Tollimur in caelum curvato gurgite, et idem
subducta ad manes imos descendimus unda

We are carried up to the heaven by the circling wave,
and immediately the wave subsiding, we descend to the lowest depths

VIRGIL (70 BC–19 BC), Aeneid, 3, 564

9
Wave Functions

The fundamental quantity obtained by solving a path integral is the time evolution amplitude or propagator of a system $(\mathbf{x}_b t_b|\mathbf{x}_a t_a)$. In Schrödinger quantum mechanics, on the other hand, one has direct access on the energy spectrum and the wave functions of a system [see (1.94)]. This chapter will explain how to extract this information from the time evolution amplitude $(\mathbf{x}_b t_b|\mathbf{x}_a t_a)$. The crucial quantity for this purpose the Fourier transform of $(\mathbf{x}_b t_b|\mathbf{x}_a t_a)$, the fixed-energy amplitude introduced in Eq. (1.317):

$$(\mathbf{x}_b|\mathbf{x}_a)_E \;=\; \int_{t_a}^{\infty} dt_b \exp\{iE(t_b-t_a)/\hbar\}\,(\mathbf{x}_b t_b|\mathbf{x}_b t_a), \tag{9.1}$$

which contains as much information on the system as $(\mathbf{x}_b t_b|\mathbf{x}_a t_a)$, and gives, in particular, a direct access to the energy spectrum and the wave functions of the system. This is done via the the spectral decomposition (1.325).

Alternatively, we can work with the causal propagator at imaginary time,

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \int \frac{d^Dp}{(2\pi\hbar)^D}\exp\left[\frac{i}{\hbar}\mathbf{p}(\mathbf{x}_b-\mathbf{x}_a)-\frac{\mathbf{p}^2}{2M\hbar}(\tau_b-\tau_a)\right], \tag{9.2}$$

and calculate the fixed-energy amplitude by the Laplace transformation

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -i\int_{\tau_a}^{\infty} d\tau_b \exp\{E(\tau_b-\tau_a)/\hbar\}\,(\mathbf{x}_b\tau_b|\mathbf{x}_b\tau_a). \tag{9.3}$$

9.1 Free Particle in D Dimensions

For a free particle in D dimensions, the fixed-energy amplitude was calculated in Eqs. (1.348) and (1.355). It will be instructive to rederive the same result once more using the development in Section 8.5.1. Here we start directly from the spectral representation (1.325), which for a free particle takes the explicit form Eq. (1.344):

$$(\mathbf{x}_b|\mathbf{x}_a)_E = \int \frac{d^Dk}{(2\pi)^D} e^{i\mathbf{k}(\mathbf{x}_b-\mathbf{x}_a)}\frac{i\hbar}{E-\hbar^2\mathbf{k}^2/2M+i\eta}. \tag{9.4}$$

The momentum integral can now be done as follows. The exponential function $\exp(i\mathbf{k}\mathbf{R})$ is written as $\exp(ikR\cos\vartheta)$, where $\mathbf{R}$ is the distance vector $\mathbf{x}_b - \mathbf{x}_a$ and ϑ the angle between $\mathbf{k}$ and $\mathbf{R}$. Then we use formula (8.100) with the coefficients (8.101) and the hyperspherical harmonics $Y_{l\mathbf{m}}(\hat{\mathbf{x}})$ of Eq. (8.126) and expand

$$e^{i\mathbf{k}\mathbf{R}} = \sum_{l=0} a_l(ikR) \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\hat{\mathbf{k}}) Y^*_{l\mathbf{m}}(\hat{\mathbf{R}}). \tag{9.5}$$

The integral over $\mathbf{k}$ follows now directly from the decomposition into size and direction $d^Dk = dkd\hat{\mathbf{k}}$, and the orthogonality property (8.115) of the hyperspherical harmonics, according to which

$$\int d\hat{\mathbf{k}}\, Y_{l\mathbf{m}}(\hat{\mathbf{k}}) = \delta_{l0}\delta_{\mathbf{m}0}\sqrt{S_D}, \tag{9.6}$$

with S_D of Eq. (1.558). Since $Y_{0\mathbf{0}}(\hat{\mathbf{x}}) = 1/\sqrt{S_D}$, we obtain

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -\frac{2Mi}{(2\pi)^D}\int_0^\infty dk k^{D-1}\frac{1}{k^2+\kappa^2} a_0(ikR), \tag{9.7}$$

where

$$\kappa \equiv \sqrt{-2ME/\hbar^2}, \tag{9.8}$$

as in (1.346). Inserting $a_0(ikR)$ from (8.101),

$$a_0(ikR) = (2\pi)^{D/2} J_{D/2-1}(kR)/(kR)^{D/2-1}, \tag{9.9}$$

we find

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -\frac{2Mi}{(2\pi)^{D/2}} R^{1-D/2}\int_0^\infty dk k^{D/2}\frac{1}{k^2+\kappa^2} J_{D/2-1}(kR). \tag{9.10}$$

The integral

$$\int_0^\infty dk \frac{k^{\nu+1}}{(k^2+a^2)^{\mu+1}} J_\nu(kb) = \frac{a^{\nu-\mu}b^\mu}{2^\mu\Gamma(\mu+1)} K_{\nu-\mu}(ab) \tag{9.11}$$

yields once more the fixed-energy amplitude (1.347).

In two dimensions, the amplitude (1.347) becomes [recall (1.350)]

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -\frac{iM}{\pi\hbar} K_0(\kappa|\mathbf{x}_b - \mathbf{x}_a|). \tag{9.12}$$

It can be decomposed into partial waves by inserting

$$|\mathbf{x}_b - \mathbf{x}_a| = \sqrt{r_b^2 + r_a^2 - 2r_ar_b\cos(\varphi_b - \varphi_a)}. \tag{9.13}$$

Then a well-known addition theorem for Bessel functions yields the expansion

$$K_0(\kappa|\mathbf{x}_b - \mathbf{x}_a|) = \sum_{m=-\infty}^{\infty} I_m(\kappa r_<) K_m(\kappa r_>) e^{im(\varphi_b-\varphi_a)}, \tag{9.14}$$

where $r_<$ and $r_>$ are the smaller and larger values of r_a and r_b, respectively. Hence the fixed-energy amplitude turns into

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -\frac{2iM}{\hbar}\sum_m I_m(\kappa r_<)K_m(\kappa r_>)\frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}. \tag{9.15}$$

This is an analytic function in the complex E-plane. The parameter κ is real for $E<0$. For $E>0$, the square root (9.8) allows for two imaginary solutions, $\kappa^{\pm} \equiv e^{\mp i\pi/2}k \equiv e^{\mp i\pi/2}\sqrt{2ME}/\hbar$, so that the amplitude has a right-hand cut. Its discontinuity specifies the continuum of free-particle states. On top of the cut, we use the analytic continuation formulas (valid for $-\pi/2 < \arg z \leq \pi$)[1]

$$\begin{aligned} I_\mu(e^{-i\pi/2}z) &= e^{-i\pi\mu/2}J_\mu(z), \\ K_\mu(e^{-i\pi/2}z) &= i\frac{\pi}{2}e^{i\pi\mu/2}H_\mu^{(1)}(z), \end{aligned} \tag{9.16}$$

to find the fixed-energy amplitude above the cut

$$(\mathbf{x}_b|\mathbf{x}_a)_{E+i\eta} = \frac{\pi M}{\hbar}\sum_m J_m(kr_<)H_m^{(1)}(kr_>)\frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}. \tag{9.17}$$

The reflection properties[2]

$$\begin{aligned} H_\mu^{(1)}(e^{i\pi}z) &= -H_{-\mu}^{(2)}(z) \equiv -e^{-i\pi\mu}H_\mu^{(2)}(z), \\ J_\mu(e^{i\pi}z) &= e^{i\pi\mu}J_\mu(z) \end{aligned} \tag{9.18}$$

yield the amplitude below the cut

$$(\mathbf{x}_b|\mathbf{x}_a)_{E-i\eta} = -\frac{\pi M}{\hbar}\sum_m J_m(kr_<)H_m^{(2)}(kr_>)\frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}. \tag{9.19}$$

The discontinuity across the cut follows from the relation

$$J_\mu(z) = \frac{1}{2}\left[H_\mu^{(1)}(z) + H_\mu^{(2)}(z)\right] \tag{9.20}$$

and reads

$$\operatorname{disc}(\mathbf{x}_b|\mathbf{x}_a)_E = \frac{2\pi M}{\hbar}\sum_{m=-\infty}^{\infty} J_m(kr_b)J_m(kr_a)\frac{1}{2\pi}e^{im(\varphi_b-\varphi_b)}. \tag{9.21}$$

According to (1.330), the integral over the discontinuity yields the completeness relation

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\operatorname{disc}(\mathbf{x}_b|\mathbf{x}_a)_E \\ &\quad= \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\frac{2\pi M}{\hbar}\sum_{m=-\infty}^{\infty} J_m(kr_b)J_m(kr_a)\frac{1}{2\pi}e^{im(\varphi_b-\varphi_b)} = \delta(\mathbf{x}_b-\mathbf{x}_a). \end{aligned} \tag{9.22}$$

[1] I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formulas 8.406.1 and 8.407.1.
[2] ibid., Formulas 8.476.1 and 8.476.8.

After replacing the energy integral by a k-integral,

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{2\pi M}{\hbar} = \int_0^{\infty} dkk, \tag{9.23}$$

we can also write

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \operatorname{disc}(\mathbf{x}_b|\mathbf{x}_a)_E &= \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{x}_b-\mathbf{x}_a)} = \frac{1}{2\pi}\int_0^{\infty} dkk J_0(k|\mathbf{x}_b-\mathbf{x}_a|) \\ &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} dkk J_m(kr_b) J_m(kr_a) \frac{1}{2\pi} e^{im(\varphi_b-\varphi_a)} \\ &= \frac{1}{\sqrt{r_b r_a}} \delta(r_b-r_a)\delta(\varphi_b-\varphi_a).\end{aligned} \tag{9.24}$$

The last two lines exhibit the well-known completeness relation of the radial wave functions of a free particle.[3]

9.2 Harmonic Oscillator in D Dimensions

The wave functions of the one-dimensional harmonic oscillator have already been derived in Section 2.6 from a spectral decomposition of the time evolution amplitude. This was possible with the help of Mehler's formula. In D dimensions, the fixed-energy amplitude is the best starting point for determining the wave functions. We take the radial propagator (8.142) obtained from the angular momentum decomposition (8.141) or, for the sake of greater generality, the radial amplitude (8.144) with an additional centrifugal barrier, continue it to imaginary time $\tau = it$, and go over to its Laplace transform

$$\begin{aligned}(r_b|r_a)_{E,l} = -i\sqrt{M\omega/\hbar}\sqrt{M\omega r_b r_a/\hbar} \int_{\tau_a}^{\infty} d\tau_b \; e^{E(\tau_b-\tau_a)/\hbar} \frac{1}{\sin[\omega(\tau_b-\tau_a)]} \\ \times e^{-\frac{1}{\hbar}\frac{M\omega}{2}\coth[\omega(\tau_b-\tau_a)](r_b^2+r_a^2)} I_\mu\left(\frac{M\omega r_b r_a}{\hbar\sinh[\omega(\tau_b-\tau_a)]}\right).\end{aligned} \tag{9.25}$$

To evaluate the τ-integral we make use of a standard integral formula for Bessel functions[4]

$$\begin{aligned}&\int_0^{\infty} dx\,[\coth(x/2)]^{2\nu}\, e^{-\beta\cosh x} J_\mu(\alpha\sinh x) \\ &\quad = \frac{\Gamma((1+\mu)/2-\nu)}{\alpha\Gamma(\mu+1)} W_{\nu,\mu/2}\left(\sqrt{\alpha^2+\beta^2}+\beta\right)\, M_{-\nu,\mu/2}\left(\sqrt{\alpha^2+\beta^2}-\beta\right),\end{aligned} \tag{9.26}$$

where $W_{\nu,\mu/2}(z)$, $M_{-\nu,\mu/2}(z)$ are the Whittaker functions. The formula is valid for

$$\operatorname{Re}\beta > |\operatorname{Re}\alpha|, \qquad \operatorname{Re}(\mu/2-\nu) > -\frac{1}{2}.$$

[3] Compare with Eqs. (3.112) and (3.139) in J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, 1975.

[4] I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 6.669.1.

By a change of variables

$$\sqrt{\alpha^2+\beta^2}\pm\beta = t\alpha_{b,a},$$

and

$$\sinh x = (\sinh y)^{-1},\ \ \cosh x = \coth y,\ \ \coth(x/2) = e^y,\ \ \coth x = \cosh y,$$

with

$$dx = dy/\sinh y,$$

Eq. (9.26) goes over into

$$\int_0^\infty \frac{dy}{\sinh y} e^{2\nu y} \exp\left[-\tfrac{1}{2}t(\alpha_b-\alpha_a)\coth y\right] J_\mu\left(t\frac{\sqrt{\alpha_b\alpha_a}}{\sinh y}\right)$$
$$= \frac{\Gamma\left((1+\mu)/2-\nu\right)}{t\sqrt{\alpha_b\alpha_a}\Gamma(\mu+1)} W_{\nu,\mu/2}(t\alpha_b)M_{-\nu,\mu/2}(-t\alpha_a). \tag{9.27}$$

Using the identity

$$M_{-\nu,\mu/2}(z) \equiv e^{-i(\mu+1)\pi/2} M_{\nu,\mu/2}(-z) \tag{9.28}$$

and changing the sign of α_a in (9.27), this can be turned into

$$\int_0^\infty \frac{dy}{\sinh y} e^{2\nu y} \exp\left[-\tfrac{1}{2}t(\alpha_b+\alpha_a)\coth y\right] I_\mu\left(\frac{t\sqrt{\alpha_b\alpha_a}}{\sinh y}\right)$$
$$= \frac{\Gamma\left((1+\mu)/2-\nu\right)}{t\sqrt{\alpha_b\alpha_a}\Gamma(\mu+1)} W_{\nu,\mu/2}(t\alpha_b)M_{\nu,\mu/2}(t\alpha_a), \tag{9.29}$$

with the range of validity

$$\begin{aligned}&\alpha_b>\alpha_a>0, \quad \mathrm{Re}\left[(1+\mu)/2-\nu\right]>0,\\ &\mathrm{Re}\, t>0, \quad |\arg t|<\pi.\end{aligned} \tag{9.30}$$

Setting

$$y=\omega(t_b-t_a),\quad \alpha_b=\frac{M}{\hbar}\omega r_b^2,\quad \alpha_a=\frac{M\omega}{\hbar}r_a^2,\quad \nu=E/2\omega\hbar \tag{9.31}$$

in (9.29) brings the radial amplitude (9.25) to the form (valid for $r_b>r_a$)

$$(r_b|r_a)_{E,l} = -i\frac{1}{\omega}\frac{1}{\sqrt{r_br_a}}\frac{\Gamma((1+\mu)/2-\nu)}{\Gamma(\mu+1)} W_{\nu,\mu/2}\left(\frac{M\omega}{\hbar}r_b^2\right) M_{\nu,\mu/2}\left(\frac{M\omega}{\hbar}r_a^2\right). \tag{9.32}$$

The Gamma function has poles at

$$\nu=\nu_r\equiv(1+\mu)/2+n_r \tag{9.33}$$

for integer values of the so-called radial quantum number of the system $n_r = 0,1,2,\dots$. The poles have the form

$$\Gamma\left((1+\mu)/2-\nu\right) \overset{\nu\sim\nu_r}{\sim} -\frac{(-1)^{n_r}}{n_r!}\frac{1}{\nu-\nu_r}. \tag{9.34}$$

Inserting here the particular value of the parameter μ for the D-dimensional oscillator which is $\mu = D/2 + l - 1$, and remembering that $\nu = E/2\omega\hbar$, we find the energy spectrum

$$E = \hbar\omega\left(2n_r + l + D/2\right). \tag{9.35}$$

The principal quantum number is defined by

$$n \equiv 2n_r + l \tag{9.36}$$

and the energy depends on it as follows:

$$E_n = \hbar\omega(n + D/2). \tag{9.37}$$

For a fixed principal quantum number $n = 2n_r + l$, the angular momentum runs through $l = 0, 2, \ldots, n$ for even, and $l = 1, 3, \ldots, n$ for odd n. There are $d_n = (n + D - 1)!/(D-1)!n!$ degenerate levels. From the residues $1/(\nu - \nu_r) \sim 2\hbar\omega/(E - E_n)$, we extract the product of radial wave functions at given n_r, l:

$$\begin{aligned} R_{n_r l}(r_b)\, R_{n_r l}(r_a) = {} & \frac{1}{\sqrt{r_b r_a}} \frac{2(-1)^{n_r}}{\Gamma(\mu+1) n_r!} \\ & \times W_{(1+\mu)/2+n_r,\frac{\mu}{2}}(M\omega {r_b}^2/\hbar) M_{(1+\mu)/2+n_r,\frac{\mu}{2}}(M\omega {r_a}^2/\hbar). \end{aligned} \tag{9.38}$$

It is now convenient to express the Whittaker functions in terms of the confluent hypergeometric or Kummer functions:[5]

$$W_{(1+\mu)/2+n_r,\frac{\mu}{2}}(z) = e^{-z/2} z^{(1+\mu)/2} U(-n_r, 1+\mu, z), \tag{9.39}$$

$$M_{(1+\mu)/2+n_r,\frac{\mu}{2}}(z) = e^{-z/2} z^{(1+\mu)/2} M(-n_r, 1+\mu, z). \tag{9.40}$$

The latter equation follows from the relation

$$M_{-(1+\mu)/2-n_r,\frac{\mu}{2}}(z) = e^{-z/2} z^{(1+\mu)/2} M(1+\mu+n_r, 1+\mu, z), \tag{9.41}$$

after replacing $n_r \to -n_r - \mu - 1$.

For completeness, we also mention the identity

$$M(a,b,z) = e^z M(b-a, b, -z), \tag{9.42}$$

so that

$$M(1+\mu+n_r, 1+\mu, z) = e^z M(-n_r, 1+\mu, -z). \tag{9.43}$$

This permits us to rewrite (9.41) as

$$M_{-(1+\mu)/2-n_r,\frac{\mu}{2}}(z) = e^{z/2} z^{(1+\mu)/2} M(-n_r, 1+\mu, -z), \tag{9.44}$$

which turns into (9.40) by using (9.28) and appropriately changing the indices.

[5] I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 9.220.2.

The Kummer function $M(a,b,z)$ has the power series

$$M(a,b,z) \equiv {}_1F_1(a;b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z}{z!} + \ldots, \tag{9.45}$$

showing that $M_{(1+\mu)/2+n_r,\frac{\mu}{2}}(M\omega {r_a}^2/2\hbar)$ is an exponential $e^{-M\omega r_a^2/\hbar}$ times a polynomial in r_a of order $2n_r$. A similar expression is obtained for the other factor $W_{(1+\mu)/2+n_r,\frac{\mu}{2}}(M\omega {r_b}^2/\hbar)$ of Eq. (9.39). Indeed, the Kummer function $U(a,b,z)$ is related to $M(a,b,z)$ by[6]

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right]. \tag{9.46}$$

Since $a = -n_r$ with integer n_r and $1/\Gamma(a) = 0$, we see that only the first term in the brackets is present. Then the identity

$$\Gamma(-\mu)\Gamma(1+\mu) = \pi/\sin[\pi(1+\mu)]$$

leads to the relation

$$U(-n_r, 1+\mu, z) = \frac{\Gamma(-\mu)}{\Gamma(-n_r-\mu)}M(-n_r, 1+\mu, z), \tag{9.47}$$

which is a polynomial in z of order n_r. Thus we have the useful formula

$$\begin{aligned} W_{(1+\mu)/2+n_r,\frac{\mu}{2}}(z_b)M_{(1+\mu)/2+n_r,\frac{\mu}{2}}(z_a) &= \frac{\Gamma(-\mu)}{\Gamma(-n_r-\mu)}e^{-(z_b+z_a)/2} \\ &\times (z_b z_a)^{(1+\mu)/2} M(-n_r,1+\mu,z_b)M(-n_r,1+\mu,z_a). \end{aligned} \tag{9.48}$$

We can therefore re-express Eq. (9.38) as

$$\begin{aligned} R_{n_r l}(r_b)\, R_{n_r l}(r_a) &= \sqrt{\frac{M\omega}{\hbar}}\frac{2(-)^{n_r}\Gamma(-\mu)}{\Gamma(-n_r-\mu)\Gamma(1+\mu)\; n_r!} \\ &\times e^{-M\omega(r_b^2+r_a^2)/2\hbar}\,(M\omega r_b r_a/\hbar)^{1/2+\mu} \\ &\times M(-n_r,1+\mu,M\omega r_b^2/\hbar)M(-n_r,1+\mu,M\omega r_a^2/\hbar). \end{aligned} \tag{9.49}$$

We now insert

$$\frac{(-)^{n_r}\Gamma(-\mu)}{\Gamma(-n_r-\mu)} = \frac{\Gamma(n_r+1+\mu)}{\Gamma(1+\mu)}, \tag{9.50}$$

setting $\mu = D/2 + l - 1$, and identify the wave functions as

$$\begin{aligned} R_{n_r l}(r) = C_{n_r l}\; &(M\omega/\hbar)^{1/4}(M\omega r^2/\hbar)^{l/2+(D-1)/4} \\ &\times e^{-M\omega r^2/2\hbar}M(-n_r, l+D/2, M\omega r^2/\hbar), \end{aligned} \tag{9.51}$$

[6]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965, Formula 13.1.3.

with the normalization factor

$$C_{n_r l} = \frac{\sqrt{2}}{\Gamma(1+\mu)} \sqrt{\frac{(n_r+\mu)!}{n_r!}}. \tag{9.52}$$

By introducing the Laguerre polynomials [7]

$$L_n^\mu(z) \equiv \frac{(n+\mu)!}{n!\mu!} M(-n, \mu+1, z), \tag{9.53}$$

and using the integral formula[8]

$$\int_0^\infty dz e^{-z} z^\mu L_n^\mu(z) L_{n'}^\mu(z) = \delta_{nn'} \frac{(n+\mu)!}{n!}, \tag{9.54}$$

we find that the radial wave functions satisfy the orthonormality relation

$$\int_0^\infty dr R_{n_r l}(r) R_{n_r' l}(r) = \delta_{n_r n_r'}. \tag{9.55}$$

The radial imaginary-time evolution amplitude has now the spectral representation

$$(r_b \tau_b | r_a \tau_a)_l = \sum_{n_r=0}^\infty R_{n_r l}(r_b) R_{n_r l}(r_a) e^{-E_n(\tau_b - \tau_a)/\hbar}, \tag{9.56}$$

with the energies

$$E_n = \hbar\omega(n + D/2) = \hbar\omega\left(2n_r + l + D/2\right). \tag{9.57}$$

The full causal propagator is given, as in (8.91), by

$$(\mathbf{x}_b \tau_b | \mathbf{x}_a \tau_a) = \frac{1}{(r_b r_a)^{(D-1)/2}} \sum_{l=0}^\infty (r_b \tau_b | r_a \tau_a)_l \sum_m Y_{lm}(\hat{\mathbf{x}}_b) Y_{lm}^*(\hat{\mathbf{x}}_a). \tag{9.58}$$

From this, we extract the wave functions

$$\psi_{n_r lm}(\mathbf{x}) = \frac{1}{r^{(D-1)/2}} R_{n_r l}(r) Y_{lm}(\hat{\mathbf{x}}). \tag{9.59}$$

They have the threshold behavior r^l near the origin.

The one-dimensional oscillator may be viewed as a special case of these formulas. For $D = 1$, the partial wave expansion amounts to a separation into even and odd wave functions. There are two "spherical harmonics",

$$Y_{\mathrm{e},\o}(\hat{x}) = \frac{1}{\sqrt{2}}(\Theta(x) \pm \Theta(-x)), \tag{9.60}$$

[7] I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 8.970 (our definition differs from that in L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Pergamon, London, 1965, Eq. (d.13). The relation is $L_n^\mu = (-)^\mu/(n+\mu)! L_{n+\mu}^{\mathrm{L.L.}\,\mu}$).

[8] ibid., Formula 7.414.3.

and the amplitude has the decomposition

$$(x_b\tau_b|x_a\tau_a) = (r_b\tau_b|r_a\tau_a)_{\rm e} Y_{\rm e}(\hat{x}_b)Y_{\rm e}(\hat{x}_a) + (r_b\tau_b|r_a\tau_a)_{\o} Y_{\o}(\hat{x}_b)Y_{\o}(\hat{x}_a), \tag{9.61}$$

with the "radial" amplitudes

$$(r_b\tau_b|r_a\tau_a)_{{\rm e},\o} = (x_b\tau_b|x_a\tau_a) \pm (-x_b\tau_b|x_a\tau_a). \tag{9.62}$$

These are known from Eq. (2.177) to be

$$\begin{aligned}(r_b\tau_b|r_a\tau_a)_{{\rm e},\o} &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{M\omega/\hbar}{\sinh[\omega(\tau_b-\tau_a)]}} \\ &\times \exp\left[-\frac{M\omega}{2\hbar}(r_b^2+r_a^2)\cot\omega(\tau_b-\tau_a)\right] 2\left\{\begin{array}{c}\cosh\\ \sinh\end{array}\right\}(M\omega r_b r_a/\hbar)\,.\end{aligned} \tag{9.63}$$

The two cases coincide with the integrand of (9.25) for $l = 0$ and 1, respectively, since $\mu = l + D/2 - 1$ takes the values $\pm 1/2$ and

$$\sqrt{z}I_{\mp\frac{1}{2}}(z) = \frac{2}{2\pi}\left\{\begin{array}{l}\cosh z\ ,\\ \sinh z\ .\end{array}\right. \tag{9.64}$$

The associated energy spectrum (9.35) is

$$E = \left\{\begin{array}{ll}\hbar\omega(2n_r+\frac{1}{2}) & \text{even},\\ \hbar\omega(2n_r+\frac{3}{2}) & \text{odd},\end{array}\right. \tag{9.65}$$

with the radial quantum number $n_r = 0, 1, 2, \dots$. The two cases follow the single formula

$$E = \hbar\omega(n+\tfrac{1}{2}), \tag{9.66}$$

where the principal quantum number $n = 0, 1, 2, \dots$ is related to n_r by $n = 2n_r$ and $n = 2n_r + 1$, respectively. The radial wave functions (9.51) become

$$R_{n_r,{\rm e}}(r) = (M\omega/\hbar)^{1/4}\sqrt{\frac{2\Gamma(n_r+\frac{1}{2})}{\pi n_r!}}M(-n_r,\tfrac{1}{2},M\omega r^2/\hbar), \tag{9.67}$$

$$R_{n_r,\o}(r) = (M\omega/\hbar)^{1/4}\sqrt{\frac{2\Gamma(n_r+\frac{3}{2})}{(\pi/4)n_r!}}\sqrt{\frac{M\omega r^2}{\hbar}}M(-n_r,\tfrac{3}{2},M\omega r^2/\hbar). \tag{9.68}$$

The special Kummer functions appearing here are Hermite polynomials

$$M(-n,\tfrac{1}{2},x^2) = \frac{n!}{(2n)!}(-)^n H_{2n}(x), \tag{9.69}$$

$$M(-n,\tfrac{3}{2},x^2) = \frac{n!}{(2n+1)!}(-)^n H_{2n+1}(x)/2\sqrt{x}. \tag{9.70}$$

Using the identity

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = (2\pi)^{1/2} 2^{-2z+1/2}\Gamma(2z), \tag{9.71}$$

we obtain in either case the radial wave functions [to be compared with the one-dimensional wave functions (2.302)]

$$R_n(r) = N_n\sqrt{2}\lambda_\omega^{-1/2} e^{-r^2/2\lambda_\omega^2} H_n(r/\lambda_\omega), \quad n = 0,1,2,\dots \tag{9.72}$$

with

$$\lambda_\omega \equiv \sqrt{\frac{\hbar}{M\omega}}, \qquad N_n = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}. \tag{9.73}$$

This formula holds for both even and odd wave functions with $n_r = 2n$ and $n_r = 2n+1$, respectively. It is easy to check that they possess the correct normalization $\int_0^\infty dr R_n^2(r) = 1$. Note that the "spherical harmonics" (9.60) remove a factor $\sqrt{2}$ in (9.72), but compensate for this by extending the $x > 0$ integration to the entire x-axis by the "one-dimensional angular integration".

9.3 Free Particle from $\omega \to 0$ -Limit of Oscillator

The results obtained for the D-dimensional harmonic oscillator in the last section can be used to find the amplitude and wave functions of a free particle in D dimensions in radial coordinates. This is done by taking the limit $\omega \to 0$ at fixed energy E. In the amplitude (9.32) with $W_{\nu,\mu/2}(z), M_{\nu,\mu/2}(z)$ substituted according to (9.39), we rewrite n_r as $(E/\omega\hbar - l - 1)/2$ and go to the limit $\omega \to 0$ at a fixed energy E. Replacing $M\omega r^2/\hbar$ by $k^2r^2/2n_r \equiv z/n_r$ (where $z = k^2r^2/2$, and using $E = p^2/2M = \hbar^2k^2/2M$), we apply the limiting formulas[9]

$$\begin{aligned}\lim_{n_r\to\infty}&\{\Gamma(1-n_r-b)U(-a,b,\mp z/n_r)\}\\ &= z^{-\frac{1}{2}(b-1)}\left\{\begin{array}{l} 2K_{b-1}(2\sqrt{z}) \\ -i\pi e^{i\pi b}H_{b-1}^{(1)}(2\sqrt{z}) \quad (\mathrm{Im}\, z > 0),\end{array}\right.\end{aligned} \tag{9.74}$$

$$\lim_{n_r\to\infty} M(-a,b,\mp z/n_r)/\Gamma(b) = z^{-\frac{1}{2}(b-1)}\left\{\begin{array}{l} I_{b-1}(2\sqrt{z}) \\ J_{b-1}(2\sqrt{z}),\end{array}\right. \tag{9.75}$$

and obtain the radial wave functions directly from (9.51) and (9.75):

$$R_{n_rl}(r) \xrightarrow{n_r\to\infty} C_{n_rl}(M\omega r^2/\hbar)^{(\mu/2+1/2)}\left(k^2r^2/2\right)^{-\mu/2}\Gamma(1+\mu)J_\mu(kr), \tag{9.76}$$

where

$$C_{n_rl} \xrightarrow{n_r\to\infty} \sqrt{2}\left(\frac{E}{2\hbar\omega}\right)^{\mu/2}\frac{1}{\Gamma(1+\mu)}. \tag{9.77}$$

Hence

$$R_{n_rl}(r) \xrightarrow{n_r\to\infty} r^{1/2}\sqrt{M\omega/\hbar}\sqrt{2}J_\mu(kr). \tag{9.78}$$

[9] M. Abramowitz and I. Stegun, *op. cit.*, Formulas 13.3.1–13.3.4.

Inserting these wave functions into the radial time evolution amplitude

$$(r_b\tau_b|r_a\tau_a)_l = \sum_{n_r} R_{n_r l}(r_b)R_{n_r l}(r_b)e^{-E_n(\tau_b-\tau_a)/\hbar}, \tag{9.79}$$

and replacing the sum over n_r by the integral $\int_0^\infty dk\ \hbar k/M\omega$ [in accordance with the $n_r \to \infty$ limit of $E_{n_r} = \omega\hbar(2n_r + l + D/2) \to \hbar^2k^2/2M$], we obtain the spectral representation of the free-particle propagator

$$(r_b\tau_b|r_a\tau_a)_\mu = \sqrt{r_b r_a}\int_0^\infty dk\, kJ_\mu(kr_b)J_\mu(kr_a)e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)}. \tag{9.80}$$

For comparison, we derive the same results directly from the initial spectral representation (9.2) in one dimension

$$(x_b\tau_b|x_a\tau_a) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\left[ik(x_b-x_a) - \frac{\hbar k^2}{2M}(\tau_b-\tau_a)\right]. \tag{9.81}$$

Its "angular decomposition" is a decomposition with respect to even and odd wave functions

$$\begin{aligned}(r_b\tau_b|r_a\tau_a)_{\mathrm{e},\o} &= \int_0^\infty\frac{dk}{\pi}[\cos k(r_b-r_a) \pm \cos k(r_b+r_a)]e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)}\\ &= 2\int_0^\infty\frac{dk}{\pi}\left\{\begin{array}{cc}\cos kr_b & \cos kr_a\\ \sin kr_b & \sin kr_a\end{array}\right\}e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)}.\end{aligned} \tag{9.82}$$

In D dimensions we use the expansion (8.100) for $e^{i\mathbf{kx}}$ to calculate the amplitude in the radial form

$$\begin{aligned}(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) &= \int\frac{d^Dk}{(2\pi)^D}e^{i\mathbf{k}(\mathbf{x}_b-\mathbf{x}_a)}e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)}\\ &= \frac{1}{(2\pi)^{D/2}}\int_0^\infty dk k^{2\nu}\frac{1}{(kR)^\nu}J_\nu(kR)e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)},\end{aligned} \tag{9.83}$$

with $\nu \equiv D/2-1$. With the help of the addition theorem for Bessel functions[10] (8.188) we rewrite

$$\frac{1}{(kR)^\nu}J_\nu(kR) = \frac{2^\nu\Gamma(\nu)}{(k^2r_br_a)^\nu}\sum_{l=0}^\infty(\nu+l)J_{\nu+l}(kr_b)J_{\nu+l}(kr_a)C_l^{(\nu)}(\Delta\vartheta) \tag{9.84}$$

and expand further according to

$$\frac{1}{(kR)^\nu}J_\nu(kR) = \frac{(2\pi)^{D/2}}{(k^2r_br_a)^\nu}\sum_{l=0}^\infty J_{\nu+l}(kr_b)J_{\nu+l}(kr_a)\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\hat{\mathbf{x}}_b)Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_a), \tag{9.85}$$

to obtain the radial amplitude

$$(r_b\tau_b|r_a\tau_a)_l = \sqrt{r_br_a}\int_0^\infty dk k J_{\nu+l}(kr_b)J_{\nu+l}(kr_a)e^{-\frac{\hbar k^2}{2M}(\tau_b-\tau_a)}, \tag{9.86}$$

[10] I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 8.532.

just as in (9.80).

For $D=1$, this reduces to (9.82) using the particular Bessel functions

$$\sqrt{z}J_{\mp 1/2}(z) = \frac{2}{\sqrt{2\pi}}\left\{\begin{array}{c}\cos z\\ \sin z\end{array}\right\}. \tag{9.87}$$

9.4 Charged Particle in Uniform Magnetic Field

Let us also find the wave functions of a charged particle in a magnetic field. The amplitude was calculated in Section 2.18. Again we work with the imaginary-time version. Factorizing out the free motion along the direction of the magnetic field, we write

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = (z_b\tau_b|z_a\tau_a)(\mathbf{x}_b^\perp\tau_b|\mathbf{x}_a^\perp\tau_a), \tag{9.88}$$

with

$$(z_b\tau_b|z_a\tau_a) = \frac{1}{\sqrt{2\pi\hbar(\tau_b-\tau_a)/M}}\exp\left\{-\frac{M}{2\hbar}\frac{(z_b-z_a)^2}{\tau_b-\tau_a}\right\}, \tag{9.89}$$

and have for the amplitude in the transverse direction

$$(\mathbf{x}_b^\perp\tau_b|\mathbf{x}_a^\perp\tau_a) = \frac{M}{2\pi\hbar(\tau_b-\tau_a)}\frac{\omega(\tau_b-\tau_a)/2}{\sinh\left[\omega(\tau_b-\tau_a)/2\right]}\exp\left[-\mathcal{A}_l^\perp/\hbar\right], \tag{9.90}$$

with the classical transverse action

$$\mathcal{A}_l^\perp = \frac{M\omega}{2}\left\{\tfrac{1}{2}\coth\left[\omega(\tau_b-\tau_a)/2\right](\mathbf{x}_b^\perp-\mathbf{x}_a^\perp)^2 + \mathbf{x}_a^\perp\times\mathbf{x}_b^\perp\right\}. \tag{9.91}$$

This result is valid if the vector potential is chosen as

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{x}. \tag{9.92}$$

In the other gauge with

$$\mathbf{A} = (0, Bx, 0), \tag{9.93}$$

there is an extra surface term, and $\mathcal{A}_{\rm cl}^\perp$ is replaced by

$$\tilde{\mathcal{A}}_{\rm cl}^\perp = \mathcal{A}_{\rm cl}^\perp + \frac{M\omega}{2}(x_by_b - x_ay_a). \tag{9.94}$$

The calculation of the wave functions is quite different in these two gauges. In the gauge (9.93) we merely recall the expressions (2.663) and (2.665) and write down the integral representation

$$(\mathbf{x}_b^\perp\tau_b|\mathbf{x}_a^\perp\tau_a) \;=\; \int\frac{dp_y}{2\pi\hbar}e^{ip_y(y_b-y_a)/\hbar}(x_b\tau_b|x_a\tau_a)_{x_0=p_y/M\omega}, \tag{9.95}$$

with the oscillator amplitude in the x-direction

$$(x_b\tau_b|x_a\tau_a)_{x_0} = \sqrt{\frac{M\omega}{2\pi\hbar\sinh[\omega(\tau_b-\tau_a)]}}\exp\left(-\frac{1}{\hbar}\mathcal{A}_{\rm cl}^{\rm os}\right), \tag{9.96}$$

and the classical oscillator action centered around x_0

$$\mathcal{A}_{\rm cl}^{\rm os} = \frac{M\omega}{2\sinh[\omega(\tau_b-\tau_a)]}\{[(x_b-x_0)^2+(x_a-x_0)^2]\cosh[\omega(\tau_b-\tau_b)] - 2(x_b-x_0)(x_a-x_0)\}. \tag{9.97}$$

The spectral representation of the amplitude (9.96) is then

$$(x_a\tau_b|x_a\tau_a)_{x_0} = \sum_{n=0}^{\infty}\psi_n(x_b-x_0)\psi_n(x_a-x_0)e^{-\left(n+\frac{1}{2}\right)\omega(\tau_b-\tau_a)}, \tag{9.98}$$

where $\psi_n(x)$ are the oscillator wave functions (2.302). This leads to the spectral representation of the full amplitude (9.88)

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a) = \int\frac{dp_z}{2\pi\hbar}\int\frac{dp_y}{2\pi\hbar}e^{ip_z(z_b-z_a)/\hbar} \times\sum_{n=0}^{\infty}\psi_n(x_b-p_y/M\omega)\psi_n(x_a-p_y/M\omega)e^{-(n+\frac{1}{2})\omega(\tau_b-\tau_a)}. \tag{9.99}$$

The combination of a sum and two integrals exhibits the complete set of wave functions of a particle in a uniform magnetic field. Note that the energy

$$E_n = (n+\tfrac{1}{2})\hbar\omega \tag{9.100}$$

is highly degenerate; it does not depend on p_y.

In the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{x}$, the spectral decomposition looks quite different. To derive it, the transverse Euclidean action is written down in radial coordinates [compare Eq. (2.669)] as

$$\mathcal{A}_{\rm cl}^{\perp} = \frac{M}{2}\left\{\frac{\omega}{2}\coth\left[\omega(\tau_b-\tau_a)/2\right]\left[{r_b}^2+{r_a}^2-2r_br_a\cos(\varphi_b-\varphi_a)\right] - i\omega r_br_a\sin(\varphi_b-\varphi_a)\right\}. \tag{9.101}$$

This can be rearranged to

$$\mathcal{A}_{\rm cl}^{\perp} = \frac{M}{2}\frac{\omega}{2}\coth\left[\omega(\tau_b-\tau_a)/2\right]({r_b}^2+{r_a}^2) -\frac{M}{2}\frac{\omega}{\sinh\left[\omega(\tau_b-\tau_a)/2\right]}\cos\left[\varphi_b-\varphi_a-i\omega(\tau_b-\tau_a)/2\right]. \tag{9.102}$$

We now expand $e^{-\mathcal{A}_{\rm cl}^{\perp}/\hbar}$ into a series of Bessel functions using (8.5)

$$e^{-\mathcal{A}_{\rm cl}^{\perp}/\hbar} = \exp\left\{-\frac{M}{2\hbar}\frac{\omega}{2}\coth\left[\omega(\tau_b-\tau_a)/2\right]({r_b}^2+{r_a}^2)\right\} \times\sum_{m=-\infty}^{\infty}I_m\left(\frac{M\omega}{2\hbar}\frac{r_br_a}{\sinh\left[\omega(\tau_b-\tau_a)/2\right]}\right)e^{m\omega(\tau_b-\tau_a)/2}e^{im(\varphi_b-\varphi_a)}. \tag{9.103}$$

The fluctuation factor is the same as before. Hence we obtain the angular decomposition of the transverse amplitude

$$(\mathbf{x}_b^{\perp}\tau_b|\mathbf{x}_a^{\perp}\tau_a) = \frac{1}{\sqrt{r_b r_a}}\sum_m (r_b\tau_b|r_a\tau_a)_m \frac{1}{2\pi}e^{im(\varphi_b-\varphi_a)}, \tag{9.104}$$

where

$$(r_b\tau_b|r_a\tau_a)_m = \sqrt{r_b r_a}\frac{M\omega}{2\hbar\eta}\frac{\eta}{\sinh\eta}\exp\left[-\frac{M}{2\hbar}\frac{\omega}{2}\coth\eta(r_b^2+r_a^2)\right] I_m\left(\frac{M\omega r_b r_a}{2\hbar\sinh\eta}\right)e^{m\eta}, \tag{9.105}$$

with

$$\eta \equiv \omega(\tau_b-\tau_a)/2. \tag{9.106}$$

To find the spectral representation we go to the fixed-energy amplitude

$$\begin{aligned}(r_b|r_a)_{m,E} &= -i\int_{\tau_a}^{\infty} d\tau_b e^{E(\tau_b-\tau_a)/\hbar}\,(r_b\tau_b|r_a\tau_a)_m \\ &= -i\sqrt{r_b r_a}\frac{M}{\hbar}\int_0^{\infty} d\eta \;\; e^{2\nu\eta}\frac{1}{\sinh\eta}e^{-(M\omega/4\hbar)\coth\eta(r_b^2+r_a^2)}I_m\left(\frac{M\omega r_b r_a}{2\hbar\sinh\eta}\right).\end{aligned} \tag{9.107}$$

The integral is done with the help of formula (9.29) and yields

$$\begin{aligned}(r_b|r_a)_{m,E} \;=\; & -i\sqrt{r_b r_a}\frac{M}{\hbar}\frac{\Gamma\left(\frac{1}{2}-\nu+\frac{|m|}{2}\right)}{(M\omega/2\hbar)r_b r_a\Gamma(|m|+1)} \\ & \times W_{\nu,|m|/2}\left(\frac{M\omega}{2\hbar}r_b^2\right)M_{\nu,|m|/2}\left(\frac{M\omega}{2\hbar}r_a^2\right),\end{aligned} \tag{9.108}$$

with

$$\nu \equiv \frac{E}{\omega\hbar}+\frac{m}{2}. \tag{9.109}$$

The Gamma function $\Gamma(1/2-\nu-|m|/2)$ has poles at

$$\nu = \nu_r \equiv n_r + \frac{1}{2} + \frac{|m|}{2} \tag{9.110}$$

of the form

$$\Gamma(1/2-\nu-|m|/2) \approx -\frac{1}{n_r!}\frac{(-1)^{n_r}}{\nu-\nu_r} \sim -\frac{(-1)^{n_r}}{n_r!}\frac{\omega\hbar}{E-E_{n_r m}}. \tag{9.111}$$

The poles lie at the energies

$$E_{n_r m} = \hbar\omega\left(n_r+\frac{1}{2}+\frac{|m|}{2}-\frac{m}{2}\right). \tag{9.112}$$

These are the well-known Landau levels of a particle in a uniform magnetic field. The Whittaker functions at the poles are (for $m>0$)

$$\begin{aligned} M_{-\nu,m/2}(z) &= e^{z/2}z^{\frac{1+m}{2}}M(-n_r,1+m,-z), \qquad (9.113)\\ W_{\nu,m/2}(z) &= e^{-z/2}z^{\frac{1+m}{2}}(-)^{n_r}\frac{(n_r+m)!}{m!}M(-n_r,1+m,z). \qquad (9.114)\end{aligned}$$

The fixed-energy amplitude near the poles is therefore

$$(r_b|r_a)_{m,E} \sim \frac{i\hbar}{E-E_{n_rm}}R_{n_rm}(r_b)R_{n_rm}(r_a), \qquad (9.115)$$

with the radial wave functions[11]

$$\begin{aligned} R_{n_rm}(r) &= \sqrt{r}\left(\frac{M\omega}{\hbar}\right)^{1/2}\sqrt{\frac{(n_r+|m|)!}{n_r!}}\frac{1}{|m|!} \qquad (9.116)\\ &\quad\times\exp\left(-\frac{M\omega}{4\hbar}r^2\right)\left(\frac{M\omega}{2\hbar}r^2\right)^{|m|/2}M\left(-n_r,1+|m|,\frac{M\omega}{2\hbar}r^2\right).\end{aligned}$$

Using Eq. (9.53), they can be expressed in terms of Laguerre polynomials $L_n^\alpha(z)$:

$$\begin{aligned} R_{n_rm} &= \sqrt{r}\left(\frac{M\omega}{\hbar}\right)^{1/2}\sqrt{\frac{n_r!}{(n_r+|m|)!}}\exp\left(-\frac{M\omega}{4\hbar}r^2\right)\\ &\quad\times\left(\frac{M\omega}{2\hbar}r^2\right)^{|m|/2}L_{n_r}^{|m|}\left(\frac{M\omega}{2\hbar}r^2\right). \qquad (9.117)\end{aligned}$$

The integral (9.54) ensures the orthonormality of the radial wave functions

$$\int_0^\infty dr R_{n_rm}(r)R_{n_r'm}(r) = \delta_{n_rn_r'}. \qquad (9.118)$$

A Laplace transformation of the fixed-energy amplitude (9.108) gives, via the residue theorem, the spectral representation of the radial time evolution amplitude

$$(r_b\tau_b|r_a\tau_a) = \sum_{n_rm}R_{n_rm}(r_b)R_{n_rm}(r_a)e^{-E_{n_rm}(\tau_b-\tau_a)/\hbar}, \qquad (9.119)$$

with the energies (9.112). The full wave functions in the transverse subspace are, of course,

$$\psi_{n_rm}(\mathbf{x}) = \frac{1}{\sqrt{r}}R_{n_rm}(r)\frac{e^{im\varphi}}{\sqrt{2\pi}}. \qquad (9.120)$$

Comparing the energies (9.112) with (9.100), we identify the principal quantum number n as

$$n \equiv n_r + \frac{|m|}{2} - \frac{m}{2}. \qquad (9.121)$$

[11]Compare with L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Pergamon, London, 1965, p. 427.

Note that the infinite degeneracy of the energy levels observed in (9.100) with respect to p_y is now present with respect to m. This energy does not depend on m for $m \geq 0$. The somewhat awkward m-dependence of the energy can be avoided by introducing, instead of m, another quantum number n' related to n, m by

$$m = n' - n. \tag{9.122}$$

The states are then labeled by n, n' with both n and n' taking the values $0, 1, 2, 3, \dots$. For $n' < n$, one has $n' = n_r$ and $m = n' - n < 0$, whereas for $n' \geq n$ one has $n = n_r$ and $m = n' - n \geq 0$. There exists a natural way of generating the wave functions $\psi_{n_r m}(\mathbf{x})$ such that they appear immediately with the quantum numbers n, n'. For this we introduce the Landau radius

$$a = \sqrt{\frac{2\hbar}{M\omega}} = \sqrt{\frac{2\hbar c}{eB}} \tag{9.123}$$

as a length parameter and define the dimensionless transverse coordinates

$$z = (x + iy)/\sqrt{2}a, \qquad z^* = (x - iy)/\sqrt{2}a. \tag{9.124}$$

It is then possible to prove that the $\psi_{n_r m}$'s coincide with the wave functions

$$\psi_{n,n'}(z, z^*) = N_{n,n'} e^{z^* z} \left(-\frac{1}{\sqrt{2}}\partial_{z*}\right)^n \left(-\frac{1}{\sqrt{2}}\partial_z\right)^{n'} e^{-2z^* z}. \tag{9.125}$$

The normalization constants are obtained by observing that the differential operators

$$\begin{aligned} e^{z^* z}\left(-\frac{1}{\sqrt{2}}\partial_{z^*}\right)e^{-z^* z} &= \frac{1}{\sqrt{2}}(-\partial_{z^*} + z), \\ e^{z^* z}\left(-\frac{1}{\sqrt{2}}\partial_z\right)e^{-z^* z} &= \frac{1}{\sqrt{2}}(-\partial_z + z^*) \end{aligned} \tag{9.126}$$

behave algebraically like two independent creation operators

$$\begin{aligned} \hat{a}^\dagger &= \frac{1}{\sqrt{2}}(-\partial_{z^*} + z), \\ \hat{b}^\dagger &= \frac{1}{\sqrt{2}}(-\partial_z + z^*), \end{aligned} \tag{9.127}$$

whose conjugate annihilation operators are

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}(\partial_z + z^*), \\ \hat{b} &= \frac{1}{\sqrt{2}}(\partial_{z^*} + z). \end{aligned} \tag{9.128}$$

The ground state wave function annihilated by these is

$$\psi_{0,0}(z,z^*) = \langle z,z^*|0\rangle \propto e^{-z^*z}. \tag{9.129}$$

We can therefore write the complete set of wave functions as

$$\psi_{n,n'}(z,z^*) = N_{nn'}\hat{a}^{\dagger n}\hat{b}^{\dagger n'}\psi_{0,0}(z,z^*). \tag{9.130}$$

Using the fact that $\hat{a}^{\dagger *} = \hat{b}^{\dagger}$, $\hat{b}^{\dagger *} = \hat{a}^{\dagger}$, and that partial integrations turn $\hat{b}^{\dagger},\hat{a}^{\dagger}$ into $\hat{a},\hat{b}$, respectively, the normalization integral can be rewritten as

$$\begin{aligned}
&\int dx\,dy\;\psi_{n_1,n'_1}(z,z^*)\psi_{n_2,n'_2}(z,z^*)\\
&\quad= N_{n_1n'_1}N_{n_2n'_2}\int dxdy\left[(a^\dagger)^{n_1}(b^\dagger)^{n'_1}e^{-z^*z}\right]\left[(a^\dagger)^{n_2}(b^\dagger)^{n'_2}e^{-z^*z}\right]\\
&\quad= N_{n_1n'_1}N_{n_2n'_2}\int dxdye^{-2z^*z}\left(a^{n_1}b^{n'_1}a^{\dagger n_2}b^{\dagger n'_2}\right).
\end{aligned} \tag{9.131}$$

Here the commutation relations between $\hat{a}^\dagger,\hat{b}^\dagger,\hat{a},\hat{b}$ serve to reduce the parentheses in the last line to

$$n_1!n_2!\;\;\delta_{n_1n'_1}\delta_{n_2n'_2}. \tag{9.132}$$

The trivial integral

$$\int dxdy\;e^{-2z^*z} = \pi\int dr^2\;e^{-r^2/a^2} = \pi a^2 \tag{9.133}$$

shows that the normalization constants are

$$N_{n,n'} = \frac{1}{\sqrt{\pi a^2 n!n'!}}. \tag{9.134}$$

Let us prove the equality of ψ_{n_rm} and $\psi_{n,n'}$ up to a possible overall phase. For this we first observe that z,∂_{z^*} and z^*,∂_z carry phase factors $e^{i\varphi}$ and $e^{-i\varphi}$, respectively, so that the two wave functions have obviously the azimuthal quantum number $m=n-n'$. Second, we make sure that the energies coincide by considering the Schrödinger equation corresponding to the action (2.643)

$$\frac{1}{2M}\left(-i\hbar\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)^2\psi = E\psi. \tag{9.135}$$

In the gauge where

$$\mathbf{A} = (0,Bx,0),$$

it reads

$$-\frac{\hbar^2}{2M}\left[{\partial_x}^2+(\partial_y-i\frac{eB}{c}x)^2+{\partial_z}^2\right]\psi = E\psi, \tag{9.136}$$

and the wave functions can be taken from Eq. (9.99). In the gauge where

$$\mathbf{A} = \left(-By/2,Bx/2,0\right), \tag{9.137}$$

on the other hand, the Schrödinger equation becomes, in cylindrical coordinates,

$$\left[-\frac{\hbar^2}{2M}\left(\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\varphi^2+\partial_z^2\right)-\frac{ie\hbar B}{2Mc}\partial_\varphi+\frac{e^2B^2}{8Mc^2}r^2\right]\psi(r,z,\varphi)$$
$$=E\psi(r,z,\varphi). \tag{9.138}$$

Employing a reduced radial coordinate $\rho=r/a$ and factorizing out a plane wave in the z-direction, $e^{ip_z z/\hbar}$, this takes the form

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{a^2}{\hbar^2}(2ME-p_z^2)-\rho^2-2i\partial_\varphi-\frac{1}{\rho^2}\partial_\varphi^2\right]\psi(r,\varphi)=0. \tag{9.139}$$

The solutions are

$$\psi_{n_r m}(r,\varphi)\propto e^{im\varphi}e^{-\rho^2/2}\rho^{|m|/2}M\left(-n_r,|m|+\frac{1}{2},\rho\right), \tag{9.140}$$

where the confluent hypergeometric functions $M\left(-n_r,|m|+\frac{1}{2},\rho\right)$ are polynomials for integer values of the radial quantum number

$$n_r=n+\frac{1}{2}m-\frac{1}{2}|m|-\frac{1}{2}, \tag{9.141}$$

as in (9.116). The energy is related to the principal quantum number by

$$n+\frac{1}{2}\equiv\frac{a^2}{\hbar^2}\left(2ME-p_z^2\right). \tag{9.142}$$

Since

$$\frac{2Ma^2}{\hbar^2}=\frac{1}{\hbar\omega}, \tag{9.143}$$

the energy is

$$E=\left(n+\frac{1}{2}\right)\hbar\omega+\frac{p_z^2}{2M}. \tag{9.144}$$

We now observe that the Schrödinger equation (9.139) can be expressed in terms of the creation and annihilation operators (9.127), (9.128) as

$$4\left[-(a^\dagger a+1/2)+\frac{1}{\hbar\omega}\left(E-\frac{p_z^2}{2M}\right)\right]\psi(z,z^*)=0. \tag{9.145}$$

This proves that the algebraically constructed wave functions $\psi_{n,n'}$ in (9.130) coincide with the wave functions $\psi_{n_r m}$ of (9.116) and (9.140), up to an irrelevant phase. Note that the energy depends only on the number of a-quanta; it is independent of the number of b-quanta.

9.5 Dirac δ-Function Potential

For a particle in a Dirac δ-function potential, the fixed-energy amplitudes $(\mathbf{x}_b|\mathbf{x}_a)_E$ can be calculated by performing a perturbation expansion around a free-particle amplitude and summing it up exactly. For any time-independent potential $V(\mathbf{x})$, in addition to a harmonic potential $M\omega^2\mathbf{x}^2/2$, the perturbation expansion in Eq. (3.478) can be Laplace-transformed in the imaginary time via (9.3) to find

$$\begin{aligned}(\mathbf{x}_b|\mathbf{x}_a)_E &= (\mathbf{x}_b|\mathbf{x}_a)_{\omega,E} - \frac{i}{\hbar}\int d^Dx_1(\mathbf{x}_b|\mathbf{x}_1)_{\omega,E}V(\mathbf{x}_1)(\mathbf{x}_1|\mathbf{x}_a)_{\omega,E} \\ &+ -\frac{1}{\hbar^2}\int d^Dx_2\int d^Dx_1(\mathbf{x}_b|\mathbf{x}_2)_{\omega,E}V(\mathbf{x}_2)(\mathbf{x}_2|\mathbf{x}_1)_{\omega,E}V(\mathbf{x}_1)(\mathbf{x}_1|\mathbf{x}_a)_{\omega,E} \\ &+ \dots\,. \end{aligned} \tag{9.146}$$

If the potential is a Dirac δ-function centered around $\mathbf{X}$,

$$V(\mathbf{x}) = g\,\delta^{(D)}(\mathbf{x}-\mathbf{X}), \qquad g \equiv \frac{\hbar^2}{Ml^{2-D}}, \tag{9.147}$$

this series simplifies to

$$(\mathbf{x}_b|\mathbf{x}_a)_E=(\mathbf{x}_b|\mathbf{x}_a)_{\omega,E}-\frac{ig}{\hbar}g(\mathbf{x}_b|\mathbf{X})_{\omega,E}(\mathbf{X}|\mathbf{x}_a)_{\omega,E}-\frac{g^2}{\hbar^2}(\mathbf{x}_b|\mathbf{X})_{\omega,E}(\mathbf{X}|\mathbf{X})_{\omega,E}(\mathbf{X}|\mathbf{x}_a)_{\omega,E}+\dots, \tag{9.148}$$

and can be summed up to

$$(\mathbf{x}_b|\mathbf{x}_a)_E = (\mathbf{x}_b|\mathbf{x}_a)_{\omega,E} - i\frac{g}{\hbar}\frac{(\mathbf{x}_b|\mathbf{X})_{\omega,E}(\mathbf{X}|\mathbf{x}_a)_{\omega,E}}{1+i\dfrac{g}{\hbar}(\mathbf{X}|\mathbf{X})_{\omega,E}}. \tag{9.149}$$

This is, incidentally, true if a δ-function potential is added to an arbitrary solvable fixed-energy amplitude, not just the harmonic one.

If the δ-function is the only potential, we use formula (9.149) with $\omega = 0$, so that $(\mathbf{x}_b|\mathbf{x}_a)_{0,E}$ reduces to the fixed-energy amplitude (9.12) of a free particle, and obtain directly

$$\begin{aligned}(\mathbf{x}_b|\mathbf{x}_a)_E &= -2i\frac{M}{\hbar}\frac{\kappa^{D-2}}{(2\pi)^{D/2}}\frac{K_{D/2-1}(\kappa R)}{(\kappa R)^{D/2-1}} \\ &- \frac{ig}{\hbar}\frac{i\dfrac{2M}{\hbar}\dfrac{\kappa^{D-2}}{(2\pi)^{D/2}}\dfrac{K_{D/2-1}(\kappa R_b)}{(\kappa R_b)^{D/2-1}}\times i\dfrac{2M}{\hbar}\dfrac{\kappa^{D-2}}{(2\pi)^{D/2}}\dfrac{K_{D/2-1}(\kappa R_a)}{(\kappa R_a)^{D/2-1}}}{1-\dfrac{g}{\hbar}\dfrac{2M}{\hbar}\dfrac{\kappa^{D-2}}{\pi^{D/2}}\dfrac{K_{D/2-1}(\kappa\delta)}{(\kappa\delta)^{D/2-1}}}, \end{aligned} \tag{9.150}$$

where $R \equiv |\mathbf{x}_b - \mathbf{x}_a|$ and $R_{a,b} \equiv |\mathbf{x}_{a,b} - \mathbf{X}|$, and δ is an infinitesimal distance regularizing a possible singularity at zero-distance. In $D = 1$ dimension, this reduces to

$$(x_b|x_a)_E = -i\frac{M}{\hbar}\frac{1}{\kappa}e^{-\kappa R} + i\frac{M}{\hbar\kappa}e^{\kappa(R_b+R_a)}\frac{1}{l\kappa+1}, \tag{9.151}$$

For an attractive potential with $l < 0$, the second term can be written as

$$-i\frac{1}{\kappa l^2}\frac{\hbar}{E+\hbar^2/2Ml^2}e^{\kappa(R_b+R_a)}, \tag{9.152}$$

exhibiting a pole at the bound-state energy $E_B = -\hbar/2Ml^2$. In its neighborhood, the pole contribution reads

$$\frac{2}{l}e^{-(R_b+R_a)/l}\,\frac{i\hbar}{E+\hbar^2/2Ml^2}\,. \tag{9.153}$$

This has precisely the spectral form (1.325) with the normalized bound-state wave function

$$\psi_B(x) = \sqrt{\frac{2}{l}}\,e^{-|x-X|/l}. \tag{9.154}$$

In $D = 3$ dimensions, the amplitude (9.150) becomes

$$(\mathbf{x}_b|\mathbf{x}_a)_E \;=\; -i\frac{M}{\hbar}\frac{1}{2\pi R}e^{-\kappa R} + i\frac{M}{\hbar}\frac{e^{\kappa R_b}}{2\pi R_b}\frac{e^{\kappa R_a}}{2\pi R_a}\frac{1}{1/l+e^{-\kappa\delta}/2\pi\delta}. \tag{9.155}$$

In the limit $\delta \to 0$, the denominator requires renormalization. We introduce a renormalized coupling length scale

$$\frac{1}{l_r} \equiv 1 + \frac{1}{2\pi\delta}, \tag{9.156}$$

and rewrite the last factor in (9.155) as

$$\frac{1}{1/l_r - \kappa/2\pi}. \tag{9.157}$$

For $l_r < 0$, this has a pole at the bound-state energy $E_B = -4\pi^2\hbar^2/2Ml_R^2$ of the form

$$-l_r E_B \frac{1}{E-E_B}. \tag{9.158}$$

The total pole term in (9.155) can therefore be written as

$$\psi_B(\mathbf{x}_b)\psi_B^*(\mathbf{x}_a)\frac{i\hbar}{E-E_B}, \tag{9.159}$$

with $\kappa_B = \sqrt{2ME_B}/\hbar = 2\pi/l_r$ and the normalized bound-state wave functions

$$\psi_B(\mathbf{x}) = \left(\frac{\kappa_B^2}{4\pi^2}\right)^{1/4}\frac{e^{-\kappa_B|\mathbf{x}-\mathbf{X}|}}{r}. \tag{9.160}$$

In $D = 2$ dimensions, the situation is more subtle. It is useful to consider the amplitude (9.150) in $D = 2 + \epsilon$ dimensions where one has

$$\begin{aligned}(\mathbf{x}_b|\mathbf{x}_a)_E &= -i\frac{M}{\hbar}\frac{1}{\pi}\frac{K_{\epsilon/2}(\kappa R)}{(2\pi\kappa R)^{\epsilon/2}} \\ &+ i\frac{M^2}{\hbar^2\pi^2}\frac{1}{(2\pi\kappa R_b)^{\epsilon/2}}\frac{1}{(2\pi\kappa R_a)^{\epsilon/2}}\frac{K_{\epsilon/2}(\kappa R_b)K_{\epsilon/2}(\kappa R_a)}{\dfrac{\hbar}{g}+\dfrac{M}{\hbar\pi}\dfrac{1}{(2\pi\kappa\delta)^{\epsilon/2}}K_{\epsilon/2}(\kappa\epsilon)}. \end{aligned} \tag{9.161}$$

Inserting here $K_{\epsilon/2}(\kappa\delta) \approx (1/2)\Gamma(\epsilon/2)(\kappa\delta/2)^{-\epsilon/2}$, the denominator becomes

$$\frac{\hbar}{g} + \frac{M}{2\hbar\pi}\frac{\Gamma(\epsilon/2)}{(\pi\kappa\delta)^{\epsilon/2}} \approx \frac{\hbar}{g} + \frac{M}{2\hbar\pi}\frac{2}{\epsilon}\left[1 - \frac{\epsilon}{2}\log(\pi\kappa\delta)\right]. \tag{9.162}$$

Here we introduce a renormalized coupling constant

$$\frac{1}{g_r} = \frac{1}{g} + \frac{M}{\hbar^2}\frac{1}{\epsilon}, \tag{9.163}$$

and rewrite the right-hand side as

$$\frac{\hbar}{g_r} - \frac{M}{2\hbar\pi}\log\pi\kappa\delta. \tag{9.164}$$

This has a pole at

$$\kappa_B = \frac{1}{\pi\delta}e^{2\hbar^2\pi/Mg_r}, \tag{9.165}$$

indicating a bound-state pole of energy $E_B = -\hbar^2\kappa_B^2/2M$.

We can now go to the limit of $D = 2$ dimensions and find that the pole term in (9.161) has the form

$$\psi_B(\mathbf{x}_b)\psi_B^*(\mathbf{x}_a)\frac{i\hbar}{E - E_B}, \tag{9.166}$$

with the normalized bound-state wave function

$$\psi_B(\mathbf{x}) = \frac{\kappa_B}{\sqrt{\pi}}K_0(\kappa_B|\mathbf{x} - \mathbf{X}|). \tag{9.167}$$

Notes and References

The wave functions derived in this chapter from the time evolution amplitude should be compared with those given in standard textbooks on quantum mechanics, such as
L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Pergamon, London, 1965. The charged particle in a magnetic field is treated in §111.
The δ-function potential was studied via path integrals by
C. Grosche, Phys. Rev. Letters, **71**, 1 (1993).