Curvature, Coarea, and Stern's inequality

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Curvature

Definition

Let (M, g) be a smooth Riemannian manifold, and let ∇ be its Levi-Civita connection. Recall that we can define a tensor $R : T_pM \times T_pM \times T_pM \to T_pM$ as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1}$$

Alternate Form

We may also consider R as a tensor $R : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$. In this form, we have

$$\begin{aligned} R(X,Y,Z,W) &= g(R(X,Y)Z, W) && (2) \\ &= g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\right) && (3) \end{aligned}$$

Ricci Curvature

Definition

Let (M, g) be a smooth Riemannian manifold, and let ∇ be its Levi-Civita connection. We may define a tensor $Rc : T_pM \times T_pM \to \mathbb{R}$ as follows. Take any orthonormal basis $\{E_i\}_{i=1}^m$ of T_pM, and compute

$$Rc(X, Y) = \sum_{i=1} R(X, E_i, E_i, Y). \tag{4}$$

Scalar Curvature

Definition

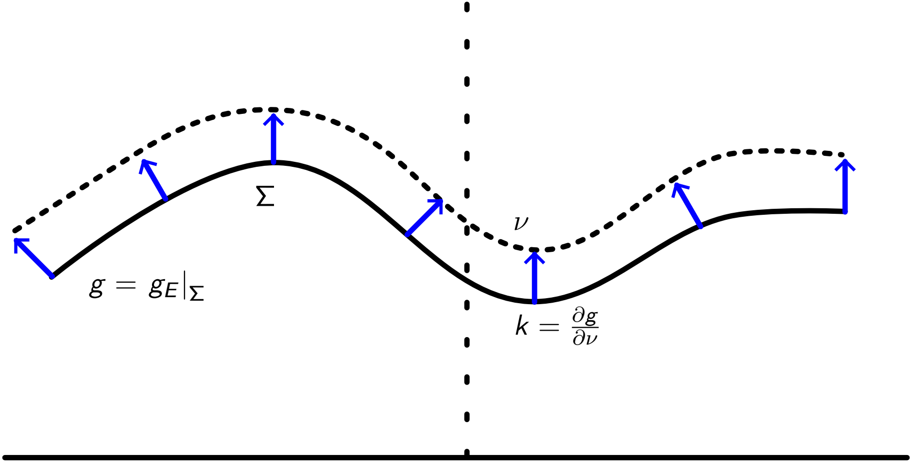
Let (M, g) be a smooth Riemannian manifold, and let ∇ be its Levi-Civita connection. We define the scalar curvature $S : M \to \mathbb{R}$ as follows. For each $p \in M$ let $\{E_i\}_{i=1}^m$ be an orthonormal basis for T_pM. Then, we have $S(p) = \sum_i Rc(E_i, E_i)$.

Second Fundamental Form

Definition

Let (M, g) be a smooth m-dimesnional Riemannian manifold, let $\Sigma \subset M$ be a smooth embedded oriented $m-1$ dimensional submanifold of M, and let ν be a normal vector field to M. Then, we may define a tensor $k : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ as follows

$$k(X, Y) = g\left(\nabla_X \nu, Y\right). \tag{5}$$


Σ
ν
$g = g_E|_\Sigma$
$k = \frac{\partial g}{\partial \nu}$

Mean Curvature

Definition

Let (M, g) be a smooth m-dimesnional Riemannian manifold, let $\Sigma \subset M$ be a smooth embedded oriented $m-1$ dimensional submanifold of M, and let k be the second fundamental form of Σ. Then, we define the mean-curvature of Σ to be the function $H_\Sigma(p) = \sum_i g\left(\nabla_{E_i}\nu, E_i\right)$, where $\{E_i\}_{i=1}^{m-1}$ is any orthonormal basis of $T_p\Sigma$.

Covariant Derivative of Forms

Definition

Let (M, g) be a smooth Riemannian manifold, let X be a smooth vector field, and ω a smooth covector field, and let ∇ be the Levi-Civita connection on (M, g). Then, define $\nabla_X \omega$ to be the unique smooth one-form such that for all vector fields Y on M we have

$$(\nabla_X \omega)(Y) : X(\omega(Y)) - \omega(\nabla_X Y). \tag{6}$$

Second Fundamental Form for Level Sets

Lemma

Let (M, g) *be a smooth Riemannian manifold, and let* $f : M \to \mathbb{S}$ *be a smooth function. For convenience, denote* $f^* d\theta$ *simply by* df*. Then, we have*

$$k_\Sigma = |df|^{-1} \nabla df \big|_\Sigma \tag{7}$$

The Schoen-Yau trick

Lemma

Let (M, g) be a smooth m-dimensional Riemannian manifold, and let $\Sigma \subset M$ be an oriented embedded $m-1$ dimensional submanifold of M. Finally, denote the scalar curvature of (M, g) be S_M and the scalar curvature of the induced metric on Σ by S_Σ, and let ν be a unit normal vector field to Σ. Then, we have

$$Rc(\nu, \nu) = \frac{1}{2}\left(S_M - S_\Sigma + H_\Sigma^2 - |k_\Sigma|^2\right) \tag{8}$$

Divergence

Definition

Let (M, g) be a smooth Riemannian manifold, and let ∇ be its Levi-Civita connection. Given a smooth vector field X on M and $p \in M$ we define the divergence of X at p as follows. Take any orthonormal basis $\{E_i\}_{i=1}^m$ of T_pM and compute

$$\sum_{i=1}^{m} g\left(\nabla_{E_i} X, E_i\right). \tag{9}$$

The Laplacian

Definition

Let (M, g) be a smooth Riemannian manifold, let ∇ be its Levi-Civita connection, and let $f : M \to \mathbb{R}$ be a smooth function. Then, the Laplacian of f is

$$\Delta f = \mathrm{div}(\nabla f). \tag{10}$$

Exercise

Lemma

Let (M, g) be a smooth closed Riemannian manifold, and let vol_g denote its volume form. Then, we have that

$$d(df \lrcorner \mathrm{vol}_g) = \Delta f \cdot \mathrm{vol}_g. \tag{11}$$

Definition

Let (M, g) be a smooth Riemannian manifold, let ∇ be its Levi-Civita connection. Then, given a smooth one-form ω we can calculate its divergence at a point $p \in M$ as follows. Take any orthonormal basis $\{E_i\}_{i=1}^m$ of T_pM and calculate

$$\sum_{i=1}(\nabla_{E_i}\omega)(E_i) = \sum_{i=1}\Big(E_i\left(\omega(E_i)\right) - \omega(\nabla_{E_i}E_i)\Big) \tag{12}$$

Lemma

Let (M, g) be a closed smooth Riemannian manifold, and let ω be a smooth one-form on M. Then, we have that

$$\delta\omega = -\mathrm{div}_g(\omega). \tag{13}$$

Recall

A closed form ω is harmonic if and only if $\delta\omega = 0$.

Bochner Identity

Lemma

Let (M, g) be a smooth Riemannian manifold, and let h be an harmonic one-form. Then, we have that

$$\Delta \frac{1}{2}|h|_g^2 = |\nabla h|_g^2 + Rc(h^\sharp, h^\sharp) \tag{14}$$

In Euclidean Space

Proof.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an harmonic function. Then, we calculate

$$\Delta|\nabla f|^2 = \sum_k \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^k} \sum_i \left(\frac{\partial f}{\partial x^i} \right)^2 \tag{15}$$

$$= 2 \sum_{ki} \left(\frac{\partial^2 f}{\partial x^k \partial x^i} \right)^2 + 2 \sum_{ki} \frac{\partial f}{\partial x^i} \frac{\partial^3 f}{\partial x^k \partial x^k \partial x^i} \tag{16}$$

□

Lemma

Let ω be an harmonic representative of an element in $H^1(M;\mathbb{Z})_{\mathbb{R}}$. Let $u : M \to \mathbb{S}$ be the map it generates, and let $d\theta$ be the standard 1-form on $\mathbb{S}$. Then $\omega = u^* d\theta$. Furthermore, for any $\theta_0 \in \mathbb{S}$ which is a regular value of u we have that $\omega^\sharp$ is orthogonal to $u^{-1}\{\theta_0\}$ and nowhere zero.

Proof.

Let V be a vector in $u^{-1}\{\theta_0\}$, and observe that by definition

$$g(V,\omega^\sharp) = \omega(V) = d\theta(Tu(V)) = d\theta(0) = 0. \tag{17}$$

$\omega^\sharp$ doesn't vanish on $u^{-1}\{\theta_0\}$ precisely because θ_0 is assumed to be a regular value of u. $\square$

Lemma

Let (M, g) be a smooth closed oriented Riemannian manifold, let ω be an harmonic one-form such that $[\omega] \in H^1(M; \mathbb{Z})_{\mathbb{R}}$, and let $u : M \to \mathbb{S}$ be the map it generates. Finally, let θ_0 be a regular value of u. Then, on the embedded submanifold $u^{-1}\{\theta_0\}$ we have

$$\Delta \frac{1}{2}|\omega|_g^2 = |\nabla\omega|^2 + |\omega|^2 Rc(\nu, \nu) \tag{18}$$

Notation

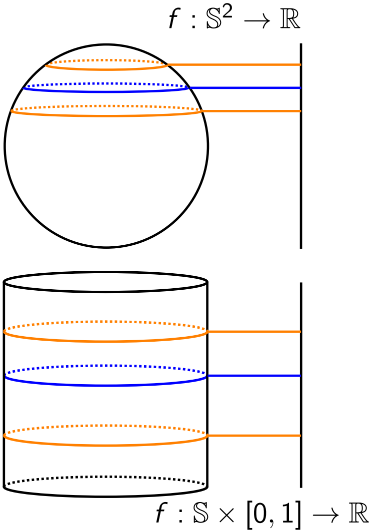
Definition

Let (M, g) be a smooth Riemannian manifold, and let $A \subset M$ be a subset of M. Then, we set $\mathcal{H}^k_g(A)$ to be the k-dimensional Hausdorff measure of A. If A is an embedded and oriented k-dimensional submanifold of M, then this is just the k-area of A.

Notation

To save space, we will sometimes write $|A|_{k,g}$ to denote $\mathcal{H}^k_g(A)$.

The Idea



The Coarea Formula

Theorem

Let (M, g) *be a smooth m-dimensional Riemannian manifold, and let* $f : M \to \mathbb{R}$ *be a smooth (actually Lipschitz is enough) function. Then, we have that*

$$\int_M |\nabla f|_g \mathrm{vol}_g = \int_{-\infty}^{\infty} \mathcal{H}^{m-1}\left(f^{-1}\{t\}\right) dt. \tag{19}$$

Intuition

Why it might be true

Suppose that for $f : M \to \mathbb{R}$ we have that $\frac{\nabla f}{|\nabla f|^2}$ is a smooth vector field, let $\mathrm{Fl} : M \times [0,1] \to M$ be its flow, and let $\Sigma_0 = f^{-1}\{0\}$. Then, $\mathrm{Fl}|_{\Sigma_0} : \Sigma_0 \times [0,1] \to M$ is a diffeomorphism, and $\mathrm{Fl}(\Sigma_0, t) = f^{-1}\{t\}$.

Stern's Inequality

Theorem

Let (M, g) be a closed 3-dimensional Riemannian manifold, and let $u : M \to \mathbb{S}$ be such that du is harmonic and $[du] \neq 0 \in H^1(M; \mathbb{Z})_{\mathbb{R}}$. Then, we have that

$$\int_{\mathbb{S}^1} \chi(u^{-1}\{\theta\}) d\theta \geq \frac{1}{2} \int_M S_M |du| + \frac{|\nabla du|^2}{|du|}. \tag{20}$$

Proof.

$$\Delta\frac{1}{2}|du|_g^2 = |\nabla du|^2 + |du|^2(S_M - R_\Sigma + H_\Sigma^2 - |k_\Sigma|^2) \tag{21}$$

We know that $k_\Sigma = |du|^{-1}\nabla du$:

$$|h|^2|k_\Sigma|^2 = |\nabla du|^2 - 2|d|du||^2 + \nabla du(\nu,\nu)^2 \tag{22}$$

$\square$

Proof.

From definitions, we have

$$|du|H_\Sigma = \sum_i^{m-1} g(\nabla_{E_i} du, E_i) = \mathrm{div}_g(du) - \nabla du(\nu,\nu) = -\nabla du(\nu,\nu). \quad (23)$$

So $|du|^2(H_\Sigma^2 - |k_\Sigma|^2) = 2|d|du||^2 - |\nabla du|^2$. Putting everything together gives

$$2Rc(du,du) = |du|^2(S_M - S_\Sigma) + 2|d|du||^2 - |\nabla du|^2 \quad (24)$$

$\square$

Proof.

$$\Delta|du| = \frac{1}{|du|}\Delta|du|^2 - |du|^{-2}g(d|du|, du). \tag{25}$$

$$\Delta|du| = \frac{1}{|du|}\left(|\nabla du|^2 + \frac{|du|^2}{2}(S_M - S_R) + |d|du||^2 - \frac{1}{2}|\nabla du|^2\right) \tag{26}$$
$$- |du|^{-2}g(d|du|, du) \tag{27}$$

$\square$

Proof.

Using Cauchy-Schwarz we get

$$\Delta|du| \geq \frac{|\nabla du|^2}{2|du|} + \frac{|du|}{2} S_M - \frac{|du|}{2} S_\Sigma. \tag{28}$$

Integrating gives

$$0 \geq \int_M \frac{|\nabla du|^2}{2|du|} + \frac{|du|}{2} S_M - \frac{|du|}{2} S_\Sigma. \tag{29}$$

Proof.

From the coarea formula, we have that

$$\int_M |du| S_\Sigma = \int_{\mathbb{S}^1} \int_{u^{-1}\{\theta\}} S_\Sigma. \tag{30}$$

This is equal to

$$\int_{\mathbb{S}^1} 2\chi\left(u^{-1}\{\theta\}\right). \tag{31}$$

$\square$

Frame Title

Lemma

Let (M, g) *be a closed smooth Riemannian manifold, and let* $\omega \in \Omega^k(M)$ *be a smooth* k*-form. Then, we have that* $\pm\delta = \star d \star$.

Proof.

A direct calculation. □

Corollary for Tori

Lemma

Let g be a Riemannian metric on $\mathbb{T}^3$, and let $u : \mathbb{T}^3 \to \mathbb{S}$ be a map such that du is an harmonic one-form, and $[du] \neq 0 \in H^1(M; \mathbb{Z})_{\mathbb{R}}$. Then, we have that

$$-\int_{\mathbb{T}^3} S|du| \geq \int_{\mathbb{T}^3} \frac{|\nabla du|^2}{|du|}. \tag{32}$$

Proof.

Since we have

$$2\int_{\mathbb{S}} \chi(u^{-1}\{\theta\})d\theta \geq \int_{\mathbb{T}^3} \frac{|\nabla du|^2}{|du|} + S|du|. \tag{33}$$

This comes down to estimating $\chi(u^{-1}\{\theta\})$.

Proof.

Consider the 2−form $\star du$, which on $u^{-1}\{\theta\}$ is $|du|A_{u^{-1}\{\theta\}}$. Thus, we see that

$$\int_{u^{-1}\{\theta\}} \star du > 0. \tag{34}$$

On the otherhand, since du is harmonic, we have $d \star du = 0$. Therefore, if $u^{-1}\{\theta\} = \partial\Omega$, then, we would have

$$\int_{u^{-1}\{\theta\}} \star du = 0. \tag{35}$$