega_\alpha(k\varepsilon) = 0$$

(we do not have to impose this relation; it is a consequence of considering forms on the manifold M_n). The volume element (normalized to unity) is

$$\frac{1}{C_0^n}[\omega_1(\varepsilon)\wedge\omega_2(\varepsilon)\wedge\omega_3(\varepsilon)]\wedge\dots\wedge[\omega_1(k\varepsilon)\wedge\omega_2(k\varepsilon)\wedge\omega_3(k\varepsilon)]\wedge\dots\wedge[\omega_1(t)\wedge\omega_2(t)\wedge\omega_3(t)]$$

and the Berezin integral will be just the volume integral over M_n. It is then clear that we can repeat all the steps of sect. 3 to obtain the Weyl propagator corresponding to eq. (3.9):

$$\exp[-tQ] = \tag{4.8a}$$

$$= \frac{1}{C_0^\infty}\int_{SU(2)^\infty}\exp\left[i\boldsymbol{\sigma}\cdot\int_0^t\boldsymbol{\omega}(s)\right]\exp\left[ie\int_0^t \boldsymbol{A}(\cdot-\xi(t)+\xi(s))\,\delta\xi(s)\right]\exp[-\partial\cdot\xi(t)]\,,$$

or perhaps in a more standard form

$$\lim_{n\to\infty}\frac{1}{C_0^n}\int_{SU(2)^n}\prod_{k=1}^{n}\exp[i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}(k\varepsilon)]\dots \tag{4.8b}$$

and there is no longer a Berezin integral; this formula is just the usual integral of a $3n$-form on the $3n$-manifold $SU(2)^n$. The factor C_0^n is a normalization, more or less like the $(2\pi)^{n/2}$ appearing in the Feynman path integral.

Moreover, the spin contribution $\exp[i\boldsymbol{\sigma}\cdot\boldsymbol{\omega}(k\varepsilon)]$ is also $\exp[g_k^{-1}\,\mathrm{d}g_k]$, where g_k is an element of the k-th $SU(2)_k$ group in the product $M_n = SU(2)\times\ldots\times SU(2)$ and $g_k^{-1}\,\mathrm{d}g_k$ is the canonical invariant form of $SU(2)_k$.

4'6. *Interpretation of the supersymmetric rule.* – Our question is *why can we interpret even products of ω_α's, in particular the increments*

$$\delta\xi_\alpha(k\varepsilon) = \frac{\varepsilon}{2}\sum \varepsilon_{\alpha\beta\gamma}\,\omega_\beta(k\varepsilon)\wedge\omega_\gamma(k\varepsilon) \tag{4.9}$$

as translation operators in the underlying $\mathfrak{R}^3$ space? According to the supersymmetric rule of Berezin this is the definition of a supermanifold, but we want to understand this rule in terms of the $SU(2)$ geometry we have associated with the Grassmann algebra. We proceed in several steps.

1*st step.* Consider the Lie algebra $su(2)$ of $SU(2)$ as a three-dimensional Euclidean space with the orthonormal basis $i\sigma_1, i\sigma_2, i\sigma_3$. This, by the way, is the Euclidean structure coming from the negative of the Killing form in $su(2)$, the Killing form being defined as

$$B(M, M') = \mathrm{Tr}\,(\mathrm{ad}\,M\,\mathrm{ad}\,M')\,,\qquad M, M'\in su(2)\,,$$

and $\mathrm{ad}\,M(N) = [M, N]$. Because $SU(2)$ is compact, it is well known that B is a *negative* definite quadratic form and $-B$ is a scalar product. An orthonormal basis is $i\sigma_1, i\sigma_2, i\sigma_3$.

Moreover, let us consider the adjoint representation of $SU(2)$ on its Lie algebra $su(2)$ given by

$$\mathrm{Ad}\,g_0(M) = g_0^{-1}Mg_0\,,\qquad M\in su(2),\;\; g_0\in SU(2)\,.$$

It is clear that $B(M, M')$ is invariant under $\mathrm{Ad}\,g_0$:

$$B(\mathrm{Ad}\,g_0(M),\;\mathrm{Ad}\,g_0(M')) = B(M, M')\,,$$

so that $\mathrm{Ad}\,g_0$ is an isometry of $su(2)$ with respect to its Euclidean structure; we write in the orthonormal basis $i\sigma_\alpha$

$$(\mathrm{Ad}\,g_0)(i\sigma_\alpha) = \sum_\beta \mu_{\alpha\beta}(g_0)(i\sigma_\beta)\,, \tag{4.10}$$

where $\mu_{\alpha\beta}(g_0)$ is an orthogonal 3×3 matrix and thus is in $SO(3)$. We have a mapping $g_0 \in SU(2) \to \mathrm{Ad}\, g_0 \in SO(3)$ which is the usual covering mapping from $SU(2)$ to $SO(3)$.

2nd step. We know that the canonical form $g^{-1}\mathrm{d}g$ is left-invariant. But it is not right-invariant, in fact

$$(gg_0)^{-1}\, d(gg_0) = g_0^{-1}(g^{-1}\,\mathrm{d}g)\, g_0 \equiv (\mathrm{Ad}\, g_0)(g^{-1}\,\mathrm{d}g)\,.$$

Moreover, right multiplication has the same effect on products of forms:

$$\begin{aligned}(4.11)\qquad & (gg_0)^{-1}\, d(gg_0) \wedge (gg_0)\, d(gg_0) = \\ & \qquad = g_0^{-1}(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g)\, g_0 \equiv (\mathrm{Ad}\, g_0)(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g)\,,\end{aligned}$$

where in the second member of these formulae $\mathrm{Ad}\, g_0$ acts as the adjoint representation of $SU(2)$ on the Lie algebra part of $g^{-1}\mathrm{d}g$ or $g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g$. (Recall that by the remark at the end of subsect. 4˙3, $g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g$ has its value in $su(2)$, which is not the case for $g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g$ which has its value in the scalars.) Recall also that

$$(4.12)\qquad g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g = -\sum_{\alpha\beta\gamma} i\sigma_\alpha\, \omega_\beta \wedge \omega_\gamma\, \varepsilon_{\alpha\beta\gamma}$$

and thus using (4.12) and (4.10)

$$(4.13)\qquad (\mathrm{Ad}\, g_0)(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g) = \sum_\rho i\sigma_\rho \sum_{\alpha\beta\gamma} \mu_{\alpha\rho}(g_0)\, \varepsilon_{\alpha\beta\gamma}(\omega_\beta \wedge \omega_\gamma)(g_0, \mathrm{d}g)\,,$$

where in the last term of the second member we have stressed the dependence of the ω_β's on g and $\mathrm{d}g$. If we compare (4.13) with the first member of (4.11) we obtain the following identity:

$$(4.14)\qquad \sum_\rho i\sigma_\rho \sum_{\beta,\gamma} \varepsilon_{\rho\beta\gamma}(\omega_\beta \wedge \omega_\gamma)(gg_0, (\mathrm{d}g)\, g_0) = \sum_\rho i\sigma_\rho \sum_{\alpha\beta\gamma} ({}^t\mu(g_0))_{\rho\alpha}\, \varepsilon_{\alpha\beta\gamma}(\omega_\beta \wedge \omega_\gamma)(g, \mathrm{d}g)\,.$$

If we identify the coefficient of $i\sigma_\rho$, this means that a right translation by g_0 [inside the argument in the $\omega_\beta(g, \mathrm{d}g)$] is exactly equivalent to making a $SO(3)$ rotation with the matrix ${}^t\mu(g_0)$. Or, if one prefers, one can come back to the increment

$$\delta\xi_\alpha(g, \mathrm{d}g) = \frac{\varepsilon}{2} \sum \varepsilon_{\alpha\beta\gamma}(\omega_\beta \wedge \omega_\gamma)(g, \mathrm{d}g)\,.$$

We have the following identity:

$$(4.15)\qquad \delta\xi_\alpha(gg_0, (\mathrm{d}g)\, g_0) = \sum_\rho ({}^t\mu(g_0))_{\alpha\rho}\, \delta\xi_\rho(g, \mathrm{d}g)\,,$$

where the time argument $k\varepsilon$ in $\delta\xi_\alpha(k\varepsilon)$ has been suppressed.

3rd step. Let us now come back to our functional integral giving the Weyl propagator and for simplicity assume that the electromagnetic field is zero. If ψ is a 2-component spinor, the action of the propagator on ψ can be simply written because the rightmost term in the propagator (4.8) is a translation operator. Thus

$$(\exp[-tQ]\psi)(\boldsymbol{x}) = \lim_{n\to\infty} \frac{1}{C_0^n} \int_{SU(2)^n} \prod_{k=1}^{n} \exp[g_k^{-1}\,\mathrm{d}g_k]\,\psi(\boldsymbol{x} + \sum \delta\xi(k\varepsilon)).$$

For clarity, we make the notation a bit more explicit:

$$(4.16)\qquad (\exp[-tQ]\psi)(\boldsymbol{x}) =$$
$$= \lim_{n\to\infty} \frac{1}{C_0^n} \int_{SU(2)^n} \exp[g_n^{-1}\,\mathrm{d}g_n]\ldots\exp[g_1^{-1}\,\mathrm{d}g_1]\,\psi\left(\boldsymbol{x} + \sum_{k=1}^{n} \delta\xi(k\varepsilon, g_k, \mathrm{d}g_k)\right).$$

We have stressed the dependence of $\delta\xi(k\varepsilon)$ on g_k and $\mathrm{d}g_k$ which are hidden inside the $\omega_\beta \wedge \omega_\gamma \ldots$. Now, in (4.16), let us rotate $\boldsymbol{x}$ by an element of $SO(3)$. We can always choose this element to be of the form $\mu(g_0)$. So the argument of ψ in (4.16) is $\mu(g_0)\boldsymbol{x}$. Note that, since $\mu(g_0) \in SO(3)$, $\mu(g_0)\,{}^t\mu(g_0) = 1$. We write inside the fundamental integral

$$\psi\left(\mu(g_0)\boldsymbol{x} + \sum_{k=1}^{n} \delta\xi(k\varepsilon, g_k, \mathrm{d}g_k)\right) = \psi\left(\mu(g_0)\left(\boldsymbol{x} + \sum_{k=1}^{m} {}^t\mu(g_0)\,\delta\xi(k\varepsilon, g_k, \mathrm{d}g_k)\right)\right),$$

which by (4.15) is

$$(4.17)\qquad \psi\left(\mu(g_0)\boldsymbol{x} + \sum_{k=1}^{n} \delta\xi(k\varepsilon, g_k, \mathrm{d}g_k)\right) = \psi\left(\mu(g_0)\left(\boldsymbol{x} + \sum_{k=1}^{n} \delta\xi(k\varepsilon, g_k g_0, d(g_k g_0))\right)\right).$$

We use (4.17) repeatedly to replace $\psi(\mu(g_0)\boldsymbol{x} + \sum_{k=1}^{n} \delta\xi(k\varepsilon))$ within the functional integral (4.16) [for $(\exp[-tQ]\psi)(\mu(g_0)\boldsymbol{x})$] and recall that the functional integral is just an infinite number of integrals over independent copies of $SU(2)$ with respect to the volume element of $SU(2)$. But the volume element in $SU(2)$ is both left and *right* invariant, so

$$(4.18)\qquad (\exp[-tQ]\psi)(\mu(g_0)\boldsymbol{x}) = \lim_{n\to\infty} \frac{1}{C_0^{n+1}} \int_{SU(2)^n} \exp[(\gamma_n g_0^{-1})^{-1}\,\mathrm{d}(\gamma_n g_0^{-1})]\ldots$$
$$\ldots\exp[(\gamma_1 g_0^{-1})^{-1}\,\mathrm{d}(\gamma_1 g_0^{-1})]\,\psi\left(\mu(g_0)\left(\boldsymbol{x} + \sum_{k=1}^{m} \delta\xi(k\varepsilon, \gamma_k, \mathrm{d}\gamma_k)\right)\right),$$

where we have made the changes of variables

$$\gamma_k = g_k g_0, \qquad k = 1, \ldots, n.$$

But

$$\exp[(\gamma_k g_0^{-1})^{-1} \mathrm{d}(\gamma_k g_0^{-1})] = g_0 \exp[\gamma_k^{-1} \mathrm{d}\gamma_k] g_0^{-1}$$

and $\psi(\mu(g_0)\boldsymbol{y}) = g_0 \psi(\boldsymbol{y})$ (by the transformation in law of spinors) so that finally

$$(\exp[-tQ]\psi)(\mu(g_0)\boldsymbol{x}) = g_0(\exp[-tQ]\psi)(\boldsymbol{x}), \tag{4.19}$$

which is the correct transformation formula proving the covariance (with respect to rotation) of $\exp[-tQ]$ using its functional integral form.

4·7. *Two remarks.*

1) As one can see, the main step is the transformation formula (4.15) for the infinitesimal increments of the Grassmann process. This is the formula that lies behind the interpretation of $\delta\xi$ as a standard spatial variable.

2) The electromagnetic field is easy to treat because of the common transformation laws of the vectors $\boldsymbol{A}$ and $\delta\xi$, namely the action of the $SO(3)$ element $\mu(g)$. Thus $\boldsymbol{A}\cdot\delta\xi$ is invariant.

Moreover, all this seems to be understandable only because we rewrote the Berezin integral as an ordinary integral over $SU(2)$. In the context of abstract Grassmann variables and Grassmann integration, we do not know how to reproduce the preceding interpretation. The point is that the ω_α are $\omega_\alpha(g, \mathrm{d}g)$, functions of g and $\mathrm{d}g$ for $g \in SU(2)$.

4·8. *Paths.* – Equations (4.15) and (4.19) show that the $SU(2)$ geometry introduced as a concrete realization of the Grassmann integration bears a natural relation to the $\mathfrak{R}^3$ coordinate space in which one ordinarily conceives the motion of the Dirac particle. The formal supersymmetric correspondence «$\theta \times \theta \sim x$» is thus interpreted geometrically. There is a further step in this direction that we would like to take, namely a representation of the propagator as a sum over paths, *paths in an ordinary manifold*. The integral (4.8) suggests that one should look for a path in $SU(2)$ which will in turn, through (4.9), impose a path in $\mathfrak{R}^3$. Combined $\mathfrak{R}^3 - SU(2)$ paths have been considered before [14] for path integration, but the novel feature of the present work is the subtle intertwining of the joint motion (*i.e.* (4.9)). We have in mind motion on a fiber bundle combining $\mathfrak{R}^3$ and $SU(2)$ and we hope to report progress on this formulation in future publications.

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[14] L. S. SCHULMAN: *Nucl. Phys. B*, 18, 595 (1970).

5. – Discussion.

From ref. (3) it follows that the motion of a Dirac electron can be described in the following way: The particle travels as a massless polarized particle with 180° flips of direction and helicity occurring at random times. For the propagator one sums over all such possible histories with a pure imaginary flip rate whose magnitude is the particle's inverse mass (in natural units). From ref. (7) one can go beyond this description: The between-flips propagation can be written as a «sum over paths» in a space of Grassmann variables where the «sum» refers to Berezin integration. The coordinate space motion is derived from the supersymmetric correspondence relating products of Grassmann variables and ordinary space.

In the present paper we give a new presentation of the earlier work and take it one step further. The Grassmann variables are interpreted as forms on $SU(2)$ and the Berezin integral becomes an ordinary integral. The supersymmetric correspondence then acquires geometrical meaning. We check this by the application of a rotation to the coordinate space and by seeing how this is manifested as a rotation on the $SU(2)$ spaces introduced for the Grassmann variable interpretation.

Although we are pleased to have found a geometrical interpretation for the Grassmann variables of ref. (7), we do not feel that this necessarily provides a key to interpreting Grassmann variables in all the many contexts in which they now appear in physics. This is because our correspondence makes use of special $SU(2)$ properties, in particular the relevance of $SU(2)$ to ordinary space $\mathfrak{R}^3$.

Finally, in subsect. 4·8 we suggested a desired step beyond the present work, namely a formulation of the Dirac particle path integrals as paths—possibly constrained—on an ordinary manifold.

* * *

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● RIASSUNTO

Si considera l'integrale funzionale per una particella che ubbidisce all'equazione di Dirac. È stato mostrato, in precedenti lavori, che tale sistema può essere descritto come particella senza massa che 1) si muove di moto casuale a velocità complessa i/m e che 2) il suo moto fra i «flips» può essere scritto come una somma sui cammini per un processo stocastico su variabili di Grassman. Si propone l'estensione di un precedente lavoro sulla base di un'interpretazione geometrica delle variabili di Grassmann come forme su $SU(2)$. Tale interpretazione chiarisce la corrispondenza supersimmetrica che mette in relazione i prodotti delle variabili di Grassmann con le coordinate spaziali.

Интеграл по траекториям для уравнения Дирака. Интерпретация переменных Грассмана.

Резюме (*). — Предлагатся функциональный интеграл для частицы, удовлетворяющей уравнению Дирака. В предыдущей работе мы показали, что 1) такая частица может быть описана, как безмассовая частица, хаотически изменяющая направление и спиральность при комплексной скорости i/m, и 2) её распространение между изменениями ориентации может быть записано, как сумма по траекториям для переменной Грассмана, определяющей стохастический процесс. В этой работе предлагается обобщение результатов предыдущей статьи, что обеспечивает геометрическую интерпретацию переменных Грассмана, как выражений на SU_2. Эта интерпретация позволяет нам установить суперсимметричное соответствие, связывающее произведение переменных Грассмана с пространственными координатами.

(*) *Переведено редакцией.*