Feynman Path Integrals in Quantum Mechanics

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Abstract

This text is written as a report to the seminar course in theoretical physics at KTH, Stockholm. The idea of this work is to show Quantum Mechanics from a different perspective: based on the Path Integral formalism, originally worked out by R.P. Feynman in 1948. The mathematical equivalence to the familiar formulation shall be shown. In addition some other applications and methods of Path Integrals in non-relativistic Quantum Mechanics are discussed.

Contents

1 Introduction

The standard formulations of Quantum Mechanics were developed by Schrödinger, Heisenberg and others in the 1920s. Already 1933 Dirac published a paper where he suggests $\exp(iS/\hbar)$ to correspond to the propagator, where S is the classical action. Feynman developed this idea, concerning other paths than only the classical one and published a third complete formulation of Quantum Mechanics in 1948 ([1]). As we will see, this formulation provides a much more intuitive introduction to the quantum theory. First of all we will have to state the fundamental concepts of Quantum Mechanics. Since they are well known to the reader I will do that very shortly but in a way which is useful for the development of the theory in later sections.

2 The Path Integral Formulation of Quantum Mechanics

The goal of this section is to introduce Quantum Mechanics in a completely different way from how it is usually done in textbooks.

2.1 Probability and Probability Amplitude

Let's start with the famous imaginary double-slit experiment: We have an electron source in some point A, a screen B with two holes through which the electrons might pass, and a detector plane C. We define $P_1(x)$ as the probability that, if we close hole number 2, an electron is detected at the position x in C. Analogously for $P_2(x)$. It is a well known experimental fact that $P(x)$, the probability to arrive at x with both holes open, is *not* equal to $(P_1(x) + P_2(x))$ as long as we do not detect which hole the electron has gone through. We can state a correct law for $P(x)$ mathematically as follows: there are complex numbers Φ_1 and Φ_2 such that

$$P = |\Phi|^2 \text{ , where } \quad \Phi = \Phi_1 + \Phi_2, \quad P_1 = |\Phi_1|^2, \quad P_2 = |\Phi_2|^2 \tag{1}$$

We call $\Phi(x)$ the *probability amplitude* for arrival at x.

2.2 Probability Amplitude for a Path

Instead of only having one screen and two holes as in 2.1, we can think about inserting more screens and having several holes in each screen. The number of possible paths (each of which will have some "partial" amplitude $\Phi_i(x)$) from A to some point x on C increases. The total amplitude $\Phi(x)$ for an electron starting in A to reach this point x on C in a certain time T will be the sum of all partial amplitudes $\Phi_i(x)$. To formulate this more formally, we shall limit ourselves to a one-dimensional problem. The generalization to several dimensions is obvious.

We define $P(b,a)$ as the probability to go from a point x_a at the time t_a to the point x_b at t_b. We now know that we can write the probability $P(b,a) = |K(b,a)|^2$ of an amplitude $K(b,a)$ to go from a to b. This amplitude is the sum of all the partial amplitudes[1], one for each possible path from (x_a, t_a) to (x_b, t_b). We now write the partial amplitudes as $\Phi[x(t)]$.

$$K(b,a) = \sum_{\text{all paths from } a \text{ to } b} \Phi[x(t)] \tag{2}$$

The question is how each path $x(t)$ contributes to the total amplitude. It is so to speak the central point in the Path Integral theory of Quantum Mechanics that Feynman postulates[2]: The paths contribute equally in magnitude, but the phase of their contribution

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[1] This corresponds to Feynmans postulate I in [1].
[2] Postulate II in [1]

is the classical action S in units of the quantum of action $\hbar$, i.e.

$$\Phi[x(t)] = const \ e^{(i/\hbar)S[x(t)]} \tag{3}$$

where the classical action S is defined by:

$$S[x(t)] \ = \int_{t_a}^{t_b} L(\dot{x}, x, t) \ \mathrm{d}t \tag{4}$$

2.3   The Path Integral and the Wave Function

The main question is, how to perform the sum over all paths. We divide the time interval $(t_b - t_a)$ in N intervals of length ϵ. For each path $x(t)$ we can write $x_i = x(t_i)$ $[N\epsilon = t_b - t_a, \quad \epsilon = t_{i+1} - t_i, \quad t_0 = t_a, \quad t_N = t_b, \quad x_0 = x_a, \quad x_N = x_b]$ and get as a result for "the sum over all paths":

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A} \int \int \cdots \int e^{(i/\hbar)S[b,a]} \ \frac{\mathrm{d}x_1}{A} \frac{\mathrm{d}x_2}{A} \cdots \frac{\mathrm{d}x_{N-1}}{A} \tag{5}$$

with some normalization factor A, which we need for the convergence of the whole expression. Its value shall be determined later (but it will naturally depend on ϵ).[3]
$K(b,a)$ is called the Kernel of the motion. We will denote it as

$$K(b,a) = \int_a^b e^{(i/\hbar)S[x(t)]} \ \mathcal{D}x(t) \tag{6}$$

and call it in this notation the *configuration space path integral* or Feynman path integral.

For $K(b,a)$ there is a rule for combining amplitudes for events occuring in succession in time:

$$K(b,a) = \int_{x_c} K(b,c)K(c,a)\mathrm{d}x_c \ , \ \text{if} \quad t_a < t_c < t_b \tag{7}$$

[this rule follows from the fact $S[b,a] = S[b,c] + S[c,a]$ (which is true from the definition of the action. It holds for any point c lying on the path from a to b). To prove (7), simply insert the relation above in Eq. (6).]
This procedure can be extended from only having one to $N-1$ intermediate steps between a and b. We get the expression

$$K(b,a) = \int_{x_1}\int_{x_2}\cdots\int_{x_{N-1}} K(b,N{-}1)K(N{-}1,N{-}2)\cdots K(i{+}1,i)\cdots K(1,a)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{N-1}. \tag{8}$$

If we compare now to (5), we find for the kernel for two points separated by an infinitesimal time interval ϵ, which is correct to first order[4] in ϵ:

$$K(i+1,i) = \frac{1}{A}\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{x_{i+1}-x_i}{\epsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right)\right] \tag{9}$$

Dealing with Quantum Mechanics we are however more used to have some wave function $\Psi(x,t)$ rather than the kernel[5]. We can derive a expression for $\Psi(x,t)$ from (7):

$$\Psi(x_2,t_2) = \int\limits_{-\infty}^{\infty} K(x_2,t_2;x_1,t_1)\Psi(x_1,t_1)\mathrm{d}x_1 \tag{10}$$

In physical terms: The total amplitude to arrive at (x_2,t_2) is equal the sum over all possible values of x_1 of the amplitude to be at x_1(at a fixed time t_1) multiplyed by the amplitude to go from 1 to 2.

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[3]To be precise: In (5) the action is defined by $S = \sum_i S(x_{i+1}, x_i)$, where $S(x_{i+1}, x_i) = \min \int_{t_i}^{t_{i+1}} L(\dot{x}(t), x(t))\mathrm{d}x$, which obviously corresponds to the classical action on the interval (x_i, x_{i+1}).
[4]Detailed calculations and Feynmans own "justification" are found in [1]
[5]Actually the kernel $K(x_2,t_2;x_1,t_1) = \Psi(x_2,t_2)$ is a wave function as well.

2.3.1 The Free Particle

It might be good to see a kernel once in its very explicit form. Maybe it will even be helpfull in later sections. For the free particle case we simply have to put $L = m\dot{x}^2/2$ in (9) and insert this in (8). I do not write down the calculations, since it is basically solving a set of Gaussian integrals and can certainly be found in any of the references.

$$K(b,a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} \tag{11}$$

2.4 Classical Limit

In the classical limit, i.e. when $\hbar \to 0$, only paths that lie very close to the classical one give significant contribution to the probability amplitude.

One can see this in the following way: Let's look at any two neighbouring paths $x(t)$ and $x'(t)$ which contribute to the path integral (6). We can write $x'(t) = x(t) + \eta(t)$, with $\eta(t)$ small. For the action we get:

$$S[x'(t)] = S[x(t) + \eta(t)] = S[x(t)] + \int \eta(t)\frac{\delta S[x(t)]}{\delta x(t)}\mathrm{d}t + O(\eta^2) \tag{12}$$

The contribution of the two paths to the path integral is:

$$A = e^{iS[x(t)]/\hbar} + e^{iS[x'(t)]/\hbar} \simeq e^{iS[x(t)]/\hbar}\left(1 + \exp\frac{i}{\hbar}\int \eta(t)\frac{\delta S[x(t)]}{\delta x(t)}\mathrm{d}t\right), \tag{13}$$

where we have neglected terms of order η^2. The phase difference between the contributions is approximately $\hbar^{-1}\int(\eta(t)\delta S[x(t)]/\delta x(t))\mathrm{d}t$. Even if δS gets very small, in the limit $\hbar \to 0$ the phase difference is huge and consequently on average destructive interference between neighbouring paths occurs! There is an exception for the classical path. Since δS at $x_c(t)$ vanishes, the action is of the form $S[x_c + \eta] = S[x_c] + O(\eta^2)$ and even for very small values of $\hbar$ constructive interference results.

3 Equivalence to Schrödingers Operator Notation

In the classical formulation of Quantum Mechanics the Schrödinger equation is postulated. In this section we will show, that it is possible to derive the Schrödinger equation from the postulates made in part 2.

3.1 Derivation of the Schrödinger Equation

Let's start with Eq. (10) for the wave function. We set $t_1 = t$ and $t_2 = t + \epsilon$ and use for the kernel the expression derived in (9). In words, for a short time interval ϵ the action is approximately ϵ times the lagrangian for this interval. We get

$$\Psi(x, t + \epsilon) = \int\limits_{-\infty}^{\infty} \frac{1}{A}\exp\left(\epsilon\frac{i}{\hbar}L\left(\frac{x-y}{\epsilon}, \frac{x+y}{2}, t\right)\right)\Psi(y,t)\mathrm{d}y \tag{14}$$

Observe that this equation is not exact. But it needs only be true to first order to ϵ, since (5) still holds[6]; i.e. we shall derive the Schrödinger equation assuming (14) to be true in first order to ϵ. This can be done in the following way[7]: write the Lagrangian in the form $L = m\dot{x}^2/2 - V(x,t)$ and substitute $y = x + \eta$. The integration over y becomes an integral

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[6] For a finite time interval T the number of factors is T/ϵ. If an error of order ϵ^2 is made in each factor, the resulting error is of order ϵ and vanishes in the limit

[7] Intermediate steps are left out here. The are found in [3]

over η. The resulting integrals are Gaussian and can be solved explicitely. Expanding[8] to first order in ϵ and second order to η leeds then to the equation

$$\Psi + \epsilon \frac{\partial \Psi}{\partial t} = \Psi - \frac{i\epsilon}{\hbar} V \Psi - \frac{\hbar \epsilon}{2im} \frac{\partial^2 \Psi}{\partial x^2} \quad . \tag{15}$$

(15) is true in first order to ϵ, if Ψ satisfies [defining $H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V$] the differential Equation

$$-\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = H\Psi \quad , \tag{16}$$

which is the Schrödinger equation!

3.2 Schrödinger Equation for the Kernel

The kernel $K(2,1)$ introduced by (6) is only defined for $t_2 > t_1$. Since $K(2,1)$, considered as a function of the variables 2, is a special wave function it is clear that (16) is satisfied:

$$-\frac{\hbar}{i} \frac{\partial \Psi}{\partial t_2} K(2,1) - H_2 K(2,1) = 0 \qquad \text{for } t_2 > t_1 \tag{17}$$

Where H_2 acts on the variables 2 only.

If we define $K(2,1) = 0$ for $t_2 < t_1$, (17) is also valid for $t_2 < t_1$ but $K(2,1)$ gets discontinous at $t_2 = t_1$. One can show, that the kernel satisfies

$$-\frac{\hbar}{i} \frac{\partial \Psi}{\partial t_2} K(2,1) - H_2 K(2,1) = -\frac{\hbar}{i} \delta(x_2 - x_1) \delta(t_2 - t_1) \tag{18}$$

Remark:

We introduced $K(2,1)$ as the probability amplitude. Another possibility is to start from the usual formulation of Quantum Mechanics. The kernel can then be defined as the Green's function for the Schrödinger equation. This means: one *defines* $K(2,1)$ to be the solution of Eq. (18). This is equivalent[9] to the following definition in the better known Dirac representation: $K(2,1) = \langle x_2 | \hat{U}(t) | x_1 \rangle$, where $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ as long as the Hamiltonian is time-independent.

Finally we remark that $K(2,1)$ only depends on the time difference $(t_2 - t_1)$ [as long as the Hamiltonian is time-independent].

4 Pertubation Theory

The idea of Pertubation theory in the path integral formulation of Quantum Mechanics is quite simple. We look at a particle moving in some potential[10] $V(x,t)$. If we insert (4) in (6) and expand the exponential, we get

$$K_V(2,1) = K_0(2,1) + K_1(2,1) + K_2(2,1) + \cdots \quad , \text{ where} \tag{19}$$

$$K_0(2,1) = \int_1^2 \left[\exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} \frac{m\dot{x}^2}{2} dt \right) \right] \mathcal{D}x(t) \tag{20}$$

$$K_1(2,1) = -\frac{i}{\hbar} \int_1^2 \left[\exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} \frac{m\dot{x}^2}{2} dt \right) \right] \int_{t_1}^{t_2} V[x(s),s] ds \, \mathcal{D}x(t) \tag{21}$$

$$K_2(2,1) = -\frac{1}{2\hbar^2} \int_1^2 \left[\exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} \frac{m\dot{x}^2}{2} dt \right) \right] \left[\int_{t_1}^{t_2} V[x(s),s] ds \right]^2 \mathcal{D}x(t) \tag{22}$$

$$\vdots$$

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[8]Demanding this expansion to be true in the limit $\epsilon \to 0$, the promised condition in A appears, with the result $A = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{1/2}$.

[9]This follows from "normal" Quantum Mechanics. Discussed in [4], [5]

[10]The index V in Eq. (19) shall just symbolize that the particle is moving in some potential $V(x,t)$

K_0 is just the free-particle kernel (11). If we interchange the order of integration over x and x(t) we get for K_1:

$$K_1(2,1) = -\frac{i}{\hbar} \int_{t_1}^{t_2} F(s)\mathrm{d}s \quad, \text{ where} \tag{23}$$

$$F(s) = \int_1^2 \left[\exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2}\frac{m\dot{x}^2}{2}\mathrm{d}t\right)\right] V[x(s),s] \; \mathcal{D}x(t) \tag{24}$$

This expression $F(s)$ can be interpreted as free propagation from t_1 until some intermediate time s and again from s until t_2, only weighting each path with a characteristic factor $V[x(s),s]$. Consequently we can write $F(s)$ in the form

$$F(s) = \int_{-\infty}^{\infty} K_0(x_2,t_2;x_s,s)V(x_s,s)K_0(x_s,s;x_1,t_1)\mathrm{d}x_s \tag{25}$$

and with (23) the pertubation term gives

$$K_1(2,1) = -\frac{i}{\hbar}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} K_0(x_2,t_2;x_s,s)V(x_s,s)K_0(x_s,s;x_1,t_1)\mathrm{d}x_s\mathrm{d}s \tag{26}$$

The interpretation is the following: $K_0(2,1)$ gives us the amplitude that the particle is moving all the way from 1 to 2 without being affected by $V(x,t)$ at all. $K_1(2,1)$ is the amplitude that the particle is scattered once at any time s between 1 and 2. $K_2(2,1)$ will then be the amplitude to be scattered twice, and so on. This can be developed by similar steps as above. A very interesting feature of the kernel $K_V(2,1)$ is that it fulfills the following integral equation:

$$K_V(2,1) = K_0(2,1) - \frac{i}{\hbar}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} K_0(x_2,t_2;x_s,s)V(x_s,s)K_V(x_s,s;x_1,t_1)\mathrm{d}x_s\mathrm{d}s \tag{27}$$

We will not further look at that. It shall just be said, that this has some applications in scattering theory.

5 Path Integrals in Phase Space

As we have seen in the end of 3.2, an equivalent definition for the kernel is $K(2,1) = \langle x_2|\hat{U}(t)|x_1\rangle$. With this definition and making use of Eq. (8) (Dividing $(t_1 - t_2)$ in N intervals of length ϵ; we then assume $\hat{H}$ to be time independent on such an interval, i.e. $K(i, i+1) = \langle x_i|e^{-i\hat{H}\epsilon/\hbar}|x_{i+1}\rangle)$ one finds the following expression for $K(2,1)$ [see [7]]:

$$K(2,1) = \lim_{\epsilon\to 0}\int \prod_{i=1}^{N-1}\mathrm{d}x_i \int \prod_{j=0}^{N-1}\frac{\mathrm{d}p_j}{2\pi\hbar}\exp\left(\epsilon\frac{i}{\hbar}\sum_{k=0}^{N-1}(p_k\dot{x}_k - H(p_k,\bar{x}_k))\right), \tag{28}$$

where $\dot{x}_k = \frac{1}{\epsilon}(x_{k+1} - x_k)$ and $\bar{x}_k = \frac{1}{2}(x_k + x_{k+1})$. Similar to (6) we write that as

$$K(b,a) = \int_a^b \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b}(p\dot{x} - H(p,x))\mathrm{d}t\right] \mathcal{D}x(t)\frac{\mathcal{D}p(t)}{2\pi\hbar} \tag{29}$$

and call it the *phase space path integral*.

The phase space path integral is somehow more general than the Feynman path integral. For the case where the Hamiltonian has its standard form ($H = p^2/2m + V(x,t)$) it is easy to show the equivalence to the configuration space path integral. One inserts[11] the standard form of the Hamiltonian in (28) (the proper definition of (29)) and gets (5) by straightforward calculation, which then can be written in the form of (6).

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[11]For the explicit calculations look again in [7].

6 Path Integrals and Topology

6.1 The Aharonov-Bohm effect

Let's "abuse" the Aharonov-Bohm effect to illustrate a first example of a non-simply-connected configuration space. Take the double slit experiment from section 2.1 and put a perfectly shielded magnetic flux between the holes. Classically, because of the perfect shields, an electron does not feel any force. Due to the magnetig flux, we have to replace the Lagrangian L by L':

$$L(\dot{\mathbf{x}}, \mathbf{x}) \to L'(\dot{\mathbf{x}}, \mathbf{x}) = L(\dot{\mathbf{x}}, \mathbf{x}) - \frac{e}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{x}) \text{ , where } \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{30}$$

This changes the action S of a path by the amount $-\frac{e}{c} \int dt \, \mathbf{v} \cdot \mathbf{A}(\mathbf{x}) = -\frac{e}{c} \int dt \frac{d\mathbf{x}}{dt} \cdot \mathbf{A}(\mathbf{x}(t))$, which is the line integral of $\mathbf{A}$ taken along the path. For any two paths $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, which go through hole 1 and 2 respectively we get

$$\int_{\mathbf{x}_2(t)} d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) - \int_{\mathbf{x}_1(t)} d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) = \oint d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) = \Phi \quad , \tag{31}$$

where Φ is the flux inside the closed loop. Note that the value of the integral does not depend on the details of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$! We might write the total amplitude $K' = K_1' + K_2'$, where K_1' (K_2') is the sum over all paths through slit 1 (2). $K = K_1 + K_2$ denotes the case, when the magnetic field ist turned off ($\Phi = 0$). We can write for K_1':

$$K_1' = \int_{\text{slit 1}} \mathcal{D}\mathbf{x} \, e^{i(S[\mathbf{x}] - (e/c) \int d\mathbf{x} \cdot \mathbf{A})/\hbar} = e^{-ie \int_1 d\mathbf{x} \cdot \mathbf{A}/\hbar c} K_1, \tag{32}$$

where we have pulled out the line integral factor since it is the same for all paths through slit 1. Thus for the total amplitude:

$$\begin{align} K' &= e^{-ie \int_1 d\mathbf{x} \cdot \mathbf{A}/\hbar c} K_1 + e^{-ie \int_2 d\mathbf{x} \cdot \mathbf{A}/\hbar c} K_2 \tag{33} \\ &= e^{-ie \int_1 d\mathbf{x} \cdot \mathbf{A}/\hbar c} \left(K_1 + e^{-ie \oint d\mathbf{x} \cdot \mathbf{A}/\hbar c} K_2 \right) \tag{34} \\ &= e^{-ie \int_1 d\mathbf{x} \cdot \mathbf{A}/\hbar c} \left(K_1 + e^{-ie\Phi/\hbar c} K_2 \right) \tag{35} \end{align}$$

The first factor is just an irrelevant overall phase. The interference pattern is determined by the phase $-e\Phi/\hbar c$ and can be changed by varying the magnetic flux.

6.2 Path Integrals with Topological Constraints

There are many examples for path integrals for spaces with topological constraints, such as a particle in a box, on a half-line or generally in a half-space. Here we look at a **particle on a cirle**. On a circle our coordinate is $\varphi, 0 \leq \varphi \leq 2\pi$ with $\varphi = 0$ and $\varphi = 2\pi$ identified. A path in this system is a continuous function $\varphi(t)$ with the identification above. We can divide the set of all paths into subsets of paths with the same winding number. We can therefore write (take $\hbar = 1$ for this section) the kernel

$$K(\varphi_2, t_2; \varphi_1, t_1) = \sum_{\substack{\varphi(t_1)=\varphi_1 \\ \varphi(t_2)=\varphi_2}} e^{iS[\varphi(t)]} = \sum_{n=-\infty}^{\infty} \sum_{\varphi(t) \in g_n} e^{iS[\varphi(t)]} \quad , \text{ where} \tag{36}$$

$$\begin{align} g_n &= \{\varphi(t) | \quad t_1 \leq t \leq t_2, \; \varphi(t_1) = \varphi_1, \; \varphi(t_2) = \varphi_2, \\ & \qquad \varphi(t) \text{ is continuous and has winding number } n\} \tag{37} \end{align}$$

We now assume that each term in the sum (36) individually satisfies the Schrödinger equation[12]. We see that in this case

$$\sum_n A_n \sum_{\varphi \in g_n} e^{iS[\varphi(t)]}, \qquad A_n \in \mathbf{C} \tag{38}$$

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[12] A justification for that can be found in [5]

can be a solution for the kernel as well. From periodicity follows the condition $A_{n+1} = e^{i\delta} A_n$, $\delta \in \mathbf{R}$. With $A_0 = 1$ we get $A_n = e^{in\delta}$ and can then write:

$$K(\varphi_2, t_2; \varphi_1, t_1) = \sum_{n=-\infty}^{\infty} A_n K_n(\varphi_2, t_2; \varphi_1, t_1), \tag{39}$$

where $K_n(\varphi_2, t_2; \varphi_1, t_1)$ is the kernel for all paths with n loops. For each K_n we can carry the Lagrangian[13] from S^1 to $\mathbf{R}$ and, for a free particle, use[14] (11). Inserting this in (39) we obtain the kernel for a free particle moving on a circle:

$$K(\varphi_2, t_2; \varphi_1, t_1) = \sum_{n=-\infty}^{\infty} \left(\frac{I}{2\pi i (t_2 - t_1)} \right)^{1/2} \exp\left[in\delta + \frac{iI}{2(t_2 - t_1)} ((\varphi_2 - \varphi_1) - 2n\pi)^2 \right] \tag{40}$$

7   Statistical Mechanics

In this last section we just want to see, how Statistical Mechanics and path integrals in Quantum Mechanics are related. The central object in Statistical Mechanics is the partition function. It is defined by

$$Z := \sum_j e^{-\beta E_j} = \sum_j \langle j | e^{-\beta \hat{H}} | j \rangle = \mathrm{Tr} e^{-\beta \hat{H}}, \tag{41}$$

where $\beta = 1/k_B T$ and E_j is the energy of the state $|j\rangle$. Recall the alternative definition for the kernel: $K(x_2, t_2; x_1, t_1) = \langle x_2 | e^{-i\hat{H}(t_2 - t_1)/\hbar} | x_1 \rangle$. Let's see what happens, if we make the substitution $(t_2 - t_1) := -i\beta\hbar$, where β is real:

$$K(x_2, -i\beta\hbar; x_1, 0) = \langle x_2 | e^{-i\hat{H}(-i\beta\hbar)/\hbar} | x_1 \rangle = \langle x_2 | e^{-\beta\hat{H}} \sum_j |j\rangle\langle j| | x_1 \rangle \tag{42}$$

$$= \sum_j e^{-\beta E_j} \langle x_2 | j \rangle \langle j | x_1 \rangle = \sum_j e^{-\beta E_j} \langle j | x_1 \rangle \langle x_2 | j \rangle \tag{43}$$

where we used the completenessrelation $\sum_j |j\rangle\langle j| = 1$. Putting $x = x_1 = x_2$ and integrating over x, we get:

$$\int K(x, -i\beta\hbar; x, 0) \mathrm{d}x = \sum_j e^{-\beta E_j} \langle j | \underbrace{\int \mathrm{d}x |x\rangle\langle x|}_{=1} |j\rangle = Z \tag{44}$$

This equation tells us how the propagator evaluated at **negative imaginary time** is related to the partition function! This can be of practical use: It gives us a tool to easily calculate the partition function of a physical system of which we know the path integral (kernel).

8   Summary

This was a short introduction in the path integrals in Quantum Mechanics. I hope it gave a vague overview over the field even if the work is not complete at all. Especially the treatment of measurements and operators in the path integral formalism was completely left out. But still the basic ideas and some applications were presented. In quantum field theory path integrals will play an even more important role and I hope (for myself) that the acquired knowledge will be of use! In the last section it was shown how different fields of physics, such as statistical mechanics and Quantum Mechanics, can be linked by path integrals. A similar relation can be made between statistical field theory and quantum field theory.

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[13]We can do that, since we can define a smooth mapping from $\mathbf{R}$ to S^1. For details consult [5]!
[14]Since $L = I\dot{\varphi}^2/2$, we just set $m = I$.

References

[1] R.P. Feynman: *Space-Time Approach to Non-Relativistic Quantum Mechanics* Rev.Mod.Phys. 20 (1948) 367

[2] R.P. Feynman: *Statistical Mechanics: A set of lectures* McGraw-Hill, 1965

[3] R.P. Feynman and A.R. Hibbs: *Quantum Mechanics and Path Integrals* McGraw-Hill, 1965

[4] H. Kleinert: *Path Integrals in Qunatum Mechanics, Statistics and Polymer Physics* World Scientific, Singapore, 1990

[5] L.S. Schulman: *Techniques and Applications of Path Integration* John Wiley and Sons, 1981

[6] M. Chaichian: *Path integrals in physics. Vol. 1, Stochastic Processes and Quantum Mechanics* Institute of Physics, Bristol, 2001

[7] R. MacKenzie: *Path Integral Methods and Applications* quant-ph/0004090

[8] C. Grosche: *An Introduction into the Feynman Path Integral* hep-th/9302097