Theorem 3.3.5

Theorem 3.3.5 (Midpoint Theorem). *Every segment has a unique midpoint.*

Proof. Consider the segment $\overline{BC}$. Let A be a point off the line $\overleftrightarrow{BC}$. Let D be a point such that $A - C - D$. Then $\angle DCB$ is an exterior angle to the triangle $\triangle ABC$ with remote interior angle $\angle B$. Then by the Exterior Angle Theorem, $\angle DCB > \angle B$. Thus, there exists a point $E \in \angle^\circ DCB$ such that $\angle B \cong ECB$. By segment translation we may assume that $\overline{CE} \cong \overline{AB}$.

Since $E \in \angle^\circ DCB$, we know that $E \in H^\circ(\overleftrightarrow{BC}, D)$. On the other hand, A and D are on opposite sides of $\overleftrightarrow{BC}$ since $A - C - D$. So E and A are on opposite sides of $\overleftrightarrow{BC}$ by Plane Separation. Let $F \in \overline{AE} \cap \overleftrightarrow{BC}$. In particular, $A - F - E$.

We also have that $E \in H^\circ(\overleftrightarrow{CD}, B) = H^\circ(\overleftrightarrow{AC}, B)$, since $E \in \angle^\circ DCB$. Since $\angle ABC$ and $\angle ECB$ are congruent alternate interior angles to the pair of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$, we conclude that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$ by the Alternate Interior Angle Theorem. In particular $\overline{AB} \parallel \overline{CE}$. So, $E \in H^\circ(\overleftrightarrow{AB}, C)$, which shows that $E \in \angle^\circ BAC$. The Between-Cross Lemma then shows that $B - F - C$.

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