*Every path hath a puddle*
George Herbert, Outlandish Proverbs, 1640

# 8

# Path Integrals in Polar and Spherical Coordinates

Many physical systems possess rotational symmetry. In operator quantum mechanics, this property is of great help in finding wave functions and energies of a system. If a rotationally symmetric Schrödinger equation is transformed to spherical coordinates, it separates into a radial and several angular differential equations. The latter are universal and have well-known solutions. Only the radial equation contains specific information on the dynamics of the system. Being an ordinary one-dimensional Schrödinger equation, it can be solved with the usual techniques.

In the path integral approach, a similar coordinate transformation is possible, although it makes things initially more complicated rather than simpler. First, the use of non-Cartesian coordinates causes nontrivial problems of the kind observed in Chapter 6, where the configuration space was topologically constrained. Such problems can be solved as in Chapter 6 using the knowledge of the correct procedure in Cartesian coordinates. A second complication is more severe: When studying a system at a given angular momentum, the presence of a centrifugal barrier destroys the possibility of setting up a time-sliced path integral of the Feynman type as in Chapter 2. The recent solution of the latter problem has paved the way for two major advances in path integration which will be presented in Chapters 10, 11, and 12.

## 8.1 Angular Decomposition in Two Dimensions

Consider a two-dimensional quantum-mechanical system with rotational invariance. In Schrödinger quantum mechanics, it is convenient to introduce polar coordinates

$$\mathbf{x} = r(\cos\varphi, \sin\varphi), \tag{8.1}$$

and to split the differential equation into a radial and an azimuthal one which are solved separately. Let us try to follow the same approach in path integrals. To avoid the complications associated with path integrals in the canonical formulation [1],all calculations will be done in the Lagrange formulation. It will, moreover, be advantageous to work with the imaginary-time amplitude (the thermal density matrix) to avoid carrying around factors of $i$. Thus we start out with the path integral

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \int \mathcal{D}^2x(\tau)\exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau \left[\frac{M}{2}\dot{\mathbf{x}}^2 + V(r)\right]\right\}. \tag{8.2}$$

It is time-sliced in the standard way into a product of integrals

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) \approx \prod_{n=1}^{N}\left[\int \frac{d^2x_n}{2\pi\hbar\epsilon/M}\right]\exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon^2}(\mathbf{x}_n - \mathbf{x}_{n-1})^2 + V(r_n)\right]\right\}. \tag{8.3}$$

When going over to polar coordinates, the measure of integration changes to

$$\prod_{n=1}^{N}\int_0^\infty dr_n r_n \int_0^{2\pi}\frac{d\varphi_n}{2\pi\hbar\epsilon/M}$$

and the kinetic term becomes

$$\exp\left[-\frac{1}{\hbar}\frac{M}{2\epsilon}(\mathbf{x}_n-\mathbf{x}_{n-1})^2\right] = \exp\left\{-\frac{1}{\hbar}\frac{M}{2\epsilon}\left[r_n^2 + r_{n-1}^2 - 2r_nr_{n-1}\cos(\varphi_n-\varphi_{n-1})\right]\right\}. \tag{8.4}$$

To do the $\varphi_n$-integrals, it is useful to expand (8.4) into a factorized series using the formula

$$e^{a\cos\varphi} = \sum_{m=-\infty}^{\infty} I_m(a)e^{im\varphi}, \tag{8.5}$$

where $I_m(z)$ are the modified Bessel functions. Then (8.4) becomes

$$\begin{aligned}\exp&\left[-\frac{1}{\hbar}\frac{M}{2\epsilon}(\mathbf{x}_n - \mathbf{x}_{n-1})^2\right]\\ &= \exp\left[-\frac{1}{\hbar}\frac{M}{2\epsilon}(r_n^2 + r_{n-1}^2)\right]\sum_{m=-\infty}^{\infty} I_m\left(\frac{M}{\hbar\epsilon}r_nr_{n-1}\right)e^{im(\varphi_n-\varphi_{n-1})}.\end{aligned} \tag{8.6}$$

In the discretized path integral (8.3), there are $N+1$ factors of this type. The $N$-integrations over the $\varphi_n$-variables can now be performed and produce $N$ Kronecker $\delta$'s:

$$\prod_{n=1}^{N} 2\pi\delta_{m_n,m_{n-1}}. \tag{8.7}$$

These can be used to eliminate all but one of the sums over $m$, so that we arrive at the amplitude

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) &\approx \frac{2\pi}{2\pi\hbar\epsilon/M}\prod_{n=1}^{N}\left[\int_0^\infty\frac{dr_nr_n2\pi}{2\pi\hbar\epsilon/M}\right]\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}\\ &\times\prod_{n=1}^{N+1}\left\{\exp\left[-\frac{1}{\hbar}\frac{M}{2\epsilon}(r_n^2 + r_{n-1}^2)\right]I_m\left(\frac{M}{\hbar\epsilon}r_nr_{n-1}\right)\right\}\exp\left[-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}V(r_n)\right].\end{aligned} \tag{8.8}$$

We now define the *radial time evolution amplitudes* by the following expansion with respect to the azimuthal quantum numbers $m$:

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{r_b r_a}}(r_b\tau_b|r_a\tau_a)_m \frac{1}{2\pi} e^{im(\varphi_b-\varphi_a)}. \tag{8.9}$$

The amplitudes $(r_b\tau_b|r_a\tau_a)_m$ are obviously given by the *radial path integral*

$$(r_b\tau_b|r_a\tau_a)_m \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}} \prod_{n=1}^{N}\left[\int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}}\right] \\ \times \prod_{n=1}^{N+1}\left[\exp\left\{-\frac{M}{2\epsilon\hbar}(r_n-r_{n-1})^2\right\}\tilde{I}_m\left(\frac{Mr_nr_{n-1}}{\hbar\epsilon}\right)\right]\exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}V(r_n)\right\}. \tag{8.10}$$

Here we have introduced slightly different modified Bessel functions

$$\tilde{I}_m(z) \equiv \sqrt{2\pi z}e^{-z}I_m(z). \tag{8.11}$$

They will also be called "Bessel functions", for short. They have the asymptotic behavior

$$\tilde{I}_m(z) \xrightarrow{z\to\infty} 1-\frac{m^2-1/4}{2z}+\ldots = e^{-\frac{m^2-1/4}{2z}}+\ldots\,, \tag{8.12}$$

$$\tilde{I}_m(z) \xrightarrow{z\to 0} 2\sqrt{\pi}(z/2)^{m+1/2}+\ldots\,. \tag{8.13}$$

In the case of a free particle with $V(r) = 0$, it is easy to perform all the intermediate integrals over $r_n$ in (8.10). Two neighboring figures in the product require the integral

$$\int_0^\infty dr' \exp\left(-\frac{r''^2}{2\epsilon_2}-\frac{r^2}{2\epsilon_1}\right)\exp\left[-r'^2\left(\frac{1}{2\epsilon_2}+\frac{1}{2\epsilon_1}\right)\right]\frac{\sqrt{r''r'}}{\epsilon_2}I_m\left(\frac{r''r'}{\epsilon_2}\right)\frac{\sqrt{r'r}}{\epsilon_1}I_m\left(\frac{r'r}{\epsilon_1}\right) \\ = \exp\left[-\frac{r''^2+r^2}{2\left(\epsilon_1+\epsilon_2\right)}\right]\frac{\sqrt{r''r}}{\epsilon_1+\epsilon_2}I_m\left(\frac{r''r}{\epsilon_1+\epsilon_2}\right). \tag{8.14}$$

For simplicity, the units in this formula are $M = 1, \hbar = 1$. The right-hand side of (8.14) follows directly from the formula

$$\int_0^\infty drre^{-r^2/\epsilon}I_\nu\left(\beta r\right)I_\nu\left(\alpha r\right) = \frac{\epsilon}{2}e^{(\alpha^2+\beta^2)\epsilon/4}I_\nu\left(\epsilon\alpha\beta/2\right), \tag{8.15}$$

after identifying $\epsilon, \alpha, \beta$ as

$$\begin{aligned} \epsilon &= 2\epsilon_1\epsilon_2/\left(\epsilon_1+\epsilon_2\right), \\ \alpha &= r/\epsilon_1, \\ \beta &= r''/\epsilon_2. \end{aligned} \tag{8.16}$$

Thus, the integrals in (8.10) with $V(r) = 0$ can successively be performed yielding the thermal amplitude for $\tau_b > \tau_a$:

$$(r_b\tau_b|r_a\tau_a)_m = \frac{M}{\hbar}\frac{\sqrt{r_b r_a}}{\tau_b - \tau_a}\exp\left[-\frac{M}{2\hbar}\frac{r_b^2 + r_a^2}{(\tau_b - \tau_a)}\right] I_m\left(\frac{M}{\hbar}\frac{r_b r_a}{\tau_b - \tau_a}\right). \tag{8.17}$$

Note that the same result could have been obtained more directly from the imaginary-time amplitude of a free particle in two dimensions,

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \frac{M}{2\pi\hbar(\tau_b - \tau_a)}\exp\left[-\frac{M}{2\hbar}\frac{(\mathbf{x}_b - \mathbf{x}_a)^2}{\tau_b - \tau_a}\right], \tag{8.18}$$

by rewriting the right-hand side as

$$\frac{M}{2\pi\hbar\,(\tau_b - \tau_a)}\exp\left(-\frac{M}{2\hbar}\frac{r_b^2 + r_a^2}{\tau_b - \tau_a}\right)\exp\left[\frac{M}{2\hbar}\frac{r_b r_a}{\tau_b - \tau_a}\cos\left(\varphi_b - \varphi_a\right)\right],$$

and expanding the second exponential according to (8.5) into the series

$$\sum_{m=-\infty}^{\infty} I_m\left(\frac{M}{\hbar}\frac{r_b r_a}{\tau_b - \tau_a}\right) e^{im(\varphi_b - \varphi_a)}. \tag{8.19}$$

A comparison of the coefficients with those in (8.9) gives the radial amplitudes (8.17).

Due to (8.14), the radial amplitude satisfies a fundamental composition law corresponding to (2.4), which reads for $\tau_b > \tau > \tau_a$

$$\int_0^\infty drr\,(r_b\tau_b|r\;\tau)_m\,(r\;\tau|r_a\tau_a)_m = (r_b\tau_b|r_a\tau_a)_m\,. \tag{8.20}$$

## 8.2 Trouble with Feynman's Path Integral Formula in Radial Coordinates

In the above calculation we have shown that the expression (8.10) is certainly the correct radial path integral. It is, however, not of the Feynman type. In operator quantum mechanics we learn that the action of a particle moving in a potential $V(r)$ at a fixed angular momentum $L_3 = m\hbar$ contains a centrifugal barrier $\hbar^2(m^2 - 1/4)/2Mr^2$ and reads

$$\mathcal{A}_m = \int_{\tau_a}^{\tau_b} d\tau\left(\frac{M}{2}\dot{r}^2 + \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r^2} + V(r)\right). \tag{8.21}$$

This is shown by separating the Hamiltonian operator into radial and azimuthal coordinates, over fixing the azimuthal angular momentum $L_3$, and choosing for it the quantum-mechanical value $\hbar m$. According to Feynman's rules, the radial amplitude therefore should simply be given by the path integral

$$(r_b\tau_b|r_a\tau_a)_m = \int_0^\infty \mathcal{D}r\exp\left(-\frac{1}{\hbar}\mathcal{A}_m\right). \tag{8.22}$$

The reader may object to using the word "classical" in the presence of a term proportional to $\hbar^2$ in the action. In this section, however, $\hbar m$ is merely meant to be a parameter specifying the azimuthal momentum $p_\varphi \equiv \hbar m$ in the classical centrifugal barrier $p_\varphi^2/2Mr^2$. It is parametrized in terms of a dimensionless number $m$ which does not necessarily have the integer values required by the quantization of the azimuthal motion.

By naively time-slicing (8.22) according to Feynman's rules of Section 2.1 we would have defined it by the finite-$N$ expression

$$(r_b\tau_b|r_a\tau_a)_m \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}}\prod_{n=1}^{N}\left[\int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}}\right] \times \exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon}\left(r_n-r_{n-1}\right)^2+\epsilon\frac{\hbar^2}{2M}\frac{m^2-1/4}{r_n r_{n-1}}+\epsilon V\left(r_n\right)\right]\right\}. \tag{8.23}$$

Actually, the denominators in the centrifugal barrier could have been chosen to be $r_n^2$. This would make a negligible difference for small $\epsilon$. Note that in contrast to a standard Feynman path integral in one dimension, the integrations over $r$ cover only the semi-axis $r \geq 0$ rather than the complete $r$-axis. This represents no problem since we have learned in Chapter 6 how to treat such half-spaces.

The expression (8.23) is now a place for an unpleasant surprise: For $m = 0$, a time-sliced Feynman path integral formula cannot possibly exist since the potential has an abyss at small $r$. This leads to a phenomenon which will be referred to as the *path collapse*, to be understood physically and resolved later in Chapter 12. At this place we merely point out the mathematical origin of the problem, by comparing the naively time-sliced expression (8.23) with the certainly correct one (8.10). The singularity would be of no consequence if the two expressions were to converge towards each other in the continuum limit $\epsilon \to 0$. At first sight, this seems to be the case. After all, $\epsilon$ is assumed to be infinitesimally small so that we may replace the "Bessel function" $\tilde{I}_m\left(Mr_nr_{n-1}/\hbar\epsilon\right)$ by its asymptotic form (8.12),

$$\tilde{I}_m\left(\frac{Mr_nr_{n-1}}{\hbar\epsilon}\right) \xrightarrow{\epsilon\to 0} \exp\left(-\epsilon\frac{\hbar}{2M}\frac{m^2-1/4}{r_nr_{n-1}}\right). \tag{8.24}$$

For a fixed set of $r_n$, i.e., for a given path, the continuum limit $\epsilon \to 0$ makes the integrands (8.10) and (8.23) coincide, the difference being of the order $\epsilon^2$. Unfortunately, the path integral requires the limit to be taken *after* the integrations over the $dr_n$. The integrals, however, do not exist at $m = 0$. For paths moving very close to the singularity at $r = 0$, the approximation (8.24) breaks down. In fact, the large-$z$ expansion

$$\tilde{I}_m(z) = 1 - \frac{m^2-1/4}{2z} + \frac{(m^2-1/4)(m^2-1/9)}{2!(2z)^2} - \ldots \tag{8.25}$$

with $z = Mr_nr_{n-1}/\hbar\epsilon$ is never convergent even for a very small $\epsilon$. The series shows only an asymptotic convergence (more on this subject in Section 17.9). If we want to evaluate $\tilde{I}_m(z) = \sqrt{2\pi z}e^{-z}I_m(z)$ for all $z$ we have to use the convergent power series expansion of $I_m(z)$ around $z = 0$:

$$I_m(z) = \left(\frac{z}{2}\right)^m \left[\frac{1}{0!m!} + \frac{1}{1!(m+1)!}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(m+2)!}\left(\frac{z}{2}\right)^4 + \dots\right].$$

It is known from the Schrödinger theory that the leading power $z^m$ determines the threshold behavior of the quantum-mechanical particle distribution near the origin. This is qualitatively different from the exponentially small distribution $\exp\left[-\epsilon\hbar(m^2 - 1/4)/2Mr^2\right]$ contained in each time slice of the Feynman formula (8.23) for $|m| > 1/2$.

The root of these troubles is an anomalous behavior in the high-temperature limit of the partition function. In this limit, the imaginary time difference $\tau_b - \tau_a = \hbar/k_BT$ is very small and it is usually sufficient to keep only a single slice in a time-sliced path integral (see Sections 2.9, 2.13). If this were true also here, the formula (8.23) would lead, in the absence of a potential $V(r)$, to the classical particle distribution [compare (2.349)]

$$(r_a\ \tau_a + \epsilon|r_a\tau_a) = \frac{1}{\sqrt{2\pi\hbar\epsilon/M}} \exp\left(-\epsilon\frac{\hbar}{2M}\frac{m^2 - 1/4}{r_a^2}\right). \tag{8.26}$$

If we subtract the barrier-free distribution, this amounts to the classical partition function[1]

$$Z_{\rm cl} = \int_0^\infty \frac{dr}{\sqrt{2\pi a}}\left[\exp\left(-a\frac{m^2 - 1/4}{2r^2}\right) - 1\right] = -\frac{1}{2}\sqrt{m^2 - \frac{1}{4}}, \tag{8.27}$$

where we have abbreviated the factor $\epsilon\hbar/M$ by $a$. The integral is temperature-independent and converges only for $|m| > 1/2$.

Compare this result with the proper high-temperature limit of the exact partition function calculated with the use of (8.17) and with the same subtraction as before. It reads for *all* $T$

$$Z = \frac{1}{2}\int_0^\infty dz e^{-z}[I_m(z) - I_{1/2}(z)]. \tag{8.28}$$

As in the classical expression, there is no temperature dependence. The integral

$$\int_0^\infty dz e^{-\alpha z} I_\mu(z) = (\alpha^2 - 1)^{-1/2}(\alpha + \sqrt{\alpha^2 - 1})^{-\mu} \tag{8.29}$$

[see formula (2.475)] converges for arbitrary real $\nu$ and $\alpha > 1$, and gives in the limit $\alpha \to 1$

$$Z = -\frac{1}{2}(m - 1/2). \tag{8.30}$$

[1] This follows by expanding the formula $\int_0^\infty dx e^{-a/x^2 - bx^2} = \sqrt{\pi/4b}e^{-2\sqrt{ab}}$ in powers of $\sqrt{b}$, subtracting the $a = 0$ -term, and taking the limit $b \to 0$.

This is different from the classical result (8.27) and agrees with it only in the limit of large $m$.[2]

Thus we conclude that a time-sliced path integral containing a centrifugal barrier can only give the correct amplitude when using the Euclidean action

$$\tilde{\mathcal{A}}_m^N = \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon}(r_n - r_{n-1})^2 - \hbar \log \tilde{I}_m \left( \frac{M}{\hbar\epsilon} r_n r_{n-1} \right) \right], \tag{8.31}$$

in which the neighborhood of the singularity is treated quantum-mechanically. The naively time-sliced classical action in (8.23) is of no use. The centrifugal barrier renders therefore a counterexample to Feynman rules of path integration according to which quantum-mechanical amplitudes should be obtainable from a sum over all histories of exponentials which involve only the classical expression for the short-time actions.

It is easy to see where the derivation of the time-sliced path integral in Section 2.1 breaks down. There, the basic ingredient was the Trotter product formula (2.26), which for imaginary time reads

$$e^{-\beta(\hat{T}+\hat{V})} = \lim_{N\to\infty} \left( e^{-\epsilon \hat{T}} e^{-\epsilon \hat{V}} \right)^{N+1}, \quad \epsilon \equiv \beta/(N+1). \tag{8.32}$$

An exact identity is $e^{-\beta(\hat{T}+\hat{V})} \equiv \left( e^{-\epsilon \hat{T}} e^{-\epsilon \hat{V}} e^{i\epsilon^2 X} \right)^{N+1}$ with $X$ given in Eq. (2.10) consisting of a sum of higher and higher commutators between $\hat{V}$ and $\hat{T}$. The Trotter formula neglects these commutators. In the presence of a centrifugal barrier, however, this is not permitted. Although the neglected commutators carry increasing powers of the small quantity $\epsilon$, they are more and more divergent at $r = 0$, like $\epsilon^n/r^{2\pi}$. The same terms occur in the asymptotic expansion (8.25) of the "Bessel function". In the proper action (8.31), these terms are present.

It should be noted that for a Hamiltonian possessing a centrifugal barrier $V_{\rm cb}$ in addition to an arbitrary smooth potential $V$ , i.e., for a Hamiltonian operator of the form $\hat{H} = \hat{T} + \hat{V}_{\rm cf} + \hat{V}$, the Trotter formula *is* applicable in the form

$$e^{-\beta H} = \lim_{N\to\infty} \left( e^{-\epsilon(\hat{T}+\hat{V}_{\rm cb})} e^{-\epsilon \hat{V}} \right)^{N+1}, \quad \epsilon \equiv \beta/(N+1). \tag{8.33}$$

It leads to a valid time-sliced path integral formula

$$(r_b\tau_b|r_a\tau_a)_m \approx \prod_{n=1}^{N} \left[ \int_0^\infty dr_n \right] \prod_{n=1}^{N+1} \left[ \int \frac{dp_n}{2\pi\hbar} \right] \exp\left\{ -\frac{1}{\hbar} \tilde{\mathcal{A}}_m^N[p,r] \right\}, \tag{8.34}$$

with the sliced action

$$\tilde{\mathcal{A}}_m^N[p,r] = \sum_{n=1}^{N+1} \left[ -ip_n(r_n - r_{n-1}) + \epsilon \frac{p_n^2}{2M} - \hbar \log \tilde{I}_m \left( \frac{M}{\hbar\epsilon} r_n r_{n-1} \right) + \epsilon V(r_n) \right]. \tag{8.35}$$

[2]If $m$ were not merely a dimensionless number parametrizing an arbitrary centrifugal barrier with fixed $p_\varphi^2/2Mr^2$, as it is in this section, the classical limit at a fixed $p_\varphi$ would eliminate the problem since for $\hbar \to 0$, the number $m$ would become infinitely large, leading to the correct high-temperature limit of the partition function. For a fixed finite $m$, however, the discrepancy is unavoidable.

After integrating out the momenta, this becomes

$$(r_b\tau_b|r_a\tau_a)_m \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}} \prod_{n=1}^{N} \left[\int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}}\right] \exp\left\{-\frac{1}{\hbar}\tilde{\mathcal{A}}_m^N[r]\right\}, \tag{8.36}$$

with the action

$$\tilde{A}_m^N[r,\dot{r}] = \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon}(r_n - r_{n-1})^2 - \hbar\log\tilde{I}_m\left(\frac{Mr_nr_{n-1}}{\hbar\epsilon}\right) + \epsilon V(r_n)\right]. \tag{8.37}$$

The path integral formula (8.36) can in principle be used to find the amplitude for a fixed angular momentum of some solvable systems.

An example is the radial harmonic oscillator at an angular momentum $m$, although it should be noted that this particularly simple example does not really require calculating the integrals in (8.36). The result can be found much more simply from a direct angular momentum decomposition of the amplitude (2.177). After a continuation to imaginary times $t = -i\tau$ and an expansion of part of the exponent with the help of (8.5), it reads for $D = 2$

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) &= \frac{1}{2\pi}\frac{M\omega}{\hbar\sinh[\omega(\tau_b-\tau_a)]}e^{-\frac{M\omega}{2\hbar}\coth[\omega(\tau_b-\tau_a)](r_b^2+r_a^2)}\\ &\times\sum_{m=-\infty}^{\infty} I_m\left(\frac{M\omega r_b r_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right)e^{im(\varphi_b-\varphi_a)}.\end{aligned} \tag{8.38}$$

By comparison with (8.9), we extract the radial amplitude

$$(r_b\tau_b|r_a\tau_a)_m = \frac{M}{\hbar}\frac{\omega\sqrt{r_br_a}}{\coth[\omega(\tau_b-\tau_a)]}I_m\left(\frac{M\omega r_b r_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right). \tag{8.39}$$

The limit $\omega \to 0$ gives the free-particle result

$$(r_b\tau_b|r_a\tau_a)_m = \frac{M}{\hbar}\frac{\sqrt{r_br_a}}{\tau_b-\tau_a}I_m\left(\frac{M}{\hbar}\frac{r_br_a}{\tau_b-\tau_a}\right). \tag{8.40}$$

## 8.3 Cautionary Remarks

It is important to emphasize that we obtained the correct amplitudes by performing the time slicing in Cartesian coordinates followed by the transformation to the polar coordinates in the time-sliced expression. Otherwise we would have easily missed the factor $-1/4$ in the centrifugal barrier. To see what can go wrong let us proceed illegally and do the change of variables in the initial continuous action. Thus we try to calculate the path integral

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) &= \sum_{l=-\infty,\infty}\int_0^\infty \mathcal{D}r\, r\int_{-\infty}^{\infty}\mathcal{D}\varphi\\ &\times \exp\left\{\frac{1}{\hbar}\int_{\tau_a}^{\tau_b}d\tau\left[\frac{M}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)\right]\right\}\Bigg|_{\varphi(\tau_b)=\varphi_b+2\pi l}.\end{aligned} \tag{8.41}$$

The summation over all periodic repetitions of the final azimuthal angle $\varphi$ accounts for its multivaluedness according to the rules of Section 6.1. If the expression (8.41) is time-sliced straightforwardly it reads

$$\sum_{\varphi_{N+1}=\varphi_b+2\pi l, l=-\infty,\infty} \frac{1}{2\pi\hbar\epsilon/M} \prod_{n=1}^{N} \left[\int_0^\infty \frac{dr_n r_n}{\sqrt{2\pi\hbar\epsilon/M}} \int_{-\infty}^{\infty} \frac{d\varphi_n}{\sqrt{2\pi\hbar\epsilon/M}}\right]$$
$$\times \exp\left\{-\frac{1}{\hbar}\epsilon \sum_{n=1}^{N+1} \left[\frac{M}{2\epsilon^2}[(r_n - r_{n-1})^2 + r_n^2(\varphi_n - \varphi_{n-1})^2] + V(r_n)\right]\right\}, \tag{8.42}$$

where $r_b = r_{N+1}$, $\varphi_a = \varphi_0, r_a = r_0$. The integrals can be treated as follows. We introduce the momentum integrals over $(p_\varphi)_n$ conjugate to $\varphi_n$, writing for each $n$ [with the short notation $(p_\varphi)_n \equiv p_n$]

$$\frac{r_n}{\sqrt{2\pi\hbar\epsilon/M}} e^{-(M/2\hbar\epsilon)r_n^2(\varphi_n-\varphi_{n-1})^2} = \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp\left\{-\frac{1}{\hbar}\frac{\epsilon}{2M}\frac{p_n^2}{r_n^2} + \frac{i}{\hbar}p_n(\varphi_n - \varphi_{n-1})\right\}. \tag{8.43}$$

After this, the integrals over $\varphi_n$ $(n = 1 \ldots N)$ in (8.42) enforce all $p_n$ to be equal to each other, i.e., $p_n \equiv p$ for $n = 1, \ldots, N+1$, and we remain with

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) \approx \frac{1}{2\pi\hbar\epsilon/M}\frac{1}{r_b} \sum_{l-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{(i/\hbar)p(\varphi_b-\varphi_a+2\pi l)} \prod_{n=1}^{N} \left[\int_0^\infty \frac{dr_n}{2\pi\hbar\epsilon/M}\right]$$
$$\times \exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2}\frac{(r_n - r_{n-1})^2}{\epsilon^2} + \frac{p^2}{2Mr_n^2} + V(r_n)\right]\right\}. \tag{8.44}$$

Performing the sum over $l$ with the help of Poisson's formula (6.9) changes the integral over $dp/2\pi\hbar$ into a sum over integers $m$ and yields the angular momentum decomposition (8.9) with the partial-wave amplitudes

$$(r_b\tau_b|r_a\tau_a)_m \approx \sqrt{r_b r_a}\frac{1}{2\pi\hbar\epsilon/M}\frac{1}{r_b}\prod_{n=1}^{N}\left[\int_0^\infty \frac{dr_n}{2\pi\hbar\epsilon/M}\right]$$
$$\times \exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2}\frac{(r_n - r_{n-1})^2}{\epsilon^2} + \frac{\hbar^2}{2M}\frac{m^2}{r_n^2} + V(r_n)\right]\right\}. \tag{8.45}$$

This wrong result differs from the correct one in Eq. (8.10) in three respect. First, it does not possess the proper centrifugal term $-\hbar \log \tilde{I}_m(Mr_br_a/\hbar\epsilon)$. Second, there is a spurious overall factor $\sqrt{r_a/r_b}$. Third, in comparison with the limiting expression (8.24), the centrifugal barrier lacks the term $1/4$.

It is possible to restore the term $1/4$ by observing that in the time-sliced expression, the factor $\sqrt{r_b/r_a}$ is equal to

$$\prod_{n=1}^{N+1}\sqrt{\frac{r_b}{r_a}} = \prod_{n=1}^{N+1}\left(1 + \frac{r_n - r_{n-1}}{r_{n-1}}\right)^{-1/2} = \prod_{n=1}^{N+1}\left(1 - \frac{1}{2}\frac{r_n - r_{n-1}}{\sqrt{r_n r_{n-1}}} + \frac{1}{8}\frac{(r_n - r_{n-1})^2}{r_n r_{n-1}} - \ldots\right)$$

$$= \exp\left(-\frac{1}{2}\sum_{n=1}^{N+1}\frac{r_n - r_{n-1}}{\sqrt{r_n r_{n-1}}} + \ldots\right). \tag{8.46}$$

This, in turn, can be incorporated into the kinetic term of (8.45) via a quadratic completion leading to

$$\exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2}\left(\frac{r_n - r_{n-1}}{\epsilon} + i\frac{\hbar}{2M\sqrt{r_n r_{n-1}}}\right)^2 + \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r_n^2}\right]\right\}. \tag{8.47}$$

The centrifugal barrier is now correct, but the kinetic term is wrong. In fact, it does not even correspond to a Hermitian Hamiltonian operator, as can be seen by introducing momentum integrations and completing the square to

$$\exp\left\{-ip_n(r_n - r_{n-1}) + \frac{\epsilon}{\hbar}\left[\frac{p_n^2}{2M} - i\hbar\frac{p_n}{2Mr_n}\right]\right\}. \tag{8.48}$$

The last term is an imaginary energy. Only by dropping it artificially would the time-sliced action acquire the Feynman form (8.23), while still being beset with the problem of nonexistence for $m = 0$ (path collapse) and the nonuniform convergence of the path integrations to be solved in Chapter 12.

The lesson of this is the following: A naive time slicing *cannot* be performed in curvilinear coordinates. It can safely be done in the Cartesian formulation.

Fortunately, a systematic modification of the naive slicing rules has recently been found which makes them applicable to non-Cartesian systems. This will be shown in Chapters 10 and 11.

In the sequel it is useful to maintain, as far as possible, the naive notation for the radial path integral (8.23) and the continuum limit of the action (8.21). The places where care has to be taken in the time-slicing process will be emphasized by setting the centrifugal barrier in quotation marks and defining

$$\text{``}\epsilon\frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r_n r_{n-1}}\text{''} \equiv -\hbar\log\tilde{I}_m\left(\frac{Mr_n r_{n-1}}{\hbar\epsilon}\right). \tag{8.49}$$

Thus we shall write the properly sliced action (8.37) as

$$\tilde{A}_m^N[r, \dot{r}] = \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon}(r_n - r_{n-1})^2 + \text{``}\epsilon\frac{\hbar^2}{2\mu}\frac{m^2 - 1/4}{r_n r_{n-1}}\text{''} + \epsilon V(r_n)\right], \tag{8.50}$$

and emphasize the need for the non-naive time slicing of the continuum action correspondingly:

$$\mathcal{A}_m = \int_{\tau_a}^{\tau_b} d\tau\left[\frac{M}{2}\dot{r}^2 + \text{``}\frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r^2}\text{''} + V(r)\right]. \tag{8.51}$$

## 8.4 Time Slicing Corrections

It is interesting to find the origin of the above difficulties. For this purpose, we take the Cartesian kinetic terms expressed in terms of polar coordinates

$$(\mathbf{x}_n - \mathbf{x}_{n-1})^2 = (r_n - r_{n-1})^2 + 2r_n r_{n-1}[1 - \cos(\varphi_n - \varphi_{n-1})], \tag{8.52}$$

and treat it perturbatively in the coordinate differences [2]. Expanding the cosine into a power series we obtain the time-sliced action

$$\mathcal{A}^N = \frac{M}{2\epsilon}\sum_{n=1}^{N+1}(\mathbf{x}_n - \mathbf{x}_{n-1})^2 = \frac{M}{2\epsilon}\sum_{n=1}^{N+1}\Big\{(r_n - r_{n-1})^2 + 2r_n r_{n-1}\left[\frac{1}{2!}(\varphi_n - \varphi_{n-1})^2 - \frac{1}{4!}(\varphi_n - \varphi_{n-1})^4 + \dots\right]\Big\}. \tag{8.53}$$

In contrast to the naively time-sliced expression (8.42), we now keep the quartic term $(\varphi_n - \varphi_{n-1})^4$. To see how it contributes, consider a single intermediate integral

$$\int \frac{d\varphi_{n-1}}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)\left[(\varphi_n-\varphi_{n-1})^2 + a_4(\varphi_n-\varphi_{n-1})^4+\dots\right]}. \tag{8.54}$$

The first term in the exponent restricts the width of the fluctuations of the difference $\varphi_n - \varphi_{n-1}$ to

$$\langle(\varphi_n - \varphi_{n-1})^2\rangle_0 = \frac{\epsilon}{a}. \tag{8.55}$$

If we rescale the arguments, $\varphi_n \to \sqrt{\epsilon}u_n$, the integral takes the form

$$\int \frac{du_{n-1}}{\sqrt{2\pi/a}} e^{-(a/2)[(u_n-u_{n-1})^2+\epsilon a_4(u_n-u_{n-1})^4+\dots]}. \tag{8.56}$$

This shows that each higher power in the difference $u_n - u_{n-1}$ is suppressed by an additional factor $\sqrt{\epsilon}$. We now expand the integrand in powers of $\sqrt{\epsilon}$ and use the integrals

$$\int \frac{du}{\sqrt{2\pi/a}} e^{-(a/2)u^2} \left\{\begin{array}{c} u^2 \\ u^4 \\ \vdots \\ u^{2n} \end{array}\right\} = \left\{\begin{array}{r} a^{-1} \\ 3a^{-2} \\ \vdots \\ (2n-1)!!a^{-n} \end{array}\right\}, \tag{8.57}$$

with odd powers of $u$ giving trivially 0, to find an expansion of the integral (8.56). It begins as follows:

$$1 - \epsilon\,\frac{a}{2}a_4\; 3a^{-2} + \mathcal{O}(\epsilon^2). \tag{8.58}$$

This can be thought of as coming from the equivalent integral

$$\int \frac{d\varphi_{n-1}}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)(\varphi_n-\varphi_{n-1})^2 - 3a_4\epsilon/2a+\dots}\;. \tag{8.59}$$

The quartic term in (8.54),

$$\Delta A = \frac{a}{2\epsilon} a_4 (\varphi_n - \varphi_{n-1})^4, \tag{8.60}$$

has generated an effective action-like term in the exponent:

$$A_{\rm eff} = \epsilon \frac{3a_4}{2a}. \tag{8.61}$$

This is obviously due to the expectation value $\langle \Delta A \rangle_0$ of the quartic term and we can record, for later use, the perturbative formula

$$A_{\rm eff} = \langle \Delta A \rangle_0. \tag{8.62}$$

If $u$ is a vector in $D$ dimensions to be denoted by $\mathbf{u}$, with the quadratic term being $(a/2\epsilon)(\mathbf{u}_n - \mathbf{u}_{n-1})^2$, the integrals (8.57) are replaced by

$$\int \frac{d^D u}{\sqrt{2\pi/a}^D} e^{-(a/2)\mathbf{u}^2} \left\{ \begin{array}{c} u_i u_j \\ u_i u_j u_k u_l \\ \vdots \\ u_{i_1} \cdot \ldots \cdot u_{i_{2n}} \end{array} \right\} = \left\{ \begin{array}{l} a^{-1}\delta_{ij} \\ a^{-2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ \vdots \\ a^{-n}\delta_{i_1 \ldots i_{2n}} \end{array} \right\}, \tag{8.63}$$

where $\delta_{i_1 \ldots i_{2n}}$ will be referred to as *contraction tensors*, defined recursively by the relation

$$\delta_{i_1 \ldots i_{2n}} = \delta_{i_1 i_2}\delta_{i_3 i_4 \ldots i_{2n}} + \delta_{i_1 i_3}\delta_{i_2 i_4 \ldots i_{2n}} + \ldots + \delta_{i_1 i_{2n}}\delta_{i_2 i_3 \ldots i_{2n-1}}. \tag{8.64}$$

A comparison with the Wick expansion (3.306) shows that this recursion relation amounts to $\delta_{i_1 \ldots i_{2n}}$ possessing a Wick-like expansion into the sum of products of Kronecker $\delta$'s, each representing a pair contraction. Indeed, the integral formulas (8.63) can be derived by adding a source term $\mathbf{j} \cdot \mathbf{u}$ to the exponent in the integrand, completing the square, and differentiating the resulting $e^{(1/2a)\mathbf{j}^2}$ with respect to the "current" components $j_i$. For vectors $\varphi_i$, a possible quartic term in the exponent of (8.54) may have the form

$$\Delta A = \frac{a}{2\epsilon}(a_4)_{ijkl}(\varphi_n - \varphi_{n-1})_i(\varphi_n - \varphi_{n-1})_j(\varphi_n - \varphi_{n-1})_k(\varphi_n - \varphi_{n-1})_l. \tag{8.65}$$

Then the factor $3a_4$ in $A_{\rm eff}$ of Eq. (8.61) is replaced by the three contractions

$$(a_4)_{ijkl}(\delta_{ij}\delta_{kl} + 2 \text{ more pair terms}).$$

Applying the simple result (8.61) to the action (8.53), where $a = (M/\hbar)2r_n r_{n-1}$ and $a_4 = -1/4!$, we find that the naively time-sliced kinetic term of the $\varphi$ field

$$\frac{M}{2\epsilon} r_n r_{n-1} (\varphi_n - \varphi_{n-1})^2$$

is extended to

$$\frac{M}{2\epsilon}r_n r_{n-1}(\varphi_n-\varphi_{n-1})^2-\epsilon\hbar^2\frac{1/4}{2Mr_nr_{n-1}}+\dots\,.$$

Thus, the lowest perturbative correction due to the fourth-order expansion term of $\cos(\varphi_n-\varphi_{n-1})$ supplies precisely the 1/4-term in the centrifugal barrier which was missing in (8.45). Proceeding in this fashion, the higher powers in the expansion of $\cos(\varphi_n-\varphi_{n-1})$ give higher and higher contributions $(\epsilon/r_nr_{n-1})^n$. Eventually, they would of course produce the entire asymptotic expansion of the "Bessel function" in the correct time-sliced action (8.37).

Note that the failure of this series to converge destroys the justification for truncating the perturbation series after any finite number of terms. In particular, the knowledge of the large-order behavior (the "tail end" of the series) [3] is needed to recover the correct threshold behavior $\propto r^m$ in the amplitudes observed in (8.26).

The reader may rightfully object that the integral (8.56) should really contain an exponential factor $\exp\left[-(a/2)(u_{n-1}-u_{n-2})^2\right]$ from the adjacent time slice which also contains the variable $u_{n-1}$ and which has been ignored in the integral (8.56) over $u_{n-1}$. In fact, with the abbreviations $\bar{u}_{n-1}\equiv(u_n+u_{n-2})/2,\ \delta\equiv u_{n-1}-\bar{u}_{n-1},\ \Delta\equiv u_n-u_{n-2}$, the complete integrand containing the variable $u_{n-1}$ can be written as

$$\int\frac{d\delta}{\sqrt{2\pi/a}}e^{-a\delta^2}e^{-a\Delta^2/4}\left\{1-\frac{a}{2}\epsilon a_4[(-\delta+\Delta/2)^4+(\delta+\Delta/2)^4]+\dots\right\}. \tag{8.66}$$

When doing the integral over $\delta$, each even power $\delta^{2n}$ gives a factor $(1/2a)(2n-1)!!$ and we observe that the mean value of the fluctuating $u_{n-1}$ is different from what it was above, when we singled out the expression (8.56) and ignored the $u_{n-1}$ dependence of the adjacent integral. Instead of $u_n$ in (8.56), the mean value of $u_{n-1}$ is now the average position of the neighbors, $\bar{u}_{n-1}=(u_n+u_{n-2})/2$. Moreover, instead of the width of the $u_{n-1}$ fluctuations being $\langle(u_n-\bar{u}_{n-1})^2\rangle_0\sim 1/a$, as in (8.55), it is now given by half this value:

$$\langle(u_{n-1}-\bar{u}_{n-1})^2\rangle_0=\frac{1}{2a}. \tag{8.67}$$

At first, these observations seem to invalidate the above perturbative evaluation of (8.56). Fortunately, this objection ignores an important fact which cancels the apparent mistake, and the result (8.62) of the sloppy derivation is correct after all. The argument goes as follows: The integrand of a single time slice is a sharply peaked function of the coordinate difference whose width is of the order $\epsilon$ and goes to zero in the continuum limit. If such a function is integrated together with some *smooth* amplitude, it is sensitive only to a small neighborhood of a point in the amplitude. The sharply peaked function is a would-be $\delta$-function that can be effectively replaced by a $\delta$-function plus correction terms which contain increasing derivatives of $\delta$-functions multiplied by corresponding powers of $\sqrt{\epsilon}$. Indeed, let us take the integrand of the model integral (8.54),

$$\frac{1}{\sqrt{2\pi\epsilon/a}}e^{-(a/2\epsilon)\left[(\varphi_n-\varphi_{n-1})^2+a_4(\varphi_n-\varphi_{n-1})^4+\dots\right]}, \tag{8.68}$$

and integrate it over $\varphi_{n-1}$ together with a smooth amplitude $\psi(\varphi_{n-1})$ which plays the same role of a test function in mathematics [recall Eq. (1.162)]:

$$\int_{-\infty}^{\infty} \frac{d\varphi_{n-1}}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)\left[(\varphi_n-\varphi_{n-1})^2+a_4(\varphi_n-\varphi_{n-1})^4+\dots\right]}\psi(\varphi_{n-1}). \tag{8.69}$$

For small $\epsilon$, we expand $\psi(\varphi_{n-1})$ around $\varphi_n$,

$$\psi(\varphi_{n-1}) = \psi(\varphi_n) - (\varphi_n - \varphi_{n-1})\psi'(\varphi_n) + \frac{1}{2}(\varphi_n - \varphi_{n-1})^2\psi''(\varphi_n) + \dots\ , \tag{8.70}$$

and (8.69) becomes

$$(1 - A_{\text{eff}})\psi(\varphi_n) + \frac{\epsilon}{2a}\psi''(\varphi_n) + \dots\ . \tag{8.71}$$

This shows that the amplitude for a single time slice, when integrated together with a smooth amplitude, can be expanded into a series consisting of a $\delta$-function and its derivatives:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)\left[(\varphi_n-\varphi_{n-1})^2+a_4(\varphi_n-\varphi_{n-1})^4+\dots\right]}\\
&\qquad = (1 - A_{\text{eff}})\delta(\varphi_n - \varphi_{n-1}) + \frac{\epsilon}{2a}\delta''(\varphi_n - \varphi_{n-1}) + \dots\ .
\end{aligned} \tag{8.72}$$

The right-hand side may be viewed as the result of a simpler would-be $\delta$-function

$$\frac{1}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)(\varphi_n-\varphi_{n-1})^2} e^{-A_{\text{eff}}}, \tag{8.73}$$

correct up to terms of order $\epsilon$. This is precisely what we found in (8.59).

The problems observed above arise only if the would-be $\delta$-function in (8.72) is integrated together with another sharply peaked neighbor function which is itself a would-be $\delta$-function. Indeed, in the theory of distributions, it is strictly forbidden to form integrals over products of two proper distributions. For the would-be distributions at hand the rule is not quite as strict and integrals over products can be formed. The crucial expansion (8.72), however, is no longer applicable if the accompanying function is a would-be $\delta$-function, and a more careful treatment is required.

The correctness of formula (8.62) derives from the fact that each time slice has, for sufficiently small $\epsilon$, a large number of neighbors at earlier and later times. If the integrals are done for all these neighbors, they render a smooth amplitude before and a smooth amplitude after the slice under consideration. Thus, each intermediate integral in the time-sliced product contains a would-be $\delta$-function multiplied on the right- and left-hand side with a *smooth* amplitude. In each such integral, the replacement (8.59) and thus formula (8.62) is correct. The only exceptions are time slices near the endpoints. Their integrals possess the above subtleties. The relative number of these, however, goes to zero in the continuum limit $\epsilon \to 0$. Hence they do not change the final result (8.62).

For completeness, let us state that the presence of a cubic term in the single-sliced action (8.68) has the following $\delta$-function expansion

$$\frac{1}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)\left[(\varphi_n-\varphi_{n-1})^2+a_3(\varphi_n-\varphi_{n-1})^3+a_4(\varphi_n-\varphi_{n-1})^4+\ldots\right]} \tag{8.74}$$
$$= (1-A_{\rm eff})\delta(\varphi_n-\varphi_{n-1}) + 3a_3\frac{\epsilon}{2a}\delta'(\varphi_n-\varphi_{n-1}) + \frac{\epsilon}{2a}\delta''(\varphi_n-\varphi_{n-1}) + \ldots$$

corresponding to an "effective action"

$$A_{\rm eff} = \frac{\epsilon}{2a}\left[3a_4 - \frac{15}{4}a_3^2\right]. \tag{8.75}$$

Using (8.75), the left-hand side of (8.74) can also be replaced by the would-be $\delta$-function

$$\frac{1}{\sqrt{2\pi\epsilon/a}} e^{-(a/2\epsilon)(\varphi_n-\varphi_{n-1})^2} e^{-A_{\rm eff}}[1 - \frac{3a_3}{2}(\varphi_n-\varphi_{n-1}) + \ldots], \tag{8.76}$$

which has the same leading terms in the $\delta$-function expansion.

In $D$ dimensions, the term $3a_3(\varphi_n-\varphi_{n-1})$ has the general form

$$[(a_3)_{ijj} + (a_3)_{jij} + (a_3)_{jji}](\varphi_n-\varphi_{n-1})_i,$$

and the term $15a_3^2$ in $A_{\rm eff}$ becomes

$$(a_3)_{ijk}(a_3)_{i'j'k'}(\delta_{ii'}\delta_{jj'}\delta_{kk'} + 14 \text{ more pair terms}).$$

## 8.5 Angular Decomposition in Three and More Dimensions

Let us now extend the two-dimensional development of Section 8.2 and study the radial path integrals of particles moving in three and more dimensions. Consider the amplitude for a rotationally invariant action in $D$ dimensions

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \int \mathcal{D}^D x(\tau) \exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b}\left[\frac{M}{2}\dot{\mathbf{x}}^2 + V(r)\right]\right\}. \tag{8.77}$$

By time-slicing this in Cartesian coordinates, the kinetic term gives an integrand

$$\exp\left[-\frac{1}{\hbar}\frac{M}{2\epsilon}\sum_{n=1}^{N+1}(r_n^2 + r_{n-1}^2 - 2r_nr_{n-1}\cos\Delta\vartheta_n)\right], \tag{8.78}$$

where $\Delta\vartheta_n$ is the relative angle between the vectors $\mathbf{x}_n$ and $\mathbf{x}_{n-1}$.

### 8.5.1 Three Dimensions

In three dimensions, we go over to the spherical coordinates

$$\mathbf{x} = r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \tag{8.79}$$

and write

$$\cos\Delta\vartheta_n = \cos\theta_n\cos\theta_{n-1} + \sin\theta_n\sin\theta_{n-1}\cos(\varphi_n - \varphi_{n-1}). \tag{8.80}$$

The integration measure in the time-sliced version of (8.77),

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/M}^3}\prod_{n=1}^{N}\int\frac{d^3x_n}{\sqrt{2\pi\hbar\epsilon/M}^3}, \tag{8.81}$$

becomes

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/M}^3}\prod_{n=1}^{N}\int\frac{dr_n r_n^2 d\cos\theta_n d\varphi_n}{\sqrt{2\pi\hbar\epsilon/M}^3}. \tag{8.82}$$

To perform the integrals, we use the spherical analog of the expansion (8.5)

$$e^{h\cos\Delta\vartheta_n} = \sqrt{\frac{\pi}{2h}}\sum_{l=0}^{\infty} I_{l+1/2}(h)(2l+1)P_l(\cos\Delta\vartheta_n), \tag{8.83}$$

where $P_l(z)$ are the Legendre polynomials. These, in turn, can be decomposed into spherical harmonics

$$Y_{lm}(\theta,\varphi) = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}, \tag{8.84}$$

with the help of the addition theorem

$$\frac{2l+1}{4\pi}P_l(\cos\Delta\vartheta_n) = \sum_{m=-l}^{l} Y_{lm}(\theta_n,\varphi_n)Y_{lm}^*(\theta_{n-1},\varphi_{n-1}), \tag{8.85}$$

the sum running over all azimuthal (magnetic) quantum numbers $m$. The right-hand side of $P_l^m(z)$ contains the associated Legendre polynomials

$$P_l^m(z) = \frac{(1-z^2)^{m/2}}{2^l l!}\frac{(l-m)!}{(l+m)!}\frac{d^{l+m}}{dz^{l+m}}(z^2-1)^l, \tag{8.86}$$

which are solutions of the differential equation[3]

$$\left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{m^2}{\sin^2\theta}\right] P_l^m(\cos\theta) = l(l+1)P_l^m(\cos\theta). \tag{8.87}$$

[3]Note that $y_l^m(\cos\theta) = \sqrt{\sin\theta}P_l^m(\cos\theta)$ satisfies $\left[-\frac{d^2}{d\theta^2} - \frac{1}{4} + \frac{m^2-1/4}{\sin^2\theta}\right] y_l^m = l(l+1)y_l^m$. This differential equation will be used later in Eq. (8.197).

Thus, the expansion (8.83) becomes

$$e^{h\cos\Delta\vartheta_n} = \sqrt{\frac{\pi}{2h}} 4\pi \sum_{l=0}^{\infty} I_{l+1/2}(h) \sum_{m=-l}^{l} Y_{lm}(\theta_n, \varphi_n) Y_{lm}^*(\theta_{n-1}, \varphi_{n-1}). \tag{8.88}$$

Inserted into (8.78), it leads to the time-sliced path integral

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) \approx\; & \frac{4\pi}{\sqrt{2\pi\hbar\epsilon/M}^3} \prod_{n=1}^{N} \left[\int_0^\infty \frac{dr_n r_n^2 d\cos\theta_n d\varphi_n 4\pi}{\sqrt{2\pi\hbar\epsilon/M}^3}\right] \\ \times & \prod_{n=1}^{N+1} \left[\left(\frac{\hbar\epsilon\pi}{2Mr_nr_{n-1}}\right)^{1/2} \sum_{l_n=0}^{\infty} \sum_{m_n=-l_n}^{l_n} I_{l_n+1/2}\left(\frac{M}{\hbar\epsilon} r_n r_{n-1}\right)\right. \\ \times & \left. Y_{l_nm_n}(\theta_n, \varphi_n) Y_{l_nm_n}^*(\theta_{n-1}, \varphi_{n-1})\right] \exp\left\{-\frac{\epsilon}{\hbar} \sum_{n=1}^{N+1} \left[\frac{M}{2} \frac{r_n^2 + r_{n-1}^2}{\epsilon^2} + V(r_n)\right]\right\}. \end{aligned} \tag{8.89}$$

The intermediate $\varphi_n$- and $\cos\theta_n$-integrals can now all be done using the orthogonality relation

$$\int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\varphi\; Y_{lm}^*(\theta, \varphi) Y_{l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'}. \tag{8.90}$$

Each $\varphi_n$-integral yields a product of Kronecker symbols $\delta_{l_nl_{n-1}}\delta_{m_nm_{n-1}}$. Only the initial and the final spherical harmonics survive, $Y_{l_{N+1}m_{N+1}}$ and $Y_{l_0m_0}^*$, since they are not subject to integration. Thus we arrive at the angular momentum decomposition

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r_br_a} (r_b\tau_b|r_a\tau_a)_l Y_{lm}(\theta_b, \varphi_b) Y_{lm}^*(\theta_a, \varphi_a), \tag{8.91}$$

with the radial amplitude

$$\begin{aligned}(r_b\tau_b|r_a\tau_a)_l \approx & \frac{4\pi r_br_a}{\sqrt{2\pi\hbar\epsilon/M}^3} \prod_{n=1}^{N} \left[\int \frac{dr_n r_n^2 4\pi}{\sqrt{2\pi\hbar\epsilon/M}^3}\right] \prod_{n=1}^{N+1} \left[\frac{\hbar\epsilon}{2Mr_nr_{n-1}}\right] \\ & \times \prod_{n=1}^{N+1} \left[\tilde{I}_{l+1/2}\left(\frac{M}{\hbar\epsilon} r_n r_{n-1}\right)\right] \exp\left\{-\frac{\epsilon}{\hbar} \sum_{n=1}^{N+1} \left[\frac{M}{2\epsilon^2}(r_n - r_{n-1})^2 + V(r_n)\right]\right\}. \end{aligned} \tag{8.92}$$

The factors $\prod_a^N r_n^2 / \prod_a^{N+1} r_nr_{n-1}$ pile up to $1/r_br_a$ and cancel the prefactor $r_br_a$. Together with the remaining product, the integration measure takes the usual one-dimensional form

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/M}} \prod_{n=1}^{N} \left[\int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}}\right]. \tag{8.93}$$

If we were to use here the large-argument limit (8.24) of the Bessel function, the integrand would become $\exp(-\mathcal{A}_l^N/\hbar)$, with the time-sliced radial action

$$\mathcal{A}_l^N = \epsilon \sum_{n=1}^{N+1} \left[\frac{M}{2\epsilon^2}(r_n - r_{n-1})^2 + \frac{\hbar^2}{2M} \frac{l(l+1)}{r_nr_{n-1}} + V(r_n)\right]. \tag{8.94}$$

The associated radial path integral

$$(r_b\tau_b|r_a\tau_a)_l \approx \frac{1}{\sqrt{2\pi\epsilon\hbar/M}}\prod_{n=1}^{N}\left[\int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}}\right]\exp\left(-\frac{1}{\hbar}\mathcal{A}_l^N\right) \tag{8.95}$$

agrees precisely with what would have been obtained by naively time-slicing the continuum path integral

$$(r_b\tau_b|r_a\tau_a)_l = \int_0^\infty \mathcal{D}r(\tau)e^{-\frac{1}{\hbar}\mathcal{A}_l[r]}, \tag{8.96}$$

with the radial action

$$\mathcal{A}_l[r] = \int_{\tau_a}^{\tau_b} d\tau\left[\frac{M}{2}\dot{r}^2 + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2} + V(r)\right]. \tag{8.97}$$

In particular, this would contain the correct centrifugal barrier

$$V_{\text{cf}} = \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}. \tag{8.98}$$

However, as we know from the discussion in Section 8.2, Eq. (8.95) is incorrect and must be replaced by (8.92), due to the non uniformity of the continuum limit $\epsilon \to 0$ in the integrand of (8.92).

## 8.5.2 *D* Dimensions

The generalization to $D$ dimensions is straightforward. The main place where the dimension enters is the expansion of

$$e^{h\cos\Delta\vartheta_n} = e^{-\frac{M}{\hbar\epsilon}r_n r_{n-1}\cos\Delta\vartheta_n}, \tag{8.99}$$

in which $\Delta\vartheta_n$ is the relative angle between $D$-dimensional vectors $\mathbf{x}_n$ and $\mathbf{x}_{n-1}$. The expansion reads [compare with (8.5) and (8.83)]

$$e^{h\cos\Delta\vartheta_n} = \sum_{l=0}^{\infty} a_l(h)\frac{l+D/2-1}{D/2-1}\frac{1}{S_D}C_l^{(D/2-1)}(\cos\Delta\vartheta_n), \tag{8.100}$$

where $S_D$ is the surface of a unit sphere in $D$ dimensions (1.558), and

$$\begin{aligned} a_l(h) &\equiv (2\pi)^{D/2}h^{1-D/2}I_{l+D/2-1}(h) \\ &\equiv e^h\tilde{a}_l(h) = e^h\left(\frac{2\pi}{h}\right)^{(D-1)/2}\tilde{I}_{l+D/2-1}(h). \end{aligned} \tag{8.101}$$

The functions $C_l^{(\alpha)}(\cos\vartheta)$ are the ultra-spherical Gegenbauer polynomials, defined by the expansion

$$\frac{1}{(1-2t\alpha+\alpha^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}\alpha^n. \tag{8.102}$$

The expansion (8.100) follows from the completeness of the polynomials $C_l^{(\nu)}(\cos\vartheta)$ at fixed $\nu$, using the integration formulas[4]

$$\int_0^\pi d\vartheta \sin^\nu \vartheta e^{h\cos\vartheta} C_l^{(\nu)}(\cos\vartheta) = \pi \frac{2^{1-\nu}\Gamma(2\nu+l)}{l!\Gamma(\nu)} h^{-\nu} I_{\nu+l}(h), \tag{8.103}$$

$$\int_0^\pi d\vartheta \sin^\nu \vartheta C_l^{(\nu)}(\cos\vartheta) C_{l'}^{(\nu)}(\cos\vartheta) = \pi \frac{2^{1-2\nu}\Gamma(2\nu+l)}{l!(l+\nu)\Gamma(\nu)^2}\delta_{ll'}. \tag{8.104}$$

The Gegenbauer polynomials are related to Jacobi polynomials, which are defined in terms of hypergeometric functions (1.453) by[5]

$$P_l^{(\alpha,\beta)}(z) \equiv \frac{1}{l!}\frac{\Gamma(l+1+\beta)}{\Gamma(1+\beta)} F(-l, l+1+\alpha+\beta; 1+\beta; (1-z)/2). \tag{8.105}$$

The relation is

$$C_l^{(\nu)}(z) = \frac{\Gamma(2\nu+l)\Gamma(\nu+1/2)}{\Gamma(2\nu)\Gamma(\nu+l+1/2)} P_l^{(\nu-1/2,\nu-1/2)}(z). \tag{8.106}$$

This follows from the relation[6]

$$C_l^{(\nu)}(z) = \frac{1}{l!}\frac{\Gamma(l+2\nu)}{\Gamma(2\nu)} F(-l, l+2\nu; 1/2+\nu; (1-z)/2). \tag{8.107}$$

For $D=2$ and 3, one has[7]

$$\lim_{\nu\to 0}\frac{1}{\nu} C_l^{(\nu)}(\cos\vartheta) = \frac{1}{2l}\cos l\vartheta, \tag{8.108}$$

$$C_l^{(1/2)}(\cos\vartheta) = P_l^{(0,0)}(\cos\vartheta) = P_l(\cos\vartheta), \tag{8.109}$$

and the expansion (8.100) reduces to (8.5) and (8.7), respectively.
For $D=4$

$$C_l^{(1)}(\cos\vartheta) = \frac{\sin(l+1)\beta}{\sin\beta}. \tag{8.110}$$

According to an addition theorem, the Gegenbauer polynomials can be decomposed into a sum of pairs of $D$-dimensional ultra-spherical harmonics $Y_{l\mathbf{m}}(\hat{\mathbf{x}})$.[8] The label $\mathbf{m}$ stands collectively for the set of *magnetic* quantum numbers

[4] I. S. Gradshteyn and I. M. Ryzhik, op. cit., Formulas 7.321 and 7.313.
[5] M. Abramowitz and I. Stegun, *op. cit.*, Formula 15.4.6.
[6] ibid., Formula 15.4.5.
[7] I.S. Gradshteyn and I.M. Ryzhik, op. cit.,*ibid.*, Formula 8.934.4.
[8] See H. Bateman, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953, Vol. II, Ch. XI; N.H. Vilenkin, *Special Functions and the Theory of Group Representations*, Am. Math. Soc., Providence, RI, 1968.

$m_1, m_2, m_3, ..., m_{D-1}$ with $1 \leq m_1 \leq m_2 \leq \ldots \leq |m_{D-2}|$. The direction $\hat{\mathbf{x}}$ of a vector $\mathbf{x}$ is specified by $D-1$ polar angles

$$\begin{aligned} \hat{x}^1 &= \sin\varphi_{D-1}\cdots\sin\varphi_1, \\ \hat{x}^2 &= \sin\varphi_{D-1}\cdots\cos\varphi_1, \\ &\vdots \\ \hat{x}^D &= \cos\varphi_{D-1}, \end{aligned} \tag{8.111}$$

with the ranges

$$0 \leq \varphi_1 < 2\pi, \tag{8.112}$$

$$0 \leq \varphi_i < \pi, \quad i \neq 1. \tag{8.113}$$

The ultra-spherical harmonics $Y_{l\mathbf{m}}(\hat{\mathbf{x}})$ form an orthonormal and complete set of functions on the $D$-dimensional unit sphere. For a fixed quantum number $l$ of total angular momentum, the label $\mathbf{m}$ can take

$$d_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!} \tag{8.114}$$

different values. The functions are orthonormal,

$$\int d\hat{\mathbf{x}}\; Y^*_{l\mathbf{m}}(\hat{\mathbf{x}})Y_{l'\mathbf{m}'}(\hat{\mathbf{x}}) = \delta_{ll'}\delta_{\mathbf{m}\mathbf{m}'}, \tag{8.115}$$

with $\int d\hat{\mathbf{x}}$ denoting the integral over the surface of the unit sphere:

$$\int d\hat{\mathbf{x}} \;=\int d\varphi_{D-1}\sin^{D-2}\varphi_{D-1}\int d\varphi_{D-2}\sin^{D-3}\varphi_{D-2}\cdots\int d\varphi_2\sin\varphi_2\int d\varphi_1\;. \tag{8.116}$$

By evaluating this integral over a unit integrand[9] we find $S_D = 2\pi^{D/2}/\Gamma(D/2)$ as anticipated in Eq. (1.558). Since $Y_{\mathbf{00}}(\hat{\mathbf{x}})$ is independent of $\hat{\mathbf{x}}$, the integral (8.115) implies that $Y_{\mathbf{00}}(\hat{\mathbf{x}}) = 1/\sqrt{S_D}$.

Note that the integral over the unit sphere in $D$-dimensions can be decomposed recursively into an angular integration with respect to any selected direction, say $\hat{\mathbf{u}}$, in the space followed by an integral over a sphere of radius $\sin\varphi_{D-1}$ in the remaining $D-1$-dimensional space to $\hat{\mathbf{u}}$. If $\hat{\mathbf{x}}^\perp$ denotes the unit vector covering the directions in this remaining space, one decomposes $\hat{\mathbf{x}} = (\cos\varphi_D\hat{\mathbf{u}} + \sin\varphi_D\hat{\mathbf{x}})$, and can factorize the integral measure as

$$\int d^{D-1}\hat{\mathbf{x}} \;=\int d\varphi_{D-1}\sin^{D-2}\varphi_{D-1}\int d^{D-2}\hat{\mathbf{x}}_\perp\;. \tag{8.117}$$

For clarity, the dimensionalities of initial and remaining surfaces are marked as superscripts on the measure symbols $d^{D-1}\hat{\mathbf{x}}$ and $d^{D-1}\hat{\mathbf{x}}^\perp$.

[9]With the help of the integral formula $\int_0^\pi d\varphi\,\sin^k\varphi = \sqrt{\pi}\Gamma((k+1)/2)/\Gamma((k+2)/2)$ we find $S_D = \prod_{k=0}^{D-1}\int_0^\pi d\varphi_k\,\sin^k\varphi_k = 2\pi^{D/2}\prod_{k=1}^{D-2}\Gamma((k+1)/2)/\prod_{k=1}^{D-2}\Gamma((k+2)/2) = 2\pi^{D/2}/\Gamma(D/2)$.

For the surface of the sphere, this corresponds to the recursion relation

$$S_D = \frac{\sqrt{\pi}\Gamma((D-1)/2)}{\Gamma(D/2)} \times S_{D-1}, \tag{8.118}$$

which is solved by $S_D = 2\pi^{D/2}/\Gamma(D)$.

In four dimensions, the unit vectors $\hat{\mathbf{x}}$ have a parametrization in terms of polar angles

$$\hat{\mathbf{x}} = (\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\varphi, \sin\theta\sin\psi\sin\varphi), \tag{8.119}$$

with the integration measure

$$d\hat{\mathbf{x}} = d\theta\sin^2\theta d\psi\sin\psi\, d\varphi. \tag{8.120}$$

It is, however, more convenient to go over to another parametrization in terms of the three Euler angles which are normally used in the kinematic description of the spinning top. In terms of these, the unit vectors have the components

$$\begin{aligned} \hat{x}^1 &= \cos(\theta/2)\cos[(\varphi+\gamma)/2], \\ \hat{x}^2 &= -\cos(\theta/2)\sin[(\varphi+\gamma)/2], \\ \hat{x}^3 &= \sin(\theta/2)\cos[(\varphi-\gamma)/2], \\ \hat{x}^4 &= \sin(\theta/2)\sin[(\varphi-\gamma)/2], \end{aligned} \tag{8.121}$$

with the angles covering the intervals

$$\theta \in [0,\pi), \quad \varphi \in [0, 2\pi), \quad \gamma \in [-2\pi, 2\pi). \tag{8.122}$$

We have renamed the usual Euler angles $\alpha, \beta, \gamma$ introduced in Section 1.15 calling them $\varphi, \theta, \gamma$, since the formulas to be derived for them will be used in a later application in Chapter 13 [see Eq. (13.102)]. There the first two Euler angles coincide with the polar angles $\varphi, \theta$ of a position vector in a three-dimensional space. It is important to note that for a description of the entire surface of the sphere, the range of the angle $\gamma$ must be twice as large as for the classical spinning top. The associated group space belongs to the covering group, of the rotation group which is equivalent to the group of unimodular matrices in two dimensions called SU(2). It is defined by the matrices

$$g(\varphi,\theta,\gamma) = \exp(i\varphi\sigma_3/2)\exp(i\theta\sigma_2/2)\exp(i\gamma\sigma_3/2), \tag{8.123}$$

where $\sigma_i$ are the Pauli spin matrices (1.448). In this parametrization, the integration measure reads

$$d\hat{\mathbf{x}} = \frac{1}{8} d\theta\sin\theta\, d\varphi\, d\gamma. \tag{8.124}$$

When integrated over the surface, the two measures give the same result $S_4 = 2\pi^2$. The Euler parametrization has the advantage of allowing the spherical harmonics in four dimensions to be expressed in terms of the well-known representation functions of the rotation group introduced in (1.445), (1.446):

$$Y_{l,m_1,m_2}(\hat{\mathbf{x}}) = \sqrt{\frac{l+1}{2\pi^2}}\mathcal{D}^{l/2}_{m_1m_2}(\varphi,\theta,\gamma) = \sqrt{\frac{l+1}{2\pi^2}}d^{l/2}_{m_1m_2}(\theta)e^{i(m_1\varphi+m_2\gamma)}. \tag{8.125}$$

For even and odd $l$, the numbers $m_1, m_2$ are both integer or half-integer, respectively.

In arbitrary dimensions $D>2$, the ultra-spherical Gegenbauer polynomials satisfy the following addition theorem

$$\frac{2l+D-2}{D-2}\frac{1}{S_D}C_l^{(D/2-1)}(\cos\Delta\vartheta_n) = \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\hat{\mathbf{x}}_n)Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_{n-1}). \tag{8.126}$$

For $D=3$, this reduces properly to the well-known addition theorem for the spherical harmonics

$$\frac{1}{4\pi}(2l+1)P_l(\cos\Delta\vartheta_n) = \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{x}}_n)Y^*_{lm}(\hat{\mathbf{x}}_{n-1}). \tag{8.127}$$

For $D=4$, it becomes[10]

$$\frac{l+1}{2\pi^2}C_l^{(1)}(\cos\Delta\vartheta_n) = \frac{l+1}{2\pi^2}\sum_{m_1,m_2=-l/2}^{l/2}\mathcal{D}^{l/2}_{m_1m_2}(\varphi_n,\theta_n,\gamma_n)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_{n-1},\theta_{n-1},\gamma_{n-1}), \tag{8.128}$$

where the angle $\Delta\vartheta_n$ is related to the Euler angles of the vectors $\mathbf{x}_n,\ \mathbf{x}_{n-1}$ by

$$\begin{aligned}\cos\Delta\vartheta_n &= \cos(\theta_n/2)\cos(\theta_{n-1}/2)\cos[(\varphi_n-\varphi_{n-1}+\gamma_n-\gamma_{n-1})/2]\\ &\quad+\sin(\theta_n/2)\sin(\theta_{n-1}/2)\cos[(\varphi_n-\varphi_{n-1}-\gamma_n+\gamma_{n-1})/2].\end{aligned} \tag{8.129}$$

Using (8.126), we can rewrite the expansion (8.100) in the form

$$e^{h(\cos\Delta\vartheta_n-1)} = \sum_{l=0}^{\infty}\tilde{a}_l(h)\sum_{\mathbf{m}} Y_{l\mathbf{m}}(\hat{\mathbf{x}}_n)Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_{n-1}). \tag{8.130}$$

This is now valid for any dimension $D$, including the case $D=2$ where the left-hand side of (8.126) involves the limiting procedure (8.108). We shall see in Chapter 9 in connection with Eq. (9.61) that it also makes sense to apply this expansion to the case $D=1$ where the "partial-wave expansion" degenerates into a separation of even and odd wave functions. In four dimensions, we shall mostly prefer the expansion

$$e^{h(\cos\Delta\vartheta_n-1)} = \sum_{l=0}^{\infty}\tilde{a}_l(h)\frac{l+1}{2\pi^2}\sum_{m_1,m_2=-l/2}^{l/2}\mathcal{D}^{l/2}_{m_1m_2}(\varphi_n,\theta_n,\gamma_n)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_{n-1},\theta_{n-1},\gamma_{n-1}), \tag{8.131}$$

[10]Note that $C_l^{(1)}(\cos\Delta\vartheta_n)$ coincides with the trace over the representation functions (1.446) of the rotation group, i.e., it is equal to $\sum_{m=-l/2}^{l/2} d^{l/2}_{m,m}(\Delta\vartheta_n)$.

where the sum over $m_1, m_2$ runs for even and odd $l$ over integer and half-integer numbers, respectively.

The reduction of the time evolution amplitude in $D$ dimensions to a radial path integral proceeds from here on in the same way as in two and three dimensions. The generalization of (8.89) reads

$$
\begin{aligned}
(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) \approx & \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^D} \prod_{n=1}^{N} \left[ \int_0^\infty \frac{dr_n r_n^{D-1} d\hat{\mathbf{x}}_n}{\sqrt{2\pi\hbar\epsilon/M}^D} \right] \\
& \times \prod_{n=1}^{N+1} \left[ \left( \frac{2\pi\hbar\epsilon}{M r_n r_{n-1}} \right)^{(D-1)/2} \sum_{l_n=0}^{\infty} \tilde{I}_{D/2-1+l_n} \left( \frac{M}{\hbar\epsilon} r_n r_{n-1} \right) \right. \\
& \times \left. \sum_{\mathbf{m}_n} Y_{l_n \mathbf{m}_n}(\hat{\mathbf{x}}_n) Y^*_{l_n \mathbf{m}_n}(\hat{\mathbf{x}}_{n-1}) \right] \exp\left\{ -\frac{\epsilon}{\hbar} \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon^2} (r_n - r_{n-1})^2 + V(r_n) \right] \right\}.
\end{aligned}
\tag{8.132}
$$

By performing the angular integrals and using the orthogonality relations (8.115), the product of sums over $l_n, \mathbf{m}_n$ reduces to a single sum over $l, \mathbf{m}$, just as in the three-dimensional amplitude (8.91). The result is the spherical decomposition

$$
(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \frac{1}{(r_b r_a)^{(D-1)/2}} \sum_{l=0}^{\infty} (r_b\tau_b|r_a\tau_a)_l \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\hat{\mathbf{x}}_b) Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_a),
\tag{8.133}
$$

where $(r_b\tau_b|r_a\tau_a)_l$ is the purely radial amplitude

$$
(r_b\tau_b|r_a\tau_a)_l \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}} \prod_{n=1}^{N} \left[ \int_0^\infty \frac{dr_n}{\sqrt{2\pi\hbar\epsilon/M}} \right] \exp\left\{ -\frac{1}{\hbar} \mathcal{A}_l^N[r] \right\},
\tag{8.134}
$$

with the time-sliced action

$$
\mathcal{A}_l^N[r] \;=\; \epsilon \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon^2} (r_n - r_{n-1})^2 - \frac{\hbar}{\epsilon} \log \tilde{I}_{l+D/2-1} \left( \frac{M}{\hbar\epsilon} r_n r_{n-1} \right) + V(r_n) \right].
\tag{8.135}
$$

As before, the product $\prod_{n=1}^{N+1} 1/(r_n r_{n-1})^{(D-1)/2}$ has removed the product $\prod_{n=1}^{N} r_n^{D-1}$ in the measure as well as the factor $(r_b r_a)^{(D-1)/2}$ in front of it, leaving only the standard one-dimensional measure of integration.

In the continuum limit $\epsilon \to 0$, the asymptotic expression (8.24) for the Bessel function brings the action to the form

$$
\mathcal{A}_l^N[r, \bar{r}] \approx \epsilon \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon^2} (r_n - r_{n-1})^2 + \frac{\hbar^2}{2M} \frac{(l + D/2 - 1)^2 - 1/4}{r_n r_{n-1}} + V(r_n) \right].
\tag{8.136}
$$

This looks again like the time-sliced version of the radial path integral in $D$ dimensions

$$
(r_b\tau_b|r_a\tau_a)_l = \int \mathcal{D}r(\tau) \exp\left\{ -\frac{1}{\hbar} \mathcal{A}_l[r] \right\},
\tag{8.137}
$$

with the continuum action

$$\mathcal{A}_l[r] = \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{M}{2}\dot{r}^2 + \text{``}\frac{\hbar^2}{2M}\frac{(l+D/2-1)^2-1/4\text{''}}{r^2} + V(r) \right]. \tag{8.138}$$

As in Eq. (8.50), we have written the centrifugal barrier as

$$\text{``}\frac{\hbar^2}{2Mr^2}[(l+D/2-1)^2-1/4]\text{''}, \tag{8.139}$$

to emphasize the subtleties of the time-sliced radial path integral, with the understanding that the time-sliced barrier reads [as in (8.51)]

$$\text{``}\frac{\epsilon\hbar^2}{2Mr_nr_{n-1}}[(l+D/2-1)^2-1/4)]\text{''} \equiv -\hbar\log\tilde{I}_{l+D/2-1}(\frac{M}{\hbar\epsilon}r_nr_{n-1}). \tag{8.140}$$

## 8.6 Radial Path Integral for Harmonic Oscillator and Free Particle in $D$ Dimensions

For the harmonic oscillator and the free particle, there is no need to perform the radial path integral (8.134) with the action (8.135). As in (8.38), we simply take the known amplitude in $D$ dimensions, (2.177), continue it to imaginary times $t = -i\tau$, and expand it with the help of (8.130):

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) &= \frac{1}{\sqrt{2\pi\hbar/M}^D}\sqrt{\frac{\omega}{\sinh[\omega(\tau_b-\tau_a)]}}^D \\ &\times \exp\left\{-\frac{1}{\hbar}\frac{M\omega}{\sinh[\omega(\tau_b-\tau_a)]}(r_b^2+r_a^2)\cosh[\omega(\tau_b-\tau_a)]\right\} \\ &\times \sum_{l=0}^{\infty} a_l\left(\frac{M\omega r_br_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right)\sum_{\mathbf{m}} Y_{l\mathbf{m}}(\hat{\mathbf{x}}_b)Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_a).\end{aligned} \tag{8.141}$$

Comparing this with Eq. (8.133) and remembering (8.101), we identify the radial amplitude as

$$\begin{aligned}(r_b\tau_b|r_a\tau_a)_l &= \frac{M}{\hbar}\frac{\omega\sqrt{r_br_a}^{D-1}}{\sinh[\omega(\tau_b-\tau_a)]} \\ &\times e^{-(M\omega/2\hbar)\coth[\omega(\tau_b-\tau_a)](r_b^2+r_a^2)}I_{l+D/2-1}\left(\frac{M\omega r_br_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right),\end{aligned} \tag{8.142}$$

generalizing (8.39). The limit $\omega \to 0$ yields the amplitude for a free particle

$$(r_b\tau_b|r_a\tau_a)_l = \frac{M}{\hbar}\frac{\sqrt{r_br_a}^{D-1}}{(\tau_b-\tau_a)}e^{-M(r_b^2+r_a^2)/2\hbar(\tau_b-\tau_a)}I_{l+D/2-1}\left(\frac{Mr_br_a}{\hbar(\tau_b-\tau_a)}\right). \tag{8.143}$$

Comparing this with (8.40) on the one hand and Eqs. (8.140), (8.138) with (8.49), (8.51) on the other hand, we conclude: An analytical continuation in $D$ yields the path integral for a linear oscillator in the presence of an arbitrary $1/r^2$-potential as follows:

$$
\begin{aligned}
(r_b\tau_b|r_a\tau_a)_l &= \int_0^\infty \mathcal{D}r(\tau)\exp\left[-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau\left(\frac{M}{2}\dot{r}^2 + \text{“}\frac{\hbar^2}{2M}\frac{\mu^2-1/4}{r^2}\text{”} + \frac{M}{2}\omega^2 r^2\right)\right] \\
&= \frac{M}{\hbar}\frac{\omega\sqrt{r_b r_a}^{D-1}}{\sinh[\omega(\tau_b-\tau_a)]} e^{-(M\omega/2\hbar)\coth[\omega(\tau_b-\tau_a)](r_b^2+r_a^2)} I_\mu\left(\frac{M\omega r_b r_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right).
\end{aligned}
\tag{8.144}
$$

Here $\mu$ is some strength parameter which initially takes the values $\mu = l + D/2 - 1$ with integer $l$ and $D$. By analytic continuation, the range of validity is extended to all real $\mu > 0$. The justification for the continuation procedure follows from the fact that the integral formula (8.14) holds for arbitrary $m = \mu \geq 0$. The amplitude (8.144) satisfies therefore the fundamental composition law (8.20) for all real $m = \mu \geq 0$. The harmonic oscillator with an arbitrary extra centrifugal barrier potential

$$
V_{\rm extra}(r) = \hbar^2\frac{l_{\rm extra}^2}{2Mr^2} \tag{8.145}
$$

has therefore the radial amplitude (8.144) with

$$
\mu = \sqrt{(l+D/2-1)^2 + l_{\rm extra}^2}\,. \tag{8.146}
$$

For a finite number $N+1$ of time slices, the radial amplitude is known from the angular momentum expansion of the finite-$N$ oscillator amplitude (2.199) in its obvious extension to $D$ dimensions. It can also be calculated directly as in Appendix 2B by a successive integration of (8.132), using formula (8.14). The iteration formulas are the Euclidean analogs of those derived in Appendix 2B, with the prefactor of the amplitude being $2\pi\mathcal{N}_1^2\mathcal{N}_{N+1}^2\sqrt{r_b r_a}$, with the exponent $-a_{N+1}(r_b^2+r_a^2)/\hbar$, and with the argument of the Bessel function $2b_{N+1}r_b r_a/\hbar$. In this way we obtain precisely the expression (8.144), except that $\sinh[\omega(\tau_b-\tau_a)]$ is replaced by $\sinh[\tilde{\omega}(N+1)\epsilon]\epsilon/\sinh\tilde{\omega}\epsilon$ and $\cosh[\omega(\tau_b-\tau_a)]$ by $\cosh[\tilde{\omega}(N+1)\epsilon]$.

## 8.7 Particle *near* the Surface of a Sphere in $D$ Dimensions

With the insight gained in the previous sections, it is straightforward to calculate exactly a certain class of auxiliary path integrals. They involve only angular variables and will be called path integrals of a point particle moving *near* the surface of a sphere in $D$ dimensions. The resulting amplitudes lead eventually to the physically more relevant amplitudes describing the behavior of a particle *on* the surface of a sphere.

On the surface of a sphere of radius $r$, the position of the particle as a function of time is specified by a unit vector $\mathbf{u}(t)$. The Euclidean action is

$$
\mathcal{A} = \frac{M}{2}r^2\int_{\tau_a}^{\tau_b} d\tau\ \dot{\mathbf{u}}^2(\tau). \tag{8.147}
$$

The precise way of time-slicing this action is not known from previous discussions. It cannot be *deduced* from the time-sliced action in Cartesian coordinates, nor from its angular momentum decomposition. A new geometric feature makes the previous procedures inapplicable: The surface of a sphere is a Riemannian space with nonzero intrinsic curvature. Sections 1.13 to 1.15 have shown that the motion in a curved space does not follow the canonical quantization rules of operator quantum mechanics. The same problem is encountered here in another form: Right in the beginning, we are not allowed to time-slice the action (8.147) in a straightforward way. The correct slicing is found in two steps. First we use the experience gained with the angular momentum decomposition of time-sliced amplitudes in a Euclidean space to introduce and solve the earlier mentioned auxiliary time-sliced path integral *near* the surface of the sphere. In a second step we shall implement certain corrections to properly describe the action *on* the sphere. At the end, we have to construct the correct measure of path integration which will not be what one naively expects. To set up the auxiliary path integral *near* the surface of a sphere we observe that the kinetic term of a time slice in $D$ dimensions

$$\frac{M}{2\epsilon}\sum_{n=1}^{N+1}(r_n^2+r_{n-1}^2-2r_nr_{n-1}\cos\Delta\vartheta_n) \tag{8.148}$$

decomposes into radial and angular parts as

$$-\frac{M}{2\epsilon}\sum_{n=1}^{N+1}(r_n^2+r_{n-1}^2-2r_nr_{n-1})+\frac{M}{2\epsilon}\sum_{n=1}^{N+1}2r_nr_{n-1}(1-\cos\Delta\vartheta_n). \tag{8.149}$$

The angular factor can be written as

$$-\frac{M}{2\epsilon}\sum_{n=1}^{N+1}r_nr_{n-1}(\hat{\mathbf{x}}_n-\hat{\mathbf{x}}_{n-1})^2, \tag{8.150}$$

where $\hat{\mathbf{x}}_n$, $\hat{\mathbf{x}}_{n-1}$ are the unit vectors pointing in the directions of $\mathbf{x}_n$, $\mathbf{x}_{n-1}$ [recall (8.111)]. Restricting all radial variables $r_n$ to the surface of a sphere of a fixed radius $r$ and identifying $\hat{\mathbf{x}}$ with $\mathbf{u}$ leads us directly to the time-sliced path integral *near* the surface of the sphere in $D$ dimensions:

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)\approx\frac{1}{\sqrt{2\pi\hbar\epsilon/Mr^2}^{D-1}}\prod_{n=1}^{N}\left[\int\frac{d\mathbf{u}_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}^{D-1}}\right]\exp\left(-\frac{1}{\hbar}\mathcal{A}^N\right), \tag{8.151}$$

with the sliced action

$$\mathcal{A}^N=\frac{M}{2\epsilon}r^2\sum_{n=1}^{N+1}(\mathbf{u}_n-\mathbf{u}_{n-1})^2. \tag{8.152}$$

The measure $d\mathbf{u}_n$ denotes infinitesimal surface elements on the sphere in $D$ dimensions [recall (8.116)]. Note that although the endpoints $\mathbf{u}_n$ lie all on the sphere, the paths remain only *near* the sphere since the path sections between the points leave the surface and traverse the embedding space along a straight line. This will be studied further in Section 8.8.

As mentioned above, this amplitude can be solved exactly. In fact, for each time interval $\epsilon$, the exponential

$$\exp\left[-\frac{Mr^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right]=\exp\left[-\frac{Mr^2}{\hbar\epsilon}(1-\cos\Delta\vartheta_n)\right] \tag{8.153}$$

can be expanded into spherical harmonics according to formulas (8.100)–(8.101),

$$\begin{aligned}\exp\left[-\frac{Mr^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right]&=\sum_{l=0}^{\infty}\tilde{a}_l(h)\frac{l+D/2-1}{D/2-1}\frac{1}{S_D}C_l^{(D/2-1)}(\cos\Delta\vartheta_n)\\&=\sum_{l=0}^{\infty}\tilde{a}_l(h)\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\mathbf{u}_n)Y^*_{l\mathbf{m}}(\mathbf{u}_{n-1}),\end{aligned} \tag{8.154}$$

where

$$\tilde{a}_l(h)=\left(\frac{2\pi}{h}\right)^{(D-1)/2}\tilde{I}_{l+D/2-1}(h),\quad h=\frac{Mr^2}{\hbar\epsilon}. \tag{8.155}$$

For each adjacent pair $(n+1,n),(n,n-1)$ of such factors in the sliced path integral, the integration over the intermediate $\mathbf{u}_n$ variable can be done using the orthogonality relation (8.115). In this way, (8.151) produces the time-sliced amplitude

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)=\left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2}\sum_{l=0}^{\infty}\tilde{a}_l(h)^{N+1}\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\mathbf{u}_b)Y^*_{l\mathbf{m}}(\mathbf{u}_a). \tag{8.156}$$

We now go to the continuum limit $N\to\infty$, $\epsilon=(\tau_b-\tau_a)/(N+1)\to 0$, where [recall (8.11)]

$$\begin{aligned}\left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2}\tilde{a}_l(h)^{N+1}&=\left[\tilde{I}_{l+D/2-1}\left(\frac{Mr^2}{\hbar\epsilon}\right)\right]^{N+1}\\&\xrightarrow{\epsilon\to 0}\exp\left\{-(\tau_b-\tau_a)\hbar\frac{(l+D/2-1)^2-1/4}{2Mr^2}\right\}.\end{aligned} \tag{8.157}$$

Thus, the final time evolution amplitude for the motion *near* the surface of the sphere is

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)=\sum_{l=0}^{\infty}\exp\left[-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right]\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\mathbf{u}_b)Y^*_{l\mathbf{m}}(\mathbf{u}_a), \tag{8.158}$$

with

$$L_2\equiv(l+D/2-1)^2-1/4. \tag{8.159}$$

For $D=3$, this amounts to an expansion in terms of associated Legendre polynomials

$$\begin{aligned}(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)=&\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\exp\left\{-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right\}\\&\times\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_b)P_l^m(\cos\theta_a)e^{im(\varphi_b-\varphi_a)}.\end{aligned} \tag{8.160}$$

If the initial point lies at the north pole of the sphere, this simplifies to

$$(\mathbf{u}_b\tau_b|\hat{\mathbf{z}}_a\tau_a) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \exp\left[-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right] P_l(\cos\theta_b)P_l(1), \tag{8.161}$$

where $P_l(1)=1$. By rotational invariance the same result holds for arbitrary directions of $\mathbf{u}_a$, if $\theta_b$ is replaced by the difference angle $\vartheta$ between $\mathbf{u}_b$ and $\mathbf{u}_a$.
In four dimensions, the most convenient expansion uses again the representation functions of the rotation group, so that (8.158) reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_{l=0}^{\infty} \exp\left[-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right] \tag{8.162}$$
$$\times \frac{l+1}{2\pi^2} \sum_{m_1,m_2=-l/2}^{l/2} \mathcal{D}^{l/2}_{m_1m_2}(\varphi_b,\theta_b,\gamma_b)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_a,\theta_a,\gamma_a).$$

These results will be needed in Sections 8.9 and 10.4 to calculate the amplitudes *on* the surface of a sphere. First, however, we extract some more information from the amplitudes *near* the surface of the sphere.

## 8.8 Angular Barriers *near* the Surface of a Sphere

In Section 8.5 we have projected the path integral of a free particle in three dimensions into a state of fixed angular momentum $l$ finding a radial path integral containing a singular potential, the centrifugal barrier. This could not be treated via the standard time-slicing formalism. The projection of the path integral, however, supplied us with a valid time-sliced action and yielded the correct amplitude. A similar situation occurs if we project the path integral near the surface of a sphere into a fixed azimuthal quantum number $m$. The physics very near the poles of a sphere is almost the same as that on the tangential surfaces at the poles. Thus, at a fixed two-dimensional angular momentum, the tangential surfaces contain centrifugal barriers. We expect analogous centrifugal barriers at a fixed azimuthal quantum number $m$ near the poles of a sphere at a fixed azimuthal quantum number $m$. These will be called *angular barriers*.

### 8.8.1 Angular Barriers in Three Dimensions

Consider first the case $D=3$ where the azimuthal decomposition is

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_m (\sin\theta_b\tau_b|\sin\theta_a\tau_a)_m \frac{1}{2\pi} e^{im(\varphi_b-\varphi_a)}. \tag{8.163}$$

It is convenient to introduce also the differently normalized amplitude

$$(\theta_b\tau_b|\theta_a\tau_a)_m \equiv \sqrt{\sin\theta_b\sin\theta_a}(\sin\theta_b\tau_b|\sin\theta_a\tau_a)_m, \tag{8.164}$$

in terms of which the expansion reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_m \frac{1}{\sqrt{\sin\theta_b \sin\theta_a}}(\theta_b\tau_b|\theta_a\tau_a)_m \frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}. \tag{8.165}$$

While the amplitude $(\sin\theta_b\tau_b|\sin\theta_a\tau_a)_m$ has the equal-time limit

$$(\sin\theta_b\tau|\sin\theta_a\tau)_m = \frac{1}{\sin\theta_a}\delta(\theta_b-\theta_a) \tag{8.166}$$

corresponding to the invariant measure of the $\theta$-integration on the surface of the sphere $\int d\theta \sin\theta$, the new amplitude $(\theta_b\tau_b|\theta_a\tau)_m$ has the limit

$$(\theta_b\tau|\theta_a\tau)_m = \delta(\theta_b-\theta_a) \tag{8.167}$$

with a simple $\delta$-function, just as for a particle moving on the coordinate interval $\theta \in (0, 2\pi)$ with an integration measure $\int d\theta$. The renormalization is analogous to that of the radial amplitudes in (8.9).

The projected amplitude can immediately be read off from Eq. (8.158):

$$\begin{aligned}(\theta_b\tau_b|\theta_a\tau_a)_m &= \sqrt{\sin\theta_b \sin\theta_a} \\ &\times \sum_{l=m}^{\infty} \exp\left[-\frac{\hbar l(l+1)}{2Mr^2}(\tau_b-\tau_a)\right] 2\pi Y_{lm}(\theta_b,0)Y^*_{lm}(\theta_a,0).\end{aligned} \tag{8.168}$$

In terms of associated Legendre polynomials [recall (8.84)], this reads

$$\begin{aligned}(\theta_b\tau_b|\theta_a\tau_a)_m = \sqrt{\sin\theta_b \sin\theta_a} \sum_{l=m}^{\infty} \exp\left\{-\frac{\hbar l(l+1)}{2Mr^2}(\tau_b-\tau_a)\right\} \\ \times \frac{(2l+1)}{2}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_b)P_l^m(\cos\theta_a).\end{aligned} \tag{8.169}$$

Let us look at the time-sliced path integral associated with this amplitude. We start from Eq. (8.151) for $D = 3$,

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) \approx \frac{1}{2\pi\hbar\epsilon/Mr^2} \prod_{n=1}^{N}\left[\int \frac{d\cos\theta_n d\varphi_n}{2\pi\hbar\epsilon/Mr^2}\right] \exp\left(-\frac{1}{\hbar}\mathcal{A}^N\right), \tag{8.170}$$

and use the addition theorem

$$\cos\Delta\vartheta_n = \cos\theta_n\cos\theta_{n-1} + \sin\theta_n\sin\theta_{n-1}\cos(\varphi_n-\varphi_{n-1}) \tag{8.171}$$

to expand the exponent as

$$\begin{aligned}\exp\left[-\frac{Mr^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right] &= \exp\left[-\frac{Mr^2}{\hbar\epsilon}(1-\cos\Delta\vartheta_n)\right] \\ = \exp&\left[-\frac{Mr^2}{\hbar\epsilon}(1-\cos\theta_n\cos\theta_{n-1}-\sin\theta_n\sin\theta_{n-1})\right] \\ \times&\frac{1}{\sqrt{2\pi h_n}}\sum_{m=-\infty}^{\infty}\tilde{I}_m(h_n)e^{im(\varphi_n-\varphi_{n-1})},\end{aligned} \tag{8.172}$$

where $h_n$ is defined as

$$h_n \equiv \frac{Mr^2}{\hbar\epsilon} \sin\theta_n \sin\theta_{n-1}. \tag{8.173}$$

By doing successively the $\varphi_n$-integrations, we wind up with the path integral for the projected amplitude

$$(\theta_b\tau_b|\theta_a\tau_a)_m \approx \frac{1}{\sqrt{2\pi\epsilon\hbar/Mr^2}} \prod_{n=1}^{N} \left[\int_0^\pi \frac{d\theta_n}{\sqrt{2\pi\epsilon\hbar/Mr^2}}\right] \exp\left(-\frac{1}{\hbar}\mathcal{A}_m^N\right), \tag{8.174}$$

where $\mathcal{A}_m^N$ is the sliced action

$$\mathcal{A}_m^N = \sum_{n=1}^{N+1} \left\{\frac{Mr^2}{\epsilon}[1-\cos(\theta_n-\theta_{n-1})] - \hbar \log \tilde{I}_m(h_n)\right\}. \tag{8.175}$$

For small $\epsilon$, this can be approximated (setting $\Delta\theta_n \equiv \theta_n - \theta_{n-1}$) by

$$\mathcal{A}_m^N \approx \epsilon \sum_{n=1}^{N+1} \left\{\frac{Mr^2}{2\epsilon^2}\left[(\Delta\theta_n)^2 - \frac{1}{12}(\Delta\theta_n)^4 + \ldots\right] + \frac{\hbar^2}{2Mr^2}\frac{m^2-1/4}{\sin\theta_n \sin\theta_{n-1}}\right\}, \tag{8.176}$$

with the continuum limit

$$\mathcal{A}_m = \int_{\tau_a}^{\tau_b} d\tau \left(\frac{Mr^2}{2}\dot{\theta}^2 - \frac{\hbar^2}{8Mr^2} + \frac{\hbar^2}{2Mr^2}\frac{m^2-1/4}{\sin^2\theta}\right). \tag{8.177}$$

This action has a $1/\sin^2\theta$ -singularity at $\theta = 0$ and $\theta = \pi$, i.e., at the north and south poles of the sphere, whose similarity with the $1/r^2$-singularity of the centrifugal barrier justifies the name "angular barriers".

By analogy with the problems discussed in Section 8.2, the amplitude (8.174) with the naively time-sliced action (8.176) does not exist for $m = 0$, this being the path collapse problem to be solved in Chapter 12. With the full time-sliced action (8.175), however, the path integral is stable for all $m$. In this stable expression, the successive integration of the intermediate variables using formula (8.14) gives certainly the correct result (8.169).

To do such a calculation, we start out from the product of integrals (8.174) and expand in each factor $I_m(h_n)$ with the help of the addition theorem

$$\begin{aligned}
&\sqrt{\frac{2\zeta}{\pi}} e^{\zeta\cos\theta_n\cos\theta_{n-1}} I_m(\zeta\sin\theta_n\sin\theta_{n-1}) \\
&\qquad = \sum_{l=m}^{\infty} I_{l+1/2}(\zeta)(2l+1)\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_n)P_l^m(\cos\theta_{n-1}),
\end{aligned} \tag{8.178}$$

where $\zeta \equiv Mr^2/\hbar\epsilon$. This theorem follows immediately from a comparison of two expansions

$$\begin{aligned}
e^{-\zeta(1-\cos\Delta\theta_n)} &= e^{-\zeta[1-\cos\theta_n\cos\theta_{n-1}-\sin\theta_n\sin\theta_{n-1}\cos(\varphi_n-\varphi_{n-1})]} \qquad (8.179)\\
&\times \frac{1}{\sqrt{2\pi\zeta\sin\theta_n\sin\theta_{n-1}}} \sum_{m=-\infty}^{\infty} \tilde{I}_m(\zeta\sin\theta_n\sin\theta_{n-1}) e^{im(\varphi_n-\varphi_{n-1})},\\
e^{-\zeta(1-\cos\Delta\theta_n)} &= e^{-\zeta}\sqrt{\frac{\pi}{2\zeta}} \sum_{l=0}^{\infty}(2l+1)I_{l+1/2}(\zeta)P_l(\cos\Delta\theta_n). \qquad (8.180)
\end{aligned}$$

The former is obtained with the help (8.5), the second is taken from (8.83). After the comparison, the Legendre polynomialis expanded via the addition theorem (8.85), which we rewrite with (8.84) as

$$P_l(\cos\Delta\theta_n) = \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\theta_n) P_l^m(\theta_{n-1}) e^{im(\varphi_n - \varphi_{n-1})}. \tag{8.181}$$

We now recall the orthogonality relation (8.50), rewritten as

$$\int_{-1}^{1} \frac{d\cos\theta}{\sin^2\theta} P_l^m(\cos\theta) P_{l'}^m(\cos\theta) = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{ll'}. \tag{8.182}$$

This allows us to do all angular integrations in (8.175). The result

$$(\theta_b\tau_b|\theta_a\tau_a)_m = \sqrt{\sin\theta_b \sin\theta_a} \sum_{l=m}^{\infty} [\tilde{I}_{m+l+1/2}(\zeta)]^{N+1} \times \frac{(2l+1)}{2}\frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta_b) P_l^m(\cos\theta_a) \tag{8.183}$$

is the solution of the time-sliced path integral (8.174).

In the continuum limit, $[\tilde{I}_{m+l+1/2}(\zeta)]^{N+1}$ is dominated by the leading asymptotic term of (8.12) so that

$$[\tilde{I}_{m+l+1/2}(\zeta)]^{N+1} \approx \exp\left[-\frac{\hbar}{2Mr^2} L_2(\tau_b - \tau_a)\right], \tag{8.184}$$

leading to the previously found expression (8.169).

We have gone through this calculation in detail for the following purpose. Later applications will require an analytic continuation of the path integral from integer values of $m$ to arbitrary real values $\mu \geq 0$. With the present calculation, such a continuation is immediately possible by rewriting (8.183) with the help of the relation

$$P_l^m(z) = (-)^m P_l^{-m} \frac{(l+m)!}{(l-m)!} \tag{8.185}$$

as

$$(\theta_b\tau_b|\theta_a\tau_a)_\mu = \sqrt{\sin\theta_b \sin\theta_a} \sum_{n=0}^{\infty} [\tilde{I}_{n+\mu+1/2}(\zeta)]^{N+1} \times \frac{(2n+2\mu+1)}{2}\frac{(n+2\mu)!}{n!} P_{n+\mu}^{-\mu}(\cos\theta_b) P_{n+\mu}^{-\mu}(\cos\theta_a). \tag{8.186}$$

Here, $\mu$ can be an arbitrary real number if the factorials $(n+2\mu)!$ and $n!$ are defined as $\Gamma(n+2\mu+1)$ and $\Gamma(n+1)$. In the continuum limit, (8.186) becomes

$$\begin{aligned}(\theta_b\tau_b|\theta_a\tau_a)_\mu &= \sqrt{\sin\theta_b \sin\theta_a} \sum_{n=0}^{\infty} \exp\left[-\frac{\hbar(n+\mu)(n+\mu+1)}{2Mr^2}(\tau_b - \tau_a)\right] \\ &\times \frac{(2n+2\mu+1)}{2}\frac{(n+2\mu)!}{n!} P_{n+\mu}^{-\mu}(\cos\theta_b) P_{n+\mu}^{-\mu}(\cos\theta_a).\end{aligned} \tag{8.187}$$

We prove this to solve the time-sliced path integral (8.174) for arbitrary real values of $m = \mu$ [4] by using the addition theorem[11]

$$(\sin\alpha\sin\beta)^{-\mu} J_\mu(z\sin\alpha\sin\beta) e^{iz\cos\alpha\cos\beta} = \frac{2^{2\mu+1}\Gamma^2(\mu+1/2)}{\sqrt{2\pi z}} \times \sum_{n=0}^{\infty} \frac{i^n n!(n+\mu+1/2)}{\Gamma(n+2\mu+1)} J_{n+\mu+1/2}(z) C_n^{(\mu+1/2)}(\cos\alpha) C_n^{(\mu+1/2)}(\cos\beta). \quad (8.188)$$

After substituting $z$ by $\zeta e^{-i\pi/2}$ this turns into

$$(\sin\alpha\sin\beta)^{-\mu} I_\mu(\zeta\sin\alpha\sin\beta)\exp(\zeta\cos\alpha\cos\beta) = \frac{2^{2\mu+1}\Gamma^2(\mu+1/2)}{\sqrt{2\pi\zeta}} \times \sum_{n=0}^{\infty} \frac{n!(n+\mu+1/2)}{\Gamma(n+2\mu+1)} I_{n+\mu+1/2}(\zeta) C_n^{(\mu+1/2)}(\cos\alpha) C_n^{(\mu+1/2)}(\cos\beta). \quad (8.189)$$

The Gegenbauer polynomials $C_n^{(\mu+1/2)}(z)$ can be expressed, for arbitrary $\mu$, by means of Eq. (8.106) in terms of Jacobi polynomials $P_n^{(\mu,\mu)}$, and these further in terms of Legendre functions $P_{n+\mu}^{-\mu}$, using the formula

$$P_n^{(\mu,\mu)}(z) = (-2)^\mu \frac{(n+\mu)!}{n!}(1-z^2)^{-\mu/2} P_{n+\mu}^{-\mu}(z). \quad (8.190)$$

Thus[12]

$$C_n^{(\mu+1/2)}(z) = \frac{\Gamma(n+2\mu+1)\Gamma(\mu+1)}{\Gamma(2\mu+1)n!}\left[\frac{1-z^2}{4}\right]^{-\mu/2} P_{n+\mu}^{-\mu}(z). \quad (8.191)$$

We can now perform the integrations in the time-sliced path integral by means of the known continuation of the orthogonality relation (8.182) to arbitrary real values of $\mu$:

$$\int_{-1}^{1} \frac{d\cos\theta}{\sin^2\theta} P_{n+\mu}^{-\mu}(\cos\theta) P_{n'+\mu}^{-\mu}(\cos\theta) = \frac{n!}{(n+2\mu)!}\frac{2}{2n+1\mu+1}\delta_{nn'}. \quad (8.192)$$

Note that for noninteger $\mu$, the Legendre functions $P_{n+m}^{-m}(\cos\theta)$ are no longer polynomials as in (1.421). Instead, they are defined in terms of the hypergeometric function as follows:

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\mu/2} F(-\nu,\nu+1;1-\mu;(1-z)/2). \quad (8.193)$$

The integral formula (8.192) is a consequence of the orthogonality of the Gegenbauer polynomials (8.104), which is applied here in the form

$$\int_{-1}^{1} dz\,(1-z^2)^\mu C_n^{(\mu+1/2)}(z) C_{n'}^{(\mu+1/2)}(z) = \delta_{nn'}\frac{\pi 2^{-2\mu}\Gamma(2\mu+2+n)}{n!(n+\mu)[\Gamma(\mu+1/2)]^2}. \quad (8.194)$$

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[11] G.N. Watson, *Theory of Bessel Functions*, Cambridge University Press, 1952, Ch. 11.6, Eq. (11.9).

[12] I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 8.936.

Using (8.189), (8.191) and (8.192), the integrals in the product (8.207) can all be performed as before, resulting in the amplitude (8.186) with the continuum limit (8.187), both valid for arbitrary real values of $m = \mu \geq 0$.

The continuation to arbitrary real values of $\mu$ has an important application: The action (8.177) of the projected motion of a particle near the surface of the sphere coincides with the action of a particle moving in the so-called Pöschl-Teller potential [5]:

$$V(\theta) = \frac{\hbar^2}{2Mr^2}\frac{s(s+1)}{\sin^2\theta} \tag{8.195}$$

with the strength parameter $s = m - 1/2$. After the continuation of arbitrary real $m = \mu \geq 0$, the amplitude (8.187) describes this system for any potential strength. This fact will be discussed further in Chapter 14 where we develop a general method for solving a variety of nontrivial path integrals.

Note that the amplitude $(\sin\theta_b\tau_b|\sin\theta_a\tau_a)_m$ satisfies the Schrödinger equation

$$\begin{aligned}\left[\frac{\hbar^2}{Mr^2}\left(-\frac{1}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\frac{m^2}{2\sin^2\theta}\right)+\hbar\partial_\tau\right](\sin\theta\;\tau|\sin\theta_a\tau_a)_m&\\ =\hbar\delta(\tau-\tau_a)\delta(\cos\theta-\cos\theta_a).&\end{aligned} \tag{8.196}$$

This follows from the differential equation obeyed by the Legendre polynomials $P_l^m(\cos\theta)$ in (8.87). The new amplitude $(\theta\,\tau|\theta_a\tau_a)_m$, on the other hand, satisfies the equation [corresponding to that of $\sqrt{\sin\theta}P_l^m(\cos\theta)$ in the footnote to Eq. (8.87)]

$$\left[\frac{\hbar^2}{Mr^2}\left(-\frac{1}{2}\frac{d}{d\theta^2}-\frac{1}{8}+\frac{m^2-1/4}{2\sin^2\theta}\right)+\hbar\partial_\tau\right](\theta\,\tau|\theta_a\tau_a)_m = \hbar\delta(\tau-\tau_a)\delta(\theta-\theta_a). \tag{8.197}$$

### 8.8.2 Angular Barriers in Four Dimensions

In four dimensions, the angular momentum decomposition reads in terms of Euler angles [see (8.162)]

$$\begin{aligned}(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = &\sum_{l=0}^{\infty}\exp\left\{-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right\}\\ &\times\frac{l+1}{2\pi^2}\sum_{m_1m_2=-l/2}^{l/2} d^{l/2}_{m_1m_2}(\theta_b)d^{l/2}_{m_1m_2}(\theta_a)e^{im_1(\varphi_b-\varphi_a)+im_2(\gamma_b-\gamma_a)},\end{aligned} \tag{8.198}$$

with

$$L_2 \equiv (l+1)^2 - 1/4 = 4(l/2)(l/2+1) + 3/4 \tag{8.199}$$

and $m_1, m_2$ running over integers or half-integers depending on $l/2$. We now define the projected amplitudes by the expansion

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = 8\sum_{m_1m_2}(\sin\theta_b\tau_b|\sin\theta_a\tau_a)_{m_1m_2}\frac{1}{2\pi}e^{im_1(\varphi_b-\varphi_a)}\frac{1}{4\pi}e^{im_2(\gamma_b-\gamma_a)}. \tag{8.200}$$

As in (8.164), it is again convenient to introduce the differently normalized amplitude $(\theta_b\tau_b|\theta_a\tau_a)_m$ defined by

$$(\theta_b\tau_b|\theta_a\tau_a)_{m_1m_2} \equiv \sqrt{\sin\theta_b\sin\theta_a}(\sin\theta_b\tau_b|\sin\theta_a\tau_a)_{m_1m_2}, \tag{8.201}$$

in terms of which the expansion becomes [compare (8.163)]

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_m \frac{8}{\sqrt{\sin\theta_b\sin\theta_a}}(\theta_b\tau_b|\theta_a\tau_a)_{m_1m_2}\frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}\frac{1}{4\pi}e^{im(\gamma_b-\gamma_a)}. \tag{8.202}$$

A comparison with (8.198) gives immediately the projected amplitude

$$\begin{aligned}(\theta_b\tau_b|\theta_a\tau_a)_{m_1m_2} &= \sqrt{\sin\theta_b\sin\theta_a} \\ &\times\sum_l^\infty \exp\left\{-\frac{\hbar[(l+1)^2-1/4]}{2Mr^2}(\tau_b-\tau_a)\right\}\frac{l+1}{2}d^{l/2}_{m_1m_2}(\theta_b)d^{l/2}_{m_1m_2}(\theta_a),\end{aligned} \tag{8.203}$$

in which $l$ is summed in even steps from the larger value of $|2m_1|, |2m_2|$ to infinity.

Let us write down the time-sliced path integral leading to this amplitude. According to (8.151)–(8.153), it is given by

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/Mr^2}^3}\prod_{n=1}^N\left[\int_0^\pi\int_0^{2\pi}\int_0^{4\pi}\frac{d\theta_n\sin\theta_n d\varphi_n d\gamma_n}{8\sqrt{2\pi\hbar\epsilon/Mr^2}^3}\right]\exp\left(-\frac{1}{\hbar}\mathcal{A}^N\right). \tag{8.204}$$

In each time slice we make use of the addition theorem (8.129) and expand the exponent with (8.6) as

$$\begin{aligned}\exp\left[-\frac{Mr^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right] &= \exp\left[-\frac{Mr^2}{\hbar\epsilon}(1-\cos\Delta\vartheta_n)\right] \\ = \exp&\left\{-\frac{Mr^2}{2\hbar\epsilon}[1-\cos(\theta_n/2)\cos(\theta_{n-1}/2)-\sin(\theta_n/2)\sin(\theta_{n-1}/2)]\right\} \\ \times&\frac{1}{\sqrt{2\pi h_n^c}}\frac{1}{\sqrt{4\pi h_n^s}}\sum_{m_1,m_2=-\infty}^{\infty}\tilde{I}_{|m_1+m_2|}(h_n^c)\tilde{I}_{|m_1-m_2|}(h_n^s) \\ &\times\exp\{im_1(\varphi_n-\varphi_{n-1})+im_2(\gamma_n-\gamma_{n-1})\},\end{aligned} \tag{8.205}$$

where $h_n^c$ and $h_n^s$ are given by

$$h_n^c = \frac{Mr^2}{\hbar\epsilon}\cos(\theta_n/2)\cos(\theta_{n-1}/2), \qquad h_n^s = \frac{Mr^2}{\hbar\epsilon}\sin(\theta_n/2)\sin(\theta_{n-1}/2). \tag{8.206}$$

By doing successively the $\varphi_n$- and $\gamma_n$-integrations, we wind up with the path integral for the projected amplitude

$$(\theta_b\tau_b|\theta_a\tau_a)_{m_1m_2} \approx \frac{1}{\sqrt{2\pi\epsilon\hbar/4Mr^2}}\prod_{n=1}^N\left[\int_0^\pi\frac{d\theta_n}{\sqrt{2\pi\epsilon\hbar/4Mr^2}}\right]\exp\left\{-\frac{1}{\hbar}\mathcal{A}^N_{m_1m2}\right\}, \tag{8.207}$$

where $\mathcal{A}^N_{m_1m_2}$ is the sliced action

$$\mathcal{A}^N_{m_1m_2} = \sum_{n=1}^{N+1}\left\{\frac{Mr^2}{\epsilon}[1-\cos[(\theta_n-\theta_{n-1})/2]\right.$$
$$\left. -\hbar\log\tilde{I}_{|m_1+m_2|}(h_n^c)-\hbar\log\tilde{I}_{|m_1-m_2|}(h_n^s)\right\}. \tag{8.208}$$

For small $\epsilon$, this can be approximated (setting $\Delta\theta_n\equiv\theta_n-\theta_{n-1}$) by

$$\mathcal{A}^N_{m_1m_2}\to\epsilon\sum_{n=1}^{N+1}\left\{\frac{Mr^2}{2\epsilon^2}[(\Delta\theta_n/2)^2-\frac{1}{12}(\Delta\theta_n/2)^4+\ldots]\right.$$
$$\left.+\frac{\hbar^2}{2Mr^2}\frac{(m_1+m_2)^2-1/4}{\cos(\theta_n/2)\cos(\theta_{n-1}/2)}+\frac{\hbar^2}{2Mr^2}\frac{(m_1-m_2)^2-1/4}{\sin(\theta_n/2)\sin(\theta_{n-1}/2)}\right\}, \tag{8.209}$$

with the continuum limit

$$\mathcal{A}_{m_1m_2}=\int_{\tau_a}^{\tau_b}d\tau\left(\frac{Mr^2}{8}\dot{\theta}^2-\frac{\hbar^2}{8Mr^2}+\frac{\hbar^2}{2Mr^2}\frac{|m_1+m_2|^2-1/4}{\cos^2(\theta/2)}\right.$$
$$\left.+\frac{\hbar^2}{2Mr^2}\frac{|m_1-m_2|^2-1/4}{\sin^2(\theta/2)}\right). \tag{8.210}$$

After introducing the auxiliary mass

$$\mu=M/4 \tag{8.211}$$

and rearranging the potential terms, we can write the action equivalently as

$$\mathcal{A}_{m_1m_2}=\int_{\tau_a}^{\tau_b}d\tau\left(\frac{\mu r^2}{2}\dot{\theta}^2-\frac{\hbar^2}{32\mu r^2}+\frac{\hbar^2}{2\mu r^2}\frac{m_1^2+m_2^2-1/4-2m_1m_2\cos\theta}{\sin^2\theta}\right). \tag{8.212}$$

Just as in the previous system, this action contains an angular barrier $1/\sin^2\theta$ at $\theta=0$, and $\theta=\pi$, so that the amplitude (8.207) with the naively time-sliced action (8.176) does not exist for $m_1=m_2$ or $m_1=-m_2$, due to path collapse. Only with the properly time-sliced action (8.208) is the path integral stable and solvable by successive integrations with the result (8.203).

As before, the path integral (8.207) is initially only defined and solved by (8.203) if both $m_1$ and $m_2$ have integer or half-integer values. The path integral and its solution can, however, be continued to arbitrary real values of $m_1=\mu_1\geq 0$ and its $m_2=\mu_2\geq 0$. For this we rewrite (8.203) in the form [4]

$$(\theta_b\tau_b|\theta_a\tau_a)_{\mu_1\mu_2}=\sqrt{\sin\theta_b\sin\theta_a} \tag{8.213}$$
$$\times\sum_{n=0}^{\infty}\exp\left\{\frac{\hbar[(n+\mu_1+1)^2-1/4]}{2Mr^2}(\tau_b-\tau_a)\right\}\frac{n+\mu_1+1}{2}d^{n+\mu_1}_{\mu_1\mu_2}(\theta_b)d^{n+\mu_1}_{\mu_1\mu_2}(\theta_a),$$

assuming that $\mu_1 \geq \mu_2$. The products of the rotation functions $d^{\lambda}_{\mu_1\mu_2}(\theta)$ have a well-defined analytic continuation to arbitrary real values of the indices $\mu_1, \mu_2, \lambda$, as can be seen by expressing them in terms of Jacobi polynomials via formula (1.446).

To perform the path integral in the analytically continued case, we use the expansion valid for all $\mu_+, \mu_-$,[13]

$$
\begin{aligned}
\frac{z}{2} J_{\mu_+}(z\cos\alpha\cos\beta) J_{\mu_-}(z\sin\alpha\sin\beta) &= \cos^{\mu_+}\alpha\cos^{\mu_+}\beta\sin^{\mu_-}\alpha\sin^{\mu_-}\beta \\
&\times \sum_{n=0}^{\infty}(-1)^n(\mu_+ + \mu_- + 2n + 1) J_{\mu_+ + \mu_- + 2n+1}(z) \\
&\qquad \times \frac{\Gamma(n+\mu_+ + \mu_- + 1)\Gamma(n+\mu_- + 1)}{n!\Gamma(n+\mu_+ + 1)[\Gamma(\mu_- + 1)]^2} \\
&\qquad \times F(-n, n+\mu_+ + \mu_- + 1; \mu_- + 1; \sin^2\alpha) \\
&\qquad \times F(-n, n+\mu_+ + \mu_- + 1; \mu_- + 1; \sin^2\beta)
\end{aligned}
\tag{8.214}
$$

with $\zeta \equiv Mr^2/\hbar\epsilon$. The hypergeometric functions appearing on the right-hand side have a first argument with a negative integer value. They are therefore proportional to the Jacobi polynomials $P_n^{(\mu_-,\mu_+)}$:

$$
P_n^{(\mu_-,\mu_+)}(x) = \frac{1}{n!}\frac{\Gamma(n+\mu_- + 1)}{\Gamma(\mu_- + 1)} F\left(-n, n+\mu_+ + \mu_- + 1; 1+\mu_-; (1-x)/2\right) \tag{8.215}
$$

[recall (1.446) and the identity $P_n^{(\mu_-,\mu_+)}(x) = (-)^n P_n^{(\mu_-,\mu_+)}(-x)$]. Inserting $z = i\zeta$, $\alpha = \theta_n/2$, $\beta = \theta_{n-1}/2$, and expressing the Jacobi polynomials in terms of rotation functions continued to real-valued $\mu_1, \mu_2$, we obtain from (8.214) for $\mu_1 \geq \mu_2$

$$
\begin{aligned}
& I_{\mu_+}\left(\zeta\cos\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}\right) I_{\mu_-}\left(\zeta\sin\frac{\theta_n}{2}\sin\frac{\theta_{n-1}}{2}\right) \\
&= \frac{4}{\zeta}\sum_{n=0}^{\infty} I_{2n+\mu_+ + \mu_- + 1}(\zeta)\frac{(n+\mu_1+\mu_2)!(n+\mu_1-\mu_2)!}{(n+2\mu_1)!n!} d^{n+\mu_1}_{\mu_1\mu_2}(\theta_n) d^{n+\mu_1}_{\mu_1\mu_2}(\theta_{n-1}).
\end{aligned}
\tag{8.216}
$$

Now we make use of the orthogonality relation [compare (1.455)]

$$
\int_{-1}^{1} d\cos\theta \; d^{n+\mu_1}_{\mu_1\mu_2}(\theta) d^{n'+\mu_1}_{\mu_1\mu_2}(\theta) = \delta_{nn'}\frac{2}{2n+1}, \tag{8.217}
$$

which for real $\mu_1, \mu_2$ follows from the corresponding relation for Jacobi polynomials[14]

$$
\begin{aligned}
&\int_{-1}^{1} dx\,(1-x)^{\mu_-}(1+x)^{\mu_+} P_n^{(\mu_-,\mu_+)}(x) P_{n'}^{(\mu_-,\mu_+)}(x) \\
&\qquad = \delta_{nn'}\frac{2^{\mu_+ + \mu_- + 1}\Gamma(\mu_+ + n + 1)\Gamma(\mu_- + n + 1)}{n!(\mu_+ + \mu_- + 1 + 2n)\Gamma(\mu_+ + \mu_- + n + 1)},
\end{aligned}
\tag{8.218}
$$

[13] G.N. Watson, op. cit., Chapter 11.6, Gl. (11.6), (1).

[14] I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 7.391.

valid for $\operatorname{Re}\mu_+ > -1$ , $\operatorname{Re}\mu_- > -1$. Performing all $\theta_n$-integrations in (8.207) yields the time-sliced amplitude

$$(\theta_b\,\tau_b|\theta_a\,\tau_a)_{\mu_1\mu_2} = \sqrt{\sin\theta_b\sin\theta_a}\sum_{n=0}^{\infty}\left[\tilde{I}_{2n+\mu_++\mu_-+1}(\zeta)\right]^{N+1} d^{n+\mu_1}_{\mu_1\mu_2}(\theta_b)d^{n+\mu_1}_{\mu_1\mu_2}(\theta_a), \tag{8.219}$$

valid for all real $\mu_1 \geq \mu_2 \geq 0$. In the continuum limit, this becomes

$$(\theta_b\,\tau_b|\theta_a\,\tau_a)_{\mu_1\mu_2} = \sqrt{\sin\theta_b\sin\theta_a}\sum_{n=0}^{\infty} e^{-E_n(\tau_b-\tau_a)/\hbar} d^{n+\mu_1}_{\mu_1\mu_2}(\theta_b)d^{n+\mu_1}_{\mu_1\mu_2}(\theta_a), \tag{8.220}$$

with

$$E_n = \frac{\hbar}{2Mr^2}[(2n+\mu_++\mu_-+1)^2 - 1/4], \tag{8.221}$$

which proves (8.213).

Apart from the projected motion of a particle near the surface of the sphere, the amplitude (8.213) describes also a particle moving in the general Pöschl-Teller potential [5]

$$V_{\mathcal{PT}'}(\theta) = \frac{\hbar^2}{2Mr^2}\left[\frac{s_1(s_1+1)}{\sin^2(\theta/2)} + \frac{s_2(s_2+1)}{\cos^2(\theta/2)}\right]. \tag{8.222}$$

Due to the analytic continuation to arbitrary real $m_1$, $m_2$ the parameters $s_1$ and $s_2$ are arbitrary with the potential strength parameters $s_1 = m_1 + m_2 - 1/2$ and $s_2 = m_1 - m_2 - 1/2$. This will be discussed further in Chapter 14.

Recalling the differential equation (1.454) satisfied by the rotation functions $d^{l/2}_{m_1m_2}(\theta)$, we see that the original projected amplitude (8.207) obeys the Schrödinger equation

$$\left[\frac{\hbar^2}{2\mu r^2}\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \frac{3}{16} + \frac{m_1^2+m_2^2-2m_1m_2\cos\theta}{\sin^2\theta}\right) + \hbar\partial_\tau\right]$$
$$\times\,(\sin\theta\,\tau|\sin\theta_a\tau_a)_{m_1m_2} = \hbar\delta(\tau-\tau_a)\delta(\cos\theta-\cos\theta_a). \tag{8.223}$$

The extra term $3/16$ is necessary to account for the energy difference between the motion near the surface of a sphere in four dimensions, whose energy is $(\hbar^2/2\mu r^2)[(l/2)(l/2+1)+3/16]$ [see (8.158)], and that of a symmetric spinning top with angular momentum $L = l/2$ in three dimensions, whose energy is $(\hbar^2/2\mu r^2)(l/2)(l/2+1)$, as shown in the next section in detail.

The amplitude $(\theta_b\tau_b|\theta_a\tau_a)_{m_1m_2}$ defined in (8.201) satisfies the differential equation

$$\left[\frac{\hbar^2}{2\mu r^2}\left(-\frac{d^2}{d\theta^2} - \frac{1}{16} + \frac{m_1^2+m_2^2-1/4-2m_1m_2\cos\theta}{\sin^2\theta}\right) + \hbar\partial_\tau\right]$$
$$\times\,(\theta\,\tau|\theta_a\tau_a)_{m_1m_2} = \hbar\delta(\tau-\tau_a)\delta(\theta-\theta_a). \tag{8.224}$$

This is, of course, precisely the Schrödinger equation associated with the action (8.212).

## 8.9 Motion on a Sphere in $D$ Dimensions

The wave functions in the time evolution amplitude *near* the surface of a sphere are also correct for the motion *on* a sphere. This is not true for the energies, for which the amplitude (8.158) gives

$$E_l = \frac{\hbar^2}{2Mr^2}(L_2^2)_l, \tag{8.225}$$

with

$$(L_2^2)_l = (l + D/2 - 1)^2 - 1/4, \quad l = 0, 1, 2, \dots\ . \tag{8.226}$$

As we know from Section 1.14, the energies should be equal to

$$E_l = \frac{\hbar^2}{2Mr^2}(\hat{L}^2)_l, \tag{8.227}$$

where $(\hat{L}^2)_l$ denotes the eigenvalues of the square of the angular momentum operator. In $D$ dimensions, the eigenvalues are known from the Schrödinger theory to be

$$(\hat{L}^2)_l = l(l + D - 2), \quad l = 0, 1, 2, \dots\ . \tag{8.228}$$

Apart from the trivial case $D = 1$, the two energies are equal only for $D = 3$, where $(L_2^2)_l \equiv (\hat{L}^2)_l = l(l + 1)$. For all other dimensions, we shall have to remove the difference

$$\Delta(L_2^2)_l \equiv \hat{L}^2 - L_2^2 = \frac{1}{4} - \left(\frac{D}{2} - 1\right)^2 = -\frac{(D-1)(D-3)}{4}. \tag{8.229}$$

The simplest nontrivial case where the difference appears is for $D = 2$ where the role of $l$ is played by the magnetic quantum number $m$ and $(L_2^2)_m = m^2 - 1/4$, whereas the correct energies should be proportional to $(\hat{L}^2)_m = m^2$.

Two changes are necessary in the time-sliced path integral to find the correct energies. First, the time-sliced action (8.152) must be modified to measure the proper distance on the surface rather than the Euclidean distance in the embedding space. Second, we will have to correct the measure of path integration. The modification of the action is simply

$$\mathcal{A}^N_{\text{on sphere}} = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \frac{(\Delta\vartheta_n)^2}{2}, \tag{8.230}$$

in addition to

$$\mathcal{A}^N = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} (1 - \cos\Delta\vartheta_n). \tag{8.231}$$

Since the time-sliced path integral was solved exactly with the latter action, it is convenient to expand the true action around the solvable one as follows:

$$\mathcal{A}^N_{\text{on sphere}} = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \left[(1 - \cos\Delta\vartheta_n) + \frac{1}{24}(\Delta\vartheta_n)^4 - \dots\right]. \tag{8.232}$$

There is no need to go to higher than the fourth order in $\Delta\vartheta_n$ , since these do not contribute to the relevant order $\epsilon$. For $D = 2$, the correction of the action is sufficient to transform the path integral *near* the surface of the sphere into one *on* the sphere, which in this reduced dimension is merely a circle. On a circle, $\Delta\vartheta_n = \varphi_n - \varphi_{n-1}$ and the measure of path integration becomes

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/Mr^2}} \prod_{n=1}^{N+1} \int_{-\pi/2}^{\pi/2} \frac{d\varphi_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}}. \tag{8.233}$$

The quartic term $(\Delta\vartheta_n)^4 = (\varphi_n - \varphi_{n-1})^4$ can be replaced according to the rules of perturbation theory by its expectation [see (8.62)]

$$\langle(\Delta\vartheta_n)^4\rangle_0 = 3\frac{\epsilon\hbar}{Mr^2}. \tag{8.234}$$

The correction term in the action

$$\Delta_{\rm qu}\mathcal{A}^N = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \frac{1}{24}(\Delta\vartheta_n)^4 \tag{8.235}$$

has, therefore, the expectation

$$\langle\Delta_{\rm qu}\mathcal{A}^N\rangle_0 = (N+1)\epsilon\frac{\hbar^2/4}{2Mr^2}. \tag{8.236}$$

This supplies precisely the missing term which raises the energy from the *near*-the-surface value $E_m = \hbar^2(m^2 - 1/4)/2Mr^2$ to the proper *on*-the-sphere value $E_m = \hbar^2 m^2/2Mr^2$.

In higher dimensions, the path integral near the surface of a sphere requires a second correction. The difference (8.229) between $\hat{L}^2$ and $L_2^2$ is negative. Since the expectation of the quartic correction term alone is always positive, it can certainly not explain the difference.[15] Let us calculate first its contribution at arbitrary $D$. For very small $\epsilon$, the fluctuations near the surface of the sphere lie close to the $D-1$ -dimensional tangent space. Let $\Delta\mathbf{x}_n$ be the coordinates in this space. Then we can write the quartic correction term as

$$\Delta_{\rm qu}\mathcal{A}^N \approx \frac{M}{\epsilon} \sum_{n=1}^{N+1} \frac{1}{24r^2}(\Delta\mathbf{x}_n)^4, \tag{8.237}$$

where the components $(\Delta\mathbf{x}_n)_i$ have the correlations

$$\langle(\Delta\mathbf{x}_n)_i(\Delta\mathbf{x}_n)_j\rangle_0 = \frac{\hbar\epsilon}{M}\delta_{ij}. \tag{8.238}$$

[15] This was claimed by G. Junker and A. Inomata, in *Path Integrals from* meV *to* MeV, edited by M.C. Gutzwiller, A. Inomata, J.R. Klauder, and L. Streit (World Scientific, Singapore, 1986), p.333.

Thus, according to the rule (8.62), $\Delta_{\rm qu}\mathcal{A}^N$ has the expectation

$$\langle\Delta\mathcal{A}^N\rangle_0 = (N+1)\epsilon\frac{\hbar^2}{2Mr^2}\Delta_{\rm qu}L_2^2, \tag{8.239}$$

where $\Delta_{\rm qu}L_2^2$ is the contribution of the quartic term to the value $L_2^2$:

$$\Delta_{\rm qu}L_2^2 = \frac{D^2-1}{12}. \tag{8.240}$$

This result is obtained using the contraction rules for the tensor

$$\langle\Delta x_i\Delta x_j\Delta x_k\Delta x_l\rangle_0 = \left(\frac{\epsilon\hbar}{M}\right)^2(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{8.241}$$

which follow from the integrals (8.63).

Incidentally, the same result can also be derived in a more pedestrian way: The term $(\Delta\mathbf{x}_n)^4$ can be decomposed into $D-1$ quartic terms of the individual components $\Delta x_{ni}$, and $(D-1)(D-2)$ mixed quadratic terms $(\Delta x_{ni})^2(\Delta x_{nj})^2$ with $i \neq j$. The former have an expectation $(D-1)\cdot 3(\epsilon\hbar/Mr)^2$, the latter $(D-1)(D-2)\cdot(\epsilon\hbar/Mr)^2$. When inserted into (8.237), they lead to (8.239).

Thus we remain with a final difference in $D$ dimensions:

$$\Delta_{\rm f}L_2^2 = \Delta L_2^2 - \Delta_{\rm qu}L_2^2 = -\frac{1}{3}(D-1)(D-2). \tag{8.242}$$

This difference can be removed only by the measure of the path integral. Near the sphere we have used the measure

$$\prod_{n=1}^N\left[\int\frac{d^{D-1}\mathbf{u}_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}^{D-1}}\right]. \tag{8.243}$$

In Chapter 10 we shall argue that this measure is incorrect. We shall find that the measure (8.243) receives a correction factor

$$\prod_{n=1}^N\left[1+\frac{D-2}{6}(\Delta\vartheta_n)^2\right] \tag{8.244}$$

[see the factor $(1+i\Delta\mathcal{A}_J^\epsilon)$ of Eq. (10.151)]. Setting $(\Delta\vartheta_n)^2=(\Delta\mathbf{x}_n/r)^2$, the expectation of this factor becomes

$$\prod_{n=1}^N\left[1+\frac{(D-2)(D-1)}{6r^2}\frac{\epsilon\hbar}{M}\right] \tag{8.245}$$

corresponding to a correction term in the action

$$\langle\Delta\mathcal{A}_f^N\rangle_0 = (N+1)\epsilon\frac{\hbar^2}{2Mr^2}\Delta_f L_2^2, \tag{8.246}$$

with $\Delta_f L_2^2$ given by (8.242). This explains the remaining difference between the eigenvalues $(L_2)_l$ and $(\hat{L})_l^2$.

In summary, the time evolution amplitude on the $D$-dimensional sphere reads [6]

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_{l=0}^{\infty} \exp\left[-\frac{\hbar\hat{L}^2}{2Mr^2}(\tau_b-\tau_a)\right]\sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_b)Y_{l\mathbf{m}}^*(\mathbf{u}_a), \tag{8.247}$$

with

$$\hat{L}^2 = l(l+D-2), \tag{8.248}$$

which are precisely the eigenvalues of the squared angular momentum operator of Schrödinger quantum mechanics. For $D=3$ and $D=4$, the amplitude (8.247) coincides with the more specific representations (8.161) and (8.162), if $L_2^2$ is replaced by $\hat{L}^2$.

Finally, let us emphasize that in contrast to the amplitude (8.158) *near* the surface of the sphere, the normalization of the amplitude (8.247) *on* the sphere is

$$\int d^{D-1}\mathbf{u}_b\,(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = 1. \tag{8.249}$$

This follows from the integral

$$\begin{aligned}\int d^{D-1}\mathbf{u}_b\sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_b)Y_{l\mathbf{m}}^*(\mathbf{u}_a) &= \delta_{l0}\int d^{D-1}\mathbf{u}_b\,Y_{\mathbf{00}}(\mathbf{u}_b)Y_{\mathbf{00}}^*(\mathbf{u}_a)\\ &= \delta_{l0}\int d^{D-1}\mathbf{u}_b\,1/S_D = \delta_{l0}.\end{aligned} \tag{8.250}$$

This is in contrast to the amplitude near the surface which satisfies

$$\int d^{D-1}\mathbf{u}_b\,(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \exp\left[-\frac{(D/2-1)^2-1/4}{2\mu r^2}(\tau_b-\tau_a)\right]. \tag{8.251}$$

We end this section with the following observation. In the continuum, the Euclidean path integral on the surface of a sphere can be rewritten as a path integral in flat space with an auxiliary path integral over a Lagrange multiplier $\lambda(\tau)$ in the form[16]

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \int_{-i\infty}^{i\infty}\int \mathcal{D}^2x(\tau)\frac{\mathcal{D}\lambda(\tau)}{2\pi i\hbar}\exp\left(-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b}d\tau\left\{\frac{M}{2}\dot{\mathbf{x}}^2+\frac{\lambda(\tau)}{2r}[x^2(\tau)-r^2]\right\}\right). \tag{8.252}$$

A naive time slicing of this expression would *not* yield the correct energy spectrum on the sphere. The slicing would lead to the product of integrals

$$(\mathbf{u}_bt_b|\mathbf{u}_at_a) \approx \prod_{n=1}^{N}\left[\int d^Dx_n\right]\prod_{n=1}^{N}\left[\int\frac{d\lambda_n}{2\pi i\hbar/\epsilon}\right]\exp\left(\frac{i}{\hbar}\mathcal{A}^N\right), \tag{8.253}$$

[16]The field-theoretic generalization of this path integral, in which $\tau$ is replaced by a $d$-dimensional spatial vector $\mathbf{x}$, is known as the O($D$)-symmetric *nonlinear $\sigma$-model* in $d$ dimensions. In statistical mechanics it corresponds to the well-studied classical O($D$) Heisenberg model in $d$ dimensions.

with $\mathbf{u} \equiv \mathbf{x}/|\mathbf{x}|$ and the time-sliced action

$$\mathcal{A}^N = \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon}(\mathbf{x}_n - \mathbf{x}_{n-1})^2 + \epsilon \frac{\lambda_n}{2r}(\mathbf{x}_n^2 - r^2) \right]. \tag{8.254}$$

Integrating out the $\lambda_n$'s would produce precisely the expression (8.151) with the action (8.152) *near* the surface of the sphere. The $\delta$-functions arising from the $\lambda_n$-integrations would force only the intermediate positions $\mathbf{x}_n$ to lie on the sphere; the sliced kinetic terms, however, would not correspond to the geodesic distance. Also, the measure of path integration would be wrong.

## 8.10 Path Integrals on Group Spaces

In Section 8.3, we have observed that the surface of a sphere in four dimensions is equivalent to the covering group of rotations in three dimensions, i.e., with the group SU(2). Since we have learned how to write down an exactly solvable time-sliced path integral *near* and *on* the surface of the sphere, the equivalence opens up the possibility of performing path integrals for the motion of a mechanical system *near* and *on* the group space of SU(2). The most important system to which the path integral on the group space of SU(2) can be applied is the spinning top, whose Schrödinger quantum mechanics was discussed in Section 1.15. Exploiting the above equivalence we are able to describe the same quantum mechanics in terms of path integrals. The theory to be developed for this particular system will, after a suitable generalization, be applicable to systems whose dynamics evolves on any group space.

First, we discuss the path integral *near* the group space using the exact result of the path integral *near* the surface of the sphere in four dimensions. The crucial observation is the following: The time-sliced action near the surface

$$\mathcal{A}^N = \frac{M}{2\epsilon} r^2 \sum_{n=1}^{N+1} (\mathbf{u}_n - \mathbf{u}_{n-1})^2 = \frac{Mr^2}{\epsilon} \sum_{n=1}^{N+1} (1 - \cos \Delta\vartheta_n) \tag{8.255}$$

can be rewritten in terms of the group elements $g(\varphi, \theta, \gamma)$ defined in Eq. (8.123) as

$$\mathcal{A}^N = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \left[ 1 - \frac{1}{2} \mathrm{tr}(g_n g_{n-1}^{-1}) \right], \tag{8.256}$$

with the obvious notation

$$g_n = g(\varphi_n, \theta_n, \gamma_n). \tag{8.257}$$

This follows after using the explicit matrix form for $g$, which reads

$$\begin{aligned} g(\varphi, \theta, \gamma) &= \exp(i\varphi\sigma_3/2) \exp(i\theta\sigma_2/2) \exp(i\gamma\sigma_3/2) \qquad (8.258) \\ &= \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}. \end{aligned}$$

After a little algebra we find

$$\frac{1}{2}\mathrm{tr}(g_n g_{n-1}^{-1}) = \cos(\theta_n/2)\cos(\theta_{n-1}/2)\cos[(\varphi_n-\varphi_{n-1}+\gamma_n-\gamma_{n-1})/2] + \sin(\theta_n/2)\sin(\theta_{n-1}/2)\cos[(\varphi_n-\varphi_{n-1}-\gamma_n+\gamma_{n-1})/2], \tag{8.259}$$

just as in (8.129). The invariant group integration measure is usually defined to be normalized to unity, i.e.,

$$\int dg \equiv \frac{1}{16\pi^2}\int_0^\pi\int_0^{2\pi}\int_0^{4\pi} d\theta\sin\theta d\varphi d\gamma = \frac{1}{2\pi^2}\int d^3\mathbf{u} = 1. \tag{8.260}$$

We shall renormalize the time evolution amplitude $(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)$ near the surface of the four-dimensional sphere accordingly, making it a properly normalized amplitude for the corresponding group elements $(g_b\tau_b|g_a\tau_a)$. Thus we define

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) \equiv \frac{1}{2\pi^2}(g_b\tau_b|g_a\tau_a). \tag{8.261}$$

The path integral (8.151) then turns into the following path integral for the motion near the group space [compare also (8.204)]:

$$(g_b\tau_b|g_a\tau_a) \approx \frac{2\pi^2}{\sqrt{2\pi\hbar\epsilon/Mr^2}^3}\prod_{n=1}^N\left[\int\frac{2\pi^2 dg_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}^3}\right]\exp\left(-\frac{1}{\hbar}\mathcal{A}^N\right). \tag{8.262}$$

Let us integrate this expression within the group space language. For this we expand the exponential as in (8.131):

$$\exp\left\{-\frac{Mr^2}{\hbar\epsilon}\left[1-\frac{1}{2}\mathrm{tr}(g_n g_{n-1}^{-1})\right]\right\} = \sum_{l=0}^\infty \tilde{a}_l(h)\frac{l+1}{2\pi^2}C_l^{(1)}(\cos\Delta\vartheta_n) \tag{8.263}$$

$$= \sum_{l=0}^\infty \tilde{a}_l(h)\frac{l+1}{2\pi^2}\sum_{m_1,m_2=-l/2}^{l/2}\mathcal{D}^{l/2}_{m_1m_2}(\varphi_n,\theta_n,\gamma_n)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_{n-1},\theta_{n-1},\gamma_{n-1}).$$

In general terms, the right-hand side corresponds to the well-known *character expansion* for the group SU(2):

$$\exp\left[\frac{h}{2}\mathrm{tr}(g_n g_{n-1}^{-1})\right] = \frac{1}{h}\sum_{l=0}^\infty(l+1)I_{l+1}(h)\chi^{(l/2)}(g_n g_{n-1}^{-1}). \tag{8.264}$$

Here $\chi^{l/2}(g)$ are the so-called *characters*, the traces of the representation matrices of the group element $g$, i.e.,

$$\chi^{(l/2)}(g) = \mathcal{D}^{l/2}_{mm}(g). \tag{8.265}$$

The relation between the two expansions is obvious if we use the representation properties of the $\mathcal{D}^{l/2}_{m_1m_2}$ functions and their unitarity to write

$$\chi^{(l/2)}(g_n g_{n-1}^{-1}) = \mathcal{D}^{l/2}_{mm'}(g_n)\mathcal{D}^{l/2*}_{mm'}(g_{n-1}). \tag{8.266}$$

This leads directly to (8.263) [see also the footnote to (8.128)]. Having done the character expansion in each time slice, the intermediate group integrations can all be performed using the orthogonality relations of group characters

$$\int dg \chi^{(L)}(g_1 g^{-1})\chi^{(L')}(g g_2^{-1}) = \delta_{LL'}\frac{1}{d_L}\chi^{(L)}(g_1 g_2^{-1}). \tag{8.267}$$

The result of the integrations is, of course, the same amplitude as before in (8.162):

$$\begin{aligned}(g_b\tau_b|g_a\tau_a) = \sum_{l=0}^{\infty} &\exp\left[-\frac{\hbar L_2}{2Mr^2}(\tau_b-\tau_a)\right] \\ \times (l+1) &\sum_{m_1,m_2=-l/2}^{l/2} \mathcal{D}^{l/2}_{m_1m_2}(\varphi_n,\theta_n,\gamma_n)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_{n-1},\theta_{n-1},\gamma_{n-1}).\end{aligned} \tag{8.268}$$

Given this amplitude *near* the group space we can find the amplitude for the motion *on* the group space, by adding to the energy near the sphere $E = \hbar^2[(l/2+1)^2 - 1/4]/2Mr^2$ the correction $\Delta E = \hbar^2 \Delta L_2^2/2Mr^2$ associated with Eq. (8.229). For $D=4$, $L_2^2 = (l/2)(l/2+1)+3/4$ has to be replaced by $\hat{L}^2 = L_2^2 + \Delta L_2^2 = (l/2)(l/2+1)$, and the energy changes by

$$\Delta E = -\frac{3\hbar^2}{8M}. \tag{8.269}$$

Otherwise the amplitude is the same as in (8.268) [6].

Character expansions of the exponential of the type (8.264) and the orthogonality relation (8.267) are general properties of group representations. The above time-sliced path integral can therefore serve as a prototype for the quantum mechanics of other systems moving *near* or *on* more general group spaces than SU(2).

Note that there is no problem in proceeding similarly with noncompact groups [7]. In this case we would start out with a treatment of the path integral *near* and *on* the surface of a hyperboloid rather than a sphere in four dimensions. The solution would correspond to the path integral near and on the group space of the covering group SU(1,1) of the Lorentz group O(2,1). The main difference with respect to the above treatment would be the appearance of hyperbolic functions of the second Euler angle $\theta$ rather than trigonometric functions.

An important family of noncompact groups whose path integral can be obtained in this way are the Euclidean groups [8] consisting of rotations and translations. Their *Lie algebra* comprises the momentum operators $\hat{\mathbf{p}}$, whose representation on the spatial wave functions has the Schrödinger form $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$. Thus, the canonical commutation rules in a Euclidean space form part of the representation algebra of these groups. Within a Euclidean group, the separation of the path integral into a radial and an azimuthal part is an important tool in obtaining all group representations.

## 8.11 Path Integral of Spinning Top

We are now also in a position to solve the time-sliced path integral of a spinning top by reducing it to the previous case of a particle moving on the group space SU(2). Only in one respect is the spinning top different: the equivalent "particle" does not move on the covering space SU(2) of the rotation group, but on the rotation group O(3) itself. The angular configurations with Euler angles $\gamma$ and $\gamma+2\pi$ are physically indistinguishable. The physical states form a representation space of O(3) and the time evolution amplitude must reflect this. The simplest possibility to incorporate the O(3) topology is to add the two amplitudes leading from the initial configuration $\varphi_a, \theta_a, \gamma_a$ to the two identical final ones $\varphi_b, \theta_b, \gamma_b$ and $\varphi_b, \theta_b, \gamma_b + 2\pi$. This yields the amplitude of the spinning top:

$$
\begin{aligned}
&(\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a)_{\rm top} \\
&\quad= (\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a) + (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a).
\end{aligned}
\tag{8.270}
$$

The sum eliminates all half-integer representation functions $D^{l/2}_{mm'}(\theta)$ in the expansion (8.268) of the amplitude.

Instead of the sum we could have also formed another representation of the operation $\gamma \to \gamma + 2\pi$, the antisymmetric combination

$$
\begin{aligned}
&(\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a)_{\rm fermionic} \\
&\quad= (\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a) - (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a).
\end{aligned}
\tag{8.271}
$$

Here the expansion (8.268) retains only the half-integer angular momenta $l/2$. As discussed in Chapter 7, half-integer angular momenta are associated with fermions such as electrons, protons, muons, and neutrinos. This is indicated by the subscript "fermionic". In spite of this, the above amplitude cannot be used to describe a single fermion since this has only one fixed spin $l/2$, while (8.271) contains *all* possible fermionic spins at the same time.

In principle, there is no problem in also treating the non-spherical top. In the formulation *near* the group space, the gradient term in the action,

$$
\frac{1}{\epsilon^2}\left[1 - \frac{1}{2}\mathrm{tr}(g_n g_{n-1}^{-1})\right],
\tag{8.272}
$$

has to be separated into time-sliced versions of the different angular velocities. In the continuum these are defined by

$$
\omega_a = -i\,\mathrm{tr}\,(\dot{\sigma}_a g^{-1}), \quad a = \xi, \eta, \zeta.
\tag{8.273}
$$

The gradient term (8.272) has the symmetric continuum limit $\dot{\omega}_a^2$. With the different moments of inertia $I_\xi, I_\eta, I_\zeta$, the asymmetric sliced gradient term reads

$$
\begin{aligned}
\frac{1}{\epsilon^2}\Bigg\{ I_\xi \left[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_\xi g_{n-1}^{-1})\right] &+ I_\eta \left[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_\eta g_{n-1}^{-1})\right] \\
&+\ I_\zeta \left[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_\zeta g_{n-1}^{-1})\right]\Bigg\},
\end{aligned}
\tag{8.274}
$$

rather than (8.272). The amplitude *near* the top is an appropriate generalization of (8.268). The calculation of the correction term $\Delta E$, however, is more complicated than before and remains to be done, following the rules explained above.

## 8.12 Path Integral of Spinning Particle

The path integral of a particle on the surface of a sphere contains states of all integer angular momenta $l = 1, 2, 3, \dots$ . The path integral on the group space SU(2) contains also all half-integer spins $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ .

The question arises whether it is possible to set up a path integral which contains only a single spinning particle, for instance of spin $s = 1/2$. Thus we need a path integral which for each time slice spans precisely one irreducible representation space of the rotation group, consisting of the $2s+1$ states $|s\, s_3\rangle$ for $s_3 = -s, \dots, s$. In order to sum over paths, we must parametrize this space in terms of a continuous variable. This is possible by selecting a particular spin state, for example the state $|ss\rangle$ pointing in the $z$-direction, and rotating it into an arbitrary direction $\mathbf{u} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ with the help of some rotation, for instance

$$|\theta\,\varphi\rangle \equiv R^s(\theta,\varphi)|ss\rangle \equiv e^{-iS_3\varphi}e^{-iS_2\theta}|ss\rangle, \tag{8.275}$$

where $S_i$ are matrix generators of the rotation group of spin $s$, which satisfy the commutation rules of the generators $\hat{L}_i$ in (1.414). The states (8.275) are nonabelian versions of the coherent states (7.343). They can be expanded into the $2s+1$ spin states $|s\, s_3\rangle$ as follows:

$$|\theta\,\varphi\rangle = \sum_{s_3=-s}^{s} |s\, s_3\rangle\langle s\, s_3|\hat{R}(\theta,\varphi)|ss\rangle = \sum_{s_3=-s}^{s} |s\, s_3\rangle e^{-is_3\varphi} d^s_{s_3\, s}(\theta), \tag{8.276}$$

where $d^j_{mm'}(\theta)$ are the representation matrices of $e^{-iS_2\theta}$ with angular momentum $j$ given in Eq. (1.446). For $s = 1/2$, where the matrix $d^j_{mm'}$ has the form (1.447), the states (8.276) are

$$|\theta\varphi\rangle = e^{i\varphi/2}\cos\theta/2|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - e^{-i\varphi/2}\sin\theta/2|\tfrac{1}{2}\; -\tfrac{1}{2}\rangle. \tag{8.277}$$

At this point it is useful to introduce the so-called *monopole spherical harmonics* defined by

$$Y^j_{m\,q}(\theta,\varphi) \equiv \sqrt{\frac{2j+1}{4\pi}}\, e^{im\varphi} d^j_{m\,q}(\theta). \tag{8.278}$$

Comparison with Eq. (8.125) shows that these are simply the ultra-spherical harmonics $Y_{j\,m\,q}(\hat{\mathbf{x}})$ with vanishing third Euler angle $\gamma$. They satisfy the orthogonality relation

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, Y^{j\,*}_{m\,q}(\theta,\varphi) Y^{j'}_{m'q}(\theta,\varphi) = \delta_{jj'}\delta_{mm'}, \tag{8.279}$$

and the completeness relation

$$\sum_{j,m} Y^{j}_{m\,q}(\theta,\varphi)Y^{j\,*}_{m\,q}(\theta',\varphi') = \delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'). \tag{8.280}$$

We define now the covariant looking states

$$|\mathbf{u}\rangle \equiv \sqrt{\frac{2j+1}{4\pi}}|\theta\,\varphi\rangle = \sum_{s_3=-s}^{s} |s\,s_3\rangle\, Y^{s\,*}_{s_3\,s}(\theta,\varphi), \tag{8.281}$$

and write the angular integral as an integral over the surface of the unit sphere:

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi = \int d^3u\,\delta(\mathbf{u}^2-1) \equiv \int d\mathbf{u}. \tag{8.282}$$

From (8.279) we deduce that the states $|\mathbf{u}\rangle$ are complete in the space of spin-$s$ states:

$$\begin{aligned}\int d\mathbf{u}\,|\mathbf{u}\rangle\langle\mathbf{u}| &= \sum_{s_3=-s}^{s}\sum_{s'_3=-s}^{s}|s\,s_3\rangle \int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\varphi Y^{s\,*}_{s_3\,s}(\theta,\varphi)Y^{s}_{s'_3 s}(\theta,\varphi)\langle s s'_3| \\ &= \sum_{s_3=-s}^{s}|s\,s_3\rangle\,\langle s\,s_3| = 1^s. \end{aligned}\tag{8.283}$$

The states are not orthogonal, however. Writing $Y^{s}_{s_3\,s}(\theta,\varphi)$ as $Y^{s}_{s_3\,s}(\mathbf{u})$, we see that

$$\langle\mathbf{u}|\mathbf{u}'\rangle = \sum_{s_3=-s}^{s}\sum_{s'_3=-s}^{s} Y^{s}_{s_3\,s}(\mathbf{u})\langle s\,s_3|s s'_3\rangle Y^{s}_{s'_3 s}(\mathbf{u}') = \sum_{s_3=-s}^{s} Y^{s}_{s_3\,s}(\mathbf{u})Y^{s\,*}_{s_3\,s}(\mathbf{u}'). \tag{8.284}$$

The right-hand side can be calculated as follows:

$$\begin{aligned}\frac{2j+1}{4\pi}\langle ss|e^{i\theta S_2}e^{i\varphi S_3}e^{-i\varphi' S_3}e^{-i\theta' S_2}|ss\rangle &= \frac{2j+1}{4\pi}\langle ss|e^{-isAS_3}e^{-i\beta S_2}|ss\rangle \\ = \frac{2j+1}{4\pi}e^{-isAS_3}d^s_{ss}(\beta) &= \frac{2j+1}{4\pi}e^{-isAS_3}\left(\frac{1+\mathbf{u}\cdot\mathbf{u}'}{2}\right)^s, \end{aligned}\tag{8.285}$$

where $\beta$ is the angle between $\mathbf{u}$ and $\mathbf{u}'$, and $A(\mathbf{u},\mathbf{u}',\hat{\mathbf{z}})$ is the area of the spherical triangle on the unit sphere formed by the three points in the argument. For a radius $R$, the area is equal to $R^2$ time the angular excess $E$ of the triangle, defined as the amount by which the sum of the angles in the triangle is larger that $\pi$. An explicit formula for $E$ is the spherical generalization of *Heron's formula* for the area $A$ of a triangle [9]:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s=(a+b+c)/2 = \text{semiperimeter}. \tag{8.286}$$

The angular excess on a sphere is

$$\tan\frac{E}{4} = \sqrt{\tan\frac{\phi_s}{2}\tan\frac{\phi_s-\phi_a}{2}\tan\frac{\phi_s-\phi_b}{2}\tan\frac{\phi_s-\phi_c}{2}}, \tag{8.287}$$

where $\phi_a, \phi_b, \phi_c$ are the *angular lengths* of the sides of the triangle and

$$\phi_s = (\phi_a + \phi_b + \phi_c)/2 \tag{8.288}$$

is the angular semiperimeter on the sphere.

We can now set up a path integral for the scalar product (8.285):

$$\langle \mathbf{u}_b|\mathbf{u}_a\rangle = \left[\prod_{n=1}^{N}\int d\mathbf{u}_n\right]\langle \mathbf{u}_b|\mathbf{u}_N\rangle\langle \mathbf{u}_N|\mathbf{u}_{N-1}\rangle\langle \mathbf{u}_{N-1}|\cdots|\mathbf{u}_1\rangle\langle \mathbf{u}_1|\mathbf{u}_a\rangle. \tag{8.289}$$

For large $N$, the intermediate $\mathbf{u}_n$-vectors will all lie close to their neighbors and we can write approximately

$$\langle \mathbf{u}_n|\mathbf{u}_{n-1}\rangle \approx \frac{2s+1}{4\pi}e^{i\Delta A_n}\left[1+\tfrac{1}{2}\mathbf{u}_n(\mathbf{u}_n-\mathbf{u}_{n-1})\right]^s \approx \frac{2s+1}{4\pi}e^{i\Delta A_n+\frac{s}{2}\mathbf{u}_n(\mathbf{u}_n-\mathbf{u}_{n-1})}. \tag{8.290}$$

Let us take this expression to the continuum limit. We introduce a time parameter labeling the chain of $\mathbf{u}_n$-vectors by $t_n = n\epsilon$, and find the small-$\epsilon$ approximation

$$\frac{2s+1}{4\pi}\exp\left[i\epsilon\, s\,\cos\theta\,\dot{\varphi} - \frac{s}{4}\epsilon^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)+\dots\right]. \tag{8.291}$$

The first term is obtained from the scalar product

$$\begin{aligned}
\langle \theta\,\varphi|i\partial_t|\theta\,\varphi\rangle &= \langle ss|e^{iS_2\theta}e^{iS_3\varphi}i\partial_t e^{-iS_3\varphi}e^{-iS_2\theta}|ss\rangle \\
&= \langle ss|e^{iS_2\theta}e^{iS_3\varphi}\left(\dot{\varphi}S_3e^{-iS_3\varphi}e^{-iS_2\theta}+e^{-iS_3\varphi}\dot{\theta}S_2e^{-iS_2\theta}\right)|ss\rangle \\
&= \langle ss|\left(\cos\theta S_3-\sin\theta S_1\right)\dot{\varphi}+\dot{\theta}S_2|ss\rangle = s\,\cos\theta\,\dot{\varphi}.
\end{aligned} \tag{8.292}$$

This result is actually not completely correct. The reason is that the angular variables in the states (8.275) are cyclic variables. For integer spins, $\theta$ and $\varphi$ are cyclic in $2\pi$, for half-integer spins in $4\pi$. Thus there can be jumps by $2\pi$ or $4\pi$ in these angles which do not change the states (8.275). In writing down the approximation (8.291) we must assume that we are at a safe distances from such singularities. If we get close to them, we must change the direction of the quantization axis.

Keeping this in mind we can express the scalar product in the limit $\epsilon \to 0$ by the path integral

$$\langle \mathbf{u}_b|\mathbf{u}_a\rangle = \int \mathcal{D}\mathbf{u}\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\,\hbar s\cos\theta\dot{\varphi}}, \tag{8.293}$$

where $\int\mathcal{D}\mathbf{u}$ is defined by the limit $N\to\infty$ of the product of integrals

$$\int \mathcal{D}\mathbf{u} \equiv \lim_{N\to\infty}\left[\frac{2s+1}{4\pi}\prod_{n=1}^{N}\int d\mathbf{u}_n\right]. \tag{8.294}$$

The path integral fixes the Hilbert space of the spin theory. It is the analog of the zero-Hamiltonian path integral in Eqs. (2.17) and (2.18). Comparing (8.293) with (2.18), we see that $s\hbar\cos\theta$ plays the role of a canonically conjugate momentum of the variable $\varphi$.

For a specific spin dynamics we must add, as in (2.15), a Hamiltonian $H(\cos\theta, \varphi)$, and arrive at the general path integral representation for the time evolution amplitude of a spinning particle [10]

$$(\mathbf{u}_b t_b|\mathbf{u}_a t_a) = \int \mathcal{D}\mathbf{u}\, e^{i\int_{t_a}^{t_b} dt\,[\hbar s\cos\theta\dot{\varphi} - H(\theta,\varphi)]/\hbar}. \tag{8.295}$$

The above path integral has a remarkable property which is worth emphasizing. For half-integer spins $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ it is able to describe the physics of a fermion in terms of a field theory involving a unit vector field $\mathbf{u}$ which describes the direction of the spin state:

$$\langle \mathbf{u}|\hat{\mathbf{S}}|\mathbf{u}\rangle = \langle ss|\hat{R}^{-1}(\theta,\varphi)\hat{\mathbf{S}}\hat{R}(\theta,\varphi)|ss\rangle = R_{ij}(\theta,\varphi)\langle ss|S_j|ss\rangle = s\mathbf{u}. \tag{8.296}$$

This follows from the vector property of the spin matrices $\hat{\mathbf{S}}$. The matrices $R_{ij}(\theta,\varphi)$ are the defining $3\times 3$ matrices of the rotation group (the so-called *adjoint representation*). Thus we describe a fermion in terms of a Bose field.

The above path integral is only the simplest illustration for a more general phenomenon. In 1961, Skyrme pointed out that a certain field configuration of pions is capable of behaving in many respects like a nucleon [11], in particular its fermionic properties.

In two dimensions, Bose field theories are even more powerful and can describe particles with any commutation rule, called anyons in Section 7.5. This will be shown in Chapter 16.

In Chapter 10 we shall see that the action in (8.293) can be interpreted as the action of a particle of charge $e$ on the surface of the unit sphere whose center contains a fictitious magnetic monopole of charge $g = -4\pi\hbar cs/e$. The associated vector potential $\mathbf{A}^{(g)}(\mathbf{u})$ will be given in Eq. (10A.59). Coupling this minimally to a particle of charge $e$ as in Eq. (2.635) on the surface of the sphere yields the action

$$\mathcal{A}_0 = \frac{e}{c}\int_{t_a}^{t_b} \mathbf{A}^{(g)}(\mathbf{u})\cdot\dot{\mathbf{u}} = \hbar\, s\int_{t_a}^{t_b} dt\,\cos\theta\,\dot{\varphi}\,, \tag{8.297}$$

where the magnetic flux is supplied to the monopole by two infinitesimally thin flux tubes, the famous *Dirac strings*, one from below and one from above. The field $\mathbf{A}^{(-s)}$ in (8.297) is the average of the two expressions in (10A.61) for $g = -4\pi\hbar cs/e$. We can easily change the supply line to a single string from above, by choosing the states

$$|\theta\,\varphi\rangle' \equiv \hat{R}(\theta,\varphi)|ss\rangle = e^{-iS_3\varphi}e^{-iS_2\theta}e^{iS_3\varphi}|ss\rangle = |\theta\,\varphi\rangle e^{is\varphi}, \tag{8.298}$$

rather than $|\theta\,\varphi\rangle$ of Eq. (8.275) for the construction of the path integral. The physics is the same since the string is an artifact of the choice of the quantization axis.

In terms of Cartesian coordinates, the action (8.297) with a flux supplied from the north pole can also be expressed in terms of the vector $\mathbf{u}(t)$ as [compare (10A.59)]

$$\mathcal{A}_0 = \hbar s\int dt\,\frac{\hat{\mathbf{z}}\times\mathbf{u(t)}}{1-u_z(t)}\cdot\dot{\mathbf{u}}(t). \tag{8.299}$$

This expression is singular on the north pole of the unit sphere. The singularity can be rotated into an arbitrary direction $\mathbf{n}$, leading to

$$\mathcal{A}_0 = \hbar s \int dt \, \frac{\mathbf{n} \times \mathbf{u}(\mathbf{t})}{1 - \mathbf{n} \cdot \mathbf{u}(t)} \cdot \dot{\mathbf{u}}(t). \tag{8.300}$$

If $\mathbf{u}$ gets close to $\mathbf{n}$ we must change the direction of $\mathbf{n}$. The action (8.300) is referred to as *Wess-Zumino action*.

In Chapter 10 we shall also calculate the curl of the vector potential $\mathbf{A}^{(g)}$ of a monopole of magnetic charge $g$ and find the radial magnetic field accompanied by a singular string contribution along the direction $\mathbf{n}$ of flux supply [compare (10A.54)]:

$$\mathbf{B}^{(g)} = \boldsymbol{\nabla} \times \mathbf{A}^{(g)} = g \, \frac{\mathbf{u}}{|\mathbf{u}|} - 4\pi g \int_0^\infty ds \, \hat{\mathbf{n}} \, \delta^{(3)}(\mathbf{u} - s \, \hat{\mathbf{n}}), \tag{8.301}$$

The singular contribution is an artifact of the description of the magnetic field. The line from zero to infinity is called a *Dirac string*. Since the magnetic field has no divergence, the magnetic flux emerging at the origin of the sphere must be imported from somewhere at infinity. In the field (8.302) the field is imported along the straight line in the direction $\mathbf{u}$. Indeed, we can easily check that the divergence of (8.302) is zero.

For a closed orbit, the interaction (8.297) can be rewritten by Stokes' theorem as

$$\mathcal{A}_0 = \frac{e}{c} \int dt \, \mathbf{A}^{(g)}(t) \cdot \mathbf{u}(t) = \frac{e}{c} \int d\mathbf{S} \cdot \left[ \boldsymbol{\nabla} \times \mathbf{A}^{(g)} \right] = \frac{e}{c} \int d\mathbf{S} \cdot \mathbf{B}^{(g)}, \tag{8.302}$$

where $\int d\mathbf{S}$ runs over the surface enclosed by the orbit. This surface may or may not contain the Dirac string of the monopole, in which case $\mathcal{A}_0$ differ by $4\pi ge/c$. A path integral over closed orbits of the spinning particle

$$Z_{\mathrm{QM}} = \oint d\mathbf{u} \, e^{i(\mathcal{A}_0 + \mathcal{A})/\hbar}, \tag{8.303}$$

is therefore invariant under changes of the position of the Dirac string if the monopole charge $g$ satisfies the *Dirac charge quantization condition*

$$\frac{ge}{\hbar c} = s, \tag{8.304}$$

with $s =$half-integer or integer.

Dirac was the first to realize that as a consequence of quantum mechanics, an electrically charged particle whose charge satisfies the quantization condition (8.304) sees only the radial monopole field in (8.302), not the field in the string. The string can run along *any line* $L$ without being detectable. This led him to conjecture that there could exist magnetic monopoles of a specific $g$, which would explain that all charges in nature are integer multiples of the electron charge [16]. More on this subject will be discussed in Section 16.2.

In Chapter 10 we shall learn how to define a monopole field $\mathbf{A}^{(g)}$ which is free of the artificial string singularity [see Eq. (10A.58)]. With the new definition, the divergence of $\mathbf{B}$ is a $\delta$-function at the origin:

$$\boldsymbol{\nabla} \times \mathbf{B}(\mathbf{u}) = 4\pi g \, \delta^{(3)}(\mathbf{u}). \tag{8.305}$$

## 8.13 Berry Phase

This phenomenon has a simple physical basis which can be explained most clearly by means of the following gedanken experiment. Consider a thin rod whose dynamics is described by a unit vector field $\mathbf{u}(t)$ with an action

$$\mathcal{A} = \frac{M}{2} \int dt \left[ \dot{\mathbf{u}}^2(t) - V(\mathbf{u}^2(t)) \right], \quad \mathbf{u}^2(t) \equiv 1, \tag{8.306}$$

where $\mathbf{u}(t)$ is a unit vector along the rod. This is the same Lagrangian as for a particle on a sphere as in (8.147) (recall p. 746).

Let us suppose that the thin rod is a solenoid carrying a strong magnetic field, and containing at its center a particle of spin $s = 1/2$. Then (8.306) is extended by the action [13]

$$\mathcal{A}_0 = \int dt\, \psi^*(t) \left[ i\hbar\partial_t + \gamma \mathbf{u}(t) \cdot \boldsymbol{\sigma} \right] \psi(t), \tag{8.307}$$

where $\sigma^i$ are the Pauli spin matrices (1.448). For large coupling strength $\gamma$ and sufficiently slow rotations of the solenoid, the direction of the fermion spin will always be in the ground state of the magnetic field, i.e., its direction will follow the direction of the solenoid adiabatically, pointing always along $\mathbf{u}(t)$. If we parametrize $\mathbf{u}(t)$ in terms of spherical angles $\theta(t), \phi(t)$ as $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the associated wave function satisfies $\mathbf{u}(t) \cdot \boldsymbol{\sigma}\, \psi(\mathbf{u}(t)) = (1/2)\psi(\mathbf{u}(t))$, and reads

$$\psi(\mathbf{u}(t)) = \begin{pmatrix} e^{-i\phi(t)/2} \cos\theta(t)/2 \\ e^{i\phi(t)/2} \sin\theta(t)/2 \end{pmatrix}. \tag{8.308}$$

Inserting this into (8.307) we obtain [compare (8.292)]

$$\mathcal{A}_0 = \int dt\, \psi(\mathbf{u}(t))\, i\hbar\partial_t\, \psi(\mathbf{u}(t)) = \hbar \frac{1}{2} \cos\theta(t)\, \dot{\phi}(t) \equiv \hbar\beta(t). \tag{8.309}$$

The action coincides with the previous expression (8.297). The angle $\beta(t)$ is called *Berry phase* [14].

In this simple model it is obvious why the bosonic theory of the solenoid behaves like a spin-1/2 particle: It simply inherits the physical properties of the enslaved spinor.

The reason why it is a monopole field that causes the spin-1/2 behavior will become clear in another way in Section 14.6. There we shall solve the path integral of a charged particle in a monopole field and show that it behaves like a fermion if its charge $e$ and the monopole charge $g$ have half-integer products $q \equiv eg/\hbar c$.

## 8.14 Spin Precession

The Wess-Zumino action $\mathcal{A}_0$ adds an interesting kinetic term to the equation of motion of a solenoid. Extremizing $\mathcal{A}_0 + \mathcal{A}$ yields

$$M \left( \ddot{\mathbf{u}} + \dot{\mathbf{u}}^2\, \mathbf{u} \right) = -\partial_{\mathbf{u}} V(\mathbf{u}) - \frac{\delta}{\delta \mathbf{u}(t)} \mathcal{A}_0. \tag{8.310}$$

The functional derivative of $\mathcal{A}_0$ is most easily calculated starting from the general expression in (8.297)

$$\frac{\delta}{\delta u_i(t)}\hbar\int_{t_a}^{t_b} dt\,\mathbf{A}^{(g)}\cdot\dot{\mathbf{u}} = \hbar\left(\partial_{u_i}A_j^{(g)}\dot{u}_j - \partial_t A_i^{(g)}\right) = \hbar\left[\left(\partial_{u_i}A_j^{(g)} - \partial_{u_j}A_i^{(g)}\right)\dot{u}_j\right]$$
$$= \hbar\left[\left(\boldsymbol{\nabla}\times\mathbf{A}^{(g)}\right)\times\mathbf{u}\right]_i. \tag{8.311}$$

Inserting here the curl of Eq. (8.302), while staying safely away from the singularity, the last term in (8.310) becomes $\hbar g\mathbf{u}\times\dot{\mathbf{u}}$, and the equation of motion (8.310) turns into

$$M\left(\ddot{\mathbf{u}} + \dot{\mathbf{u}}^2\,\mathbf{u}\right) = -\partial_{\mathbf{u}}V(\mathbf{u}) - \hbar g\,\mathbf{u}\times\dot{\mathbf{u}}. \tag{8.312}$$

Multiplying this vectorially by $\mathbf{u}(t)$ we find [15, 16]

$$M\mathbf{u}\times\ddot{\mathbf{u}} = -\mathbf{u}\times\partial_{\mathbf{u}}V(\mathbf{u}) - \hbar g\left[\mathbf{u}\,(\mathbf{u}\cdot\dot{\mathbf{u}}) - \dot{\mathbf{u}}\,\mathbf{u}^2\right]. \tag{8.313}$$

Since $\mathbf{u}^2 = 1$, this reduces to

$$M\mathbf{u}\times\ddot{\mathbf{u}} - \hbar g\dot{\mathbf{u}} = -\mathbf{u}\times\partial_{\mathbf{u}}V(\mathbf{u}(t)). \tag{8.314}$$

If $M$ is small we obtain for a particle of spin $s$, where $g = -s$, the so-called *Landau-Lifshitz equation* [17, 18]

$$\hbar s\,\dot{\mathbf{u}} = -\mathbf{u}\times\partial_{\mathbf{u}}V(\mathbf{u}). \tag{8.315}$$

This is a useful equation for studying magnetization fields in ferromagnetic materials.

The interaction energy of a spinning particle with an external magnetic field has the general form

$$H_{\text{int}} = -\boldsymbol{\mu}\cdot\mathbf{B}, \qquad \boldsymbol{\mu} = -\gamma\mathbf{S}, \tag{8.316}$$

where $\boldsymbol{\mu}$ is the magnetic moment, $\mathbf{S}$ the vector of spin matrices, and $\gamma$ the *gyromagnetic ratio*.

Since $\mathbf{u}$ is the direction vector of the spin, we may identify the magnetic moment of the spin $s$ as

$$\boldsymbol{\mu} = \gamma s\hbar\mathbf{u}, \tag{8.317}$$

so that the interaction energy becomes

$$H_{\text{int}} = -\gamma s\hbar\,\mathbf{u}\cdot\mathbf{B}. \tag{8.318}$$

Inserting this for $V(\mathbf{u})$ in (8.315) yields the equation of motion

$$\dot{\mathbf{u}} = -\gamma\,\mathbf{B}\times\mathbf{u}, \tag{8.319}$$

showing a rate of precession $\boldsymbol{\Omega} = -\gamma\mathbf{B}$.

For comparison we recall the derivation of this result in the conventional way from the Heisenberg equation of motion (1.272):

$$\hbar\,\dot{\mathbf{S}} = i[\hat{H}, \mathbf{S}]. \tag{8.320}$$

Inserting for $\hat{H}$ the interaction energy (8.316) and using the commutation relations (1.414) of the rotation group for the spin matrices **S**, this yields the Heisenberg equation for spin precession

$$\dot{\mathbf{S}} = -\gamma\,\mathbf{B} \times \mathbf{S}, \tag{8.321}$$

in agreement with the Landau-Lifshitz equation (8.315). This shows that the Wess-Zumino term in the action $\mathcal{A}_0 + \mathcal{A}$ has the ability to render quantum equations of motion from a classical action. This allows us, in particular, to mimic systems of half-integer spins, which are fermions, with a theory containing only a bosonic directional vector field $\mathbf{u}(t)$. This has important applications in statistical mechanics where models of interacting quantum spins for ferro- and antiferromagnets à la Heisenberg can be studied by applying field theoretic methods to vector field theories.

## Notes and References

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