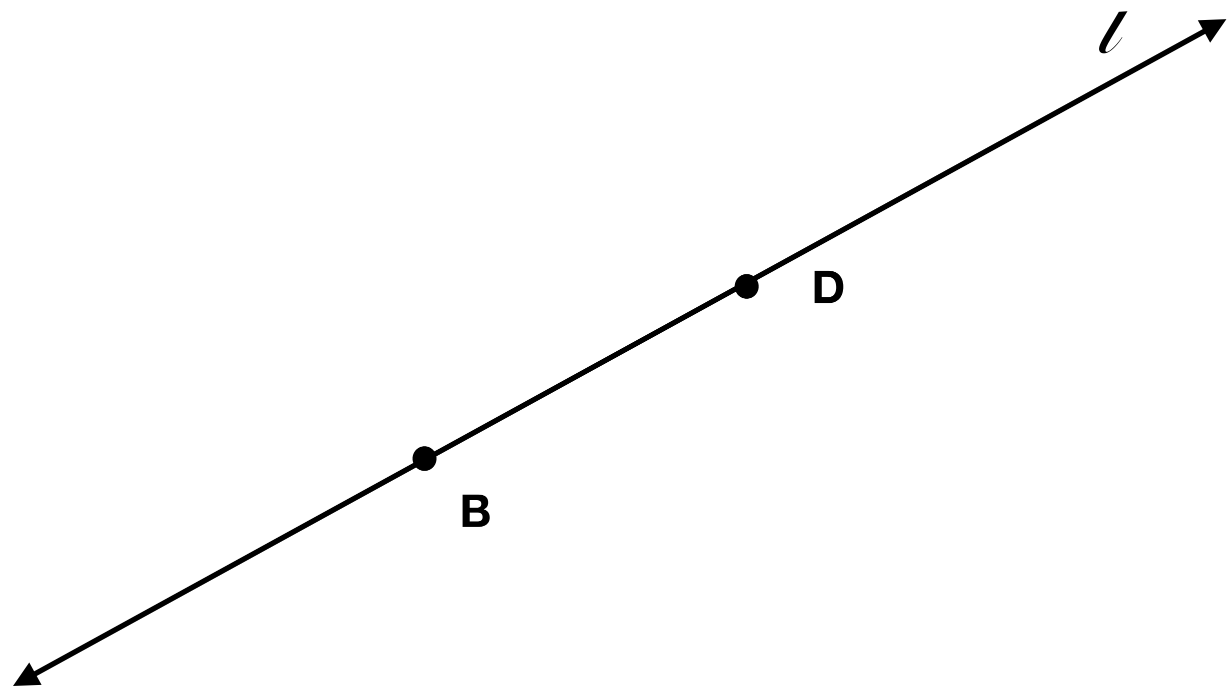
Theorem 2.4.9

Theorem 2.4.9 (Linear Density). *Given two distinct points B and D on a line ℓ in an ordered geometry, there exists points A, C, and E lying on ℓ such that $A - B - D$, $B - C - D$, and $B - D - E$.*



Proof. The existence of point E is immediate from the Extension Axiom. Likewise, the existence of A is also immediate from extension with the point B and D being switched.

Let F be a point not on ℓ and consider the line $m = \overleftrightarrow{BF}$. By Extension, there exists a point G on m such that $B - F - G$. Let $n = \overleftrightarrow{DG}$. By Extension, there exists a point H on n such that $G - D - H$. Finally, consider the line $\overleftrightarrow{FH}$. This lines intersects m at the unique point F. Thus, $B, G \notin \overleftrightarrow{FH}$. Also, $\overleftrightarrow{FH}$ intersects n at the unique point H. Thus, $D \notin \overleftrightarrow{FH}$. But as was mentioned earlier, $\overleftrightarrow{FH}$ intersects $\overline{BG}$ at F. Thus, Pasch's Axiom applies so that $\overleftrightarrow{FH}$ intersects $\overline{BD}$ or $\overline{GD}$. Again, $\overleftrightarrow{FH}$ intersects n at the unique point H, but $G - D - H$. So, $\overleftrightarrow{FH}$ does not intersect $\overline{DG}$. We conclude that $\overleftrightarrow{FH}$ intersects $\overline{BD}$ at a point called C. Therefore, $B - C - D$. $\square$

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