Analysis Disguised as Topology: Poincaré Duality and the Hodge Star Map

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Adjoint of d

Recall

Let (M, g) be a smooth closed m-dimensional Riemannian manifold. Recall that we gave the adjoint of d the special name δ; it is a map $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$.

Adjoint of d

Lemma

Let (M, g) be a smooth closed $m-$ dimensional Riemannian manifold, and let δ be the adjoint of the exterior derivative d. Then, $\delta^2 = 0$.

Proof.

By definition, for all $\omega \in \Omega^{k-2}(M)$ and $\eta \in \Omega^k(M)$ we have

$$\int_M g(\omega, \delta^2 \eta) = \int_M g(d^2\omega, \eta) = 0. \quad (1)$$

□

The Hodge Laplacian

Definition

Let (M, g) be a smooth closed $m-$ dimensional Riemannian manifold. Then, we define the Hodge-Laplacian $\Delta : \Omega^k(M) \to \Omega^k(M)$ by

$$\Delta\sigma = (d + \delta)^2\sigma = (d\delta + \delta d)\sigma. \tag{2}$$

If $\Delta\sigma = 0$, then we call σ an harmonic form.

Hodge Theory

Lemma

Let (M, g) *be a smooth closed* $m-$*dimensional Riemannian manifold. Then, every element* $w \in H^k(M)$ *has a unique harmonic representative* $\omega \in \Omega^k(M)$*. Furthermore, we have that*

$$\int_M g(\omega, \omega) = \min \left\{ \int_M g(\sigma, \sigma) : \sigma \in w \right\}. \tag{3}$$

Consequences for the integral cohomology lattice?

Recall

Recall that we defined $H^1(M;\mathbb{Z})_{\mathbb{R}}$ to be those elements of $H^1_{dR}(M)$ whose representatives ω satsify

$$\int_\gamma \omega \in \mathbb{Z} \forall \gamma \in \pi_1(M). \tag{4}$$

Observation

Since the topological properties of $H^1(M;\mathbb{Z})_{\mathbb{R}}$ are tied to $\pi_1(M)$, one may expect that the geometric properties of $H^1(M;\mathbb{Z})_{\mathbb{R}}$ are linked to the geometric properties of $\pi_1(M)$.

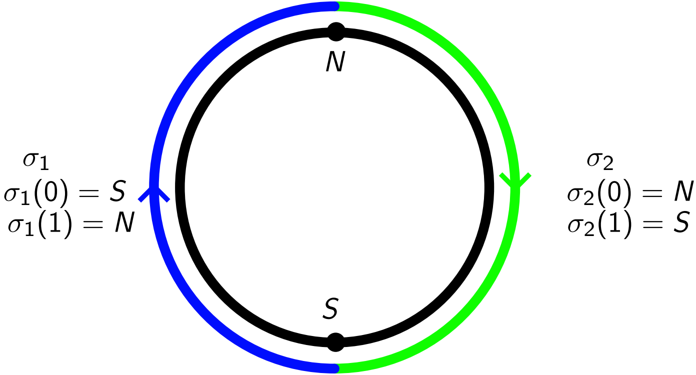
One-cycles

Definition

Let M be a smooth manifold, and let Δ^1 denote the 1-dimensional simplex $[0,1]$. Recall that given $\sigma : [0,1] \to M$ we set $\partial\sigma = \sigma(1) - \sigma(0)$. A real one cycle is a finite sum of $\sigma_i : [0,1] \to M$ with coefficients $r_i \in \mathbb{R}$ such that

$$\partial \sum_i r_i \sigma_i = \sum_i r_i \partial \sigma_i = 0. \tag{5}$$

Sketch



Real k-cycles

Definition

Let M be a smooth manifold, and let Δ^k denote the $k-$dimensional simplex. A real k-cycle is a finite sum of $\sigma_i : \Delta^k \to M$ with coefficients $r_i \in \mathbb{R}$ such that

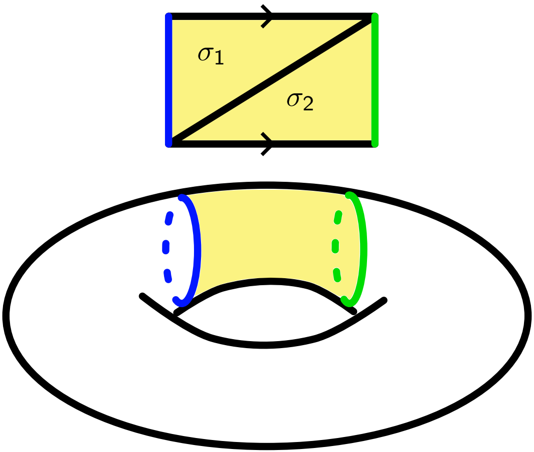
$$\partial \sum_i r_i \sigma_i = \sum_i r_i \partial \sigma_i = 0. \tag{6}$$

Definition

Let M be a smooth manifold, and let c_1 and c_2 be two real one-cycles. We say that they are homologous if there is a real two-cycle σ such that

$$\partial \sigma = c_2 - c_1. \tag{7}$$

Sketch



Real Homology

Definition

Let M be a smooth manifold. Two real k-cycles c_1 and c_2 are said to be homologous if and only if there exists a $k+1$ cycle *sigma* such that $\partial\sigma = c_2 - c_1$. We denote the collection of all k-real-homology classes of M by $H_k(M;\mathbb{R})$

Homology-Cohomology Pairing

Definition

Let M be a closed smooth manifold. Given a map $f : \Delta^k \to M$ we get a corresponding map $\Omega^k(M) \to \mathbb{R}$ given by

$$\omega \mapsto \int_{\Delta^k} f^*\omega. \tag{8}$$

Lemma

This map descends to a map $I : H_k(M;\mathbb{R}) \times H^k_{dR}(M) \to \mathbb{R}$.

Proof.

This is an application of Stoke's theorem to the relevant definitions. $\square$

The Pairing is Non-degenerate

Theorem

Let M be a closed smooth oriented manifold. Then, the map $I : H_k(M;\mathbb{R}) \times H^k_{dR}(M) \to \mathbb{R}$ is non-degenerate in the following sense.

1. *For $a \in H_k(M;\mathbb{R})$ we have $I(a,w) = 0$ for all $w \in H^k_{dR}(M)$ if and only if $a = 0$.*
2. *For $w \in H^k(M;\mathbb{R})$ we have $I(a,w) = 0$ for all $a \in H_k(M;\mathbb{R})$ if and only if $w = 0$.*

Cohomology-Cohomology Pairing

Definition

Let M be a smooth closed oriented m-dimensional manifold. We get a pairing $I_\wedge : \Omega^k(M) \times \Omega^{m-k}(M) \to \mathbb{R}$ as follows:

$$I_\wedge(\eta, \omega) = \int_M \eta \wedge \omega. \tag{9}$$

Lemma

The pairing $I_\wedge$ descends to $I_\wedge : H^k_{dR}(M) \times H^{m-k}_{dR}(M)$. Here it is non-degenerate.

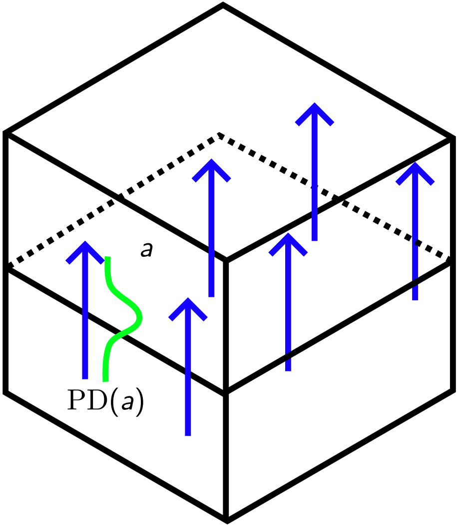
Poincaré Dual

Definition

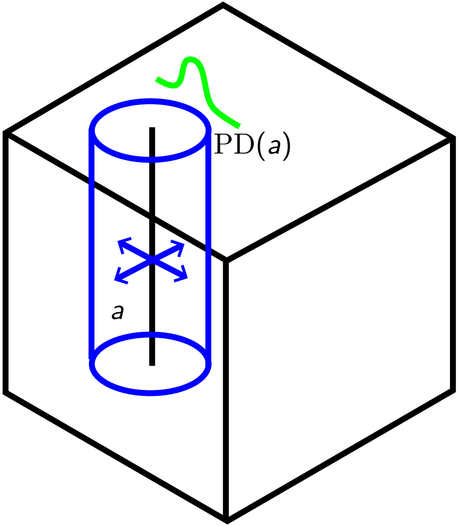
Let M be a smooth closed $m-$dimensional manifold, and consider $a \in H_k(M;\mathbb{R})$. The Poincaré Dual of a is the unique element $\mathrm{PD}(a) \in H^{m-k}_{dR}(M;\mathbb{R})$ such that for all $h \in H^k_{dR}(M)$ we have

$$I(a,h) = I_\wedge\big(\mathrm{PD}(a), h\big). \tag{10}$$

Sketch



Sketch



Poincaré Dual

Definition

Let M be a smooth closed oriented $m-$ dimensional manifold, and consider $w \in H^k_{dR}(M)$. The Poincaré Dual of w is the unique element $\mathrm{PD}(w)$ in $H_{m-k}(M;\mathbb{R})$ such that for all $h \in H_{m-k}(M;\mathbb{R})$ such that

$$I(\mathrm{PD}(w), h) = I_\wedge(w, h). \tag{11}$$

Interior Product

Definition

Let (M, g) be a smooth m-dimensional Riemannian manifold,let $k \leq m$, let $p \in M$, and $\eta \in \bigwedge_{i=1}^{k} T_p^* M$. Then, for each $l \leq m - k$ we get a map $\psi \mapsto \eta \wedge \psi$ from $\bigwedge_{i=1}^{l} T_p^* M$ to $\bigwedge_{i=1}^{l+k} T_p^* M$. Define $\eta_\lrcorner : \bigwedge_{i=1}^{l+k} T_p^* M \to \bigwedge_{i=1}^{l} T_p^* M$ to be its adjoint: for all $\omega \in \bigwedge_{i=1}^{l+k} T_p^* M$ and all $\sigma \in \bigwedge_{i=1}^{l} T_p^* M$ we have

$$g(\eta_\lrcorner(\omega), \sigma) = g(\omega, \eta \wedge \sigma). \tag{12}$$

Hodge Star Map

Definition

Let (M, g) be a smooth oriented $m-\textit{dimensional}$ Riemannian manifold, and let vol_g denote its volume form. For any $k \leq m$ we define a map $\star : \Omega^k(M) \to \Omega^{m-k}(M)$ as follows. For $\sigma \in \Omega^k(M)$ for each $p \in M$ we set

$$(\star\sigma)_p = (\sigma_p) \lrcorner (\mathrm{vol}_g)_p. \tag{13}$$

Lemma

Let (M, g) be a smooth closed oriented $m-$dimesnional manifold. Then for each $p \in M$ and all $\omega, \sigma \in \bigwedge_{i=1}^{k} T_p^* M$ we have

$$g(\omega, \sigma) \cdot (\mathrm{vol}_g)_p = \omega \wedge \star\sigma. \tag{14}$$

By integrating over M, we get

$$\int_M g(\omega, \sigma)\mathrm{vol}_g = \int_M \omega \wedge \star\sigma. \tag{15}$$

Proof.

Let $\{E_i\}_{i=1}^m$ be an orthonormal basis for T_pM, and let $\{\Theta_i\}_{i=1}^m$ be the dual basis for T_p^*M. Consider the element $\sigma = \Theta_1 \wedge \cdots \wedge \Theta_k$, and observe that $\sigma \lrcorner \mathrm{vol}_g = \Theta_{k+1} \wedge \cdots \wedge \Theta_m$. This shows that for any ω we have

$$g(\omega, \Theta_1 \wedge \cdots \wedge \Theta_k) \cdot (\mathrm{vol}_g)_p = \omega \wedge \Theta_{k+1} \cdots \wedge \Theta_m. \tag{16}$$

Similar arguments work for all of the elements Θ_J for $J \in J(k, m)$. $\square$

Corollary

The map $\star$ satisfies $\star^2 = (-1)^{k(m-k)}\mathrm{Id}$.

Isometry

Corollary

The map $\star$ is an isometry.

Proof.

$$
\begin{aligned}
g(\star\omega, \star\omega)\mathrm{vol}_g &= \star\omega \wedge \star^2\omega && (17)\\
&= (-1)^{k(m-k)} \star \omega \wedge \omega && (18)\\
&= (-1)^{k(m-k)}(-1)^{k(m-k)}\omega \wedge \star\omega && (19)\\
&= g(\omega, \omega)\mathrm{vol}_g. && (20)
\end{aligned}
$$

$\square$

Length

Definition

Let (M, g) be a smooth Riemannian manifold, and let $\sigma : [0, 1] \to M$ be a curve. Then, the length of the curve, denoted by $\mathrm{vol}_1([0, 1], \sigma^* g)$ is given by

$$\int_0^1 \sqrt{g(\dot{\sigma}, \dot{\sigma})}dt. \tag{21}$$

Observation

The integrand is the area form on $[0, 1]$ induced by the metric $\sigma^* g$ on $[0, 1]$.

Volume

Definition

Let (M, g) be a smooth closed $m-$dimensional Riemannian manifold, and let Δ^k denote the $k-$simplex Then, for $f : \Delta^k \to M$ we define $\mathrm{vol}_k(\Delta^k, f^*g)$ to be

$$\mathrm{vol}_k(\Delta^k, f^*g) = \int_{\Delta^k} \mathrm{vol}_{f^*g} \tag{22}$$

Warning

The fully accurate definition is a bit more subtle

Mass

Definition

Let (M, g) be a smooth closed $m-$dimensional Riemannian manifold. Given a class $w \in H_k(M; \mathbb{R})$ we define its k-mass to be

$$\|w\|_k = \inf \left\{ \sum_i |r_i| \mathrm{vol}_k(\Delta^k, \sigma_i^* g) : \sum_i r_i \sigma_i \in w \right\} \tag{23}$$

Lemma (Hebda)

Let (M, g) be a smooth closed oriented Riemannian manifold. For $a \in H^{m-1}_{dR}(M)$ and $\mathrm{PD}(a) \in H_1(M; \mathbb{R})$ we have

$$\|\mathrm{PD}(a)\|_1 \leq \mathrm{Vol}_g(M)^{\frac{1}{2}} C(m, 1) \inf \left\{ \left(\int_M g(\omega, \omega) \right)^{\frac{1}{2}} : \omega \in a \right\} \tag{24}$$

Proof.

Let ω be the harmonic form representing a. Then, for all closed 1-forms ϕ we have

$$I(\mathrm{PD}(a), \phi) = I_\wedge(\omega, \phi) = \int_M \omega \wedge \phi. \tag{25}$$

This RHS is equal to

$$\pm \int_M g(\star\omega, \phi). \tag{26}$$

Proof.

Taking absolute values, we get

$$
\begin{aligned}
|I(\mathrm{PD}(a), \phi)| &\leq \int_M |\star\omega|_g |\phi| && (27)\\
&\leq \|\phi\|_{L^\infty} \int_M |\star\omega| && (28)\\
&\leq \|\phi\|_{L^\infty} \mathrm{vol}_g(M)^{\frac{1}{2}} \left(\int_M g(\star\omega, \star\omega)\right)^{\frac{1}{2}}. && (29)\\
&= \|\phi\|_{L^\infty} \mathrm{vol}_g(M)^{\frac{1}{2}} \left(\int_M g(\omega, \omega)\right)^{\frac{1}{2}}. && (30)
\end{aligned}
$$

Proof.

On the otherhand, by the properties of mass, we have

$$\|\mathrm{PD}(a)\|_1 \leq C(m,1)\sup\left\{|I(\mathrm{PD}(a),\phi)| : \|\phi\|_{L^\infty} \leq 1\right\}. \tag{31}$$

$\square$