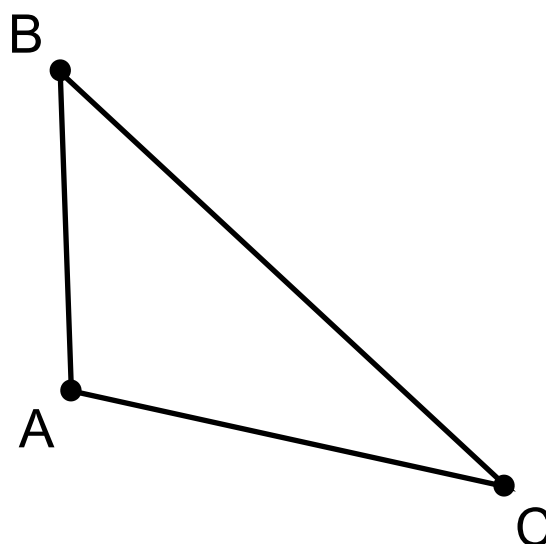




**Lemma 3.5.3**

In neutral geometry, for any  $\triangle ABC$  there exists  $\triangle A_1B_1C_1$  such that the two triangles have the same angle sums but  $m\angle A_1 \leq \frac{1}{2}(m\angle A)$ .





**Lemma 3.5.3.** *In neutral geometry, for any  $\triangle ABC$  there exists  $\triangle A_1B_1C_1$  such that the two triangles have the same angle sums but  $m\angle A_1 \leq \frac{1}{2}(m\angle A)$ .*

*Proof.* Let  $D$  be the midpoint of  $\overline{BC}$ . Let  $E$  be the point such that  $A - D - E$  and  $\overline{AD} \cong \overline{DE}$ . Since  $\angle ADC \cong \angle EDB$  as vertical angles, we conclude that  $\triangle ADC \cong \triangle EDB$  by SAS. In particular,  $\angle CAD \cong \angle BED$ ,  $\angle EBD \cong \angle ACD$ . Let  $\triangle A_1B_1C_1 = \triangle ABE$ . Also,

$$\begin{aligned}
 m\angle A + m\angle B + m\angle C &= m\angle BAC + m\angle ABC + m\angle BCA \\
 &= (m\angle CAD + m\angle BAD) + m\angle ABC + m\angle BCA \\
 &= m\angle BED + m\angle BAD + m\angle ABC + m\angle EBD \\
 &= m\angle BAD + (m\angle ABC + m\angle EBD) + m\angle BED \\
 &= m\angle BAD + m\angle ABE + m\angle BED \\
 &= m\angle A_1 + m\angle B_2 + m\angle C_2
 \end{aligned}$$

Therefore,  $\triangle ABC$  and  $\triangle A_1B_1C_1$  have the same angle sum. Since  $m\angle A = m\angle A_1 + m\angle C_1$ , where our angle measures are nonnegative real numbers, we have that  $m\angle A_1 \leq \frac{1}{2}(m\angle A)$  or  $m\angle C_1 \leq \frac{1}{2}(m\angle A)$ . Switching the labels  $A_1$  and  $C_1$  if necessary, this finishes the proof.  $\square$

# Lemma 3.5.3

In neutral geometry, for any  $\triangle ABC$  there exists  $\triangle A_1B_1C_1$  such that the two triangles have the same angle sums but  $m\angle A_1 \leq \frac{1}{2}(m\angle A)$ .