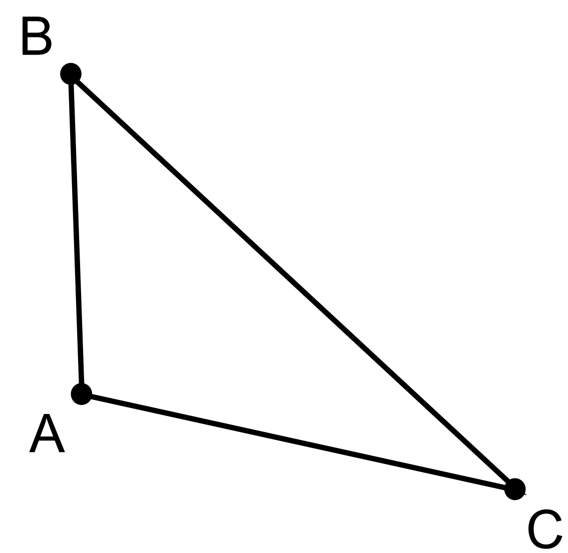
Lemma 3.5.3

Lemma 3.5.3. *In neutral geometry, for any* $\triangle ABC$ *there exists* $\triangle A_1B_1C_1$ *such that the two triangles have the same angle sums but* $m\angle A_1 \leq \frac{1}{2}(m\angle A)$.



Proof. Let D be the midpoint of $\overline{BC}$. Let E be the point such that $A-D-E$ and $\overline{AD} \cong \overline{DE}$. Since $\angle ADC \cong \angle EDB$ as vertical angles, we conclude that $\triangle ADC \cong \triangle EDB$ by SAS. In particular, $\angle CAD \cong \angle BED$, $\angle EBD \cong \angle ACD$. Let $\triangle A_1B_1C_1 = \triangle ABE$. Also,

$$\begin{aligned}
m\angle A + m\angle B + m\angle C &= m\angle BAC + m\angle ABC + m\angle BCA \\
&= (m\angle CAD + m\angle BAD) + m\angle ABC + m\angle BCA \\
&= m\angle BED + m\angle BAD + m\angle ABC + m\angle EBD \\
&= m\angle BAD + (m\angle ABC + m\angle EBD) + m\angle BED \\
&= m\angle BAD + m\angle ABE + m\angle BED \\
&= m\angle A_1 + m\angle B_2 + m\angle C_2
\end{aligned}$$

Therefore, $\triangle ABC$ and $\triangle A_1B_1C_1$ have the same angle sum. Since $m\angle A = m\angle A_1 + m\angle C_1$, where our angle measures are nonnegative real numbers, we have that $m\angle A_1 \leq \frac{1}{2}(\angle A)$ or $m\angle C_1 \leq \frac{1}{2}(\angle A)$. Switching the labels A_1 and C_1 if necessary, this finishes the proof. $\square$

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