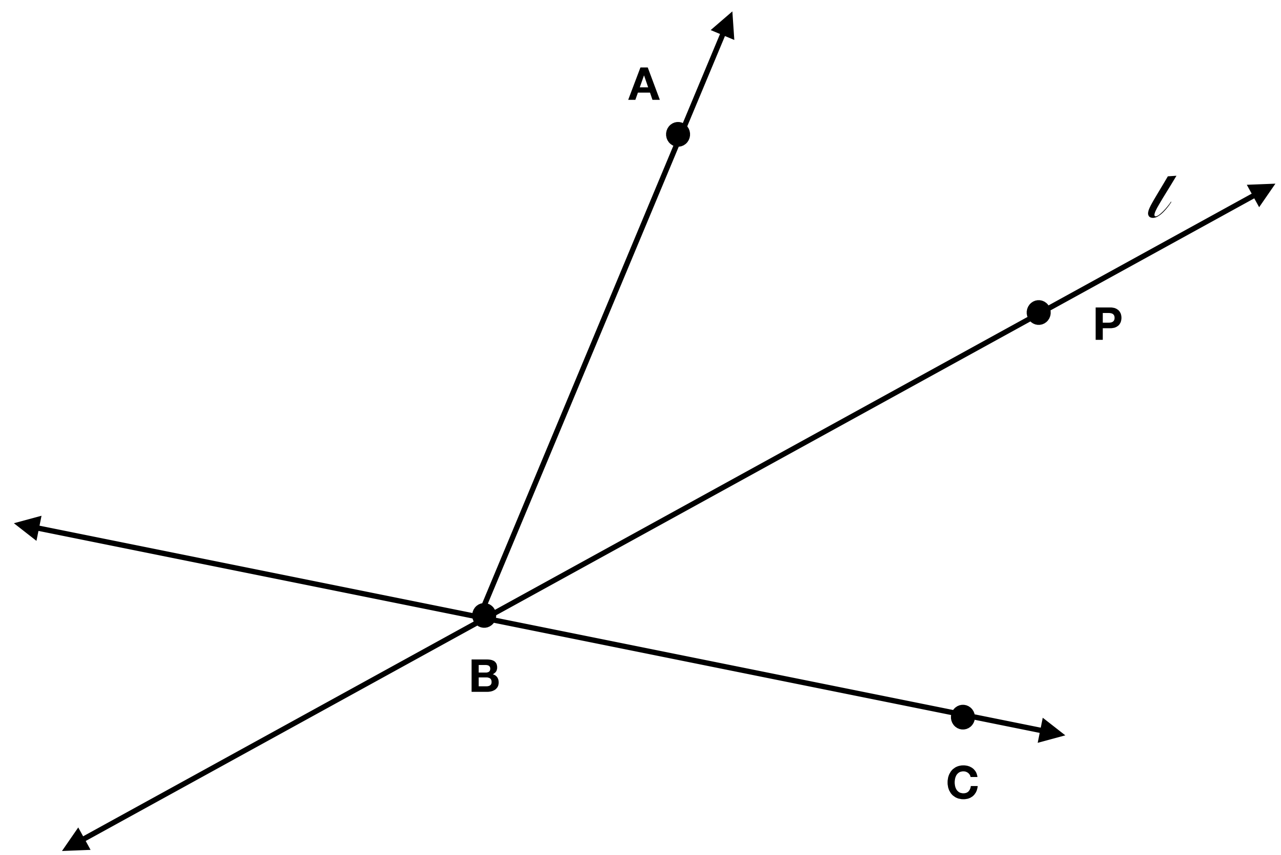
Theorem 2.4.19

Theorem 2.4.19 (Crossbar Theorem). *If a ray $\overrightarrow{BP}$ in an ordered geometry is between two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, then $\overrightarrow{BP}$ intersects $\overline{AC}$ at D between A and C.*



Proof. By Extension, there exists a point C' such that $C - B - C'$. By construction, C and C' are on opposite sides of $\ell = \overleftrightarrow{BP}$. By the previous theorem, $A \in \angle^{(]} PBC'$. Therefore, $\overline{AC'} \subseteq \angle^{(]} PBC'$ (a fact left to be proven by the reader). Therefore, $\overline{AC'}$ does not intersect ℓ. This shows that A and C' are on the same side of ℓ. By Plane Separation, we conclude that A and C are on opposite sides of ℓ. Thus $\overline{AC}$ intersects ℓ at a point D. Since A, B, and C are non-collinear, we know that B is not between A and C. In particular, $B \neq D$. The opposite ray to $\overrightarrow{BP}$ only intersects $\angle ABC$ at B, which means that it does not intersect D. Since ℓ is the union of $\overrightarrow{BP}$ and its opposite ray, we conclude that $\overrightarrow{BP}$ intersects $\overline{AC}$ at D. $\square$

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