

Differential Geometry Surfaces

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Slides credits: Mira Ben-Chen (CS468 taught in 2012 at Stanford)

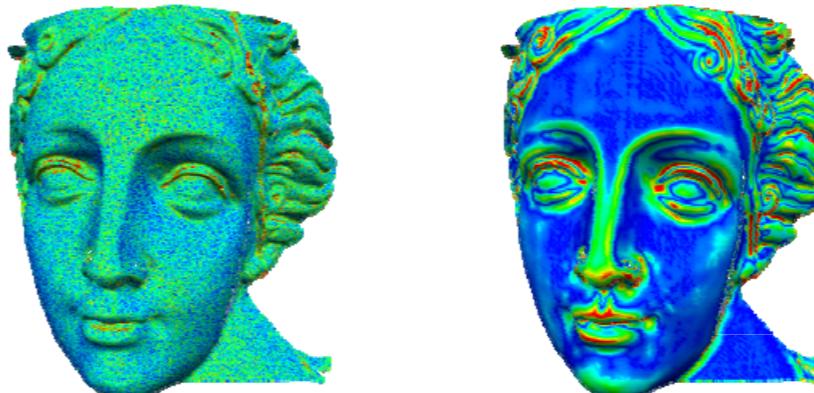
Motivation

- Understand the structure of the surface
 - Properties: smoothness, “curviness”, important directions
- How to modify the surface to change these properties
- What properties are preserved for different modifications
- The math behind the scenes for many geometry processing applications

Motivation

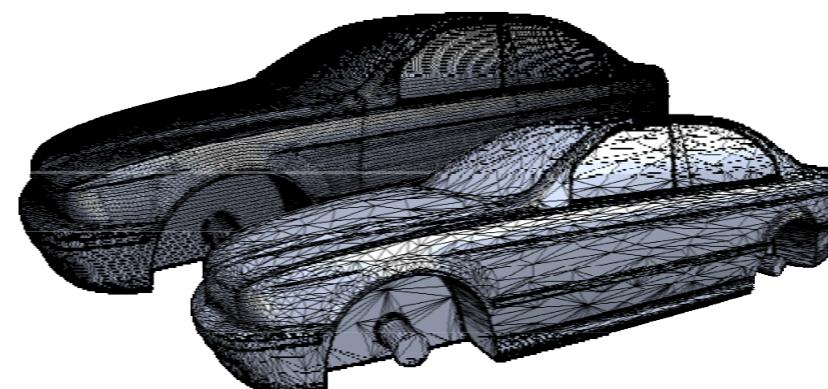
- Smoothness

→ Mesh smoothing

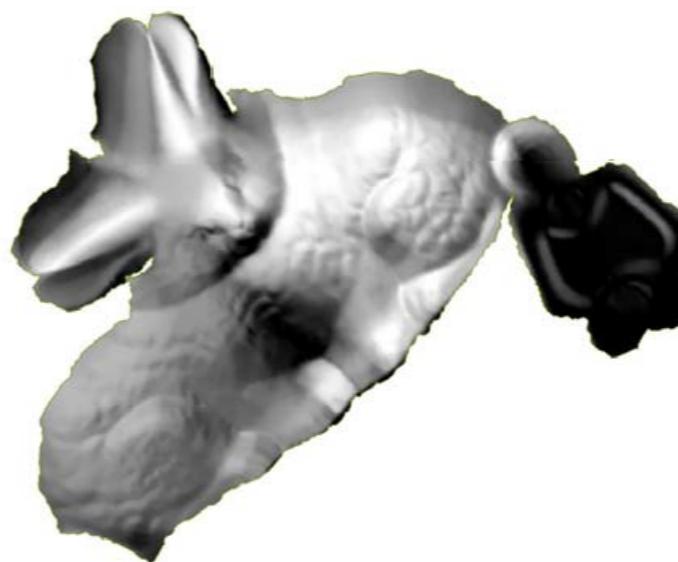


- Curvature

→ Adaptive simplification

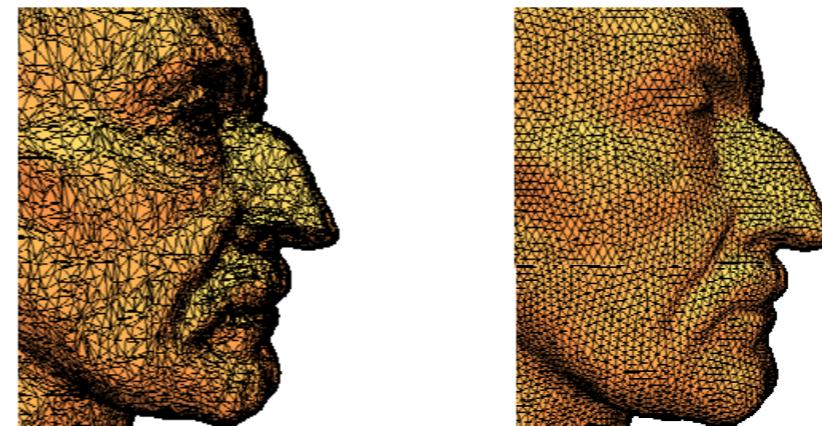


→ Parameterization

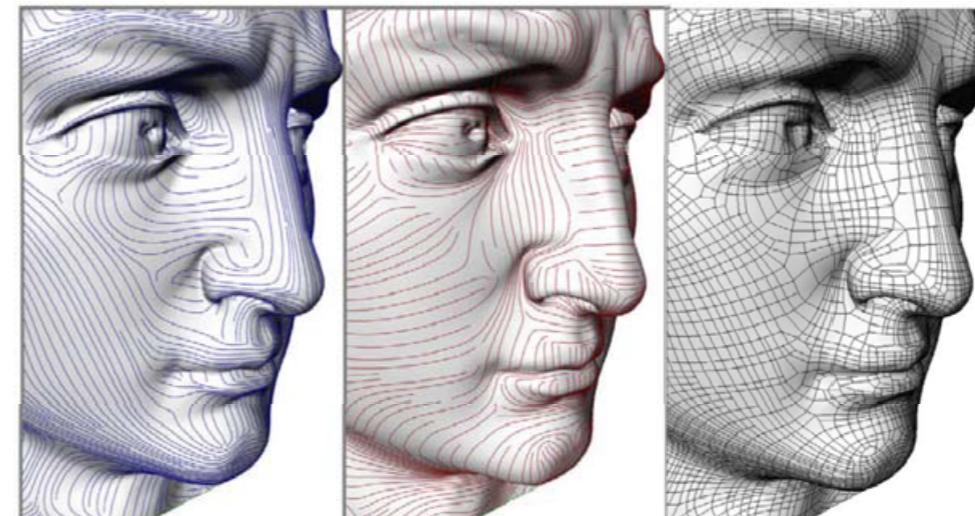


Motivation

- Triangle shape
 - Remeshing

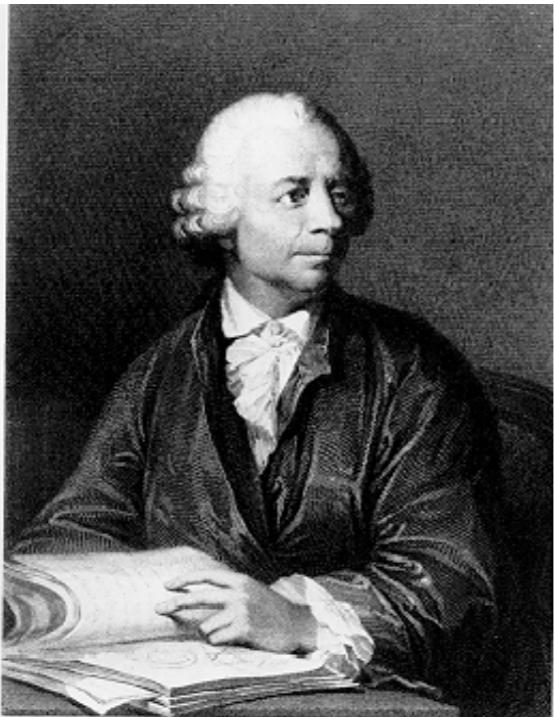


- Principal directions
 - Quad remeshing



Differential Geometry

- M.P. do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976



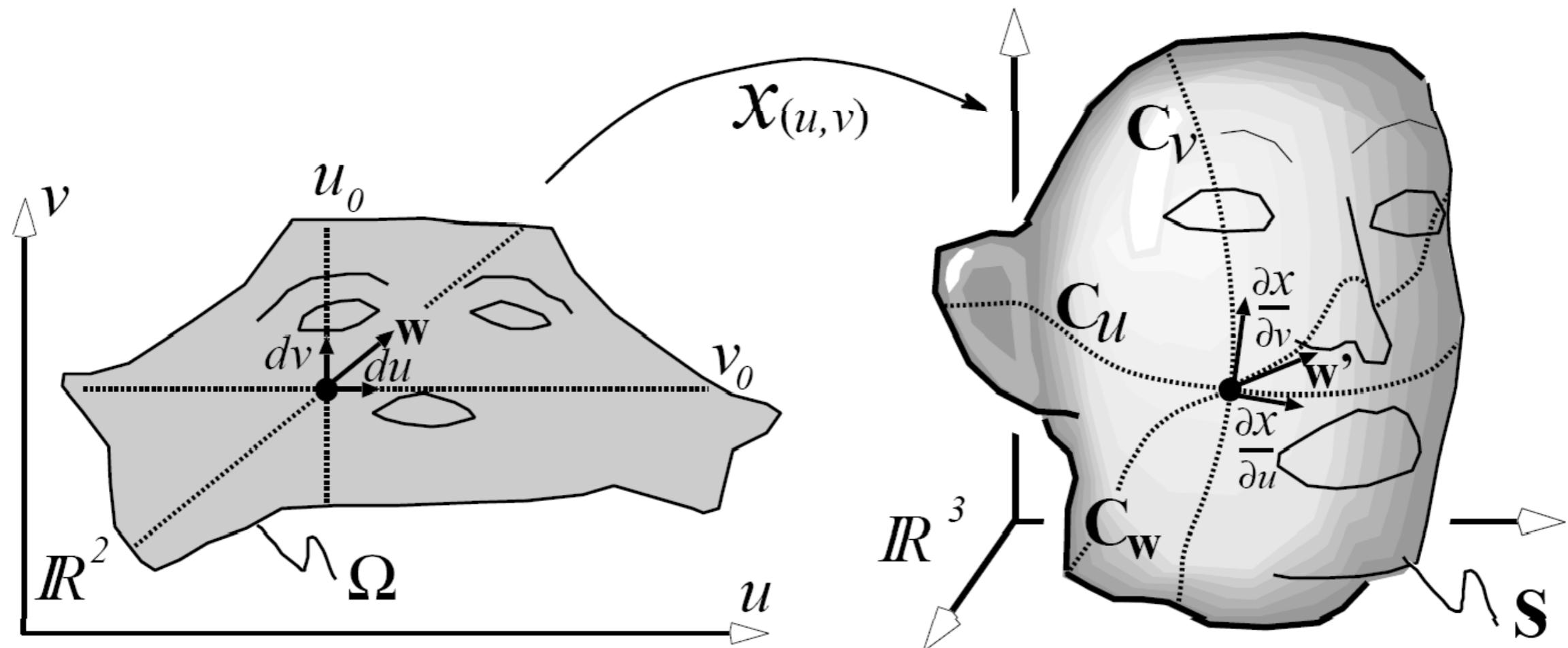
Leonard Euler (1707 - 1783)



Carl Friedrich Gauss (1777 - 1855)

Differential Geometry: Surfaces

$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, (u, v) \in \mathbb{R}^2$$



Differential Geometry: Surfaces

- Continuous surface

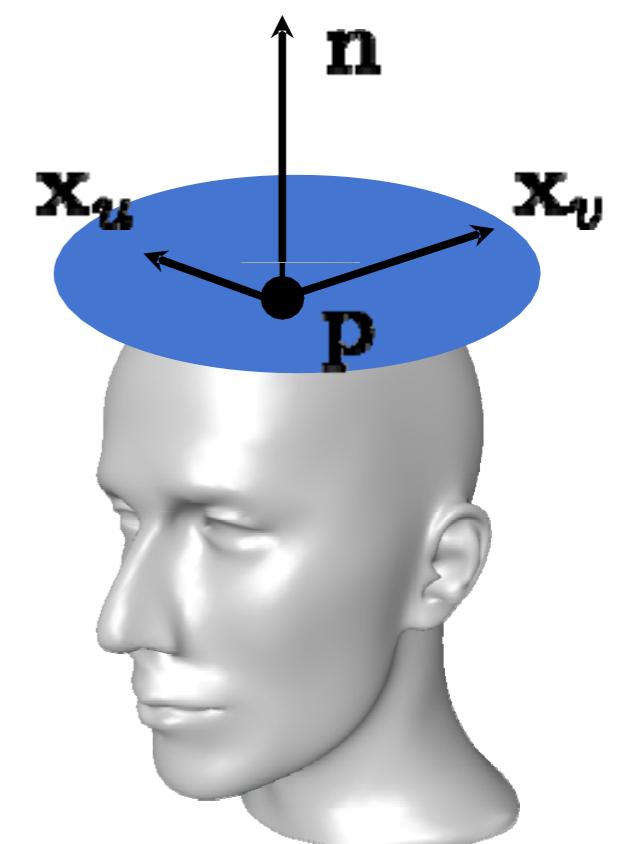
$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, (u, v) \in \mathbb{R}^2$$

- Normal vector

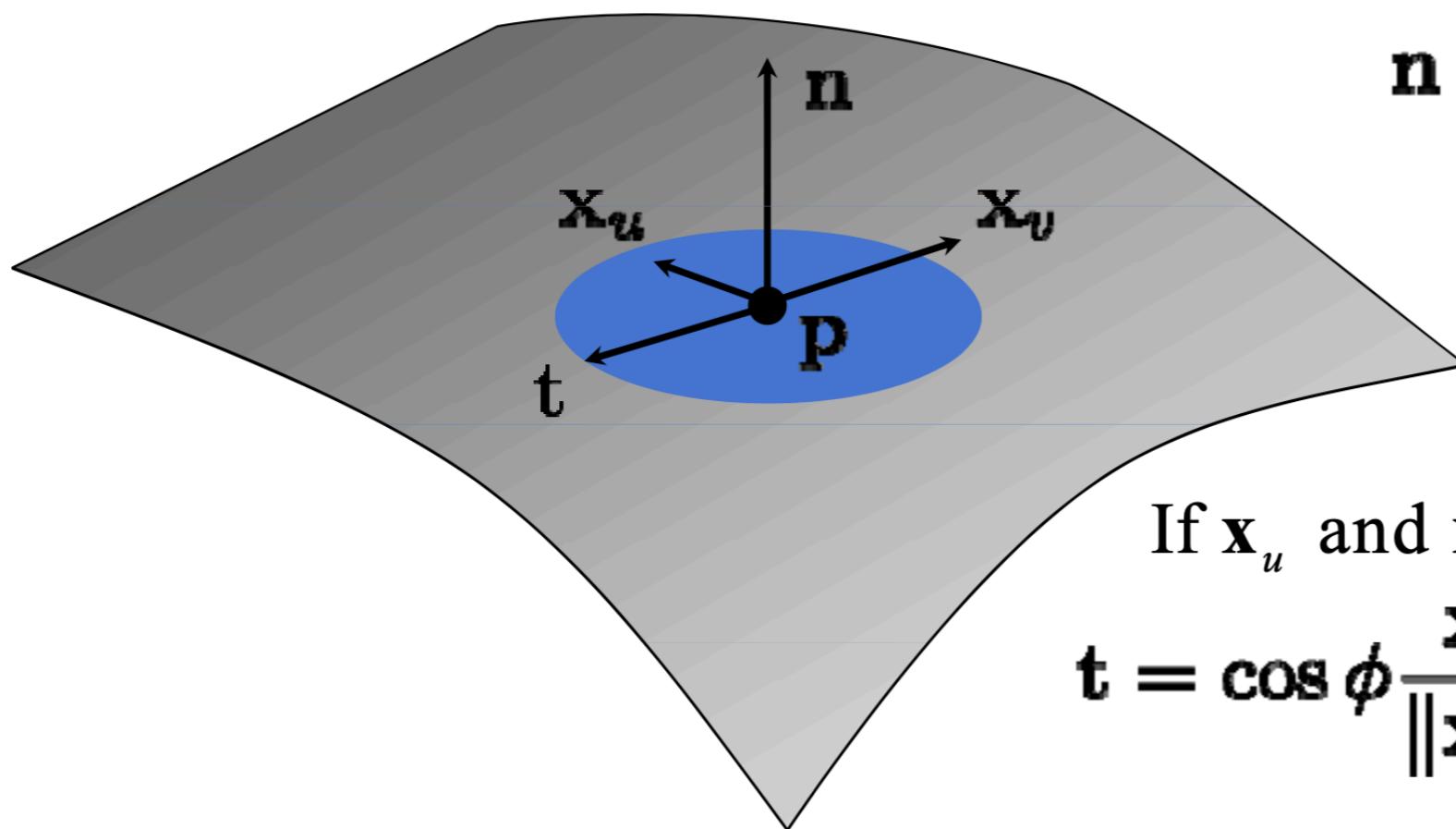
$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v) / \|\mathbf{x}_u \times \mathbf{x}_v\|$$

– assuming regular parameterization, i.e.

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$



Normal Curvature

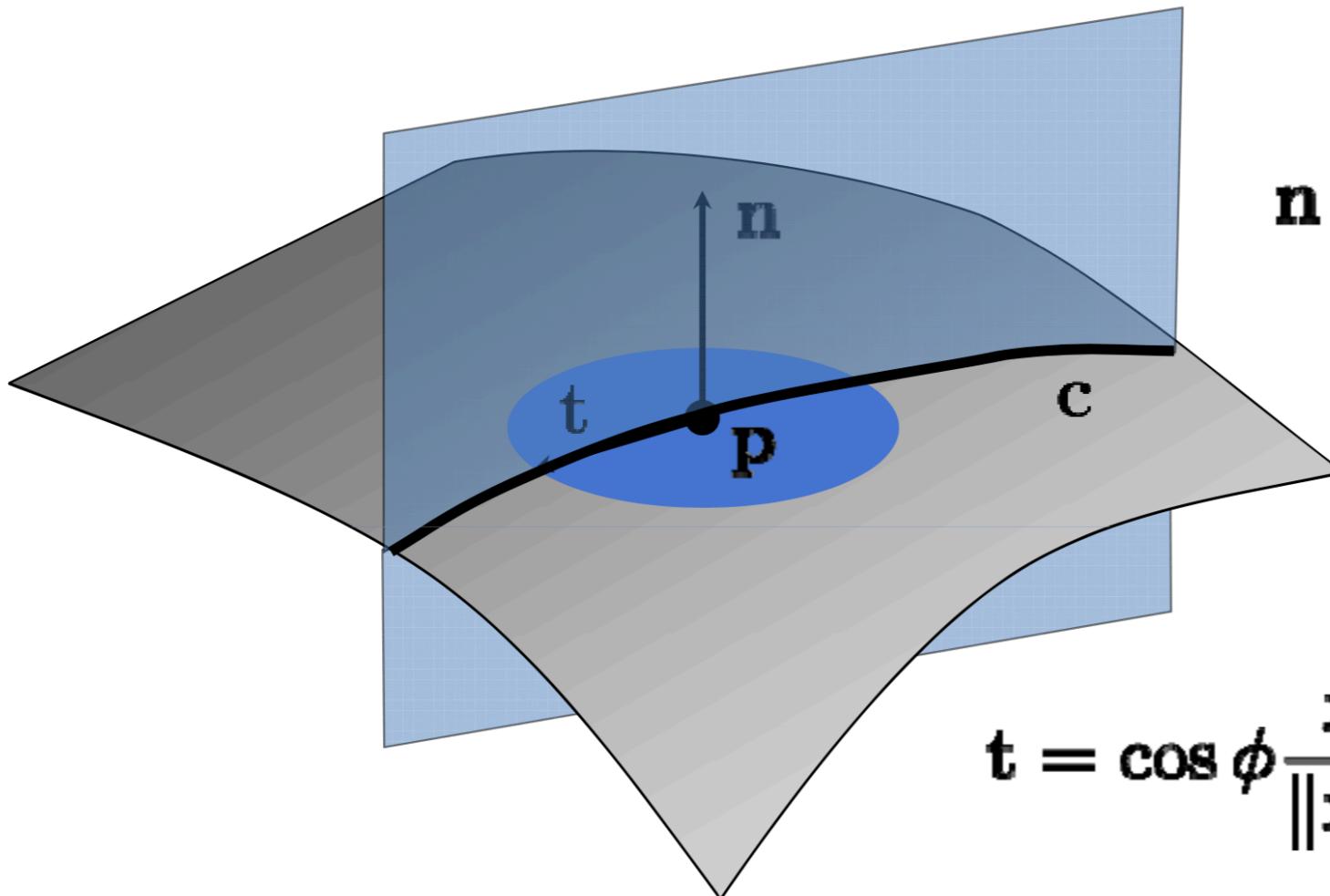


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

If \mathbf{x}_u and \mathbf{x}_v are orthogonal:

$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature



$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

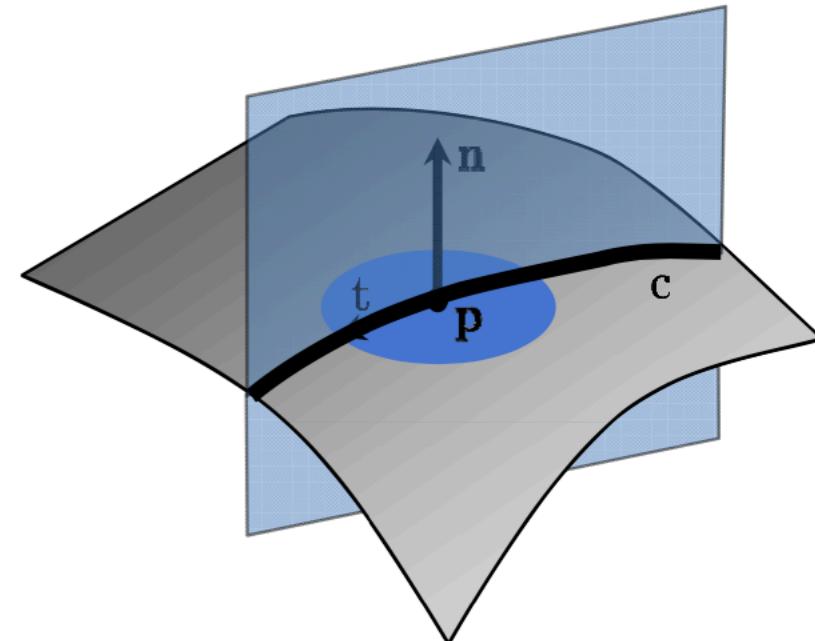
$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Surface Curvature

- Principal Curvatures
 - maximum curvature
 - minimum curvature

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$

$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$



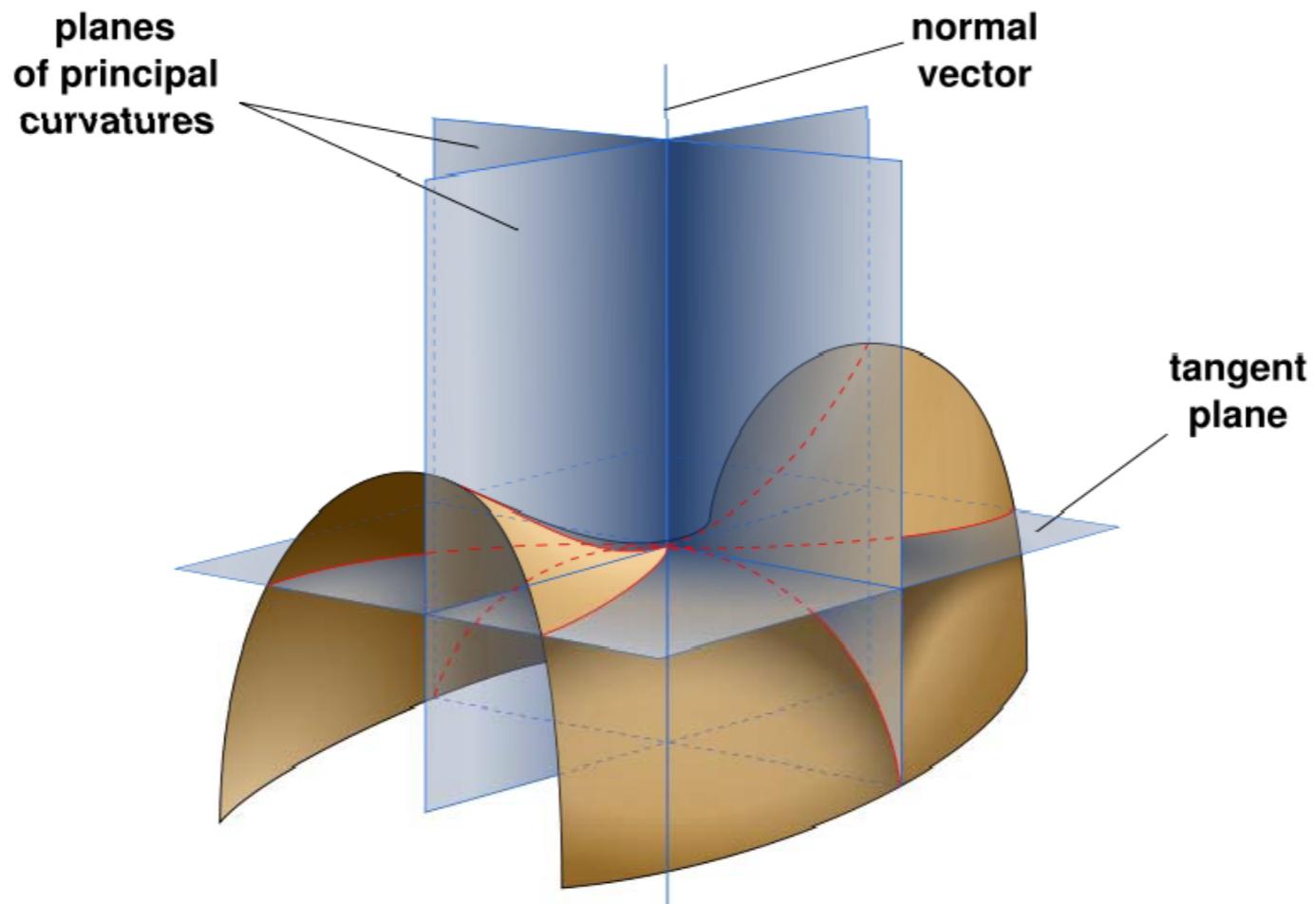
- Mean Curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi$$

- Gaussian Curvature

$$K = \kappa_1 \cdot \kappa_2$$

Principal Curvature



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta \quad \theta = \text{angle with } \kappa_1$$

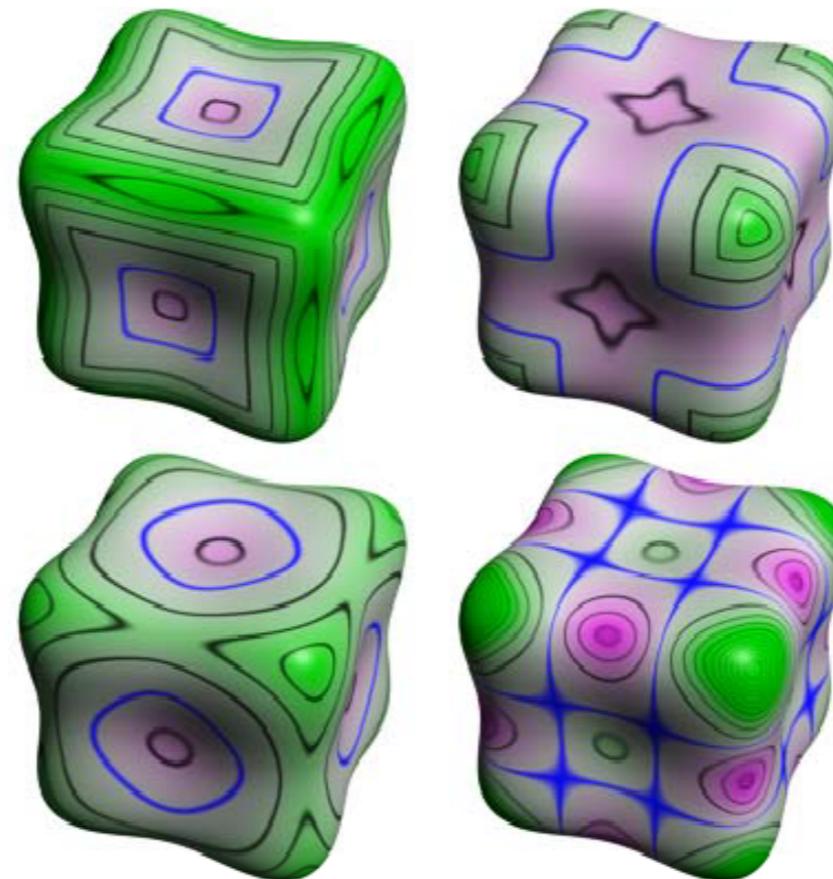
Curvature

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

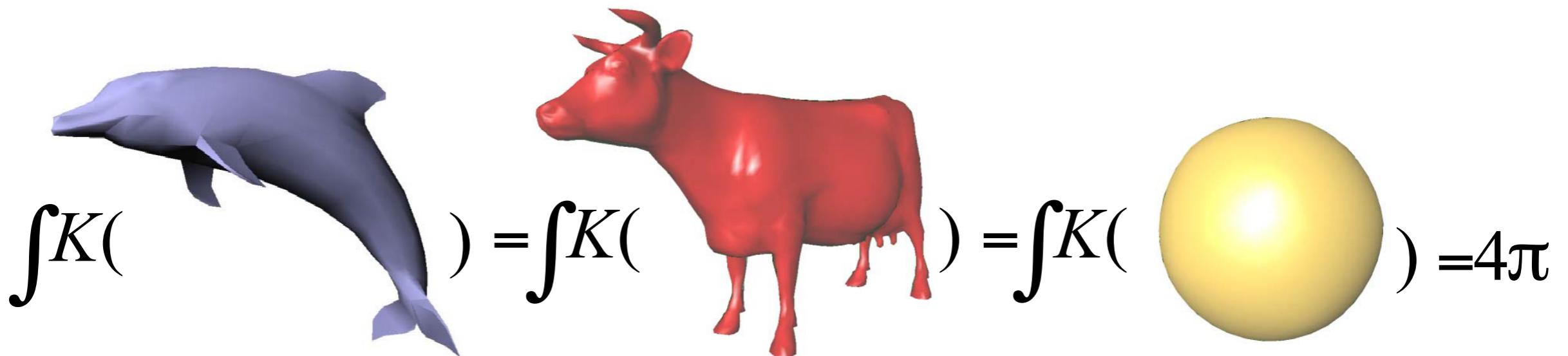
$$K = \kappa_1 \cdot \kappa_2$$



Gauss-Bonnet Theorem

For ANY closed manifold surface with Euler number
 $\chi=2-2g$:

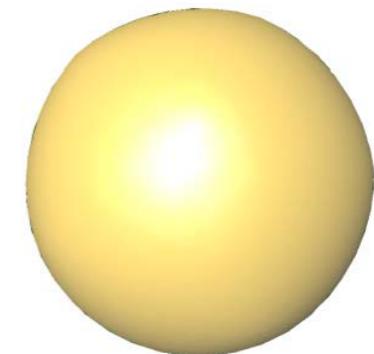
$$\int K = 2\pi\chi$$



$$\int K(\text{purple dolphin}) = ?$$



$$= \int K(\text{red cow}) = ?$$



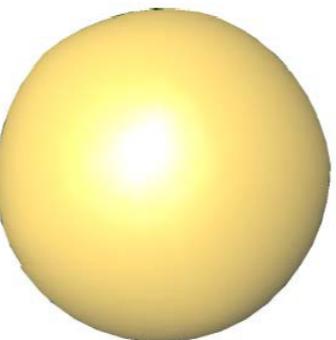
$$= \int K(\text{yellow sphere}) = 4\pi$$

Gauss-Bonnet Theorem

Example

- Sphere

- $k_1 = k_2 = 1/r$
- $K = k_1 k_2 = 1/r^2$

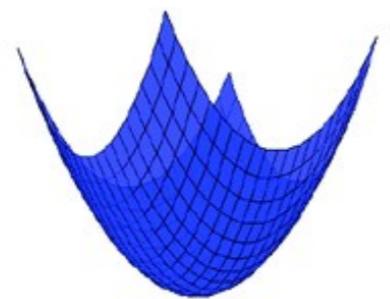


$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

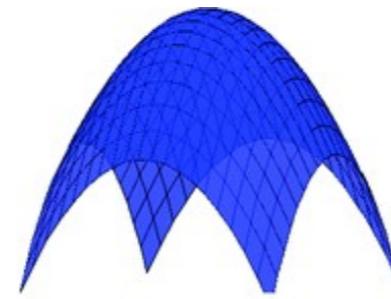
- Manipulate sphere

- New **positive** + **negative** curvature
- Cancel out!

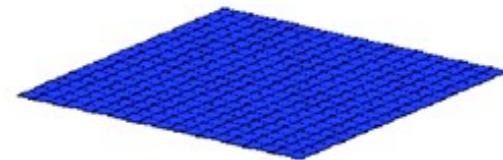
High-Level Questions



(a) $KG > 0, KH > 0$
elliptic concave



(b) $KG > 0, KH < 0$
elliptic convex



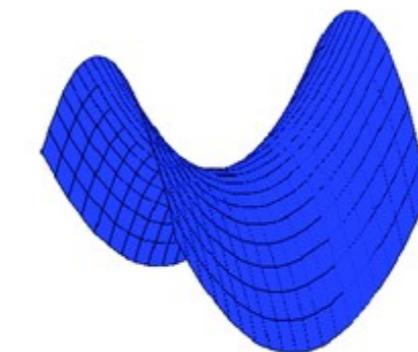
(c) $KG = 0, KH = 0$
plane



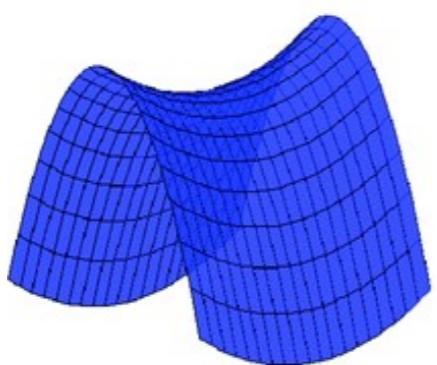
(d) $KG = 0, KH > 0$
parabolic concave



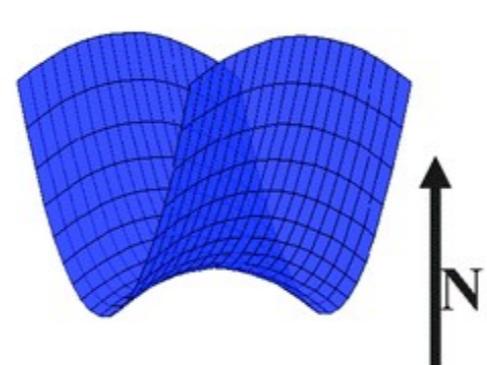
(e) $KG = 0, KH < 0$
parabolic convex



(f) $KG < 0, KH = 0$
saddle (hyperbolic)



(g) $KG < 0, KH < 0$
hyperbolic-like



(h) $KG < 0, KH > 0$
hyperbolic-like

How to
distinguish?

Recall: Frenet Frame: Curves in \mathbb{R}^3

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- **Binormal:** $T \times N$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion

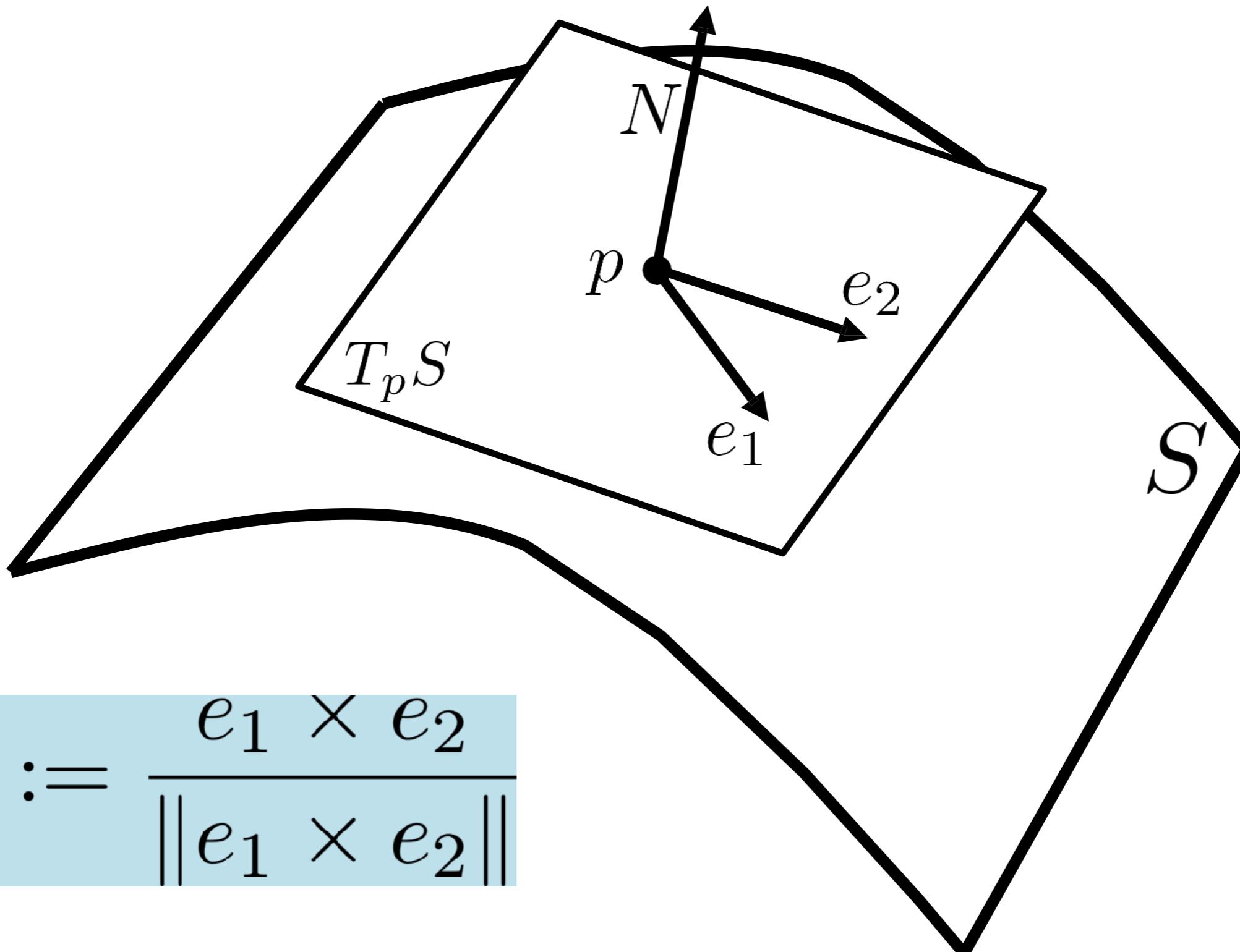
Theorem:

Curvature and torsion determine
geometry of a curve up to rigid motion.



Can curvature/torsion
of a curve help us
understand surfaces?

Unit Normal

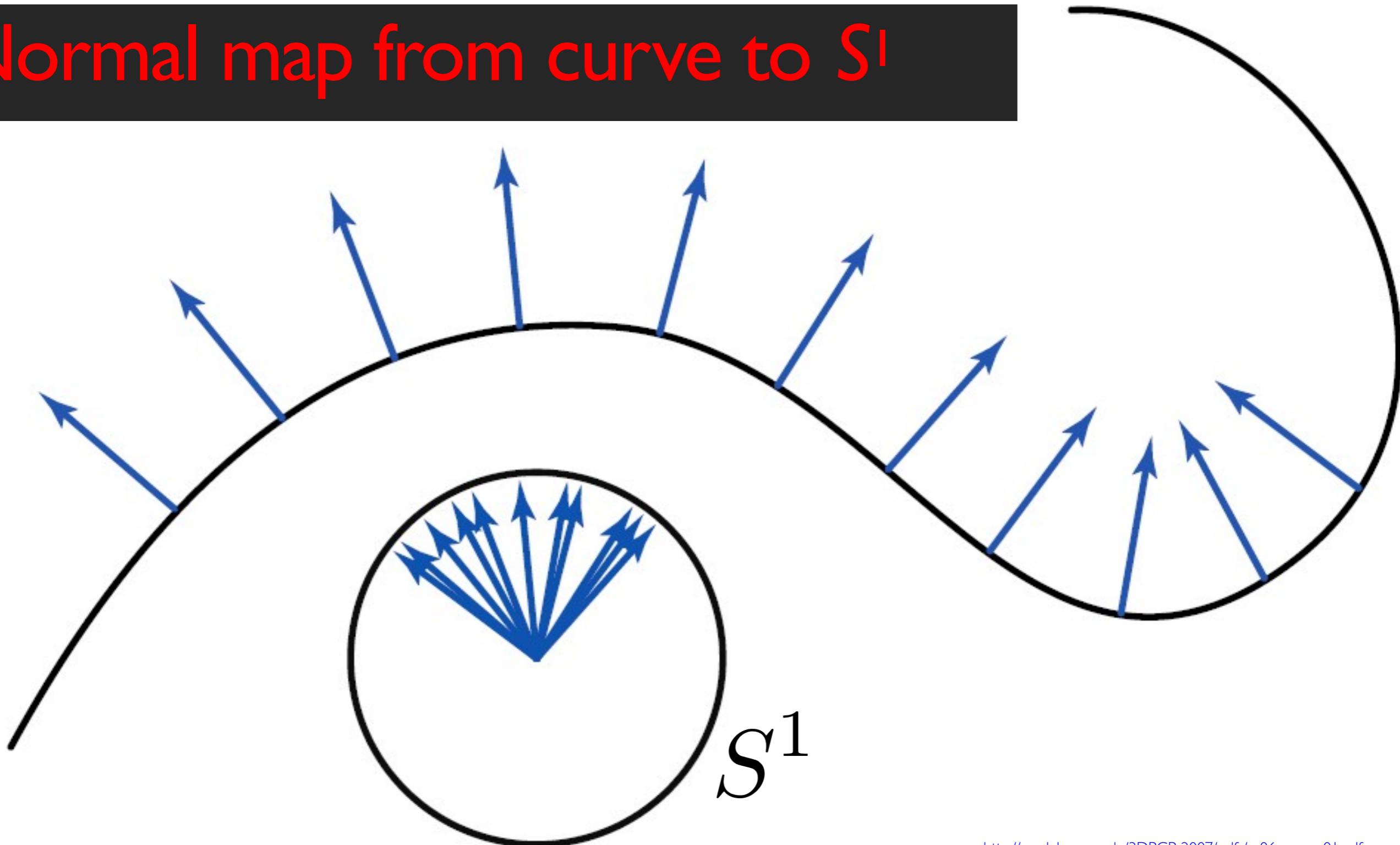


$$N := \frac{e_1 \times e_2}{\|e_1 \times e_2\|}$$

Recall:

Gauss Map

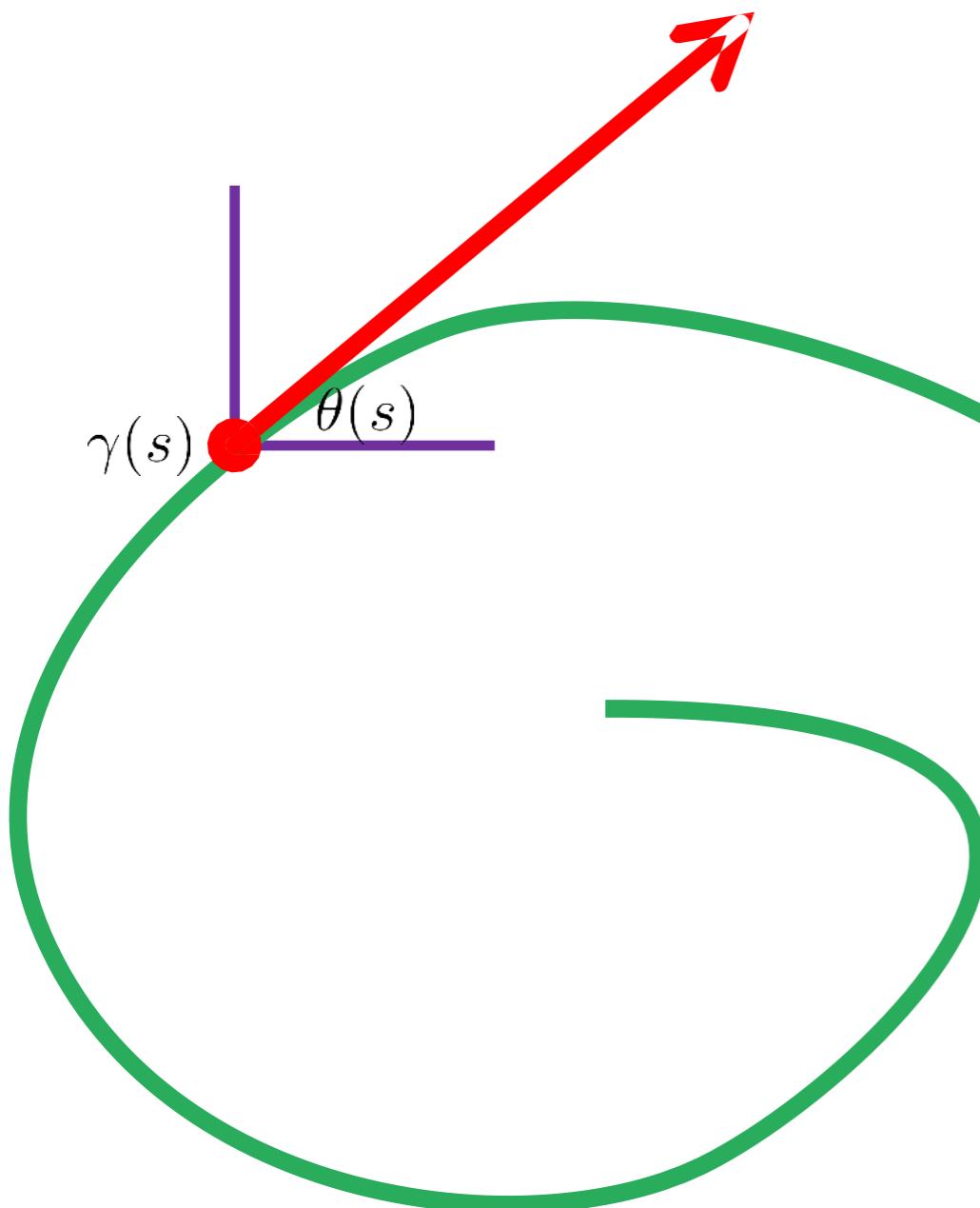
Normal map from curve to S^1



Recall:

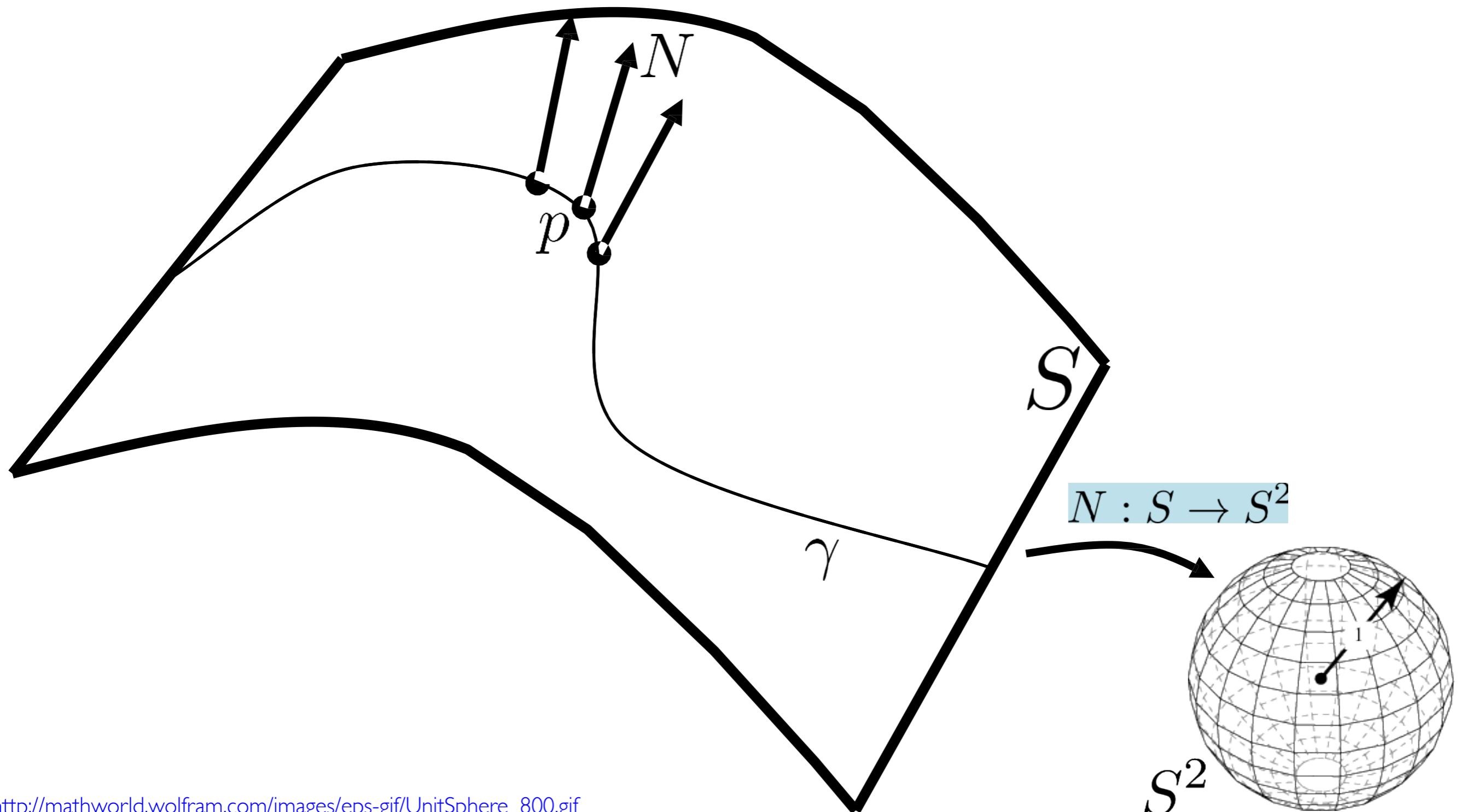
Signed Curvature on Plane Curves

$$T(s) = (\cos \theta(s), \sin \theta(s))$$



$$\begin{aligned} T'(s) &= \theta'(s)(-\sin \theta(s), \cos \theta(s)) \\ &:= \kappa(s)N(s) \end{aligned}$$

Gauss Map for Surface



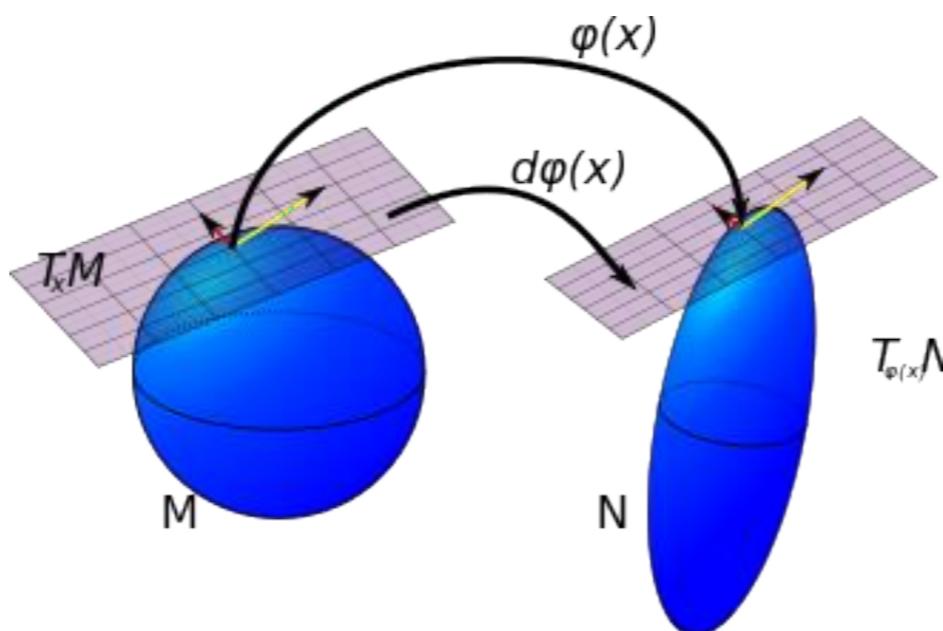
Differential of a Map

$$\varphi : M \rightarrow N$$

$$\implies d\varphi_p : T_p M \rightarrow T_{\varphi(p)} N$$

Linear map of tangent spaces

$$d\varphi_p(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$



Calculation on Board

Where is the
derivative of N?

Spoiler alert: $T_p S$

Second Fundamental Form

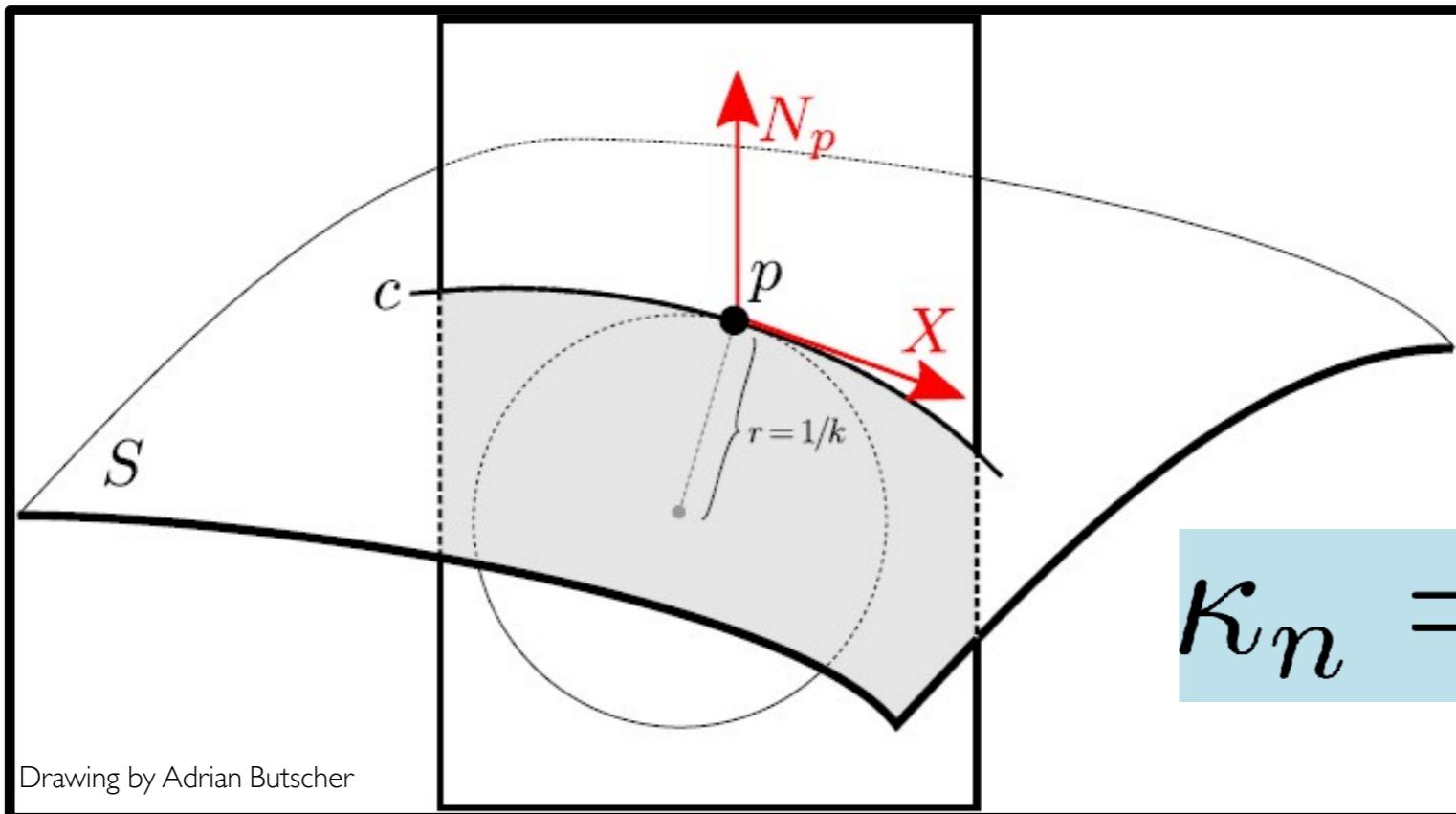
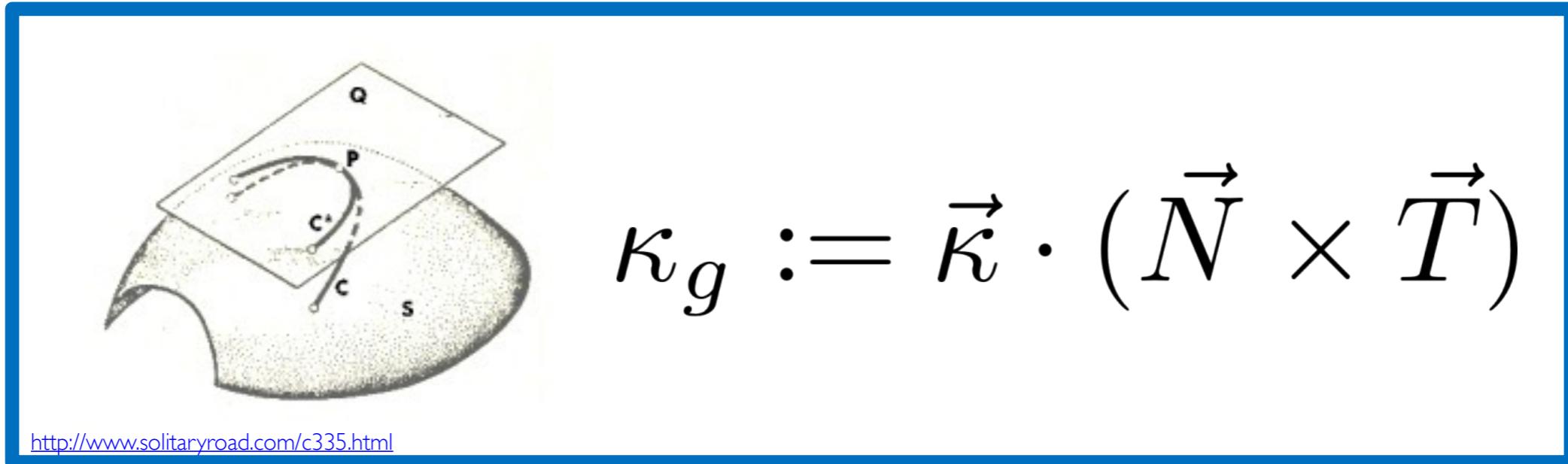
$$DN_p : T_p S \rightarrow T_p S$$



$$A_p(V, W) := -\langle DN_p(V), W \rangle$$

“Shape operator”

Relationship to Curvature of Curves



A_p is Self-Adjoint

Means that

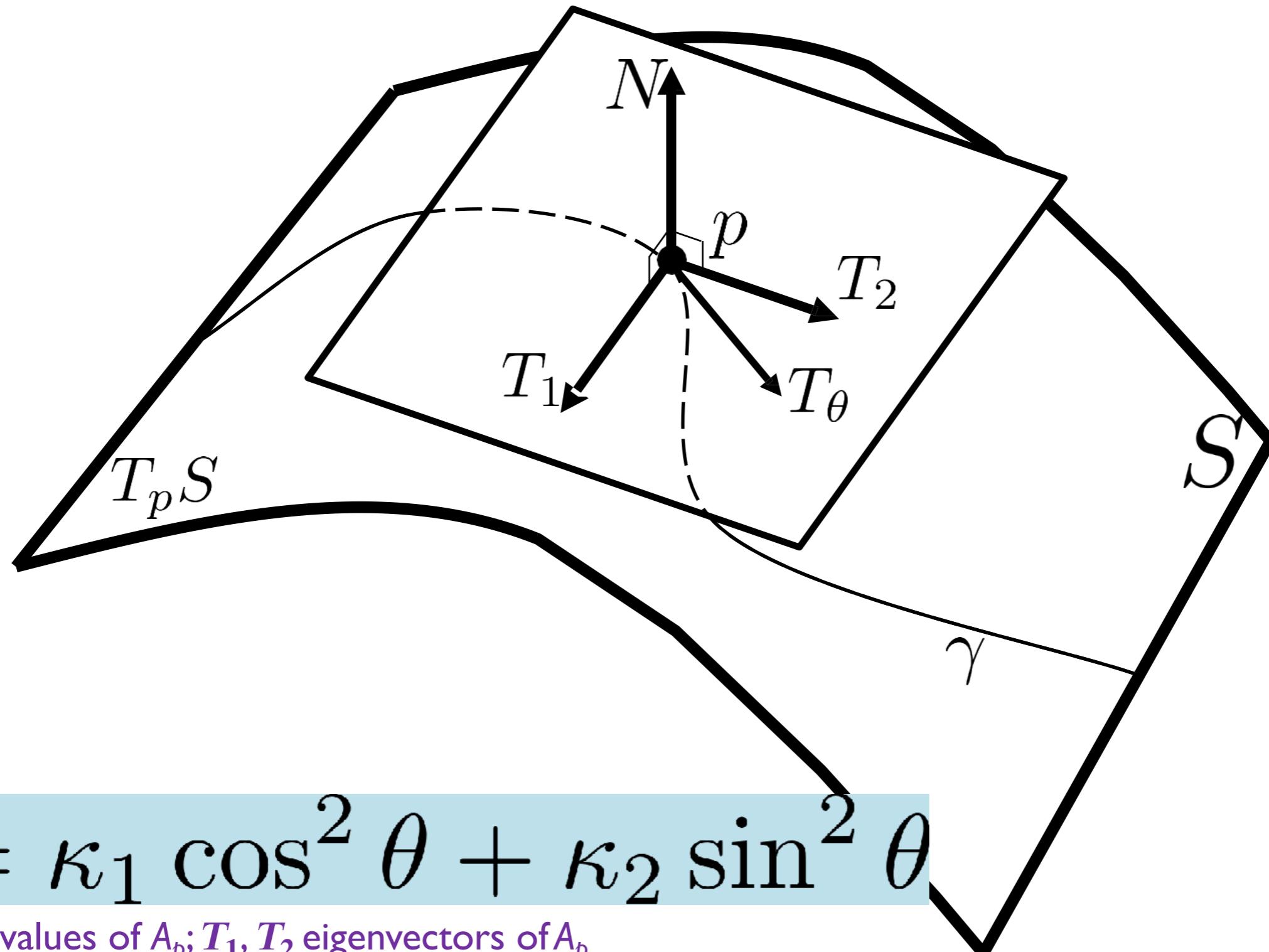
$$A_p(V, W) = - \langle DN_p(V), W \rangle = - \langle V, DN_p(W) \rangle$$

In matrix form,

$$A_p(V, W) = V^T JW \quad J \text{ symmetric}$$

Validate by yourself if interested

Principal Directions and Curvatures



$$\kappa_\theta = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$

κ_1, κ_2 eigenvalues of A_p ; T_1, T_2 eigenvectors of A_p

Principal Curvatures

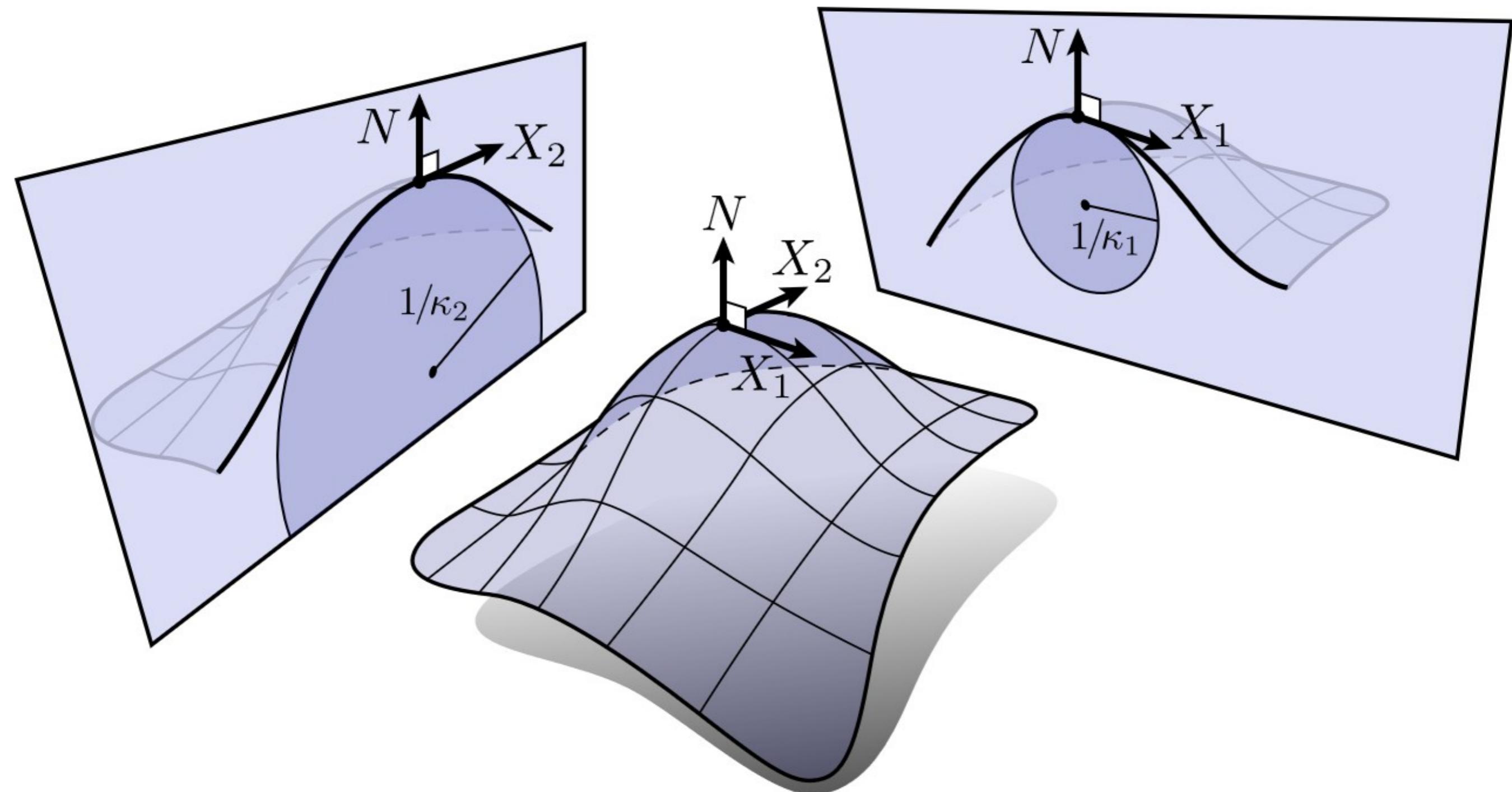
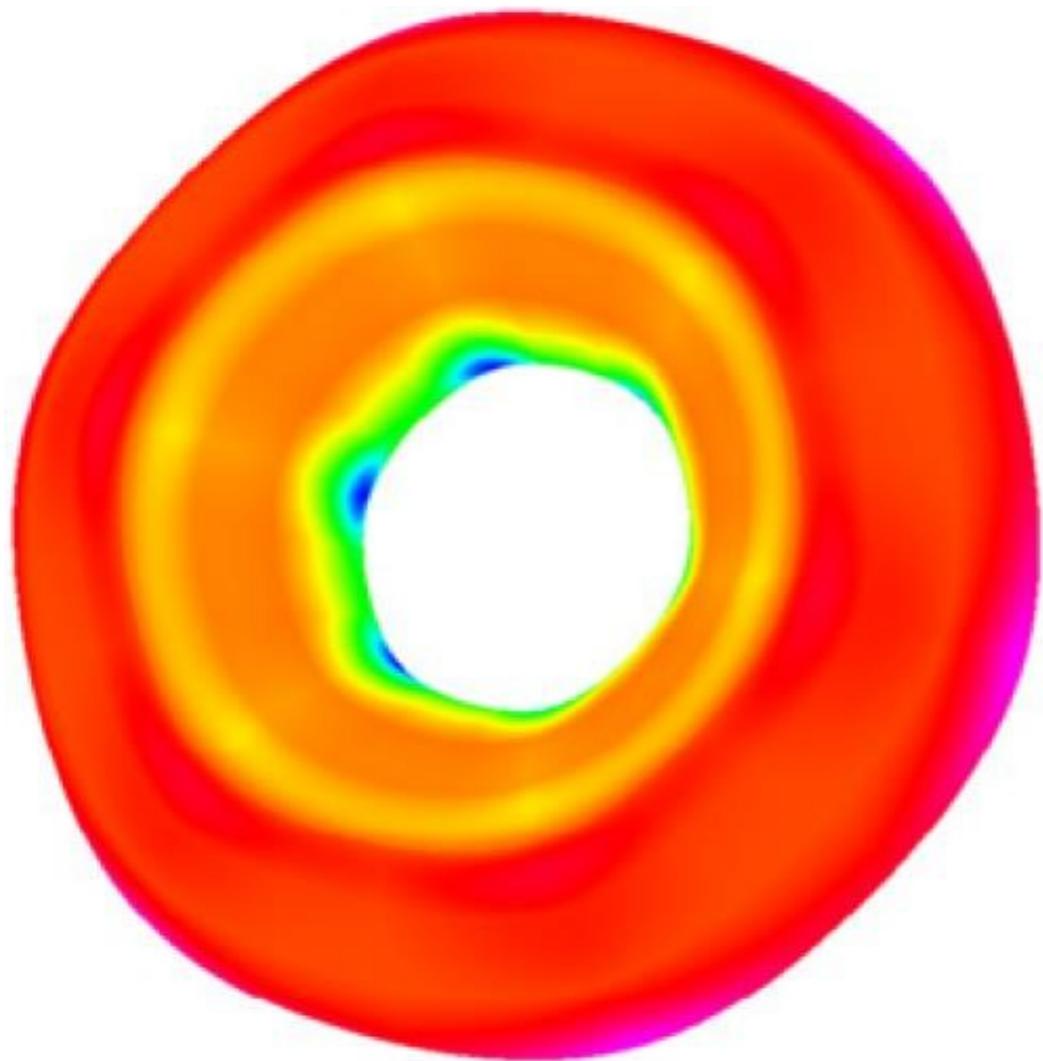


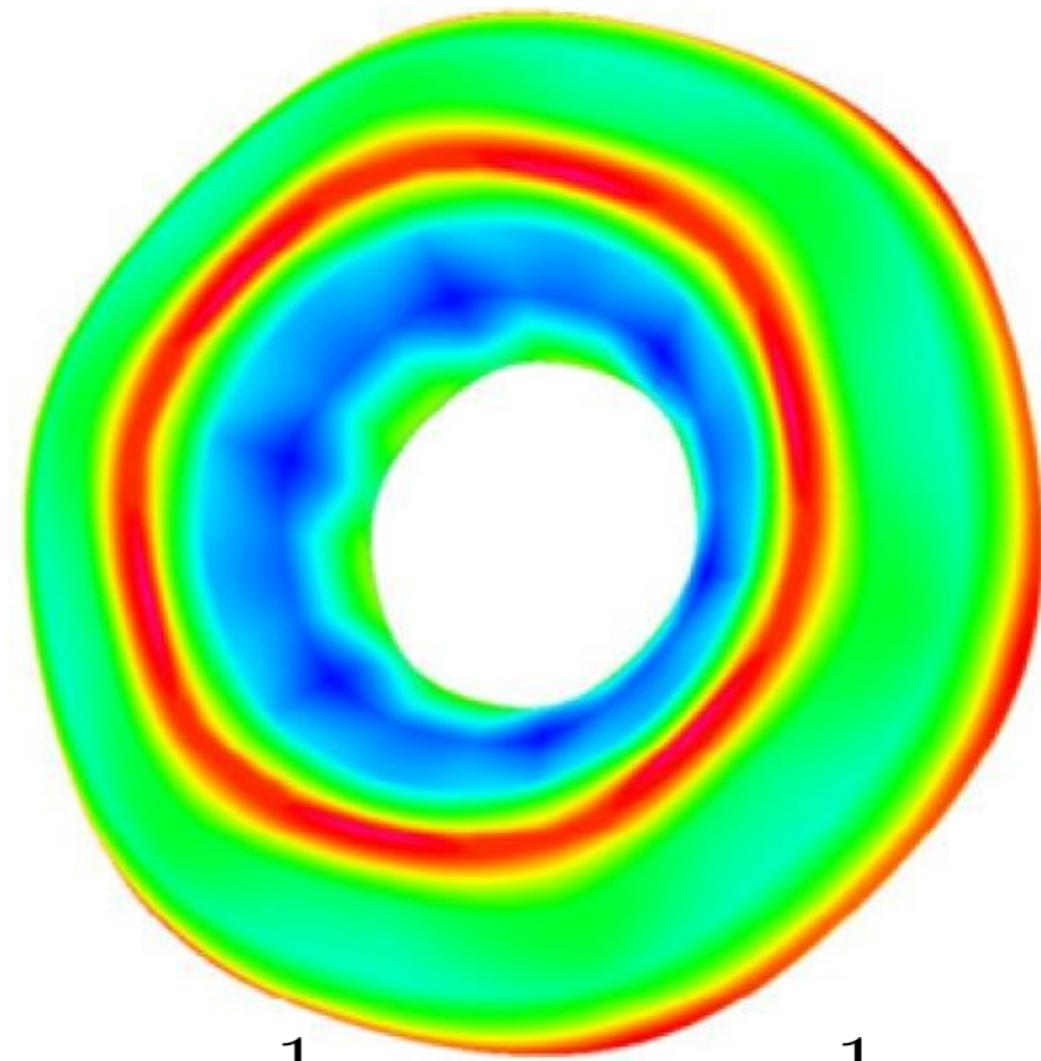
Image courtesy K. Crane, CMU

Extrinsic Curvature

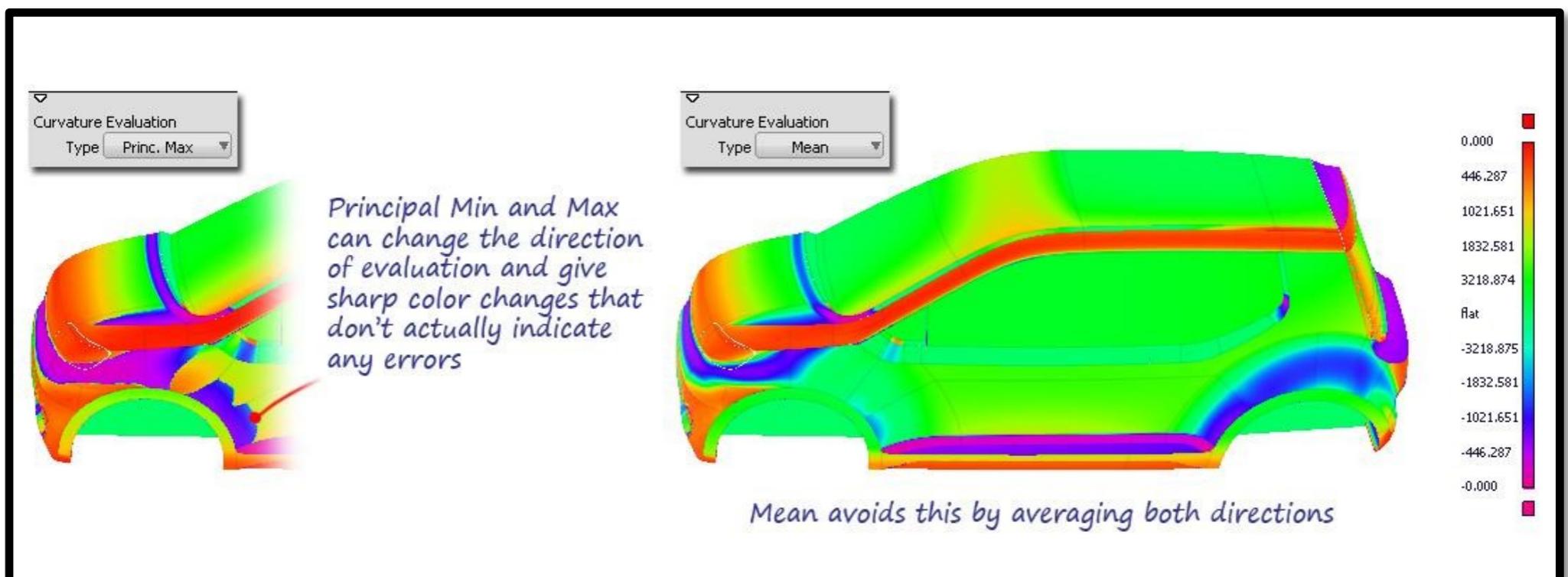
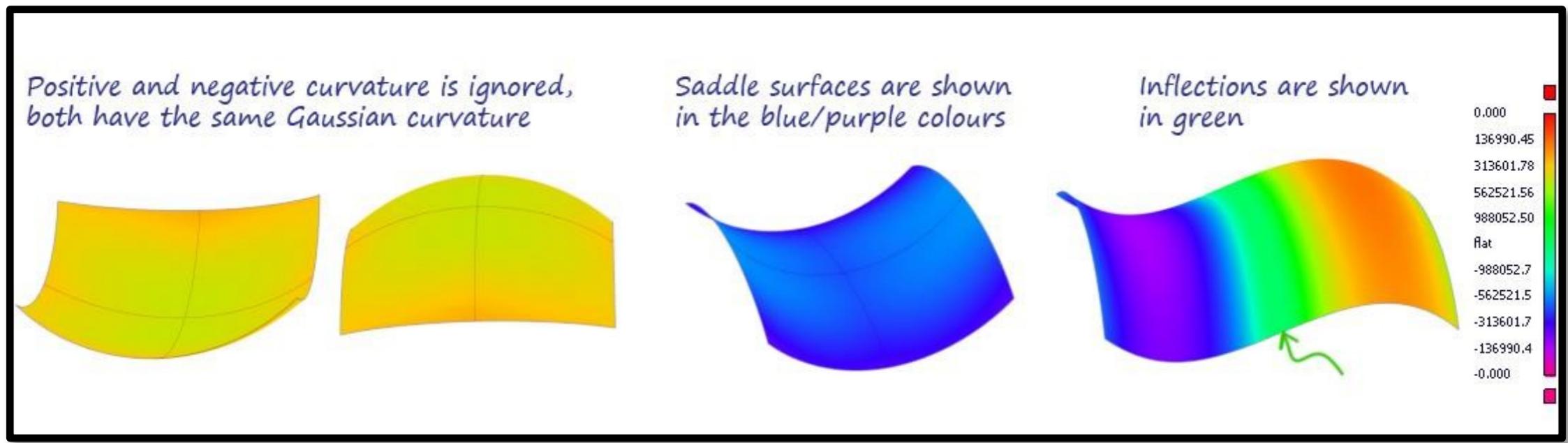


$$K := \kappa_1 \kappa_2 = \det \mathbb{II}$$

$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\text{tr } \mathbb{II}$$



Interpretation



Uniqueness Result

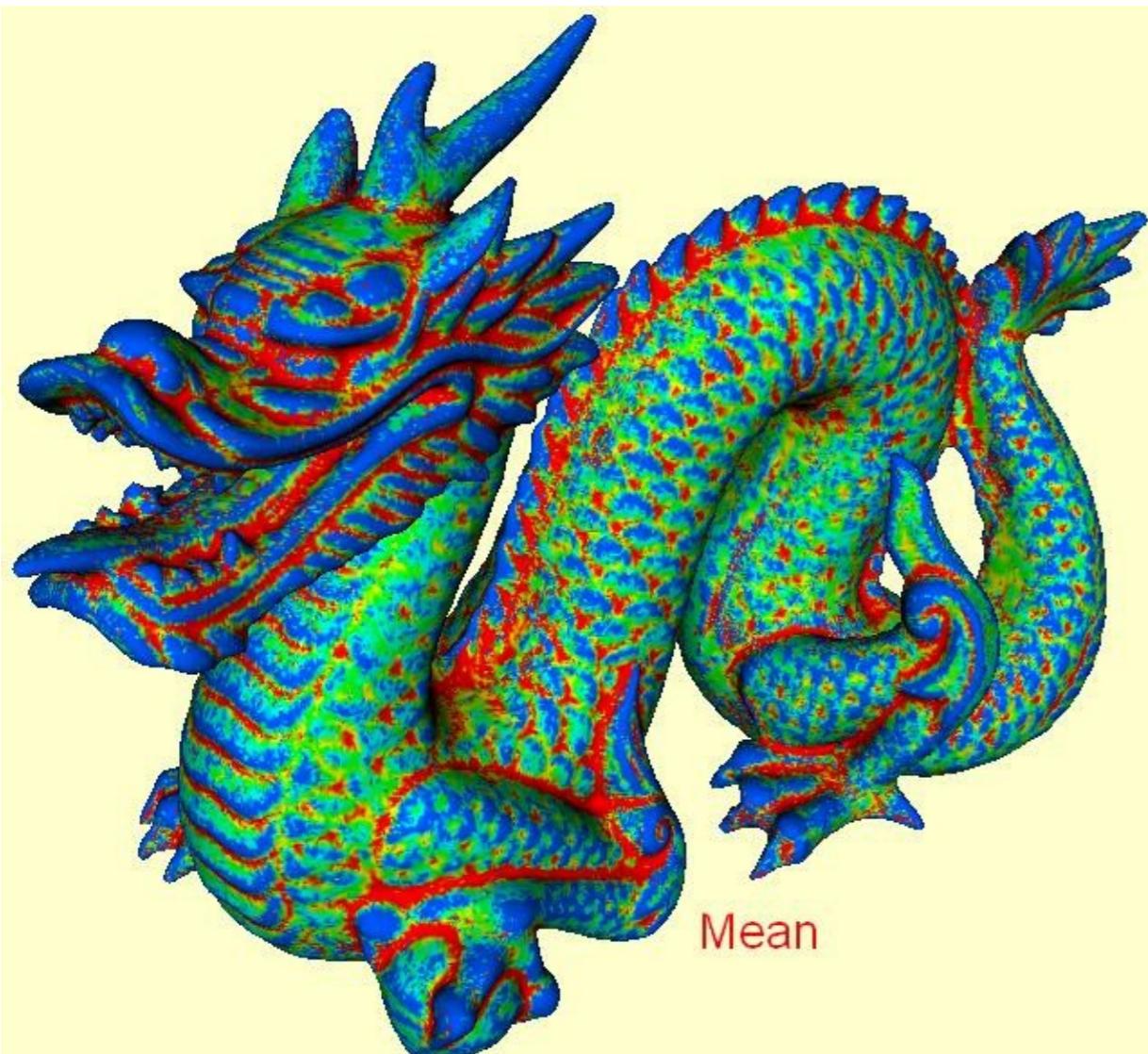
Theorem:

A smooth surface is determined up to rigid motion by its first and second fundamental forms.

Who Cares?

Curvature
completely determines
local surface geometry.

Use as a Descriptor



Smoothing and Reconstruction

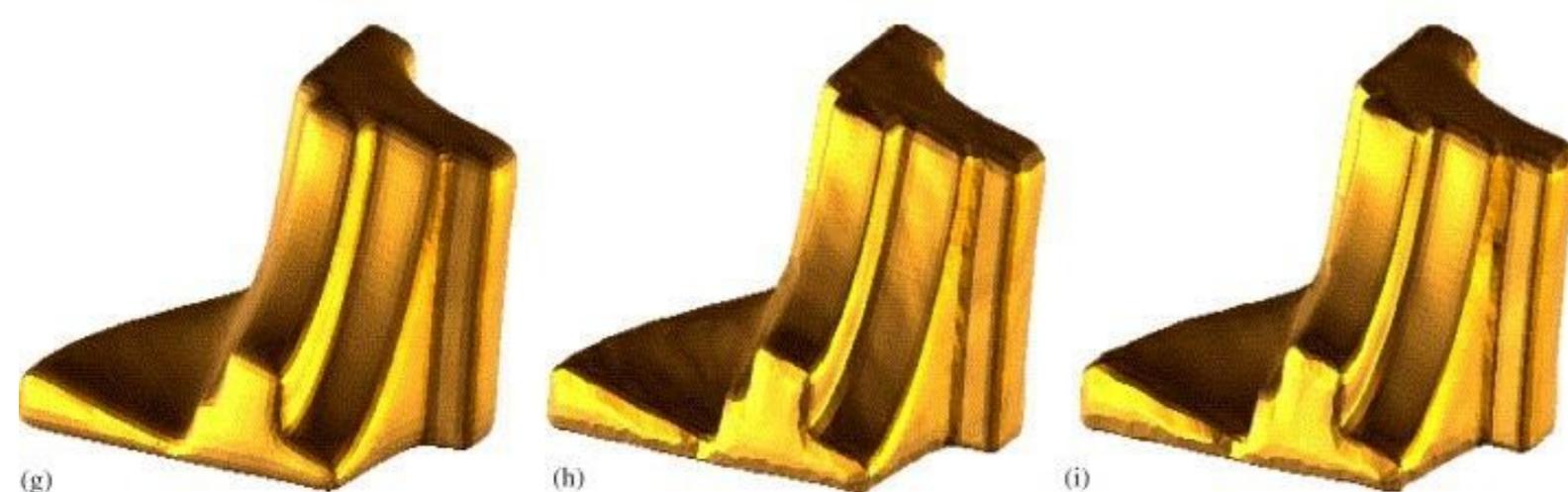
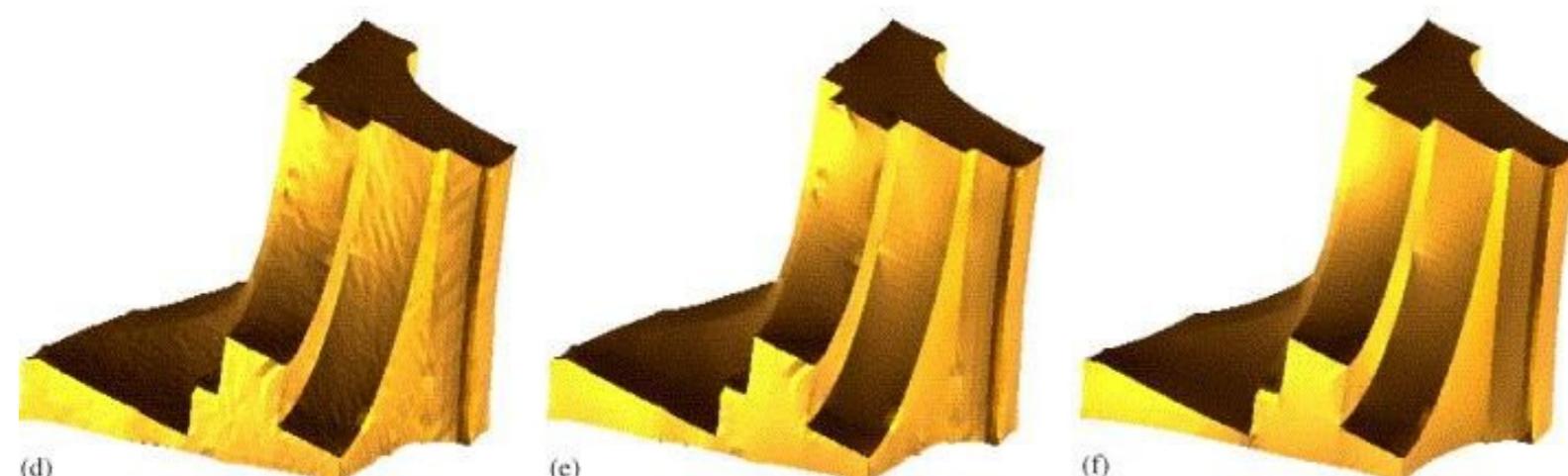
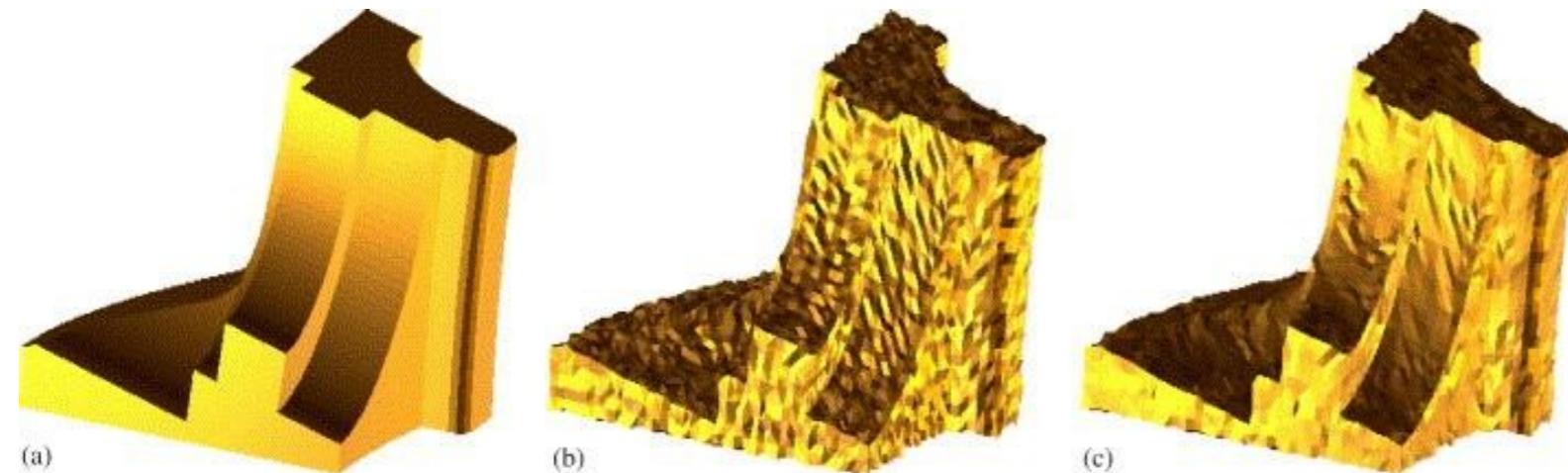


Linear Surface Reconstruction from Discrete Fundamental Forms on Triangle Meshes

Wang, Liu, and Tong

Computer Graphics Forum 31.8 (2012)

Fairness Measure



Triangular Surface Mesh Fairing via
Gaussian Curvature Flow

Zhao, Xu

Journal of Computational and Applied Mathematics 195.1-2 (2006)

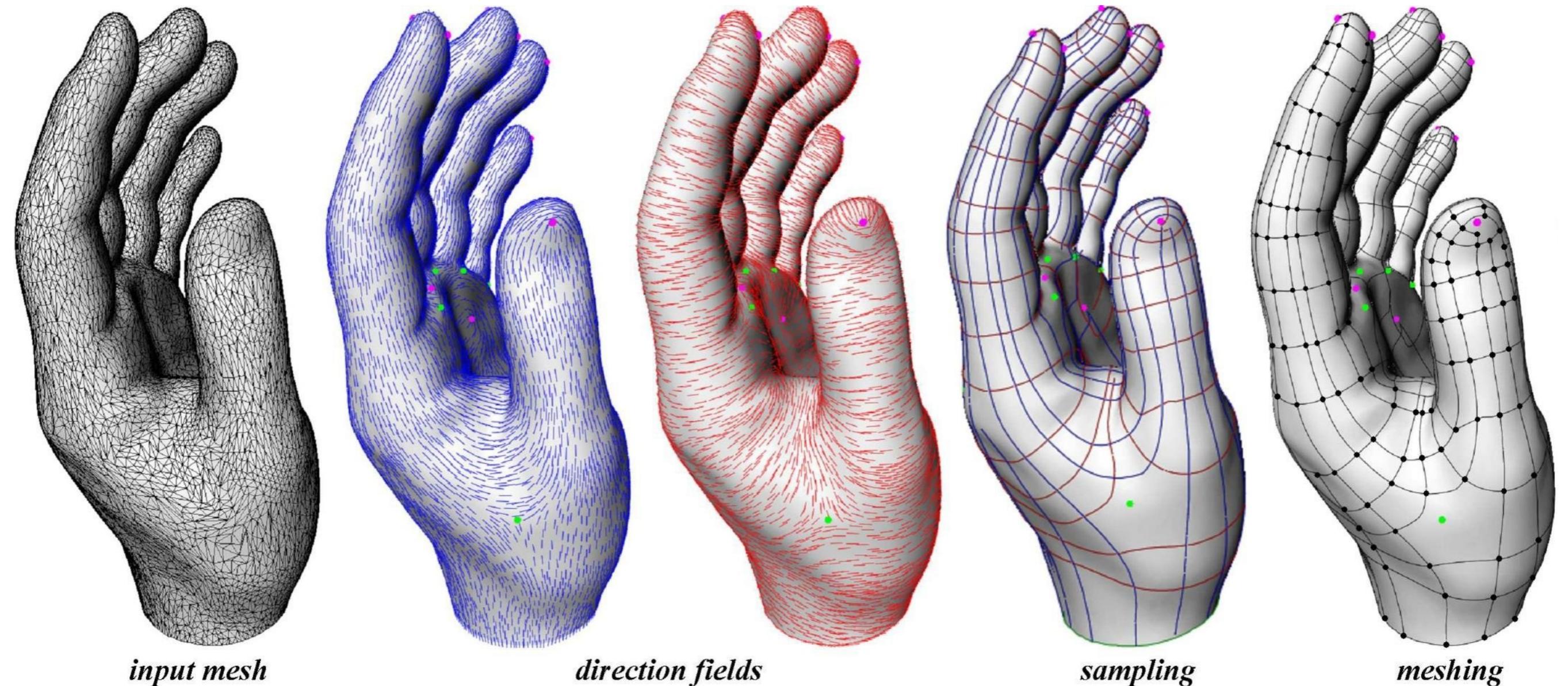
...and many more

Guiding Rendering



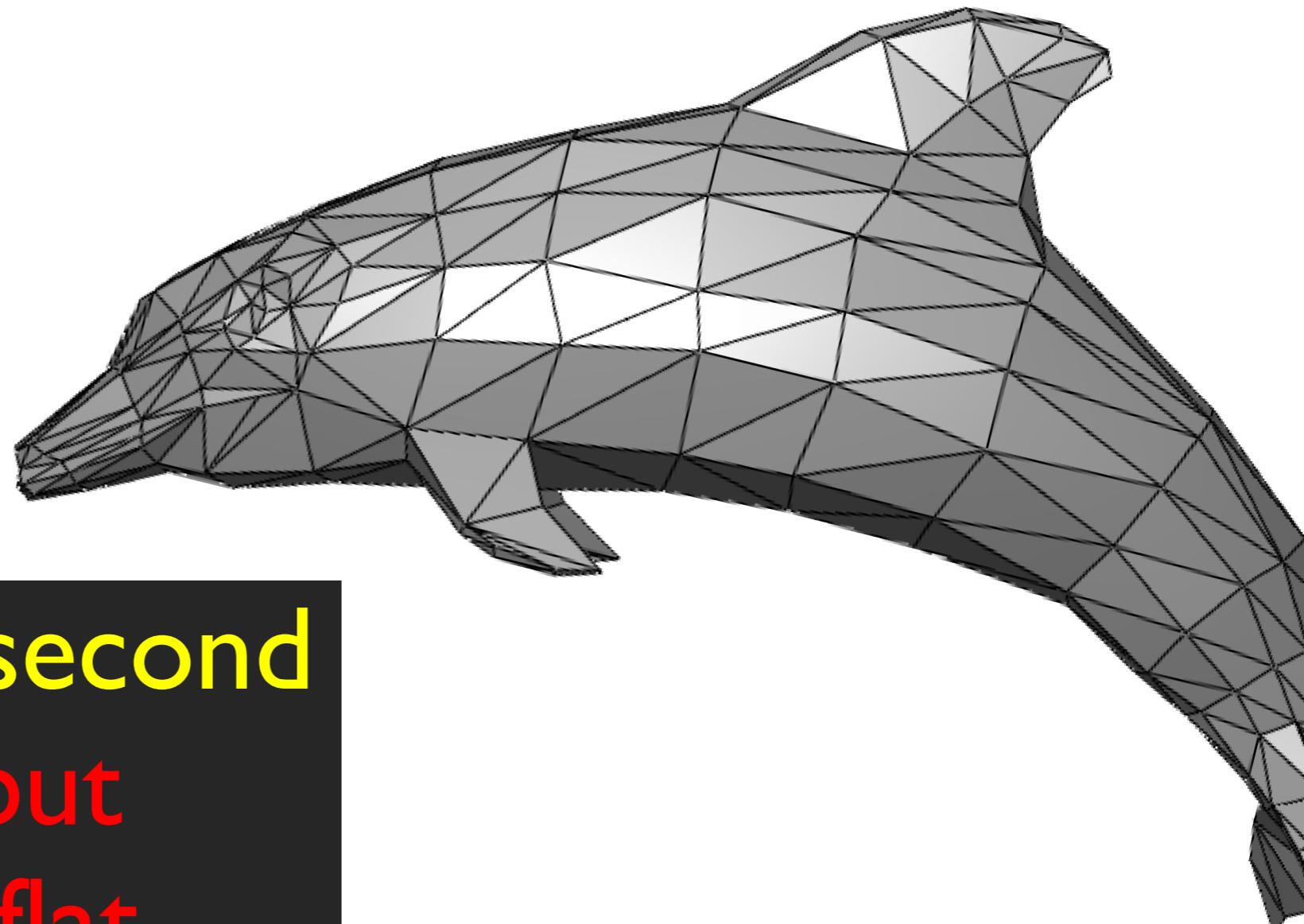
Highlight Lines for Conveying Shape
DeCarlo, Rusinkiewicz
NPAR (2007)

Guiding Meshing



Anisotropic Polygonal Remeshing
Alliez et al.
SIGGRAPH (2003)

Challenge on Meshes



Curvature is a second derivative, but triangles are flat.

Standard Citation

ESTIMATING THE TENSOR OF CURVATURE OF A SURFACE FROM A POLYHEDRAL APPROXIMATION

Gabriel Taubin

ICCV 1995

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taubin@watson.ibm.com

Abstract

Estimating principal curvatures and principal directions of a surface from a polyhedral approximation with a large number of small faces, such as those produced by iso-surface construction algorithms, has become a basic step in many computer vision algorithms. Particularly in those targeted at medical applications. In this paper we describe a method to estimate the tensor of curvature of a surface at the vertices of a polyhedral approximation. Principal curvatures and principal directions are obtained by computing in closed form the eigenvalues and eigenvectors of certain 3×3 symmetric matrices defined by integral formulas, and

mate principal curvatures at the vertices of a triangulated surface. Both this algorithm and ours are based on constructing a quadratic form at each vertex of the polyhedral surface and then computing eigenvalues (and eigenvectors) of the resulting form, but the quadratic forms are different. In our algorithm the quadratic form associated with a vertex is expressed as an integral, and is constructed in time proportional to the number of neighboring vertices. In the algorithm of Chen and Schmitt, it is the least-squares solution of an overdetermined linear system, and the complexity of constructing it is quadratic in the number of neighbors.

2. The Tensor of Curvature

Taubin Matrix

$$M := \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta T_\theta T_\theta^\top d\theta$$

$$\kappa_\theta := \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$

$$T_\theta := T_1 \cos \theta + T_2 \sin \theta$$

Taubin Matrix

$$M := \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta T_\theta T_\theta^\top d\theta$$

- Eigenvectors are N , T_1 , and T_2
- Eigenvalues are $\frac{3}{8}\kappa_1 + \frac{1}{8}\kappa_2$ and $\frac{1}{8}\kappa_1 + \frac{3}{8}\kappa_2$

Prove at home!

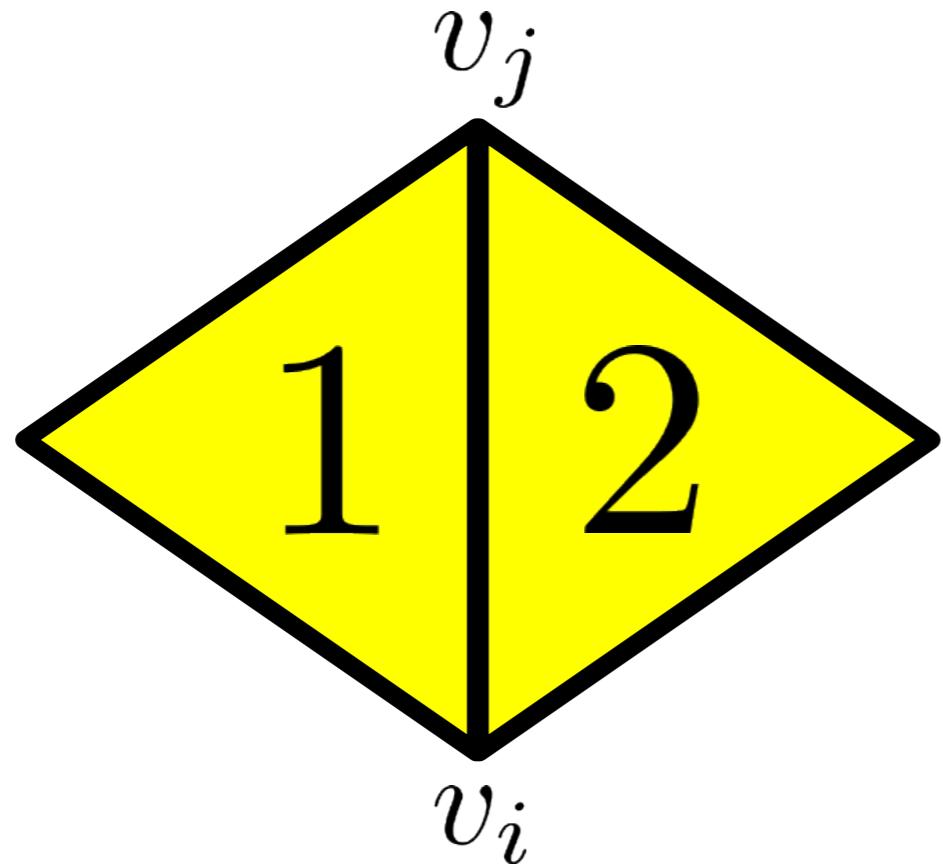
Taubin's Approximation

$$M := \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta T_\theta T_\theta^\top d\theta$$



$$\tilde{M}_{v_i} := \sum_{v_j \sim v_i} w_{ij} \kappa_{ij} T_{ij} T_{ij}^\top$$

Taubin's Approximation



Divided difference approximation

$$\tilde{M}_{v_i} := \sum_{v_j \sim v_i} w_{ij} \kappa_{ij} T_{ij} T_{ij}^\top$$