

Small Negative Curvature Implies Almost Constant Innerproducts

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Lemma

Let $V, \sigma, \eta > 0$ and $m \in \mathbb{N}$. Then there is a constant $C(V, \sigma, \eta, m)$ such that for any metric g on \mathbb{T}^3 which satisfies

- ① $|T^3|_g \leq V$;
- ② $\min\{\text{stabsys}_1(\mathbb{T}^3, g), \min\{\text{stabsys}_2(\mathbb{T}^3, g)\}\}$;
- ③ $\kappa(g, \eta)m$,

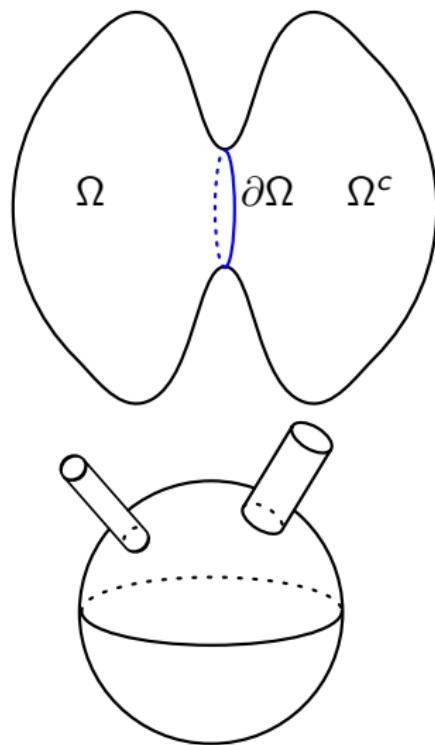
we may find a degree one map $U : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that for $i = 1, 2, 3$ we have $\max\{\|du^i\|_{L^2}, \|du^i\|_L^3\} \leq C(V, \sigma, \eta, m)$, and for $g_{ij} = g(du^i, du^j)$ we have

$$\|dg_{ij}\| \leq C(V, \sigma, \eta, m) \|S_g^-\|_{L^2}^{\frac{1}{2}}.$$

Definition

Let g be a Riemannian metric on \mathbb{T}^3 . Recall that $IN_1(\mathbb{T}^3, g)$, also called the Cheeger Constant, is defined to be

$$IN_1(\mathbb{T}^3, g) = \inf \left\{ \frac{\mathcal{H}^{n-1}(\partial\Omega)}{\min\{|\Omega|, |\Omega^c|\}} : \Omega < \mathbb{T}^3 \right\}.$$



Definition

Let g be a Riemannian metric on \mathbb{T}^3 . Recall that the Sobolev-Neumann Constant of g on \mathbb{T}^3 , denoted $SN_1(\mathbb{T}^3, g)$, is defined to be

$$SN_1(\mathbb{T}^3, g) = \inf \left\{ \frac{\int_{\mathbb{T}^3} |\nabla f| \text{vol}_g}{\min_{k \in \mathbb{R}} \int_{\mathbb{T}^3} |f - k| \text{vol}_g} : f \in W^{1,1}(\mathbb{T}^3) \right\}$$

Lemma

Let g be a metric on \mathbb{T}^3 . Then, we have that $SN_1(\mathbb{T}^3, g) = IN_1(\mathbb{T}^3, g)$.

Rough Good Set

Definition

Let g be a Riemannian metric on \mathbb{T}^3 . Given $\tau > 0$, a constant symmetric matrix a , a map $\Psi : \mathbb{T}^3 \rightarrow \mathbb{T}^3$, and $g_{ij} = g(d\Psi^i, d\Psi^j)$, we define

$$E_\Psi^p(\tau, a) = \left\{ x \in \mathbb{T} : \sum_{ij} |g_{ij} - a_{ij}|^p < \tau^p \right\}$$

Family of Metrics

Definition

Let $V, \sigma, \Lambda, \eta > 0$ and $m \in M$, and let $\mathcal{F}(V, \sigma, \Lambda, \eta, m)$ denote the collection of metrics on \mathbb{T}^3 which satisfy the following conditions.

- ① $|\mathbb{T}^3|_g \leq V$;
- ② $\min\{\text{stabsys}_1(\mathbb{T}^3, g), \text{stabsys}_2(\mathbb{T}^3, g)\}$;
- ③ $IN_1(\mathbb{T}^3, g) \geq \Lambda$;
- ④ $\kappa(g, \eta) \leq m$.

There is a Good Set

Lemma

There exists a constant $C = C(V, \sigma, \Lambda, \eta, m)$ such that for each metric g in $\mathcal{F}(V, \sigma, \Lambda, \eta, m)$ there is a degree one map $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$, and a symmetric matrix a such that

$$|E_{\mathbb{U}}^1(\tau, a)| \geq |\mathbb{T}^3|_g - \frac{C \|S_g^-\|^{\frac{1}{2}}}{\tau}$$

Proof.

- From Stern's inequality, we have $\int_{\mathbb{T}^3} |\nabla dg_{ij}| \text{vol}_g \leq C \|S_g^-\|_{L^2}^{\frac{1}{2}}$.
- From the definition of $SN_1(\mathbb{T}^3, g)$, for each ij there is an element $a_{ij} \in \mathbb{R}$ such that

$$\int_{\mathbb{T}^3} |g_{ij} - a_{ij}| \text{vol}_g \leq \frac{\|dg_{ij}\|_{L^1}}{\Lambda}.$$

- Use Chebyshev's inequality.



Corollary

There exists constants $C(V, \sigma, \Lambda, \eta, m, p)$ such that

$$|E_{\mathbb{U}}^p(\tau, a)| \geq |\mathbb{T}^3|_g - \frac{C \|S_g^-\|^{\frac{1}{2}}}{\tau}$$

Proof.

Depending on p , we can see that $E_{\mathbb{U}}^1(\frac{\tau}{C}, a) \subset E^p$. □

Idea

Define $f : \mathbb{T}^3 \rightarrow \mathbb{R}$ to be $x \mapsto \sum_{ij} (g_{ij} - a_{ij})^2$. Then f is a smooth function, and we will try to understand $E_{\mathbb{U}}^2(\tau, a) = f^{-1}[0, \sqrt{\tau}]$ using f .

Idea

In particular, we know that $\partial E^2(\sqrt{s}, a) = f^{-1}\{s\}$.

Calculation

Using the coarea formula, we integrate

$$\begin{aligned} \int_{\tau^2}^{4\tau^2} |\partial E^2(\sqrt{s}, a)| &\leq \int_{E^2(2\tau, a)} |d \sum_{ij} (g_{ij} - a_{ij})^2| \text{vol}_g \\ &= 2 \sum_{ij} \int_{E^2(2\tau, a)} |g_{ij} - a_{ij}| |dg_{ij}| \text{vol}_g. \\ &\leq C\tau \|S^-\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

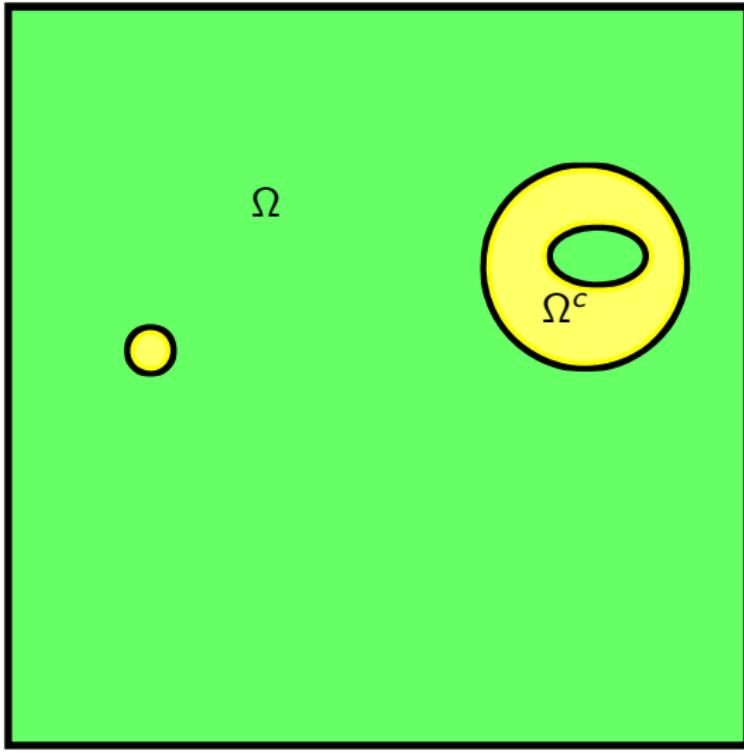
Calculation

Using Chebyshev's inequality, we see that

$$|\{s : |\partial E^2(\sqrt{s}, a)| > C\|S^-\|_{L^2}^{\frac{1}{4}}\}| < \tau\|S^-\|^{\frac{1}{4}}$$

For $\|S^-\|_{L^2} \leq \tau^4$, we see that

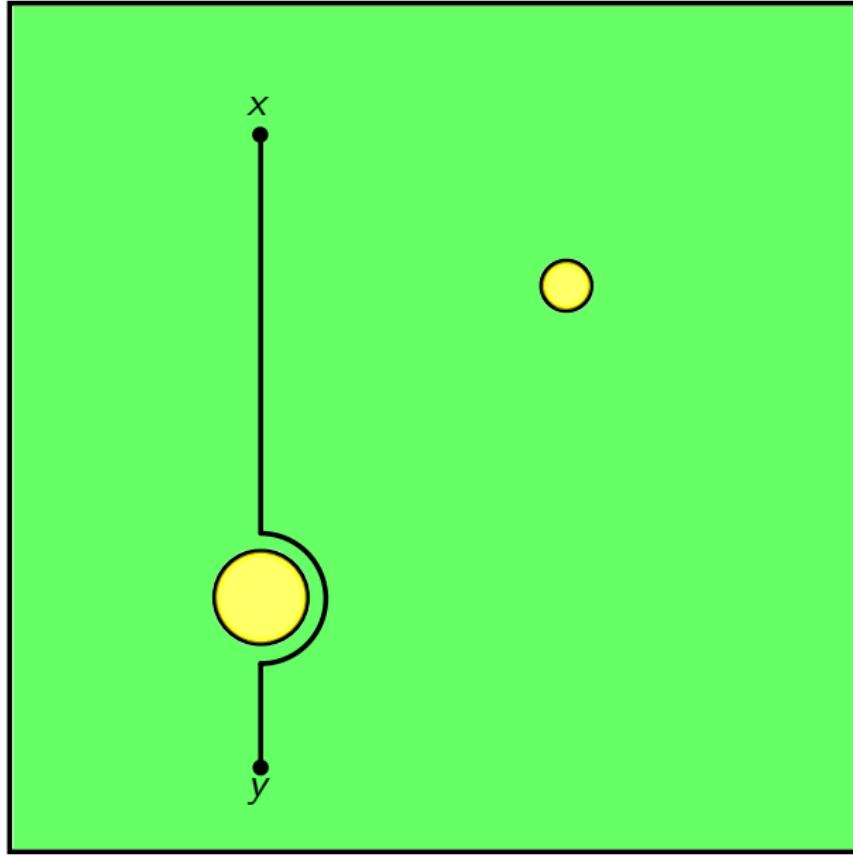
$$|\{s : |\partial E^2(\sqrt{s}, a)| < C\|S^-\|_{L^2}^{\frac{1}{4}}\}| > 3\tau^2 - \tau\|S^-\|^{\frac{1}{4}} \geq 2\tau^2.$$



Lemma

Let $V, \sigma, \Lambda, \eta > 0$ and $m \in \mathbb{N}$. Then, there exists a constant $C(V, \sigma, \Lambda, \eta, m)$ such that for each metric in \mathcal{F} we can find a degree one map $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$, a symmetric matrix a , and an open connected subset $\Omega(g) \subset \mathbb{T}^3$ with smooth boundary such that

- ① $|\Omega(g)| \geq |\mathbb{T}|_g - C\|S^-\|^{\frac{1}{4}},$
- ② $\mathcal{H}^2(\Omega(g)) \leq C\|S^-\|^{\frac{1}{4}}.$



Restricted Length Metric

Definition

Let (M, g) be a Riemannian manifold, let $\Omega \subset M$ be a connected open subset of M . Given points x and y in Ω , we define

$$\hat{d}_{\Omega}^g(x, y) = \inf\{\text{Len}(\gamma) : \gamma(0) = x, \gamma(1) = y, \gamma \subset \Omega\}.$$

We call \hat{d}_{Ω}^g the restricted length metric for Ω .

Observation

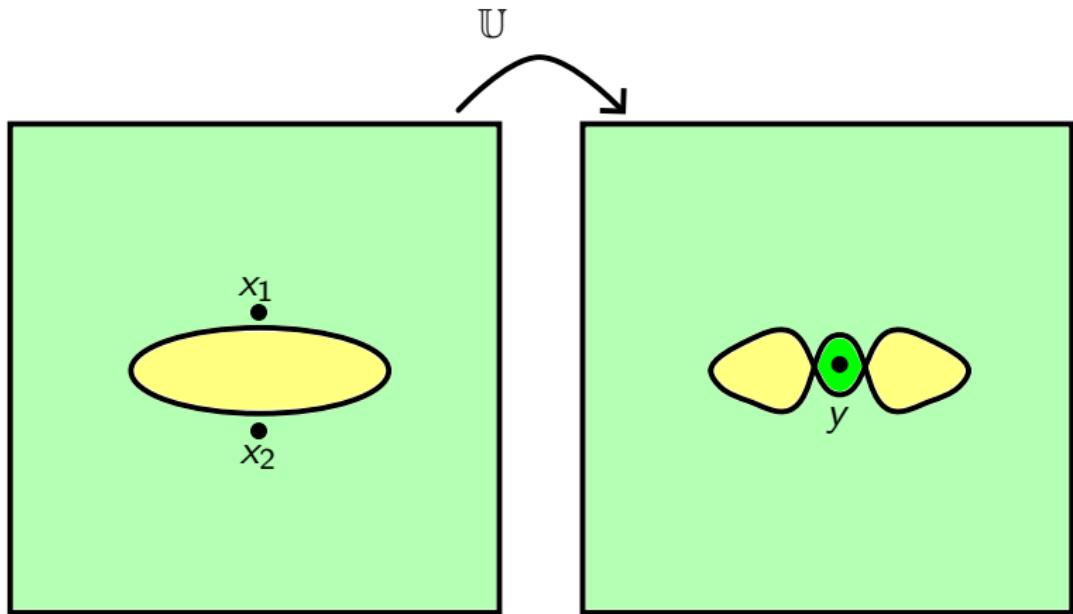
Since $\Omega \subset M$ is a connected open subset of M , for all $x, y \in \Omega$ we have that $\hat{d}_{\Omega}^g(x, y) < \infty$.

Dong-Song Convergence

Definition

Let (M, g_0) be a smooth Riemannian manifold, and let (M_i, g_i, Ω_i) be a sequence of triplets consisting of a Riemannian manifold (M_i, g_i) and an open connected subset $\Omega_i \subset M_i$. We say that the sequence (M_i, g_i, Ω_i) converges to (M, g_0) in the Dong-Song sense if and only if

$$\lim_{i \rightarrow \infty} d_{m\text{GH}} \left((M, d^{g_0}, \text{vol}_g), \left(\Omega_i, \hat{d}_{\Omega_i}^{g_i}, \text{vol}_{g_i} \right) \right) = 0.$$



Goal

We want to show that \mathbb{U} is injective on $\Omega(g)$, or at least a good subset of $\Omega(g)$.

Goal

We will begin by showing that $\det(a) > C > 0$. From the definitions, we can calculate that

$$|du^1 \wedge du^2 \wedge du^3|_g = \det(g_{ij})^{\frac{1}{2}}.$$

Since $\deg(\mathbb{U}) = 1$, we know that $1 = \int_{\mathbb{T}^3} du^1 \wedge du^2 \wedge du^3$. We want to show that $\int_{\Omega(g)^c} |du^1 \wedge du^2 \wedge du^3|$ is small.

Proof.

- Consider $\pi^{-1}(\Omega(g))$ and $\pi^{-1}(\partial\Omega(g)) = \partial\pi^{-1}(\Omega(g))$.
- Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the cutoff function for \mathbb{V}_η .
- Calculate as follows:

$$\begin{aligned}\int_{\pi^{-1}(\Omega(g)^c)} \pi^*g(f|d\hat{u}|d\hat{u}, d\hat{u}) \text{vol}_{\pi^*g} &= \int_{\partial\pi^{-1}(\Omega(g))} \hat{u}f|d\hat{u}| \frac{\partial\hat{u}}{\partial\nu} \\ &\quad - \int_{\pi^{-1}(\Omega(g)^c)} |d\hat{u}| \pi^*g(df, d\hat{u}) \\ &\quad + \int_{\pi^{-1}(\Omega(g))} f \pi^*g(d|d\hat{u}|, d\hat{u})\end{aligned}$$

Piece

$$\left| \int_{\partial\pi^{-1}(\Omega(g))} \hat{u} f |d\hat{u}| \frac{\partial \hat{u}}{\partial \nu} \text{area}_{\pi^*g} \right| \leq C \mathcal{H}^2(\partial\Omega(g)).$$

Piece

$$\left| \int_{\pi^{-1}(\Omega(g)^c)} |d\hat{u}| \pi^*g(df, d\hat{u}) \right| \leq C \int_{\Omega(g)^c} |du|_g^2 \text{vol}_g.$$

Piece

$$\left| \int_{\pi^{-1}(\Omega(g))} f \pi^*g(d|d\hat{u}|, d\hat{u}) \right| \leq C \|S^-\|_{L^2}^{\frac{1}{2}}.$$

Lemma

For the subset $\Omega(g)$ we have that $\int_{\Omega(g)^c} |du|^3 \text{vol}_g \leq C \|S^-\|_{L^2}^{\frac{1}{4}}$. In particular, we have

$$\int_{\Omega(g)} du^1 \wedge du^2 \wedge du^2 \geq 1 - C \|S^-\|_{L^2}^{\frac{1}{4}},$$

and

$$C \leq \det(a) > \frac{1}{C} - C \|S^-\|_{L^2}^{\frac{1}{4}}.$$

Finally, this means that on $\Omega(g)$, the map \mathbb{U} is a local diffeomorphism.

Goal.

Consider the function $\psi : \mathbb{T}^3 \rightarrow \mathbb{N}$ given by $\psi(y) = \sharp(\mathbb{U}^{-1}\{y\} \cap \Omega(g))$.
We want to understand $\{y : \psi(y) = 1\}$.

Lemma

Let $\mathbb{U} : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a smooth surjective map such that around each point of $\Omega(g)$ \mathbb{U} is a local diffeomorphism. Define $\psi : \mathbb{T}^3 \rightarrow \mathbb{N}$ by $\psi(y) = \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g))$. Then, the function ψ is upper semi-continuous, and if y is a point of discontinuity for ψ , then $y \in \mathbb{U}(\partial\Omega(g))$.

Calculation

From the area formula and our previous work, we have

$$\begin{aligned} 1 + C\|S_g^-\|_{L^2}^{\frac{1}{4}} &\geq \int_{\Omega(g)} J(\mathbb{U}) \text{vol}_g = \int_{\Omega(g)} |du^1 \wedge du^2 \wedge du^3| \text{vol}_g \\ &= \int_{\mathbb{U}(\Omega(g))} \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g)). \end{aligned}$$

and

$$\begin{aligned} C\|S^-\|_{L^2}^{\frac{1}{4}} &\geq \int_{\Omega(g)^c} J(\mathbb{U}) \text{vol}_g = \int_{\Omega(g)^c} |du^1 \wedge du^2 \wedge du^3| \text{vol}_g \\ &= \int_{\mathbb{U}(\Omega(g)^c)} \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g)^c) d\theta^1 d\theta^2 d\theta^3 \\ &\geq |\mathbb{U}(\Omega(g))^c| \end{aligned}$$

Calculation

- We see that $|\mathbb{U}(\Omega(g))| \geq 1 - C\|S_g^-\|_{L^2}$.
- We see that

$$\int_{\mathbb{U}(\omega(g))} \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g)) \leq 1 + C\|S_g^-\|_{L^2}^{\frac{1}{4}}.$$

- Applying Chebyshev's inequality, we get

$$|\{y : \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g)) \geq 2\}| \leq \frac{1}{2}(1 + C\|S_g^-\|_{L^2}^{\frac{1}{4}})$$

- Thus, $A(g) = \{y = \#(\mathbb{U}^{-1}\{y\} \cap \Omega(g)) < 2\}$ is open,
 $\partial A(g) \subset \mathbb{U}(\partial\Omega(g))$, and $|A(g)| \geq \frac{1}{2}(1 - 3C\|S_g^-\|_{L^2}^{\frac{1}{4}})$.

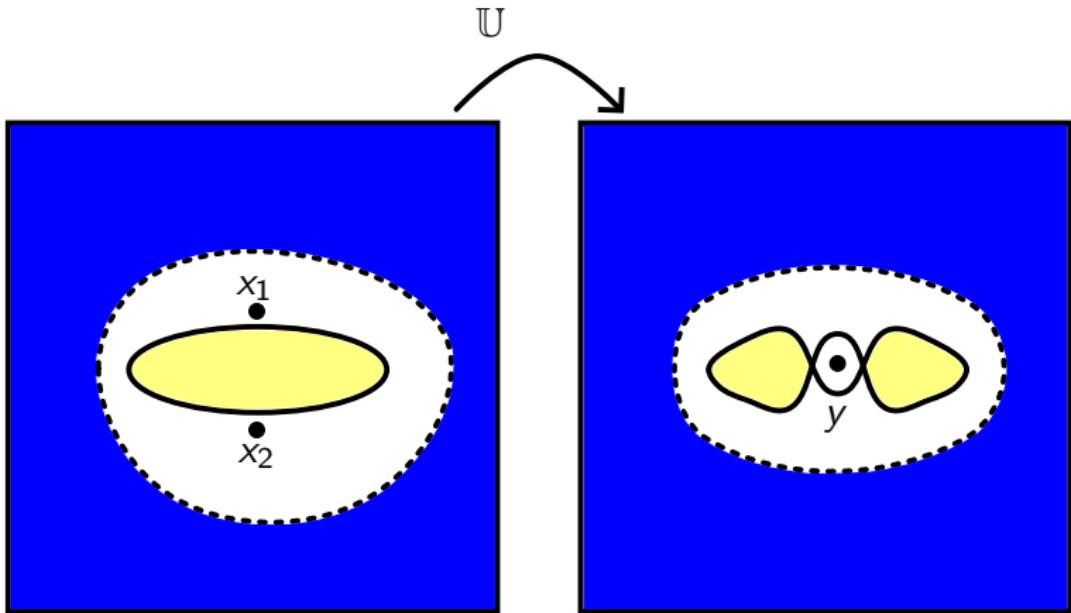
Theorem

Let $V, \sigma, \Lambda, \eta > 0$ and $m \in \mathbb{N}$. For a sequence of metrics $g_i \in \mathcal{F}(V, \sigma, \Lambda, \eta, m)$ such that $\lim_{i \rightarrow \infty} \|S_{g_i}^-\|_{L^2} = 0$, there is a subsequence $g_{i(j)}$ and a flat metric g_0 on \mathbb{T}^3 such that $g_{i(j)} \rightarrow g_0$ in the sense of Dong-Song.

Proof.

- We have established that for each g_i there is a corresponding symmetric matrix a_i and a subset $\Omega(g_i)$ on which $|(g_i)_{jk} - a_{jk}| < C\|S^-\|_{L^2}^{\frac{1}{4}}$.
- We can find a connected $A(g_i) \subset \Omega(g_i)$ such that $\partial A(g_i)$ is smooth, $|A(g_i)| \geq |\mathbb{T}^3|_{g_i} - \frac{C}{\Lambda}\|S_{g_i}^-\|_{L^2}^{\frac{1}{4}}$, and $\mathcal{H}^2(\partial A(g_i)) \leq C\|S_{g_i}^-\|_{L^2}^{\frac{1}{4}}$.





Proof.

- A subsequence of the a_i converge to a limit a_∞ . Let $g_0 = (a_\infty)_{kl} d\theta^k d\theta^l$
- Since $|g_{i(j)} - a_{i(j)}| \leq C \|S_{g_{i(j)}}^-\|$ on $A(g_{i(j)})$ we see that $|g - \mathbb{U}^* g_0| \rightarrow 0$ on $A(g_{i(j)})$:

$$d_{m\text{GH}} \left(\left(A, \hat{d}_A^{g_{i(j)}}, \text{vol}_{g_{i(j)}} \right), \left(\mathbb{U}(A), \hat{d}_{\mathbb{U}(A)}^{g_0}, \text{vol}_{g_0} \right) \right) \rightarrow 0$$

- It follows from an approximation result of C. Dong and A. Song that

$$d_{m\text{GH}} \left(\left(\mathbb{U}(A), \hat{d}_{\mathbb{U}(A)}^{g_0}, \text{vol}_{g_0} \right), \left(\mathbb{T}^3, d^{g_0}, \text{vol}_{g_0} \right) \right) \rightarrow 0.$$

