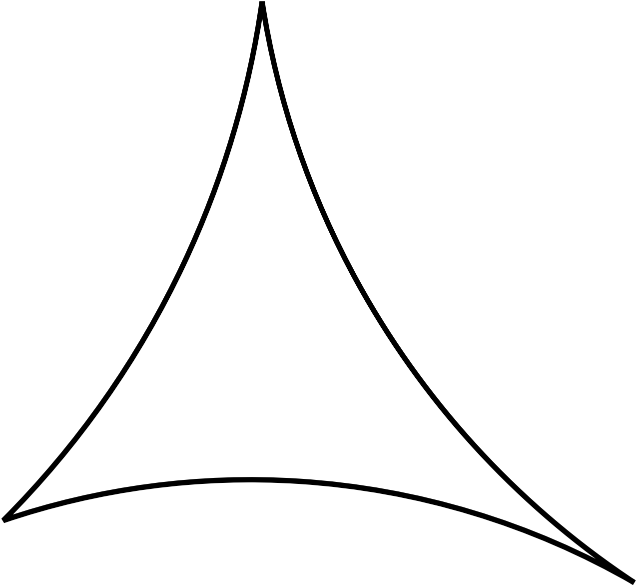
Proof. Let $\triangle ABC$, $\triangle A'B'C'$ be triangles in hyperbolic geometry such that $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. In particular, $d(\triangle ABC) = d(\triangle A'B"C')$. If any pair of sides of these two triangles are congruent, namely $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, or $\overline{BC} \cong \overline{B'C'}$, then $\triangle ABC \cong \triangle A'B'C'$ by ASA. For the sake of contradiction, we will suppose that no such pair of sides of $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Then one of the triangles has at least two sides longer than their correspondents in the other triangle. Without the loss of generality, we may suppose that $\overline{AB} > \overline{A'B'}$ and $\overline{AC} > \overline{A'C'}$. Then there exists points $A - B'' - B$ and $A - C'' - C$ such that $\overline{A'B'} \cong \overline{AB''}$ and $\overline{A'C'} \cong \overline{AC''}$. Therefore, $\triangle AB''C'' \cong \triangle A'B'C'$ by SAS. In particular, $d(\triangle AB''C'') = d(\triangle A'B'C') = d(\triangle ABC)$.

- A’

•

C’

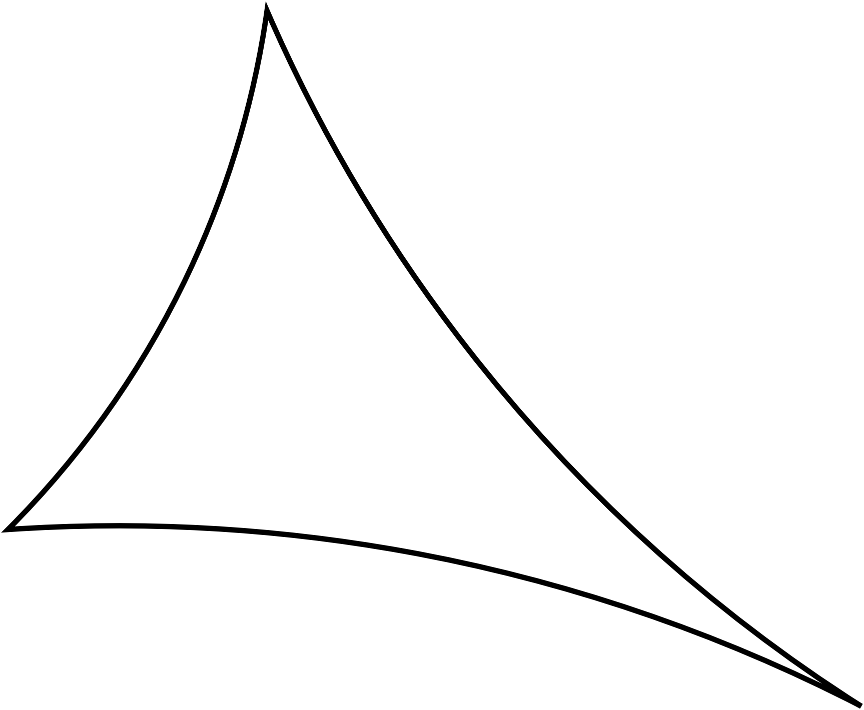
- **B’**

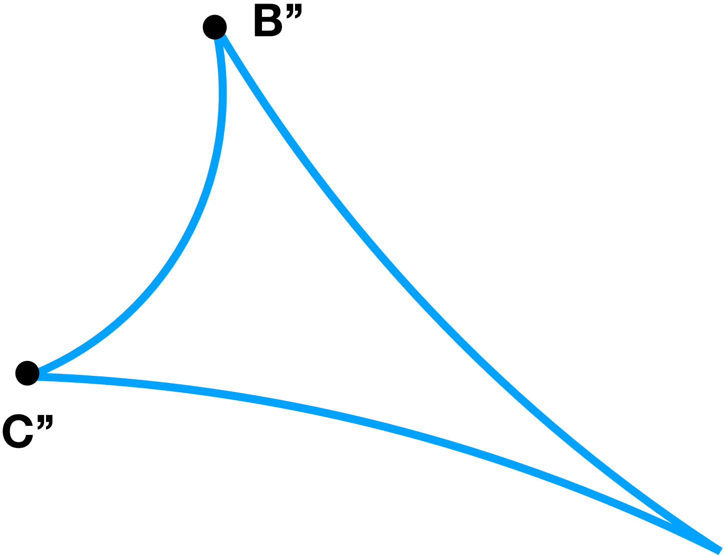


•

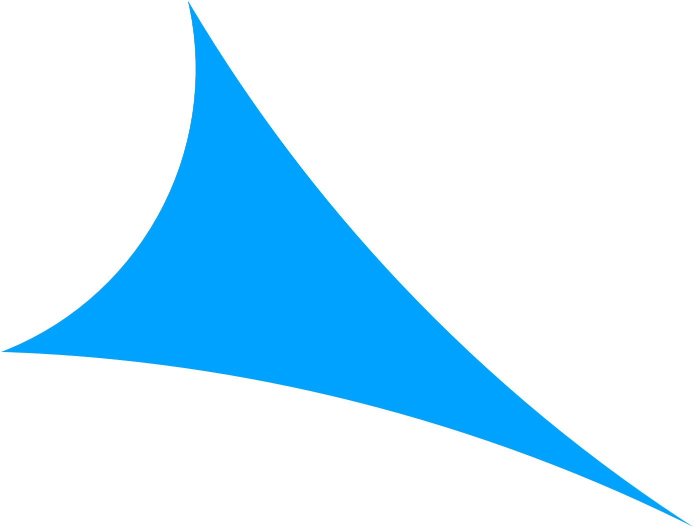
c

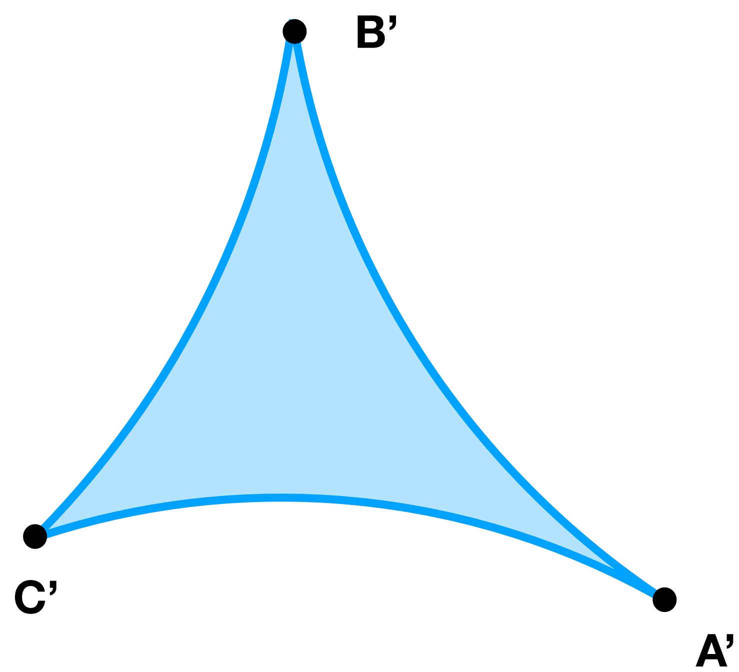
• B

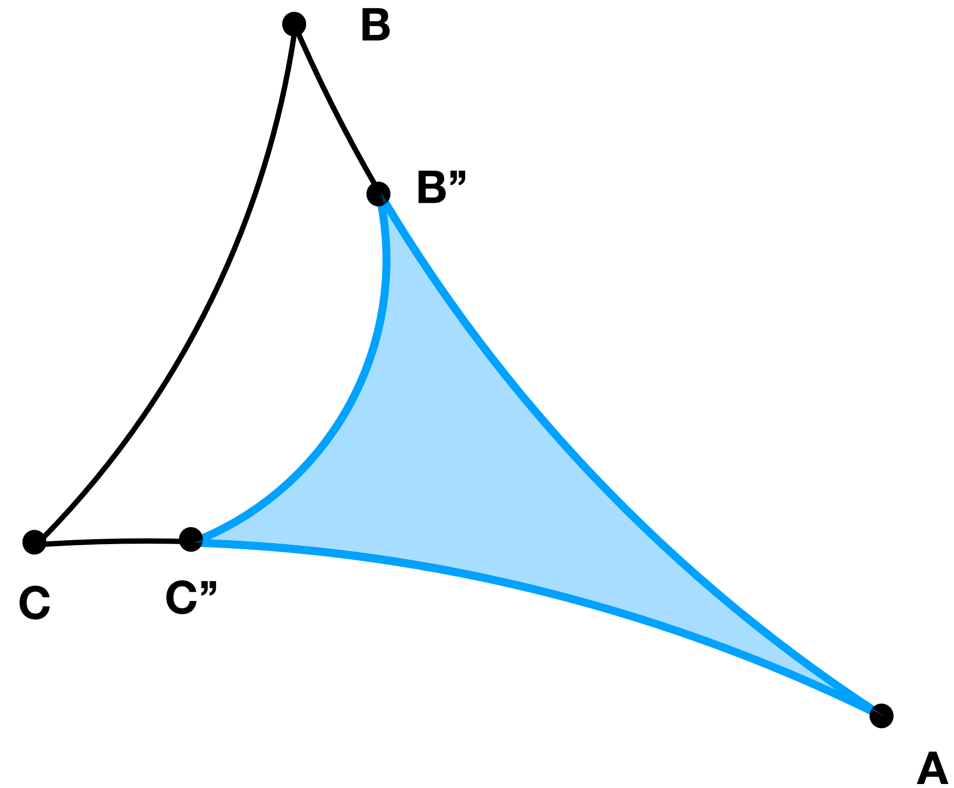



B”
C”

A







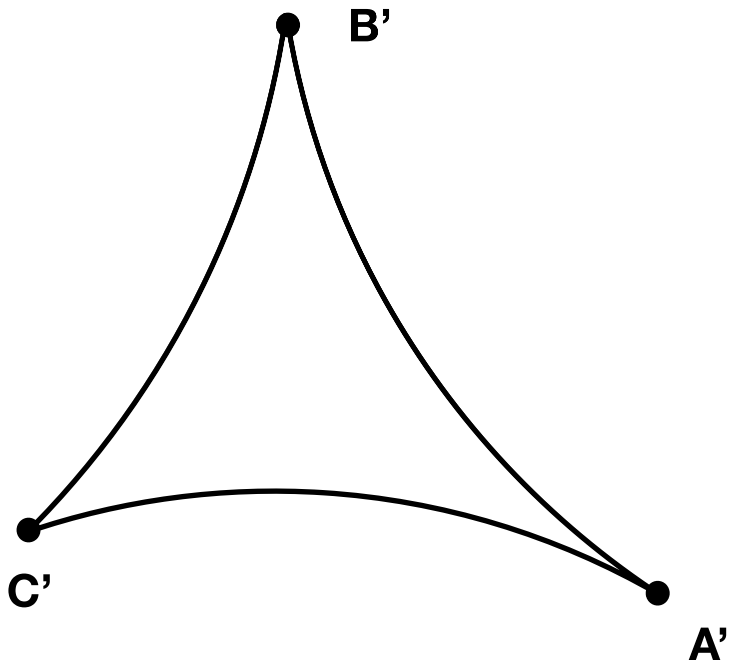
By the convexity of $\triangle ABC$, the triangle can be partitioned into the union of three triangles, namely

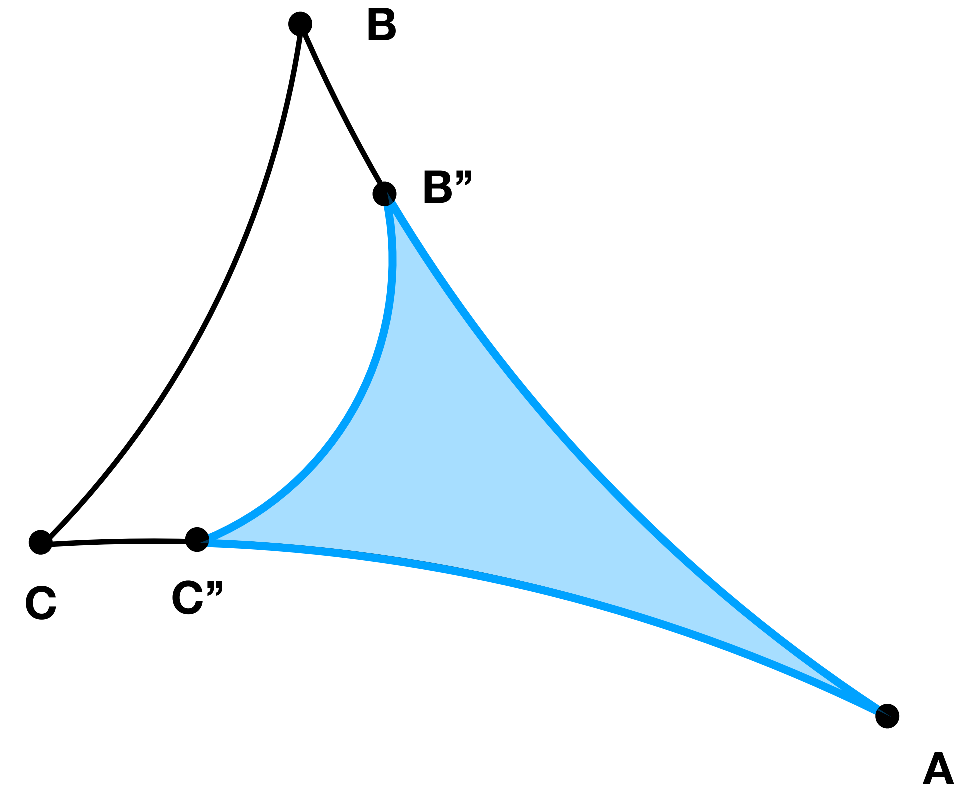
$$\triangle ABC = \triangle AB''C'' \cup \triangle B''C''B \cup \triangle BCC''.$$

The additivity of defects implies that

$$d(\triangle ABC) = d(\triangle AB''C'') + d(\triangle B''C''B) + d(\triangle BCC) = d(\triangle ABC) + d(\triangle B''C''B) + d(\triangle BCC)$$
$$\Rightarrow 0 = d(\triangle B''C''B) + d(\triangle BCC).$$

Given that defects are always positive, the above equation leads to a contradiction. $\square$





Proof. Let $\triangle ABC$, $\triangle A'B'C'$ be triangles in hyperbolic geometry such that $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. In particular, $d(\triangle ABC) = d(\triangle A'B"C')$. If any pair of sides of these two triangles are congruent, namely $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, or $\overline{BC} \cong \overline{B'C'}$, then $\triangle ABC \cong \triangle A'B'C'$ by ASA. For the sake of contradiction, we will suppose that no such pair of sides of $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Then one of the triangles has at least two sides longer than their correspondents in the other triangle. Without the loss of generality, we may suppose that $\overline{AB} > \overline{A'B'}$ and $\overline{AC} > \overline{A'C'}$. Then there exists points $A - B'' - B$ and $A - C'' - C$ such that $\overline{A'B'} \cong \overline{AB''}$ and $\overline{A'C'} \cong \overline{AC''}$. Therefore, $\triangle AB''C'' \cong \triangle A'B'C'$ by SAS. In particular, $d(\triangle AB''C'') = d(\triangle A'B'C') = d(\triangle ABC)$.