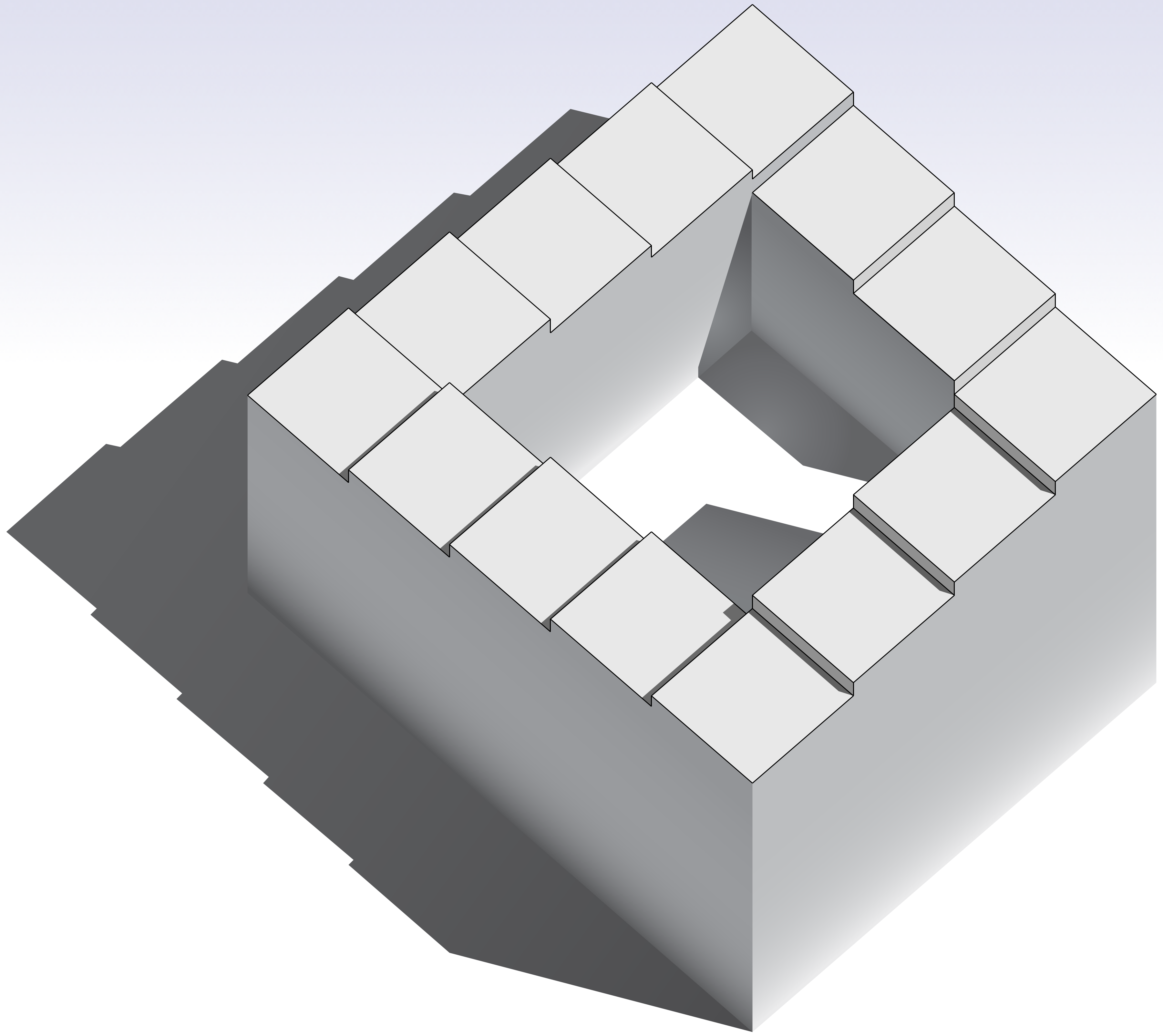
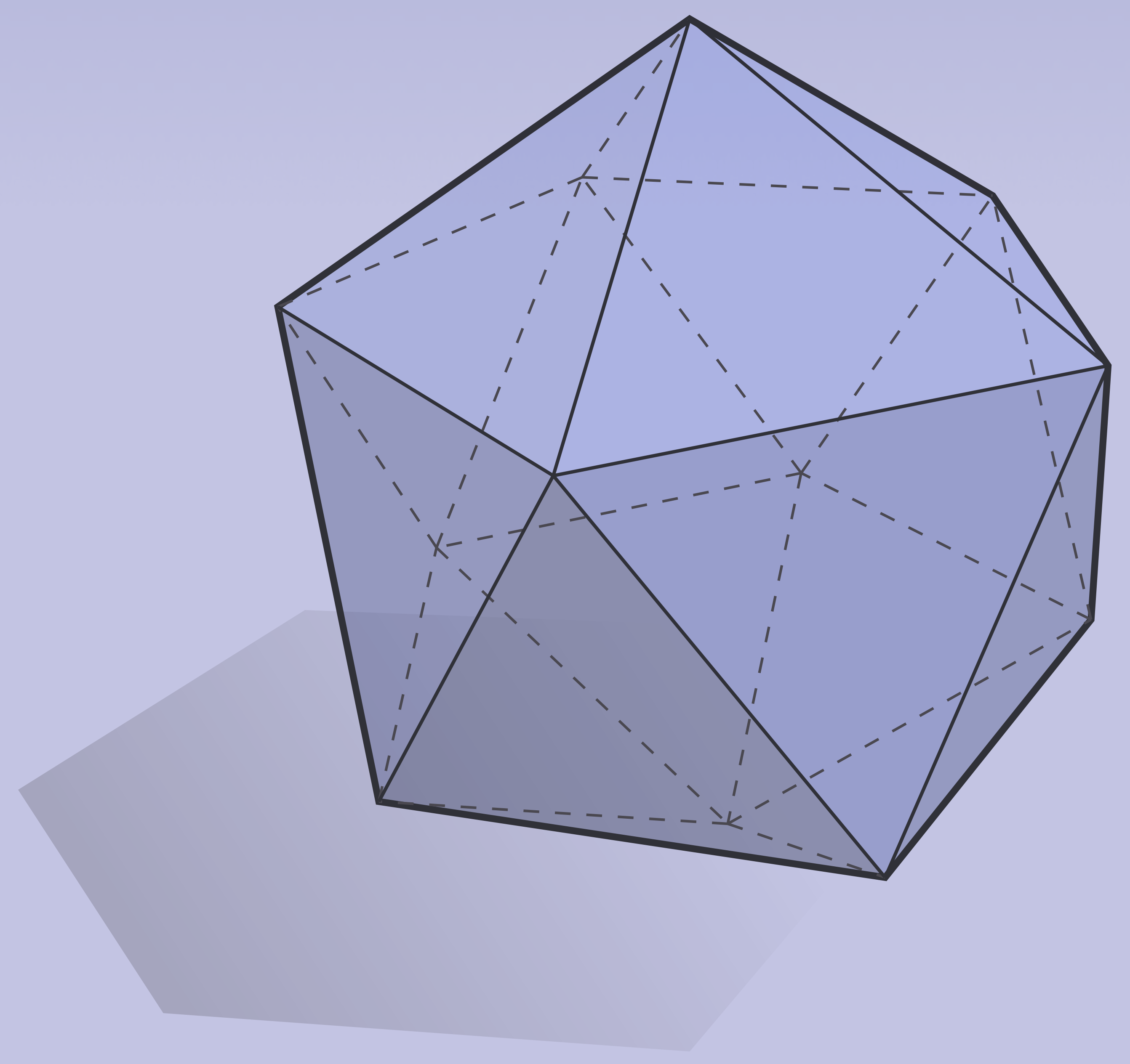


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 7:
CURVES



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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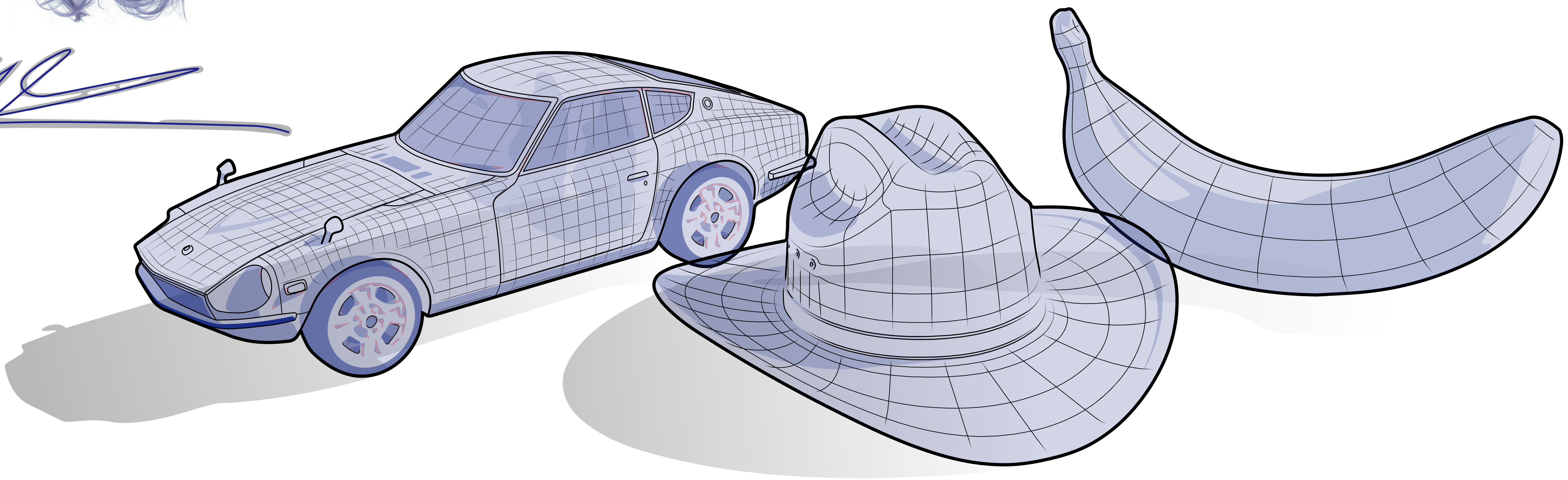
Curves & Surfaces

- Much of the geometry we encounter in life well-described by *curves* and *surfaces**

(Curves)

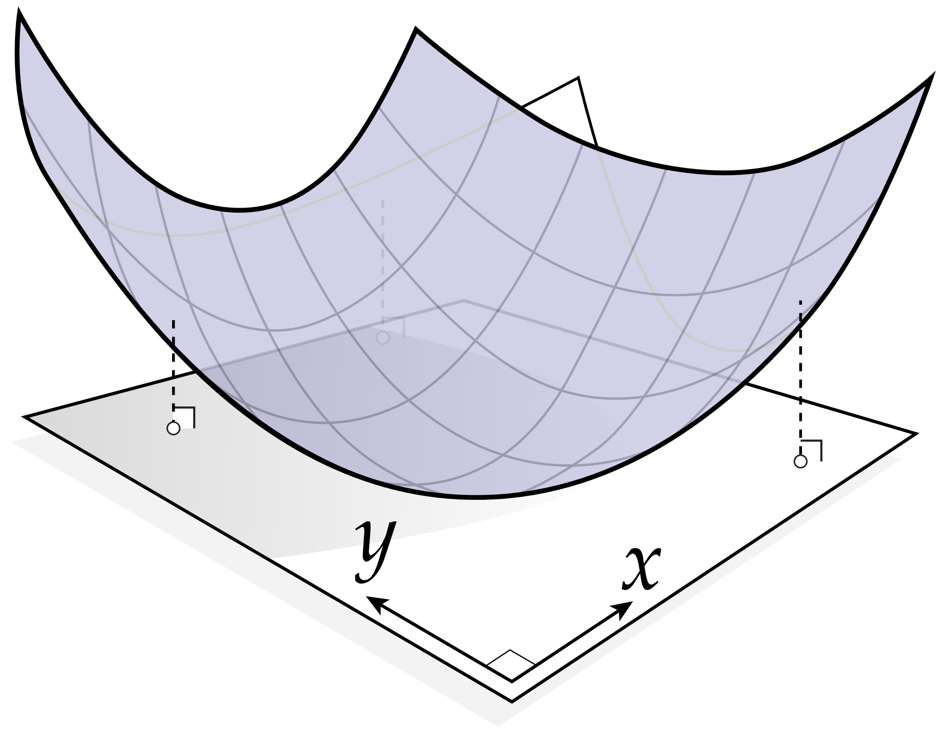


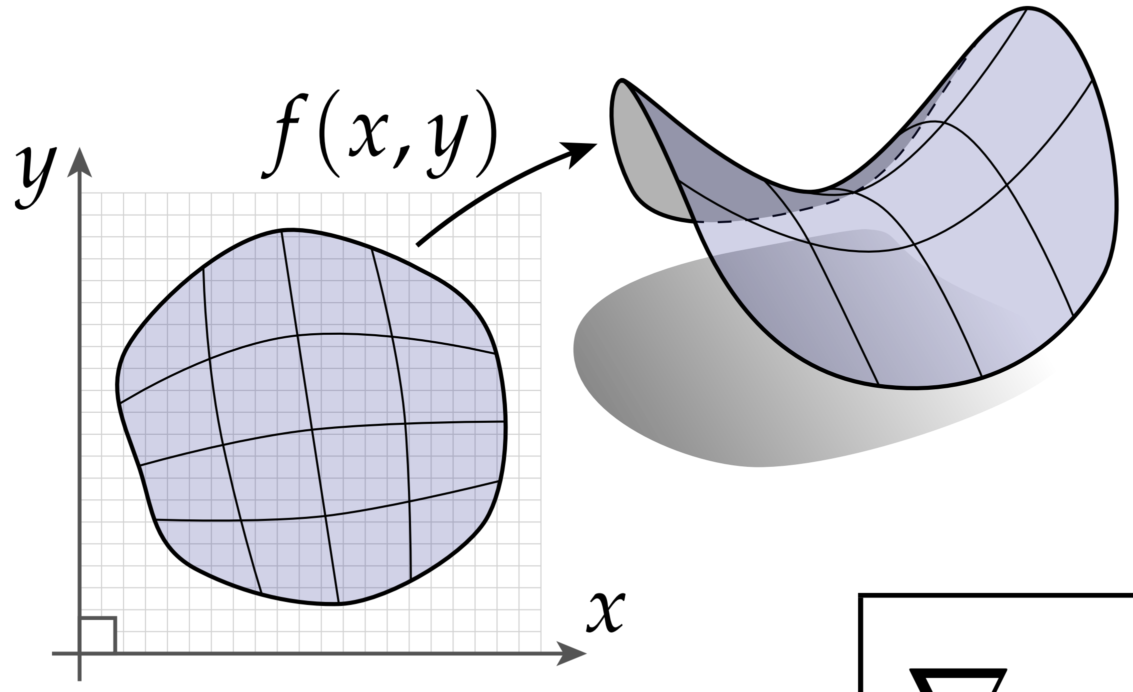
(Surfaces)

*Or solids… but the boundary of a solid is a surface!

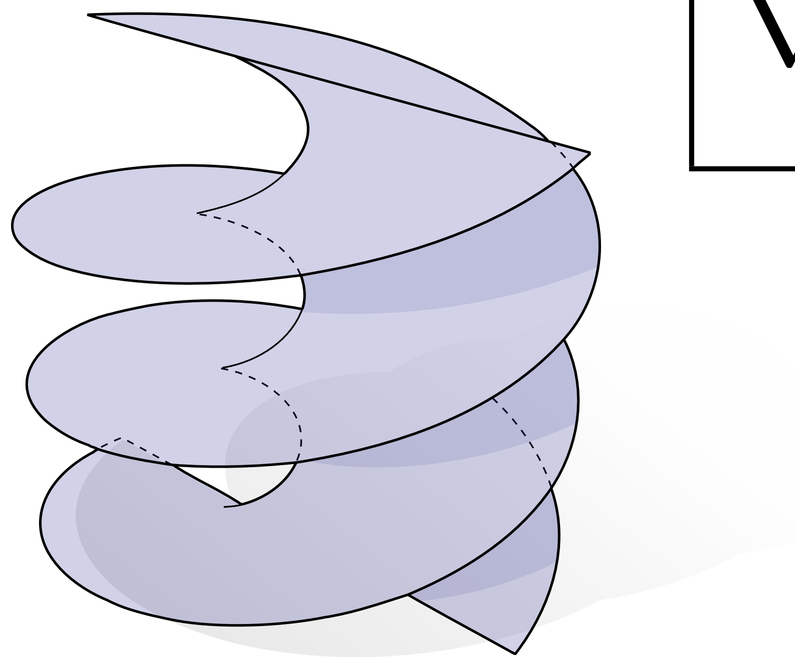
Smooth Descriptions of Surfaces

- Many ways to express the geometry of a curve or surface:
  - height function over tangent plane
  - local parameterization
  - Christoffel symbols — coordinates/indices
  - **differential forms** — “coordinate free”
  - moving frames — builds on differential forms
  - Riemann surfaces (*local*); Quaternionic functions (*global*)
- People can get very religious about these different “dialects”... best to be multilingual!
- We'll dive deep into one description (**differential forms**) and touch on others





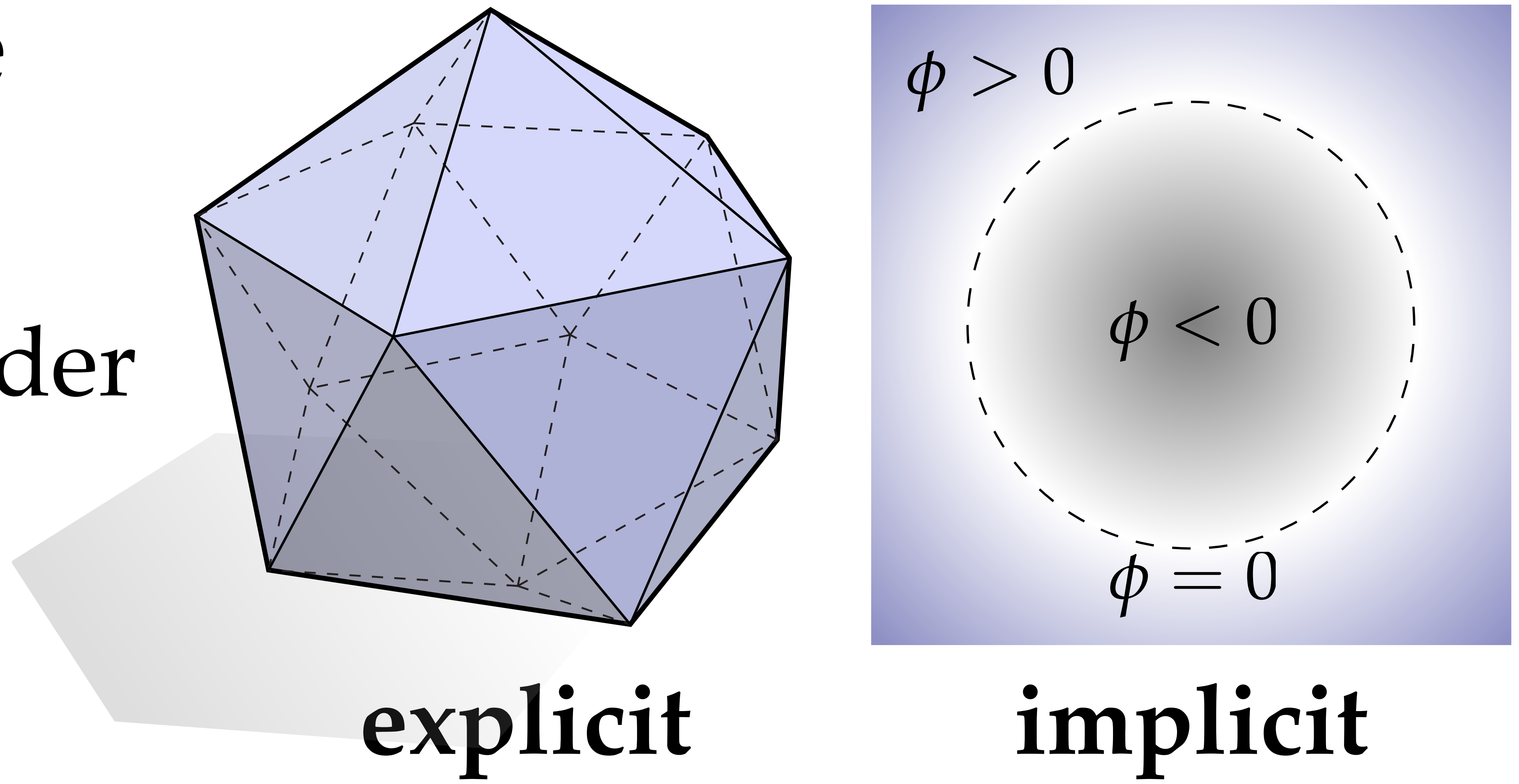
$$\boxed{\nabla_i e_j = \Gamma_{ij}^k e_k}$$



$$\boxed{\begin{array}{ccc} d\sigma & = & \omega \wedge \sigma \\ d\omega & = & \omega \wedge \omega \end{array}}$$

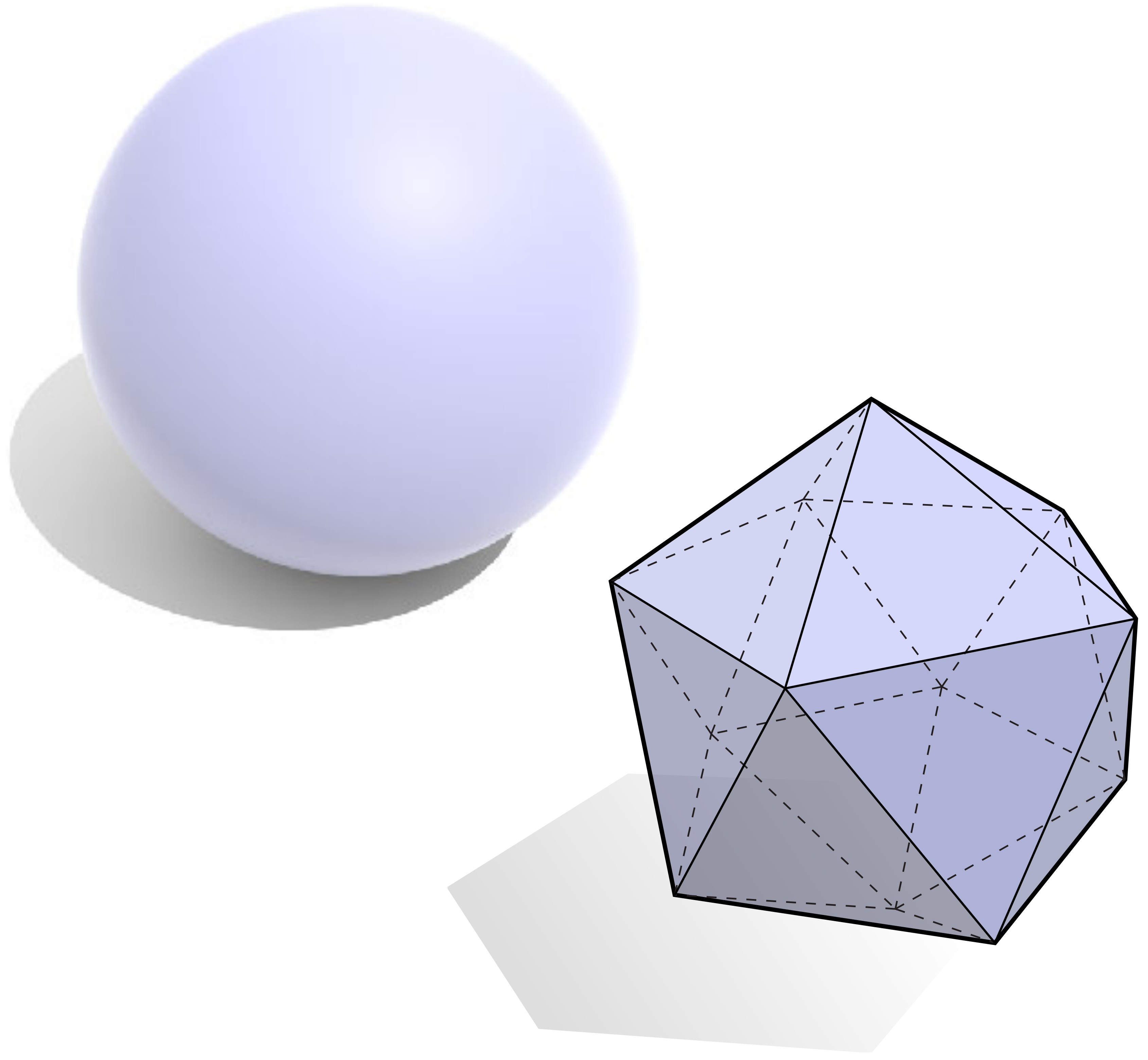
Discrete Descriptions of Curves & Surfaces

- Also *many* ways to discretize a surface
- For instance:
  - **implicit** — *e.g.*, zero set of scalar function on a grid
    - good for changing topology, high accuracy
    - expensive to store/adaptivity is harder
    - hard to solve sophisticated equations *on* surface
  - **explicit** — *e.g.*, polygonal surface mesh
    - changing topology, high-order continuity is harder
    - cheaper to store / adaptivity is much easier
    - more mature tools for equations *on* surfaces
- Don't be "religious"; use the right tool for the job!



Curves & Surfaces—Overview

- **Goal:** understand curves & surfaces from complementary smooth and discrete points of view.
- **<u>Smooth setting:</u>**
  - express geometry via differential forms
  - will first need to think about *vector-valued* forms
- **<u>Discrete setting:</u>**
  - use explicit mesh as domain
  - express geometry via discrete differential forms



- **Payoff:** will become very easy to switch back & forth between smooth setting (*scribbling in a notebook*) and discrete setting (*running algorithms on real data!*)

Vector Valued Differential Forms

Vector Valued k-Forms

- So far, we've defined a k-form as a linear map from k vectors to a real number
- For working with curves and surfaces in R^n, it will be essential to generalize this definition to *vector-valued k-forms*.
- In particular, a **vector-valued *k*-form** is a multi-linear map from k vectors in a vector space V to some other vector space U (not necessarily U=V)
  - So far, for instance, all of our forms have been R-valued k-forms on R^n *(V=R^n,U=R)*
  - A R^3-valued 2-form on R^2 would instead be a multilinear, fully-antisymmetric symmetric map from a pair of vectors u,v in R^2 to a single vector in R^3:

$$\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \qquad \alpha(u,v) = -\alpha(v,u)$$

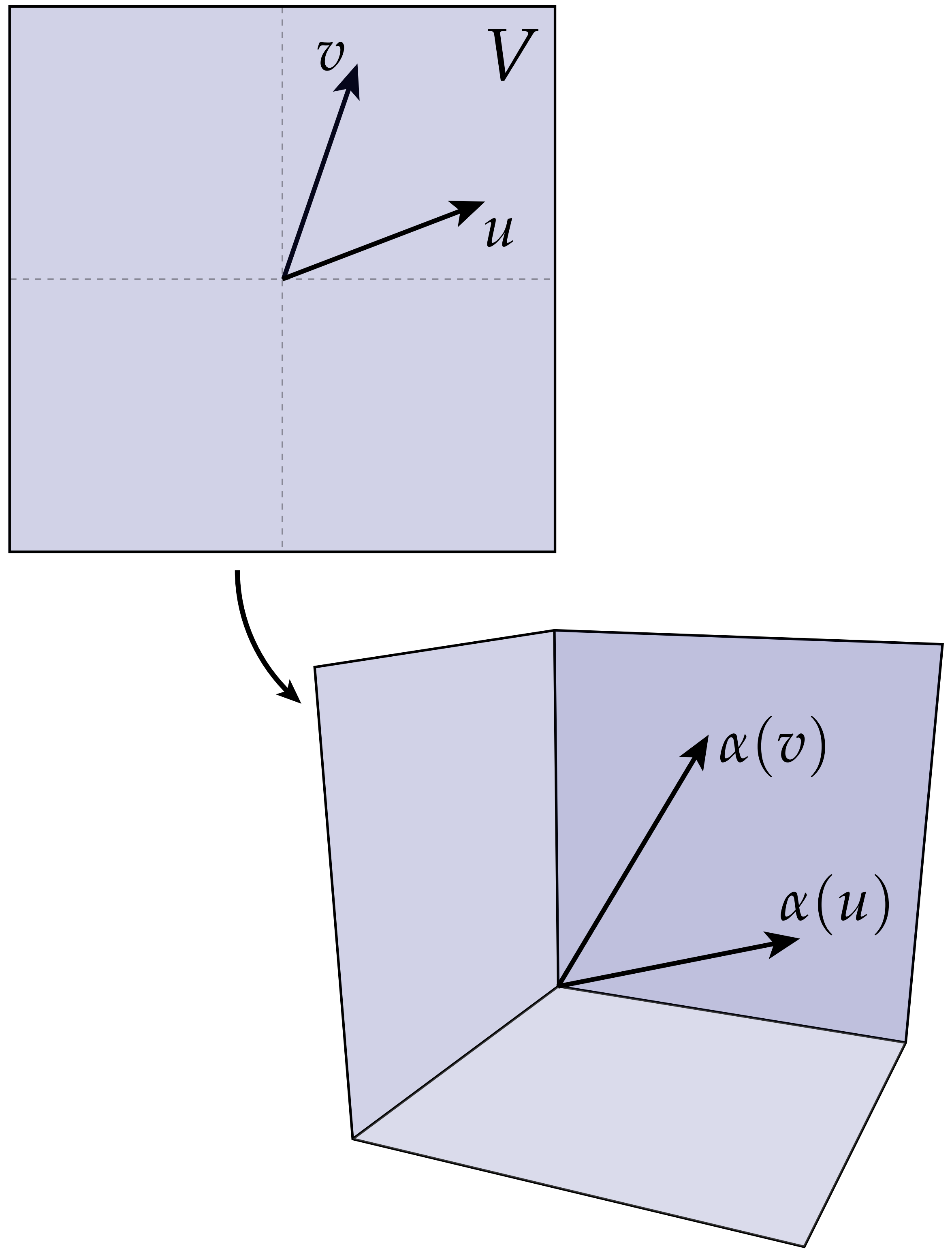
$$\alpha(au + b, w) = a\alpha(u,w) + b\alpha(v,w), \quad \forall u,v,w \in \mathbb{R}, a,b \in \mathbb{R}$$

Q: What kind of object is a R^2-valued 0-form on R^2?

Vector-Valued k-forms — Example

Consider for instance the following R^3-valued 1-form on R^2:

$$\alpha := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^1 + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} e^2$$



Q: What do we get if we evaluate this 1-form on the vector

$$u := e_1 - e_2$$

A: Evaluation is not much different from real-valued forms:

$$\alpha(u) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \overset{1}{\cancel{e^1(e_1 - e_2)}} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \overset{-1}{\cancel{e^2(e_1 - e_2)}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Key idea: coefficients just have a different type

Wedge Product of Vector-Valued k-Forms

- Most important change is how we evaluate wedge product for vector-valued forms.
- Consider for instance a pair of R^3-valued 1-forms:

$$\alpha, \beta : V \to \mathbb{R}^3$$

- To evaluate their wedge product on a pair of vectors u,v we would normally write:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

- If α and β were *real*-valued, then $\alpha(u)$, $\beta(v)$, $\alpha(v)$, $\beta(u)$, would just be *real numbers*, so we could just multiply the two pairs and take their difference.
- But what does it mean to take the "product" of two vectors from R^3?
- Many possibilities (*e.g.*, dot product), but if we want result to be an R^3-valued 2-form, the product we choose must produce another 3-vector!

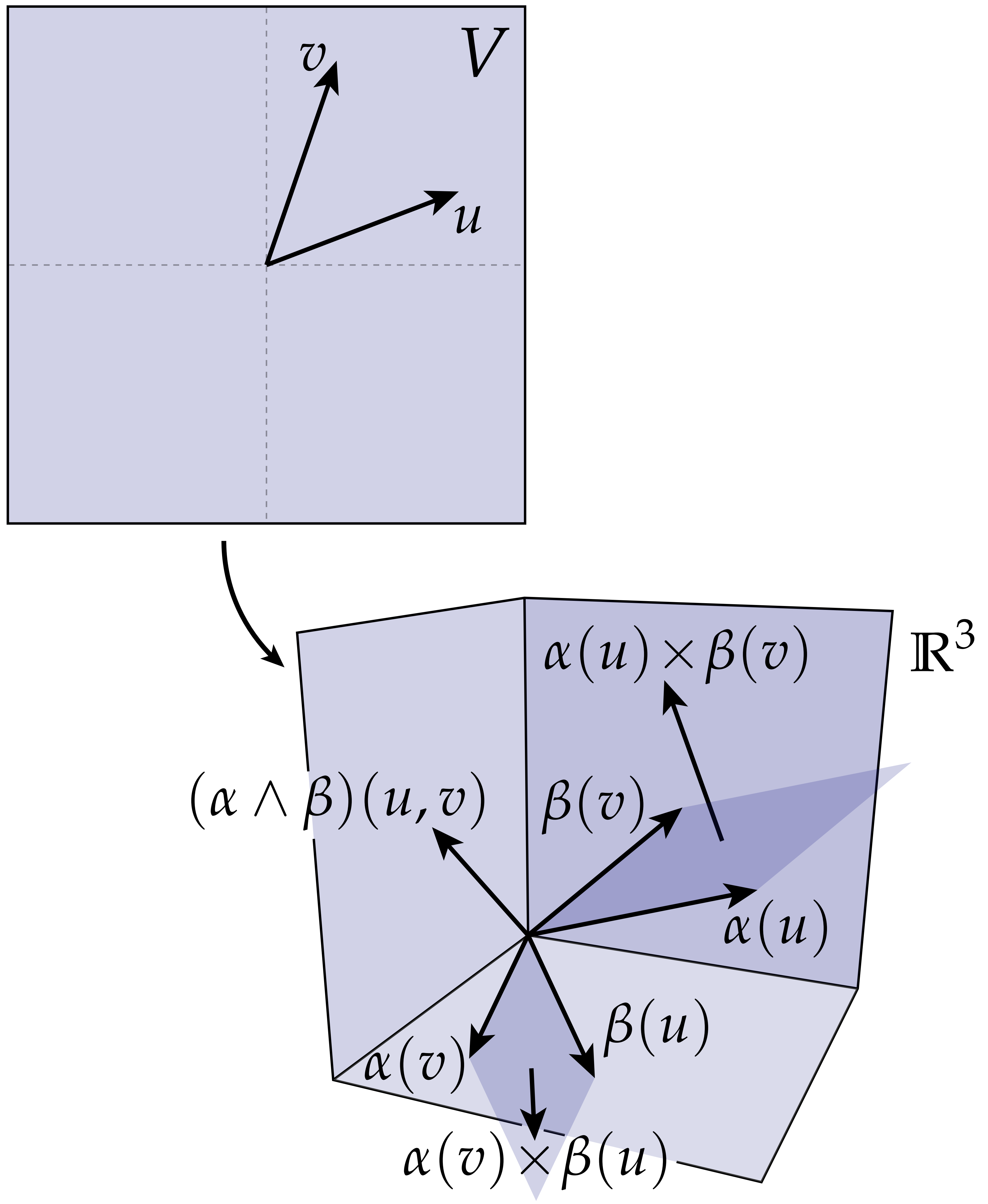
Wedge Product of R^3-Valued k-Forms

- Most common case for our study of surfaces:
  - k-forms are R^3-valued
  - use **cross product** to multiply 3-vectors

$$\alpha, \beta : V \to \mathbb{R}^3$$

$$\alpha \wedge \beta : V \times V \to \mathbb{R}^3$$

$$\boxed{(\alpha \wedge \beta)(u, v) := \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u)}$$



R^3-valued 1-forms: Antisymmetry & Symmetry

With real-valued forms, we observed antisymmetry in both the wedge product of 1-forms as well as the application of the 2-form to a pair of vectors, *i.e.*,

$$(\alpha \wedge \beta)(u,v) = -(\alpha \wedge \beta)(v,u)$$
$$(\beta \wedge \alpha)(u,v) = -(\alpha \wedge \beta)(u,v)$$

What happens w/ R^3-valued 1-forms? Since cross product is antisymmetric, we get

$$\begin{aligned}(\alpha \wedge \beta)(v,u) &= \alpha(v) \times \beta(u) - \alpha(u) \times \beta(v) \\ &= -(\alpha(u) \times \beta(v) - \alpha(v) \times \beta(u))\end{aligned}$$
$$\Rightarrow \boxed{(\alpha \wedge \beta)(u,v) = -(\alpha \wedge \beta)(v,u)}$$

(no change)

$$\begin{aligned}(\beta \wedge \alpha)(u,v) &= \beta(u) \times \alpha(v) - \beta(v) \times \alpha(u) \\ &= \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u) \\ &= (\alpha \wedge \beta)(u,v)\end{aligned}$$
$$\Rightarrow \boxed{\alpha \wedge \beta = \beta \wedge \alpha}$$

(big change!)

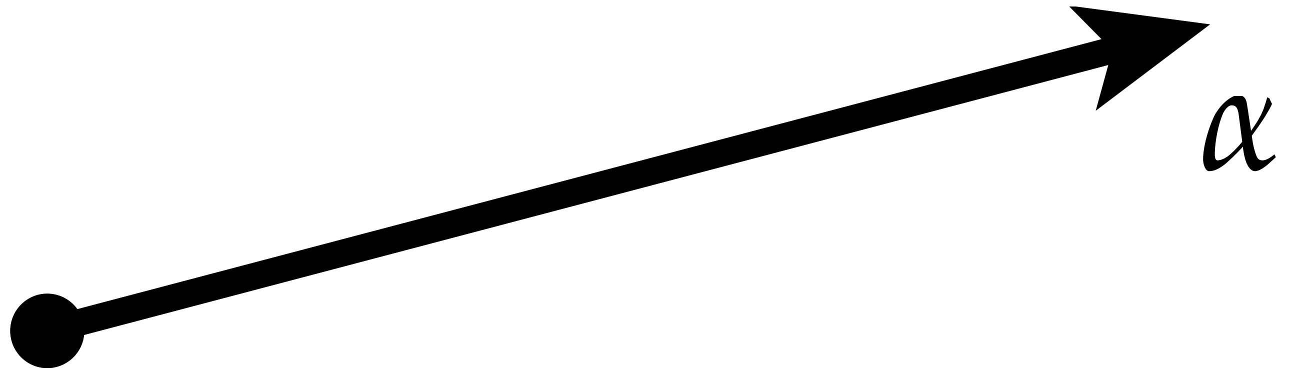
$\mathbb{R}^3$-valued 1-forms: Self-Wedge

Likewise, we saw that wedging a real-valued 1-form with itself yields zero:

$$\alpha \wedge \alpha = 0$$

Q: What was the *geometric* interpretation?

A: Parallelogram spanned by two copies of the same vector has zero area!



…But, no longer true with $(\mathbb{R}^3, \times)$-valued 1-forms:

$$(\alpha \wedge \alpha)(u, v) = \alpha(u) \times \alpha(v) - \alpha(v) \times \alpha(u) = 2\alpha(u) \times \alpha(v) \neq 0$$

Geometric meaning will become clearer as we work with surfaces.

Vector-Valued Differential k-Forms

- Just as we distinguished between a *k-form* (value at a single point) and a *differential k-form* (value at ever point in space), we will also say that a *vector-valued differential k-form* is a vector-valued k-form at each point of space.

- Just like any differential form, a vector-valued differential *k*-form gets evaluated on k vector fields $X_1, \ldots, X_k$.

- **<u>Example:</u>** an R^3-valued differential 1-form on R^2 (with coordinates u,v):

$$\alpha := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} du + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dv$$

Q: What does this 1-form do to any given vector field X on the plane?

A: It simply "copies" it to the *yz*-plane in 3D.

Exterior Derivative on Vector-Valued Forms

Unlike the wedge product, not much changes with the exterior derivative.
For instance, if we have an R^n-valued k-form we can simply imagine we have n real-valued k-forms and differentiate as usual.

Example.

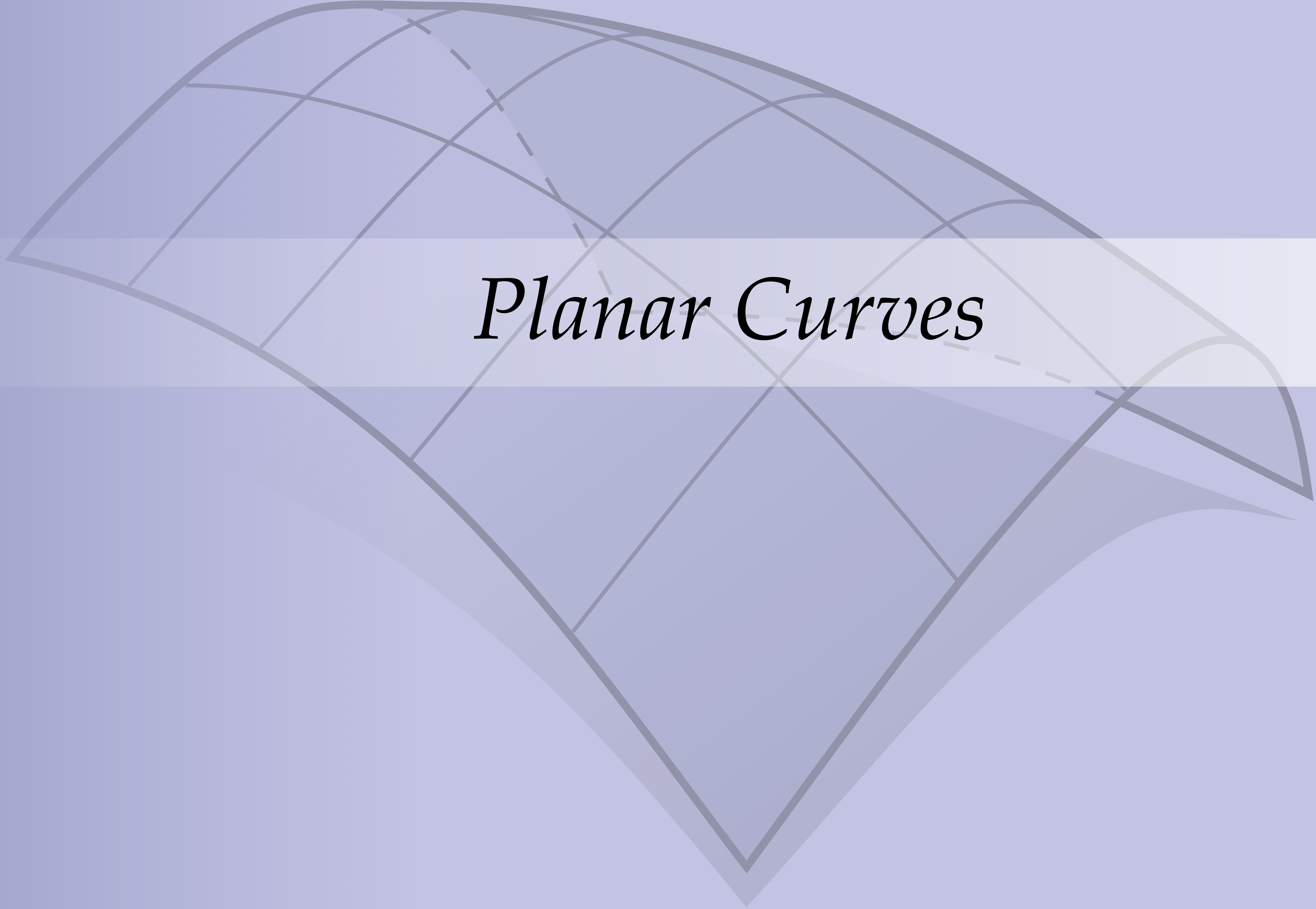
Consider an $\mathbb{R}^2$-valued differential 0-form $\phi_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix}$

Then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = \begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy$

Example.

Consider an $\mathbb{R}^2$-valued differential 1-form $\alpha_{(x,y)} := \begin{bmatrix} x^2 \\ xy \end{bmatrix} dx + \begin{bmatrix} xy \\ y^2 \end{bmatrix} dy$

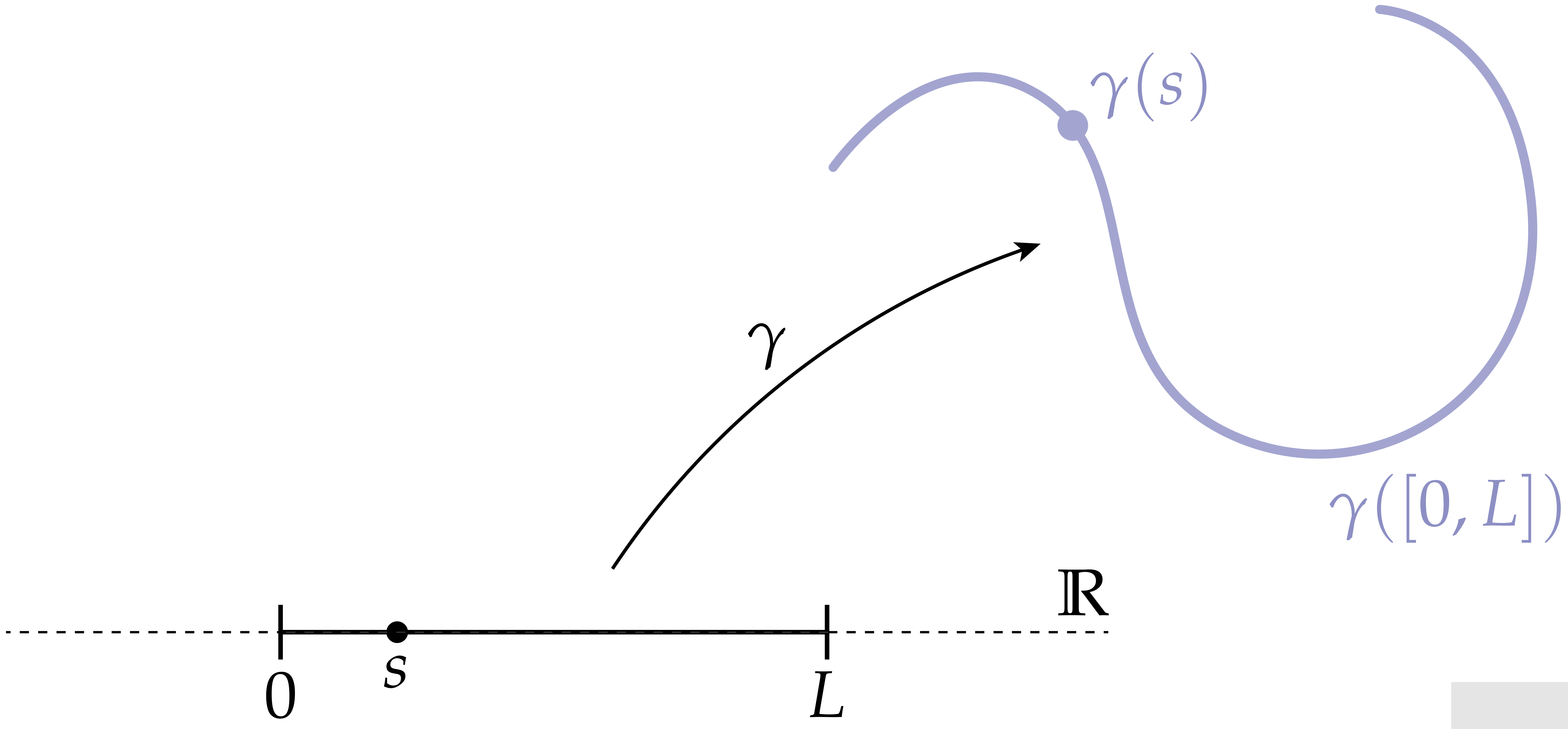
Then $d\alpha = \left(\begin{bmatrix} 2x \\ y \end{bmatrix} dx + \begin{bmatrix} 0 \\ x \end{bmatrix} dy \right) \wedge dx + \left(\begin{bmatrix} y \\ 0 \end{bmatrix} dx + \begin{bmatrix} x \\ 2y \end{bmatrix} dy \right) \wedge dy = \begin{bmatrix} y \\ -x \end{bmatrix} dx \wedge dy$



Planar Curves

Parameterized Plane Curve

- A **parameterized plane curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in the plane $\mathbb{R}^2$:



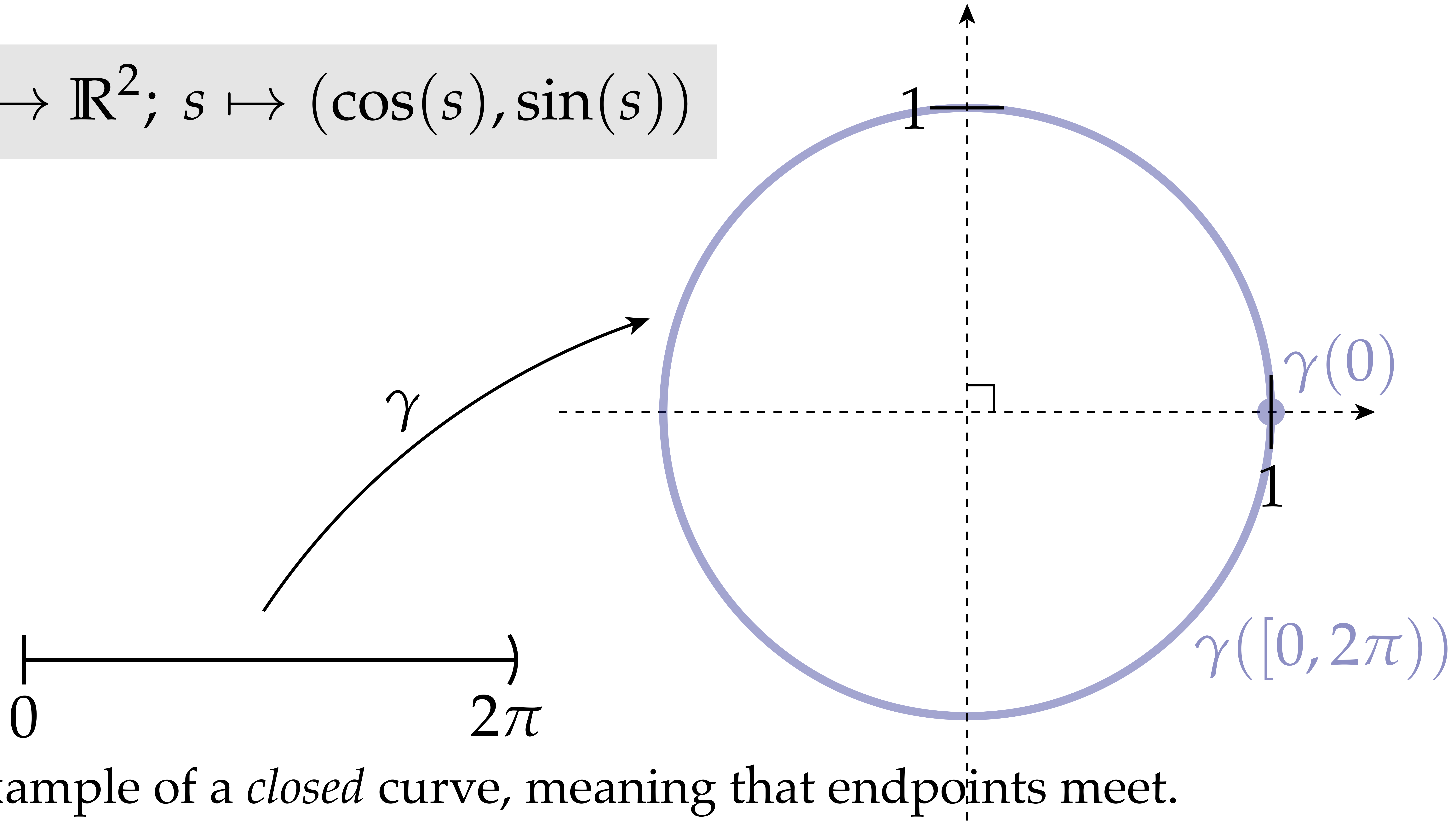
*Continuous, differentiable, smooth…

$$\gamma : [0,L] \to \mathbb{R}^2$$

Curves in the Plane—Example

- As an example, we can express a circle as a parameterized curve γ:

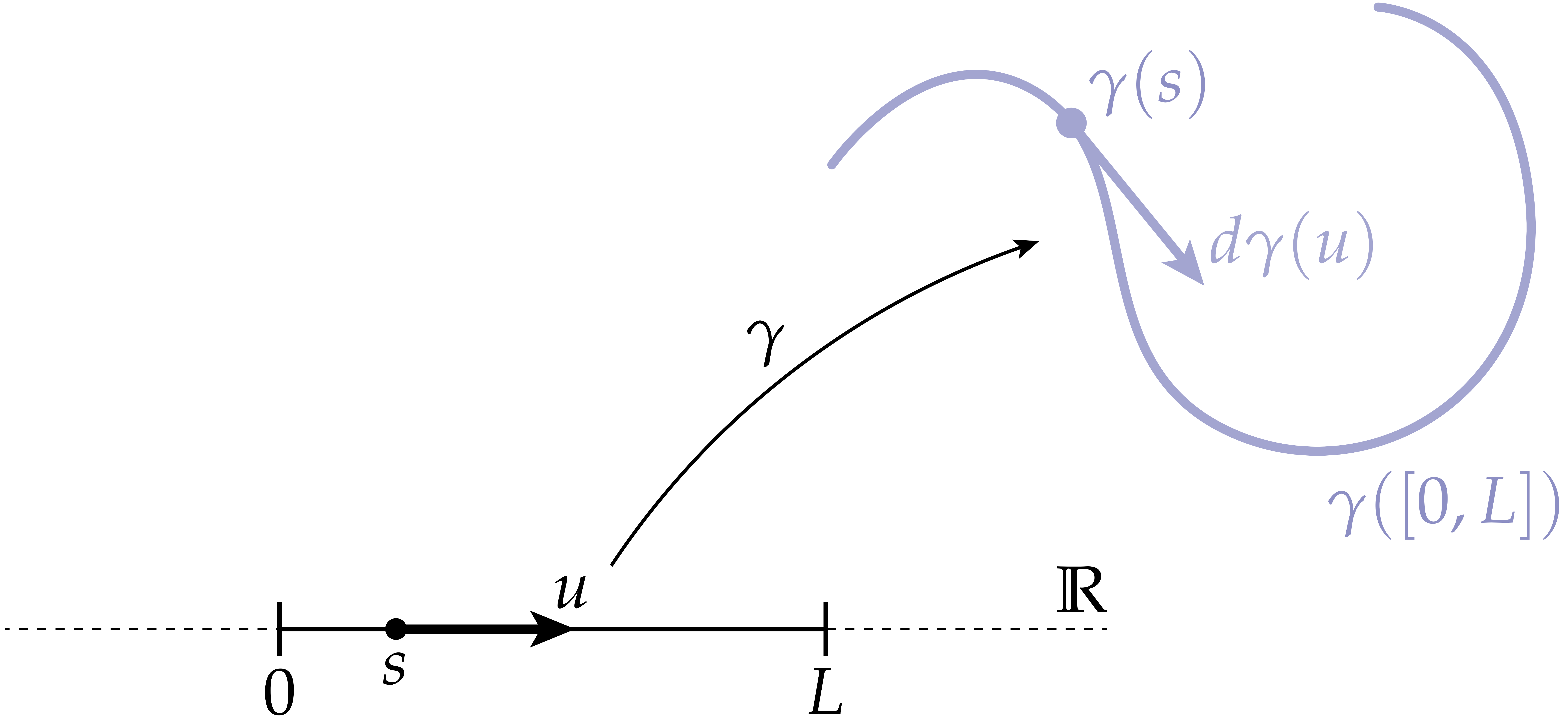
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$



The circle is an example of a *closed* curve, meaning that endpoints meet.

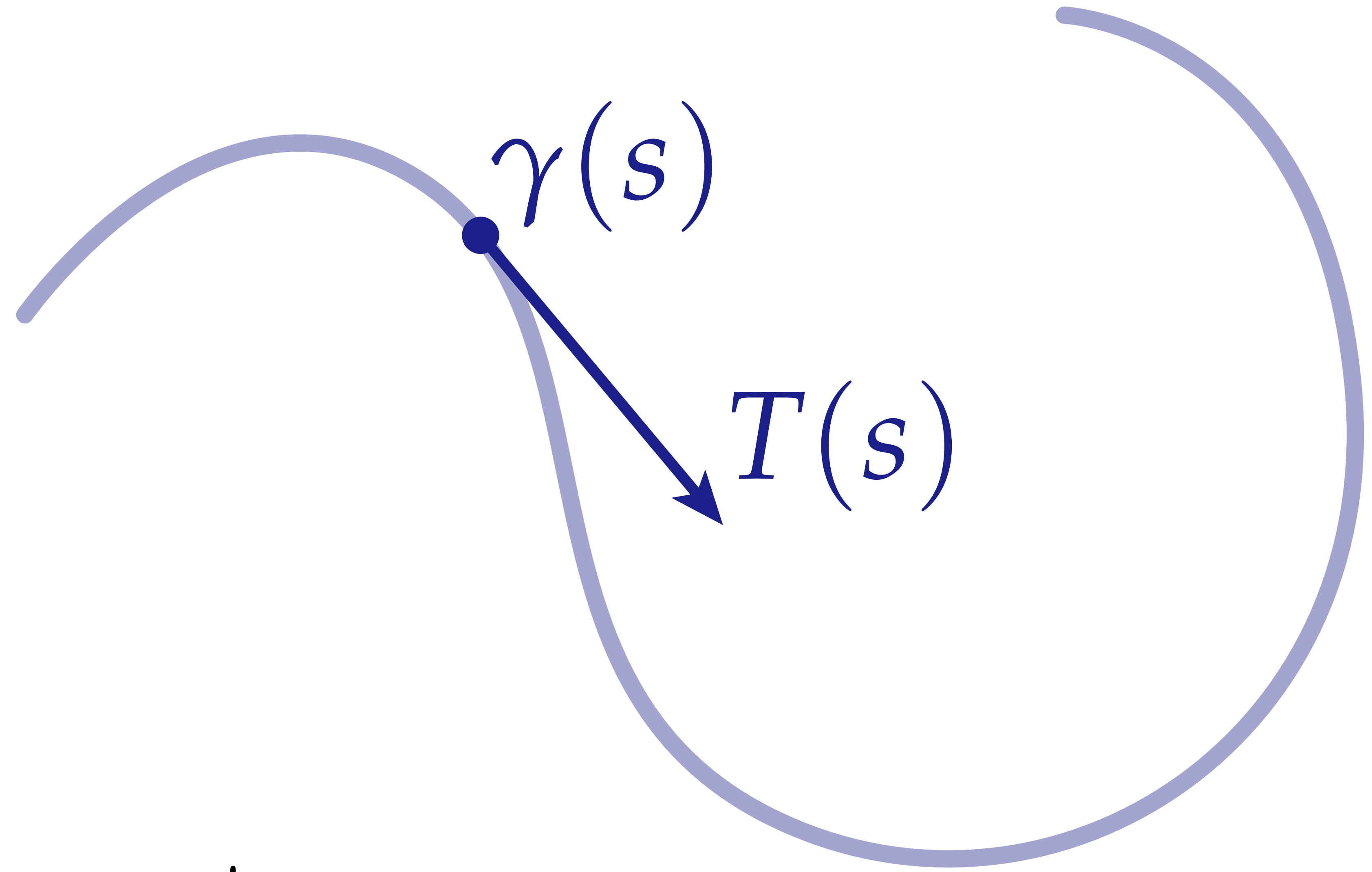
Differential of a Curve

- If we think of a parameterized curve as an R^2-valued 0-form on an interval of the real line, then the differential (or exterior derivative) says how vectors on the domain get mapped into the plane:



Tangent of a Curve

- Informally, a vector is *tangent* to a curve if it "just barely grazes" the curve.



- More formally, the **unit tangent** (or just **tangent**) of a regular curve is the map obtained by normalizing its first derivative:

$$T(s) := \tfrac{d}{ds}\gamma(s) \Big/ \left|\tfrac{d}{ds}\gamma(s)\right| = d\gamma(\tfrac{d}{ds}) \Big/ \left|d\gamma(\tfrac{d}{ds})\right|$$

- If the derivative already has unit length, then we say the curve is **arc-length parameterized** and can write the tangent as just

$$T(s) := \tfrac{d}{ds}\gamma(s) = d\gamma(\tfrac{d}{ds})$$

Tangent of a Curve—Example

- Let's compute the unit tangent of a circle:

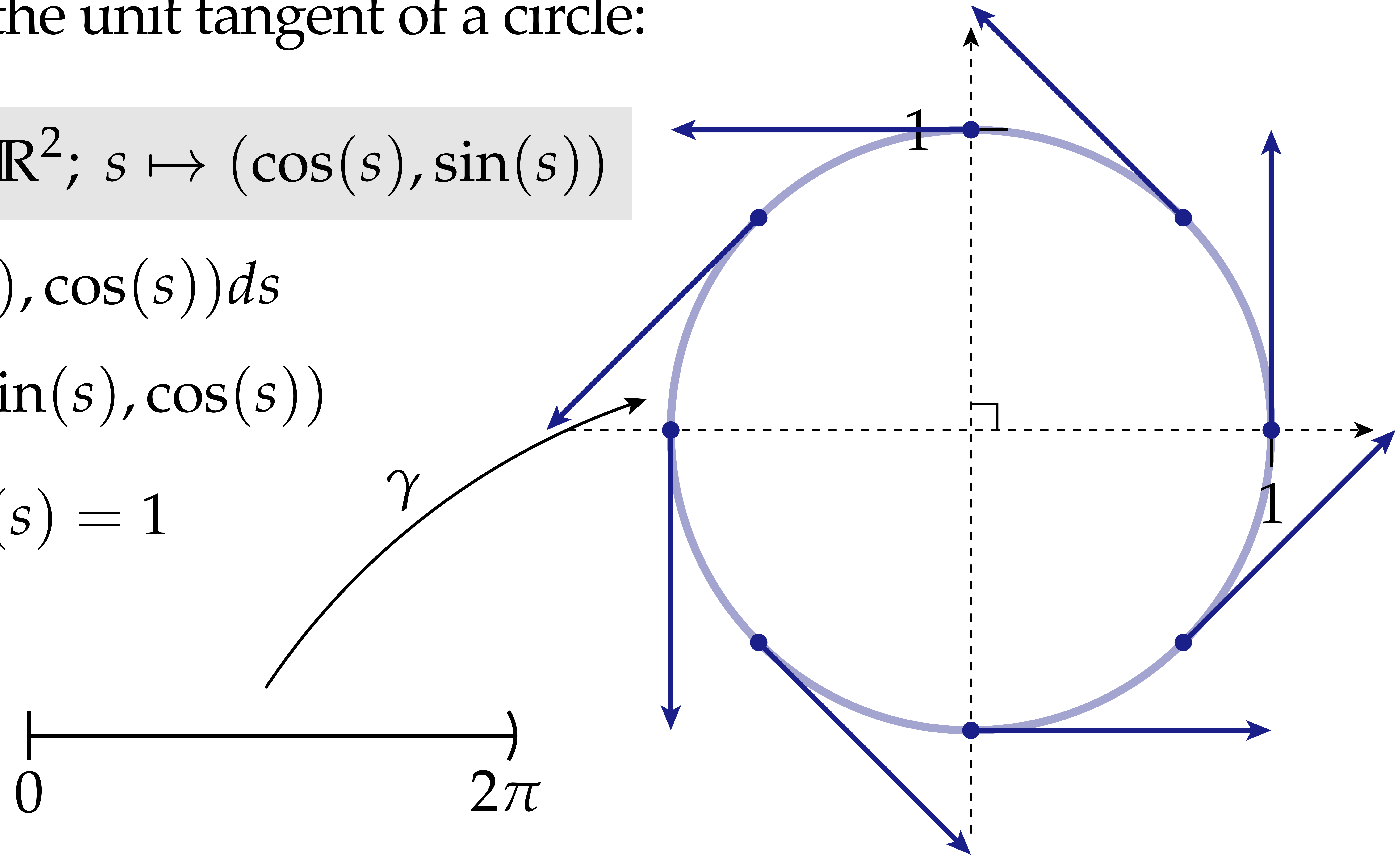
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$d\gamma = (-\sin(s), \cos(s))ds$$

$$d\gamma(\tfrac{\partial}{\partial s}) = (-\sin(s), \cos(s))$$

$$\cos^2(s) + \sin^2(s) = 1$$

$$\Rightarrow T = d\gamma(\tfrac{d}{ds})$$

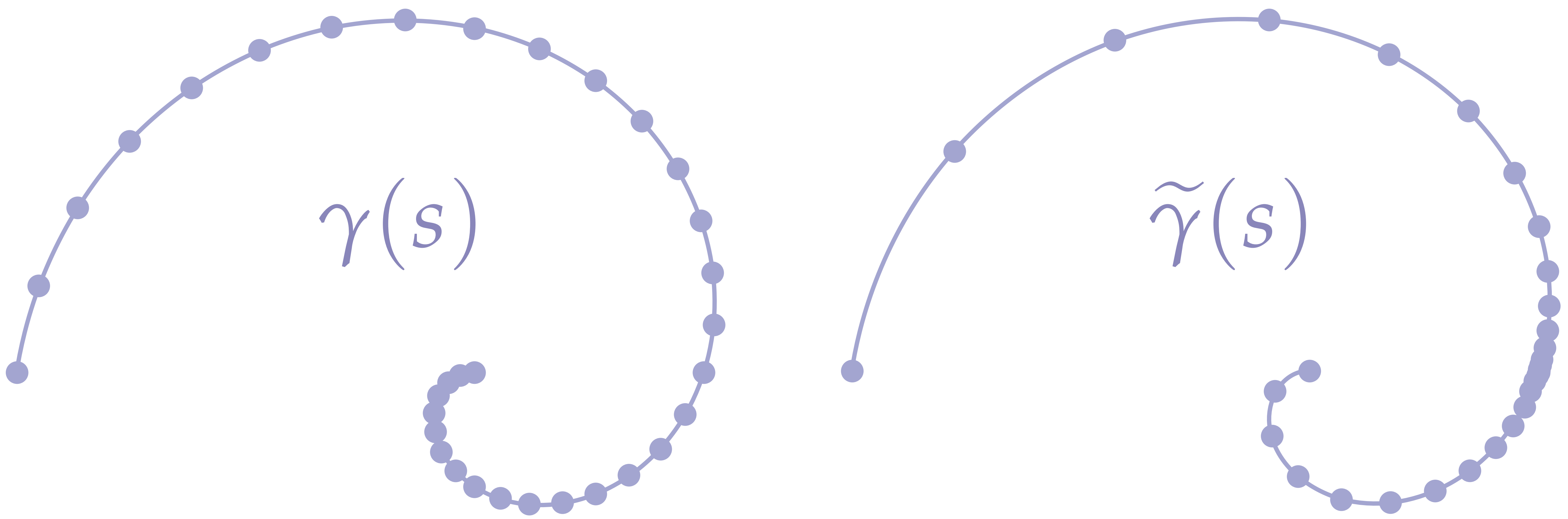


Reparameterization of a Curve

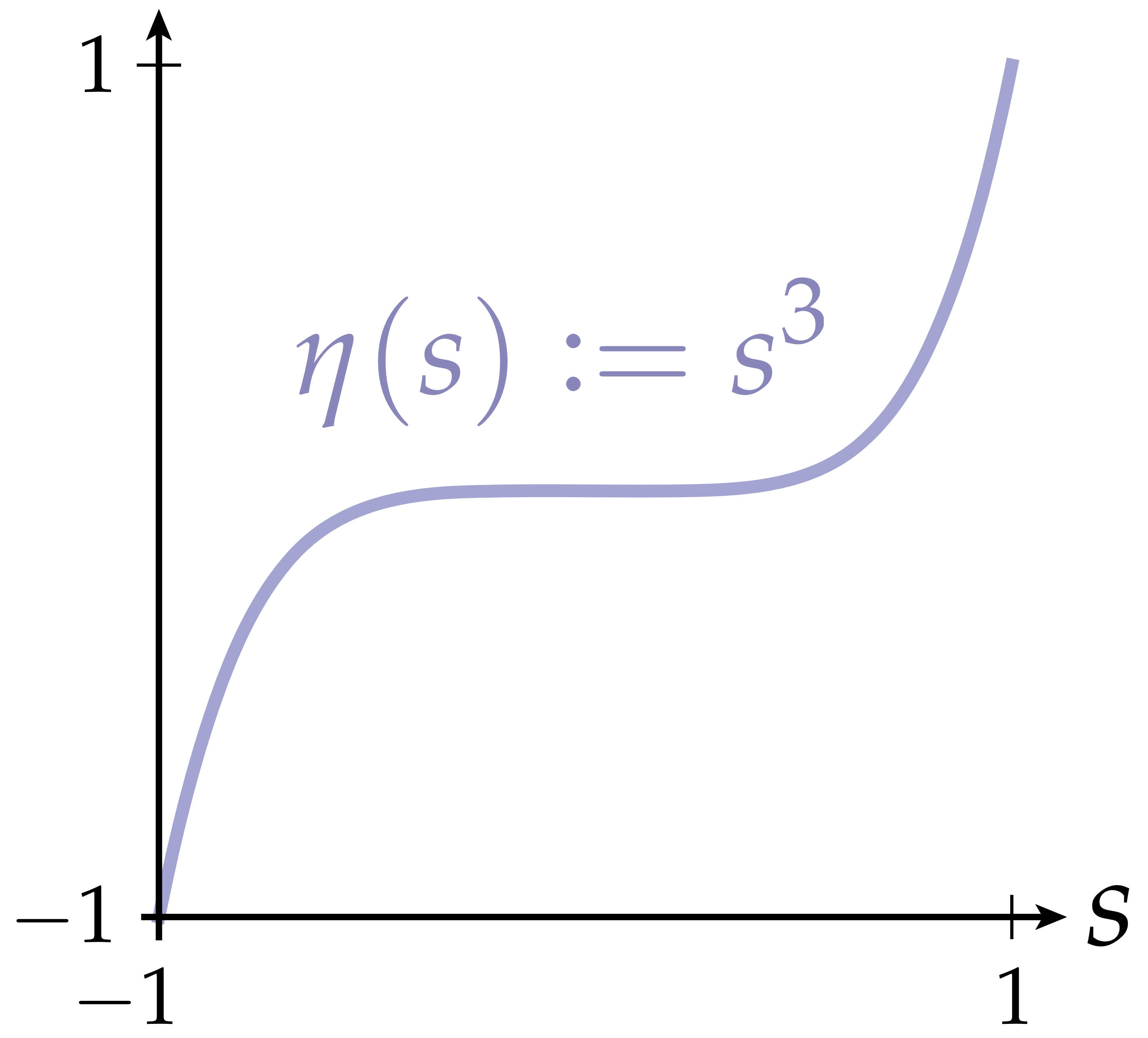
- We can *reparameterize* a curve $\gamma : \mathbb{R} \supset I \to \mathbb{R}$ by composing it with a bijection $\eta : I \to I$ to obtain a new parameterized curve

$$\tilde{\gamma}(s) := \gamma(\eta(s))$$

- The *image* of the new curve is the same, even though the map itself changes. For example:

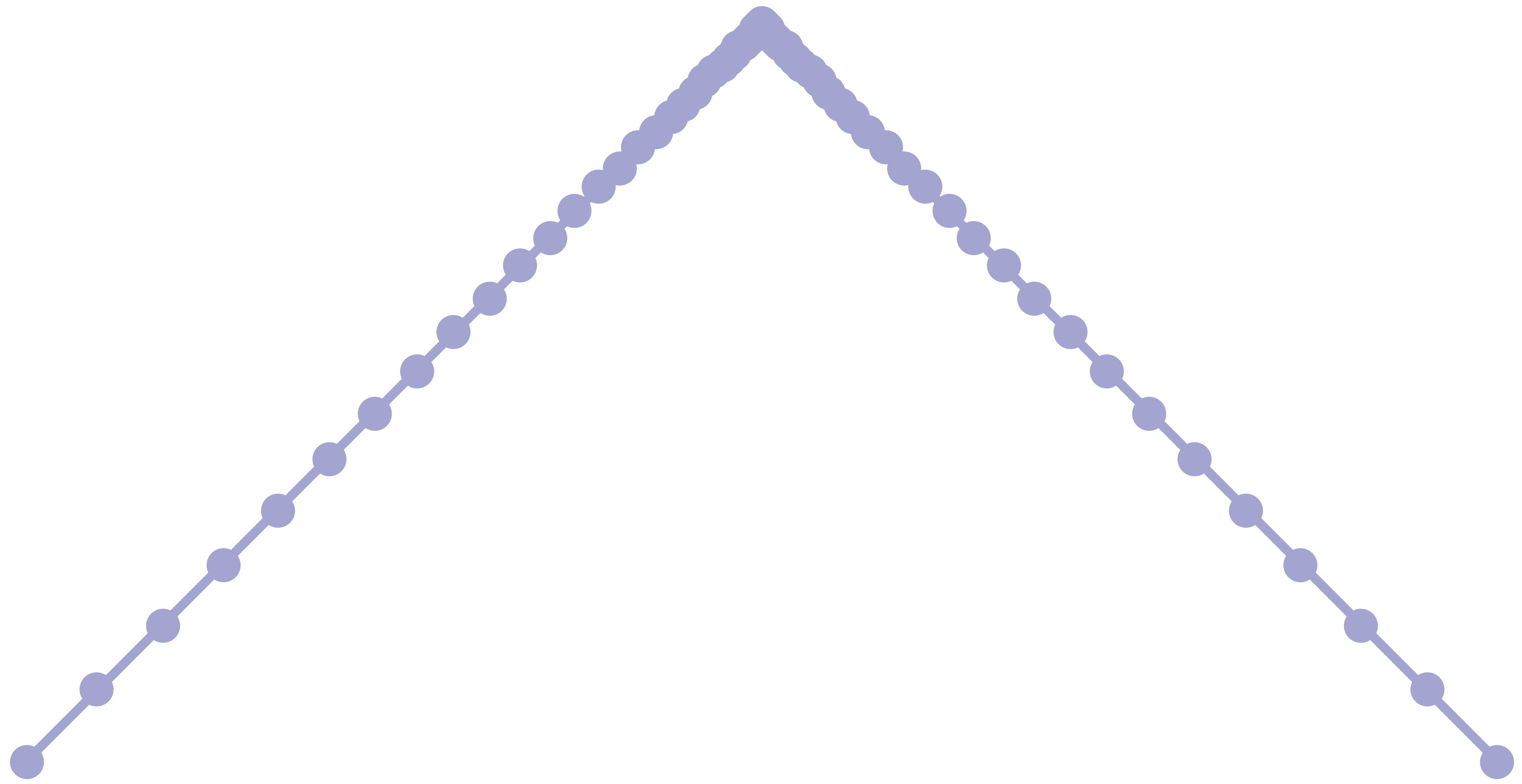


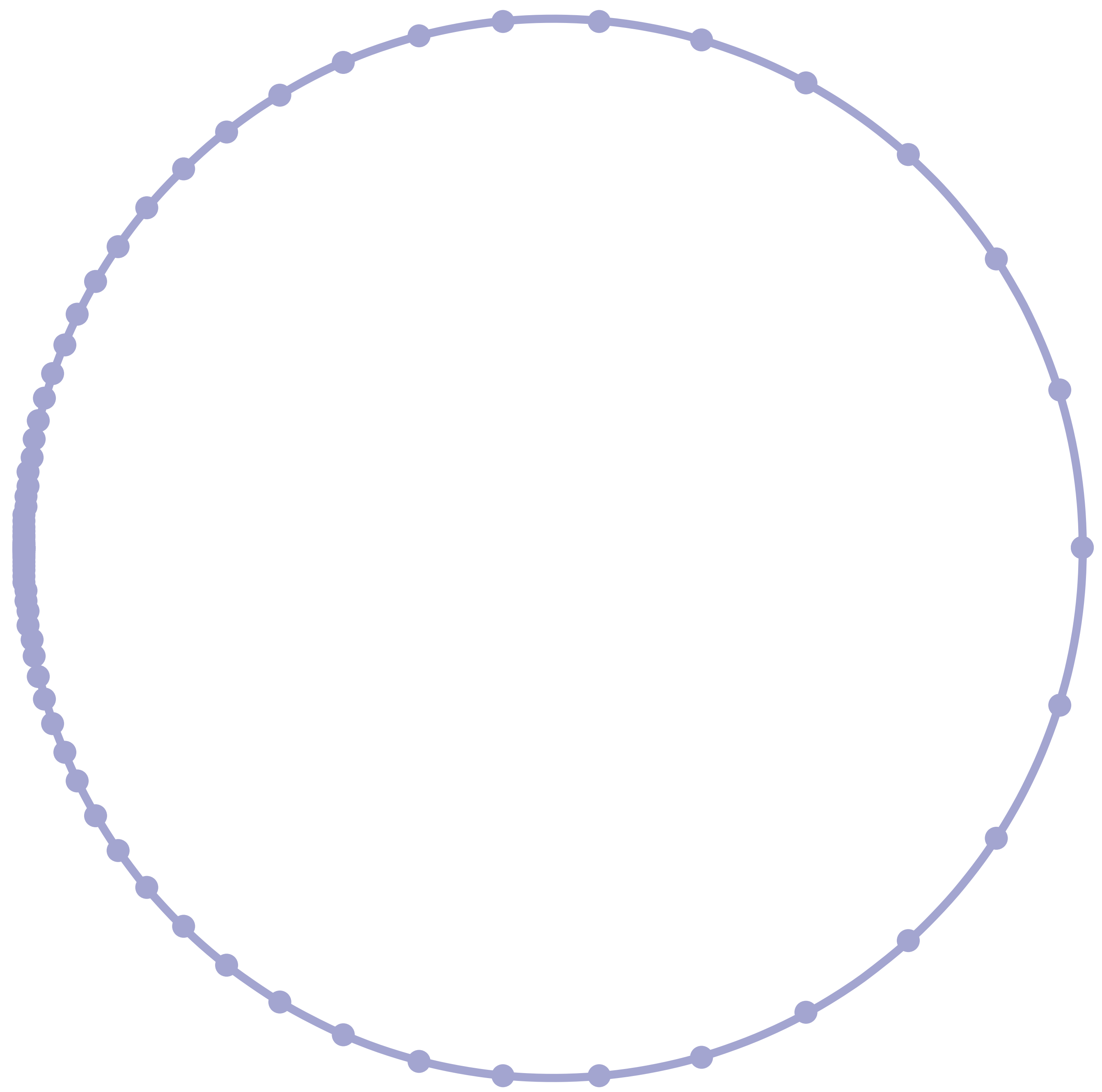
$\gamma(s) := (1+s)(\cos(\pi s), \sin(\pi s))$



Regular Curve / Immersion

- A parameterized curve is *regular* (or *immersed*) if the differential is nonzero everywhere, *i.e.*, if the curve "never slows to zero"
- Without this condition, a parameterized curve may look non-smooth but actually be differentiable everywhere, or look smooth but fail to have well-defined tangents.



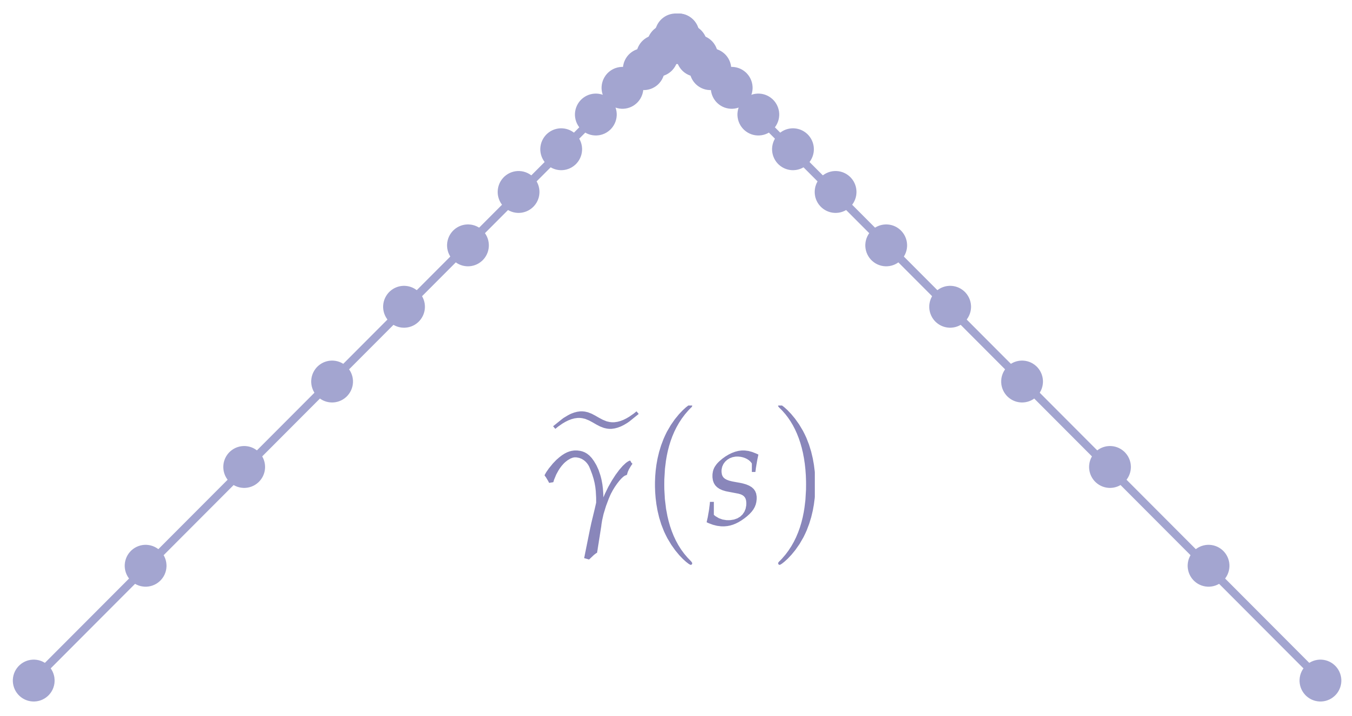


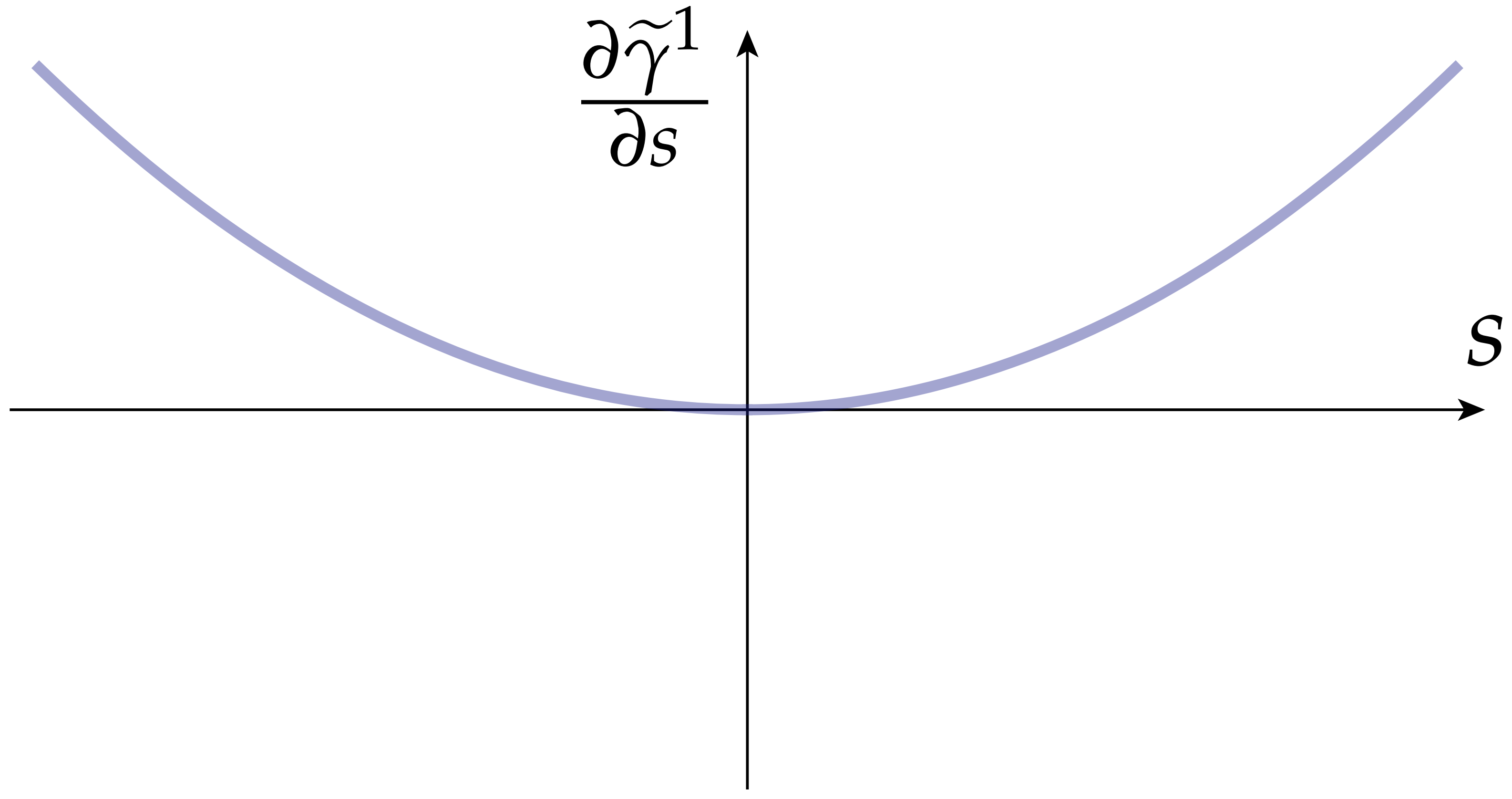
Irregular Curve—Example

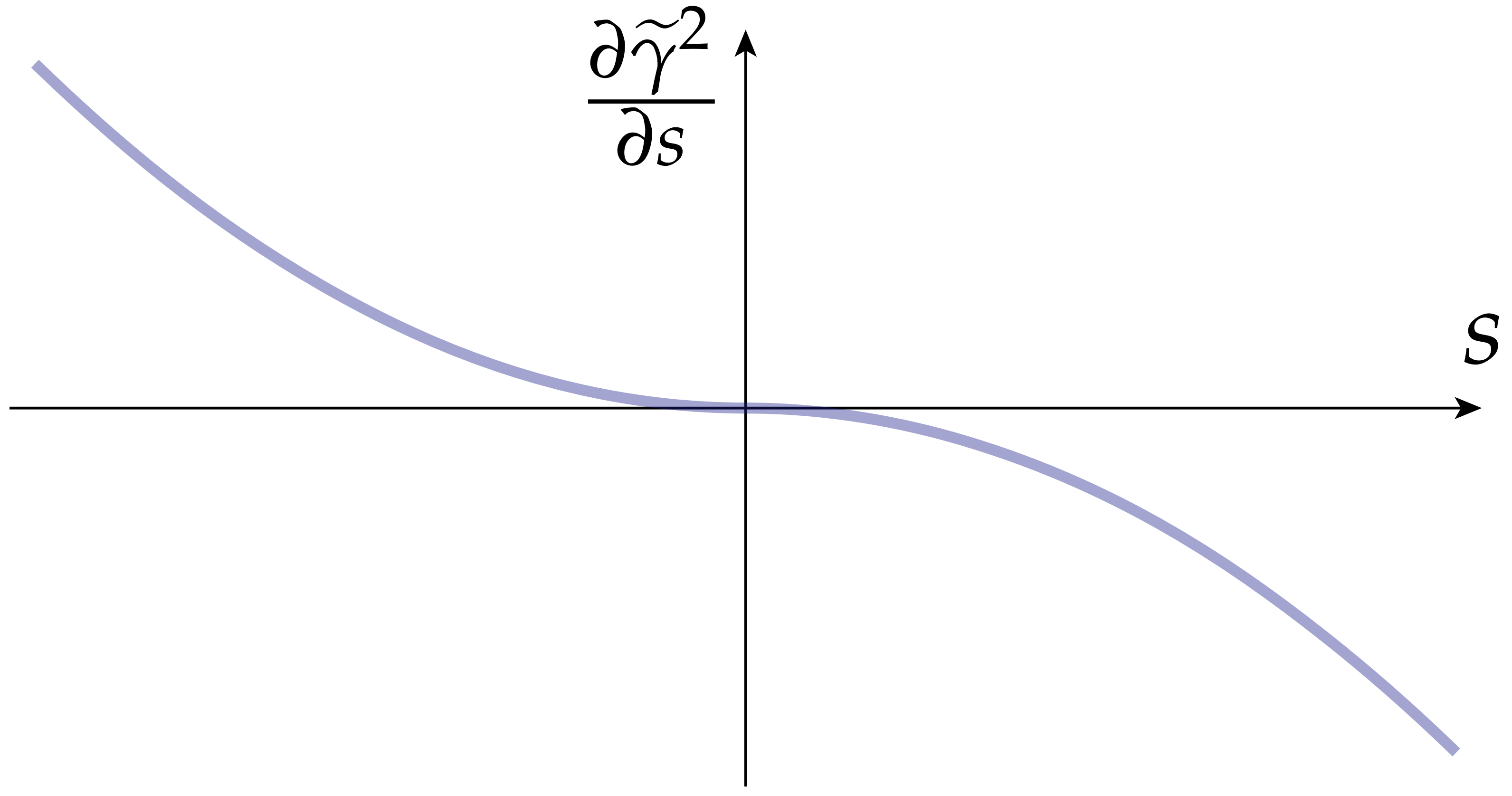
- Consider the reparameterization of a piecewise linear curve:

$$\eta(s) := s^3 \qquad \gamma(s) := \begin{cases} (s,s) & s \leq 0 \\ (s,-s) & s > 0 \end{cases} \qquad \widetilde{\gamma}(s) = \begin{cases} (s^3,s^3) & s \leq 0, \\ (s^3,-s^3) & s > 0 \end{cases}$$

- Even though the reparameterized curve has a continuous first derivative, this derivative goes to zero at $s = 0$:



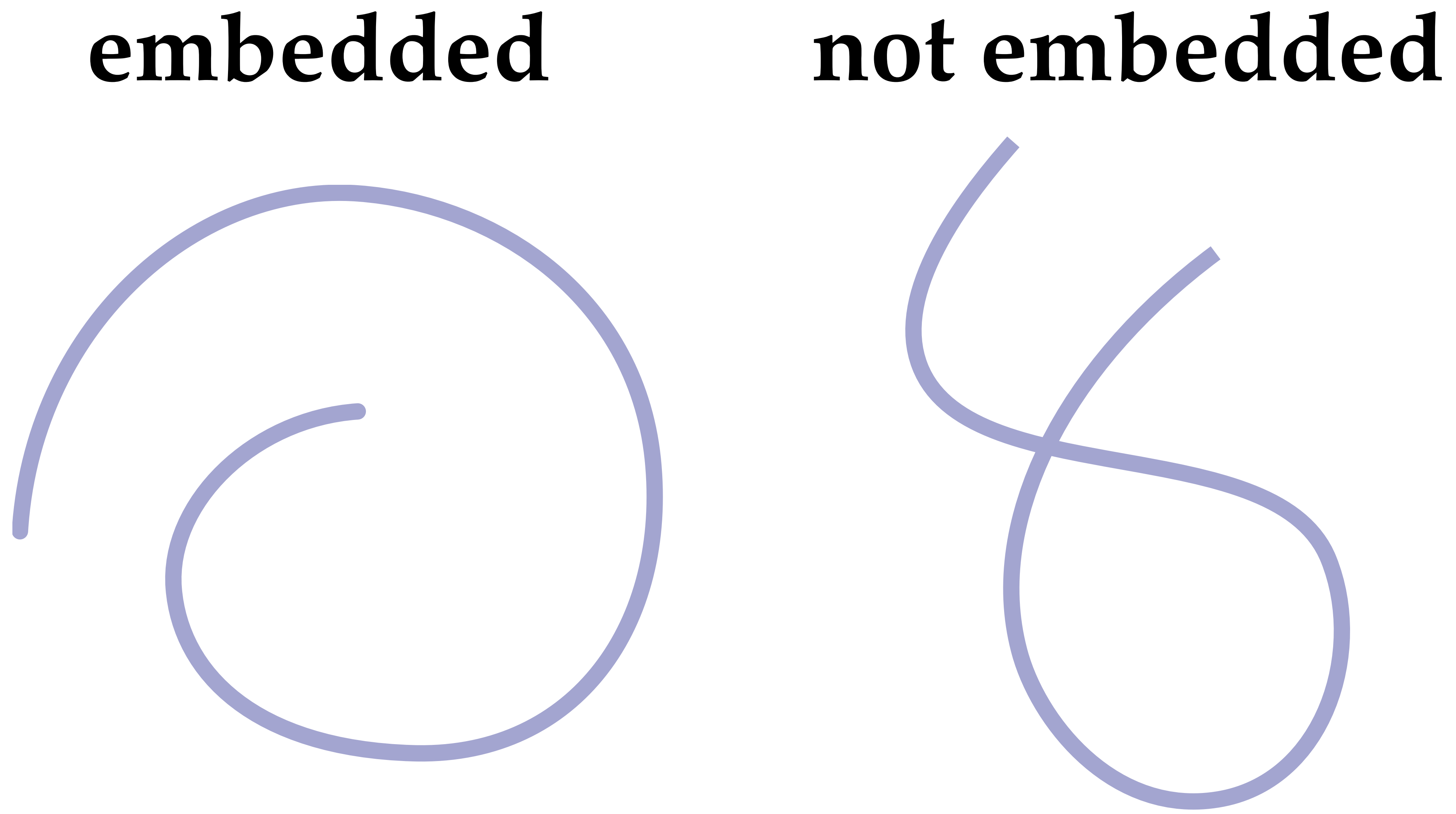


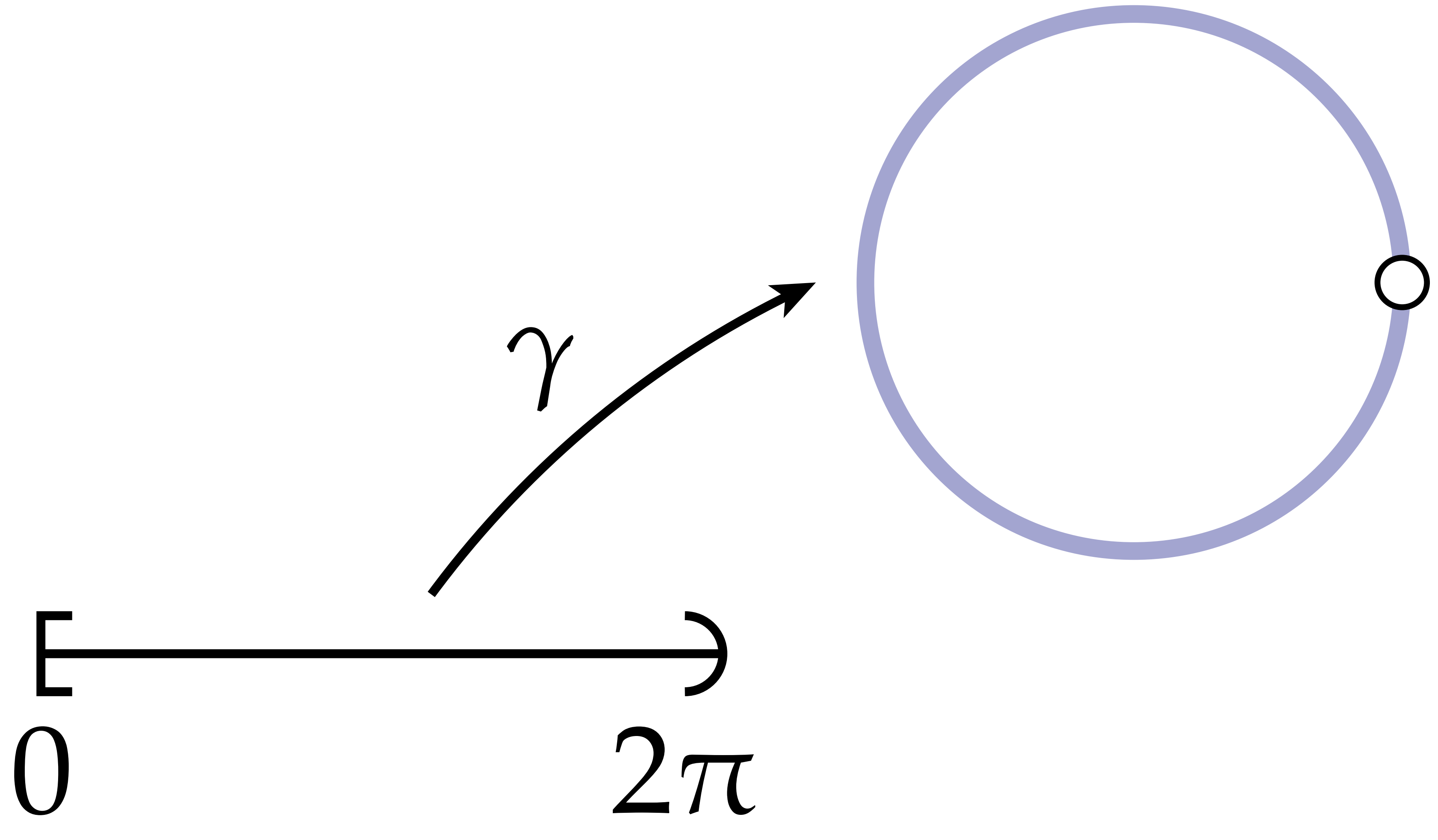


- Hence, (still) can't define tangent at zero.

Embedded Curve

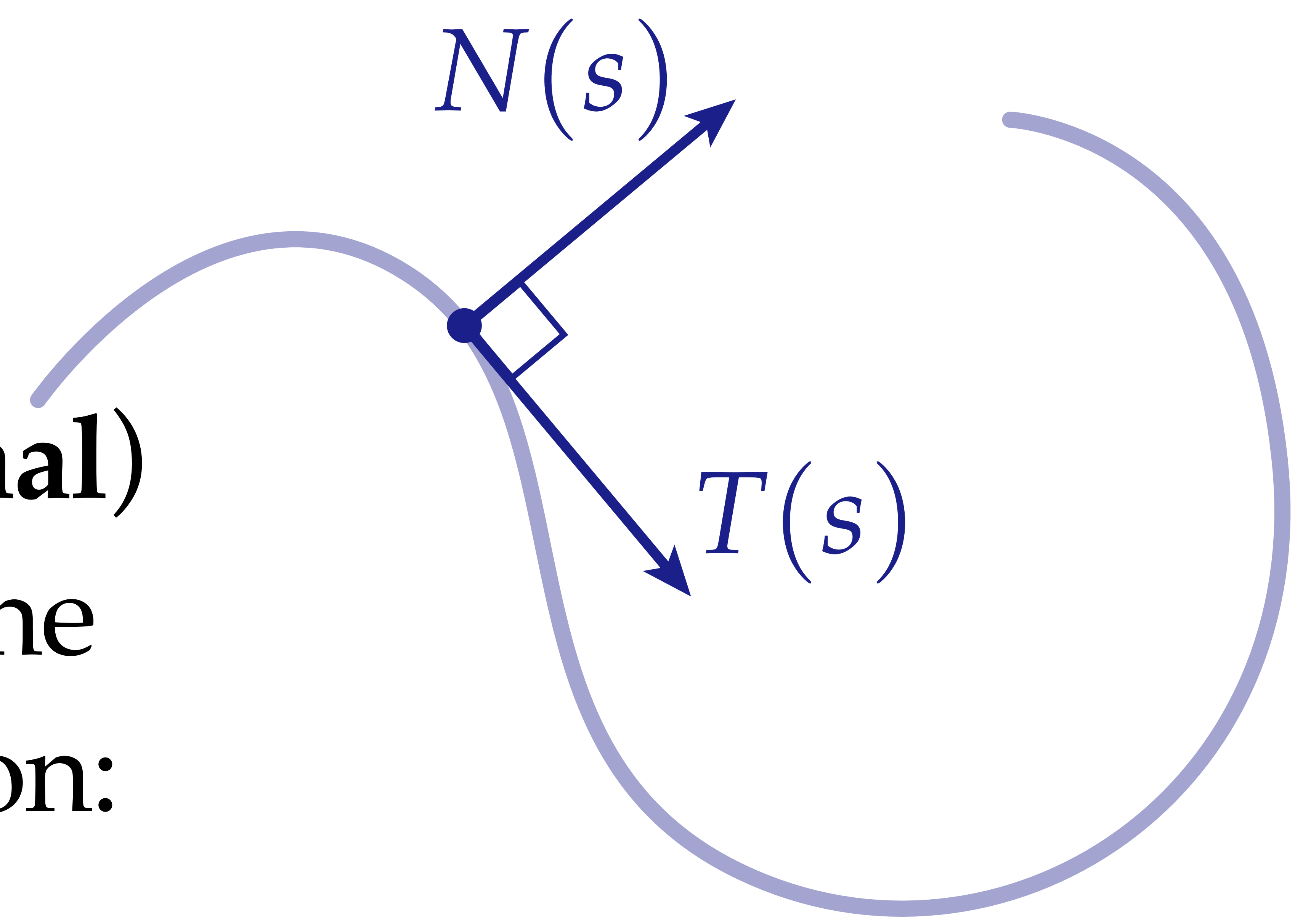
- Roughly speaking, an *embedded* curve does not cross itself
- More precisely, a curve is embedded if it is a continuous and injective map from its domain to its image, and the inverse map is also continuous
- **Q:** What's an example of a continuous injective curve that is not embedded?
- **A:** A half-open interval mapped to a circle (inverse is not continuous)





Normal of a Curve

- Informally, a vector is *normal* to a curve if it "sticks straight out" of the curve.



- More formally, the **unit normal** (or just **normal**) can be expressed as a quarter-rotation $\mathcal{J}$ of the unit tangent in the counter-clockwise direction:

$$N(s) := \mathcal{J}T(s)$$

- In coordinates (x,y), a quarter-turn can be achieved by* simply exchanging x and y, and then negating y:

$$(x, y) \overset{\mathcal{J}}{\mapsto} (-y, x)$$

*Why does this work?

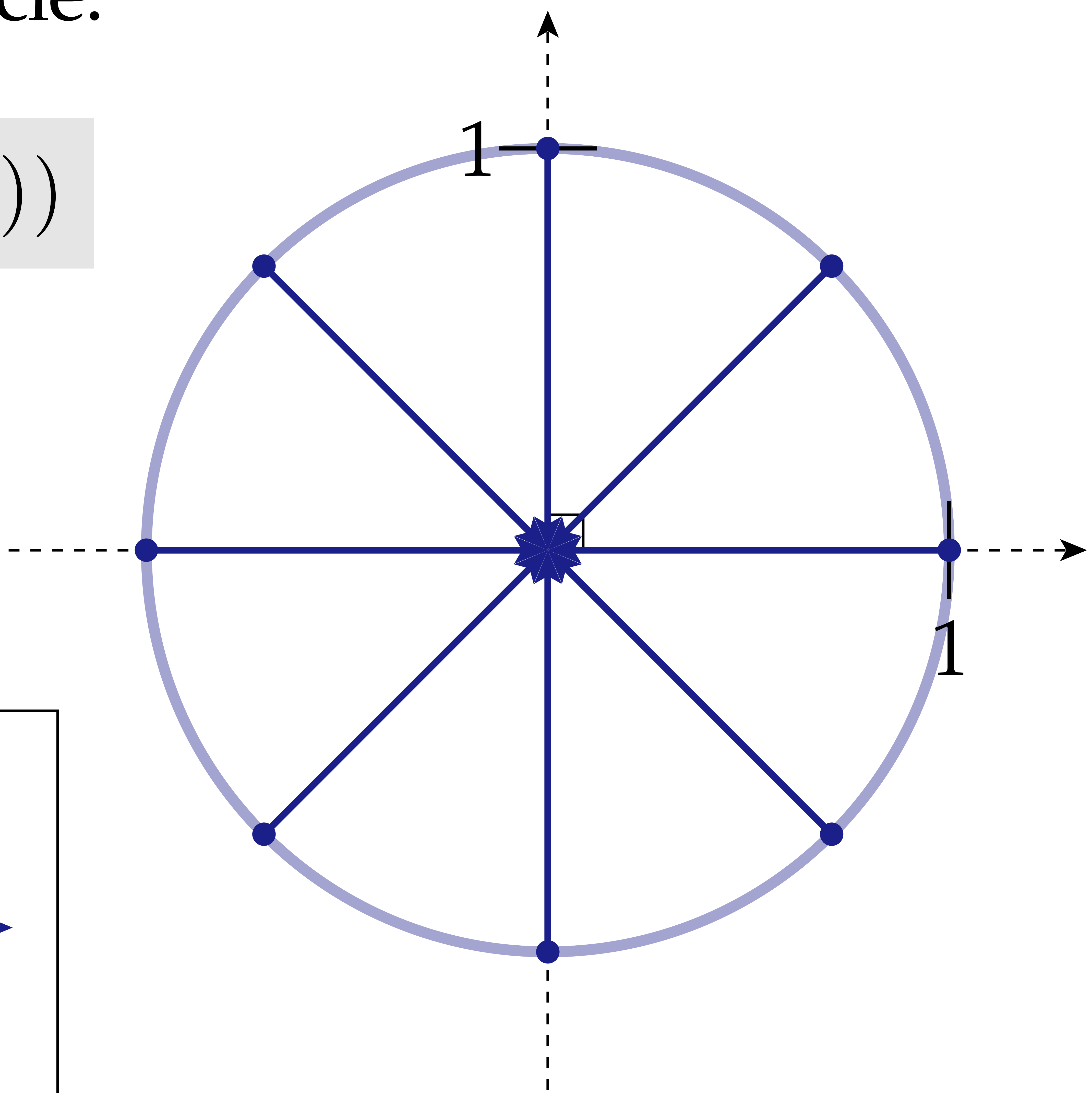
Normal of a Curve—Example

- Let's compute the unit normal of a circle:

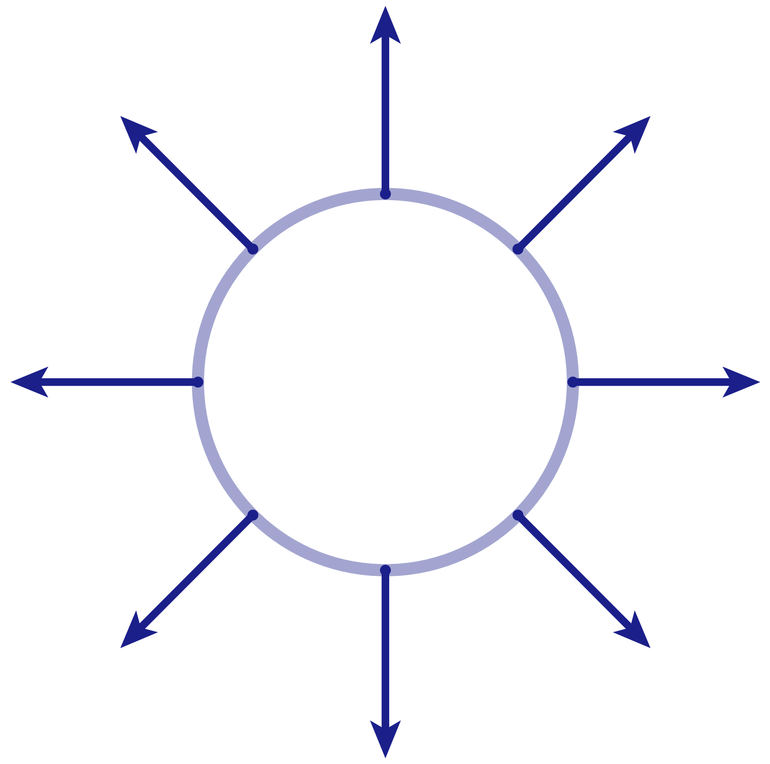
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$T(s) = (-\sin(s), \cos(s))$$

$$N(s) = \mathcal{J}T(s) = (-\cos(s), -\sin(s))$$



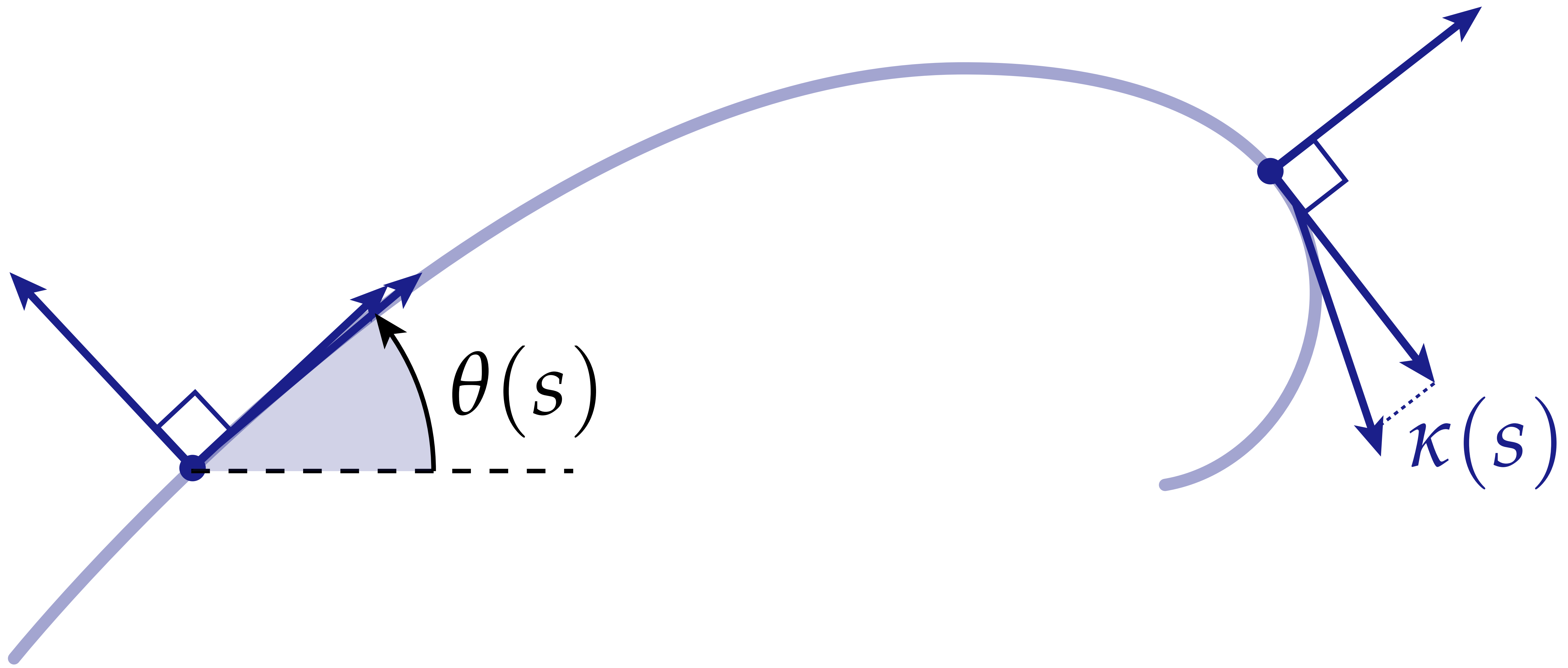
Note: could also adopt the convention $N = -\mathcal{J}T$. (Just remain consistent!)



Curvature of a Plane Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent

$$\begin{aligned}\kappa(s) &:= \langle N(s), \tfrac{d}{ds}T(s)\rangle \\ &= \langle N(s), \tfrac{d^2}{ds^2}\gamma(s)\rangle\end{aligned}$$



<u>Equivalently:</u>

$$\kappa(s) = \frac{d}{ds}\theta(s)$$

Here the angle brackets denote the usual dot product, i.e., $\langle (a,b),(x,y)\rangle := ax + by$.

Fundamental Theorem of Plane Curves

Fact. Up to rigid motions, an arc-length parameterized plane curve is uniquely determined by its curvature.

Q: Given only the curvature function, how can we recover the curve?

A: Just "invert" the two relationships $\frac{d}{ds}\theta = \kappa$, $\frac{d}{ds}\gamma = T$

First integrate curvature to get angle: $\theta(s) := \int_0^s \kappa(t)\, dt$

Then evaluate unit tangents: $T(s) := (\cos(\theta), \sin(\theta))$

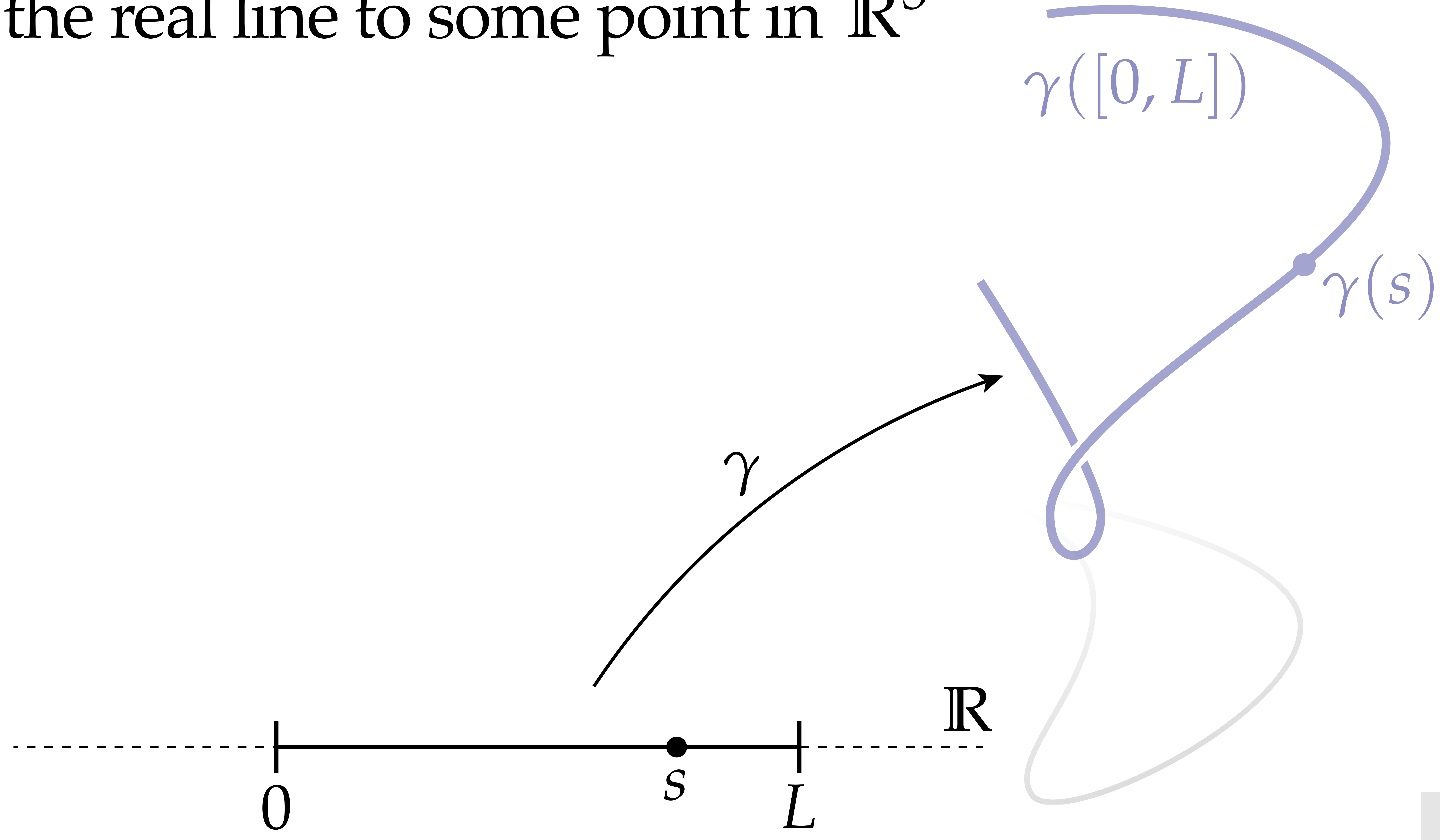
Finally, integrate tangents to get curve: $\gamma(s) := \int_0^s T(t)\, dt$

Q: What about the rigid motion? Will this work for *closed* curves?

Space Curves

Parameterized Space Curve

- A **parameterized space curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in $\mathbb{R}^3$



$\gamma : [0,L] \to \mathbb{R}^3$

*Continuous, differentiable, smooth…

Pushforward of Vectors on a Space Curve

Suppose we apply the differential of a parameterized space curve to a vector field X on its domain:

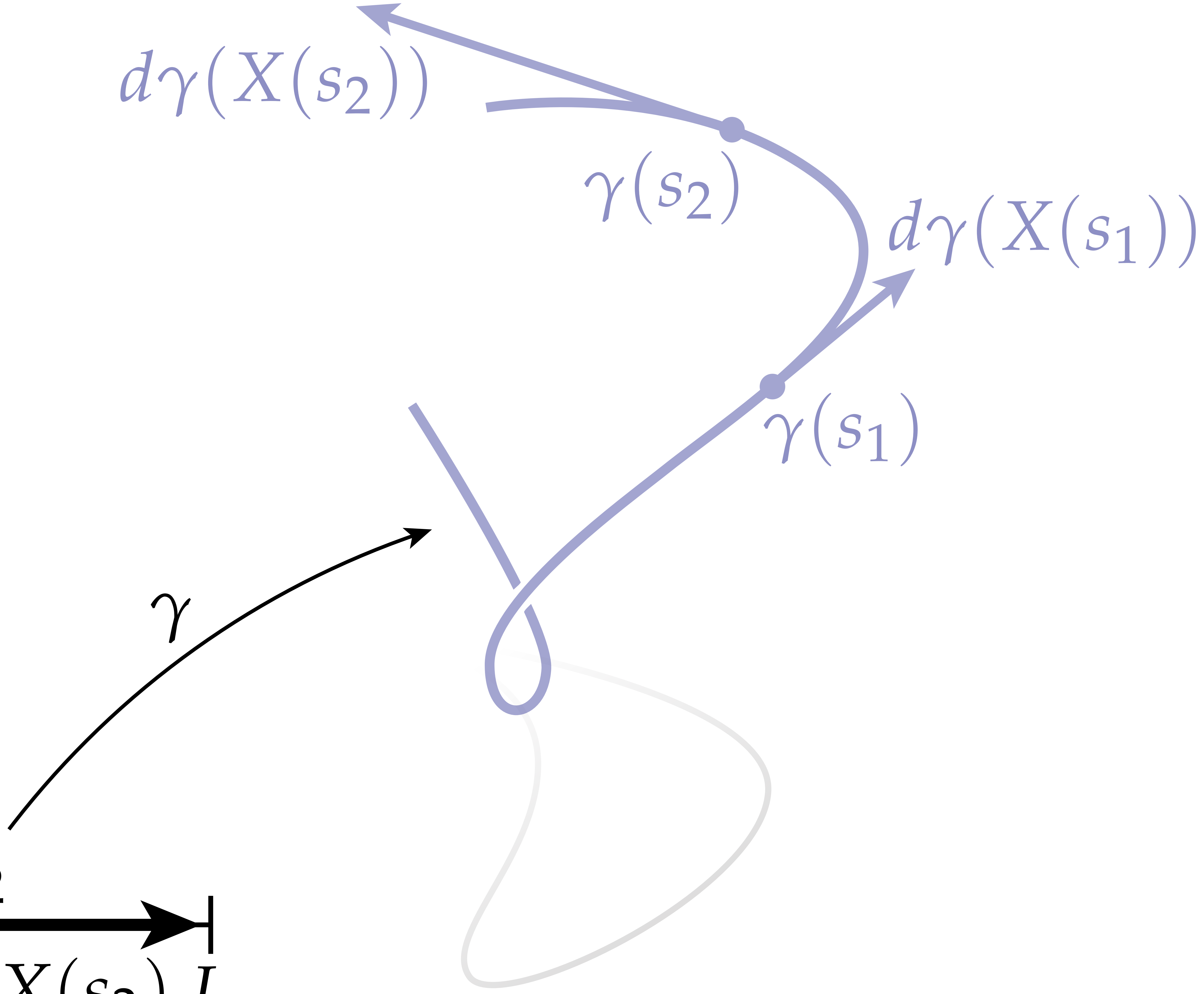
$$\gamma := (x, y, z), \quad x, y, z : [0, L] \to \mathbb{R}$$

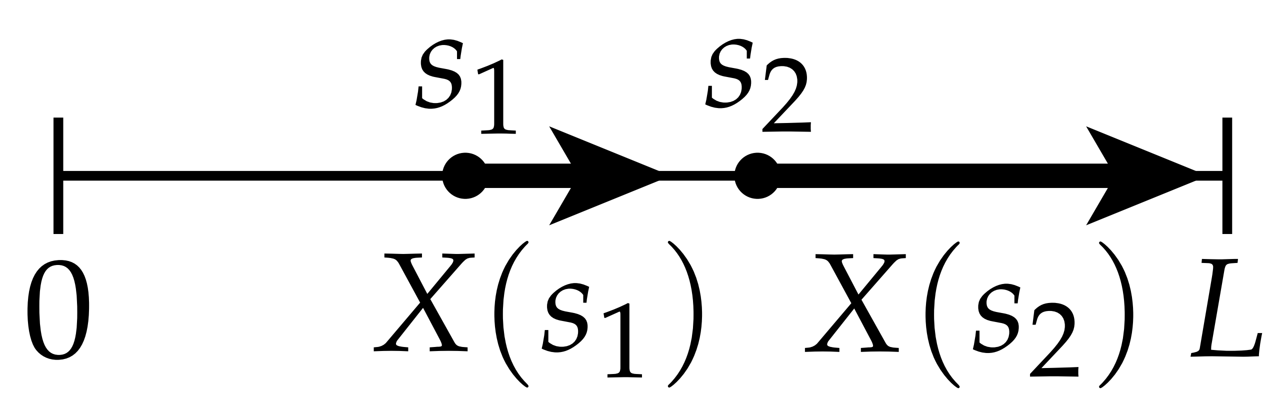
$$X := a\tfrac{\partial}{\partial s}, \quad a : [0, L] \to \mathbb{R}$$

$$d\gamma = (\tfrac{\partial x}{\partial s}, \tfrac{\partial y}{\partial s}, \tfrac{\partial z}{\partial s})ds$$

$$d\gamma(X) = a(\tfrac{\partial x}{\partial s}, \tfrac{\partial y}{\partial s}, \tfrac{\partial z}{\partial s})$$

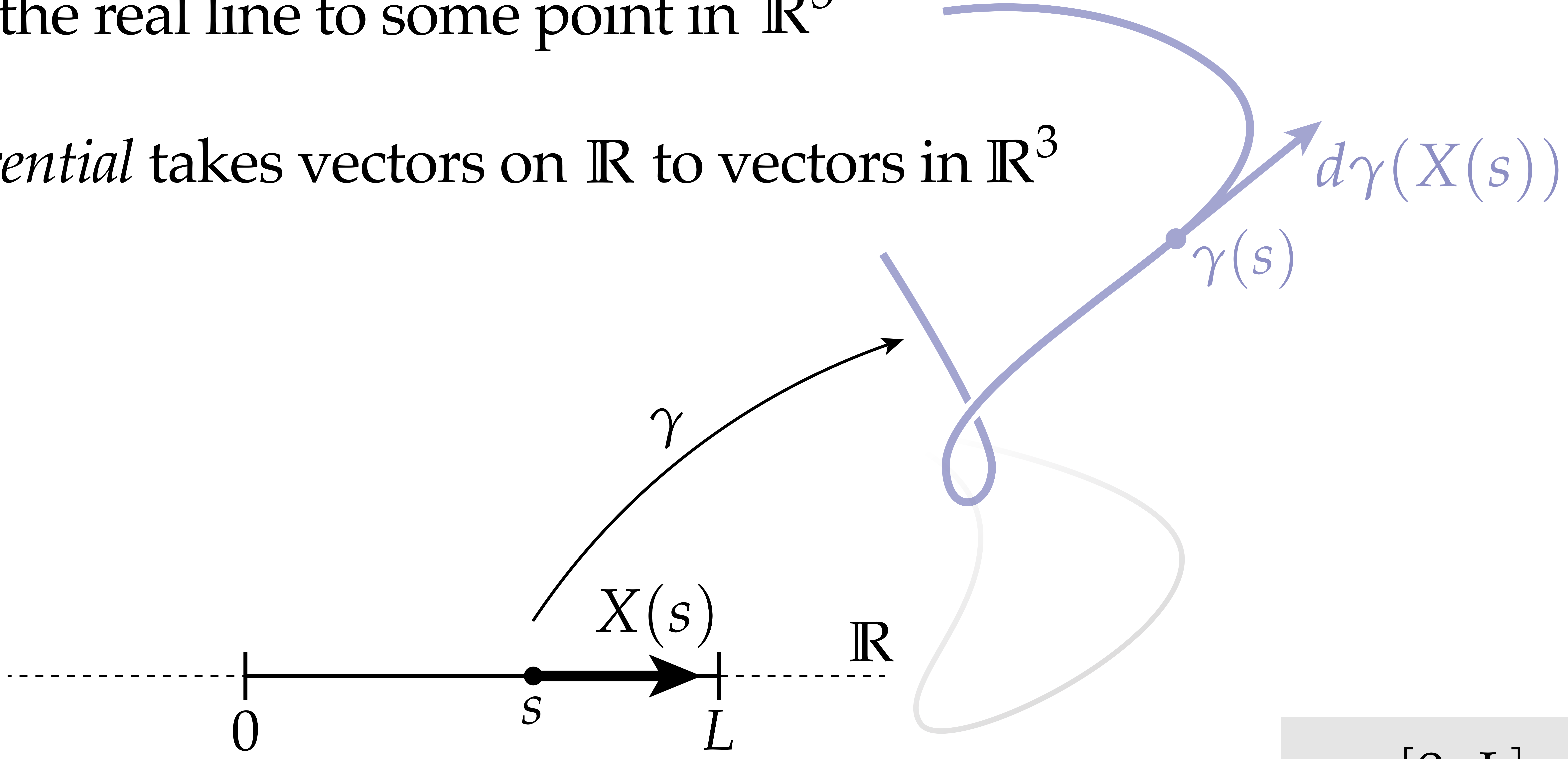
Q: What's the *geometric* meaning?





Parameterized Space Curve

- A **parameterized space curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in $\mathbb{R}^3$
- Its *differential* takes vectors on $\mathbb{R}$ to vectors in $\mathbb{R}^3$

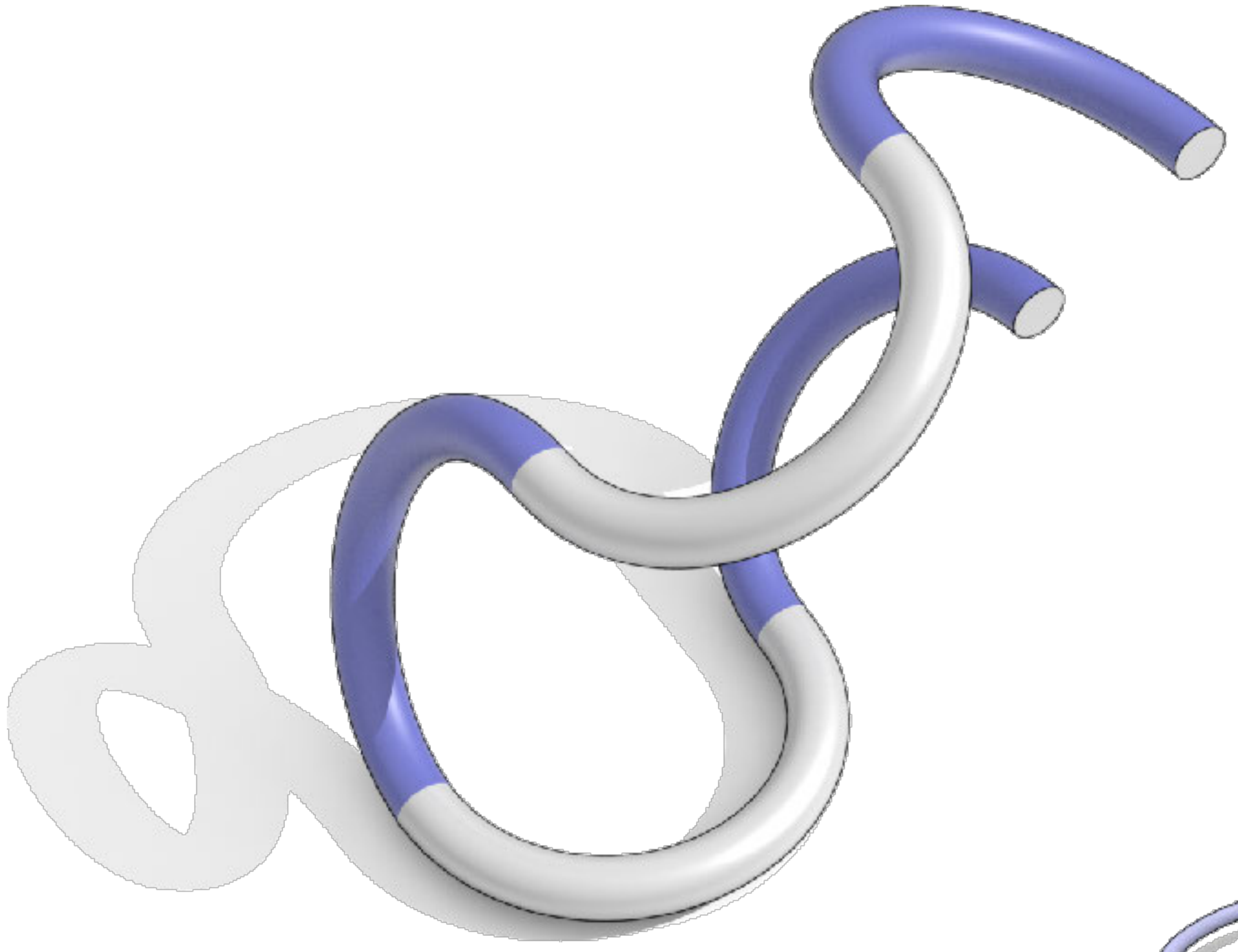


$\gamma : [0, L] \to \mathbb{R}^3$

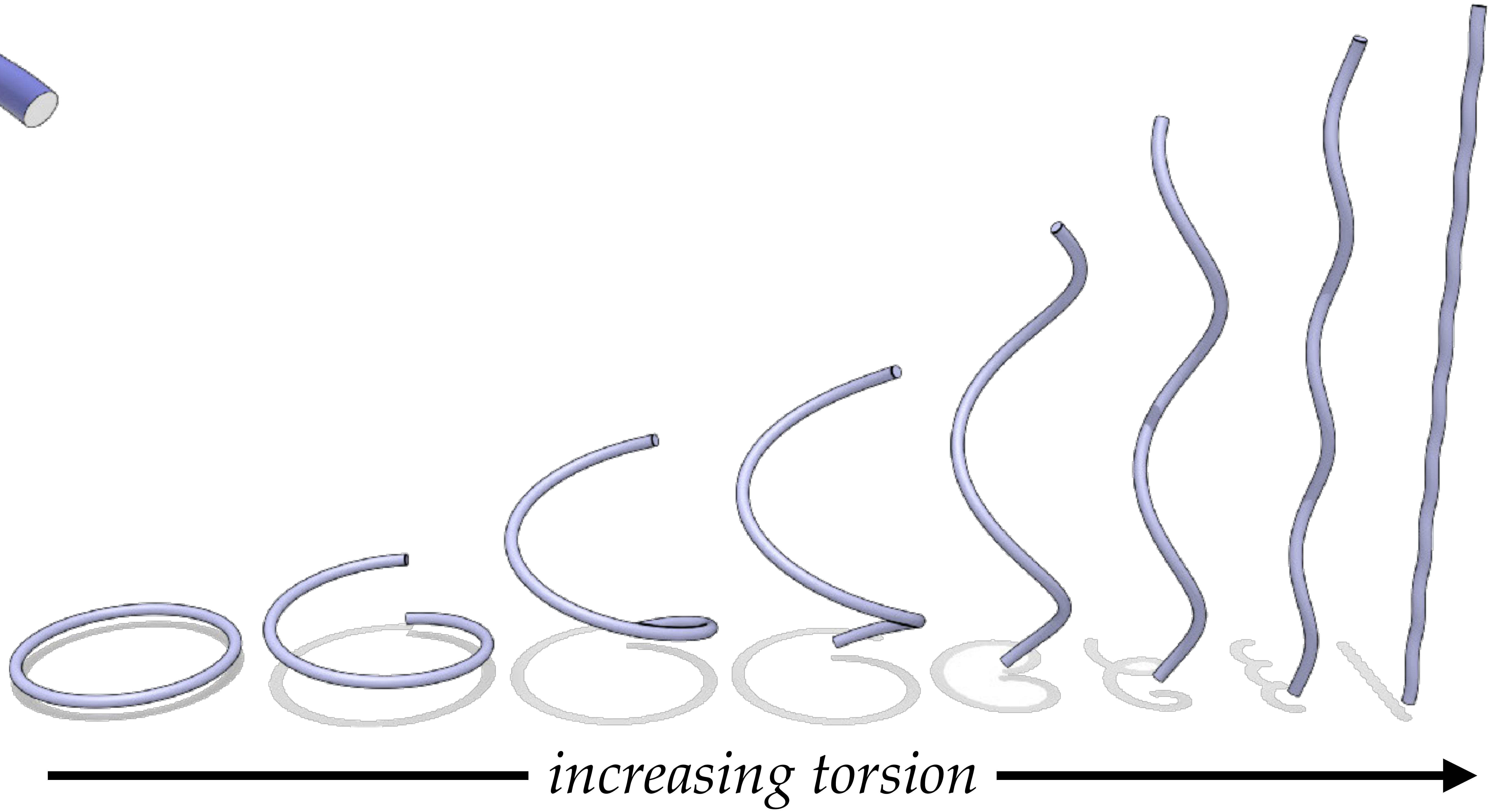
*Continuous, differentiable, smooth…

Curvature and Torsion of a Space Curve

- For a plane curve, *curvature* captured the notion of "bending"
- For a space curve we also have *torsion*, which captures "twisting"



Intuition: torsion is "out of plane bending"



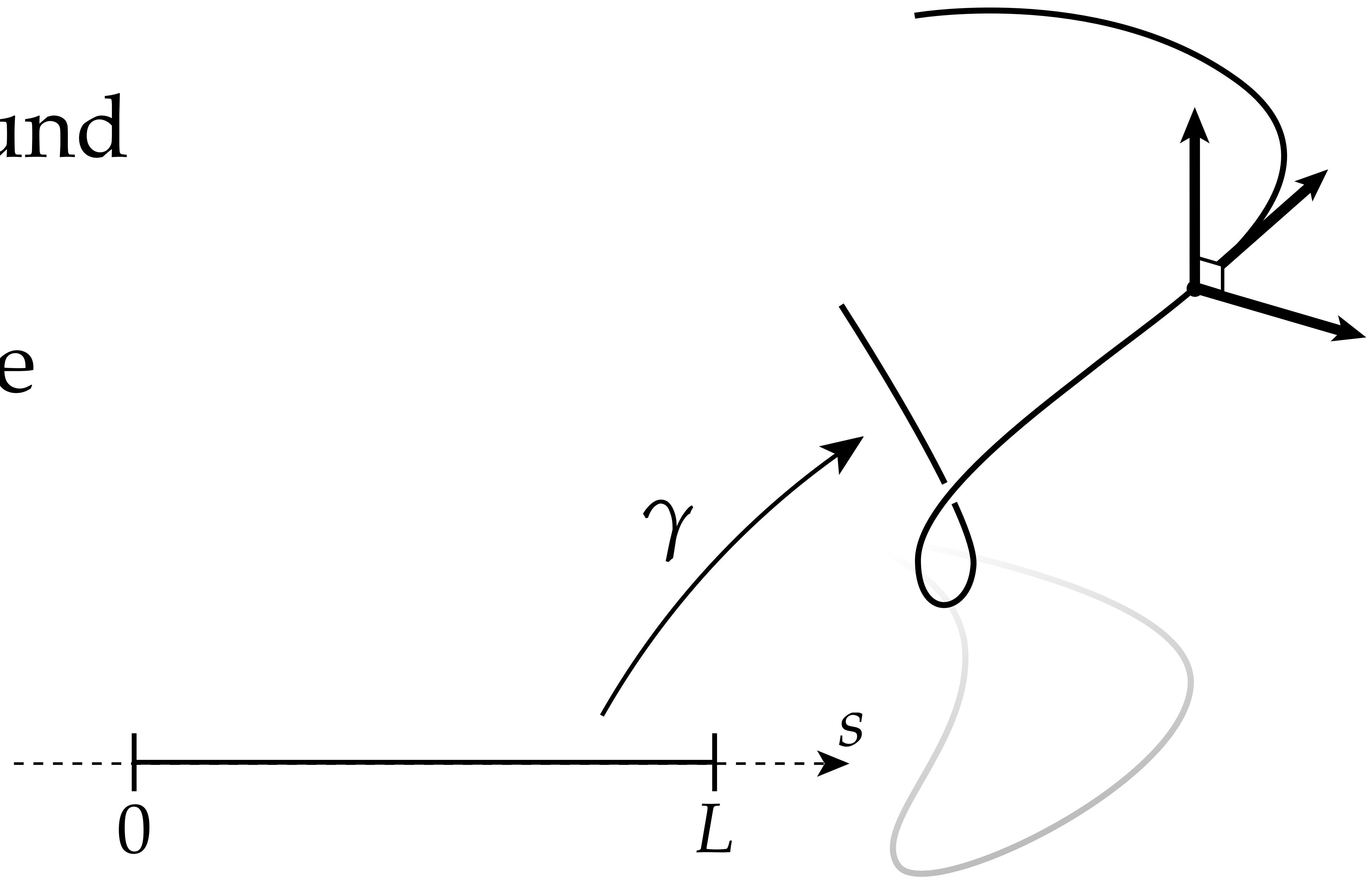
Frenet Frame

- Each point of a space curve has a natural coordinate frame called the *Frenet frame*, which depends only on the local geometry

- As in the plane, the tangent T is found by differentiating the curve, and differentiating the tangent yields the curvature times the normal N

- The binormal B then completes an orthonormal basis with T and N

$$\begin{aligned} T(s) &:= \tfrac{d}{ds}\gamma(s) \\ N(s) &:= \tfrac{d}{ds}T/|\tfrac{d}{ds}T| \\ B(s) &:= T(s) \times N(s) \end{aligned}$$



Frenet-Serret Equation

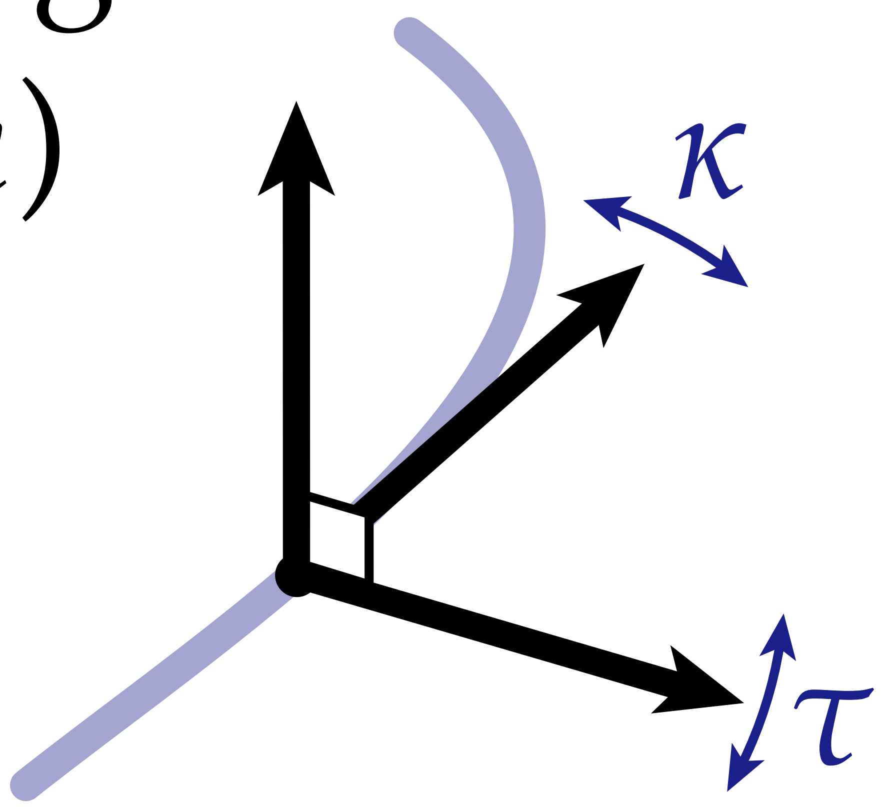
- Curvature κ and torsion τ can be defined in terms of the change in the Frenet frame as we move along the curve:

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

- Most importantly, change in the tangent describes bending (*curvature*); change in binormal describes twisting (*torsion*)

$$\kappa = -\langle N, \tfrac{d}{ds}T\rangle$$

$$\tau = \quad \langle N, \tfrac{d}{ds}B\rangle$$



Example—Helix

- Let's compute the Frenet frame, curvature, and torsion for a *helix**

$$\gamma(s) := (a\cos(s), a\sin(s), bs)$$

$$\tfrac{d}{ds}\gamma(s) = (-a\sin(s), a\cos(s), b)$$

$$|\tfrac{d}{ds}\gamma| = \sqrt{a^2+b^2} = 1$$

$$\Rightarrow T(s) = \tfrac{d}{ds}\gamma(s)$$

$$\tfrac{d}{ds}T(s) = -a(\cos(s), \sin(s), 0)$$

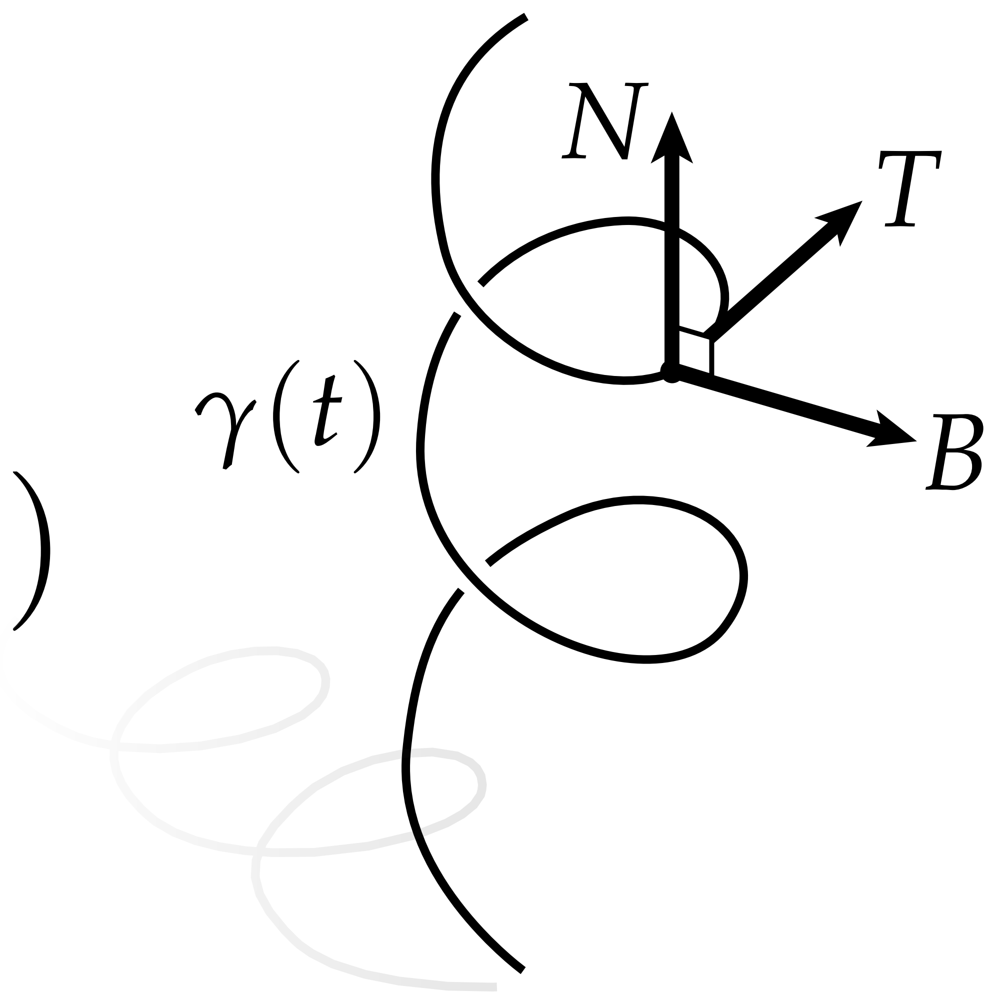
$$\Rightarrow \kappa(s) = -a,\ N(s) = (\cos(s), \sin(s), 0)$$

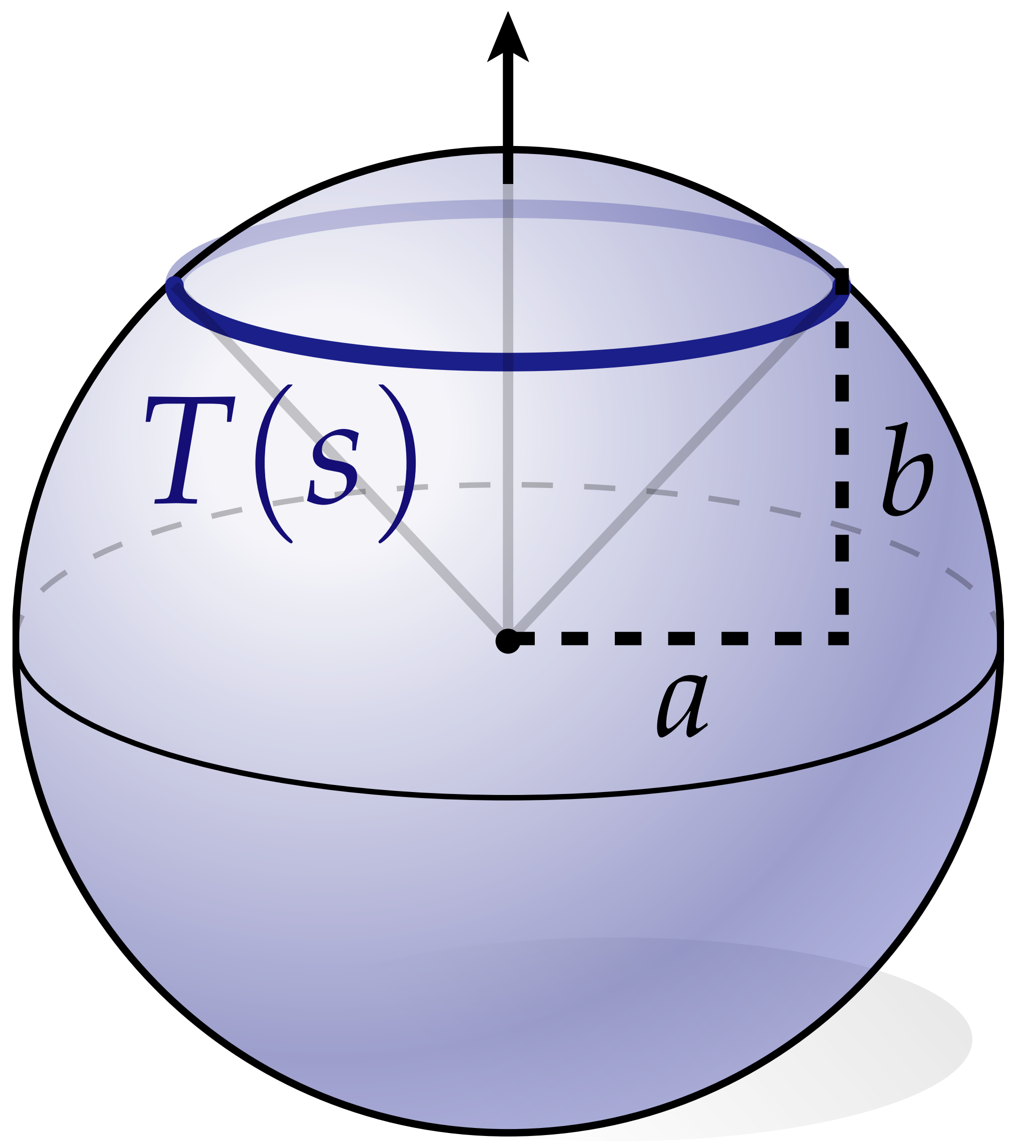
$$B(s) = T(s) \times N(s) =$$

$$(-b\sin(s), b\cos(s), -a)$$

$$\tfrac{d}{ds}B(s) = -b(\cos(s), \sin(s), 0)$$

$$\Rightarrow \tau(s) = -b$$



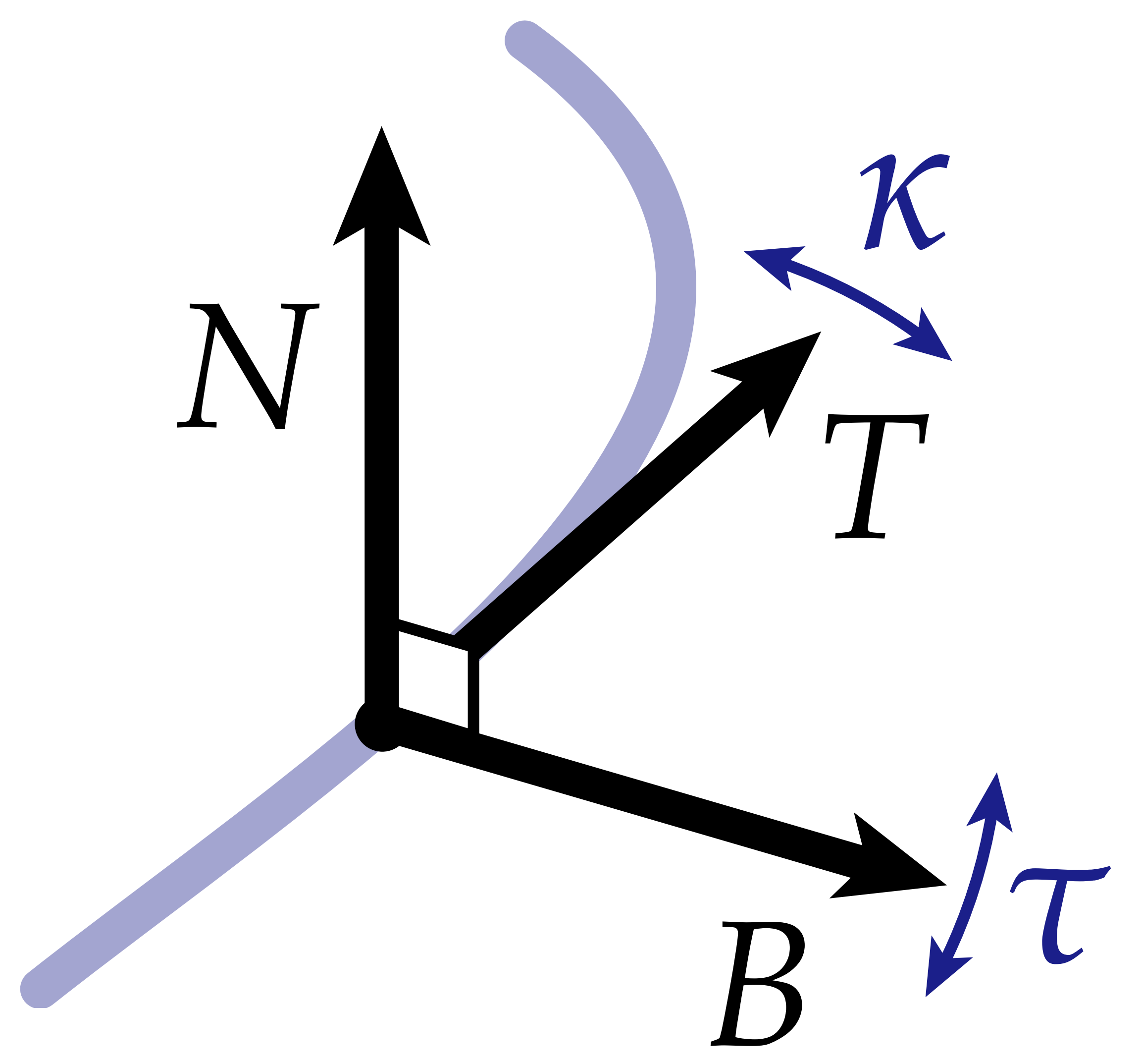


*For simplicity, let's pick a,b such that $a^2 + b^2 = 1$.

Fundamental Theorem of Space Curves TODO

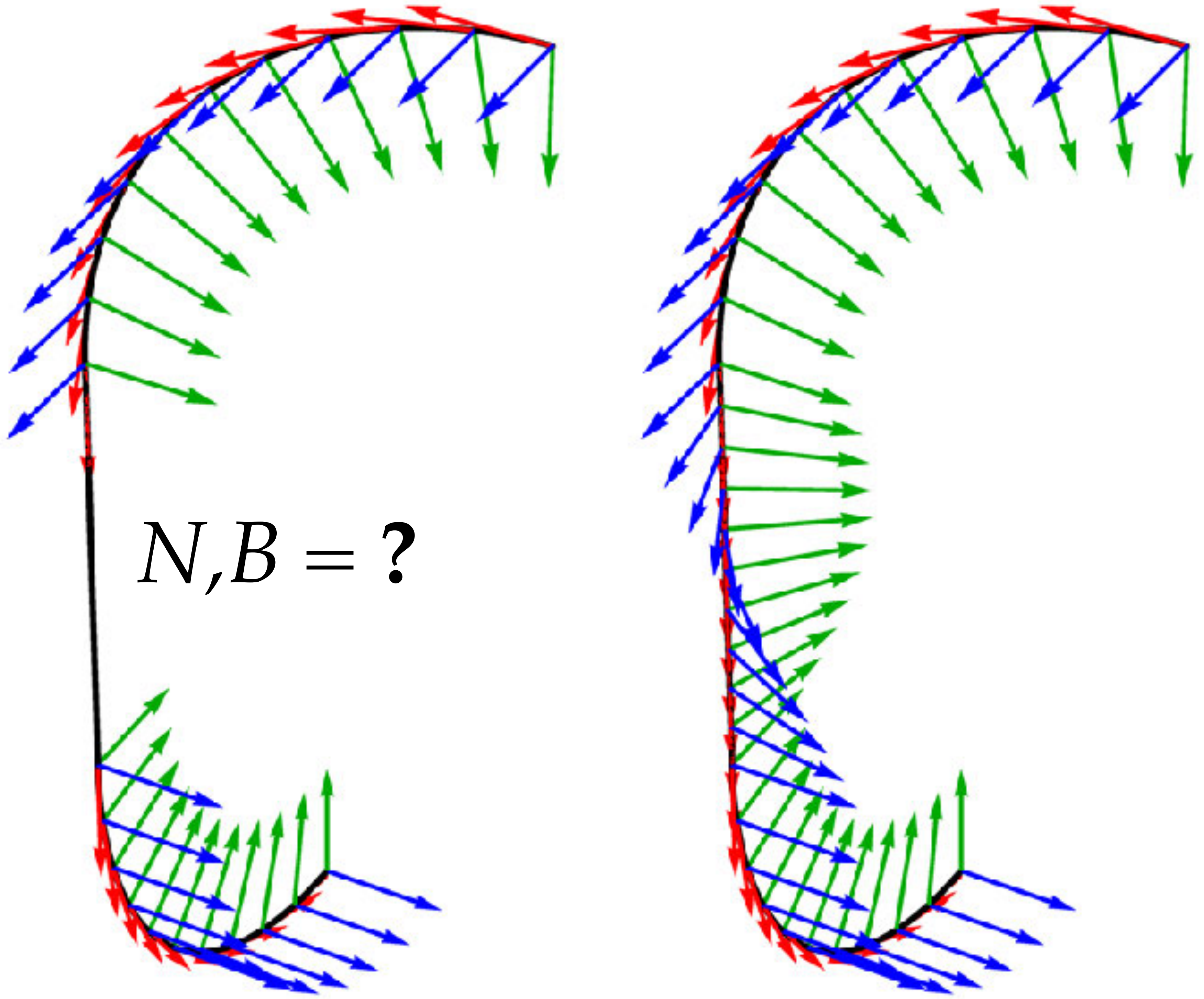
- The *fundamental theorem of space curves* tells us we can also go the other way: given the curvature and torsion of an arc-length parameterized space curve, we can recover the curve itself
- In 2D we just had to integrate a single ODE; here we integrate a system of three ODEs—namely, Frenet-Serret!

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$



Adapted Frames on Curves

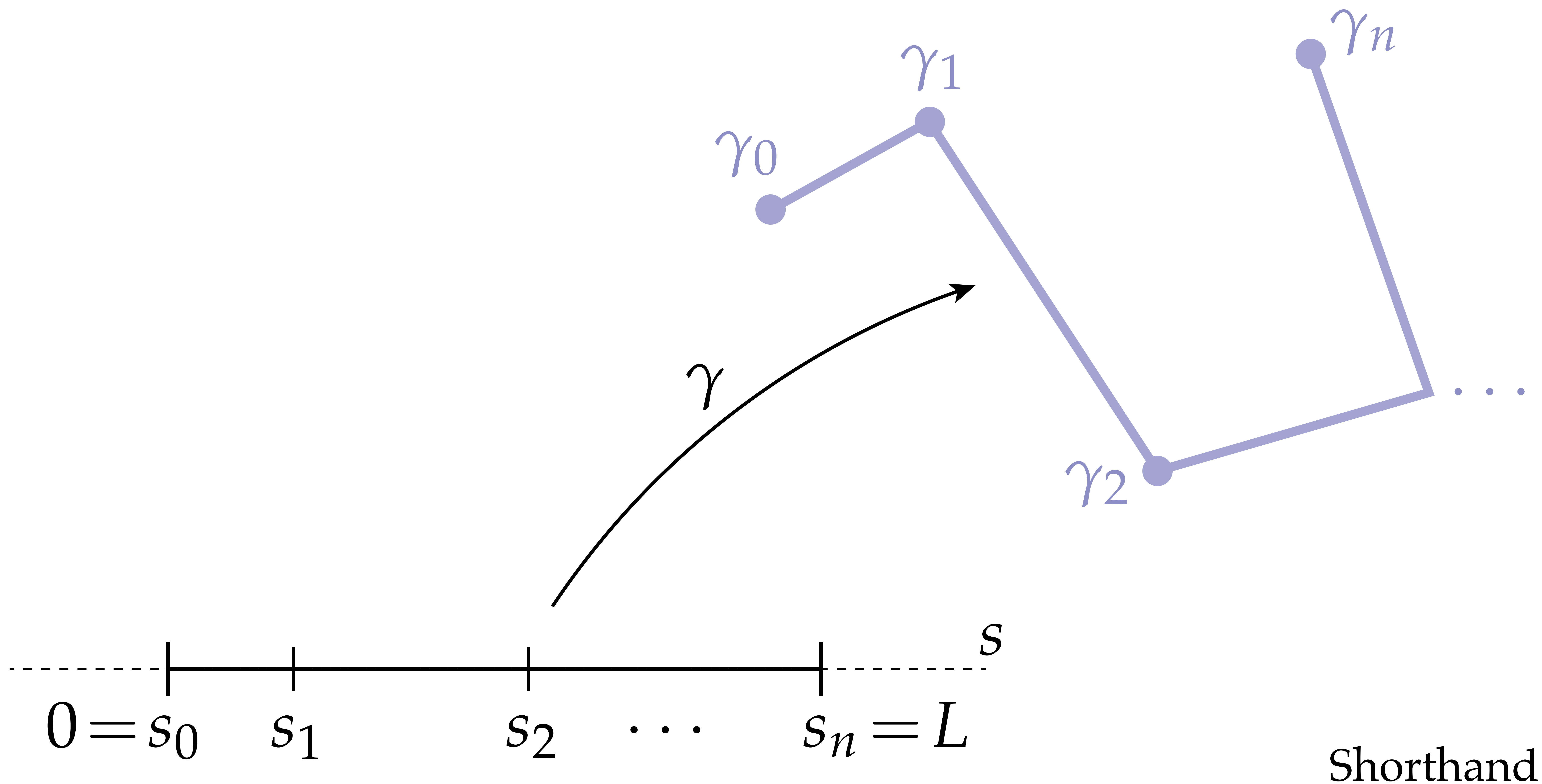
- **Q:** If our curve has a straight piece, is the Frenet frame well-defined?
- **A:** No, we don't have a clear normal/binormal (since, *e.g.*, $dT/ds = 0$)
- However, there are many ways to choose an *adapted frame*
- Any orthonormal frame including T
- *E.g.*, *least-twisting* frame (Bishop)
  - Unlike Frenet, *global* rather than *local*
- First example of *moving frames*
- (Will see more later for surfaces…)



Discrete Curves

Discrete Curves in the Plane

- We'll define a **discrete curve** as a *piecewise linear* parameterized curve, *i.e.*, a sequence of points connected by straight line segments:

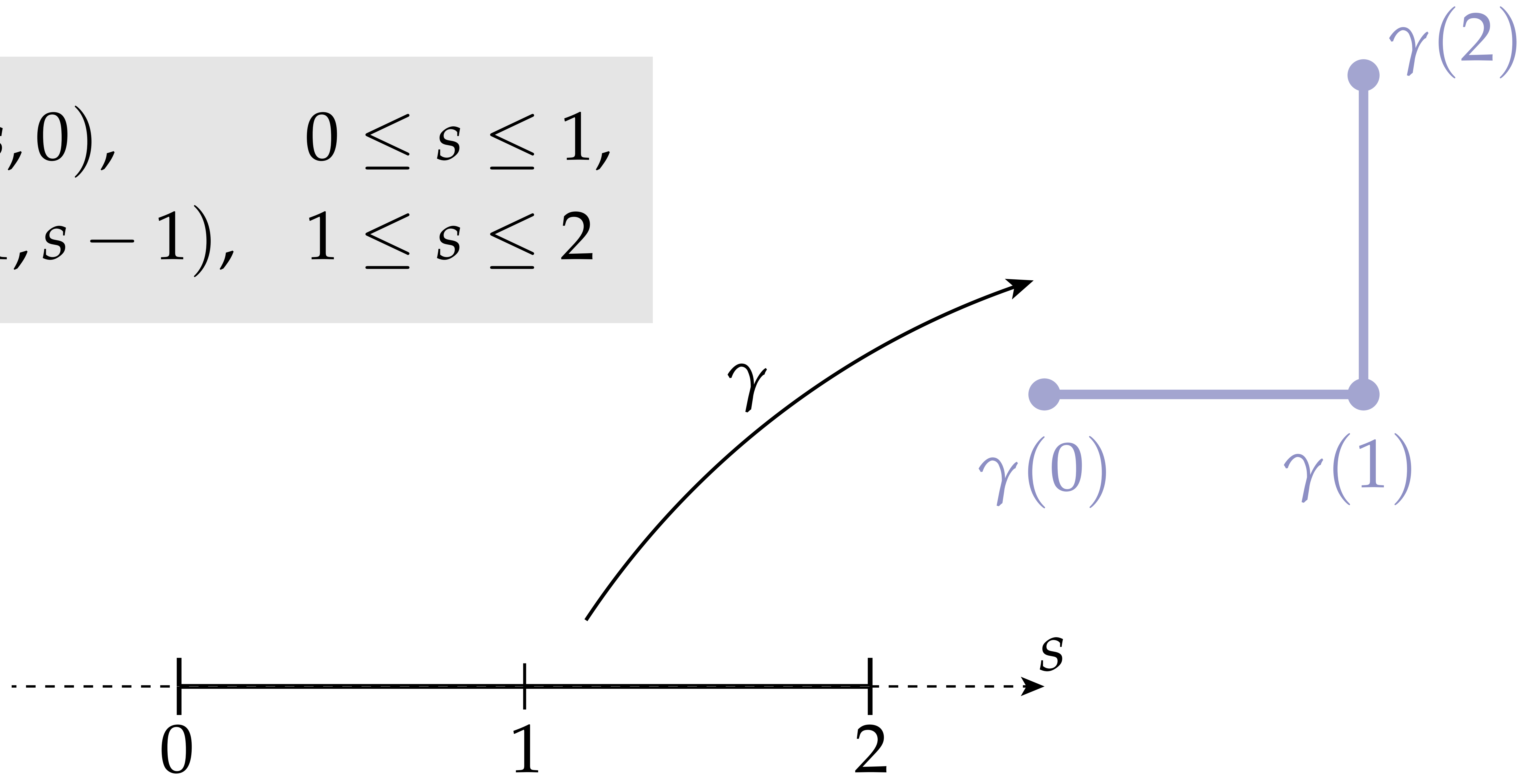


Shorthand: $\gamma_i := \gamma(s_i)$

Discrete Curves in the Plane—Example

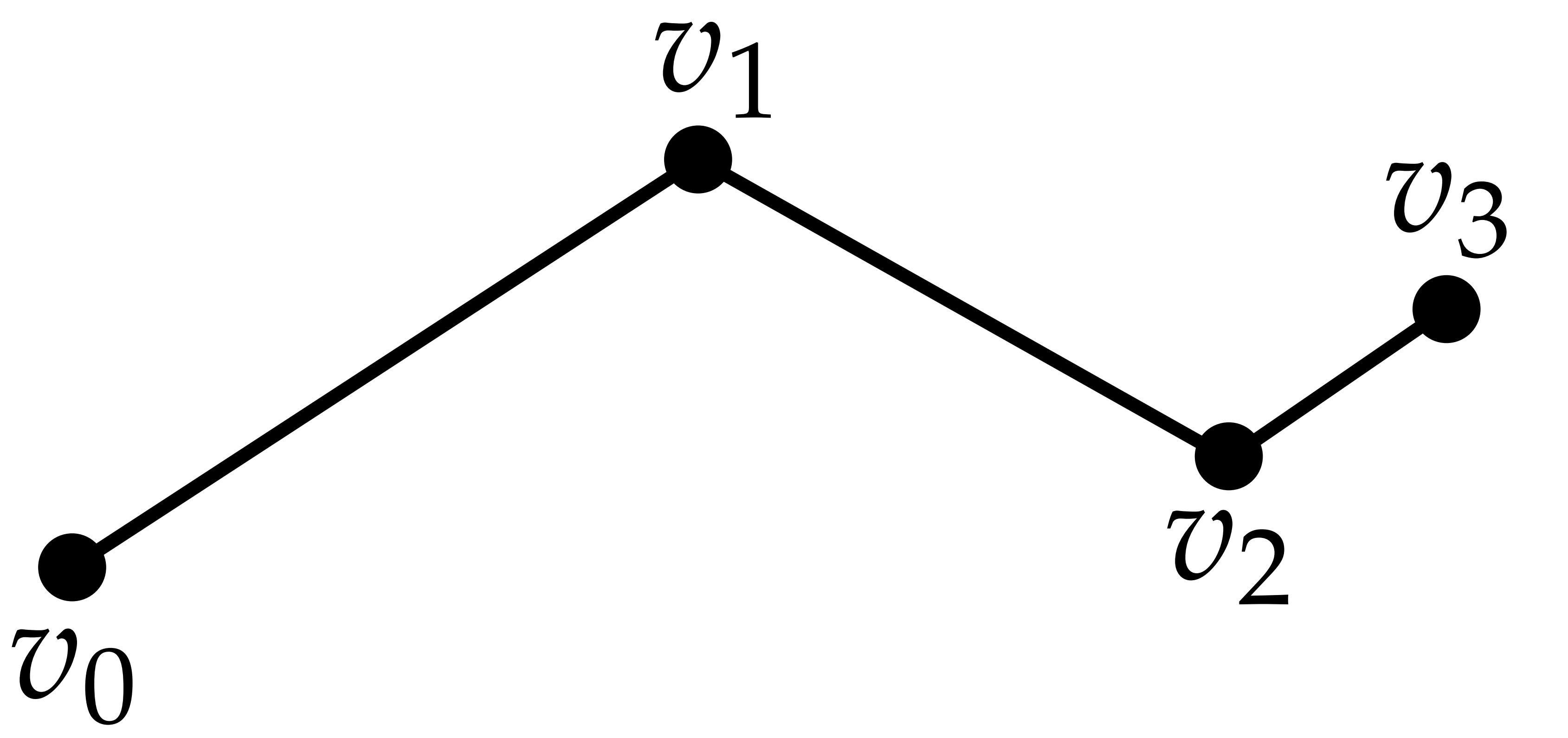
- A simple example is a curve comprised of two segments:

$$\gamma(s) := \begin{cases} (s,0), & 0 \leq s \leq 1, \\ (1, s-1), & 1 \leq s \leq 2 \end{cases}$$



Discrete Curves and Discrete Differential Forms

- Equivalently, a discrete curve is determined by a discrete, R^n-valued 0-form on a manifold simplicial 1-complex
- The 0-form values give the location of the vertices; interpolation by Whitney bases (hat functions) gives the map from each edge to R^n



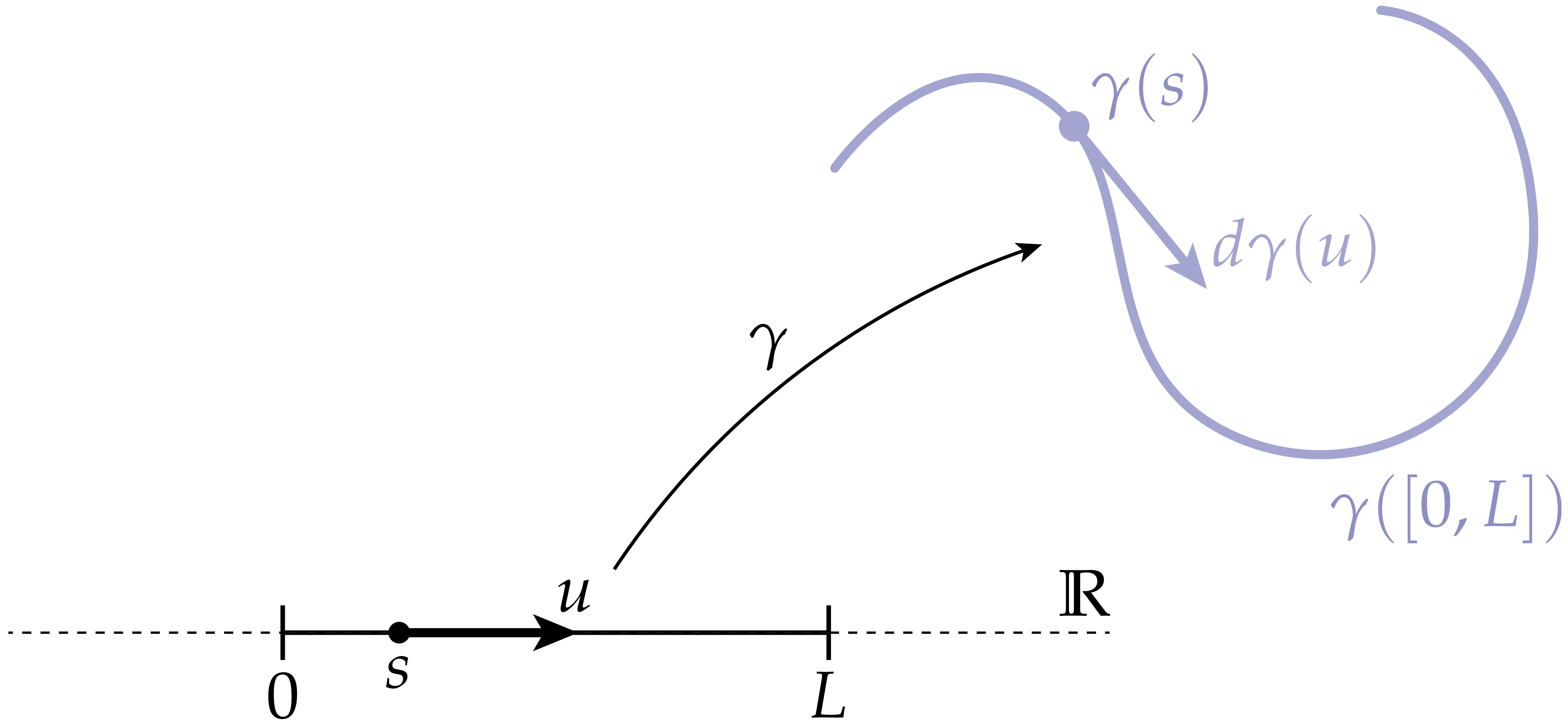
$$K = \{ (v_0,v_1), (v_1,v_2), (v_2,v_3), (v_0), (v_1), (v_2), (v_3), \emptyset \}$$

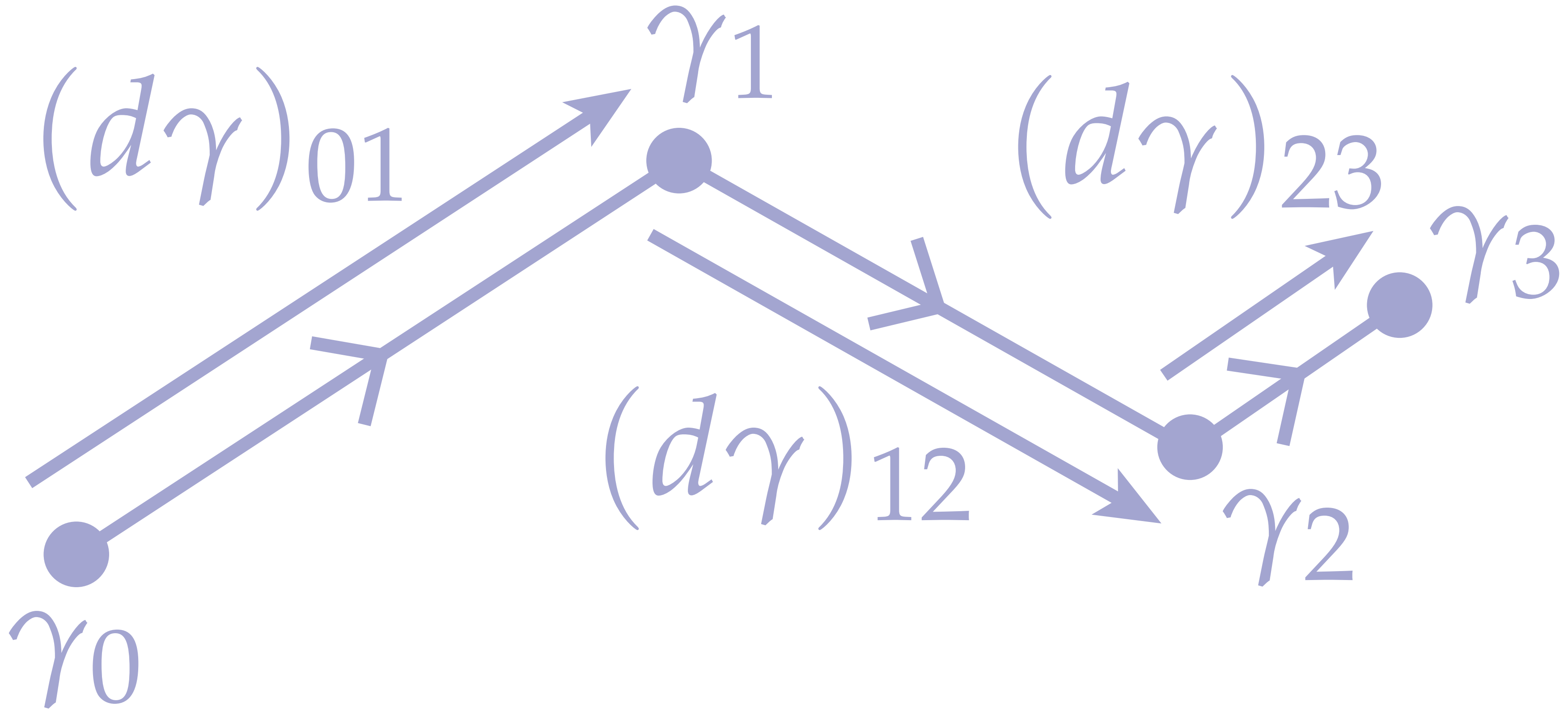
$$\begin{aligned}
\gamma(v_0) &= (33, 66) \\
\gamma(v_1) &= (79, 36) \\
\gamma(v_2) &= (118, 58) \\
\gamma(v_3) &= (134, 47)
\end{aligned}$$

Differential of a Discrete Curve

- We can now directly translate statements about **smooth** curves expressed via **smooth** exterior calculus into statements about **discrete** curves expressed using **discrete** exterior calculus

- Simple example: the *differential* just becomes the edge vectors:

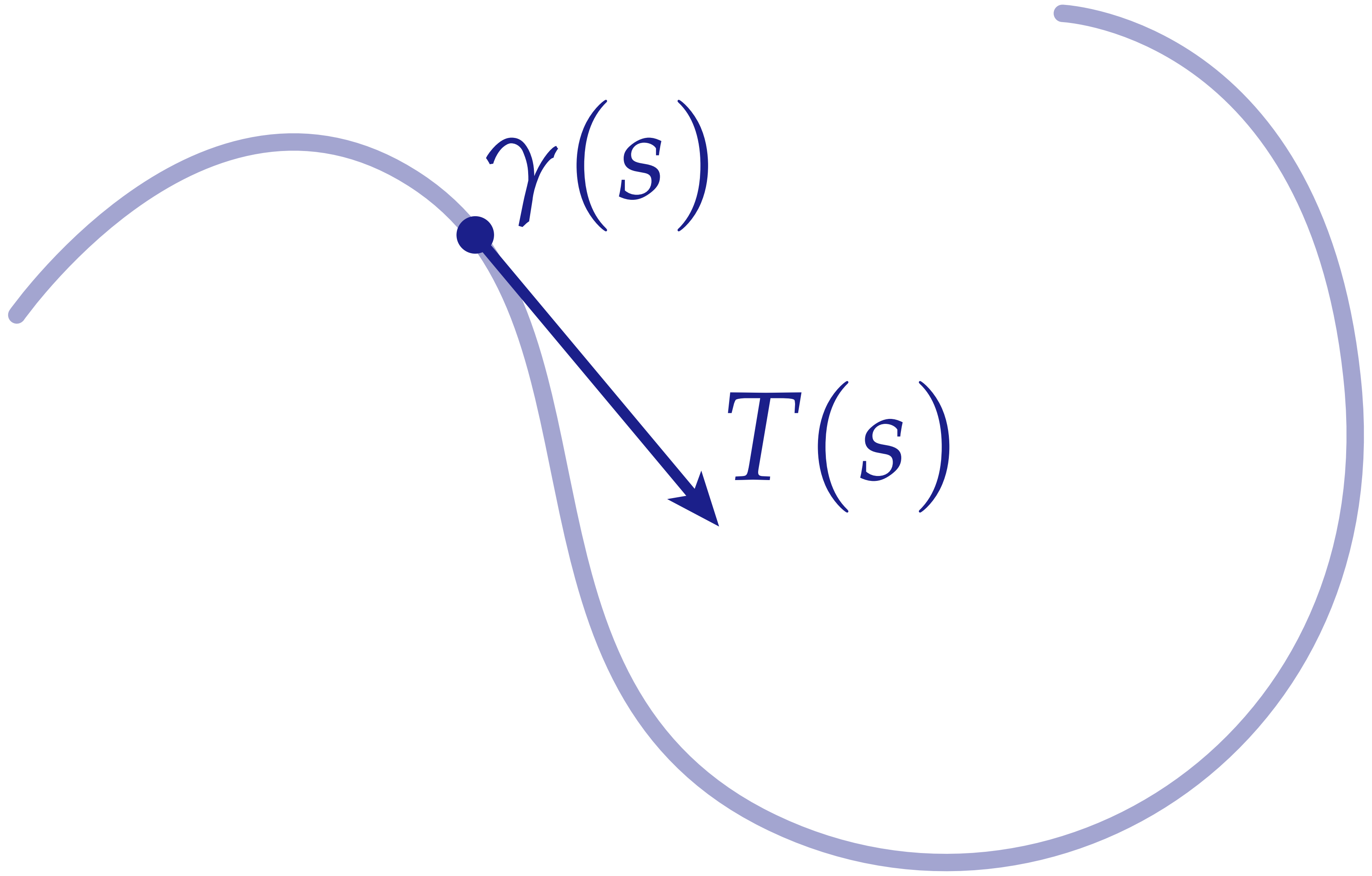


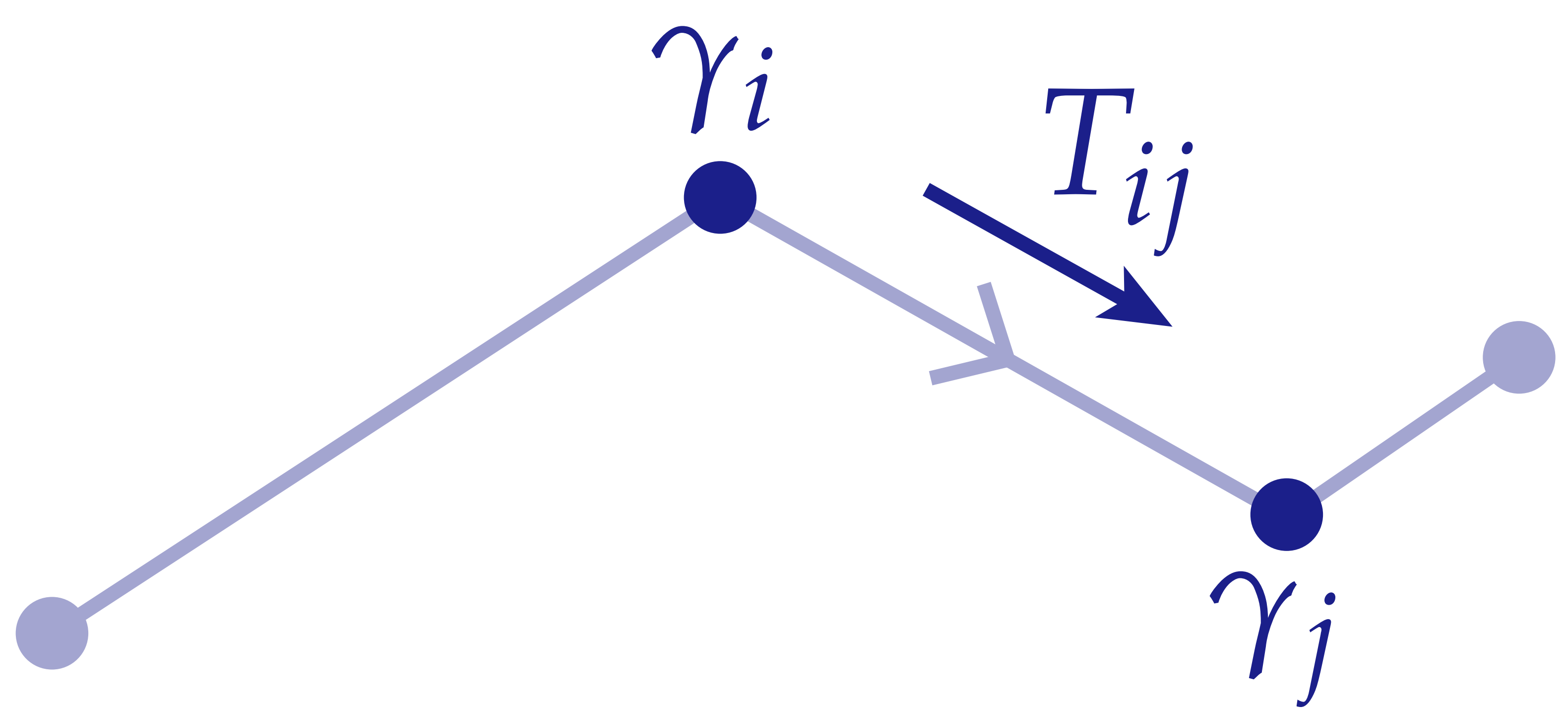


$$(d\gamma)_{ij} = \gamma_j - \gamma_i$$

Discrete Tangent

- As in smooth setting, can simply normalize differential to obtain tangents, yielding a vector per edge*





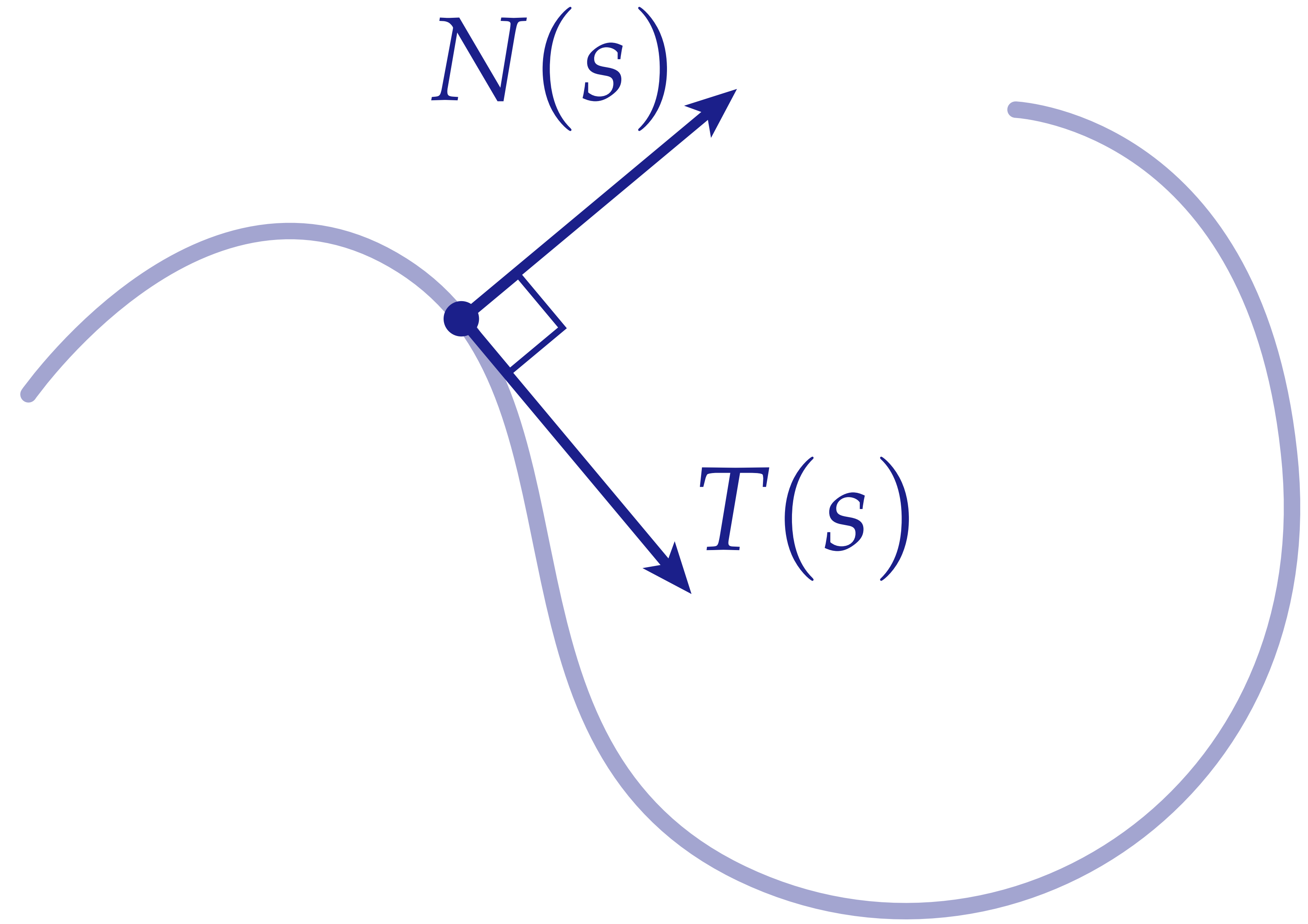
$$T(s) := d\gamma(\tfrac{d}{ds})/|d\gamma(\tfrac{d}{ds})|$$

$$T_{ij} := (d\gamma)_{ij}/|(d\gamma)_{ij}|$$

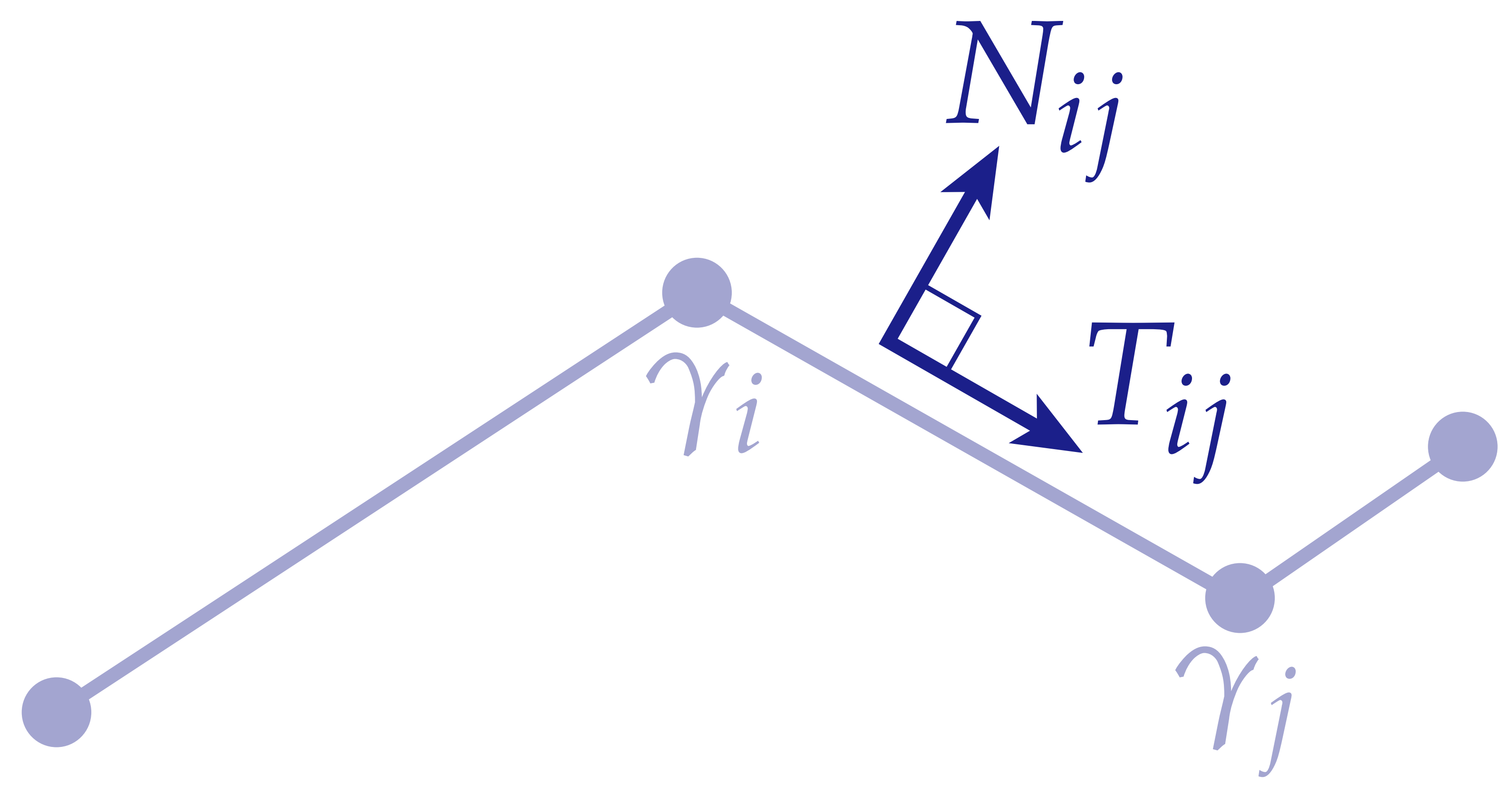
*And no definition of the tangent at vertices!

Discrete Normal

- As in the smooth setting, we can express the (discrete) normals of a planar curve as a 90-degree rotation of the (discrete) tangent:



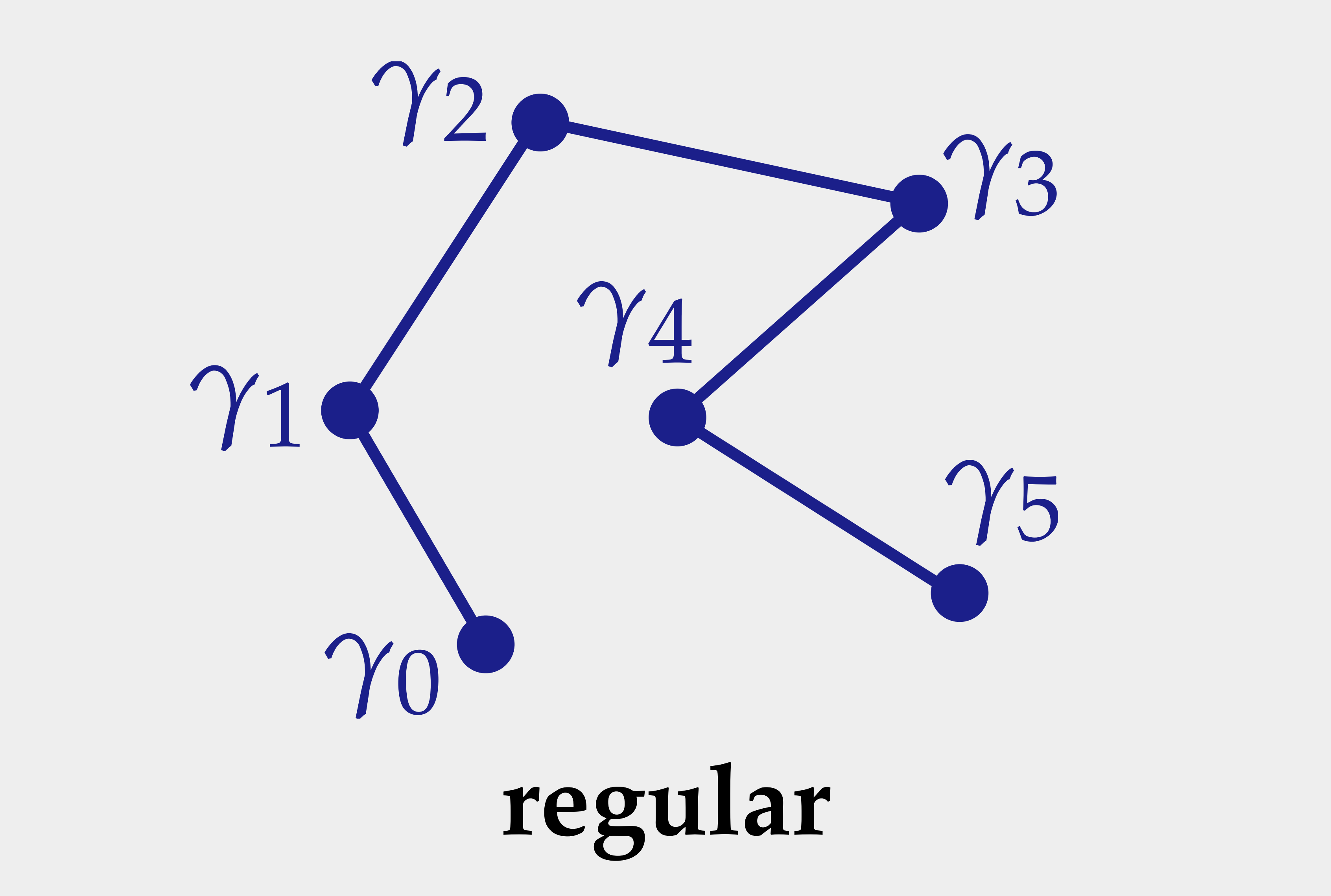
$$N(s) = \mathcal{J}T(s)$$

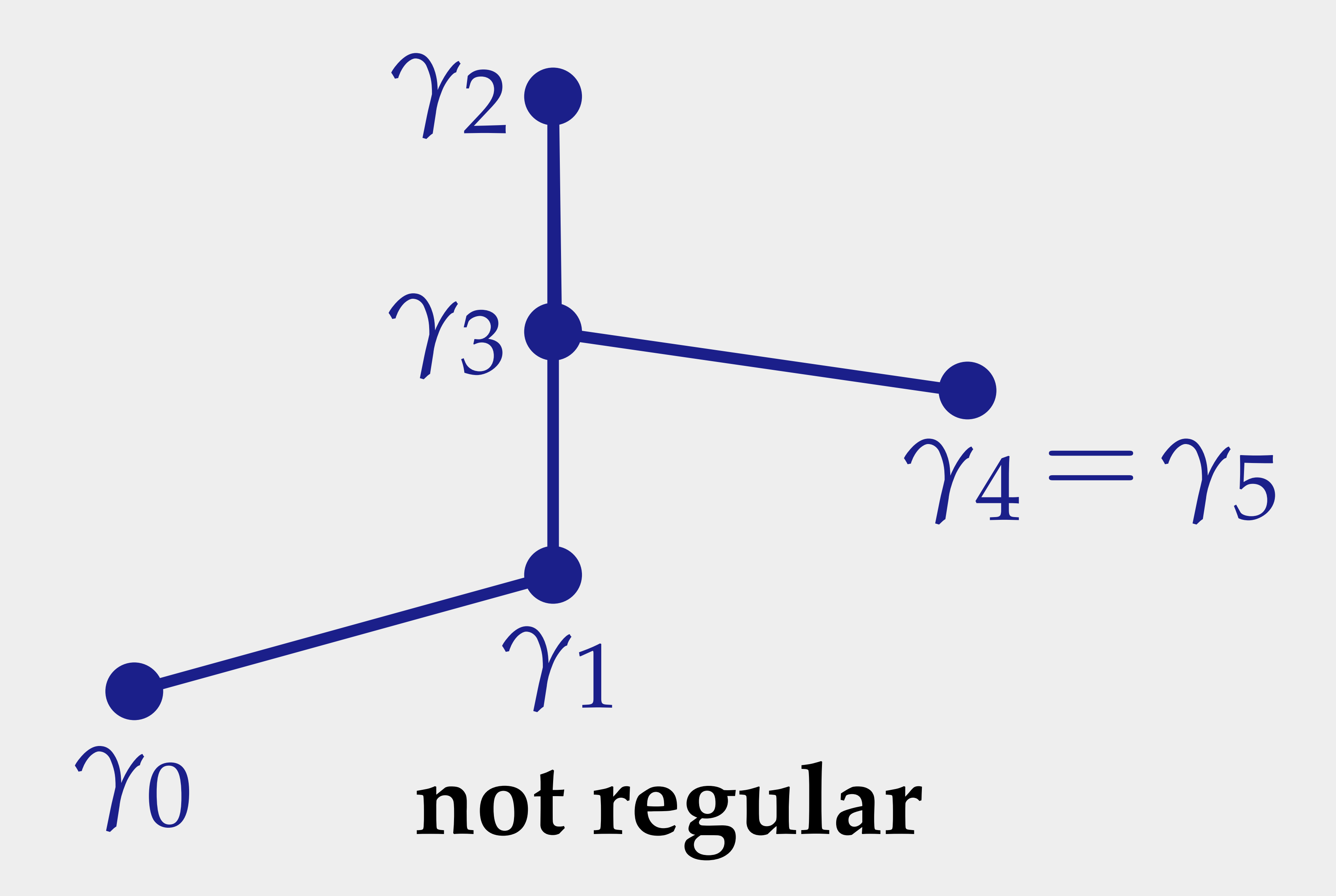


$$N_{ij} = \mathcal{J}T_{ij}$$

Regular Discrete Curve / Discrete Immersion

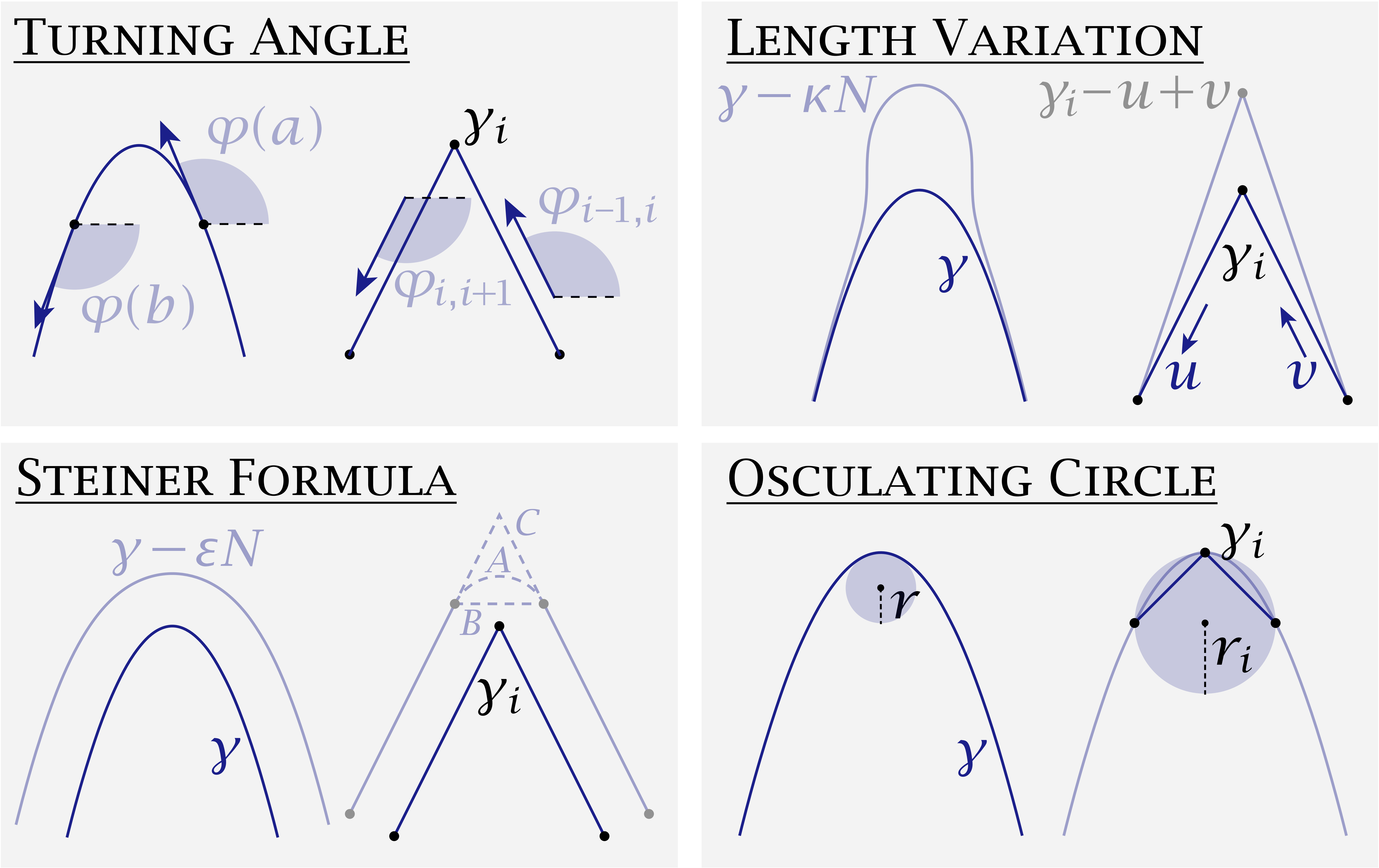
- Recall that a smooth curve is *regular* if its differential is nonzero; this condition helps avoid "bad behavior" like sharp cusps
- For a discrete curve, a nonzero differential merely prevents zero edge lengths; need something stronger to get "nice" curves
- In particular, a *regular discrete curve* or *discrete immersion* is a discrete curve that is a locally injective map
- Rules out zero edge lengths *and* zero angles





Discrete Curvature

- For a regular discrete curve, discrete curvature has several definitions

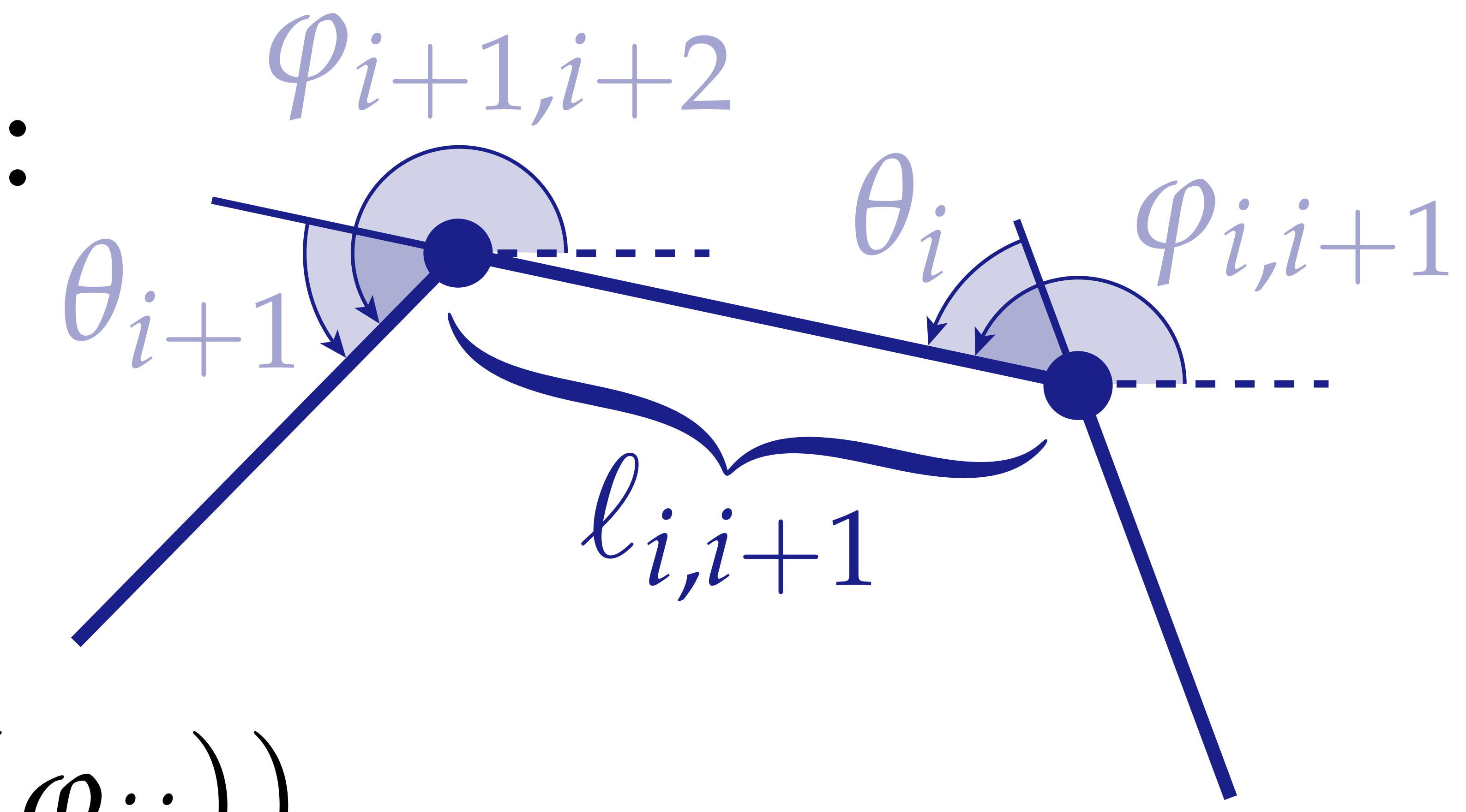


Fundamental Theorem of Discrete Plane Curves

Fact. Up to rigid motions, a regular discrete plane curve is uniquely determined by its edge lengths and turning angles.

Q: Given only this data, how can we recover the curve?

A: Mimic the procedure from the smooth setting:



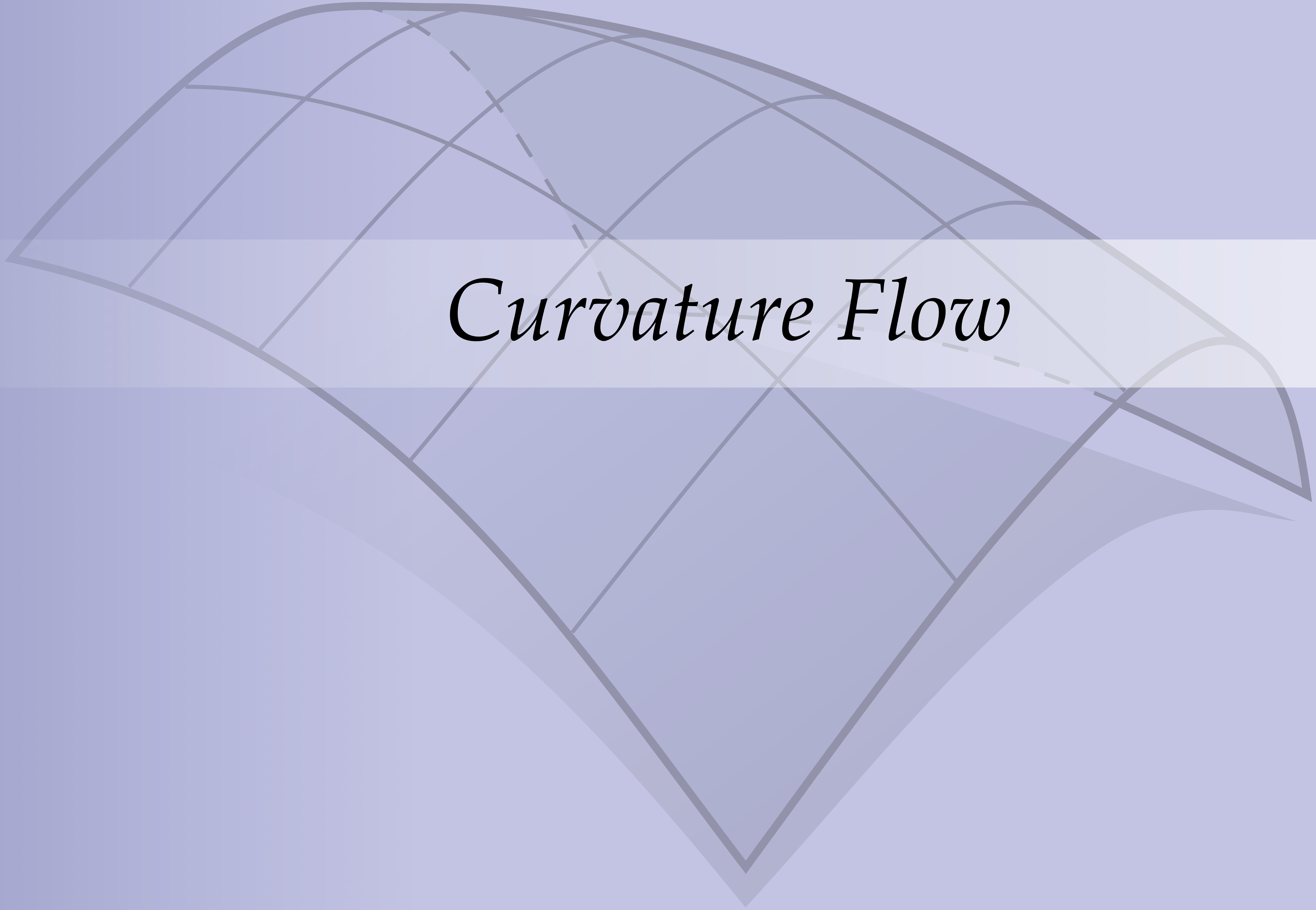
Sum curvatures to get angles: $\varphi_{i,i+1} := \sum_{k=1}^{i} \theta_k$

Evaluate unit tangents: $T_{ij} := (\cos(\varphi_{ij}), \sin(\varphi_{ij}))$

Sum tangents to get curve: $\gamma_i := \sum_{k=1}^{i} \ell_{k,k+1} T_{k,k+1}$

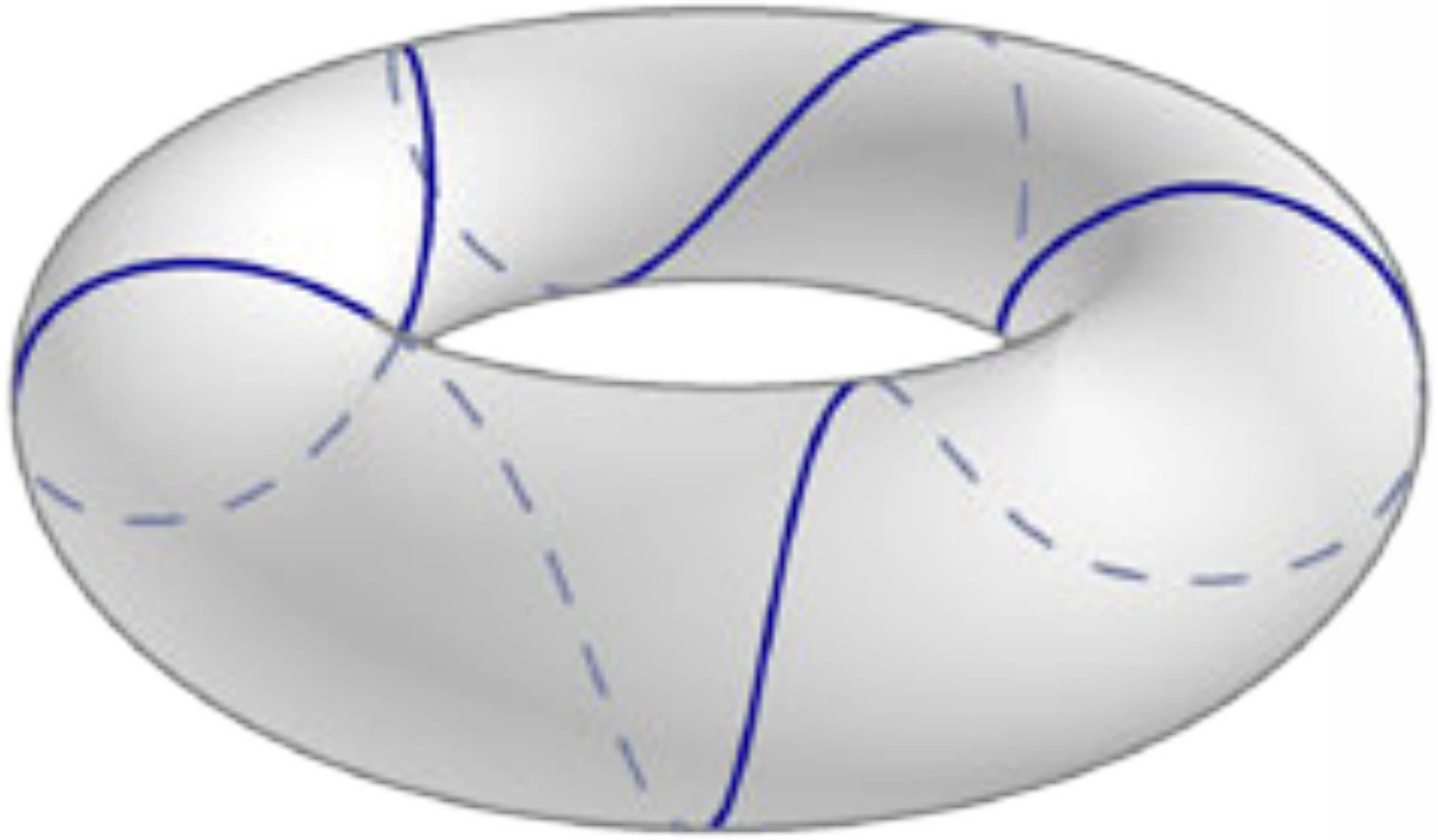
Q: Rigid motions?

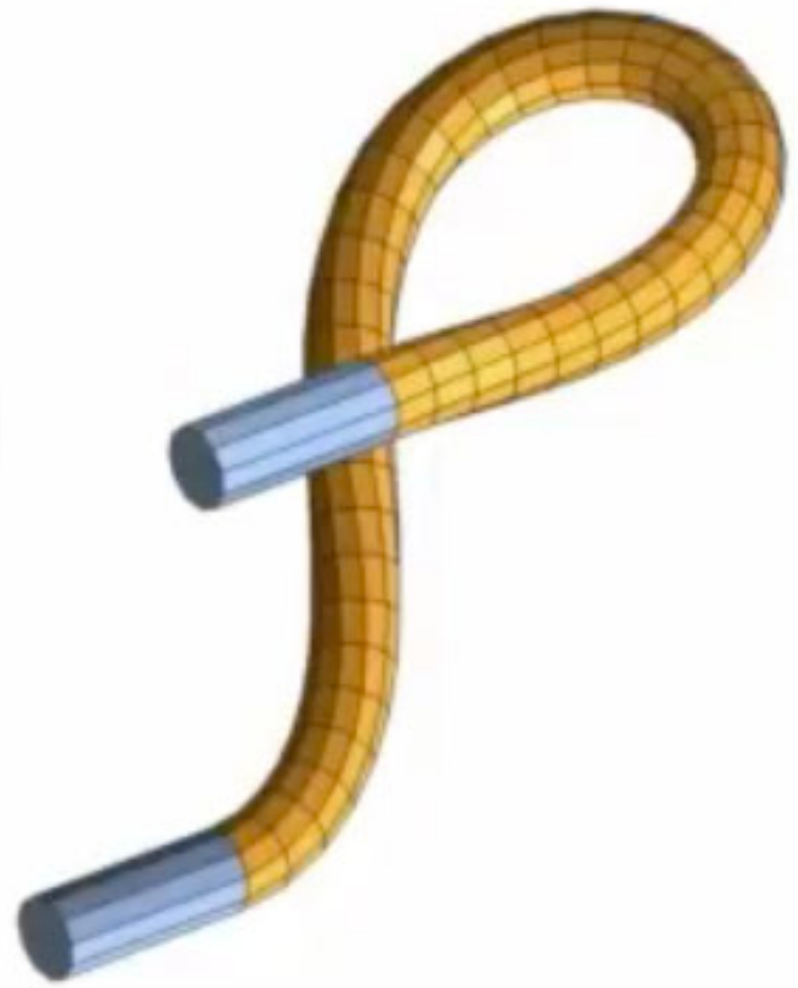
Curvature Flow

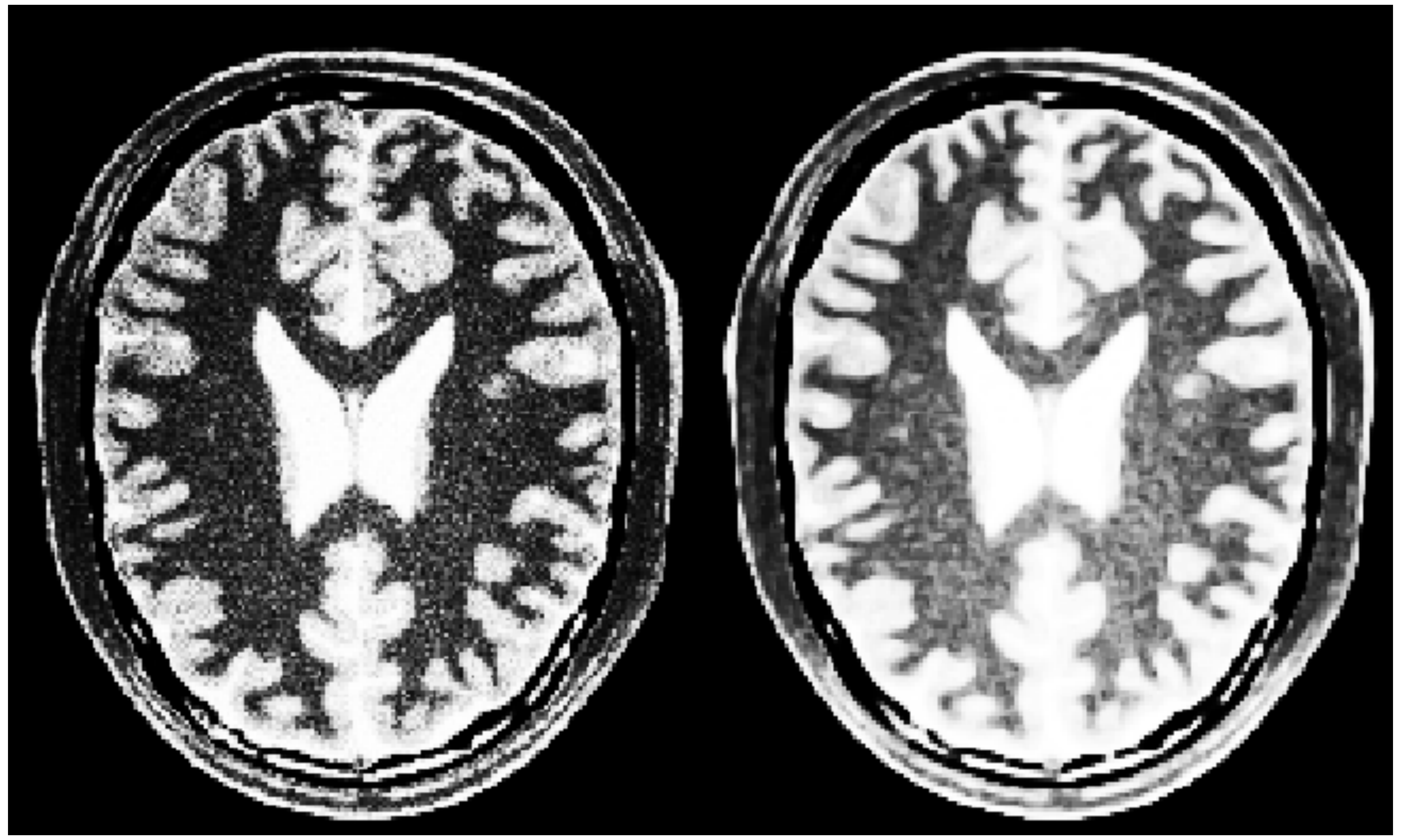


Curvature Flow on Curves

- A *curvature flow* is a time evolution of a curve (or surface) driven by some function of its curvature.
- Such flows model physical *elastic rods,* can be used to find shortest curves (*geodesics*) on surfaces, or might be used to smooth noisy data (*e.g.*, image contours).
- Two common examples: *length-shortening flow* and *elastic flow*.

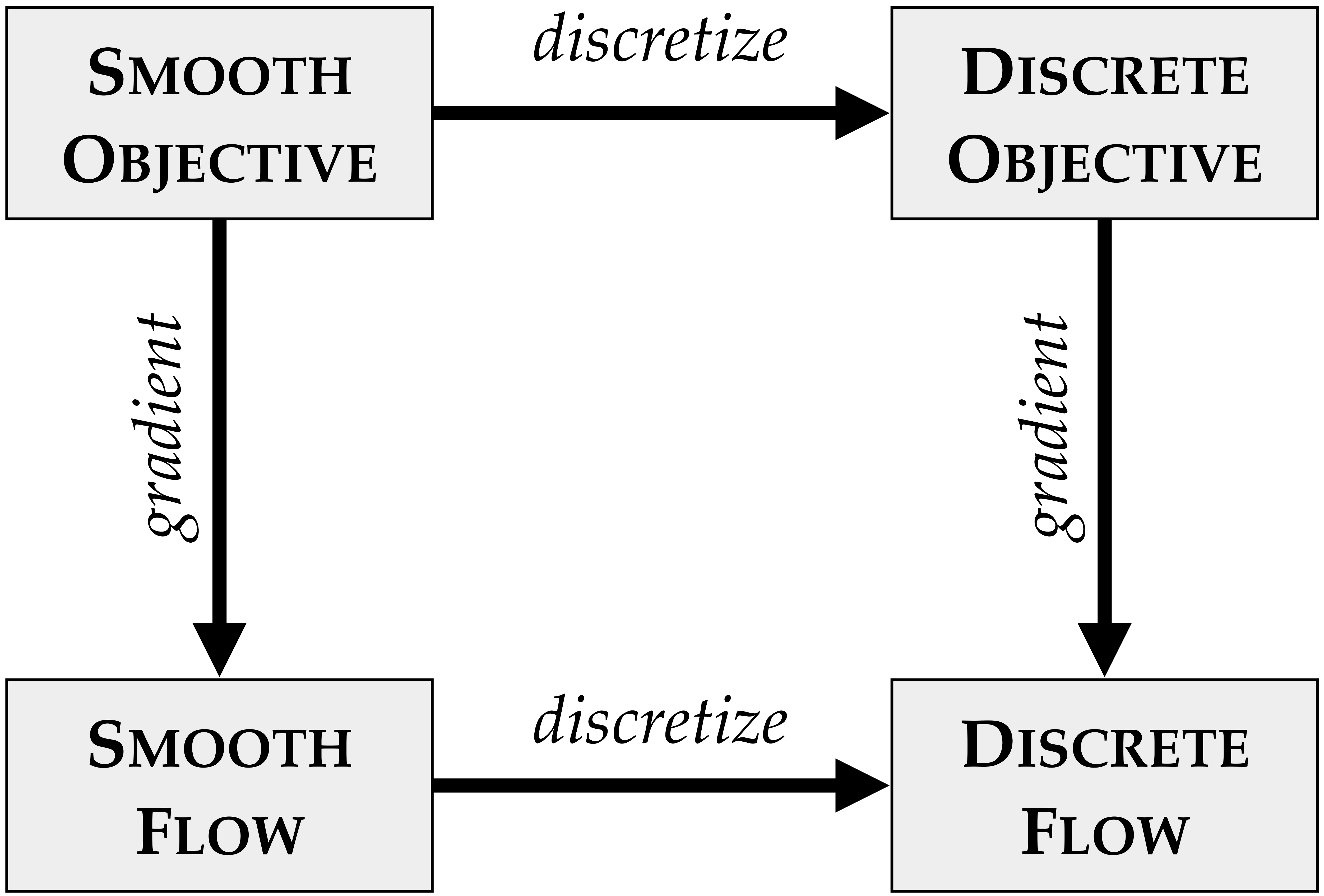






Discretizing a Gradient Flow

- Two possible paths for discretizing any gradient flow:
  1. **First** derive the gradient of the objective in the smooth setting, **then** discretize the resulting evolution equation.
  2. **First** discretize the objective itself, **then** take the gradient of the resulting discrete objective.
- In general, ***will not*** *lead to the same numerical scheme/algorithm!*



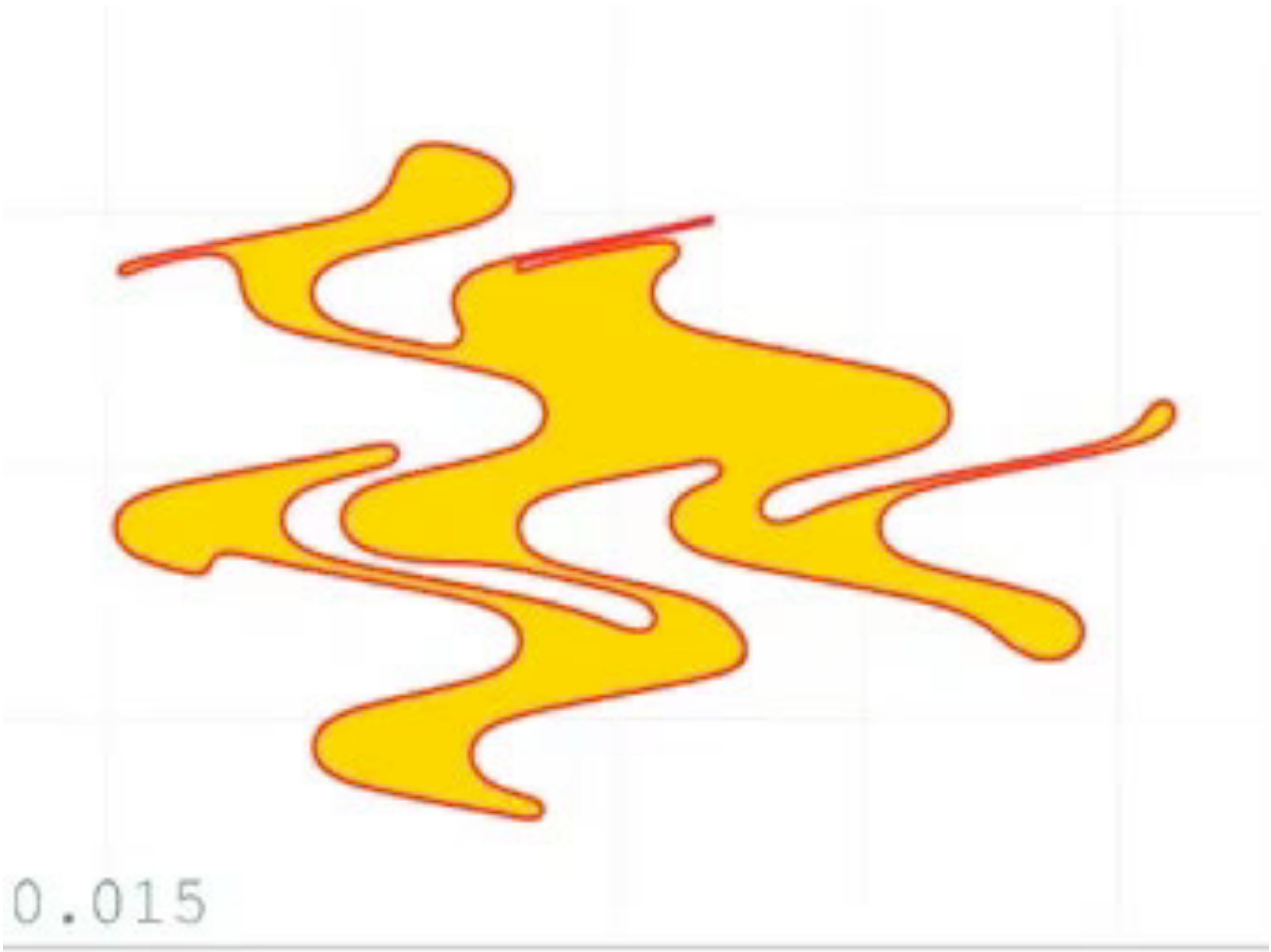
(Does **NOT** commute in general.)

Length Shortening Flow

- The objective for length shortening flow is simply the total length of the curve; the flow is then the (L^2) gradient flow.
- For closed curves, several interesting features (Gage-Grayson-Hamilton):
  - Center of mass is preserved
  - Curves flow to "round points"
  - Embedded curves remain embedded

$$\mathrm{length}(\gamma) := \int_0^L \left|\tfrac{d}{ds}\gamma\right| ds$$

$$\tfrac{d}{dt}\gamma = -\nabla_\gamma \mathrm{length}(\gamma)$$

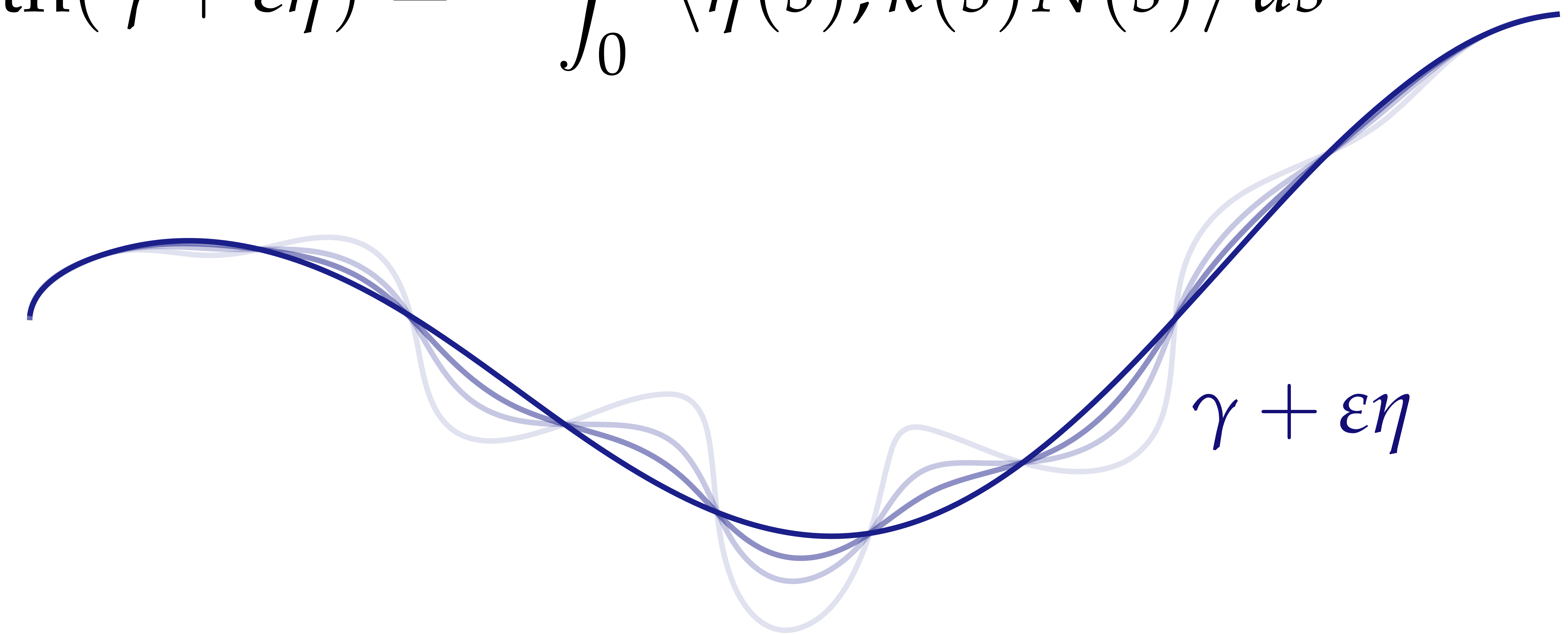


credit: Sigurd Angenent

Length Shortening Flow

Let $\mathrm{length}(\gamma)$ denote the total length of a regular plane curve $\gamma : [0, L] \to \mathbb{R}^2$, and consider a variation $\eta : [0, L] \to \mathbb{R}^2$ vanishing at endpoints. One can then show that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{length}(\gamma + \varepsilon\eta) = -\int_0^L \langle \eta(s), \kappa(s) N(s) \rangle \, ds$$



Key idea: quickest way to reduce length is to move in the direction κN.

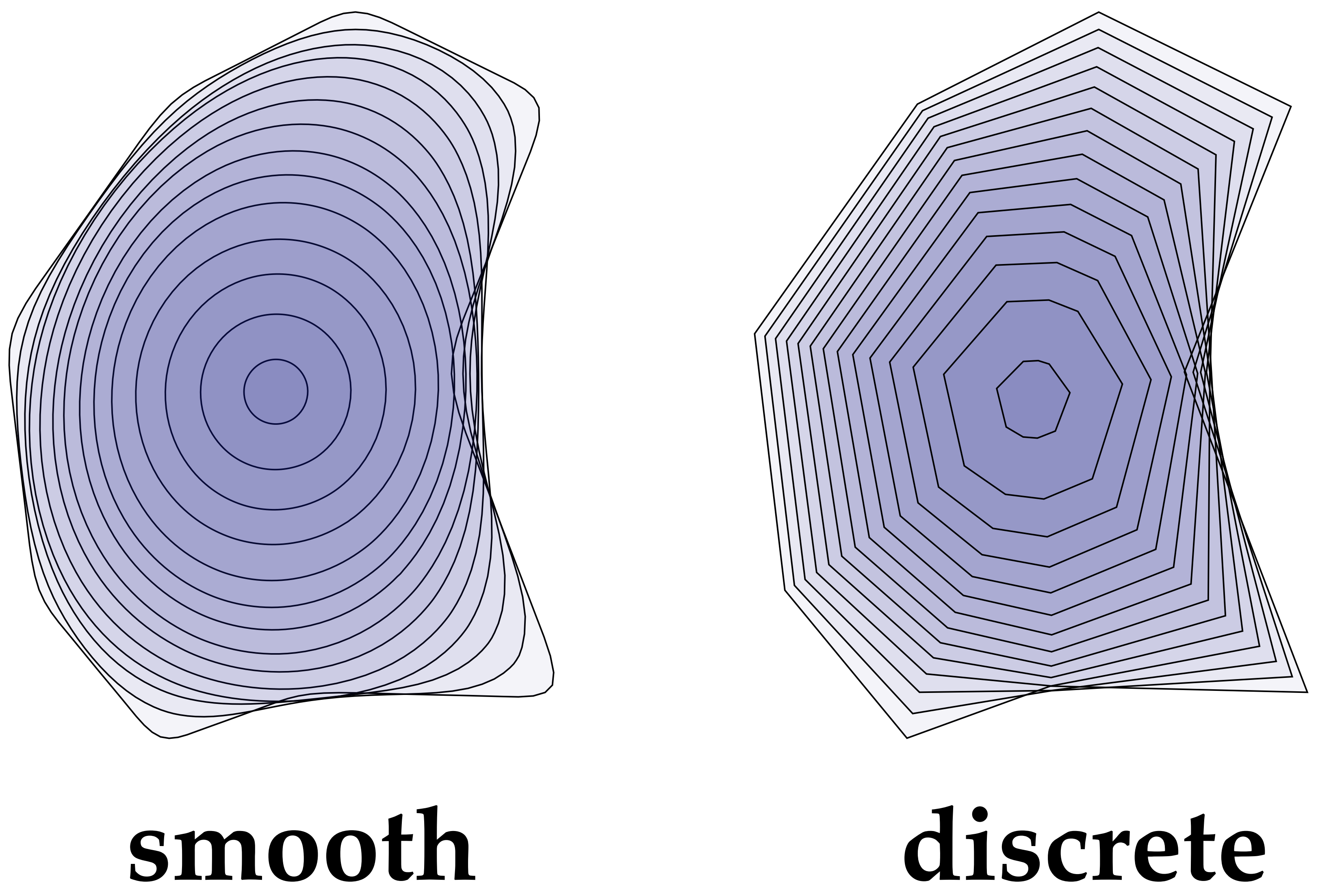
Length Shortening Flow—Forward Euler

- At each moment in time, move curve in normal direction with speed proportional to curvature
- "Smooths out" curve (*e.g.*, noise), eventually becoming circular
- Discretize by replacing time derivative with difference in time; smooth curvature with one (of many) curvatures
- Repeatedly add a little bit of κN (*"forward Euler method"*)

$$\frac{d}{dt}\gamma(s,t) = -\kappa(s,t)N(s,t)$$

$$\frac{\gamma_i^{t+1} - \gamma_i^t}{\tau} = -\kappa_i^t N_i^t$$

$$\Rightarrow \gamma_i^{t+1} = \gamma_i^t - \tau\kappa_i^t N_i^t$$

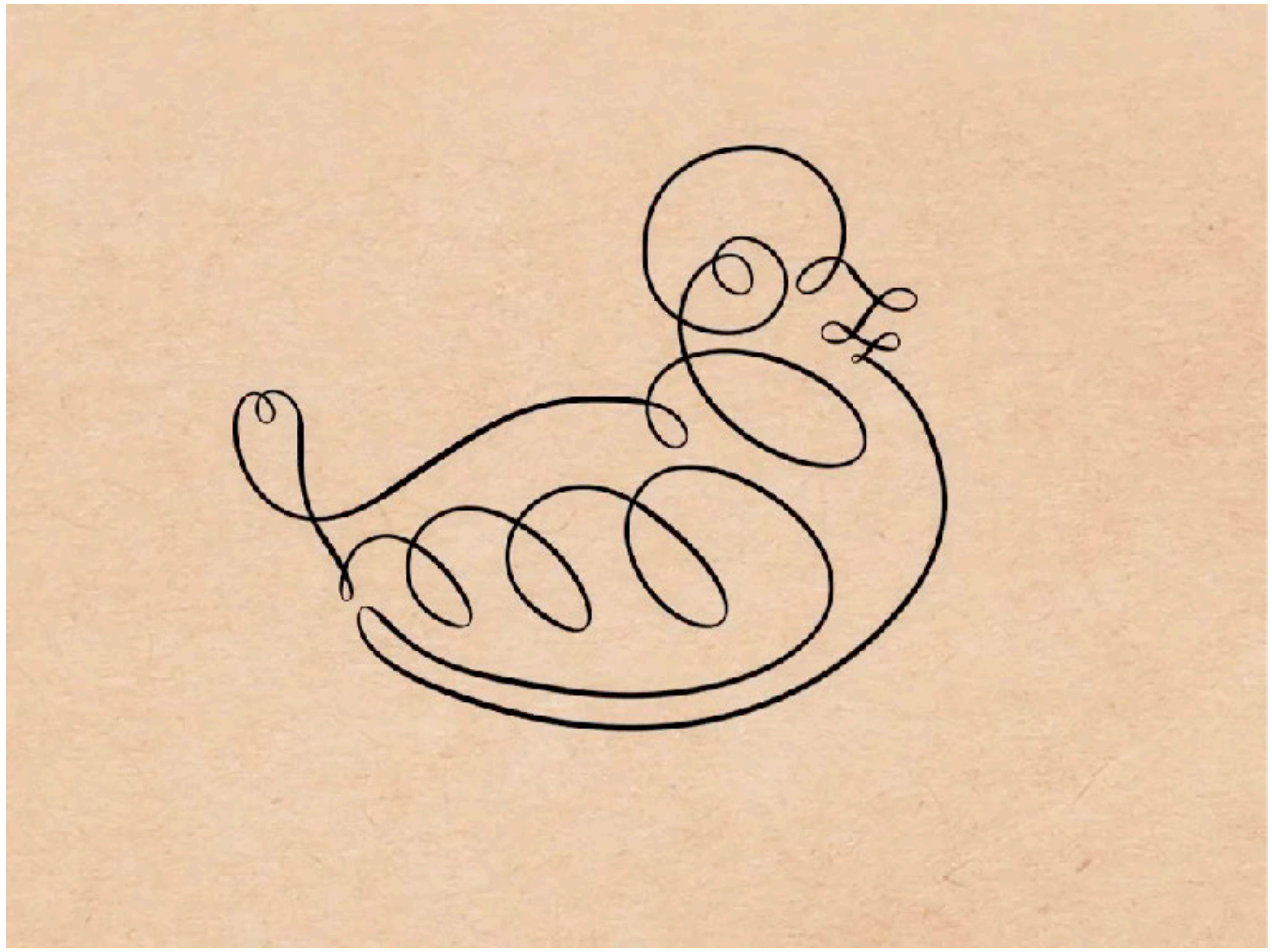


Elastic Flow

- Basic idea: rather than shrinking length, try to reduce *bending* (curvature)
- Objective is integral of squared curvature; elastic flow is then gradient flow on this objective
- Minimizers are called *elastic curves*
- More interesting w/ constraints (e.g., endpoint positions & a tangents)

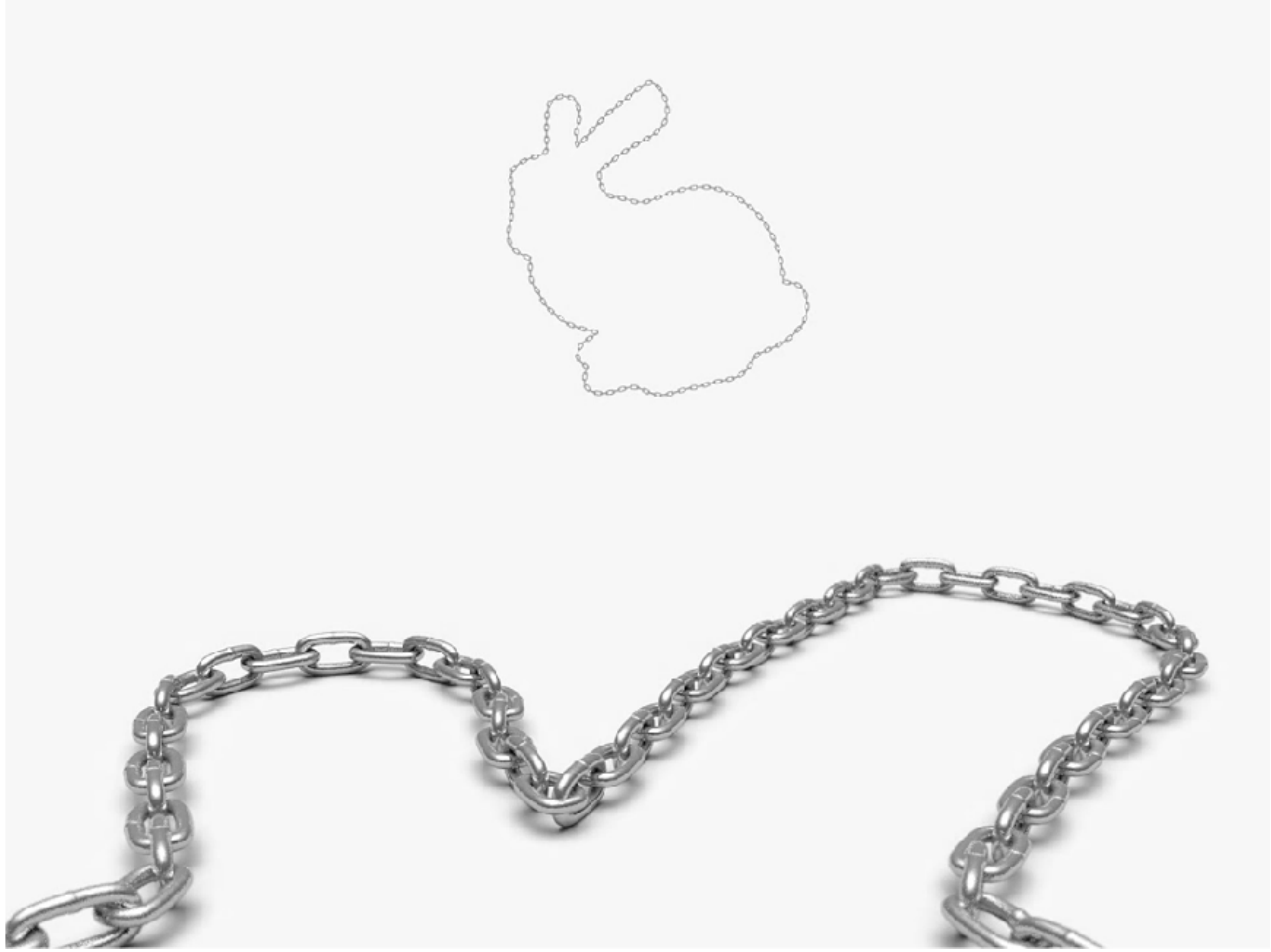
$$E(\gamma) := \int_0^L \kappa(s)^2 \, ds$$

$$\frac{d}{dt}\gamma = -\nabla_\gamma E(\gamma)$$



Isometric Elastic Flow

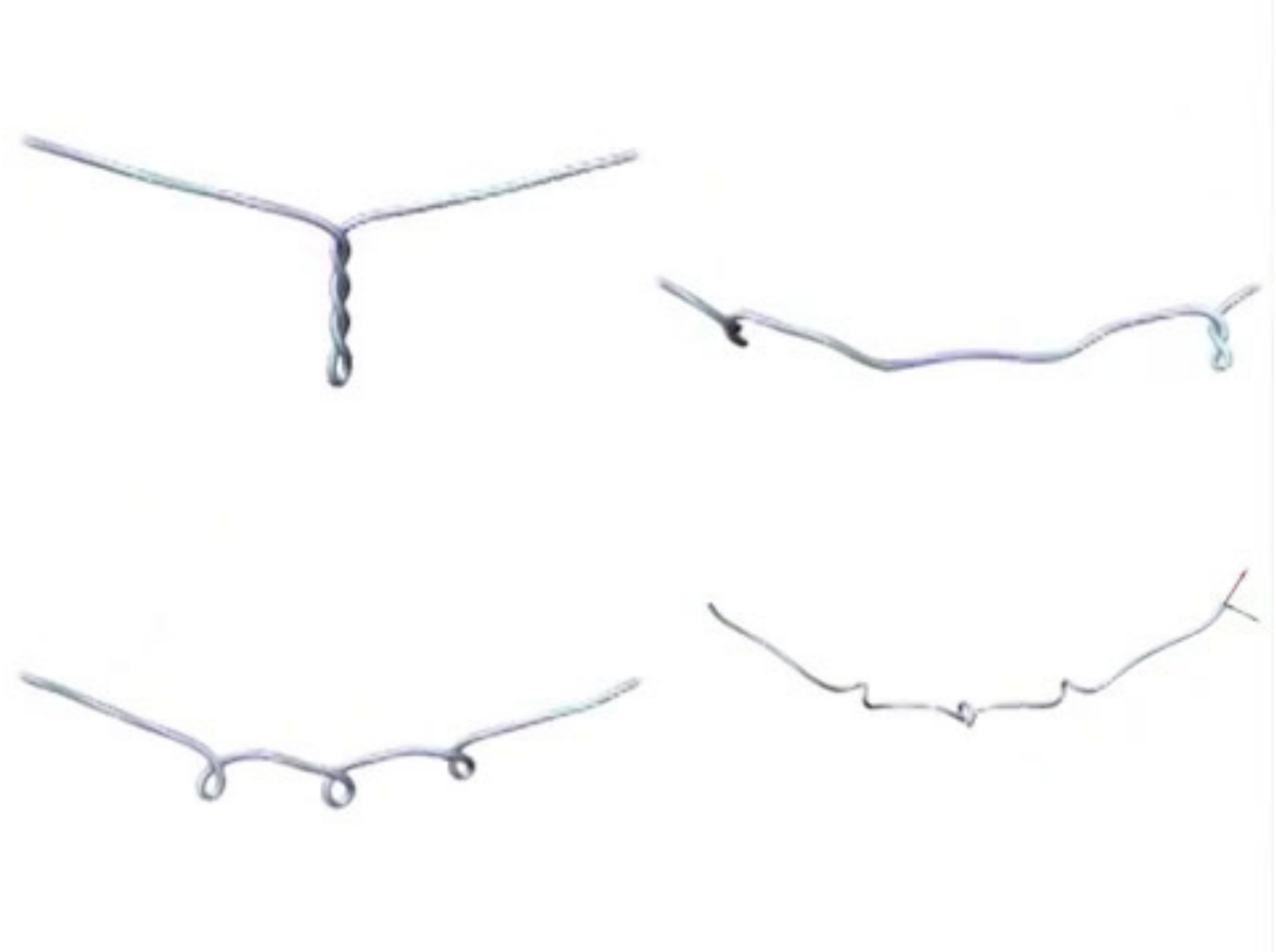
- Different way to smooth out a curve is to directly "shrink" curvature
- Discrete case: "scale down" turning angles, then use the fundamental theorem of discrete plane curves to reconstruct
- Extremely stable numerically; exactly preserves edge lengths
- Challenge: how do we make sure closed curves remain closed?



From Crane et al, *"Robust Fairing via Conformal Curvature Flow"*

Elastic Rods

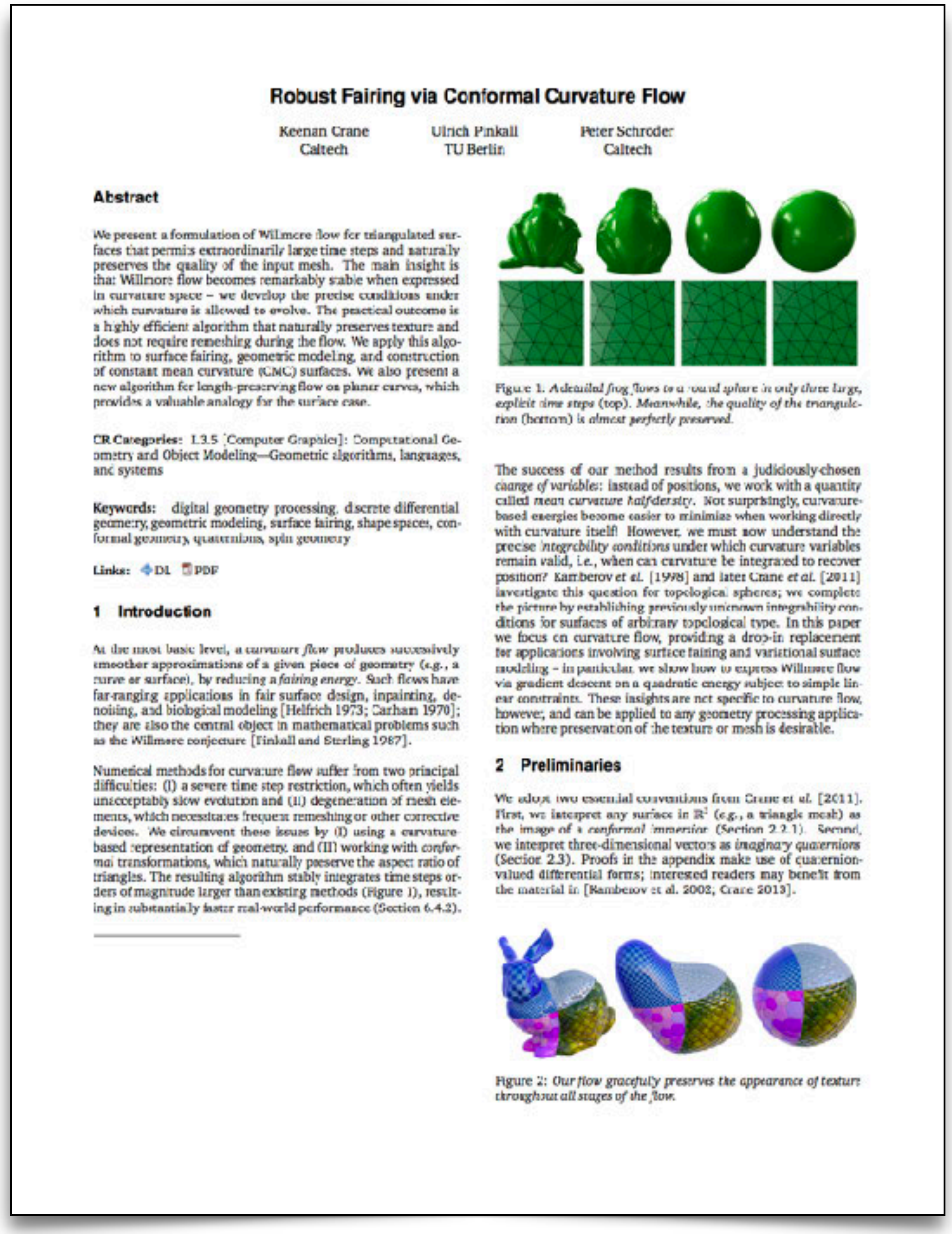
- For space curve, can also try to minimize both *curvature* **and** *torsion*
- Both in some sense measure "non-straightness" of curve
- Provides rich model of *elastic rods*
- Lots of interesting applications (simulating hair, laying cable, …)



From Bergou et al, *"Discrete Elastic Rods"*

Reading Assignment

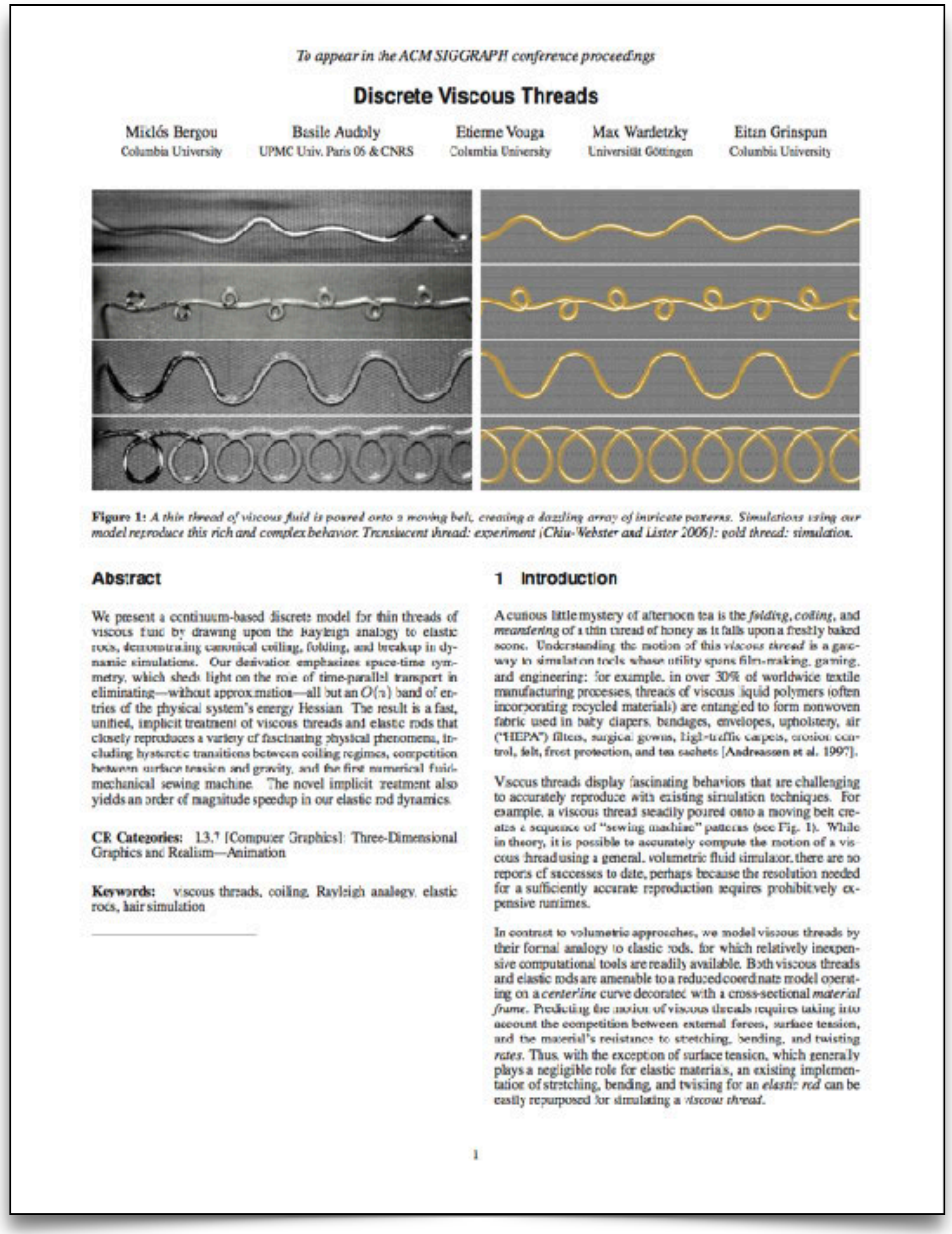
- Readings from papers on curve algorithms (will be posted online)


Robust Fairing via Conformal Curvature Flow

Keenan Crane, Caltech
Ulrich Pinkall, TU Berlin
Peter Schröder, Caltech

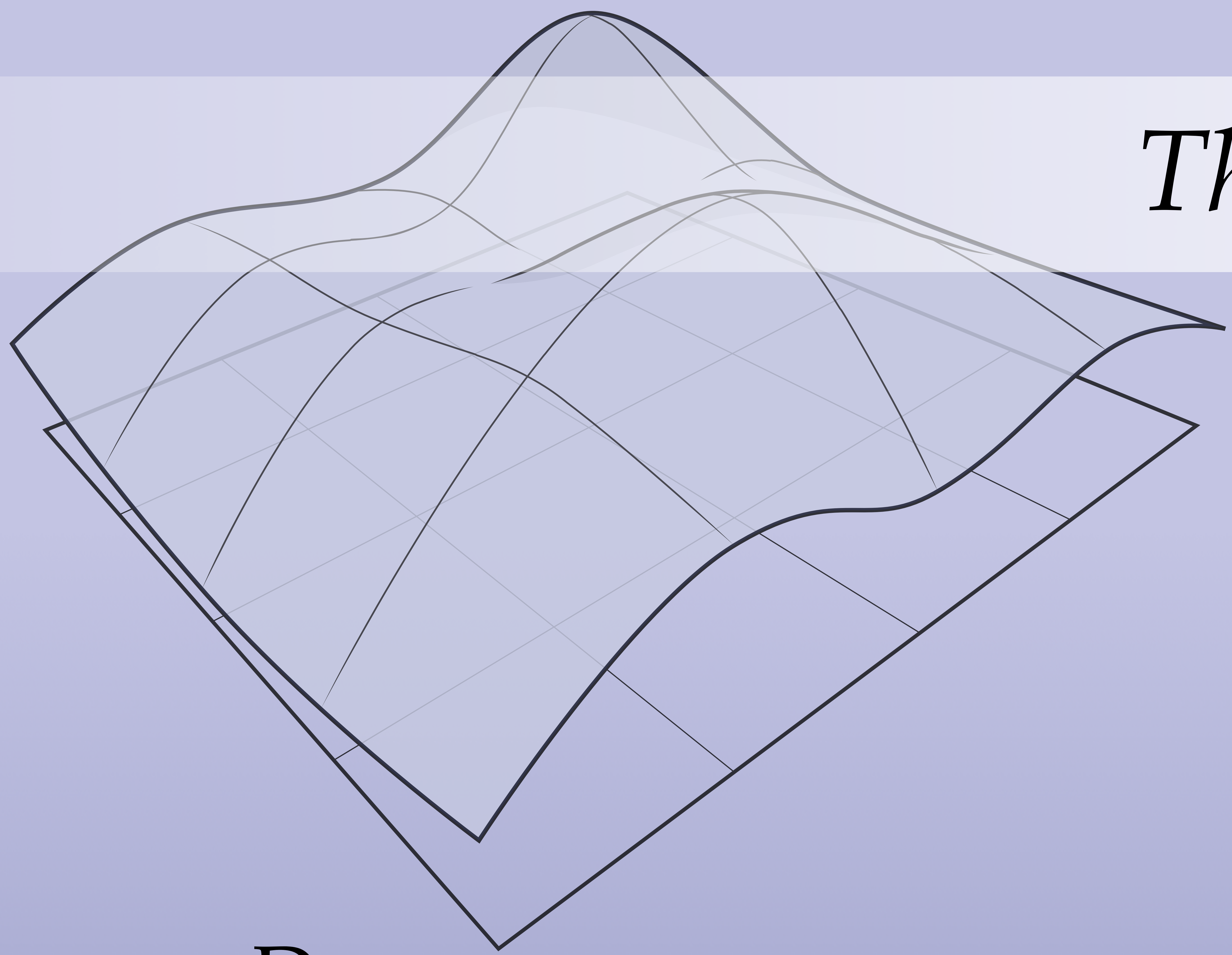

Discrete Elastic Rods

Miklós Bergou, Columbia University
Max Wardetzky, Freie Universität Berlin
Stephen Robinson, Columbia University
Basile Audoly, CNRS / UPMC Univ Paris 06
Eitan Grinspun, Columbia University


To appear in the ACM SIGGRAPH conference proceedings

Discrete Viscous Threads

Miklós Bergou, Columbia University
Basile Audoly, UPMC Univ. Paris 06 & CNRS
Etienne Vouga, Columbia University
Max Wardetzky, Universität Göttingen
Eitan Grinspun, Columbia University



Thanks!

DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017