7 The shooting method for solving BVPs

7.1 The idea of the shooting method

In the first four subsections of this lecture we will only consider BVPs that satisfy the conditions of Theorems 6.1 or 6.2 and thus are guaranteed to have a unique solution.

Suppose we want to solve a BVP with Dirichlet boundary conditions:

$$y'' = f(x, y, y'), \qquad y(a) = \alpha, \quad y(b) = \beta\,. \tag{7.1}$$

We can rewrite this BVP in the form:

$$\begin{aligned} &y' = z \\ &z' = f(x, y, z) \\ &y(a) = \alpha \\ &y(b) = \beta\,. \end{aligned} \tag{7.2}$$

The BVP (7.2) will turn into an IVP if we replace the boundary condition at $x = b$ with the condition

$$z(a) = \theta\,, \tag{7.3}$$

where θ is some number. Then we can solve the resulting IVP by any method that we have studied in Lecture 5, and obtain the value of its solution $y(b)$ at $x = b$. If $y(b) = \beta$, then we have solved the BVP. Mostly likely, however, we will find that after the first try, $y(b) \neq \beta$. Then we should choose another value for θ and try again. There is actually a strategy of how the values of θ need to be chosen. This strategy is simpler for linear BVPs, so this is the case we consider next.

7.2 Shooting method for the Dirichlet problem of linear BVPs

Thus, our immediate goal is to solve the linear BVP

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{with } Q(x) \le 0, \qquad y(a) = \alpha, \quad y(b) = \beta\,. \tag{7.4}$$

To this end, consider two auxiliary IVPs:

$$\begin{aligned} &u'' + Pu' + Qu = R, \\ &u(a) = \alpha, \quad u'(a) = 0 \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &v'' + Pv' + Qv = 0, \\ &v(a) = 0, \quad v'(a) = 1\,, \end{aligned} \tag{7.6}$$

where we omit the arguments of $P(x)$ etc. as this should cause no confusion. Next, consider the function

$$w = u + \theta v, \qquad \theta = \text{const}\,. \tag{7.7}$$

Using Eqs. (7.5) and (7.6), it is easy to see that

$$\begin{aligned} &(u + \theta v)'' + P\,(u + \theta v)' + Q\,(u + \theta v) = R, \\ &(u + \theta v)(a) = \alpha, \quad (u + \theta v)'(a) = \theta\,, \end{aligned} \tag{7.8}$$

i.e. w satisfies the IVP

$$\begin{aligned} &w'' + Pw' + Qw = R, \\ &w(a) = \alpha, \quad w'(a) = \theta\,. \end{aligned} \tag{7.9}$$

Note that the only difference between (7.9) and (7.4) is that in (7.9), we know the value of w' at $x = a$ but do *not* know whether $w(b) = \beta$. If we can choose θ in such a way that $w(b)$ *does* equal b, this will mean that we have solved the BVP (7.4).

To determine such a value of θ, we first solve the IVPs (7.5) and (7.6) by an appropriate method of Lecture 5 and find the coresponding values $u(b)$ and $v(b)$. We then choose the value $\theta = \theta_0$ by requiring that the corresponding $w(b) = \beta$, i.e.

$$w(b) = u(b) + \theta_0 v(b) = \beta\,. \tag{7.10}$$

This $w(x)$ is the solution of the BVP (7.4), because it satisfies the same ODE and the same boundary conditions at $x = a$ and $x = b$; see Eqs. (7.9) and (7.10). Equation (7.10) yields the following equation for the θ_0:

$$\theta_0 = \frac{\beta - u(b)}{v(b)}\,. \tag{7.11}$$

Thus, solving only two IVPs (7.5) and (7.6) and constructing the new function $w(x)$ according to (7.7) and (7.11), we obtain the solution to the linear BVP (7.4).

Consistency check: In (7.4), we have required that $Q(x) \le 0$, which guarantees that a unique solution of that BVP exists. What would have happened if we had overlooked to impose that requirement? Then, according to Theorem 6.2, we could have run into a situation where the BVP would have had no solutions. This would occur if

$$v(b) = 0\,. \tag{7.12}$$

But then Eqs. (7.6) and (7.12) would mean that the *homogeneous* BVP

$$\begin{aligned} &v'' + Pv' + Qv = 0, \\ &v(a) = 0, \quad v(b) = 0\,, \end{aligned} \tag{7.13}$$

must have a nontrivial[16] solution. The above considerations agree, as they should, with the Alternative Principle for the BVPs, namely: the BVP (7.4) may have no solutions if the corresponding homogeneous BVP (7.13) has nontrivial solutions.

7.3 Generalizations of the shooting method for linear BVPs

If the BVP has boundary conditions other than of the Dirichlet type, we will still proceed exactly as we did above. For example, suppose we need to solve the BVP

$$\begin{aligned} &y'' + Py' + Qy = R, \\ &y(a) = \alpha, \quad y'(b) = \beta, \end{aligned} \tag{7.14}$$

which has the Neumann boundary condition at the right end point. Denote $y_1 = y$, $y_2 = y'$, and rewrite this BVP as

$$\begin{aligned} &y_1' = y_2 \\ &y_2' = -Py_2 - Qy_1 + R \\ &y_1(a) = \alpha, \quad y_2(b) = \beta\,. \end{aligned} \tag{7.15}$$

[16] Indeed, since we also know that $v'(a) = 1$, then $v(x)$ cannot identically equal zero on $[a, b]$.

Using the vector/matrix notations with

$$\vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

we can further rewrite this BVP as

$$\vec{\mathbf{y}}\,' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \vec{\mathbf{y}} + \begin{pmatrix} 0 \\ R \end{pmatrix}, \qquad \begin{array}{l} y_1(a) = \alpha \\ y_2(b) = \beta\,. \end{array} \tag{7.16}$$

Now, in analogy with Eqs. (7.5) and (7.6), consider two auxiliary IVPs:

$$\vec{\mathbf{u}}\,' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \vec{\mathbf{u}} + \begin{pmatrix} 0 \\ R \end{pmatrix}, \qquad \vec{\mathbf{u}}(a) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}; \tag{7.17}$$

$$\vec{\mathbf{v}}\,' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \vec{\mathbf{v}}\,, \qquad \vec{\mathbf{v}}(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7.18}$$

Solve these IVPs by an appropriate method and obtain the values $\vec{\mathbf{u}}(b)$ and $\vec{\mathbf{v}}(b)$. Next, consider the vector

$$\vec{\mathbf{w}} = \vec{\mathbf{u}} + \theta\vec{\mathbf{v}}, \qquad \theta = \text{const.} \tag{7.19}$$

Using Eqs. (7.17)–(7.19), it is easy to see that this new vector satisfies the IVP

$$\vec{\mathbf{w}}\,' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \vec{\mathbf{w}} + \begin{pmatrix} 0 \\ R \end{pmatrix}, \qquad \vec{\mathbf{w}}(a) = \begin{pmatrix} \alpha \\ \theta \end{pmatrix}. \tag{7.20}$$

At $x = b$, its value is

$$\vec{\mathbf{w}}(b) = \begin{pmatrix} u_1(b) + \theta v_1(b) \\ u_2(b) + \theta v_2(b) \end{pmatrix}, \tag{7.21}$$

where u_1 is the first component of $\vec{\mathbf{u}}$ etc. From the last equation in (7.16), it follows that we are to require that

$$w_2(b) = \beta\,. \tag{7.22}$$

Equations (7.21) and (7.22) together yield

$$u_2(b) + \theta v_2(b) = \beta\,, \qquad \Rightarrow \tag{7.23}$$

$$\theta_0 = \frac{\beta - u_2(b)}{v_2(b)}\,. \tag{7.24}$$

Thus, the vector $\vec{\mathbf{w}}$ given by Eq. (7.19) where $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and θ satisfy Eqs. (7.17), (7.18), and (7.24), respectively, is the solution of the BVP (7.16).

Also, the shooting method can be used with IVPs of order higher than the second. For example, consider the BVP

$$\begin{array}{l} x^3 y''' + xy' - y = -3 + \ln x\,, \\ y(a) = \alpha, \quad y'(b) = \beta, \quad y''(b) = \gamma\,. \end{array} \tag{7.25}$$

As in the previous example, denote $y_1 = y$, $y_2 = y'$, $y_3 = y''$ and rewrite the BVP (7.25) in the matrix form:

$$\vec{\mathbf{y}}\,' = A\vec{\mathbf{y}} + \vec{\mathbf{r}}\,, \qquad \begin{array}{l} y_1(a) = \alpha \\ y_2(b) = \beta \\ y_3(b) = \gamma\,. \end{array} \tag{7.26}$$

where

$$\vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^3} & -\frac{1}{x^2} & 0 \end{pmatrix}, \quad \vec{\mathbf{r}} = \begin{pmatrix} 0 \\ 0 \\ \frac{-3+\ln x}{x^3} \end{pmatrix}.$$

Consider now *three* auxiliary IVPs:

$$\begin{array}{lll} \vec{\mathbf{u}}\,' = A\vec{\mathbf{u}} + \vec{\mathbf{r}}\,, & \vec{\mathbf{v}}\,' = A\vec{\mathbf{v}}\,, & \vec{\mathbf{w}}\,' = A\vec{\mathbf{w}}\,, \\ \vec{\mathbf{u}}(a) = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}; & \vec{\mathbf{v}}(a) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; & \vec{\mathbf{w}}(a) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{array} \tag{7.27}$$

Solve them and obtain $\vec{\mathbf{u}}(b)$, $\vec{\mathbf{v}}(b)$, and $\vec{\mathbf{w}}(b)$. Then, construct $\vec{\mathbf{z}} = \vec{\mathbf{u}} + \theta\vec{\mathbf{v}} + \phi\vec{\mathbf{w}}$, where θ and ϕ are numbers which will be determined shortly. At $x = b$, one has

$$\vec{\mathbf{z}}(b) = \begin{pmatrix} \dots \\ u_2(b) + \theta v_2(b) + \phi w_2(b) \\ u_3(b) + \theta v_3(b) + \phi w_3(b) \end{pmatrix}. \tag{7.28}$$

If we require that $\vec{\mathbf{z}}$ satisfy the BVP (7.26), we must have

$$\vec{\mathbf{z}}(b) = \begin{pmatrix} \dots \\ \beta \\ \gamma \end{pmatrix}. \tag{7.29}$$

From Eqs. (7.28) and (7.29) we form a system of two linear equations for the unknown coefficients θ and ϕ:

$$\begin{aligned} \theta v_2(b) + \phi w_2(b) &= \beta - u_2(b)\,, \\ \theta v_3(b) + \phi w_3(b) &= \gamma - u_3(b)\,. \end{aligned} \tag{7.30}$$

Solving this linear system, we obtain values θ_0 and ϕ_0 such that the corresponding $\vec{\mathbf{z}} = \vec{\mathbf{u}} + \theta_0\vec{\mathbf{v}} + \phi_0\vec{\mathbf{w}}$ solves the BVP (7.26) and hence the original BVP (7.25).

7.4 Caveat with the shooting method, and its remedy, the multiple shooting method

Here we will encounter a situation where the shooting method in its form described above does not work. We will also provide a way to modify the method so that it would be usable again.

Let us consider the BVP

$$\begin{aligned} &y'' = 30^2\,(y - 1 + 2x)\,, \\ &y(0) = 1, \quad y(b) = 1 - 2b\,; \quad b > 0. \end{aligned} \tag{7.31}$$

Its exact solution is

$$y = 1 - 2x\,; \tag{7.32}$$

by Theorem 6.2 this solution is unique. Note that the general solution of only the ODE (without boundary conditions) in (7.31) is

$$y = 1 - 2x \,+\, A\,e^{30x} + B\,e^{-30x}\,. \tag{7.33}$$

Now let us try to use the shooting method to solve the BVP (7.31). Following the lines of Sec. 7.2, we set up auxiliary IVPs

$$u'' = 30^2(u - 1 + 2x)\,, \qquad v'' = 30^2 v\,,$$
$$u(0) = 1, \quad u'(0) = 0\,; \qquad v(0) = 0, \quad v'(0) = 1\,; \tag{7.34}$$

and solve them. The *exact* solutions of (7.34) are:

$$u = 1 - 2x + \frac{1}{30}\left(e^{30x} - e^{-30x}\right), \qquad v = \frac{1}{60}\left(e^{30x} - e^{-30x}\right). \tag{7.35}$$

Then Eq. (7.11) provides the value of the auxiliary parameter θ_0:

$$\theta_0 = \frac{(1-2b) - (1 - 2\cdot b + \frac{1}{30}(e^{30\cdot b} - e^{-30\cdot b}))}{\frac{1}{60}(e^{30\cdot b} - e^{-30\cdot b})} \;=\; -2\,. \tag{7.36}$$

That is,

$$u_{(7.35)} + (-2)\cdot v_{(7.35)} \;=\; 1 - 2x \;=\; \text{exact solution}\,, \tag{7.37a}$$

or, in detail:

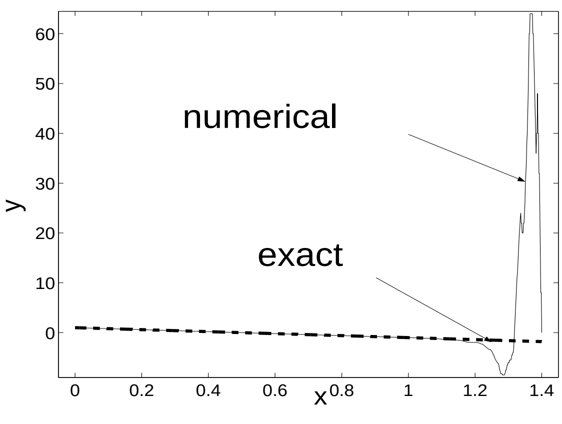
$$\left\{1 - 2x + \frac{1}{30}\left(e^{30x} - e^{-30x}\right)\right\} + (-2)\cdot\left\{\frac{1}{60}\left(e^{30x} - e^{-30x}\right)\right\} \;=\; 1 - 2x. \tag{7.37b}$$

Note that each of the terms in curly brackets in (7.37b) is a very large number. For example, for $x = 1.4$, the magnitude of the second term is:

$$\frac{1}{30}\left(e^{30\cdot 1.4} - e^{-30\cdot 1.4}\right) \approx \frac{1}{30}\cdot e^{30\cdot 1.4} \approx 5.8\cdot 10^{16}\,. \tag{7.38}$$

Now recall that Matlab's round-off error `eps` $= 2.2\cdot 10^{-16}$. This means two things:
(i) if for some numbers x and y, one has $x - y <$ `eps`, then Matlab "thinks" that $x - y = 0$;
(ii) if z is such a large number that $z > 1/$`eps`, then Matlab may compute it with an error that may be of order 1 and even larger.
(In one of the QSA's you will be asked to verify these statements.)

Statement (ii) above along with (7.38) mean that the expression on the l.h.s. of (7.37b) for $x \approx 1.4$ is evaluated by Matlab with an error of order 1 or larger. That is, the l.h.s. of (7.37b), as it is evaluated by Matlab, *deviates from the exact solution* by as much — or more — than the exact solution itself. This qualitatively explains the origin of the "burst" seen near $x = 1.4$ in the figure on the right, where the numerical solution of (7.31) with $b = 1.4$ is shown.



A reason why this burst is so large — almost two orders of magnitude greater than the exact solution, while the "pieces" $u(x)$ and $v(x)$ in (7.37) are only a factor of 2–3 greater than 1/ `eps`, — is not quite clear. Perhaps, one explanation can be that the global errors of the numerical solutions for $u(x)$ and $v(x)$ are proportional to the large number e^{30x} (see Eq. (1.16) in Lecture 1), and such large errors for $u(x)$ and $v(x)$ do not completely cancel out in (7.37). Another reason can be that the parameter θ_0 in (7.36) may be evaluated by Matlab with a round-off error: $\theta_0 = -2 + \mathsf{eps}$, and then the computed solution is

$$y = u_{(7.35)} + (-2 + \mathsf{eps}) \cdot v_{(7.35)} \;=\; 1 - 2x \;+\; \frac{\mathsf{eps}}{60}\left(e^{30x} - e^{-30x}\right)\,. \tag{7.39}$$

Again, the third term in that expression may be of order 1 or greater.

The way in which the shooting method is to be modified in order to handle the above problem is suggested by the figure on the previous page. Namely, one can see that the numerical solution is quite accurate up to the vicinity of the right end point, where $x \approx 1.4$ and where the factor e^{30x} multiplied by some small error, overtakes the exact solution. Therefore, if we split the interval [0, 1.4] into two adjoint subintervals, [0, 0.7] and [0.7, 1.4], and perform shooting on each of these subintervals, then the corresponding exponential factor, $e^{30\cdot 0.7} \sim 10^9$, will be well resolved by Matlab: $e^{30\cdot 0.7} \sim 10^9 \ll 1/\mathsf{eps} \sim 5\cdot 10^{15}$.

Below we show the implementation details of this approach, known as the multiple shooting method. (Obviously, the name comes from the fact that the shooting is performed in multiple (sub)intervals.) These details are worked out for the case of two subintervals, $[0,\, b/2]$ and $[b/2,\, b]$; a generalization for the case of more subintervals is fairly straightforward.

Consider *two* sets of auxiliary IVPs that are similar to the IVPs (7.34):

$$\underline{\text{On } [0,\, b/2]:}\qquad \begin{array}{l} u^{(1)\,\prime\prime} = 30^2(u^{(1)} - 1 + 2x)\,, \\ u^{(1)}(0) = \alpha, \quad u^{(1)\,\prime}(0) = 0\,; \end{array} \qquad \begin{array}{l} v^{(1)\,\prime\prime} = 30^2 v^{(1)}\,, \\ v^{(1)}(0) = 0, \quad v^{(1)\,\prime}(0) = 1\,; \end{array} \tag{7.40}$$

$$\underline{\text{On } [b/2,\, b]:}\qquad \begin{array}{l} u^{(2)\,\prime\prime} = 30^2(u^{(2)} - 1 + 2x)\,, \\ u^{(2)}(b/2) = \boxed{0}, \quad u^{(2)\,\prime}(b/2) = 0\,; \end{array} \qquad \begin{array}{l} v^{(2,1)\,\prime\prime} = 30^2 v^{(2,1)}\,, \\ v^{(2,1)}(b/2) = 1, \quad v^{(2,1)\,\prime}(b/2) = 0\,; \\ v^{(2,2)\,\prime\prime} = 30^2 v^{(2,2)}\,, \\ v^{(2,2)}(b/2) = 0, \quad v^{(2,2)\,\prime}(b/2) = 1\,. \end{array} \tag{7.41}$$

Note 1: The initial condition for $u^{(1)}$ at $x = 0$ is denoted as α, even though in the example considered $\alpha = 1$. This is done to emphasize that the given initial condition is always used for $u^{(1)}$ at the left end point of the original interval.

Note 2: Note that in the 2nd subinterval (and, in general, in the kth subinterval with $k \geq 2$), the initial conditions for the $u^{(k)}$ must be taken to *always* be zero. (This is stressed in the u-system in (7.41) by putting the initial condition for $u^{(2)}$ in a box.)

Continuing with solving the IVPs (7.40) and (7.41), we construct solutions

$$w^{(1)} = u^{(1)} + \theta^{(1)} v^{(1)}\,, \qquad w^{(2)} = u^{(2)} + \theta^{(2,1)} v^{(2,1)} + \theta^{(2,2)} v^{(2,2)}\,, \tag{7.42}$$

where the numbers $\theta^{(1)}$, $\theta^{(2,1)}$, and $\theta^{(2,2)}$ are to be determined. Namely, these three numbers are determined from three requirements:

$$\begin{array}{l} w^{(1)}(b/2) = w^{(2)}(b/2)\,, \\ w^{(1)\,\prime}(b/2) = w^{(2)\,\prime}(b/2)\,; \end{array} \qquad \left(\begin{array}{c}\text{The solution and its derivative} \\ \text{must be continuous at } x = b/2;\end{array}\right) \tag{7.43}$$

and

$$w^{(2)}(b) = \beta\,(= 1 - 2b)\,, \qquad \left(\begin{array}{c}\text{The solution must satisfy} \\ \text{the boundary condition at } x = b.\end{array}\right) \tag{7.44}$$

Equations (7.43) and (7.44) yield:

$$\begin{array}{lllll} w^{(1)}(b/2) = w^{(2)}(b/2) & \Rightarrow & u^{(1)}(b/2) + \theta^{(1)} v^{(1)}(b/2) & = & 0 + \theta^{(2,1)} \cdot 1 + \theta^{(2,2)} \cdot 0\,; \\ w^{(1)\prime}(b/2) = w^{(2)\prime}(b/2) & \Rightarrow & u^{(1)\prime}(b/2) + \theta^{(1)} v^{(1)\prime}(b/2) & = & 0 + \theta^{(2,1)} \cdot 0 + \theta^{(2,2)} \cdot 1\,; \\ w^{(2)}(b) = \beta & \Rightarrow & u^{(2)}(b) + \theta^{(2,1)} \cdot v^{(2,1)}(b) + \theta^{(2,2)} \cdot v^{(2,2)}(b) & = & \beta\,. \end{array} \tag{7.45}$$

In writing out the r.h.s.'s of the first two equations above, we have used the boundary conditions of the IVP (7.41).

The three equations (7.45) form a linear system for the three unknowns $\theta^{(1)}$, $\theta^{(2,1)}$, and $\theta^{(2,2)}$. (Recall that $u^{(1)}(b/2)$ etc. are known from solving the IVPs (7.40) and (7.41).) Thus, finding the $\theta^{(1)}$, $\theta^{(2,1)}$, and $\theta^{(2,2)}$ from (7.45) and substituting them back into (7.42), we obtain the solution to the original BVP (7.31).

Important note: The multiple shooting method, at least in the above form, can only be used for linear BVPs, because it is only for them that the linear superposition principle, allowing us to write

$$w = u + \theta v\,,$$

can be used.

Further reading on the multiple shooting method can be found in:

- P. Deuflhard, "Recent advances in multiple shooting techniques," in *Computation techniques for ODEs*, I. Gladwell and D.K. Sayers, eds. (Academic Press, 1980);
- J. Stoer and R. Burlish, "Introduction to numerical analysis" (Springer Verlag, 1980);
- G. Hall and J.M. Watt, "Modern numerical methods for ODEs" (Clarendon Press, 1976).

7.5 Shooting method for nonlinear BVPs

As has just been noted above, for nonlinear BVPs, the linear superposition of the auxiliary solutions u and v cannot be used. However, one can still proceed with using the shooting method by following the general guidelines of Sec. 7.1.

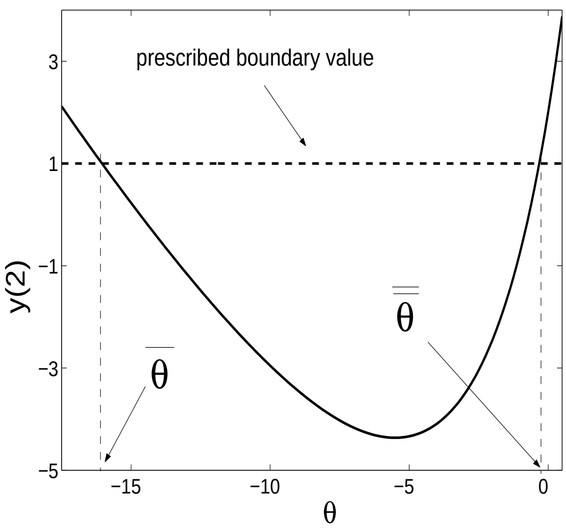
As an example, let us consider the BVP

$$\begin{aligned} y'' &= \frac{y^2}{2+x}\,, \\ y(0) &= 1, \quad y(2) = 1\,. \end{aligned} \tag{7.46}$$

We again consider the auxiliary IVP

$$\begin{aligned} y_1' &= y_2\,, \\ y_2' &= \frac{y_1^2}{2+x}\,; \\ y_1(0) &= 1, \quad y_2(0) = \theta\,. \end{aligned} \tag{7.47}$$

The idea is now to find the right value(s) of θ *iteratively*. To motivate the iteration algorithm, let us actually solve the IVP (7.47) for an (equidistant) set of θ's inside some large interval and look at the result, which is shown in the figure on the right. (The reason this particular interval of θ values is chosen is simply because the instructor knows what the result should be.) This figure shows that the boundary condition at the right end point,



$$y(2)|_{\text{as the function of } \theta} = 1\,, \tag{7.48}$$

can be considered as a *nonlinear algebraic equation* with respect to θ. Correspondingly, we can employ well-known methods of solving nonlinear algebraic equations for solving nonlinear BVPs.

Probably the simplest such a method is the secant method. Below we will show how to use it to find the values $\bar{\theta}$ and $\bar{\bar{\theta}}$, for which $y(2) = 1$ (see the figure). Suppose we have tried two values, θ_1 and θ_2, and found the corresponding values $y(2)|_{\theta_1}$ and $y(2)|_{\theta_2}$. Denote

$$F(\theta_k) = y(2)|_{\theta_k} - 1\,, \qquad k = 1, 2\,. \tag{7.49}$$

Thus, our goal is to find the roots of the equation

$$F(\theta) = 0\,. \tag{7.50}$$

Given the first two values of $F(\theta)$ at $\theta = \theta_{1,2}$, the secant method proceeds as follows:

$$\theta_{k+1} = \theta_k - \frac{F(\theta_k)}{\left[\frac{F(\theta_k)-F(\theta_{k-1})}{\theta_k-\theta_{k-1}}\right]}\,, \qquad \text{and then compute } F(\theta_{k+1}) \text{ from (7.49).} \tag{7.51}$$

The iterations are stopped when $|F(\theta_{k+1}) - F(\theta_k)|$ becomes less than a prescribed tolerance. In this manner, one will find the values $\bar{\theta}$ and $\bar{\bar{\theta}}$ and hence the corresponding two solutions of the nonlinear BVP (7.46). You will be asked to do so in a homework problem.

7.6 Broader applicability of the shooting method

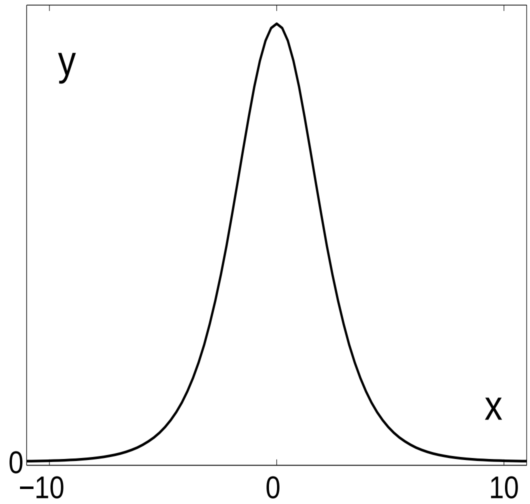
We will conclude with two remarks. The first will outline another case where the shooting method can be used. The other will mention an important case where this method *cannot* be used.

7.6.1 Shooting method for finding discrete eigenvalues

Consider a BVP

$$y'' + (2\text{sech}^2 x - \lambda^2)y = 0, \qquad x \in (-\infty, \infty), \quad y(|x| \to \infty) \to 0\,. \tag{7.52}$$

Here the term $2\text{sech}^2 x$ could be generalized to any "potential" $V(x)$ that has one or several "humps" in its central region and decays to zero as $|x| \to \infty$. Such a BVP is solvable (i.e., a $y(x)$ can be found such that $y(|x| \to \infty) \to 0$) only for some special values of λ, called the eigenvalues of this BVP. The corresponding solution, called an eigenfunction, is a localized "blob", which has some "structure" in the region where the potential is significantly different from zero and which vanishes at the ends of the infinite line. An example of such an eigenfunction is shown on the right. Note that in general, an eigenfunction may have a more complicated structure at the center than just a single "hump".



A variant of the shooting method which can find these eigenvalues is the following. First, since one cannot literally model the infinite line interval $(-\infty, \infty)$, consider the above BVP on the interval $[-R, R]$ for some reasonably large R (say, $R = 10$). For a given λ in the BVP, choose the initial conditions for the shooting as

$$y(-R) = y_0, \qquad y'(-R) = \lambda \cdot y(-R), \tag{7.53}$$

for some very small y_0 which we will discuss later. The reason behind the above relation between $y'(-R)$ and $y(-R)$ is this. Since the potential $2\text{sech}^2 x$ (almost) vanishes at $|x| = R$, then (7.52) reduces to $y'' - \lambda^2 y \approx 0$, and hence $y' \approx \lambda y$ at $x = -R$. Note that of the two possibilities $y' \approx \lambda y$ and $y' \approx -\lambda y$ which are implied by $y'' - \lambda^2 y \approx 0$, we have chosen the former, because it is its solution,

$$y = e^{\lambda x}, \tag{7.54}$$

which agrees with the behavior of the eigenfunction at the left end of the real line (see the figure above).

The constant y_0 in (7.53) can be taken as

$$y_0 = e^{-cR}, \tag{7.55}$$

where the constant c is of order one. Often one can simply take $c = 1$.

Now, compute the solution of the IVP consisting of the ODE in (7.52) and the initial condition (7.53) and record the value $y(R)$. This value can be denoted as $G(\lambda)$ since it has been obtained for a particular value of λ: $G(\lambda) \equiv y(R)|_\lambda$. Repeat this process for values of $\lambda = \lambda_{\min} + j\Delta\lambda$, $j = 0, 1, 2, \ldots$ in some specified interval $[\lambda_{\min}, \lambda_{\max}]$; as a result, one obtains a set of points representing a curve $G(\lambda)$. Those values of λ where this curve crosses zero correspond to the eigenvalues[17]. Indeed, there $y(R) = 0$, which is the approximate relation satisfied by eigenfunctions of (7.52) at $x = R$ for $R \gg 1$.

[17] In practice, one uses a more accurate method, but the description of this technical detail is outside the scope of this lecture.

7.6.2 Inapplicability of the shooting method in higher dimensions

Boundary value problems considered in this section are one-dimensional, in that they involve the derivative with respect to only one variable x. Such BVPs often arise in description of one-dimensional objects such as beams and strings. A natural generalization of these to two dimensions are plates and membranes. For example, a classic *Helmholtz equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \tag{7.56}$$

where $u = 0$ along the boundaries of a square with vertices $(x, y) = (0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, arises in the mathematical description of oscillations of a square membrane.

One could attempt to obtain a solution of this BVP by shooting from, say, the left side of this square to the right side. However, not only is this tedious to implement while accounting for all possible combinations of $\partial u/\partial x$ along the side $(x = 0,\ 0 \le y \le 1)$, but also *results of such shooting will be dominated by numerical error and will have nothing in common with the true solution.* The reason is that any IVP for certain two-dimensional equations, of which (7.56) is a particular case, are *ill-posed.* We will not go further into this issue[18] since it would substantially rely on the material studied in a course on partial differential equations. What is important to remember out of this brief discussion is that the shooting method can be used *only* for one-dimensional BVPs.

In the next lecture we will introduce alternative methods that can be used to solve BVPs both in one and many dimensions. However, we will only consider their applications in one dimension.

7.7 Questions for self-assessment

1. Explain the basic idea behind the shooting method (Sec. 7.1).

2. Why did we require that $Q(x) \le 0$ in (7.4)?

3. Verify (7.8).

4. Suppose that $Q(x) > 0$, and we have found that $v(b) = 0$, as in (7.12). Does this mean *only* that the BVP (7.4) will have no solutions, or are there other possibilities?

5. Let $Q(x) > 0$, as in the previous question, but now we have found that $v(b) \ne 0$. What possibilities for the number of solutions to the BVP do we have?

6. Verify (7.15) and (7.16).

7. Verify (7.20).

8. Verify (7.26).

9. Suppose you need to solve a 5th-order linear BVP. How many auxiliary systems do you need to consider? How many parameters (analogues of θ) do you need to introduce and then solve for?

[18] We will, however, arrive at the same conclusion, but from another view point, in Lecture 11.

10. What allows us to say that by Theorem 6.2 the solution (7.32) is unique?

11. Verify (7.36) and (7.37).

12. Verify statements (i) and (ii) found below Eq. (7.38). Specifically, do the following.
 (i) Enter in Matlab an expression `2+k*eps` where `k` is some number. Observe how Matlab's output changes as `k` increases from being less than 1 to being greater than 1.
 (ii) Enter into Matlab an expression `2.5/eps - (2.5/eps + 1)` and note the result. Vary the coefficient `2.5` (simultaneously in both terms) and, separately, the coefficient `1` and observe what happens.

13. What is a likely cause of the large deviation of the numerical solution from the exact one in the figure found below Eq. (7.38)?

14. Describe the key idea behind the multiple shooting method.

15. Suppose we use 3 subintervals for multiple shooting. How many parameters analogous to $\theta^{(1)}$, $\theta^{(2,1)}$, and $\theta^{(2,2)}$ will we need? What are the meanings of the conditions from which these parameters can be determined?

16. Verify that the r.h.s.'es in (7.45) are correct.

17. Describe the key idea behind the shooting method for nonlinear BVPs.

18. For a general nonlinear BVP, can one tell how many solutions one will find?