Felix est quantulumcunque temporis contigit, bene collocatus est

*Happy is he who has well employed the time,*
*however small the time slices may be*

SENECA (4 BC–65)

# 11

# Schrödinger Equation in General Metric-Affine Spaces

We now use the path integral representation of the last chapter to find out which Schrödinger equation is obeyed by the time evolution amplitude in a space with curvature and torsion. If there is only curvature, the result establishes the connection with the operator quantum mechanics described in Chapter 1. In particular, it will properly reproduce the energy spectra of the systems in Sections 1.14 and 1.15 — a particle on the surface of a sphere and a spinning top — which were quantized there via group commutation rules. If the space carries torsion also, the Schrödinger operator emerging from our formulation will be a *prediction*. Its correctness will be verified in Chapter 13 by an application to the path integral of the Coulomb system which can be transformed into a harmonic oscillator by a nonholonomic mapping involving curvature and torsion.

## 11.1 Integral Equation for Time Evolution Amplitude

Consider the time-sliced path integral Eq. (10.146)

$$\langle q|e^{-i(t-t')\hat{H}/\hbar}|q'\rangle = \frac{1}{\sqrt{2\pi i\hbar\epsilon/M}^D}\prod_{n=2}^{N+1}\left[\int d^D\Delta q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}\right]e^{i\sum_{n=1}^{N+1}(\mathcal{A}^\epsilon+\mathcal{A}_J^\epsilon)/\hbar}, \tag{11.1}$$

with the integrals over $\Delta q_n$ to be performed successively from $n = N$ down to $n = 1$.

Let us study the effect of the last $\Delta q_n$-integration upon the remaining product of integrals. We denote the entire product briefly by $\psi(q_{N+1}, t_{N+1}) \equiv \psi(q, t)$ and the product without the last factor by

$$\psi(q_N, t_N) = \psi(q_{N+1} - \Delta q_{N+1}, t_{N+1} - \epsilon) \equiv \psi(q - \Delta q, t - \epsilon).$$

Since the initial coordinate $q_0$ and time $t_0$ of the amplitude are kept fixed in the sequel, they are not shown in the arguments. We assume $N$ to be so large that the amplitude has had time to develop from the initial state localized at $q'$ to a smooth function of $\psi(q-\Delta q, t-\epsilon)$, smooth compared to the width of the last short-time amplitude, which is of the order $\sqrt{\hbar\epsilon\,\mathrm{tr}\,(g_{\mu\nu})/M}$.

From Eq. (11.1) we deduce the recursion relation

$$\psi(q,t) = \sqrt{g(q)}\int \frac{d^D\Delta q}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}(\mathcal{A}^\epsilon + \mathcal{A}_J^\epsilon)\right]\psi(q-\Delta q, t-\epsilon). \tag{11.2}$$

This is an integral equation

$$\psi(q,t) = \int d^D\Delta q\, K^\epsilon(q,\Delta q)\,\psi(q-\Delta q, t-\epsilon), \tag{11.3}$$

with an integral kernel

$$K^\epsilon(q,\Delta q) = \frac{\sqrt{g(q)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}(\mathcal{A}^\epsilon + \mathcal{A}_J^\epsilon)\right]. \tag{11.4}$$

The integral equation (11.3) will now be turned into a Schrödinger equation. This will be done in two ways, a short way which gives direct insight into the relevance of the different terms in the mapping (10.96), and a historic more tedious way, which is useful for comparing our path integral with previous alternative proposals in the literature (cited at the end).

### 11.1.1 From Recursion Relation to Schrödinger Equation

The evaluation of (11.3) is much easier if we take advantage of the simplicity of the integral kernel $K^\epsilon(q,\Delta q)$ and the measure when expressed in terms of the variables $\Delta x^i$. Thus we introduce into (11.3) the integration variables $\Delta\xi^\mu \equiv \Delta x^i e_i{}^\mu$, with $e_i{}^\mu$ evaluated at the postpoint $q$. The explicit relation between $\Delta\xi^\mu$ and $\Delta q^\mu$ follows directly from (10.96). In terms of $\Delta\xi^\mu$, we rewrite (11.3) as

$$\psi(q,t) \;\; = \;\; \int d^D\Delta\xi\, K_0^\epsilon(q,\Delta\xi)\psi(q-\Delta q(\Delta\xi), t-\epsilon),$$

with the zeroth-order kernel

$$K_0^\epsilon(q,\Delta\xi) = \frac{\sqrt{g(q)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}\frac{M}{2\epsilon}g_{\mu\nu}(q)\Delta\xi^\mu\Delta\xi^\nu\right] \tag{11.5}$$

of unit normalization

$$\int d^D\Delta\xi\;K_0^\epsilon(q,\Delta\xi) = 1. \tag{11.6}$$

To perform the integrals in (11.2), we expand the wave function as

$$\psi(q-\Delta q, t-\epsilon) = \left(1 - \Delta q^\mu\partial_\mu + \frac{1}{2}\Delta q^\mu\Delta q^\nu\partial_\mu\partial_\nu + \dots\right)\psi(q,t-\epsilon), \tag{11.7}$$

and the coordinate differences $\Delta q^\mu$ in powers of $\Delta\xi$ by inverting Eq. (10.96):

$$\Delta q^\lambda = \left[\Delta\xi^\lambda + \frac{1}{2!}\Gamma_{\mu\nu}{}^\lambda\Delta\xi^\mu\Delta\xi^\nu - \frac{1}{3!}(\partial_\sigma\Gamma_{\mu\nu}{}^\lambda - \Gamma_{\mu\nu}{}^\tau\Gamma_{\{\sigma\tau\}}{}^\lambda)\Delta\xi^\mu\Delta\xi^\nu\Delta\xi^\sigma + \ldots\right]. \tag{11.8}$$

All affine connections are evaluated at the postpoint $q$. Including in (11.2) only the relevant expansion terms, we find the integral equation

$$\begin{aligned} \psi(q,t) &= \int d^D\Delta\xi\, K_0^\epsilon(q,\Delta\xi) \\ &\times \left[1 - \left(\Delta\xi^\mu + \frac{1}{2!}\Gamma_{\nu\lambda}{}^\mu\Delta\xi^\nu\Delta\xi^\lambda\right)\partial_\mu + \frac{1}{2}\Delta\xi^\mu\Delta\xi^\nu\partial_\mu\partial_\nu + \ldots\right]\psi(q,t-\epsilon). \end{aligned} \tag{11.9}$$

The evaluation requires only the normalization integral (11.6) and the two-point correlation function

$$\langle\Delta\xi^\mu\Delta\xi^\nu\rangle = \int d^D\Delta\xi\, K_0^\epsilon(q,\Delta\xi)\Delta\xi^\mu\Delta\xi^\nu = \frac{i\hbar\epsilon}{M}g^{\mu\nu}(q). \tag{11.10}$$

The result is

$$\psi(q,t) = \left[1 + i\epsilon\frac{\hbar^2}{2M}\left(g^{\mu\nu}\partial_\mu\partial_\nu - \Gamma_\nu{}^{\nu\mu}\partial_\mu\right) + \ldots\right]\psi(q,t-\epsilon). \tag{11.11}$$

The differential operator in parentheses is proportional to the covariant Laplacian of the field $\psi(q,t-\epsilon)$:

$$D_\mu D^\mu\psi \equiv g^{\mu\nu}D_\mu D_\nu\psi = g^{\mu\nu}D_\mu\partial_\nu\psi = (g^{\mu\nu}\partial_\mu\partial_\nu - \Gamma_\nu{}^{\nu\mu}\partial_\mu)\psi. \tag{11.12}$$

In a space with no torsion, this is equal to the Laplace-Beltrami operator applied to the field $\psi$:

$$\Delta\psi = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu\psi. \tag{11.13}$$

In a more general space, the relation between the two operators is obtained by working out the derivatives

$$\Delta = g^{\mu\nu}\partial_\mu\partial_\nu + \left(\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\right)g^{\mu\nu}\partial_\nu + (\partial_\mu g^{\mu\nu})\partial_\nu. \tag{11.14}$$

Using

$$\begin{aligned} \left(\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\right) &= \frac{1}{2}g^{\sigma\tau}\partial_\mu g_{\sigma\tau} = \bar\Gamma_{\mu\nu}{}^\nu, \\ \partial_\mu g^{\sigma\nu} &= -g^{\sigma\lambda}g^{\nu\kappa}\partial_\mu g_{\lambda\kappa}, \\ \partial_\mu g^{\mu\nu} &= -\bar\Gamma_\mu{}^{\mu\nu} - \bar\Gamma^\nu{}_\mu{}^\mu, \end{aligned} \tag{11.15}$$

we see that

$$\frac{1}{\sqrt{g}}(\partial_\mu g^{\mu\nu}\sqrt{g}) = -\bar{\Gamma}_\mu{}^{\mu\nu}, \tag{11.16}$$

and hence

$$\Delta\psi = (g^{\mu\nu}\partial_\mu\partial_\nu - \bar{\Gamma}_\mu{}^{\mu\nu}\partial_\nu)\psi = \bar{D}_\mu\bar{D}^\mu\psi. \tag{11.17}$$

Thus, the relation between the Laplacian and the Laplace-Beltrami operator is given by

$$D_\mu D^\mu\psi = (\bar{D}_\mu\bar{D}^\mu - K_\mu{}^{\mu\nu}\partial_\nu)\psi = (\bar{D}_\mu\bar{D}^\mu - 2S^\nu\partial_\nu)\psi, \tag{11.18}$$

where $\bar{D}_\mu$ denotes the covariant derivative formed with the Riemannian affine connection, the Christoffel symbol $\bar{\Gamma}_{\mu\nu}{}^\lambda$, and $S_\mu$ is the contracted torsion

$$S_\mu \equiv S_{\mu\nu}{}^\nu. \tag{11.19}$$

As a result, the amplitude $\psi(q,t)$ in (11.2) satisfies the equation

$$\psi(q,t) = \left(1 + \frac{i\epsilon\hbar}{2M}D_\mu D^\mu\right)\psi(q,t-\epsilon) + \mathcal{O}(\epsilon^2). \tag{11.20}$$

In the limit $\epsilon \to 0$, this leads to the Schrödinger equation

$$i\hbar\partial_t\psi(q,t) = \hat{H}_0\psi(q,t), \tag{11.21}$$

where $\hat{H}_0$ is the free-particle Schrödinger operator

$$\hat{H}_0 = -\frac{\hbar^2}{2M}D_\mu D^\mu. \tag{11.22}$$

It is the naively expected generalization of the flat-space operator

$$\hat{H}_0 = -\frac{\hbar^2}{2M}\partial_i{}^2, \tag{11.23}$$

from which (11.22) arises by transforming the derivatives with respect to Cartesian coordinates $\partial_i$ to the general coordinate derivatives $\partial_\mu$ via the nonholonomic transformation

$$\partial_i = e_i{}^\mu\partial_\mu. \tag{11.24}$$

The result is

$$\partial_i{}^2 = e_i{}^\mu\partial_\mu e^{i\nu}\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu - \Gamma_\mu{}^{\mu\nu}\partial_\nu, \tag{11.25}$$

which coincides with the Laplacian $D_\mu\, D^\mu$ when applied to a scalar field. Note that the operator (11.22) contains no extra term proportional to the scalar curvature $R$ allowed by other theories.

### 11.1.2 Alternative Evaluation

For completeness, we also present an alternative evaluation of the $q$-integrals in Eq. (11.3) which is more tedious but facilitates comparison with previous work. First, the postpoint action $\mathcal{A}^\epsilon$ is conveniently split into the leading term

$$\mathcal{A}_0^\epsilon = \frac{M}{2\epsilon} g_{\mu\nu}(q)\Delta q^\mu \Delta q^\nu \tag{11.26}$$

and a remainder

$$\Delta\mathcal{A}^\epsilon \equiv \mathcal{A}^\epsilon - \mathcal{A}_0^\epsilon. \tag{11.27}$$

Correspondingly, we introduce as in (11.5) the zeroth-order kernel

$$K_0^\epsilon(q, \Delta q) = \frac{\sqrt{g(q)}}{\sqrt{2\pi i\epsilon\hbar/M}^D} \exp\left(\frac{i}{\hbar}\mathcal{A}_0^\epsilon\right), \tag{11.28}$$

with the unit normalization

$$\int d^D\Delta q \; K_0^\epsilon(q, \Delta q) = 1, \tag{11.29}$$

and expand $K^\epsilon(q, \Delta q)$ around $K_0^\epsilon(q, \Delta q)$ with a series of correction terms of higher order in $\Delta q$:

$$K^\epsilon(q, \Delta q) = K_0^\epsilon(q, \Delta q)[1 + C(\Delta q)] \equiv K_0^\epsilon(q, \Delta q)\left[1 + \sum_{n=1}^{\infty} c_n(\Delta q)^n\right]. \tag{11.30}$$

Under the smoothness assumptions above, the wave function $\psi(q - \Delta q, t - \epsilon)$ can be expanded into a Taylor series around the endpoint $q$, so that the integral equation (11.2) reads

$$\begin{aligned}\psi(q, t) \;\; &= \;\; \int d^D\Delta q \, K_0^\epsilon(q, \Delta q)\left[1 + \sum_{n=1}^{\infty} c_n(\Delta q)^n\right] \\ &\quad \times \left(1 - \Delta q^\mu\partial_\mu + \frac{1}{2}\Delta q^\mu \Delta q^\nu \partial_\mu\partial_\nu + \dots\right)\psi(q, t - \epsilon)\;.\end{aligned} \tag{11.31}$$

Due to the normalization property (11.6), the leading term simply reproduces $\psi(q, t - \epsilon)$. To calculate the correction terms $c_n(\Delta q)$, we expand

$$C(\Delta q) = \exp\left[\frac{i}{\hbar}(\Delta\mathcal{A}^\epsilon + \mathcal{A}_J^\epsilon)\right] - 1 \tag{11.32}$$

in powers of $\Delta q^\mu$. After inserting here $\Delta\mathcal{A}^\epsilon$ from (11.27) with $\mathcal{A}_0^\epsilon$ from (11.26), we expand $\mathcal{A}^\epsilon$ as in (10.107) [recalling (10.121)]. By separating the expansion for $C$ into even and odd powers of $\Delta q$,

$$C = C^{\mathrm{e}} + C^{\mathrm{o}}, \tag{11.33}$$

we find for the odd terms

$$C^{\text{ø}} = -\Gamma_{\{\mu\nu\}}{}^{\nu}\Delta q^{\mu} - \frac{i}{\hbar}\frac{M}{2\epsilon}\Gamma_{\mu\nu\lambda}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda} + \dots\ , \tag{11.34}$$

and the even terms

$$C^{\text{e}} = \sum_{a=1}^{4} C_a^{\text{e}} + \dots\ , \tag{11.35}$$

with

$$\begin{aligned}
C_1^{\text{e}} &= \frac{1}{2}[\partial_{\{\mu}\Gamma_{\nu\lambda\}}{}^{\lambda} + \Gamma_{\{\nu\kappa}{}^{\sigma}\Gamma_{\{\sigma|\mu\}\}}{}^{\kappa} + \Gamma_{\{\mu\sigma\}}{}^{\sigma}\Gamma_{\{\nu\lambda\}}{}^{\lambda} - \Gamma_{\{\nu\kappa\}}{}^{\sigma}\Gamma_{\{\mu\sigma\}}{}^{\kappa}]\Delta q^{\mu}\Delta q^{\nu},\\
C_2^{\text{e}} &= \frac{iM}{2\hbar\epsilon}\Gamma_{\{\mu\nu\}}{}^{\nu}\Gamma_{\sigma\lambda\kappa}\Delta q^{\mu}\Delta q^{\sigma}\Delta q^{\lambda}\Delta q^{\kappa},\\
C_3^{\text{e}} &= \frac{iM}{2\hbar\epsilon}\left[\frac{1}{3}g_{\kappa\tau}(\partial_{\lambda}\Gamma_{\mu\nu}{}^{\tau} + \Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\{\lambda\sigma\}}{}^{\tau}) + \frac{1}{4}\Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\lambda\kappa\sigma}\right]\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\kappa},\\
C_4^{\text{e}} &= -\frac{1}{2}\frac{M^2}{4\hbar^2\text{e}^2}\Gamma_{\mu\nu\lambda}\Gamma_{\sigma\tau\kappa}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\sigma}\Delta q^{\tau}\Delta q^{\kappa}.
\end{aligned} \tag{11.36}$$

The dots denote terms of higher order in $\Delta q^{\mu}$ which do not contribute to the limit $\epsilon \to 0$.

The evaluation now proceeds perturbatively and requires the harmonic expectation values

$$\langle \mathcal{O}(\Delta q)\rangle_0 \equiv \int d^D\Delta q\, K_0^{\epsilon}(q,\Delta q)\ \mathcal{O}(\Delta q). \tag{11.37}$$

The relevant correlation functions are

$$\langle \Delta q^{\mu}\Delta q^{\nu}\rangle_0 = \frac{i\hbar\epsilon}{M}g^{\mu\nu}, \tag{11.38}$$

$$\langle \Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\kappa}\rangle_0 = \left(\frac{i\hbar\epsilon}{M}\right)^2 g^{\mu\nu\lambda\kappa}, \tag{11.39}$$

$$\langle \Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\kappa}\Delta q^{\sigma}\Delta q^{\tau}\rangle_0 = \left(\frac{i\hbar\epsilon}{M}\right)^3 g^{\mu\nu\lambda\kappa\sigma\tau}. \tag{11.40}$$

The tensor $g^{\mu\nu\lambda\kappa}$ in the second expectation (11.39) collects three Wick contractions [recall (3.305)] and reads

$$g^{\mu\nu\lambda\kappa} \equiv g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\lambda}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\lambda}. \tag{11.41}$$

The tensor $g^{\mu\nu\lambda\kappa\sigma\tau}$ in the third expectation (11.40) collecting 15 Wick contractions is obtained recursively following the rule (3.306) by expanding

$$g^{\mu\nu\lambda\kappa\sigma\tau} = g^{\mu\nu}g^{\lambda\kappa\sigma\tau} + g^{\mu\lambda}g^{\nu\kappa\sigma\tau} + g^{\mu\kappa}g^{\nu\lambda\sigma\tau} + g^{\mu\sigma}g^{\nu\lambda\kappa\tau} + g^{\mu\tau}g^{\nu\lambda\kappa\sigma}. \tag{11.42}$$

A product of $2n$ factors $\Delta q$ results in $(2n-1)!!$ pair contractions.

Let us collect all contributions in (11.31) relevant to order $\epsilon$. Obviously, the highest derivative term of $\psi(q,t-\epsilon)$ is $\frac{1}{2}\Delta q^\mu \Delta q^\nu \partial_\mu \partial_\nu \psi(q,t-\epsilon)$. It receives only a leading contribution from $K_0^\epsilon(q,\Delta q)$,

$$i\epsilon\frac{\hbar^2}{2M}g^{\mu\nu}(q)\partial_\mu\partial_\nu\psi(q,t-\epsilon), \tag{11.43}$$

with no more corrections from $C(\Delta q)$. The term with one derivative $\partial_\mu$ on $\psi(q,t-\epsilon)$ in (11.31) becomes

$$A^\mu\partial_\mu\psi(q,t-\epsilon), \tag{11.44}$$

where $A^\mu$ is the expectation involving the odd correction terms

$$A^\mu = -\langle C^{\rm o}\Delta q^\mu\rangle_0. \tag{11.45}$$

Using the rules (11.38) and (11.39), we find

$$\begin{aligned} A^\mu &= \left\langle \left(\Gamma_{\{\lambda\nu\}}{}^\nu\Delta q^\lambda + \frac{iM}{2\hbar\epsilon}\Gamma_{\sigma\tau\lambda}\Delta q^\sigma\Delta q^\tau\Delta q^\lambda\right)\Delta q^\mu + \ldots\right\rangle_0 \\ &= i\epsilon\frac{\hbar}{M}\left[\Gamma^{\{\mu\nu\}}{}_\nu - \frac{1}{2}(\Gamma^{\mu\nu}{}_\nu + \Gamma^{\nu\mu}{}_\nu + \Gamma_\nu{}^{\nu\mu})\right] + \ldots \\ &= -i\epsilon\frac{\hbar}{2M}\Gamma_\nu{}^{\nu\mu} + \ldots\ . \end{aligned} \tag{11.46}$$

In combination with (11.43), this produces the Laplacian $D_\mu D^\mu$ of the field $\psi(q,t-\epsilon)$ as in Eq. (11.11).

We now turn to the remaining contributions in (11.31) which contain no more derivatives of $\psi(q,t-\epsilon)$. They are all due to the expectation value $\langle C^{\rm e}\rangle_0$ of the even correction terms. Let us define

$$V_{\rm eff} \equiv \frac{i\hbar}{\epsilon}\langle C\rangle_0 = \frac{i\hbar}{\epsilon}\langle C^{\rm e}\rangle_0, \tag{11.47}$$

to be called the effective potential caused by the correction terms $\langle C\rangle_0$ of (11.32). Using the expectation values (11.38)–(11.40) we find

$$V_{\rm eff} = \frac{\hbar^2}{M}v \equiv \frac{\hbar^2}{M}\sum_{A,B} v_A{}^B, \tag{11.48}$$

where the sum runs over the six terms

$$\begin{aligned} v_2{}^1 &= -\frac{1}{2}(\Gamma_{\{\mu\sigma\}}{}^\sigma\Gamma_{\{\nu\lambda\}}{}^\lambda - \Gamma_{\{\nu\kappa\}}{}^\sigma\Gamma_{\{\mu\sigma\}}{}^\kappa)g^{\mu\nu}, \\ v_2{}^2 &= \frac{1}{8}\Gamma_{\{\mu\nu\}}{}^\tau\Gamma_{\lambda\sigma\tau}g^{\mu\nu\sigma\lambda}, \\ v_2{}^3 &= \frac{1}{2}\Gamma_{\{\mu\kappa\}}{}^\kappa\Gamma_{\nu\tau\lambda}g^{\mu\nu\tau\lambda}, \\ v_2{}^4 &= -\frac{1}{8}\Gamma_{\mu\nu\lambda}\Gamma_{\sigma\tau\kappa}g^{\mu\nu\lambda\sigma\tau\kappa}, \\ v_3{}^1 &= -\frac{1}{2}(\partial_{\{\mu}\Gamma_{\nu\lambda\}}{}^\lambda + \Gamma_{\{\nu\kappa}{}^\sigma\Gamma_{\{\sigma|\mu\}\}}{}^\kappa)g^{\mu\nu}, \\ v_3{}^2 &= \frac{1}{6}g_{\mu\tau}(\partial_\kappa\Gamma_{\lambda\nu}{}^\tau + \Gamma_{\lambda\nu}{}^\sigma\Gamma_{\{\kappa\sigma\}}{}^\tau)g^{\mu\nu\lambda\kappa}. \end{aligned} \tag{11.49}$$

The subscripts 2 and 3 distinguish contributions coming from the quadratic and the cubic terms in the expansion (10.96) of $\Delta x^i$. By inserting on the right-hand sides the explicit expansions (11.42) and (11.41), we find after some algebra that the sum of all $v_A{}^B$-terms is zero. In fact, the $v_2{}^B$- and $v_3{}^B$-terms disappear separately. A simple structural reason for this is given in Appendix 11A.

Explicitly, the cancellation is rather obvious for $v_3{}^B$ after inserting (11.41). For $v_2{}^B$, the proof requires more work which is relegated to Appendix 11A.

Note that in a space without curvature and torsion, the above manipulations are equivalent to a direct transformation of the flat-space integral equation

$$\begin{aligned}\psi(\mathbf{x},t) &= \int \frac{d^D\Delta x}{\sqrt{2\pi i\epsilon\hbar/M}^D} \exp\left[i\epsilon\frac{M}{2}(\Delta x^i)^2\right]\\ &\times \left(1-\Delta x^i\partial_{x^i}+\tfrac{1}{2}\Delta x^i\Delta x^j\partial_{x^i}\partial_{x^j}+\ldots\right)\psi(\mathbf{x},t-\epsilon)\\ &= \left[1+\frac{i\epsilon\hbar}{2M}\partial_i^2+\mathcal{O}(\epsilon^2)\right]\psi(\mathbf{x},t-\epsilon), \end{aligned} \tag{11.50}$$

to the variable $\Delta q$ by a coordinate transformation. In a general metric-affine space, the wave function $\psi(q,t)$ has no counter image in $x$-space so that (11.50) cannot be used as a starting point for a nonholonomic transformation.

## 11.2 Equivalent Path Integral Representations

From the derivation of the Schrödinger equation in Subsection 11.1.1 we learn an important lesson. When deriving the transformation law (10.96) between the finite coordinate differences $\Delta x^i$ and $\Delta q^\mu$ by evaluating the integral equation (10.60) along the autoparallel, the cubic terms in $\Delta q$, which make the action and measure lengthy, can be dropped altogether. A completely equivalent path integral representation of the time evolution amplitude is obtained by transforming the flat-space path integral (10.89) into the general metric-affine one (10.146) with the help of the shortened transformation

$$\Delta x^i = e^i{}_\lambda\left(\Delta q^\lambda - \frac{1}{2!}\Gamma_{\mu\nu}{}^\lambda\Delta q^\mu\Delta q^\nu\right). \tag{11.51}$$

This has the simple Jacobian

$$J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det\left(e^i_\kappa\right)\det\left(\delta^\kappa{}_\mu - e_i{}^\kappa e^i_{\{\mu,\nu\}}\Delta q^\nu\right), \tag{11.52}$$

whose effective action reads

$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon = -e_i{}^\kappa e^i{}_{\kappa,\nu}\Delta q^\nu - \frac{1}{2}e_i{}^\mu e^i_{\{\kappa,\nu\}}e_j{}^\kappa e^j_{\{\mu,\lambda\}}\Delta q^\nu\Delta q^\lambda + \ldots\ . \tag{11.53}$$

With the help of (10.16), this is expressed in terms of the connection yielding

$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon = -\Gamma_{\{\nu\mu\}}{}^\mu\Delta q^\nu - \frac{1}{2}\Gamma_{\{\nu\kappa\}}{}^\sigma\Gamma_{\{\mu,\sigma\}}{}^\kappa\Delta q^\nu\Delta q^\mu + \ldots\ . \tag{11.54}$$

The mapping (11.51) has, however, an unattractive feature: The short-time action following from (11.51)

$$\mathcal{A}^{\epsilon} = \frac{M}{2\epsilon}(\Delta x^i)^2 = \frac{M}{2\epsilon}\Big(g_{\mu\nu}\Delta q^{\mu}\Delta q^{\nu} - \Gamma_{\mu\nu\lambda}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda} + \frac{1}{4}\Gamma_{\lambda\kappa}{}^{\sigma}\Gamma_{\mu\nu\sigma}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\kappa} + \dots\Big) \tag{11.55}$$

is no longer equal to the classical action (10.107) [recall the convention (10.121)] evaluated along the autoparallel.

This was also a feature of another mapping which is the most convenient for calculations. Instead of deriving the relation between $\Delta x^i$ and $\Delta q^{\mu}$ by evaluating (10.60) along the autoparallel, one may assume, for the moment, the absence of curvature and torsion and expand $\Delta x^i = x^i(q) - x^i(q - \Delta q)$ in powers of $\Delta q$:

$$\Delta x^i = e^i{}_{\mu}\Delta q^{\mu} - \frac{1}{2}e^i{}_{\mu,\nu}\Delta q^{\mu}\Delta q^{\nu} + \frac{1}{3!}e^i{}_{\mu,\nu\lambda}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda} + \dots\ . \tag{11.56}$$

After this, curvature and torsion are introduced by allowing the functions $x(q)$ and $\partial_{\mu}x(q)$ to be nonintegrable in the sense of the Schwartz criterion, i.e., the second derivatives of $x(q)$ and $e^i{}_{\mu}(q)$ need not commute with each other [implying that the right-hand side of (11.56) can no longer be written as $x^i(q) - x^i(q - \Delta q)$]. The expansion (11.56) is then a *definition* of the transformation from $\Delta x^i$ to $\Delta q^{\mu}$. Using the identities (10.16), (10.131), and (10.132), the transformation (11.56) turns into

$$\Delta x^i = e^i{}_{\lambda}\Big[\Delta q^{\lambda} - \frac{1}{2!}\Gamma_{\mu\nu}{}^{\lambda}\Delta q^{\mu}\Delta q^{\nu} + \frac{1}{3!}(\partial_{\sigma}\Gamma_{\mu\nu}{}^{\lambda} + \Gamma_{\mu\nu}{}^{\tau}\Gamma_{\sigma\tau}{}^{\lambda})\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\sigma} + \dots\Big]. \tag{11.57}$$

This differs from the correct one (10.96) by the third-order term

$$\Delta' x^i = \frac{1}{3!}e^i{}_{[\tau,\sigma]}e_k{}^{\tau}e^k{}_{\nu,\mu}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\sigma} = \frac{1}{3!}e^i{}_{\lambda}S_{\sigma\tau}{}^{\lambda}\Gamma_{\mu\nu}{}^{\tau}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\sigma}, \tag{11.58}$$

which vanishes if the $q$-space has no torsion. The Jacobian associated with (11.56) is

$$J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det(e^i_{\kappa})\ \det\Big(\delta^{\kappa}{}_{\mu} - e_i{}^{\kappa}e^i_{\{\mu,\nu\}}\Delta q^{\nu} + \frac{1}{2}e^{\kappa}_i e^i_{\{\mu,\nu\lambda\}}\Delta q^{\nu}\Delta q^{\lambda} + \dots\Big), \tag{11.59}$$

and corresponds to the effective action

$$\frac{i}{\hbar}\mathcal{A}^{\epsilon}_J = -e_i{}^{\kappa}e^i{}_{\kappa,\nu}\Delta q^{\nu} + \frac{1}{2}[e_i{}^{\mu}e^i_{\{\mu,\nu\lambda\}} - e_i{}^{\mu}e^i_{\{\kappa,\nu\}}e_j{}^{\kappa}e^j_{\{\mu,\lambda\}}]\Delta q^{\nu}\Delta q^{\lambda} + \dots\ . \tag{11.60}$$

With (10.16), (10.131), and (10.132), this becomes

$$\frac{i}{\hbar}\mathcal{A}^{\epsilon}_J = -\ \Gamma_{\{\nu\mu\}}{}^{\mu}\Delta q^{\nu} + \frac{1}{2}(\partial_{\{\mu}\Gamma_{\nu,\kappa\}}{}^{\kappa} + \Gamma_{\{\nu,\kappa}{}^{\sigma}\Gamma_{\mu\},\sigma}{}^{\kappa} - \Gamma_{\{\nu\kappa\}}{}^{\sigma}\Gamma_{\{\mu,\sigma\}}{}^{\kappa})\Delta q^{\nu}\Delta q^{\mu} + \dots\ , \tag{11.61}$$

which differs from the proper Jacobian action (10.145) only by one index symmetrization.

To find the short-time action following from the mapping (11.56), we form

$$\mathcal{A}^\epsilon = \frac{M}{2\epsilon}(\Delta x^i)^2 = \frac{M}{2\epsilon}\Big[g_{\mu\nu}\Delta q^\mu \Delta q^\nu - e^i{}_\mu e^i{}_{\nu,\lambda}\Delta q^\mu \Delta q^\nu \Delta q^\lambda + \left(\frac{1}{3}e^i{}_\mu e^i{}_{\nu,\lambda\kappa} + \frac{1}{4}e^i{}_{\mu,\nu} e^i{}_{\lambda,\kappa}\right)\Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \dots\Big], \tag{11.62}$$

and use the identities (10.16), (10.131), and (10.132) to obtain

$$\mathcal{A}^\epsilon = \frac{M}{2\epsilon}\Big\{g_{\mu\nu}\Delta q^\mu \Delta q^\nu - \Gamma_{\mu\nu\lambda}\Delta q^\mu \Delta q^\nu \Delta q^\lambda + \left[\frac{1}{3}g_{\mu\tau}(\partial_\kappa \Gamma_{\lambda\nu}{}^\tau + \Gamma_{\lambda\nu}{}^\delta \Gamma_{\kappa\delta}{}^\tau) + \frac{1}{4}\Gamma_{\lambda\kappa}{}^\sigma \Gamma_{\mu\nu\sigma}\right]\Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \dots \Big\}. \tag{11.63}$$

This differs from the proper short-time action (10.107) [recall the convention (10.107)] only by the absence of the symmetrization in the indices $\kappa$ and $\delta$ in the fourth term, the difference vanishing if the $q$-space has no torsion.

The equivalent path integral representation in which (11.2) contains the short-time action (11.63) and the Jacobian action (11.61) will be useful in Chapter 13 when solving the path integral of the Coulomb system.

The equivalence of different time-sliced path integral representations manifests itself in certain moment properties of the integral kernel (11.4). The derivations of the Schrödinger equation in Subsections. 11.1.1 and 11.1.2 have made use only of the following three moment properties of the kernel:

$$\int d^D\Delta q\, K^\epsilon(q,\Delta q) = 1 + \dots\ , \tag{11.64}$$

$$\int d^D\Delta q\, K^\epsilon(q,\Delta q)\Delta q^\nu = -i\epsilon\frac{\hbar}{2M}\Gamma_\mu{}^{\mu\nu} + \dots\ , \tag{11.65}$$

$$\int d^D\Delta q\, K^\epsilon(q,\Delta q)\Delta q^\mu \Delta q^\nu = i\epsilon\frac{\hbar}{M}g^{\mu\nu} + \dots\ , \tag{11.66}$$

evaluated at fixed postpoint $q$. The omitted terms indicated by the dots and all higher moments contribute to higher orders in $\epsilon$ which are irrelevant for the derivation of the differential equation obeyed by the amplitude. Any kernel $K^\epsilon(q,\Delta q)$ with these properties leads to the same Schrödinger equation. If a kernel is written as

$$K^\epsilon(q,\Delta q) = K_0^\epsilon(q,\Delta q)[1 + C(\Delta q)], \tag{11.67}$$

where $K_0^\epsilon(q,\Delta q)$ is the free-particle postpoint kernel

$$K_0^\epsilon(q,\Delta q) = \frac{\sqrt{g(q)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}g_{\mu\nu}(q)\,\Delta q^\mu\,\Delta q^\nu\right], \tag{11.68}$$

the moment properties (11.64)–(11.66) are equivalent to

$$\langle C\rangle_0 = 0+\dots\,, \tag{11.69}$$
$$\langle C\,\Delta q^\mu\rangle_0 = i\epsilon\frac{\hbar}{2M}\Gamma_\mu{}^{\mu\nu}+\dots, \tag{11.70}$$

where the expectation values are taken with respect to the kernel $K_0^\epsilon(q,\Delta q)$, the dots on the right-hand side indicating terms of the order $\epsilon^2$. Note that the third of the three moment properties is trivially true since it receives only a contribution from the leading part of the kernel $K^\epsilon(q,\Delta q)$, i.e., from $K_0^\epsilon(q,\Delta q)$. The verification of the other two requires some work, in particular the first, which is the normalization condition.

Two kernels $K_1^\epsilon, K_2^\epsilon$ are equivalent if their correction terms $C_1, C_2$ have expectations which are small of the order $\epsilon^2$:

$$\langle C_1\rangle_0 = \langle C_2\rangle_0 = \mathcal{O}(\epsilon^2), \tag{11.71}$$
$$\langle (C_1-C_2)\Delta q^\mu\rangle_0 = \mathcal{O}(\epsilon^2). \tag{11.72}$$

These are necessary and sufficient conditions for the equivalence. Many possible correction terms $C$ lead to the same moment integrals. All of them are physically equivalent, being associated with the same Schrödinger equation. The simplest possibilities are

$$K^\epsilon(q,\Delta q) = K_0^\epsilon(q,\Delta q)\left[1+\frac{1}{2}\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu\right], \tag{11.73}$$

or

$$K^\epsilon(q,\Delta q) = K_0^\epsilon(q,\Delta q)\left[1-\frac{i}{D+2}\frac{M}{2\hbar\epsilon}\Gamma_\mu{}^\mu{}_\nu\,\Delta q^\nu g_{\lambda\kappa}\Delta q^\lambda\Delta q^\kappa\right], \tag{11.74}$$

where $D$ is the space dimension. The zero-order kernel satisfies automatically the first and third moment condition, (11.64) and (11.66), while the additional term enforces precisely the second condition, (11.65), without changing the others. The equivalent kernels can also be considered as the result of working with Jacobian actions

$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon = \frac{1}{2}\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu-\frac{1}{8}(\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu)^2, \tag{11.75}$$
$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon = -\frac{i}{D+2}\frac{M}{2\hbar\epsilon}\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu\, g_{\lambda\kappa}\,\Delta q^\lambda\,\Delta q^\kappa, \tag{11.76}$$

instead of the original one (10.145). Indeed, the second term in (11.75) can further be reduced by perturbation theory to

$$\begin{aligned}-\frac{1}{8}(\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu)^2 &\rightarrow -\frac{1}{8}\Gamma_\mu{}^\mu{}_\nu\Gamma_\lambda{}^\lambda{}_\kappa\langle\Delta q^\nu\Delta q^\kappa\rangle_0\\ &= -i\epsilon\frac{\hbar}{8M}(\Gamma_\mu{}^\mu{}_\nu)^2,\end{aligned} \tag{11.77}$$

yielding an alternative and most useful form for the Jacobian action

$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon \;=\; \frac{1}{2}\Gamma_\mu{}^\mu{}_\nu\Delta q^\nu - i\epsilon\frac{\hbar}{8M}(\Gamma_\mu{}^\mu{}_\nu)^2. \tag{11.78}$$

Remarkably, this expression involves only the connection contracted in the first two indices:

$$\Gamma_\mu{}^{\mu\nu} = g^{\mu\lambda}\Gamma_{\mu\lambda}{}^\nu. \tag{11.79}$$

## 11.3 Potentials and Vector Potentials

It is straightforward to find the effect of external potentials and vector potentials upon the Schrödinger equation. For this, we merely observe that the time-sliced potential term

$$\mathcal{A}_{\rm pot}^\epsilon = \frac{e}{c}A_\mu\Delta q^\mu - \frac{e}{2c}\partial_\nu A_\mu\,\Delta q^\mu\Delta q^\nu - \epsilon V(q) + \dots \tag{11.80}$$

derived in Eq. (10.182) appears in the kernel $K^\epsilon(q,\Delta q)$ via a factor $e^{i\mathcal{A}_{\rm pot}^\epsilon/\hbar}$. This factor can be combined with the postpoint expansion of the wave function in the integral equation (11.31), which becomes

$$\begin{aligned}\psi(q,t) \;&=\; \int d^D\Delta q\, K_0^\epsilon(q,\Delta q)\,[1+C(\Delta q)]\\
&\quad\times e^{i\mathcal{A}_{\rm pot}^\epsilon/\hbar}\left(1-\Delta q^\mu\partial_\mu+\frac{1}{2}\Delta q^\mu\Delta q^\nu\partial_\mu\partial_\nu\right)\psi(q,t-\epsilon)+\dots\\
&= \int d^D\Delta q\, K_0^\epsilon(q,\Delta q)\,[1+C(\Delta q)]\Bigg[1-\Delta q^\mu\left(\partial_\mu - i\frac{e}{\hbar c}A_\mu\right)\\
&\quad+\frac{1}{2}\Delta q^\mu\Delta q^\nu\left(\partial_\mu - i\frac{e}{\hbar c}A_\mu\right)\left(\partial_\nu - i\frac{e}{\hbar c}A_\nu\right) - i\epsilon V(q)\Bigg]\psi(q,t-\epsilon)+\dots\,.\end{aligned} \tag{11.81}$$

By going through the steps of Subsection 11.1.1 or 11.1.2, we obtain the same Schrödinger equation as in (11.21),

$$i\hbar\partial_t\psi(q,t) = \hat{H}\psi(q,t). \tag{11.82}$$

The Hamiltonian operator $\hat{H}$ differs from the free operator $\hat{H}_0$ of (11.22),

$$\hat{H}_0 = -\frac{\hbar^2}{2M}D_\mu D^\mu, \tag{11.83}$$

in two ways. First, a potential energy $V(q)$ is added. Second, the covariant derivatives $D_\mu$ are replaced by

$$D_\mu^A \equiv D_\mu - i\,\frac{e}{\hbar c}A_\mu. \tag{11.84}$$

This is the Schrödinger version of the minimal substitution in Eq. (2.644).

The minimal substitution extends the covariance of $D_\mu$ with respect to coordinate changes to a covariance with respect to gauge transformations of the vector potential $A_\mu$. The subtraction of $iA_\mu/\hbar$ on the right-hand side of (11.84) reflects the fact that $P_\mu = p_\mu - A_\mu$ rather than $p^\mu$ is the gauge-invariant physical momentum of a particle in the presence of an electromagnetic field. Only the use of $P_\mu$ guarantees the gauge invariance of the electromagnetic interaction, just as in the flat-space action (2.643).

Let us briefly verify that the Schrödinger equation (11.82) with the covariant derivative (11.84) is invariant under gauge transformations. If the amplitude is multiplied by a space-dependent phase

$$\psi(q) \rightarrow e^{-i(e/\hbar c)\Lambda(q)}\psi(q), \tag{11.85}$$

the covariant derivative (11.84) is multiplied by precisely the same phase:

$$D_\mu\psi(q) \rightarrow e^{-i(e/\hbar c)\Lambda(q)}D_\mu\psi(q), \tag{11.86}$$

if the vector potential is gauge tranformed as follows:

$$A_\mu \rightarrow A_\mu + \partial_\mu\Lambda(q). \tag{11.87}$$

Under these joint transformations, the Schrödinger equation (11.82) is multiplied by an overall phase factor $e^{-i\Lambda(q)}$, and thus gauge invariant.

Adding a potential $V(q)$, the Hamilton operator in the Schrödinger equation (11.82) is therefore

$$\hat{H} = -\frac{\hbar^2}{2M}D_\mu^A D^{A\,\mu} + V(q). \tag{11.88}$$

Observe that the mixed terms containing derivative and vector potential appears in the symmetrized form

$$-\frac{\hbar}{2Mc}\left(\hat{p}_\mu A^\mu + A^\mu\hat{p}_\mu\right). \tag{11.89}$$

This corresponds to a symmetric time slicing of the interaction term $\dot{q}^\mu A_\mu$ which was derived in Section 10.5 by using the equation of motion in calculation of the short-time action. Here we see that this time slicing guarantees the gauge invariance of the Schrödinger equation.

Note further that there is no extra $R$-term in the Schrödinger equation (11.88).

## 11.4 Unitarity Problem

The appearance of the Laplace operator $D_\mu D^\mu$ in the free-particle Schrödinger equation (11.82) is in conflict with the traditional physical scalar product between two wave functions $\psi_1(q)$ and $\psi_2(q)$:

$$\langle\psi_2|\psi_1\rangle \equiv \int d^Dq\sqrt{g(q)}\psi_2^*(q)\psi_1(q). \tag{11.90}$$

In such a scalar product, only the Laplace-Beltrami operator (11.13),

$$\Delta = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu, \tag{11.91}$$

is a Hermitian operator, not the Laplacian $D_\mu D^\mu$. The bothersome term is the contracted torsion term

$$(D_\mu D^\mu - \Delta)\psi = -2S^\mu \partial_\mu \psi. \tag{11.92}$$

This term ruins the Hermiticity and thus also the unitarit y of the time evolution operator of a particle in a space with curvature and torsion.

For presently known physical systems in spaces with curvature and torsion, the unitarity problem is fortunately absent. Consider first field theories of gravity with torsion. There, the torsion field $S_{\mu\nu}{}^\lambda$ is generated by the spin current density of the fundamental matter fields. The requirement of renormalizability restricts these fields to carry spin 1/2. However, the spin current density of spin-1/2 particles happens to be a completely antisymmetric tensor.[1] This property carries over to the torsion tensor. Hence, the torsion field in the universe satisfies $S^\mu = 0$. This implies that for a particle in a universe with curvature and torsion, the Laplacian always degenerates into the Laplace-Beltrami operator, assuring unitarity after all.

In Chapter 13 we shall witness another way of escaping the unitarity problem. The path integral of the three-dimensional Coulomb system is solved by a multivalued transformation to a four-dimensional space with torsion where the physical scalar product is

$$\langle \psi_2 | \psi_1 \rangle_{\rm phys} \equiv \int d^D q \sqrt{g}\, w(q) \psi_2^*(q) \psi_1(q), \tag{11.93}$$

with some scalar weight function $w(q)$. This scalar product is different from the naive scalar product (11.90). It is, however, reparametrization-invariant, and $w(q)$ makes the Laplacian $D_\mu D^\mu$ a Hermitian operator.

The characteristic property of torsion in the transformed Coulomb system is that $S_\mu(q) = S_{\mu\nu}{}^\nu$ can be written as a gradient of a scalar function: $S_\mu(q) = \partial_\mu \sigma(q)$ [see Eq. (13.143)]. Such torsion fields admit a Hermitian Laplace operator of a scalar field in a scalar product (11.93) with the weight

$$w(q) = e^{-2\sigma(q)}. \tag{11.94}$$

Thus, the physical scalar product can be expressed in terms of the naive one as follows:

$$\langle \psi_2 | \psi_1 \rangle_{\rm phys} \equiv \int d^D q \sqrt{g(q)} e^{-2\sigma(q)} \psi_2^*(q) \psi_1(q). \tag{11.95}$$

To prove the Hermiticity, we observe that within the naive scalar product (11.93), a partial integration changes the covariant derivative $-D_\mu$ into

$$D_\mu^* \equiv (D_\mu + 2S_\mu). \tag{11.96}$$

[1] See the standard textbooks on quantum field theory, for example, Ref. [1, 2].

Consider, for example, the scalar product

$$\int d^D q \sqrt{g}\, U^{\mu\nu_1\dots\nu_n} D_\mu V_{\nu_1\dots\nu_n}. \tag{11.97}$$

A partial integration of the derivative term $\partial_\mu$ in $D_\mu$ gives

$$\text{surface term} \; - \int d^D dq [(\partial_\mu \sqrt{g}\, U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n} \\ - \sum_i \sqrt{g} U^{\mu\nu_1\dots\nu_i\dots\nu_n} \Gamma_{\mu\nu_i}{}^{\lambda_i} V_{\nu_1\dots\lambda_i\dots\nu_n}]. \tag{11.98}$$

Now we use

$$\partial_\mu \sqrt{g} = \sqrt{g}\, \bar{\Gamma}_{\mu\nu}{}^\nu = \sqrt{g}(2S_\mu + \Gamma_{\mu\nu}{}^\nu), \tag{11.99}$$

to rewrite (11.98) as

$$\text{surface term} \; - \int d^D q \sqrt{g}\, [(\partial_\mu U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n} \\ - \sum_i \Gamma_{\mu\nu_i}{}^{\lambda_i} U^{\mu\nu_1\dots\nu_i\dots\nu_n} V_{\nu_1\dots\lambda_i\dots\nu_n} \; -2S_\mu U^{\mu\nu_1\dots\nu_n} V_{\nu_1\dots\nu_n}], \tag{11.100}$$

which is equal to

$$\text{surface term} \;\; - \int d^D q \sqrt{g} (D_\mu^* U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n}. \tag{11.101}$$

In the physical scalar product (11.95), the corresponding operation is

$$\int d^D q \sqrt{g} e^{-2\sigma(q)} U^{\mu\nu_1\dots\nu_n} D_\mu V_{\nu_1\dots\nu_n} = \\ = \text{surface term} - \int d^D q \sqrt{g} (D_\mu^* e^{-2\sigma(q)} U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n} \\ = \text{surface term} - \int d^D q \sqrt{g} e^{-2\sigma(q)} (D_\mu \sqrt{g} U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n}. \tag{11.102}$$

Hence, $iD_\mu$ is a Hermitian operator, and so is the Laplacian $D_\mu D^\mu$.

For spaces with an arbitrary torsion, the correct scalar product has yet to be found. Thus the quantum equivalence principle is so far only applicable to spaces with arbitrary curvature and gradient torsion.

## 11.5 Alternative Attempts

Our procedure has to be contrasted with earlier proposals for constructing path integrals in spaces with curvature, in which torsion was always assumed to be absent. In the notable work of DeWitt,[2] the measure is taken to be proportional to

$$\prod_{n=1}^N \int dq_n \sqrt{g(q_{n-1})} = \prod_{n=1}^N \int dq_n \sqrt{g(q_n - \Delta q)} \tag{11.103}$$

[2]See Section 11.5.

so that the expansion in powers $\Delta q$ gives a Jacobian action $\mathcal{A}^{\epsilon}_{\bar{J}_0}$ of Eq. (10.133).

If one uses this action instead of the correct expression $\mathcal{A}^{\epsilon}_{J}$ of Eq. (10.145), the amplitude obeys a Schrödinger equation

$$i\hbar\partial_t\psi(q,t) = (\hat{H}_0 + V_{\rm eff})\psi(q,t), \tag{11.104}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2M}\Delta \tag{11.105}$$

contains the Laplace-Beltrami operator $\Delta$, and $V_{\rm eff}$ is an additional effective potential

$$V_{\rm eff} = \frac{\hbar^2}{6M}\bar{R}. \tag{11.106}$$

The Schrödinger equation (11.104) differs from ours in Eq. (11.21) derived by the nonholonomic mapping procedure by the extra $R$-term. The derivation is reviewed in Appendix 11A. Note that our sign convention for $\bar{R}$ is such that the surface of a sphere of radius $R$ has $\bar{R}= 2/R^2$.

In the amplitude proposed by DeWitt, the short-time amplitude carries an extra semiclassical prefactor, a curved-space version of the Van Vleck-Pauli-Morette determinant in Eq. (4.125). It contributes another term proportional to $\bar{R}$ which reduces (11.106) to $(\hbar^2/12M)\bar{R}$ (see Appendix 11B). Other path integral prescriptions lead even to additional noncovariant terms [3]. All such nonclassical terms proportional to $\hbar^2$ have to be subtracted from the classical action to arrive at the correct amplitude which satisfies the Schrödinger equation (11.104) without an extra $V_{\rm eff}$.

There are two compelling arguments in favor of our construction principle: On the one hand, if the space has only curvature and no torsion, our path integral gives the correct time evolution amplitude of a particle on the surface of a sphere in $D$ dimensions and on group spaces, as we have seen in Sections 8.7–8.9 and 10.4. In contrast to other proposals, only the classical action appears in the short-time amplitude. In spaces with constant curvature, just as in flat space, our amplitude agrees with the one obtained from operator quantum mechanics by quantizing the theory via the commutation rules of the generators of the group of motion.

In the presence of torsion the result is new. It will be tested by the integration of the path integral of the Coulomb system in Chapter 13. This requires a multivalued coordinate transformation to an auxiliary space with curvature and torsion, which reduces the system to a harmonic oscillator (see Section 13.5). The multivaleud mapping principle leads to the correct result.

The solution is so far the only indirect evidence for the question first raised by Bryce DeWitt in his fundamental 1957 paper [4], whether the Hamiltonian operator for a particle in curved space contains merely the Laplace-Beltrami operator $\Delta$ in the kinetic energy, or whether there exists an additional term proportional to $\hbar^2 R$. Recall the various older path integral literature on the subject cited in Chapter 10. From the measure generated by the nonholonomic mapping principle in Subsection 10.3.2 it follows that there is no extra $\hbar^2 R$-term. See the discussion in Section 11.5. It would, of course, be more satisfactory to have a direct experimental evidence, but so far all experimentally accessible systems in curved space have either a very small $R$ caused by gravitation, whose detection is presently impossible, or a constant $R$ which does not change level spacings, an example for the latter being the spinning symmetric and asymmetric top discussed in the context of Eq. (1.471).

## 11.6 DeWitt-Seeley Expansion of Time Evolution Amplitude

An important tool for comparing the results of path integrals in curved space with operator results of Schrödinger theory is the short-time expansion of the imaginary-time evolution amplitude $(q,\beta\mid q',0)$. In Schrödinger theory, the amplitude is given by the matrix elements of the evolution operator $e^{-\beta\hat{H}}=e^{\beta\Delta/2}$, with $\Delta$ being the Laplace-Beltrami operator (11.13). This expansion has first been given by DeWitt [6] and by Seeley [7] and reads[3]

$$(q\,\beta\mid q'\,0)=(q\mid e^{\beta\Delta/2}\mid q')=\frac{1}{\sqrt{2\pi\beta}^D}\,e^{-g_{\mu\nu}(q)\Delta q^\mu\Delta q^\nu/2\beta}\sum_{k=0}^{\infty}\beta^k a_k(q,q')\,, \tag{11.107}$$

The associated *DeWitt-Seeley expansion coefficients* $a_k(q,q')$ are, up to fourth order in $\Delta q^\mu$,

$$\begin{aligned}
a_0(q,q')&\equiv 1+\frac{1}{12}\bar{R}_{\mu\nu}\Delta q^\mu\Delta q^\nu+\left(\frac{1}{360}\bar{R}^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}\bar{R}_{\mu\sigma\nu\tau}+\frac{1}{288}\bar{R}_{\kappa\lambda}\bar{R}_{\sigma\tau}\right)\Delta q^\kappa\Delta q^\lambda\Delta q^\sigma\Delta q^\tau,\\
a_1(q,q')&\equiv\frac{1}{12}\bar{R}+\left(\frac{1}{144}\bar{R}\bar{R}_{\mu\nu}+\frac{1}{360}\bar{R}^{\kappa\lambda}\bar{R}_{\kappa\mu\lambda\nu}+\frac{1}{360}\bar{R}^{\kappa\lambda\sigma}{}_{\mu}\bar{R}_{\kappa\lambda\sigma\nu}-\frac{1}{180}\bar{R}^{\kappa}{}_{\mu}\bar{R}_{\kappa\nu}\right)\Delta q^\mu\Delta q^\nu,\\
a_2(q,q')&\equiv\frac{1}{288}\bar{R}^2+\frac{1}{720}\bar{R}^{\mu\nu\kappa\lambda}\bar{R}_{\mu\nu\kappa\lambda}-\frac{1}{720}\bar{R}^{\mu\nu}\bar{R}_{\mu\nu},
\end{aligned} \tag{11.108}$$

where $\Delta q^\mu\equiv(q-q')^\mu$. The metric at $q$ is assumed to be Minkowskian, and all curvature tensors are evaluated at $q$. For $\Delta q^\mu=0$ this simplifies to

$$(q\,\beta\mid q\,0)=\frac{1}{\sqrt{2\pi\beta}^D}\left\{1+\frac{\beta}{12}\bar{R}+\frac{\beta^2}{2}\left[\frac{1}{144}\bar{R}^2+\frac{1}{360}\left(\bar{R}^{\mu\nu\kappa\lambda}\bar{R}_{\mu\nu\kappa\lambda}-\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}\right)\right]+\dots\right\}. \tag{11.109}$$

This can also be written in the cumulant form as

$$(q\,\beta\mid q\,0)=\frac{1}{\sqrt{2\pi\beta}^D}\,\exp\left[\frac{\beta}{12}\bar{R}+\frac{\beta^2}{720}\left(\bar{R}^{\mu\nu\kappa\lambda}\bar{R}_{\mu\nu\kappa\lambda}-\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}\right)+\dots\right]. \tag{11.110}$$

[3]In the mathematical nomenclature of Footnote 16 in Chapter 2, this is a so-called *heat kernel expansion* or a *Hadamard expansion* [8], and the DeWitt-Seeley coefficients $a_k(q,q')$ are also called *Hadamard coefficients*.

The derivation goes as follows. In a neighborhood of some arbitrary point $q_0^\mu$ we expand the Laplace-Beltrami operator in normal coordinates where the metric and its determinant have the expansions (10.477) and (10.478) as

$$\Delta = \partial^2 - \frac{1}{3}\bar{R}_{ik_1jk_2}(q_0)(q-q_0)^{k_1}(q-q_0)^{k_2}\partial_\mu\partial_\nu - \frac{2}{3}\bar{R}_{\mu\nu}(q_0)(q-q_0)^\mu\partial_\nu. \quad (11.111)$$

The time evolution operator $\hat{H} = -\Delta/2$ in the exponent of Eq. (11.107) is now separated into a free part $\hat{H}_0$ and an interaction part $\hat{H}_{\rm int}$ as follows

$$\hat{H}_0 = -\frac{1}{2}\partial^2, \quad (11.112)$$

$$\hat{H}_{\rm int} = \frac{1}{6}\bar{R}_{ik_1jk_2}(q-q_0)^{k_1}(q-q_0)^{k_2}\partial_\mu\partial_\nu + \frac{1}{3}\bar{R}_{\mu\nu}(q-q_0)^\mu\partial_\nu. \quad (11.113)$$

We now recall Eq. (1.294) and see that the transition amplitude (11.107) satisfies the integral equation

$$\begin{aligned}(q\,\beta\mid q'\,0) &= \langle q\mid e^{-\beta(\hat{H}_0+\hat{H}_{\rm int})}\mid q'\rangle = \langle q\mid e^{-\beta\hat{H}_0}\left[1-\int_0^\beta d\sigma e^{\sigma\hat{H}_0}\hat{H}_{\rm int}e^{-\sigma\hat{H}}\right]\mid q'\rangle\\ &= (q\,\beta\mid q'\,0)_0 - \int_0^\beta d\sigma\int d^D\bar{q}\,(q\,\beta-\sigma\mid\bar{q}\,0)_0\,\hat{H}_{\rm int}(\bar{q})\,(\bar{q}\,\sigma\mid q\,0), \quad (11.114)\end{aligned}$$

where

$$(q\,\beta\mid q'\,0)_0 = \langle q\mid e^{-\beta\hat{H}_0}\mid q'\rangle = \frac{1}{\sqrt{2\pi\beta}^n}\,e^{-(\Delta q)^2/2\beta}. \quad (11.115)$$

To first order in $\hat{H}_{\rm int}$ we can replace $\hat{H}$ in the last exponential of Eq. (11.114) by $\hat{H}_0$ and obtain

$$(q\,\beta\mid q'\,0) \approx (q\,\beta\mid q'\,0)_0 - \int_0^\beta d\sigma\int d^D\bar{q}\;(q\;\beta-\sigma\mid\bar{q}\,0)_0\,\hat{H}_{\rm int}(\bar{q})\,(\bar{q}\,\sigma\mid q\,0)_0. \quad (11.116)$$

Inserting (11.113) and choosing $q_0 = q'$, we find

$$\begin{aligned}(q\,\beta\mid q'\,0) &= (q\,\beta\mid q'\,0)_0\left\{1+\int_0^\beta d\sigma\int\frac{d^D(\Delta\bar{q})}{\sqrt{2\pi a}^D}\,e^{-[\Delta\bar{q}-(\sigma/\beta)\,\Delta q]^2/2a}\right. \quad (11.117)\\ &\times\left.\left[-\frac{1}{6}\bar{R}_{\mu\kappa\nu\lambda}\Delta\bar{q}^\kappa\Delta\bar{q}^\lambda\left(-\frac{\delta^{\mu\nu}}{\sigma}+\frac{\Delta\bar{q}^\mu\Delta\bar{q}^\nu}{\sigma^2}\right)+\frac{1}{3}\bar{R}_{\mu\nu}\frac{\Delta\bar{q}^\mu\Delta\bar{q}^\nu}{\sigma}\right]\right\},\end{aligned}$$

where we have replaced the integrating variable $\bar{q}$ by $\Delta\bar{q} = \bar{q} - q'$ and introduced the variable $a \equiv (\beta-\sigma)\sigma/\beta$. There is initially also a term of fourth order in $\Delta\bar{q}$ which vanishes, however, because of the antisymmetry of $\bar{R}_{\mu\kappa\nu\lambda}$ in $\mu\kappa$ and $\nu\lambda$. The remaining Gaussian integrals are performed after shifting $\Delta\bar{q} \to \Delta\bar{q} + \sigma\,\Delta q/\beta$, and we obtain

$$\begin{aligned}(q\,\beta\mid q'\,0) &= (q\,\beta\mid q'\,0)_0\left\{1+\frac{1}{6}\int_0^\beta d\sigma\left[\frac{\sigma}{\beta^2}\bar{R}_{\mu\nu}(q')\Delta q^\mu\Delta q^\nu + \frac{a}{\sigma}\bar{R}(q')\right]\right\}\\ &= (q\,\beta\mid q'\,0)_0\left[1+\frac{1}{12}\bar{R}_{\mu\nu}(q')\Delta q^\mu\Delta q^\nu + \frac{\beta}{12}\bar{R}(q')\right]. \quad (11.118)\end{aligned}$$

Note that all geometrical quantities are evaluated at the initial point $q'$. They can be re-expressed in power series around the final position $q$ using the fact that in normal coordinates

$$g_{\mu\nu}(q') = g_{\mu\nu}(q) + \frac{1}{3}\bar{R}_{ik_1jk_2}(q)\Delta q^{k_1}\Delta q^{k_2} + \dots\ , \tag{11.119}$$

$$g_{\mu\nu}(q')\Delta q^\mu \Delta q^\nu = g_{\mu\nu}(q)\Delta q^\mu \Delta q^\nu, \tag{11.120}$$

the latter equation being true to all orders in $\Delta q$ due to the antisymmetry of the tensors $\bar{R}_{\mu\nu\kappa\lambda}$ in all terms of the expansion (11.119), which is just another form of writing the expansion (10.477) up to the second order in $\Delta q^\mu$.

Going back to the general coordinates, we obtain all coefficients of the expansion (11.107) linear in the curvature tensor

$$(q\,\beta \mid q'\,0) \simeq \frac{1}{\sqrt{2\pi\beta}^D}\, e^{-g_{\mu\nu}(q)\Delta q^\mu\Delta q^\nu/2\beta}\left[1 + \frac{1}{12}\bar{R}_{\mu\nu}(q)\Delta q^\mu \Delta q^\mu + \frac{\beta}{12}\bar{R}(q)\right]. \tag{11.121}$$

The higher-order terms in the Seeley-DeWitt expansion (10.512) can be derived similarly, although with much more effort [9].

A simple cross check of the expansion (10.512) to high orders is possible if we restrict the space to the surface of a sphere of radius $r$ in $D$ dimensions which has $D-1$ dimensions. Then

$$\bar{R}_{\mu\nu\kappa\lambda} = \frac{1}{r^2}\left(g_{\mu\lambda}\,g_{\nu\kappa} - g_{\mu\kappa}\,g_{\nu\lambda}\right), \qquad \mu,\nu = 1,2,\dots,D-1. \tag{11.122}$$

Contractions yield Ricci tensor and scalar curvature

$$\bar{R}_{\mu\nu} \;=\; \bar{R}_{\kappa\mu\nu}{}^{\kappa} = \frac{D-2}{r^2}\, g_{\mu\nu}, \qquad \bar{R} = \bar{R}_\mu{}^\mu = \frac{(D-1)(D-2)}{r^2} \tag{11.123}$$

and further:

$$\bar{R}^2_{\mu\nu\kappa\lambda} = \frac{2(D-1)(D-2)}{r^4}, \qquad \bar{R}^2_{\mu\nu} = \frac{(D-1)(D-2)^2}{r^4}. \tag{11.124}$$

Inserting these into (11.108), we obtain the DeWitt-Seeley short-time expansion of the diagonal amplitude for any $q$, up to order $\beta^2$:

$$(q\,\beta \mid q\,0) = \frac{1}{\sqrt{2\pi\beta}^{D-1}}\left[1 + (D-1)(D-2)\frac{\beta}{12r^2}\right.$$
$$\left. + (D-1)(D-2)(5D^2 - 17D + 18)\frac{\beta^2}{1440r^4} + \dots\right]. \tag{11.125}$$

This expansion may easily be reproduced by a simple direct calculation of the partition function for a particle on the surface of a sphere [10]

$$Z(\beta) = \sum_{l=0}^{\infty} d_l\, \exp[-l(l+D-2)\beta/2r^2]\ , \tag{11.126}$$

where $-l(l+D-2)$ are the eigenvalues of the Laplace-Beltrami operator on a sphere [recall (10.165)] and $d_l = (2l+D-2)(l+D-3)!/l!(D-2)!$ their degeneracies [recall (8.114)]. Since the space is homogeneous, the amplitude $(q\,\beta \mid q\,0)$ is obtained from this by dividing out the constant surface of a sphere:

$$(q\,\beta \mid q\,0) = \frac{\Gamma(D/2)}{2\pi^{D/2}r^{D-1}} Z(\beta). \tag{11.127}$$

For any given $D$, the sum in (11.126) easily be expanded in powers of $\beta$. As an example, take $D = 3$ where

$$Z(\beta) = \sum_{l=0}^{\infty} (2l+1)\, \exp[-l(l+1)\beta/2r^2] \;. \tag{11.128}$$

In the small-$\beta$ limit, the sum (11.128) is evaluated as follows

$$Z(\beta) = \int_0^{\infty} d\,[l(l+1)]\, \exp[-l(l+1)\beta/2r^2] + \sum_{l=0}^{\infty} (2l+1)\left[1 - l(l+1)\beta/2r^2 + \dots \quad\right]. \tag{11.129}$$

The integral is immediately done and yields

$$\int_0^{\infty} dz\, \exp(-z\beta/2r^2) = \frac{2r^2}{\beta}\,. \tag{11.130}$$

The sums are divergent but can be evaluated by analytic continuation from negative powers of $l$ to positive ones with the help of Riemann zeta functions $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, which vanishes for all even negative arguments. Thus we find

$$\sum_{l=0}^{\infty}(2l+1) \;=\; 1 + \sum_{l=1}^{\infty}(2l+1) = 1 + 2\zeta(-1) - \frac{1}{2} = \frac{1}{3}\,, \tag{11.131}$$

$$-\frac{\beta}{2r^2}\sum_{l=0}^{\infty}(2l+1)l(l+1) \;=\; -\frac{\beta}{2r^2}\sum_{l=1}^{\infty}(2l^3+l) = -\frac{\beta}{2r^2}\left[2\zeta(-3) + \zeta(-1)\right] = \frac{\beta}{30r^2}\,. \tag{11.132}$$

Substituting these into (11.129) yields

$$Z(\beta) \;=\; \frac{2r^2}{\beta}\left(1 + \frac{\beta}{6r^2} + \frac{\beta^2}{60r^4} + \dots\right). \tag{11.133}$$

Dividing out the constant surface of a sphere $4\pi r^2$ as required by Eq. (11.127), we obtain indeed the expansion (11.125) for the surface of a sphere in three dimensions.

## 11.7 Recursive Calculation

If one wants to proceed to higher orders, an efficient recursive scheme was devised by DeWitt [6]. He considers the real-time version of the amplitude (10.512)

$$(q\,t \mid q'\,0) = (q \mid e^{it\Delta/2} \mid q'), \tag{11.134}$$

which satisfies the Schrödinger equation in curved space

$$i\partial_t(q\,t\mid q'\,0)=-\frac{\Delta}{2}(q\,\beta\mid q'\,0), \tag{11.135}$$

with the initial condition

$$(q\,0\mid q'\,0)=\delta^{(D)}(q-q'). \tag{11.136}$$

To solve (11.135) he first sets up a semiclassical time evolution amplitude in curved spacetime which is a direct analog the amplitude (4.125). He considers the action (10.2) of a point particle in curved space with a metric

$$\mathcal{A}=\frac{1}{2}\int_0^t dt'\,g_{\mu\nu}(q(t'))\dot{q}^\mu(t')q^\nu(t') \tag{11.137}$$

along the geodetic trajectory (10.9) between the initial and final points $q'$ and $q$, denoting it by $A(q,q';t)$, as in Eq. (4.87). Associated with this he defines the *geodetic distance*

$$\sigma(q,q')\equiv tA(q,q';t)=\sigma(q',q). \tag{11.138}$$

This is a bilocal function called the *geodetic interval*. The classical momentum of the particle along the orbit is given by the derivative [recall Eq. (4.88)]

$$p_\mu=\partial_\mu A(q,q';t)=\partial_\mu\sigma(q,q')/t. \tag{11.139}$$

The classical path follows the Hamilton-Jacobi equation (1.65), which reads for an action satisfying (11.138):

$$\partial_t\sigma(q,q')=\frac{1}{2}\partial_\mu\sigma(q;q')\partial^\mu\sigma(q;q'). \tag{11.140}$$

The derivative $\partial_\mu\sigma(q,q')$ points in the direction of the tangent vector of the geodetic trajectory. The Hamilton-Jacobi equation states that its square length is equal to twice the geodetic interval.

Then he writes the semiclassical amplitude in curved spacetime as an obvious extension of the flat-space expression (4.125) [11]:

$$(q\,t\mid q'\,0)=(q\mid e^{it\Delta/2}\mid q')=\frac{1}{\sqrt{2\pi it}^D}\,\mathcal{D}^{1/2}(q,q')e^{i\sigma(q,q')/2t}, \tag{11.141}$$

where $\mathcal{D}$ denotes the determinant

$$\mathcal{D}\equiv\det{}_D[-\partial_\mu\partial'_\nu A(q,q';t)]=\frac{\det_D[-\partial_\mu\partial_\nu\sigma(q,q')]}{t^D}. \tag{11.142}$$

The exact amplitude has a short-time expansion

$$(q\,t\mid q'\,0)=(q\mid e^{it\Delta/2}\mid q')=\frac{1}{\sqrt{2\pi it}^D}\,\mathcal{D}^{1/2}(q,q')e^{i\sigma(q,q')/2t}\sum_{n=0}^\infty t^n\,a_n(q,q';t), \tag{11.143}$$

where for $q = q'$. $a_0(q,q') = 1$, and all other coefficients vanish. Applying the Schrödinger equation (11.135) to (11.143), he derives the recursion relation

$$\partial^\mu\sigma\partial_\mu a_0 = 0, \tag{11.144}$$
$$\partial^\mu\sigma\partial_\mu a_{n+1} + (n+1)a_{n+1} = \tilde{\mathcal{D}}^{-1/2}\bar{D}^\mu\partial_\mu(\tilde{\mathcal{D}}^{1/2}a_n), \quad n = 0,1,2,\dots\,. \tag{11.145}$$

where $\tilde{\mathcal{D}}(q,q') \equiv g^{-1/2}(q)\mathcal{D}g^{1/2}(q')$ and $\bar{D}^\mu$ is the covariant derivative of the derivative $\partial_\mu\sigma$ defined as in Eq. (10.37). He solves the lowest coefficient by the bilocal function $I(q,q')$ with the properties

$$\partial^\mu\sigma\partial_\mu I(q,q') = 0, \quad \partial^{\mu'}\sigma\partial_{\mu'}I(q,q') = 0, \quad I(q,q) = 1. \tag{11.146}$$

From this he finds the solution of (11.145) as

$$a_{n+1}(q,q') = \frac{1}{t^{n+1}}\int_0^t dt''\, t''^n\tilde{\mathcal{D}}^{-1/2}\bar{D}^\mu\partial_\mu[\tilde{\mathcal{D}}^{1/2}a_n(q(\tau''),q')]. \tag{11.147}$$

## Appendix 11A Cancellations in Effective Potential

Here we demonstrate the cancellation of the terms ${v_2}^B$ and ${v_3}^B$ in formula (11.48) for the effective potential. First we give a simple reason why the cancellation occurs *separately* for the contributions stemming from the second and third terms in the expansion (10.96) of $\Delta x^i$. Consider the model integral

$$\int\frac{d\Delta x}{\sqrt{2\pi\epsilon}}\exp\left[-\frac{(\Delta x)^2}{2\epsilon}\right], \tag{11A.1}$$

and assume that $\Delta x$ has an expansion of the type (10.96):

$$\Delta x = \Delta q[1 + a_2\Delta q + a_3(\Delta q)^2 + \dots]. \tag{11A.2}$$

The integral transforms into

$$\int\frac{d\Delta q}{\sqrt{2\pi\epsilon}}[1 + 2a_2\Delta q + 3a_3(\Delta q)^2 + \dots]\exp\left\{-\frac{(\Delta q)^2}{2\epsilon}[1 + 2a_2\Delta q + 2a_3(\Delta q)^2 + a_2^2(\Delta q)^2 + \dots]\right\}, \tag{11A.3}$$

and is evaluated perturbatively via the expansion

$$\int\frac{d\Delta q}{\sqrt{2\pi\epsilon}}\exp\left[-\frac{(\Delta q)^2}{2\epsilon}\right]\left[1 - a_2\frac{(\Delta q)^3}{\epsilon} - a_3\frac{(\Delta q)^4}{\epsilon} - a_2^2\frac{(\Delta q)^4}{2\epsilon}\right.$$
$$\left. + a_2^2\frac{(\Delta q)^6}{2\epsilon} - 2a_2^2\frac{(\Delta q)^4}{\epsilon} + 3a_3(\Delta q)^2 + \dots\right]. \tag{11A.4}$$

If $\langle\mathcal{O}\rangle_0$ denotes the harmonic expectation value

$$\langle\mathcal{O}\rangle_0 \equiv \int\frac{d\Delta q}{\sqrt{2\pi i\epsilon}}\mathcal{O}\exp[-(\Delta q)^2/2\epsilon], \tag{11A.5}$$

one has

$$\langle(\Delta q)^2\rangle_0 = \epsilon, \quad \langle(\Delta q)^4\rangle_0 = 3!\epsilon^2, \quad \langle(\Delta q)^6\rangle_0 = 5!\epsilon^3, \dots\,. \tag{11A.6}$$

Using these values we find that the $a_2$- and $a_3$-terms cancel separately. Precisely this cancellation mechanism is active in the separate cancellation of the more complicated expressions ${v_1}^B, {v_2}^B$ in Eq. (11.49).

To demonstrate the cancellations explicitly, consider first the derivative terms in ${v_3}^B$. They are

$$ {v_3}^{\partial\Gamma} = -\frac{1}{2} g^{\mu\nu}\partial_{\{\mu}\Gamma_{\nu\lambda\}}{}^{\lambda} + \frac{1}{6} g_{\mu\tau}\partial_\kappa \Gamma_{\lambda\nu}{}^{\tau}\,(g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\lambda}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\lambda}). \tag{11A.7}$$

Due to the symmetrization of the first term in $\mu\nu\lambda$, this gives zero. The cancellation of the remaining terms in ${v_3}^B$ which are quadratic in $\Gamma$ is most easily shown by writing all indices as subscripts, inserting $g_{\mu\nu\lambda\kappa}$ from (11.41), and working out the contractions.

To calculate the ${v_2}^B$-terms, it is useful to introduce the notation $\Gamma_{1\mu} \equiv \Gamma_{\mu\nu}{}^{\nu}$, $\Gamma_{2\mu} \equiv \Gamma_{\nu\mu}{}^{\nu}$, $\Gamma_{3\mu} \equiv \Gamma_{\nu}{}^{\nu}{}_{\mu}$ and similarly the matrices $\tilde\Gamma_{1\mu} \hat{=} (\Gamma_\mu)_{\lambda\kappa}$, $\tilde\Gamma^T_{1\mu} \hat{=} (\Gamma_\mu)_{\kappa\lambda}$, $\tilde\Gamma_{2\mu} \hat{=} \Gamma_{\lambda\mu\kappa}$, $\tilde\Gamma^T_{2\mu} \hat{=} \Gamma_{\kappa\mu\lambda}$, $\tilde\Gamma_{3\mu} \hat{=} \Gamma_{\lambda\kappa\mu}$, $\tilde\Gamma^T_{3\mu} \hat{=} \Gamma_{\kappa\lambda\mu}$. For contractions such as $\Gamma_{1\mu}\Gamma_{1\mu}$ we write $\Gamma_1\Gamma_1$, and for $\Gamma_{\mu\nu\lambda}\Gamma_{\mu\nu\lambda}$ we write $\tilde\Gamma_1\tilde\Gamma_1 = \tilde\Gamma_2\tilde\Gamma_2 = \tilde\Gamma_3\tilde\Gamma_3$, whichever is most convenient. Similarly, $\Gamma_{\mu\nu\lambda}\Gamma_{\lambda\mu\nu} = \tilde\Gamma_1{\tilde\Gamma_2}^T = \tilde\Gamma_2{\tilde\Gamma_3}^T = \tilde\Gamma_3\tilde\Gamma_1$. Then we work out

$$\begin{aligned}
{v_2}^1 &= -\frac{1}{8}[(\Gamma_1+\Gamma_2)^2 - \tilde\Gamma_3(\tilde\Gamma_1 + \tilde\Gamma_1^T + \tilde\Gamma_2 + \tilde\Gamma_2^T)], \qquad (11A.8)\\
{v_2}^2 &= \frac{1}{8}[\Gamma_3\Gamma_3 + \tilde\Gamma_3(\tilde\Gamma_3 + \tilde\Gamma_3^T)],\\
{v_2}^3 &= \frac{1}{4}[(\Gamma_1+\Gamma_2)^2 + \Gamma_3(\Gamma_1+\Gamma_2)],\\
{v_2}^4 &= -\frac{1}{8}[{\Gamma_1}^2 + {\Gamma_2}^2 + {\Gamma_3}^2 + 2(\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)\\
&\quad + \tilde\Gamma_3(\tilde\Gamma_1 + \tilde\Gamma_1^T + \tilde\Gamma_2 + \tilde\Gamma_2^T + \Gamma_3 + \tilde\Gamma_3^T)].
\end{aligned}$$

It is easy to check that the sum of these ${v_2}^B$-terms vanishes.

Incidentally, if the symmetrizations in (11.49) following from our Jacobian action had been absent, we would find the additional terms

$$\begin{aligned}
\Delta {v_3}^{\partial\Gamma} &= \frac{1}{6}\bar R - \frac{2}{3}\partial_\mu S^\mu + \frac{1}{6}(\tilde\Gamma_3\tilde\Gamma_2^T - \Gamma_3\Gamma_2), \qquad (11A.9)\\
\Delta {v_3}^{\Gamma^2} &= -\frac{1}{2}\tilde\Gamma_3\tilde\Gamma_2 + \frac{1}{6}(\tilde\Gamma_3\tilde\Gamma_2 + \tilde\Gamma_3\tilde\Gamma_2^T + \Gamma_3\Gamma_2), \qquad (11A.10)
\end{aligned}$$

whose sum yields the additional contribution to the ${v_3}^B$-terms

$$\Delta v_3 = \frac{1}{6}\bar R - \frac{2}{3}\partial_\mu S^\mu + \frac{2}{3}\tilde\Gamma_3\tilde S_1, \tag{11A.11}$$

after having used the identity

$$\tilde S_3\tilde\Gamma_2 = -\tilde\Gamma_3\tilde S_1. \tag{11A.12}$$

The first term in (11A.11) is the $R$-term derived by K.S. Cheng[4] as an effective potential in the Schrödinger equation.

For ${v_2}^B$, we would find the extra terms

$$\Delta {v_2}^1 = -\frac{1}{2}(\Gamma_1\Gamma_1 - \tilde\Gamma_3\tilde\Gamma_2) + \frac{1}{8}[(\Gamma_1+\Gamma_2)^2 - \tilde\Gamma_3(\tilde\Gamma_1 + \tilde\Gamma_1^T + \tilde\Gamma_2 + \tilde\Gamma_2^T)], \tag{11A.13}$$

$$\Delta {v_2}^3 = \frac{1}{4}(\Gamma_1 - \Gamma_2)(\Gamma_1+\Gamma_2+\Gamma_3), \tag{11A.14}$$

[4]K.S. Cheng, J. Math. Phys. *13*, 1723 (1972).

which add up to

$$\Delta v_2 = -\frac{1}{2}S_1 S_1 + \frac{1}{2}\Gamma_3 S_1 - \frac{1}{3}\tilde{\Gamma}_3 \tilde{S}_1 + \frac{1}{2}\tilde{S}_1\,\tilde{S}_3, \tag{11A.15}$$

where we have written $S_1$ for $S_\mu$ and used some trivial identities such as

$$\tilde{\Gamma}_3 \tilde{\Gamma}_2{}^T = \tilde{\Gamma}_3 \tilde{\Gamma}_1. \tag{11A.16}$$

Thus we would obtain an additional effective potential $V_{\text{eff}} = (\hbar^2/M)v$ with

$$v = \frac{1}{6}R - \frac{2}{3}g^{\mu\nu}\partial_\mu S_\nu - \frac{1}{2}(S_1^2 - \Gamma_3 S_1) + \frac{1}{2}\tilde{S}_1 \tilde{S}_3 - \frac{1}{3}\tilde{\Gamma}_3 \tilde{S}_1. \tag{11A.17}$$

The second and the fourth term can be combined to

$$-\frac{2}{3}D_\mu S^\mu - \frac{1}{6}\Gamma_3 S_1. \tag{11A.18}$$

Due to the presence of $\Gamma$'s in $v$, this is a noncovariant expression which cannot possibly be physically correct. In the absence of torsion, however, $v$ happens to be reparametrization-invariant, and this is the reason why the resulting effective potential $V_{\text{eff}} = \hbar^2 \bar{R}/6M$ appeared acceptable in earlier works. A procedure which has no reparametrization-invariant extension to spaces with torsion cannot be correct.

## Appendix 11B DeWitt's Amplitude

Bryce DeWitt, in his frequently quoted paper,[5] attempted to quantize the motion of a particle in a curved space using the naive measure of path integration as in Eq. (10.153), but with the short-time amplitude (11.141) with $q \equiv q_n$, $q' \equiv q_{n-1}$. However, after taking the Jacobian action $\mathcal{A}^\epsilon_{J_0}$ into account, this leads to an integral kernel $K^\epsilon(q, \Delta q)$ which differs from our correct one in Eq. (11.4) by an extra factor

$$\mathcal{D}^{1/2}(q, q')e^{-i\mathcal{A}^\epsilon_J/\hbar}/\sqrt{g(q)}. \tag{11B.1}$$

This has the postpoint expansion $1 + \frac{1}{12}\bar{R}_{\mu\nu}(q)\Delta q^\mu \Delta q^\nu + \dots$ . When treated perturbatively, the extra term is equivalent to $\epsilon\hbar\bar{R}/12M$, Thus, in order to obtain the correct amplitude without an extra $\bar{R}$-term, DeWitt had to subtract the nonclassical potential $\hbar^2\bar{R}/12M$ from the Hamiltonian. Such a correction procedure must be rejected on the grounds that it runs contrary to the very essence of the path integral approach, in which the contribution of each path is controlled entirely by the phase $e^{i\mathcal{A}/\hbar}$ with the *classical action* in the exponent.

The short-time kernel proposed fifteen years later by Cheng requires the artificial subtraction of the full effective potential (11.106) to obtain the correct amplitude (11.4).

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