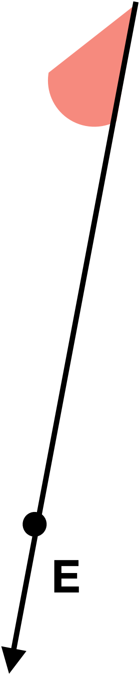
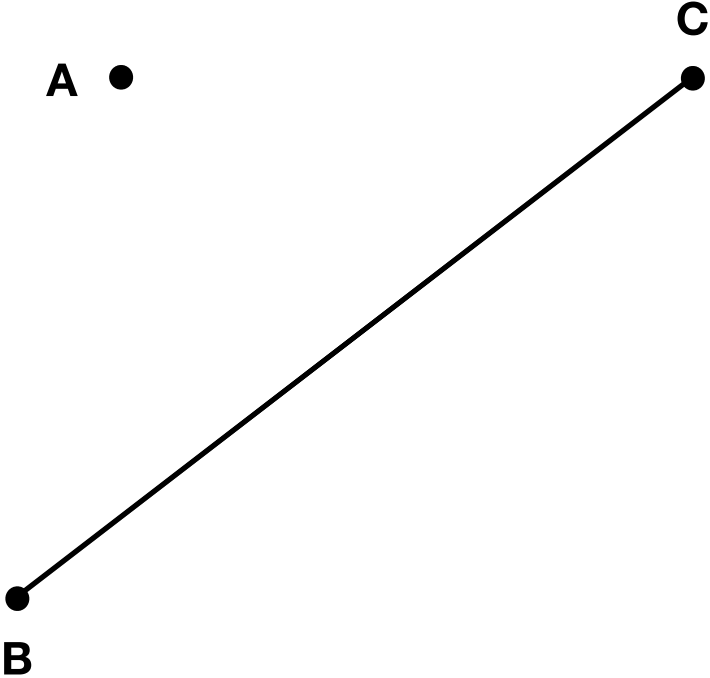
Proof. Consider the segment $\overline{BC}$. Let A be a point off the line $\overleftrightarrow{BC}$. Let D be a point such that $A - C - D$. Then $\angle DCB$ is an exterior angle to the triangle $\triangle ABC$ with remote interior angle $\angle B$. Then by the Exterior Angle Theorem, $\angle DCB > \angle B$. Thus, there exists a point $E \in \angle^\circ DCB$ such that $\angle B \cong ECB$. By segment translation we may assume that $\overline{CE} \cong \overline{AB}$.

Since $E \in \angle^\circ DCB$, we know that $E \in H^\circ(\overleftrightarrow{BC}, D)$. On the other hand, A and D are on opposite sides of $\overleftrightarrow{BC}$ since $A - C - D$. So E and A are on opposite sides of $\overleftrightarrow{BC}$ by Plane Separation. Let $F \in \overline{AE} \cap \overleftrightarrow{BC}$. In particular, $A - F - E$.

We also have that $E \in H^\circ(\overleftrightarrow{CD}, B) = H^\circ(\overleftrightarrow{AC}, B)$, since $E \in \angle^\circ DCB$. Since $\angle ABC$ and $\angle ECB$ are congruent alternate interior angles to the pair of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$, we conclude that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$ by the Alternate Interior Angle Theorem. In particular $\overline{AB} \parallel \overline{CE}$. So, $E \in H^\circ(\overleftrightarrow{AB}, C)$, which shows that $E \in \angle^\circ BAC$. The Between-Cross Lemma then shows that $B - F - C$.


D


E


A
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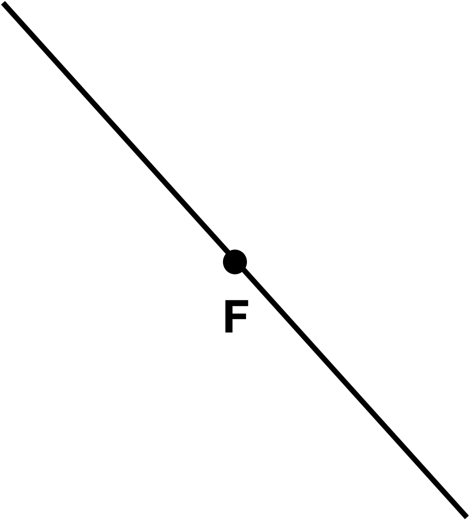
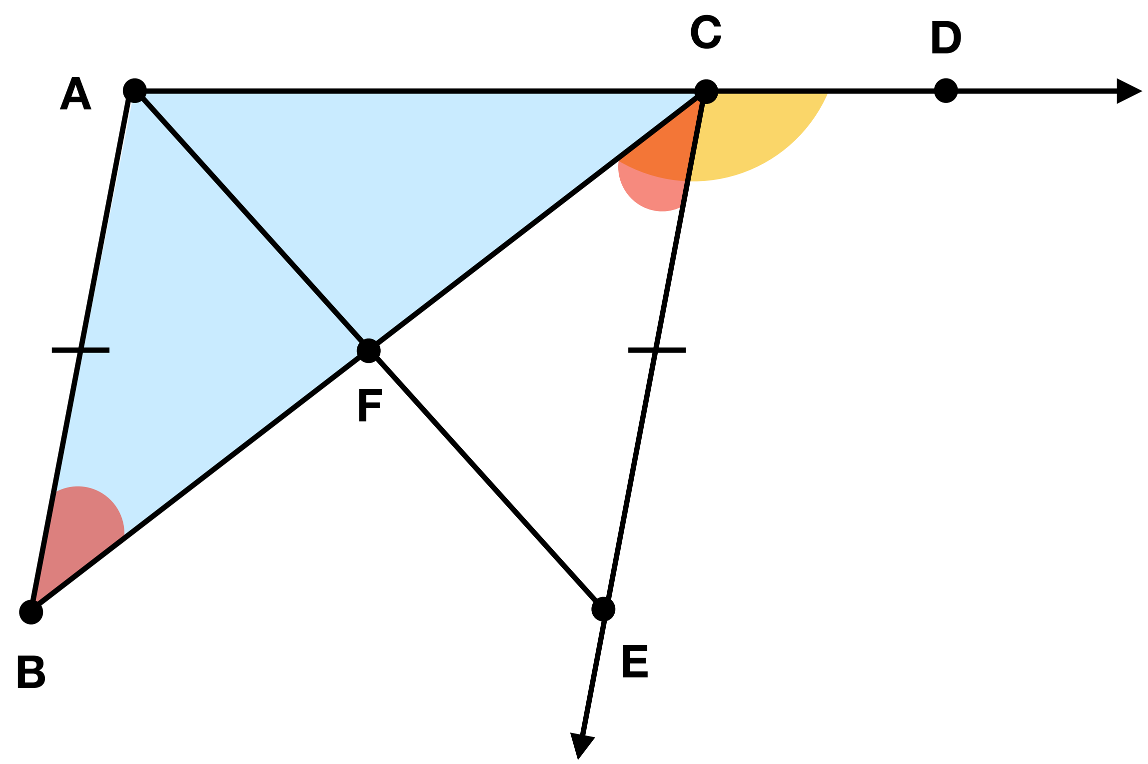
B


F

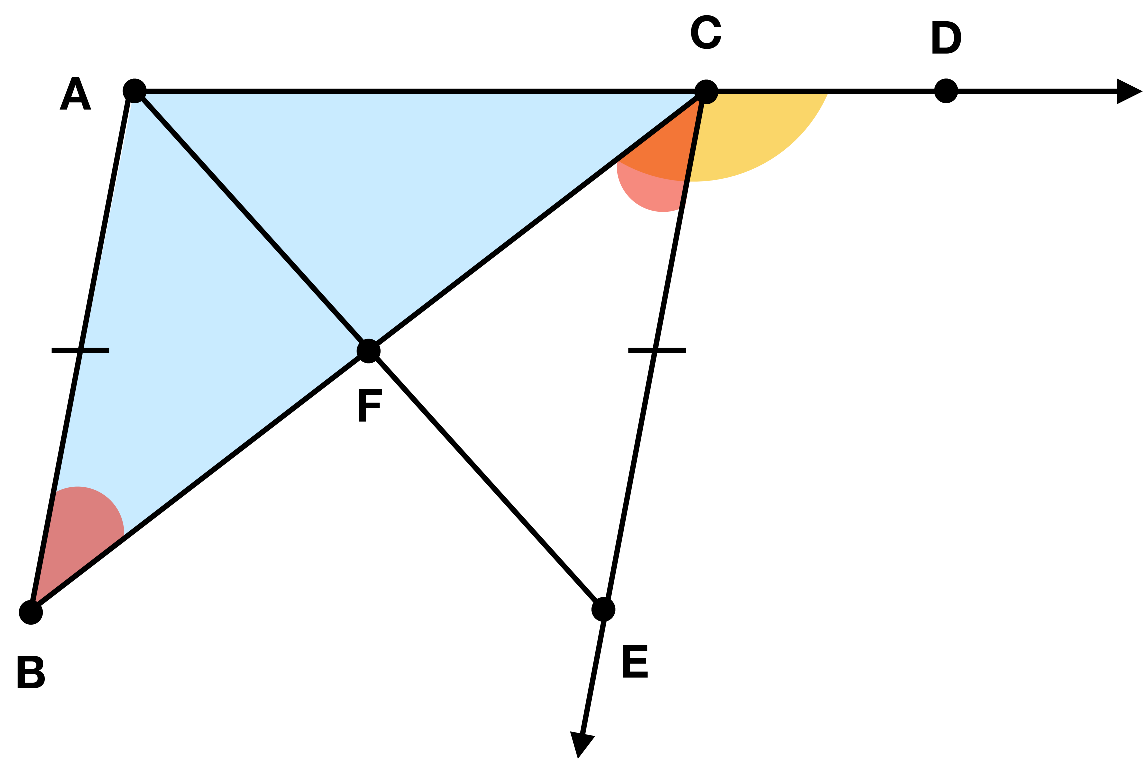


Finally, the angles $\angle AFB \cong \angle EFC$ since they are vertical angles (remember that $A - F - E$ and $B - F - C$). Therefore, $\triangle AFB \cong \triangle EFC$ by AAS. In particular, $\overline{BF} \cong \overline{FC}$. This provides the existence of midpoints.

Let B and B' be midpoints of AC, and let $\preceq$ be an ordering of $\overleftrightarrow{AC}$ such that $A \preceq C$. Without the loss of generality, we may assume that $B' \preceq B$. In particular, $A - B' - B - C$. Then

$$\overline{AB'} \leq \overline{AB} \cong \overline{BC} \leq \overline{B'C} \cong \overline{AB'}.$$

Hence, $\overline{AB'} \cong \overline{AB}$. But by the uniqueness of Segment Translation, we conclude that $B = B'$, that is, midpoints are unique. $\square$



Proof. Consider the segment $\overline{BC}$. Let A be a point off the line $\overleftrightarrow{BC}$. Let D be a point such that $A-C-D$. Then $\angle DCB$ is an exterior angle to the triangle $\triangle ABC$ with remote interior angle $\angle B$. Then by the Exterior Angle Theorem, $\angle DCB > \angle B$. Thus, there exists a point $E \in \angle^\circ DCB$ such that $\angle B \cong ECB$. By segment translation we may assume that $\overline{CE} \cong \overline{AB}$.

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