

Gromov-Hausdorff Convergence

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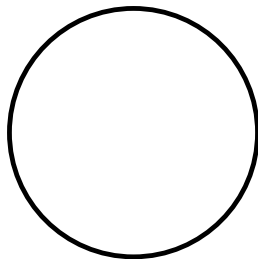
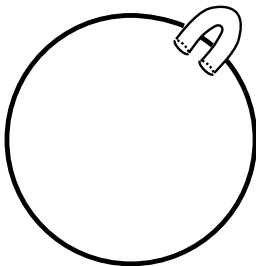
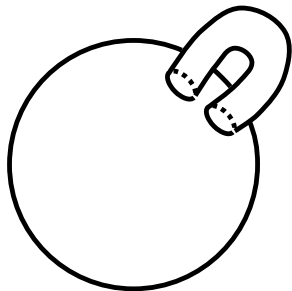
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Table of Contents

- 1 Hausdorff Distance
- 2 Gromov Hausdorff Distance
- 3 Alternate Formulations
 - Semi-metric Extensions
 - Correspondences
 - ε -isometries
- 4 Compactness Criterion

Idea



Epsilon Neighborhood

Definition

Let (X, d) be a metric space, let $x \in A$ and $A \subset X$. Then, the distance from x to A is defined to be

$$d(x, A) = \inf\{d(x, a) : a \in A\}. \quad (1)$$

Definition

Let (X, d) be a metric space, let $A \subset X$, and let $\varepsilon > 0$. The ε neighborhood of A , denoted A_ε , is the collection of all points within a distance ε of A :

$$A_\varepsilon = \{x \in X : d(x, A) < \varepsilon\}. \quad (2)$$

Hausdorff Distance

Definition

Let (X, d) be a metric space, and let $A, B \subset X$ be subsets. Then, we define the Hausdorff distance between A and B to be

$$d_H(A, B) = \inf\{\varepsilon : B \subset A_\varepsilon \text{ \& } A \subset B_\varepsilon\}. \quad (3)$$

Lemma

Let (X, d) be a metric space, and let $\text{cpt}(X)$ denote the collection of all compact subsets of X . If (X, d) is a complete metric space, then so is $(\text{cpt}(X), d_H)$. If (X, d) is a compact metric space, then so is $(\text{cpt}(X), d_H)$.

Metric Preserving Maps

Definition

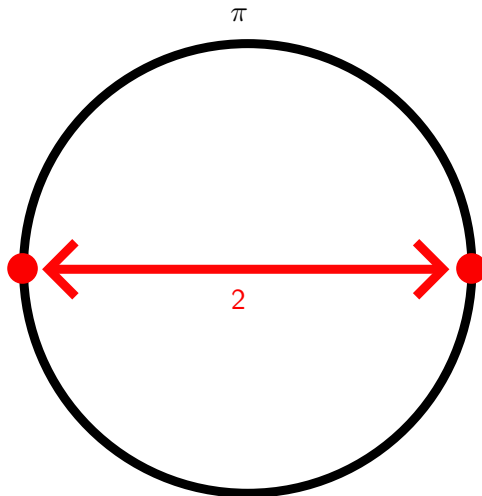
Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $\phi : X \rightarrow Y$ be a map. We call ϕ a metric preserving map (sometimes isometry) if and only if for any $x_1, x_2 \in X$ we have

$$d_X(x_1, x_2) = d_Y(\phi(x_1), \phi(x_2)). \quad (4)$$

Warning

The standard embedding of \mathbb{S}^{n-1} into R^n is not a metric preserving map in this terminology.

Geometers Beware



Gromov-Hausdorff distance

Definition

Let (X, d) and (Y, d) be two compact metric spaces. We define the the Gromov-Hausdorff distance between them as follows.

$$d_{GH}(X, Y) = \inf\{d_H(\phi(X), \psi(Y)) : \exists \phi : X \rightarrow Z, \psi : Y \rightarrow Z\},$$

where ϕ and ψ are required to be metric preserving maps.

Remark

The above makes sense for X and Y not compact, but there is a slight modification of the Gromov-Hausdorff distance which is better suited to this case.

It is a metric

Theorem

Suppose that (X, d_X) , (Y, d_Y) and (W, d_W) are three compact metric spaces. Then, we have that $d_{GH}(X, W) \leq d_{GH}(X, Y) + d_{GH}(Y, W)$. Furthermore if $d_{GH}(X, Y) = 0$ if and only if there is a surjective and injective map $\phi : X \rightarrow Y$, which is also metric preserving. We will call such a map an isometry.

Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. A semi-metric ρ on $X \sqcup Y$ is said to extend the metrics d_X and d_Y if and only if for all $x_1, x_2 \in X$ we have

$$d_X(x_1, x_2) = \rho(x_1, x_2) \quad (5)$$

and for all $y_1, y_2 \in Y$ we have

$$d_Y(y_1, y_2) = \rho(y_1, y_2). \quad (6)$$

Alternate formulation

Lemma

Let (X, d) and (Y, d) be compact metric spaces, and denote $X \sqcup Y$ by Z . Then, we have that

$$d_{GH}(X, Y) = \inf \{ \rho_H(X, Y) : \rho \text{ extends } d_X \text{ and } d_Y \} \quad (7)$$

Proof.

Let (W, d_W) be a metric space, and let $\phi : X \rightarrow W$ and $\psi : Y \rightarrow W$ be metric preserving maps. Define ρ on $X \sqcup Y$ to be $\rho|_X = d_X$, $\rho|_Y = d_Y$, and for $x \in X$ and $y \in Y$, set

$$\rho(x, y) = d_W(\phi(x), \psi(y)). \quad (8)$$



It's a metric

Lemma

Let (X_i, d_{X_i}) be compact metric spaces for $i = 1, 2, 3$. Then, we have that

$$d_{GH}(X_1, X_2) \leq d_{GH}(X_1, X_2) + d_{GH}(X_2, X_3) \quad (9)$$

Proof.

Let d_{12} be a semi-metric on $X_1 \sqcup X_2$ extending d_{X_1} and d_{X_2} , and let d_{23} be a semi-metric on $X_2 \sqcup X_3$ extending d_{X_2} and d_{X_3} . Then, for $x_1 \in X_1$ and $x_3 \in X_3$ set

$$d_{13}(x_1, x_3) = \inf \{ d_{12}(x_1, x_2) + d_{23}(x_2, x_3) : x_2 \in X_2 \}. \quad (10)$$



Correspondences

Definition

Let X and Y be two sets, a correspondence between them is a set $\mathcal{R} \subset X \times Y$ such that for every $x \in X$ there is at least one $y \in Y$ such that $(x, y) \in \mathcal{R}$, and for every $y \in Y$ there is at least one $x \in X$ such that $(x, y) \in \mathcal{R}$.

Definition

Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and let \mathcal{R} be a correspondence between them. Then, the distortion of \mathcal{R} is

$$\text{dis}(\mathcal{R}) = \sup\{|d_X(x_1, x_2) - d_Y(y_1, y_2)| : (x_1, y_1), (x_2, y_2) \in \mathcal{R}\} \quad (11)$$

Alternate formulation

Lemma

Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and let $\mathcal{R}(X, Y)$ denote the collection of all correspondences between them. Then, we have that

$$d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis}(\mathcal{R}) : \mathcal{R} \in \mathcal{R}(X, Y) \} \quad (12)$$

Intuition

Correspondences can be thought of as functions with multiple values. Suppose for each $x \in X$ we arbitrarily choose $y \in Y$ such that $(x, y) \in \mathcal{R}$. Then, the result is a map $f : X \rightarrow Y$.

Alternate formulation

Proof.

- 1 If $d_{GH}(X, Y) < \varepsilon$, then there exists a metric space (Z, d_Z) and metric preserving maps ϕ and ψ such that $d_H(\phi(X), \psi(Y)) < \varepsilon$.
- 2 Set $\mathcal{R} = \{(x, y) \in X \times Y : d_Z(\phi(x), \psi(y)) < \varepsilon\}$, then we see from the triangle inequality that $\text{dis}(\mathcal{R}) < 2\varepsilon$.
- 3 On $X \sqcup Y$ for $x \in X$ and $y \in Y$ set
$$d(x, y) = \inf\{d_X(x, x') + \varepsilon + d_Y(y', y) : (x', y') \in \mathcal{R}\}.$$
- 4 If $x \in X$, pick $y \in Y$ such that $(x, y) \in \mathcal{R}$. This shows that $X \in Y_\varepsilon$.



Epsilon nets

Definition

Let (X, d) be a metric space, and let $N \subset X$. We call N an ε -net if and only if for all $x \in X$ there exists an element $n_x \in N$ such that $d(x, n_x) < \varepsilon$.

Observation

The above is equivalent to saying that $d_H(X, N) < \rho$.

Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \rightarrow Y$. We call f an ε -isometry if and only if it satisfies the following two conditions. First, for all x_1, x_2 in X we have

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon. \quad (13)$$

Second, we require that the set $f(X)$ is an ε -net in Y .

Alternate formulation

Lemma

Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and suppose $\varepsilon > 0$. Then, we have the following.

- 1 If $d_{GH}(X, Y) < \varepsilon$, then there is a 2ε isometry from X to Y .
- 2 If there is an ε -isometry from X to Y , then $d_{GH}(X, Y) < 2\varepsilon$.

Proof.

We can produce a map f from a correspondence \mathcal{R} . Suppose we have an ε -isometry f . Let $\mathcal{R} = \{(x, y) : d_Y(f(x), y) < \varepsilon\}$. □

Lemma

Let $\{(X_i, d_{X_i})\}_{i \in \mathbb{N}}$ be a sequence of compact metric spaces such that $\sup_i \text{diam}(X_i) < \infty$. Suppose that for every $\varepsilon > 0$ there exists $N(\varepsilon) \subset \mathbb{N}$ such that for each X_i there is an ε -net with $N(\varepsilon)$ elements. Then, there is a compact metric space (X_∞, d_∞) and a sub-sequence $(X_{i(j)}, d_{X_{i(j)}})$ such that

$$\lim_{j \rightarrow \infty} d_{GH}(X_{i(j)}, X_\infty) = 0. \quad (14)$$

Compactness

Proof.

- 1 Let $\mathcal{N}_i(\varepsilon) \subset X_i$ be an ε -net with $N(\varepsilon)$ elements. That is, $\mathcal{N}_i(\varepsilon) = \{x_j^i\}_{j=1}^{N(\varepsilon)} \subset X_i$.
- 2 Let $d_{jk}^i = d_{X_i}(x_j^i, x_k^i)$. These are uniformly bounded, and so there is a matrix d_{jk}^∞ and a subsequence $i(l)$ such that

$$\lim_{l \rightarrow \infty} d_{jk}^{i(l)} = d_{jk}^\infty.$$

- 3 Let $\varepsilon \rightarrow 0$, and apply a diagonal argument.



Finite point principle

Observation

If a property can be expressed using finitely many points, then it is well behaved with respect to the Gromov-Hausdorff metric.