

N. L. Carothers' Real Analysis - A partial solutions manual

George Lydakis

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# **Part I**

# **Metric Spaces**



# Chapter 1

## Calculus Review

### 1.1 The Real Numbers

#### Exercise 1

If  $A$  is a nonempty subset of  $\mathbb{R}$  that is bounded below, show that  $A$  has a greatest lower bound. That is, show that there is a number  $m \in \mathbb{R}$  satisfying: (i)  $m$  is a lower bound for  $A$ ; and (ii) if  $x$  is a lower bound for  $A$ , then  $x \leq m$ . [Hint: Consider the set  $-A = \{-a : a \in A\}$  and show that  $m = -\sup(-A)$  works.]

*Solution.*

As pointed out in the hint, let  $m = \sup(-A)$ . Let also  $B = \{-a : a \in A\}$  and  $b$  be any element of  $B$ . Since  $A$  is non-empty,  $B$  is non-empty too. By the definition of  $B$ ,  $b = -a$  for some  $a \in A$ . Because  $A$  is bounded below, it holds that there exists  $l$  such that  $l \leq a$ . This means that  $-l \geq -a \implies -l \geq b$ . Therefore,  $-l$  is an upper bound for  $B$ . By the completeness of real numbers,  $B$  has a supremum,  $m$ . For any  $a \in A$ ,  $-a \in B$ , therefore  $-a \leq m \implies a \geq -m$ . This means that  $-m$  is a lower bound for  $A$ . For any lower bound  $m'$  of  $A$ , it holds that  $m' \leq a$  for all  $a \in A$ . This means that  $-m' \geq -a$  for all  $-a \in B$ . But then  $-m'$  is an upper bound for  $B$ , thus by the definition of the supremum,  $-m' \geq m \implies m' \leq -m$ . We have thus shown that  $-m$  is the greatest lower bound of  $A$ .

#### Exercise 3

Establish the following apparently different (but “fancier”) characterization of the supremum. Let  $A$  be a non-empty set of  $\mathbb{R}$  that is bounded above. Prove that  $s = \sup A$  if and only if (i)  $s$  is an upper bound for  $A$ , and (ii) for every  $\epsilon > 0$  there is an  $a \in A$  such that  $a > s - \epsilon$ . State and prove the corresponding result for the infimum of an non-empty subset of  $\mathbb{R}$  that is bounded below.

*Solution.*

$\implies$  : Suppose first that  $s$  is the supremum of  $A$ . By the definition of the supremum,  $s$  is indeed an upper bound for  $A$ . Suppose now that there exists  $\epsilon > 0$  such that for every  $a \in A$  it is the case that  $a \leq s - \epsilon$ . Observe that this means that  $s - \epsilon$  is an upper bound for  $A$ , while it also holds that  $s - \epsilon < s$ . But this contradicts the definition of  $s$  being the supremum of  $A$ .

We have arrived at a contradiction, and thus the negation of the assumed statement must hold. Namely, it must be the case that for every  $\epsilon > 0$ , there exists  $a \in A$  such that  $a > s - \epsilon$ .

$\impliedby$  : Suppose now that  $s$  is an upper bound for  $A$  and that for every  $\epsilon > 0$  there is an  $a \in A$  such that  $a > s - \epsilon$ . Suppose that  $s$  is not the supremum of  $A$ . By the completeness of the real numbers,  $A$  must have a supremum  $t$ . Because  $s$  is an upper bound for  $A$ , it must hold that  $t < s$ . Set  $\epsilon = s - t > 0$ . By the definition of  $s$ , there exists  $a \in A$  such that  $a > s - \epsilon = s - (s - t) = t$ . This contradicts the fact that  $t$  is an upper bound for  $A$ . Therefore,  $s$  must be the supremum of  $A$ .

The corresponding result for the infimum is that a number  $s$  is the infimum of  $A \subset \mathbb{R}$ , with  $A$  non-empty and bounded below, if and only if (i)  $s$  is a lower bound for  $A$  and (ii) for every  $\epsilon > 0$ , there exists  $a \in A$  such that  $a < s + \epsilon$ . To prove this, we have that:

$\implies$ : Suppose  $s$  is the infimum of  $A$ . Then  $s$  is a lower bound for  $A$ . Suppose that there exists  $\epsilon > 0$  such that for every  $a \in A$ ,  $a \geq s + \epsilon$ . But then  $s + \epsilon > s$  is a lower bound for  $A$ , contradiction. The negation of our assumption leads to the desired property for  $s$ .

$\impliedby$ : Now suppose  $s$  is a lower bound for  $A$  and for every  $\epsilon > 0$ , there exists  $a \in A$  such that  $a < s + \epsilon$ . Suppose  $s$  is not the infimum of  $A$ , and  $t$  is instead. Because  $s$  is a lower bound,  $s < t$ . Set  $\epsilon = t - s$ . Then there exists  $a \in A$  such that  $a < s + \epsilon = s + (t - s) = t$ . But this contradicts the fact that  $t$  is a lower bound for  $A$ .

Note that this second proof can also be done by relating the infimum of  $A$  to the supremum of  $-A$ , but since we have not yet proved this (exercise 1) we do not use it here.

#### Exercise 4

Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded above. Show that there is a sequence of elements  $x_n$  of  $A$  that converges to  $\sup A$ .

*Solution.*

Recall from exercise 3 that if  $s = \sup A$ , it holds that for every  $\epsilon > 0$  there exists  $a \in A$  such that  $a > s - \epsilon$ . Consider then the sequence that is formed by taking  $x_i$  be an element of  $A$  for which it holds that  $x_i > s - \frac{1}{i}$ . For any  $\epsilon > 0$ , take  $M = \lceil \frac{1}{\epsilon} \rceil$ . It then holds that  $\frac{1}{M} \leq \epsilon$ . Additionally, it is true by the definition of the sequence that  $x_M > s - \lceil \frac{1}{M} \rceil \geq s - \epsilon \implies \epsilon > s - x_M$ . Also, by the definition of the supremum,  $s - x_M \geq 0$ , thus  $|s - x_M| < \epsilon$ . Now, for any  $j > M$  we have that  $x_j > s - \frac{1}{j} \implies \frac{1}{j} > s - x_j$ . By the definition of the supremum,  $s - x_j \geq 0$ , thus  $|s - x_j| < \frac{1}{j} < \frac{1}{M} \leq \epsilon$ .

We have thus precisely proved that the limit of the sequence  $x_i$  is the supremum  $s$  of  $A$ .

#### Exercise 5

Suppose that  $a_n \leq b$  for all  $n$  and that  $a = \lim_{n \rightarrow \infty} a_n$  exists. Show that  $a \leq b$ . Conclude that  $a \leq \sup\{a_n : n \in \mathbb{N}\}$ .

*Solution.*

Suppose that  $a > b$ , which means that  $a - b > 0$ . Set  $\epsilon = a - b$ . By the definition of the limit of a sequence, there exists  $M > 0$  such that for all  $n > M$ , it holds that:

$$|a - a_n| < \epsilon \implies -\epsilon < a - a_n < \epsilon \implies a < a_n + \epsilon$$

It is the case that  $a_n \leq b$  for all  $n$ , thus  $a < b + \epsilon = b + (a - b) \implies a < a$ , which is clearly a contradiction. Therefore  $a \leq b$ . Now because  $s = \sup\{a_n : n \in \mathbb{N}\}$  is by definition a number for which  $a_n \leq s$  for all  $n$ , the previous result applies, and thus  $a \leq s = \sup\{a_n : n \in \mathbb{N}\}$ .

#### Exercise 6

Prove that every convergent sequence of real numbers is bounded. Moreover, if  $a_n$  is convergent, show that  $\inf a_n \leq \lim_{n \rightarrow \infty} a_n \leq \sup a_n$ .

*Solution.*

Suppose  $a_n$  is a convergent sequence of real numbers, and suppose that it converges to  $a \in \mathbb{R}$ . By definition, for any  $\epsilon > 0$ , there exists  $M > 0$  such that for every  $n > M$  it holds that  $|a_n - a| < \epsilon \implies a_n < a + \epsilon$ . Pick any such  $\epsilon$ , e.g.  $\epsilon = 1$ . We can thus see that there exist *at most*  $M$  elements of the sequence, chosen from  $a_1, a_2, \dots, a_M$  such that  $a_i \geq a + \epsilon$ . Let then  $S$  be the —possibly empty— set of all such  $a_i$ . We can then see that the set  $S \cup \{a + \epsilon\}$  has a finite number of elements. Thus its maximum element  $s$  is well defined. Observe then that  $s \geq a_n$  for all elements of the sequence. Now recall from exercise 5 that the limit of the sequence is indeed at most equal to its supremum (which we showed is well defined, since at least one upper bound exists).

Again by the definition of the limit, for every  $\epsilon > 0$  there exists  $M > 0$  such that for  $n > M$  it holds that  $|a_n - a| < \epsilon \implies -\epsilon < a_n - a \implies a + \epsilon < a_n$ . Observe that we can thus apply a completely symmetric argument: pick, say,  $\epsilon = 1$  and then there are at most  $M$  elements such that  $a_n \leq a + \epsilon$ . By constructing  $S$  from these elements in the same way as above, the rest of the proof becomes completely symmetric, as is the part regarding  $\inf a_n \leq \lim_{n \rightarrow \infty} a_n$ .

**Exercise 8**

Given  $a < b$ , show that there are, in fact, infinitely many distinct rationals between  $a$  and  $b$ . The same goes for irrationals too.

*Solution.*

Suppose that there are only a finite number of distinct rationals between  $a$  and  $b$ , and call them  $q_1, q_2, \dots, q_n$ , listed in ascending order. This means  $a < q_1 < q_2 < \dots < q_n < b$ . But then we have that  $q_n, b$  are real numbers with  $q_n < b$ , and therefore there has to exist a rational number  $q$  such that  $q_n < q < b$ . But this is a contradiction, because  $q \neq q_i$  for all  $i$  and  $a < q < b$ . Therefore there exists an infinite number of distinct rationals between  $a$  and  $b$ .

Now suppose that there is only a finite number of irrationals  $r_1, r_2, \dots, r_n$  between  $a, b$ , listed in ascending order. However, in exercise 7 we've seen that if  $a < b$ , there exists an irrational  $x$  such that  $a < x < b$ . This means that in our case there exists an irrational  $r$  such that  $r_n < r < b$ . Clearly, this is a contradiction since  $r_i \neq r$  for all  $i$ . Therefore there exist infinitely many distinct irrationals between  $a, b$ .

**Exercise 13**

Let  $a_n \geq 0$  for all  $n$ , and let  $s_n = \sum_{i=1}^n a_i$ . Show that  $(s_n)$  converges if and only if  $(s_n)$  is bounded.

*Solution.*

$\Rightarrow$  : Assume that  $(s_n)$  converges to  $s$ . Then recall from exercise 6 that every convergent sequence is bounded, thus  $s_n$  is indeed bounded.

$\Leftarrow$  : Suppose  $(s_n)$  is bounded. Because  $a_n \geq 0$ , the sequence of partial sums  $s_n$  increases monotonically. We know then that a monotone, bounded sequence always converges, thus  $(s_n)$  converges.

**Exercise 14**

Prove that a convergent sequence is Cauchy, and that any Cauchy sequence is bounded.

*Solution.*

A sequence of real numbers is Cauchy if, for every  $\epsilon > 0$ , there is an integer  $N \geq 1$  such that  $|x_n - x_m| < \epsilon$  whenever  $n, m \geq N$ .

Let then  $a_n$  be a convergent sequence that converges to  $a$ , and select any  $\epsilon > 0$ . Then, by the definition of convergence, for  $\epsilon' = \frac{\epsilon}{2}$  there exists  $N > 0$  such that for all  $n > N$  it holds that  $|a_n - a| < \epsilon'$ . Therefore, for any two  $n, m > N$ , it holds that:

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, which is the defining property of Cauchy sequences. Now let  $a_n$  be any Cauchy sequence. Suppose that  $a_n$  is not bounded. Because  $a_n$  is Cauchy, it must hold that for  $\epsilon = 1$ , there exists  $N \geq 1$  such that for all  $n, m > N$  it holds that  $|a_n - a_m| < 1$ . Let then  $a_k$  be the first element of the sequence for which this inequality holds when setting  $n = k$  and  $m$  any other index  $m > k$ . This means that for any  $m > k$ ,  $|a_m - a_k| < 1$ . In particular, this means that  $a_m$  can never be larger than  $a_k + 1$ , therefore for all elements of the sequence after that point,  $a_k + 1$  is an upper bound. Then, because  $k$  is finite,  $m = \max\{a_1, \dots, a_k\}$  is well defined. If we then set  $s = \max\{m, a_k + 1\}$ , we see that  $s$  constitutes an upper bound for the entire sequence.

A symmetric argument can be applied to extract a lower bound for the sequence, thus any Cauchy sequence is bounded.

**Exercise 15**

Show that a Cauchy sequence with a convergent subsequence actually converges.

*Solution.*

Let  $a_n$  be a Cauchy sequence and  $b_n = a_{f(n)}$  such that  $f(j) > f(k)$  whenever  $j > k$  be a convergent subsequence of it that converges to  $b$ . Pick any  $\epsilon > 0$ . Now it must be the case that there exists  $N > 0$  such that whenever  $n > N$ ,  $|b_n - b| < \frac{\epsilon}{2}$ . Now because  $a_n$  is Cauchy, there exists  $M > 0$  such that whenever  $n, m > 0$ ,  $|a_n - a_m| < \frac{\epsilon}{2}$ . Set  $K = \max\{N, M\}$  and observe then that for any  $n > K$ , it holds that:

$$|a_n - b| < |a_n - b_{K+1} + b_{K+1} - b| \leq |a_n - b_{K+1}| + |b_{K+1} - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, where in the last step we used the fact that  $a_n$  is Cauchy (and  $b_{K+1}$  is of course an element of it) for the first term and that  $b_n$  converges to  $b$  for the second term. We have thus shown that  $a_n$  converges, and more specifically it converges to the limit of its convergent subsequence.

### Exercise 17

Given real numbers  $a$  and  $b$  establish the following formulas:  $|a + b| \leq |a| + |b|$ ,  $||a| - |b|| \leq |a - b|$ ,  $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ ,  $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

*Solution.*

- $|a + b| \leq |a| + |b|$ : one way to do this is to observe that  $\mathbb{R}$  is a vector space over the field  $\mathbb{R}$  for which standard multiplication satisfies the properties of being an inner product. Thus we can use Axler's result of the triangle inequality for inner product spaces, whose proof does not assume that the triangle inequality holds for real numbers. Another way is to do a variant of this proof "manually":

$$\begin{aligned} (a + b)^2 &= a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a| \cdot |b| = (|a| + |b|)^2 \\ \implies |a + b| &\leq |a| + |b| \end{aligned}$$

, where we used some properties of the absolute value which easily follow from its definition ( $|a|^2 = |a|^2, x \leq |x|$ ), and we took square roots at the last step, observing that  $|a| + |b|$  is always non-negative.

- $||a| - |b|| \leq |a - b|$ : Quite similar to the above, we have that:

$$(|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a| \cdot |b| \leq |a|^2 + |b|^2 - 2ab = (a - b)^2 \implies ||a| - |b|| \leq |a - b|$$

, where when taking square roots the absolute values are now necessary.

- $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ : It is either the case that  $a \geq b$  or that  $a \leq b$  (here we are using the fact that  $\mathbb{R}$  is a totally ordered set). In the first case, we have that  $\max\{a, b\} = a$  and also that  $|a - b| = a - b$ , which means  $\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = a = \max\{a, b\}$ . The second case is almost exactly the same, except for  $|a - b|$  evaluating to  $b - a$ .
- $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ : This is done in the same manner as above, again by utilizing the fact that for any two  $a, b$ , it either holds that  $a \leq b$  or that  $a \geq b$ .

### Exercise 18

- Given  $a > -1, a \neq 0$ , use induction to show that  $(1 + a)^n > 1 + na$  for any integer  $n > 1$ .
- Use (a) to show that, for any  $x > 0$ , the sequence  $(1 + \frac{x}{n})^n$  increases.
- If  $a > 0$ , show that  $(1 + a)^r > 1 + ra$  holds for any *rational* exponent  $r > 1$ .  
[Hint: If  $r = \frac{p}{q}$ , then apply (a) with  $n = q$  and (b) with  $x = ap$ .]
- Finally, show that (c) holds for any *real* exponent  $r > 1$ .

*Solution.*

- The base case of the induction is  $n = 2$ . We then have that:

$$(1 + a)^2 = 1 + 2a + a^2 > 1 + 2a$$

, since  $a$  is not zero, meaning that  $a^2 > 0$ . Suppose now that the inequality holds for  $n = k > 1$ . Then we have that:

$$(1 + a)^{k+1} = (1 + a)(1 + a)^k > (1 + a)(1 + ka) = 1 + ka + a + ka^2 = 1 + (k + 1)a + ka^2 > 1 + (k + 1)a$$

, where, crucially, we used the fact that  $a > -1$ , thus that  $a + 1 > 0$ , thus that we can multiply the inequality for  $n = k$  with  $(1 + a)$ . In the last step we again use the fact that  $a \neq 0$ , thus that  $ka^2 > 0$ .

(b) We'll work through this the same way as the book's examples, i.e., compute the ratio between two successive terms of the sequence:

$$\begin{aligned} \frac{(1 + \frac{x}{n+1})^{n+1}}{(1 + \frac{x}{n})^n} &= \left(1 + \frac{x}{n}\right) \frac{(1 + \frac{x}{n+1})^{n+1}}{(1 + \frac{x}{n})^{n+1}} = \left(1 + \frac{x}{n}\right) \left(\frac{\frac{n+1+x}{n+1}}{\frac{n+x}{n}}\right)^{n+1} \\ &= \left(1 + \frac{x}{n}\right) \left(\frac{n^2 + n + nx}{n^2 + x + n + nx}\right)^{n+1} = \left(1 + \frac{x}{n}\right) \left(1 - \frac{x}{n^2 + x + n + nx}\right)^{n+1} = \left(1 + \frac{x}{n}\right) \left(1 - \frac{x}{(x+n)(n+1)}\right)^{n+1} \end{aligned}$$

Now, for any  $x > 0$  and  $n$  positive integer we have that:

$$\begin{aligned} xn + n^2 + n > 0 &\implies xn + n^2 + n + x > x \implies (x+n)(n+1) > x \implies -(x+n)(n+1) < -x \\ &\implies -1 < -\frac{x}{(x+n)(n+1)} \end{aligned}$$

Also,  $x > 0$  thus this quantity is never zero. Therefore, using it as an  $a$  in the Bernoulli inequality (part (a)) we can obtain that:

$$\frac{(1 + \frac{x}{n+1})^{n+1}}{(1 + \frac{x}{n})^n} > \left(1 + \frac{x}{n}\right) \left(1 - (n+1) \frac{x}{(x+n)(n+1)}\right) = \left(1 + \frac{x}{n}\right) \left(1 - \frac{x}{x+n}\right) = \frac{x+n}{n} \cdot \frac{n}{x+n} = 1$$

, which means that the sequence  $\left(1 + \frac{x}{n}\right)$  does indeed increase.

(c) Consider any rational  $r > 1$ .  $r$  can be written as  $\frac{p}{q}$ , where  $p, q$  are positive integers, and in fact  $p > q$ . Setting  $x = ap$  (for this part,  $a > 0$ ) and observing that  $p > q$ , from part (b) we have that:

$$\left(1 + \frac{ap}{q}\right)^q < \left(1 + \frac{ap}{p}\right)^p \implies \left(1 + \frac{ap}{q}\right)^q < (1+a)^p$$

For  $a > 0$ , both quantities inside the parentheses are positive, and thus we can take  $q$ -th roots and obtain:

$$\left(1 + \frac{ap}{q}\right) < (1+a)^{\frac{p}{q}}$$

, which, since  $r = \frac{p}{q}$ , is the equivalent of the Bernoulli inequality for a rational exponent  $r > 1$  and  $a > 0$ .

(d) We know that we can approach any real number  $r > 1$  with a sequence of rationals. Furthermore, the Bernoulli inequality will hold for all of those rationals that are larger than 1, so as we approach  $r > 1$ , the Bernoulli inequality will hold. Now, at the limit, this strict inequality holds as a non-strict inequality, thus we can conclude that:

$$(1+a)^r \geq 1 + ra$$

Now pick a rational  $q$  such that  $1 < q < r$  (such a rational always exists). Observe that:

$$(1+a)^r = (1+a)^{\frac{rq}{q}} = ((1+a)^q)^{\frac{r}{q}} > (1+qa)^{\frac{r}{q}}$$

, where we applied the Bernoulli inequality for  $q$ . Now, however,  $\frac{r}{q}$  is a real number greater than 1, and hence we can use our non-strict inequality above to arrive at the desired result:

$$(1+a)^r > (1+qa)^{\frac{r}{q}} \geq 1 + q \frac{r}{q} a = 1 + ra \implies (1+a)^r > 1 + ra$$

### Exercise 21

Let  $p \geq 2$  be a fixed integer, and let  $0 < x < 1$ . If  $x$  has a finite-length base  $p$  decimal expansion, that is, if  $x = a_1/p + \dots + a_n/p^n$  with  $a_n \neq 0$ , prove that  $x$  has precisely two base  $p$  decimal expansions. Otherwise, show that the base  $p$  decimal expansion for  $x$  is unique. Characterize the numbers  $0 < x < 1$  that have repeating base  $p$  decimal expansions. How about eventually repeating?

*Solution.*

We consider first the decimal expansion  $0.a_1a_2\dots a_n00\dots$ , with an infinite number of trailing zeros after the  $n$ -th decimal, and call the  $i$ -th decimal here  $b_i$ . As per the book's definition, this corresponds to an infinite series defined as  $\sum_{i=1}^{\infty} b_i/p^i$ . It's clear that the limit of this sum is  $x$ , since  $x$  is precisely the sum of the first  $n$  terms, and all of the terms after  $n$  are zero. Thererfore, the expansion above indeed corresponds to  $x$ .

Now consider the decimal expansion corresponding to the series  $\frac{a_1}{p} + \frac{a_2}{p^2} + \dots + \frac{a_{n-1}}{p^{n-1}} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^k}$ . This would yield a series of decimals the first  $n$  of which would equal  $a_1, \dots, a_n - 1$  (note that because  $a_n \neq 0, a_n - 1$  causes no problems), while all decimals after that would equal  $p - 1$  (call the sequence of digits  $c_i$ ). If we consider the limit of this series, we can see that the last term is a series summing to  $\frac{1}{p^n}$ , and that the first  $n$  terms sum to  $x - \frac{1}{p^n}$ . Thus, the series as a whole tends to  $x$ , which means that it is a decimal expansion for  $x$ .

We now have to show that there is no other decimal expansion for  $x$ . We will do this by selecting a decimal expansion  $0.b_1b_2\dots b_nb_{n+1}\dots$  that is not equal to  $0.a_1a_2\dots a_n00\dots$  and show that it must necessarily equal  $0.a_1a_2\dots (a_n - 1)(p - 1)(p - 1)\dots$ . Let us then examine such a decimal expansion. Since it differs from  $0.a_1a_2\dots a_n00\dots$ , there must exist a first  $j$  such that  $a_j \neq b_j$ . Because both decimal expansions correspond to  $x$ , for the two series it must hold that:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, where we have erased from both sides the first  $j - 1$  digits that are equal (note that the set of these may be empty). Because we are manipulating convergent series, we can write:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i} = \sum_{i=j+1}^n \frac{b_i - a_i}{p^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{p^i}$$

, where we noted that after the  $n$ -th digit, all  $a_i$  are zero. Now, note that if  $b_j > a_j$ , the LHS here equals at most  $-\frac{1}{p^j}$ . At the same time, the RHS equals at least 0, which happens when  $b_i = 0, i \geq n + 1$  and all  $b_i - a_i = 0, j + 1 \leq i \leq n$ . Clearly then, they can never be equal. On the other hand, if  $b_j < a_j$  the LHS equals at least  $\frac{1}{p^j}$ . The RHS equals at most  $\sum_{i=j+1}^{\infty} \frac{p-1}{p^i} = \frac{1}{p^j}$ , and, crucially, this happens if all  $b_i = p - 1, i \geq n + 1, b_i - a_i = p - 1, j + 1 \leq i \leq n$ . Because the value is achieved when all of the digits fulfill these conditions, the RHS will be strictly smaller than the LHS in all other cases. The consequence of this is that all  $b_i$  starting at  $i = n + 1$  are equal to  $p - 1$ . Additionally, all  $a_i, j + 1 \leq i \leq n$  have to be zero. But because  $a_n \neq 0$ , it must hold that  $j + 1 > n$ . That is, the first digit where the two expansions differ must be at least at position  $n$ . If this was strictly larger than  $n$ , the LHS would not achieve its minimum value ( $a_j$  would be 0 and  $b_j$  would be  $p - 1$ ), thus it could not equal the RHS, contradiction. Thus  $j = n$ . By these observations, the only digit for which we have not yet determined a value is at position  $n$ . Recall that  $a_j - b_j$  must equal 1, which means that  $b_j = a_j - 1$ . In other words,  $b_n = a_n - 1$ , all  $b_i = a_i, i < n$  and all  $b_i = p - 1, i > n$ , leading us to conclude that there exists no third possible representation.

Now we proceed to examine an  $x$  with no finite-length base  $p$  decimal expansion. Let  $a_i, i = 1, 2, \dots$  be one of its expansions, which is necessarily infinite in length. We will show that this is unique by taking another expansion  $b_i$  and showing that it must equal  $a_i$  at all digits. Indeed, suppose that they differ, and that the first digit at which this happens is at position  $j$ . As we did above, we can write:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, and, again because the series converge, we can rewrite this as:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i}$$

If  $b_j > a_j$  we again observe that the LHS equals at most  $-\frac{1}{p^j}$ . The RHS equals at least  $\sum_{i=j+1}^{\infty} \frac{-(p-1)}{p^i} = -\frac{1}{p^j}$ , which happens if all  $b_i - a_i = -(p-1), i > j$ . This cannot otherwise be true because this minimum value is achieved when all  $b_i - a_i$  are minimized. The consequence is each  $b_i = 0, a_i = p-1, i > j$ . However, this implies that  $x$  can then be written as  $0.b_1b_2\dots b_j00\dots$ , which is a finite-length expansion since all trailing digits after  $j$  are zeros. This is a contradiction. An exactly symmetrical argument applies when  $b_j < a_j$ , leading us to conclude that  $b_j = a_j$  for all  $j$ , thus  $a_i$  is a unique expansion for  $x$ .

Now we examine  $0 < x < 1$  that have an eventually repeating base  $p$  decimal expansion. This means that there exists a finite-length “unique” first part of the number, consisting of  $m$  digits, as well as a minimum “period” of repetition  $n$  (this can also be zero, in which case the part below does not apply but the number is trivially rational), such that the number can be written as:

$$0.a_1a_2\dots a_ma_{m+1}a_{m+2}a_{m+3}\dots a_{m+n}a_{m+1}\dots a_{m+n}\dots$$

, where we have defined  $n$  as the minimum integer for which this holds. Then observe that we can write:

$$\begin{aligned} x &= (a_1a_2\dots a_m)p^{-m} + (a_{m+1}a_{m+2}\dots a_{m+n})p^{-m-n} + (a_{m+1}a_{m+2}\dots a_{m+n})p^{-m-2n} \\ &\quad + (a_{m+1}a_{m+2}\dots a_{m+n})p^{-m-3n} + \dots \\ &= \frac{a_1a_2\dots a_m}{p^m} + (a_{m+1}a_{m+2}\dots a_{m+n})\left(\sum_{i=1}^{\infty} p^{-m-in}\right) = \\ &= \frac{a_1a_2\dots a_m}{p^m} + \frac{a_{m+1}a_{m+2}\dots a_{m+n}}{p^m(p^n-1)} \end{aligned}$$

, which is a sum of two quotients of integers, and therefore is a rational number. Note that for the case of repeating expansions everything above simplifies to  $m = 0$ . Now we need to examine whether *every* rational number can be written in this way, in other words, whether every rational number has an eventually repeating (which includes the trivially “eventually repeating” finite-length digit sequences) base  $p$  decimal expansion.

Recall that a rational number can be written as the quotient of two integers. Furthermore, because we are interested in numbers in  $[0, 1]$ , we can restrict ourselves to  $x = \frac{a}{b}$  where  $a, b$  natural, non-zero numbers with  $a < b$ . We now recall that for finding a base  $p$  decimal expansion for  $x$  we can use the division algorithm iteratively. Namely, we first find the smallest power  $p_1^n$  such that  $ap_1^n \geq b$ , and then we use the division algorithm on  $ap^n, b$ . According to it, there will exist unique  $q_1, r_1, 0 \leq r_1 < b \in \mathbb{N}$  such that  $ap^{n_1} = bq_1 + r_1$ . Notice that this implies that  $\frac{a}{b} = \frac{q_1}{p_1^{n_1}} + \frac{r_1}{bp_1^{n_1}}$ . The first  $n_1 - 1$  digits of the decimal expansion of  $x$  will be zeros, since for all powers  $p^k, k < n_1$  it has to be that  $ap^k < b \implies x < \frac{1}{p^k}$ . Now notice that  $q_1 \leq \frac{ap^{n_1}}{b} < p$  because  $ap^{n_1-1} < b$  due to the choice of  $n_1$ . Therefore  $q_1$  can be thought of as the  $n_1$ -th digit in the decimal expansion of  $x$ . If  $r_1 = 0$ , the decimal expansion is finished, and finite in length. Otherwise, we can apply the same procedure to  $r_1, bp^{n_1}$  to obtain that  $r_1p^{n_2} = bp^{n_1}q_2 + r_2$ , where  $n_2$  has been chosen in the same manner as above, and signifies that the next  $n_2 - 1$  digits of  $x$  will be zero, while the  $n_2$ -th will be  $q_2$ . Now we observe that if at any point  $r_k$  becomes zero, the expansion is finite in length. Otherwise, we record this sequence of remainders. Each of them is in the range  $(0, b)$ . Therefore, after at most  $b - 1$  steps we will encounter a remainder that has been encountered before. The consequence is that after this point, the sequence of digits is fully known, because we are always performing a division with  $b$ . Therefore, from that point on the decimal expansion will indeed be infinitely repeating, thus completing the proof.

### Exercise 23

If  $a_n$  is convergent, show that  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n$ .

*Solution.*

We begin by showing  $\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n$ . Let  $L$  be the limit of  $a_n$ . By definition,  $\liminf_{n \rightarrow \infty} a_n = \sup_{n \geq 1} \{\inf\{a_n, a_{n+1}, \dots\}\}$ . Suppose first that this supremum did not exist, i.e. that it “equals infinity”. Then for every  $M > 0$  there must exist  $N > 0$  such that  $\inf\{a_N, a_{N+1}, \dots\} > M$ . But this in turn would mean that  $a_n$  is not bounded and yet converges to  $L$ , a contradiction. Now suppose that the supremum

equals  $L + \epsilon$ ,  $\epsilon > 0$ . Then  $L + \epsilon'$ ,  $\epsilon' < \epsilon$  cannot constitute an upper bound for the infimums. Thus there exists  $N > 0$  such that  $\inf\{a_N, a_{N+1}, \dots\} > L + \epsilon'$ , which means that for  $n \geq N$ ,  $a_n > L + \epsilon'$ . This is a clear contradiction of the definition of limit, since  $a_n$  cannot get more than  $\epsilon'$ -close to  $L$ . Therefore, the liminf of  $a_n$  is at most  $L$ .

Now, by the definition of the limit, for any  $\epsilon > 0$  there exists  $N > 0$  such that for  $n > N$ ,  $|a_n - L| < \epsilon \implies a_n > L - \epsilon$ . Notice that this means that  $\inf\{a_N, a_{N+1}, \dots\} \geq L - \epsilon$ , which in turn means that the liminf of  $a_n$ , as an upper bound for these infimums, must equal at least  $L - \epsilon$  for any  $\epsilon > 0$ . At the same time, it equals at most  $L$ . The only possibility then is that  $\liminf_{n \rightarrow \infty} a_n = L$ , which is what we are asked to prove.

The argument for showing that  $\limsup_{n \rightarrow \infty} a_n = L$  is exactly symmetrical.

### Exercise 24

Show that  $\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$ .

*Solution.*

We begin by the definition of  $\limsup$ :

$$\limsup_{n \rightarrow \infty} (-a_n) = \inf_{n \geq 1} \{\sup\{-a_n, -a_{n+1}, \dots\}\}$$

We recall from exercise 1 that if  $-A = \{-a, a \in A\}$  for some set  $A$ , then  $\inf A = -\sup(-A)$ . This means that for any of the sets  $\{-a_n, -a_{n+1}, \dots\}$  it holds that  $\sup\{-a_n, -a_{n+1}, \dots\} = -\inf\{a_n, a_{n+1}, \dots\}$ . Thus:

$$\limsup_{n \rightarrow \infty} (-a_n) = \inf_{n \geq 1} \{-\inf\{a_n, a_{n+1}, \dots\}\}$$

Now observe that the “outer” infimum here is taken over the set  $\{-\inf\{a_1, a_2, \dots\}, -\inf\{a_2, a_3, \dots\}, \dots\}$ . If we now set  $A = \{-\inf\{a_1, a_2, \dots\}, -\inf\{a_2, a_3, \dots\}, \dots\}$ , then  $-A = \{\inf\{a_1, a_2, \dots\}, \inf\{a_2, a_3, \dots\}, \dots\}$ . Therefore we can again apply the result of exercise 1 to get that:

$$\limsup_{n \rightarrow \infty} (-a_n) = \inf A = -\sup(-A) = -\sup\{\inf\{a_1, a_2, \dots\}, \inf\{a_2, a_3, \dots\}, \dots\} = -\liminf_{n \rightarrow \infty} a_n$$

### Exercise 25

If  $\limsup_{n \rightarrow \infty} a_n = -\infty$ , show that  $a_n$  diverges to  $-\infty$ . If  $\limsup_{n \rightarrow \infty} a_n = +\infty$ , show that  $a_n$  has a subsequence that diverges to  $+\infty$ . What happens if  $\liminf_{n \rightarrow \infty} a_n = \pm\infty$ ?

*Solution.*

We have that  $\limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} \{\sup\{a_n, a_{n+1}, \dots\}\}$ . If this “equals negative infinity”, this means that the set over which the infimum is computed is not bounded below. In other words, for any  $M < 0$ , we can find  $n \geq 1$  such that  $\sup\{a_n, a_{n+1}, \dots\} < M$ . If the least upper bound of such a set is less than  $M$ , then clearly all elements of the set must also be less than  $M$ . This means that given an  $M < 0$  we can always find an  $n$  such that  $a_k < M$  for all  $k > n$ , by appropriately picking the set as above. This means precisely that  $a_n$  diverges to  $-\infty$ .

Now let us examine what happens if  $\limsup$  “equals positive infinity”. This means that the set over which the infimum is computed is empty. To see why this is the case, suppose that it was not. Then either it has an lower bound or it does not (law of excluded middle). In the first case, the (corollary of the) Least Upper Bound axiom tells us that the infimum would have to equal some real number. In the second case, by definition we would write that the infimum equals  $-\infty$ . The only case left is thus for the set to be empty, and in this case we have defined (page 3-4 for the supremum, symmetric for the infimum) that  $\inf \emptyset = +\infty$ , since every real number is a lower bound for the empty set. For the set to be empty, it must be the case that all  $\sup\{a_n, a_{n+1}, \dots\}$ , i.e. for every  $n$ , are  $+\infty$ . This means that all of these sets are unbounded, i.e. that for every  $n$ , it is the case that  $\{a_n, a_{n+1}, \dots\}$  contains, for every  $M > 0$ , an element  $a_{n_k}$  such that  $a_{n_k} > M$ .

Form then a subsequence  $b_j$  of  $a_n$  in the following way. Select  $b_1$  to be the first element of  $\{a_1, a_2, \dots\}$  such that  $b_1 > 1$ , which is guaranteed to exist. Then select  $b_2$  to be the first element of  $\{a_2, a_3, \dots\}$  such

that  $b_2 > 2, b_2 \neq b_1$ . This is also guaranteed to exist, because if it did not, then all elements of  $a_n$  after  $b_1$  would be at most 2, which would mean that after that point  $a_n$  is bounded and then at least one of the suprema above would be a real number, in which the set over which lim sup is computed would not be empty, a contradiction. Continue this way to pick any  $b_j$  to never equal any of the previously selected elements, and then observe that this is a subsequence of  $a_n$  that indeed diverges to  $+\infty$ .

For lim inf the cases are symmetrical.

### Exercise 26

Prove the characterization of lim sup given above. That is, given a bounded sequence  $(a_n)$ , show that the number  $M = \limsup_{n \rightarrow \infty} a_n$  satisfies (\*) (page 12) and, conversely, that any number  $M$  satisfying (\*) must equal  $\limsup_{n \rightarrow \infty} a_n$ . State and prove the corresponding result for  $m = \liminf_{n \rightarrow \infty} a_n$ .

*Solution.*

Suppose  $M$  is the limit supremum of a bounded sequence  $(a_n)$ . This means that:

$$M = \inf_{n \geq 1} \{\sup\{a_n, a_{n+1}, \dots\}\}$$

Now, pick any  $\epsilon > 0$ . Recall from exercise 3 that there must exist an element  $s$  of the set of suprema such that  $s < M + \epsilon$ . This means that there must exist  $N$  such that  $s = \sup\{a_N, a_{N+1}, \dots\} < M + \epsilon$ . Now, by the definition of supremum, this means that for all  $a_n, n \geq N$  it must hold that  $a_n < M + \epsilon$ . These are clearly infinitely many elements of the sequence, and thus the inequality stated here *may* not be true only for *finitely* many elements (up to and not including  $a_N$ ).

Furthermore, suppose that  $M - \epsilon < a_n$  is true for finitely many elements of the sequence, and call  $N'$  the largest integer for which  $M - \epsilon < a_{N'}$  (well defined due to the set containing finitely many elements). Then for all  $n > N'$  it must hold that  $M - \epsilon \geq a_n$ , which means that  $M - \epsilon < M$  constitutes an upper bound for all subsequences  $a_n, a_{n+1}, \dots, n > N'$ , and thus  $\sup\{a_n, a_{n+1}, \dots\} \leq M - \epsilon$ . But by the definition of lim sup,  $M$  is the infimum of the set of suprema, yet is larger than any  $\sup\{a_n, a_{n+1}, \dots\}, n > N'$ , a contradiction. Therefore  $M - \epsilon < a_n$  has to hold for infinitely many  $n$ .

In the other direction, suppose  $M$  satisfies (\*). We need to show that  $M = \inf_{n \geq 1} \{\sup\{a_n, a_{n+1}, \dots\}\}$ . Suppose first that  $M$  is greater than the lim sup. By the definition of infimum,  $M$  cannot then constitute a lower bound for the set of suprema. Therefore there exists  $N > 0$  such that  $\sup\{a_N, a_{N+1}, \dots\} < M$ . Set then  $\epsilon = M - \sup\{a_N, a_{N+1}, \dots\}$ , in which case by (\*) there must exist infinitely many  $n$  such that  $M - \epsilon < a_n \implies \sup\{a_N, a_{N+1}, \dots\} < a_n$ . But because  $n$  are infinitely many, some of them are greater than  $N$ , and thus this inequality contradicts the definition of supremum for the subsequence starting at  $N$ .

Now suppose  $M$  is smaller than the lim sup, in which case  $\limsup_{n \rightarrow \infty} a_n > M$ . Then  $\limsup_{n \rightarrow \infty} a_n - M = \epsilon > 0$ . This furthermore means that there exists  $\epsilon' > 0, \epsilon' < \epsilon$ . Now, there must exist infinite  $n$  such that  $a_n < M + \epsilon' < M + \epsilon = \limsup_{n \rightarrow \infty} a_n$ . Observe that this means the following. First, that for some  $N > 0$ , for all  $n \geq N, M + \epsilon'$  constitutes an upper bound for all  $a_n$ , i.e.  $M + \epsilon' \geq \sup\{a_N, a_{N+1}, \dots\}$ . Second, that  $M + \epsilon' < \inf_{n \geq 1} \{\sup\{a_n, a_{n+1}, \dots\}\}$ , which means that more specifically  $M + \epsilon' < \sup\{a_N, a_{N+1}, \dots\}$ . We have arrived at a contradiction. Therefore  $M$  cannot be smaller than the lim sup, leading us to conclude that these must in fact be equal.

For lim inf, the corresponding result is as follows.  $m$  equals  $\liminf_{n \rightarrow \infty} a_n$  if and only if for every  $\epsilon > 0, a_n < m + \epsilon$  for infinitely many  $n$  and  $a_n > m - \epsilon$  for all but finitely many  $n$ . The proof would use symmetrical arguments.

**Exercise 27**

Prove that every sequence of real numbers  $(a_n)$  has a subsequence  $(a_{n_k})$  that converges to

$$\lim \sup_{n \rightarrow \infty} a_n$$

[Hint: If  $M = \lim \sup_{n \rightarrow \infty} a_n = \pm\infty$ , we must interpret the conclusion loosely; this case is handled in exercise 25. If  $M \neq \pm\infty$ , use (\*) to choose  $(a_{n_k})$  satisfying  $|a_{n_k} - M| < 1/k$ , for example. There is also a subsequence that converges to  $\lim \inf_{n \rightarrow \infty} a_n$ . Why?]

*Solution.*

Firstly, as mentioned in the hint let us categorize the cases of  $\pm\infty$ . One, if  $\lim \sup$  equals  $+\infty$  then  $(a_n)$  has a subsequence diverging to  $+\infty$ . Two, if  $\lim \sup$  equals  $-\infty$ , then the sequence itself diverges to  $-\infty$ . Now let's examine the case where  $\lim \sup$  equals some real number  $M$ . We consider the following sequence:  $\epsilon_k = \frac{1}{k}$ . By the characterization of the supremum, there exist infinitely many  $n$  such that  $M - \epsilon_1 < a_n$ , and it also holds that *for all but* finitely many  $n$ ,  $a_n < M + \epsilon_1$ . Crucially, the second observation means that there exists a maximum  $N$  for which the second inequality *does not* hold. Therefore, the second inequality holds for all  $n > N$  for some  $N$ . Additionally, the first inequality holds for infinitely many  $n$ , and thus infinitely many of those have to be greater than  $N$ . Therefore, there exist infinitely many  $n$  such that  $|a_n - M| < \epsilon_1$ . Pick the first of those and call it  $a_1$ .

Precisely because these elements are always infinite in number, for any  $k > 1$  we will always be able to pick an  $a_k$  as above with the additional constraint that  $a_k$  has not been picked before. Repeating this procedure will yield a subsequence  $(a_{n_k})$  that clearly converges to  $M$  because  $\epsilon_k \rightarrow 0$  (and also decrease monotonically). This concludes the proof.

Because a symmetric characterization exists for  $\lim \inf$ , and because a symmetric version of exercise 25 also exists for it, there will also be a subsequence that converges to  $\lim \inf$ , regardless of whether it equals  $\pm\infty$  or some real number.

**Exercise 29**

If  $(a_{n_k})$  is a convergent subsequence of  $(a_n)$ , show that  $\lim \inf_{n \rightarrow \infty} a_n \leq \lim_{k \rightarrow \infty} a_{n_k} \leq \lim \sup_{n \rightarrow \infty} a_n$ .

*Solution.*

By exercise 23, we know that  $\lim \inf_{k \rightarrow \infty} a_{n_k} = \lim \sup_{k \rightarrow \infty} a_{n_k} = \lim_{k \rightarrow \infty} a_{n_k}$ , since  $(a_{n_k})$  converges. Recall the definition of  $\lim \sup$  for  $(a_n), (a_{n_k})$ :

$$\lim \sup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} \{\sup\{a_n, a_{n+1}, \dots\}\}, \lim \sup_{k \rightarrow \infty} a_{n_k} = \inf_{k \geq 1} \{\sup\{a_{n_k}, a_{n_{k+1}}, \dots\}\}$$

We claim that the second  $\lim \sup$  equals at most the first one. To see why this is the case, observe that any  $\sup\{a_{n_k}, a_{n_{k+1}}, \dots\}$  is the supremum of a number of elements of  $(a_n)$  starting at the  $n_k$ -th one and possibly excluding some elements after that. This means that  $\{a_{n_k}, a_{n_{k+1}}, \dots\} \subset \{a_{n_k}, a_{n_{k+1}}, \dots\}$ , and thus by exercise 2,  $\sup\{a_{n_k}, a_{n_{k+1}}, \dots\} \leq \sup\{a_{n_k}, a_{n_{k+1}}, \dots\}$ . Now suppose that the second  $\lim \sup$  was larger than the first. This would mean that at least one of  $\sup\{a_n, a_{n+1}, \dots\}$  is smaller than  $\lim \sup_{k \rightarrow \infty} a_{n_k}$  (such that it cannot be the largest *lower* bound for the set of suprema of subsequences of  $(a_n)$ ). Suppose that this happens for  $n = N$ .

But then because  $(a_{n_k})$  has infinite terms, it has to be the case that for some  $k, n_k > N$ . Then  $\sup\{a_{n_k}, a_{n_{k+1}}, \dots\} \leq \sup\{a_{n_k}, a_{n_{k+1}}, \dots\} \leq \sup\{a_N, a_{N+1}, \dots\} < \lim \sup_{k \rightarrow \infty} a_{n_k}$ . The first equality here was proved above, while the second is again based on exercise 3 applied on two subsequences, one starting at  $N$  and one starting at  $n_k$ . But then  $\lim \sup_{k \rightarrow \infty} a_{n_k}$  is not a lower bound for the suprema of the subsequences of  $(a_{n_k})$ , which is a contradiction. Therefore, it has to be the case that  $\lim \sup_{n \rightarrow \infty} a_n \leq \lim \sup_{k \rightarrow \infty} a_{n_k}$ . As stated in the beginning, by the convergence of  $(a_{n_k})$  this also implies that  $\lim_{k \rightarrow \infty} a_{n_k} \leq \lim \sup_{n \rightarrow \infty} a_n$ . The proof for the inequality involving the infimum makes use of exercise 23 and a completely symmetric argument.

**Exercise 31**

If  $(a_n)$  is convergent and  $(b_n)$  is bounded, show that  $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \lim_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$ .

*Solution.*

First of all, because  $a_n$  is convergent, it is also bounded. Since  $b_n$  is bounded as well, their sum is bounded. This means that  $a_n + b_n$  has a limit supremum that is indeed a real number. Then, by exercise 27, we know that the sequence  $a_n + b_n$  has a subsequence  $a_{n_k} + b_{n_k}$  that converges to  $\limsup_{n \rightarrow \infty} (a_n + b_n)$ . Because  $a_n$  converges, the subsequence  $a_{n_k}$  that corresponds to the subsequence of the sum also converges to the limit of  $a_n$ . Consequently, the subsequence  $b_{n_k}$  that corresponds to the subsequence of the sum also converges as a difference of convergent sequences. Observe that this means that:

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) = \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k} = \lim_{n \rightarrow \infty} a_n + \lim_{k \rightarrow \infty} b_{n_k}$$

Now, from exercise 29, because  $b_{n_k}$  is a convergent subsequence of  $b_n$  we conclude that  $\lim_{k \rightarrow \infty} b_{n_k} \leq \limsup_{n \rightarrow \infty} b_n$ . Combining these two observations we obtain that:

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \lim_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

**Exercise 33**

Show that  $(x_n)$  converges to  $x \in \mathbb{R}$  if and only if every subsequence  $(x_{n_k})$  of  $(x_n)$  has a *further* subsequence  $(x_{n_{k_l}})$  that converges to  $x$ .

*Solution.*

$\implies$  : First suppose that  $(x_n)$  converges to  $x \in \mathbb{R}$ . Pick any subsequence  $(x_{n_k})$ , and pick any  $\epsilon > 0$ . There exists  $N > 0$  such that for all  $n > N$  it holds that  $|x_n - x| < \epsilon$ . Because  $(x_{n_k})$  has infinite terms, there exist infinitely many  $n_k > N$ , and for all of these, by the definition of subsequences, the above inequality holds. Therefore, any subsequence of  $(x_n)$  converges to  $x$ . Any sequence is a trivial subsequence of itself, and thus we have completed the proof in this direction.

$\impliedby$  : Conversely, suppose that every subsequence  $(x_{n_k})$  of a sequence  $(x_n)$  has a subsequence  $(x_{n_{k_l}})$  that converges to  $x \in \mathbb{R}$ . Suppose by way of contradiction that  $(x_n)$  does not converge to  $x \in \mathbb{R}$ . Then there exists  $\epsilon > 0$  such that for all  $N > 0$  there exists  $n > N$  for which  $|x_n - x| \geq \epsilon$ . Construct the following subsequence  $(x_{n_k})$  of  $(x_n)$ : the  $i$ -th term is the first element of  $(x_n)$  with  $n > i$  such that  $|x_n - x| \geq \epsilon$ . By the observation above, this is always well defined.

By construction, all elements of this  $(x_{n_k})$  are at least  $\epsilon$ -away from  $x$ . Therefore, no subsequence  $(x_{n_{k_l}})$  of  $(x_{n_k})$  can ever converge to  $x$ , which directly contradicts our hypothesis, leading us to conclude that  $(x_n)$  converges to  $x \in \mathbb{R}$ .

**Exercise 34**

Suppose that  $a_n \geq 0$  and that  $\sum_{n=1}^{\infty} a_n < \infty$ .

- (i) Show that  $\liminf_{n \rightarrow \infty} n a_n = 0$ .
- (ii) Give an example showing that  $\limsup_{n \rightarrow \infty} n a_n > 0$  is possible.

*Solution.*

(i) In order to show that  $\liminf_{n \rightarrow \infty} n a_n = 0$ , we need to show two things. One, that for every  $\epsilon > 0$ ,  $n a_n > 0 - \epsilon$  holds for all but finitely many  $n$  and two, that for every  $\epsilon > 0$ ,  $n a_n < 0 + \epsilon$  holds for infinite  $n$ . The first inequality is obvious, since  $a_n \geq 0$ . Suppose then that the second inequality does not hold. This means that there exists  $\epsilon > 0$  for which  $n a_n \geq \epsilon$  for only a finite number of  $n$ . Call these  $n_1, n_2, \dots, n_k$ . Then, observe that for all  $n > n_k$ , it must be the case that  $n a_n \geq \epsilon \implies a_n \geq \frac{\epsilon}{n}$ . Then:

$$\sum_{n>n_k} a_n \geq \sum_{n>n_k} \frac{\epsilon}{n} = \epsilon \sum_{n>n_k} \frac{1}{n}$$

We know, however, that this infinite sum diverges (since the infinite sum of all  $\frac{1}{n}$  diverges), whereas our LHS here was assumed to converge. We have thus arrived at a contradiction, which means that the second inequality must indeed hold, and thus  $\liminf_{n \rightarrow \infty} n a_n = 0$ .

(ii) Consider the sequence  $a_n = \frac{1}{n}$  for  $n = 2^k, k = 1, 2, \dots$  and  $a_n = 0$  for every other  $n$ . Observe then that the corresponding series is a geometric series, and thus converges. At the same time, observe that  $na_n = 1$  for  $n = 2^k, k = 1, 2, \dots$  and  $na_n = 0$  for every other  $n$ . Clearly then it is the case that  $\limsup n a_n = 1 > 0$ .

### Exercise 35

(The ratio test): Let  $a_n \geq 0$ .

(i) If  $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ , show that  $\sum_{n=1}^{\infty} a_n < \infty$ .

(ii) If  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$ , show that  $\sum_{n=1}^{\infty} a_n$  diverges.

(iii) Find examples of both a convergent and a divergent series having  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ .

*Solution.*

(i) Observe, first of all, that  $a_n \geq 0$ . This means that the corresponding sequence of partial sums is non-decreasing. If we can show that it is also bounded, then we will have proved that it converges, which by definition means that  $\sum_{n=1}^{\infty} a_n < \infty$ .

By our hypothesis, we have that  $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ . Call this quantity  $M$ . Then we can always find some  $\epsilon > 0$  such that  $M + \epsilon < 1$ . For this  $\epsilon$ , we apply the characterization of  $\limsup$  on the sequence  $\frac{a_{n+1}}{a_n}$ . This means that for all but finitely many  $n$ , it holds that  $\frac{a_{n+1}}{a_n} < M + \epsilon$ . Call  $N$  the largest  $n$  for which this does not hold, and then observe that we have that  $a_{N+2} < a_{N+1}(M + \epsilon)$ . This implies then that  $a_{N+3} < a_{N+2}(M + \epsilon) < a_{N+1}(M + \epsilon)^2$ . More generally, if  $N' = N + 1$  for  $k \geq 1$  we have that  $a_{N'+k} < a_{N'}(M + \epsilon)^k$ .

Then for the sequence of partial sums we have that:

$$\sum_{i=1}^n a_i = \sum_{i=1}^N a_i + \sum_{i=N'}^n a_i < \sum_{i=1}^N a_i + \sum_{k=1}^{n-N'} a_{N'}(M + \epsilon)^k$$

The first term here is clearly bounded as a finite sum. The second term can also be bounded by the *infinite* sum for  $k$ , precisely because it corresponds to a geometric series with ratio  $M + \epsilon < 1$ . But then this means that the sequence of partial sums is indeed bounded, and thus it must converge.

(ii) Call the  $\liminf m$ . Then  $M > 1$  and we can always find  $\epsilon > 0$  such that  $m - \epsilon > 1$ . Then, by the characterization of  $\liminf$ , for all but finitely many  $n$  it holds that  $\frac{a_{n+1}}{a_n} > m - \epsilon$ . Call  $N$  the first  $n$  after which (and including it) this holds. Then observe that, similarly to (i), for  $k \geq 1$  we have that  $a_{N+k} > (m - \epsilon)^k a_N$ . But because  $m - \epsilon > 1$  this implies that the terms of the sequence are not bounded, and hence the infinite series must necessarily diverge.

(iii) Consider the series corresponding to the sequence  $a_n = 1$ . Clearly, the ratio of two subsequent terms is 1, but the series obviously diverges.

Consider also the series corresponding to the sequence  $a_n = \frac{1}{n^2}$ , which we take as known that it converges to  $\frac{\pi^2}{6}$ . Observe that:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{n^2 + 2n + 1}$$

, which can easily be shown to converge to 1 as  $n \rightarrow \infty$ .

### Exercise 37

If  $(E_n)$  is a sequence of subsets of a fixed set  $S$ , we define

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} E_k \right) \text{ and } \liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k \right)$$

Show that

$$\liminf_{n \rightarrow \infty} E_n \subset \limsup_{n \rightarrow \infty} E_n \text{ and that } \liminf_{n \rightarrow \infty} (E_n^c) = (\limsup_{n \rightarrow \infty} E_n)^c$$

*Solution.*

For the first part, we have the following. Suppose  $x \in \liminf_{n \rightarrow \infty} E_n$ . We need to show that  $x \in \limsup_{n \rightarrow \infty} E_n$ . By this definition of  $\liminf$ , it must be the case that  $x \in \bigcap_{k=n}^{\infty} E_k$  for at least one  $n \geq 1$ .

This in turn means that there exists  $n \geq 1$  such that  $x \in E_n, E_{n+1}, \dots$ . Now observe that the following hold:

- For  $l \leq n$ , it is the case that  $E_n \subset \bigcup_{k=l}^{\infty} E_k$ . Therefore,  $x \in \bigcup_{k=l}^{\infty} E_k, l \leq n$ .
- For  $i \geq 1$ , it is the case that  $E_{n+i} \subset \bigcup_{k=n+i}^{\infty} E_k$ . Therefore,  $x \in \bigcup_{k=n+i}^{\infty} E_k$ . This can equivalently be written as  $x \in \bigcup_{k=l}^{\infty} E_k, l > n$ .

But then we have shown that  $x \in \bigcup_{k=n}^{\infty} E_k$  for any  $n \geq 1$ , which means that, by the definition of  $\limsup$  used here,  $x \in \limsup_{n \rightarrow \infty} E_n$  (since this is the intersection of all of these sets). Since  $x$  was selected as an arbitrary element of  $\liminf_{n \rightarrow \infty} E_n$ , we've shown that  $\liminf_{n \rightarrow \infty} E_n \subset \limsup_{n \rightarrow \infty} E_n$ .

For the second part, we will use De Morgan's laws, which we know hold for infinite unions and intersections as well (this is easy to prove using the same arguments as for finite unions and intersections). Namely, we have that:

$$\liminf_{n \rightarrow \infty} (E_n^c) = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k^c \right) = \bigcup_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} E_k \right)^c = \left( \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} E_k \right) \right)^c = (\limsup_{n \rightarrow \infty} E_n)^c$$

## 1.2 Limits and Continuity

### Exercise 40

Prove the following theorem (1.17):

Let  $f$  be a real-valued function defined in some punctured neighborhood of  $a \in \mathbb{R}$ . Then, the following are equivalent:

- There exists a number  $L$  such that  $\lim_{x \rightarrow a} f(x) = L$  (by the  $\epsilon - \delta$  definition).
- There exists a number  $L$  such that  $f(x_n) \rightarrow L$  whenever  $x_n \rightarrow a$ , where  $x_n \neq a$  for all  $n$ .
- $(f(x_n))$  converges (to something) whenever  $x_n \rightarrow a$ , where  $x_n \neq a$  for all  $n$ .

*Solution.*

(i)  $\implies$  (ii): Pick any sequence  $x_n \rightarrow a$ , with  $x_n \neq a$  for all of its terms. For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $0 < |x_n - a| < \delta$  we have that  $|f(x_n) - L| < \epsilon$ . Since  $x_n \rightarrow a$ , for this  $\delta > 0$  there exists  $N > 0$  such that for  $n > N$  we have that  $0 < |x_n - a| < \delta$ . But then it also holds that for  $n > N$ ,  $|f(x_n) - L| < \epsilon$ , which means that for any  $\epsilon > 0$  we are able to find  $N > 0$  such that the definition of the limit for  $(f(x_n))$  holds, with the limit value being  $L$ .

Therefore, whenever  $x_n \rightarrow a, x_n \neq a$ , it is also the case that  $f(x_n) \rightarrow L$ .

(ii)  $\implies$  (iii): Pick any sequence  $(f(x_n))$  such that the corresponding  $(x_n)$  converges to  $a$  and such that  $x_n \neq a$ . Then, applying (ii) one can obtain that  $f(x_n) \rightarrow L$ , which means indeed that  $(f(x_n))$  converges to *something*.

(iii)  $\implies$  (ii): Pick any two sequences  $x_n \rightarrow a, y_n \rightarrow a$ , such that  $x_n \neq a, y_n \neq a$  for all  $n$ . We then have that  $f(x_n) \rightarrow L_1, f(y_n) \rightarrow L_2$ . As pointed out in the book, construct the sequence  $x_1, y_1, x_2, y_2, \dots$  by "interlacing"  $(x_n), (y_n)$ . It's easy to see that this sequence converges to  $a$  as well. Therefore,  $f(z_n) \rightarrow L_3$ . But now observe that both  $f(x_n)$  and  $f(y_n)$  are subsequences of  $f(z_n)$ , which means more specifically that all three of them must converge to the same limit, i.e.  $L_1 = L_2 = L_3$ .

We have thus shown that for any two sequences  $x_n \rightarrow a, y_n \rightarrow a$ , the corresponding  $(f(x_n)), (f(y_n))$  always converge to the same limit, which is an equivalent way of stating (ii).

To complete the full equivalence of the theorem, we will now show that (ii)  $\implies$  (i). To do this, suppose that (i) does not hold. Then there exists  $\epsilon > 0$  such that for all  $\delta > 0$  it holds that for some  $x$  it is the case that both  $0 < |x - a| < \delta$  and  $|f(x) - L| \geq \epsilon$ . Construct a sequence of  $\delta_i = \frac{1}{i}$ , and from that construct a corresponding sequence of  $i \rightarrow x_i$ , where each  $x_i$  fulfills the above mentioned conditions. Then clearly the sequence  $(x_i)$  converges to  $a$ , but the corresponding  $(f(x_i))$  does not converge to  $L$ . This directly contradicts (ii), and we have thus shown that (ii) does indeed imply (i).

**Exercise 45**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and suppose that  $f(x) = 0$  whenever  $x$  is rational. Show that  $f(x) = 0$  for every  $x \in [a, b]$ .

*Solution.*

Pick any  $x \in [a, b]$ . If  $x \in \mathbb{Q}$ , we immediately know that  $f(x) = 0$ . If  $x \notin \mathbb{Q}$ , then we know from a previous result that we can always approach the real number  $x$  with a sequence of rational numbers  $(x_n)$ . In other words,  $x_n \rightarrow x, x_n \in \mathbb{Q}$  for all  $x_n$ . Because  $f$  is continuous, it must then hold that  $f(x_n) \rightarrow f(x)$ . Observe that the sequence of  $f(x_n)$  is a sequence of zeros, since all  $x_n \in \mathbb{Q}$ . Clearly then the limit has to be zero as well, which directly means that  $f(x) = 0$ , thus showing that  $f(x)$  is indeed the zero function on  $[a, b]$ .

**Exercise 46**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous.

- (a) If  $f(0) > 0$ , show that  $f(x) > 0$  for all  $x$  in some open interval  $(-a, a)$ .
- (b) If  $f(x) \geq 0$  for every rational  $x$ , show that  $f(x) \geq 0$  for all real  $x$ . Will this result hold with “ $\geq 0$ ” replaced by “ $> 0$ ”? Explain.

*Solution.*

(a) Suppose for the sake of contradiction that this does not hold. This is equivalent to saying that for any  $a \in \mathbb{R}$ , there exist some  $x_a \in (-a, a)$  such that  $f(x_a) \leq 0$ . Consider constructing the following sequence of  $a_i$ :  $a_i = \frac{1}{n}$ . Then, in every one of these intervals there has to exist  $x_i$  such that  $f(x_i) \leq 0$ . Observe that  $a_i \rightarrow 0$ , which means also that  $x_i \rightarrow 0$ . Since  $f$  is continuous, this means that it has to be the case that  $f(x_i) \rightarrow f(0) > 0$ . However, this is by construction a sequence of non-positive numbers, and because limits preserve non-strict inequalities, it cannot be the case that the limit of  $(f(x_i))$  is positive. We arrive at a contradiction, and therefore it must be the case that for some  $a$ ,  $f(x) > 0$  for all  $x \in (-a, a)$ .

(b) Again, for an  $x \in \mathbb{R} \setminus \mathbb{Q}$ , we can approach it with a sequence of rationals  $(x_n)$ . Then it is the case that  $f(x_n) \geq 0$  for all  $n$ , and because  $f$  is continuous it also holds that  $f(x_n) \rightarrow f(x)$ . Again, limits preserve non-strict inequalities, and thus  $f(x) \geq 0$  holds for all real  $x$ .

Now we consider whether this is true if we alter “ $\geq$ ” to “ $>$ ”. Consider the function  $f(x) = |x - \sqrt{2}|$ . Clearly, this function is continuous, and also positive whenever  $x \neq \sqrt{2}$  and zero for  $x = \sqrt{2}$ . More specifically, it’s a function that is strictly positive for rational  $x$ . However, for the real  $x = \sqrt{2}$ ,  $f(\sqrt{2}) = 0$ , thus providing a counterexample to the claim “if  $f(x) > 0$  for every rational  $x$ , show that  $f(x) > 0$  for all real  $x”.$

**Exercise 50**

Let  $D$  denote the set of rationals in  $[0, 1]$  and suppose that  $f : D \rightarrow \mathbb{R}$  is increasing. Show that there is an increasing function  $g : [0, 1] \rightarrow \mathbb{R}$  such that  $g(x) = f(x)$  whenever  $x$  is rational. [Hint: For  $x \in [0, 1]$ , define  $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}\}$ .]

*Solution.*

Define  $g$  as in the provided hint. We first show that if  $x \in \mathbb{Q}$ ,  $f(x) = g(x)$ . We have that  $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}\}$ . Recall that  $f$  is increasing, therefore, for all  $0 \leq t \leq x, t \in \mathbb{Q}$  it has to hold that  $f(t) \leq f(x)$ . Therefore  $f(x)$  is an upper bound for the set over which we are computing the supremum. For any  $s < f(x)$ , it is clear that  $s$  cannot be an upper bound for this set, because  $f(x)$  belongs in it. We conclude that  $f(x)$  is indeed the supremum, and thus that  $g(x) = f(x)$ .

Now to show that  $g$  is increasing, pick  $x_1 < x_2, x_1, x_2 \in [0, 1]$  and call the sets over which  $g(x_1), g(x_2)$  are computed  $S_1, S_2$  respectively. If  $y \in S_1$ , we have that  $y = f(t)$  for some  $0 \leq t \leq x_1, t \in \mathbb{Q}$ . Since  $x_1 < x_2$ , this means that  $y \in S_2$  as well. Therefore  $S_1 \subset S_2$ , which means that  $g(x_1) \leq g(x_2)$  by the properties of the supremum. Therefore  $g$  is indeed an increasing function.

**Exercise 51**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be increasing and define  $g : [a, b] \rightarrow \mathbb{R}$  by  $g(x) = f(x+)$  for  $a \leq x < b$ . and  $g(b) = f(b)$ . Prove that  $g$  is increasing and right-continuous.

*Solution.*

Before we begin we note that by preposition 1.19,  $f(x+)$  is well-defined for all  $x \in [a, b]$ . First we will show that  $g$  is increasing. Pick  $x_1, x_2 \in [a, b], x_1 < x_2$ . We need to show that  $g(x_1) \leq g(x_2)$ , or, equivalently, that  $f(x_1+) \leq f(x_2+)$ . We will do this in two steps.

First, we will show that for every  $x \in [a, b], f(x+) \geq f(x)$ . Suppose this was not the case for some  $x$ , which would mean that  $f(x+) < f(x) \implies 0 < f(x) - f(x+) = \epsilon$ . Let  $L = f(x+)$  to ease notation. Because  $L$  is well-defined, for this  $\epsilon$  there exists  $\delta > 0$  such that for all  $y, x < y < x + \delta$  it holds that  $|f(y) - L| < \epsilon$ . More specifically, this implies that  $f(y) < L + \epsilon = L + f(x) - L \implies f(y) < f(x)$ . This contradicts the fact that  $f$  is increasing.

Secondly, we will show that for all  $x \in [a, b], y \in [a, b], x < y$  it holds that  $f(x+) \leq f(y)$ . Suppose again that for some such  $x, y$  this does not hold, which means that  $f(x+) > f(y)$ , and again for ease of notation let  $L = f(x+)$ . Firstly, let  $\delta_1 = y - x > 0$ . Secondly, let  $\epsilon = L - f(y) > 0$ . Because  $L$  is well-defined, for this  $\epsilon$  there exists  $\delta_2 > 0$  such that for all  $z, x < z < x + \delta_2$  it holds that:

$$|f(z) - L| < \epsilon \implies f(z) > L - \epsilon = L - L + f(y) \implies f(z) > f(y)$$

Set  $\delta = \min\{\delta_1, \delta_2\}$ , which means that  $z < x + \delta \leq x + \delta_1 = y$  and also that, as shown above,  $f(z) > f(y)$ . Again, this directly contradicts the fact that  $f$  is increasing.

We can now complete the proof by observing that  $g(x_1) = f(x_1+) \leq f(x_2)$  by using the second of the two proven lemmas, and then that  $f(x_2) \leq f(x_2+) = g(x_2)$  by using the first of the two proven lemmas. Note that the last step is omitted if  $x_2 = b$ , in which case  $f(x_2) = f(b) = g(b)$ .

Now, because  $g$  is increasing in  $[a, b]$ , we know by proposition 1.19 that  $g(x+)$  always exists for  $x \in [a, b]$ . We thus only need to show that  $g(x+) = g(x)$ . Because  $f(x+)$  is well defined, for any given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for  $x < y < x + \delta$  it holds that  $|f(y) - f(x+)| < \epsilon$ . Set  $\delta' = \frac{\delta}{2} < \delta$ . Then for all  $y, x < y < x + \delta'$  it is the case that there exists  $z \in (x, x + \delta)$  such that  $y < z$ . By the second of the two lemmas above,  $g(y) = f(y+) \leq f(z) < f(x+) + \epsilon$ . By the first of the two lemmas above, it is furthermore true that  $g(y) = f(y+) \geq f(y) > f(x+) - \epsilon$ .

Putting these together we obtain that  $f(x+) - \epsilon < g(y) < f(x+) + \epsilon$ , and since  $f(x+) = g(x)$  this means that by picking  $\delta' = \frac{\delta}{2}$  the definition of right continuity for  $g$  at  $x$  is satisfied.



# Chapter 2

## Countable and Uncountable Sets

### 2.1 Equivalence and Cardinality

#### Exercise 3

Given finitely many countable sets  $A_1, \dots, A_n$ , show that  $A_1 \cup \dots \cup A_n$  and  $A_1 \times \dots \times A_n$  are countable sets.

*Solution.*

We will use induction on  $n$ :

- Base case, for  $n = 2$ : Consider two countable sets,  $A_1, A_2$ . If both of them are finite, then their union and Cartesian product are also finite, and thus trivially countable.

If exactly one is finite, say  $A_2 = \{a'_1, a'_2, \dots, a'_n\}$ , then observe that  $A_1 \cup A_2$  contains at most all elements of  $A_1$  and all elements of  $A_2$ , and possibly fewer if their intersection is not empty. In any case, suppose  $|A_1 \cap A_2| = m$ , and name  $f$  the bijection from  $A_1$  to  $\mathbb{N}$ . Then let  $S = A_2 \setminus A_1 = \{a'_{k_1}, \dots, a'_{k_{n-m}}\}$ , and  $f'(a'_{k_1}) = 1, \dots, f'(a'_{k_{n-m}}) = n - m$ , in which case  $A_1 \cup A_2 = A_1 \cup S$ , and  $A_1, S$  have an empty intersection. Let  $g : A_1 \cup S \rightarrow \mathbb{N}$  be such that:

$$g(a) = \begin{cases} f(a) + (n - m) & , a \in A_1 \\ f'(a) & , a \in S \end{cases}$$

Because  $f$  is a bijection, one can clearly see that  $g$  is surjective. Furthermore,  $g$  is one-to-one because  $f, f'$  are one-to-one and because the two “branches” of  $g$  have no overlapping values ( $\min\{f(a) + (n - m)\} = n - m + 1 > n - m = \max\{f'(a)\}$ ). Therefore  $g$  is a bijection from  $A_1 \cup A_2$  to  $\mathbb{N}$ , which means precisely that the union is countable.

Now for the case where both  $A_1, A_2$  are infinite, there exist bijections  $f_1 : A_1 \rightarrow \mathbb{N}, f_2 : A_2 \rightarrow \mathbb{N}$ . These impose orders  $f_1(a_1) = 1, f_1(a_2) = 2, \dots$  for  $a_i \in A_1$  and  $f_2(a'_1) = 1, f_2(a'_2) = 2$  for  $a'_i \in A_2$ . One can again set  $S = A_2 \setminus A_1$ , and  $f_2$  can again be used to extract an order for the elements  $s_1, s_2, \dots$  of  $S$ . If  $S$  is finite, the problem reduces to the case above. If  $S$  is also infinite, we know that  $S$  is also countable, with  $f'_2 : S \rightarrow \mathbb{N}$  the corresponding bijection. Additionally,  $A_1 \cup A_2 = A_1 \cup S = \{a_1, s_1, a_2, s_2, \dots\}$  and  $A_1 \cap S = \emptyset$ . Use the orders imposed by  $f_1, f'_2$  to sort the elements of  $A_1$  in the order  $a_1, a_2, \dots$  and the elements of  $S$  in the order  $s_1, s_2, \dots$ . Then define  $g : A_1 \cup A_2 \rightarrow \mathbb{N}$  as:

$$g(a) = \begin{cases} 2i, & a = a_i \in A_1 \\ 2i + 1, & a = s_i \in S \end{cases}$$

For  $n \in \mathbb{N}$ , if  $n$  is even the equation  $n = g(a)$  has a unique solution for  $a = a_n$ , due to the bijectivity of  $f_1$ . If  $n$  is odd, the equation  $n = g(a)$  has a unique solution for  $a = s_n$ , due to the bijectivity of  $f'_2$ . In any case,  $A_1 \cup A_2$  has been shown to be equivalent to  $\mathbb{N}$ .

For the Cartesian product, the case where both  $A_1, A_2$  are finite is again trivial. If  $A_1$  is infinite and  $A_2$  finite, then:

$$A_1 = \{a_1, a_2, \dots\}, A_2 = \{a'_1, a'_2, \dots, a'_n\}$$

Let then  $g : A_1 \times A_2 \rightarrow \mathbb{N}$ :

$$g(a_i, a'_j) = n(i-1) + (j-1), \quad j = 1, 2, \dots, n$$

Observe that the fact that  $A_1 \sim \mathbb{N}$  and  $j$  only takes a finite number of values makes  $g$  a surjection. Indeed, for  $x = k \cdot n + l, l = 0, 1, \dots, n-1$  (here we use the division algorithm for integers), one has but to set  $i = 1 + k$  (possible due to  $A_1 \sim \mathbb{N}$  and  $j = l+1$  (always possible due to the range of values  $l$  can achieve) to obtain  $g(a_i, a'_j) = x$ .

To show that  $g$  is injective, suppose  $g(a_i, a'_j) = g(a_k, a'_l)$ , and we then have that:

$$n(i-1) + (j-1) = n(k-1) + (l-1) \implies n(i-k) = l-j$$

This implies that the RHS is a multiple of  $n$ , which, because  $1 \leq l, j \leq n$  is only possible if  $l = j$ . But then we also have that  $i = k$ , thus that  $(a_i, a'_j) = (a_k, a'_l)$ , i.e. that  $g$  is injective.

$g$  is therefore a bijection, and thus  $A_1 \times A_2 \sim \mathbb{N}$ .

If both  $A_1, A_2$  are infinite, then consider the function  $g : A_1 \times A_2 \rightarrow \mathbb{N}$ :

$$g(a_i, a'_j) = 2^i(2j-1)$$

, which, because of the fact that  $A_1 \sim \mathbb{N}, A_2 \sim \mathbb{N}$  (thus  $i, j$  take all natural numbers as values), and by a proof completely analogous to  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  can be shown to be a bijection, thus proving that  $A_1 \times A_2 \sim \mathbb{N}$ .

- Inductive step: If this holds for  $n = k \geq 2$ , then observe that  $A_1 \cup \dots \cup A_{k+1} = (A_1 \cup \dots \cup A_k) \cup A_{k+1}$ , and we can therefore apply the inductive hypothesis to the sets  $A_1, \dots, A_k$ , and the base case to the union of those with  $A_k$  to obtain the statement for  $n = k+1$  as well. The same argument applies to the Cartesian product as well, thus concluding the proof.

### Exercise 5

Prove that a set is infinite if and only if it is equivalent to a proper subset of itself.

[Hint: If  $A$  is infinite and  $x \in A$ , show that  $A$  is equivalent to  $A \setminus \{x\}$ .]

*Solution.*

$\implies$  : Suppose  $A$  is infinite. Then  $A$  contains an element  $x$ , and the set  $S = A \setminus \{x\}$  must also be infinite. By exercise 4, the set  $S$  contains a countably infinite subset  $S'$ . We then have that:

$$S = S' \cup (S \setminus S'), A = S' \cup (S \setminus S') \cup \{x\}$$

, where all of the sets used in the unions are disjoint. By exercise 3, it holds that  $S' \cup \{x\} \sim S'$ , and this yields a corresponding bijection  $g : S' \cup \{x\} \rightarrow S'$ . Consider then the following function  $f : A \rightarrow S$ :

$$f(z) = \begin{cases} z, & z \in S \setminus S' \\ g(z), & z \in S' \cup \{x\} \end{cases}$$

, which, by the above observations regarding the bijectivity of  $g$  and the disjointness of the sets  $S', S \setminus S', \{x\}$ , means that  $f$  is a bijection as well, proving that  $A \sim S$ .

$\Leftarrow$  : Now suppose that a set  $A$  is equivalent to  $S$ ,  $S$  being a proper subset of  $A$ . By contradiction, suppose  $A$  is not infinite and that it contains  $n$  elements. Then  $S$  must contain at most  $n-1$  elements. By the pigeonhole principle, there cannot exist a bijection from  $A$  to  $S$ , which means that  $A \sim S$  cannot be true, a contradiction. Therefore  $A$  must be infinite.

### Exercise 8

Show that  $(0, 1)$  is equivalent to  $[0, 1]$  and to  $\mathbb{R}$ .

*Solution.*

First, observe that  $[0, 1] = \{0\} \cup \{1\} \cup (0, 1)$ , and that  $[0, 1], [0, 1), (0, 1)$  are all infinite sets. By exercise 5, we have that  $[0, 1] \sim (0, 1)$  (by explicitly picking  $x = 0$  and  $S = (0, 1)$ ). Similarly, we also have that  $[0, 1] \sim [0, 1)$  (by explicitly picking  $x = 1$  and  $S = [0, 1)$ ). The transitivity property of the equivalence relation “is equivalent to” (exercise 1) yields then that  $[0, 1] \sim (0, 1)$ .

For  $(0, 1) \sim \mathbb{R}$ , consider the function  $f : \mathbb{R} \rightarrow (0, 1)$ ,  $f(x) = \frac{1}{1+e^{-x}}$ , which is known to be both injective and surjective, thus sufficing to show that  $(0, 1) \sim \mathbb{R}$ .

**Exercise 9**

Show that  $(0, 1)$  is equivalent to the unit square  $(0, 1) \times (0, 1)$ .

[Hint: “Interlace” decimals, but carefully!]

*Solution.*

Pick any  $x \in (0, 1) \times (0, 1)$ . Then  $x = (a, b)$ ,  $a, b \in (0, 1)$ . If  $a$  or  $b$  can be written as a decimal ending in infinite 9’s (e.g.  $0.3999\dots$ ), write them in the equivalent form that features finite decimals (in this example, 0.4). Now consider the function  $f : (0, 1) \times (0, 1) \rightarrow (0, 1)$ :

$$f(0.a_1a_2a_3\dots, 0.b_1b_2b_3\dots) = 0.a_1b_1a_2b_2a_3b_3\dots$$

Observe first that for any  $x \in (0, 1)$ ,  $x$  can be written as  $x = 0.x_1x_2x_3\dots$ , and again if this can be written as ending in infinite 9’s, “round it up”. Then  $f(0.x_1x_3x_5\dots, 0.x_2x_4x_6\dots) = x$ . Furthermore, suppose that  $f(0.y_1y_2y_3\dots, 0.z_1z_2z_3\dots) = x = 0.x_1x_2x_3\dots$ . By definition, it is also the case that

$$f(0.y_1y_2y_3\dots, 0.z_1z_2z_3\dots) = 0.y_1z_1y_2z_2y_3z_3\dots$$

Because we always choose to write decimals ending in infinite 9’s as “rounded up”, there can be no ambiguities here, in the sense that the equality of these two numbers directly implies that  $y_1 = x_1, z_1 = x_2, y_2 = x_3, z_2 = x_4\dots$  (i.e. any two decimals that do not end in infinite 9’s can only be equal if all corresponding digits are equal). These two observations yields that  $f$  is bijective, thus proving the desired equivalence.

**Exercise 10**

Prove that  $(0, 1)$  can be put into one-to-one correspondence with the set of all *functions*  $f : \mathbb{N} \rightarrow \{0, 1\}$ .

*Solution.*

To begin, we observe that a function from  $\mathbb{N}$  to  $\{0, 1\}$  can be thought of a set of the form:

$$\{(0, v_0), (1, v_1), (2, v_2), \dots\}$$

, where each  $v_i$  is the value of the function on  $i$ , and as such can be either 0 or 1. We now observe that the set of all such functions can be written as  $S \cup \bar{S}$ , where  $S = \{f \in \{0, 1\}^{\mathbb{N}} \mid \exists k \in \mathbb{N} \cup \{-1\}, f(i) = 1, i > k\}$  (in other words, a function either becomes 1 “forever” after some point, or it does not). Note that from now on we will use the binary system in this exercise.

Observe that all functions that belong in  $S$  are uniquely characterized by a “finite-length prefix” of values. This means that we can list all of them: start with a prefix of length 0, list all possible  $f$  such that in the above definition  $k = -1$ , then list all  $f$  with a prefix of length 1, etc. Since each length  $N$  yields only a finite number of functions (each corresponds to a binary number with  $N$  digits such that the last digit is not 1), we conclude that  $S$  is in fact countable, and that there exists a bijection  $g : S \rightarrow \mathbb{N}$ . Define now the following mapping  $F : \{0, 1\}^{\mathbb{N}} \rightarrow (0, 1) \cup \mathbb{N}$ :

$$F(f) = \begin{cases} 0.v_0v_1v_2\dots, & f \in \bar{S} \\ g(f), & f \in S \end{cases}$$

By the bijectivity of  $g$ ,  $F$  achieves all natural numbers. Furthermore, for any  $x \in (0, 1)$ , we can always write  $x = 0.b_0b_1\dots$  such that it does not end in infinite ones, and then we can define an  $f : \mathbb{N} \rightarrow \{0, 1\}$

such that  $f(i) = b_i$ , which means that  $F(f) = x$ . We have thus shown that  $F$  is surjective. Additionally, we observe that due to our construction, if  $f_1 \in \bar{S}, f_2 \in S, F(f_1), F(f_2)$  can never be equal: one is always in  $(0, 1)$  and the other is a non-zero natural number. For  $f_1, f_2 \in \bar{S}, F(f_1) = F(f_2)$  either implies that all corresponding digits are equal, in which case clearly  $f_1 = f_2$ , or that one of them ends in infinite ones, which is impossible since  $f_1, f_2 \in \bar{S}$ . Finally, for  $f_1, f_2 \in S$ , the bijectivity of  $g$  implies that  $f_1 = f_2$ .

This shows that  $F$  is also injective, which means it is a bijection from  $\{0, 1\}^{\mathbb{N}}$  to  $(0, 1) \cup \mathbb{N}$ . Now,  $\mathbb{N}$  is clearly countable, and from exercise 6 we then have that  $(0, 1) \cup \mathbb{N} \sim (0, 1)$ , which by exercise 1 also gives us our desired result,  $\{0, 1\}^{\mathbb{N}} \sim (0, 1)$ .

### Exercise 13

Show that  $\mathbb{N}$  contains infinitely many pairwise disjoint infinite subsets.

*Solution.*

We know that there exist infinitely many primes,  $p_1, p_2, \dots$ . Consider then forming the sets  $S_i = \{p_i^k | k \in \mathbb{N} \setminus \{0\}\}$ . These are clearly infinitely many, and each of them is infinite. Suppose that two of them,  $S_i, S_j$  had a non-empty intersection. Then there exists  $x \in \mathbb{N}$  such that  $x = p_i^k = p_j^l, k, l \geq 1$ . But then this number has two different prime factorizations, which is a contradiction. Therefore all of these sets are pairwise disjoint.

### Exercise 15

Show that any collection of pairwise disjoint, nonempty open intervals in  $\mathbb{R}$  is at most countable. [Hint: Each one contains a rational!]

*Solution.*

Suppose that there exists an uncountable collection of pairwise disjoint and nonempty intervals of the form  $(a, b), a, b \in \mathbb{R}$ . As indicated in the hint, each such interval must contain a rational, regardless of whether  $a, b$  are rationals or not. Furthermore, because the intervals are all disjoint, no two such rationals can be equal. This means that there exist at least as many rationals as intervals, therefore an uncountable number of them, which we know is a contradiction.

## 2.2 The Cantor Set

### Exercise 21

Show that any ternary decimal of the form  $0.a_1a_2\dots a_n11$  (base 3), i.e., any finite-length decimal ending in two (or more) 1s is *not* an element of  $\Delta$ .

*Solution.*

For this problem we will be using the following formalization of the Cantor set  $\mathcal{C}$ :

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

We now consider a ternary decimal  $x$  of the form  $x = 0.a_1\dots a_n11$ , where  $n$  may be zero, in which case the “prefix” has zero length. Consider then what  $x$  equals:

$$\begin{aligned} x &= \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} = \frac{a_13^{n+1} + a_23^n + \dots + a_n3^2 + 3 + 1}{3^{n+2}} \\ &= \frac{1}{3^{n+1}}(a_13^n + a_23^{n-1} + \dots + a_n3 + \frac{4}{3}) = \frac{1}{3^{n+1}}(3(a_13^{n-1} + a_23^{n-2} + \dots + a_n) + \frac{4}{3}) \end{aligned}$$

Notice now that if  $y = a_13^{n-1} + a_23^{n-2} + \dots + a_n$ ,  $y$  is an integer that is at least 0 and at most  $6^{\frac{3^n-1}{2}} = 3(3^n - 1)$  (by using the geometric progression sum). But then this range in which  $y$  can move means that  $x$  is precisely equal to *one* of the left endpoints in the  $n$ -th union in the definition of  $\mathcal{C}$  plus  $\frac{4}{3}$ , which, because  $1 < \frac{4}{3} < 2$ , means that  $x$  will always lie *inside* one of the intervals that comprise this union, which means of course that it lies outside of the Cantor set.

**Exercise 22**

Show that  $\Delta$  contains no (nonempty) open intervals. In particular, show that if  $x, y \in \Delta$ , then there is some  $z \in [0, 1] \setminus \Delta$  with  $x < z < y$ . (It follows from this that  $\Delta$  is *nowhere dense*, which is another way of saying that  $\Delta$  is “small”.)

*Solution.*

Let  $x, y \in \Delta$ . Following the notation of the book, we name  $I_k$  the union of closed intervals that comprises the  $k$ -th “level” of the “tree” that creates the Cantor set. Then recall that  $\Delta = \bigcap_{k=0}^{\infty} I_k$ . Since  $x, y \in \Delta$ , it holds that  $x, y \in I_k$  for all  $k$ . We now claim that there exists  $k$  such that  $x, y$  belong in two disjoint intervals that comprise the union  $I_k$ . This will of course imply that there exists a “removed middle third” interval between  $x, y$  at the  $k$ -th level, any of whose elements fulfill the condition  $z \notin \Delta$ . We do this by contradiction. Namely, suppose that  $x, y$  belong in the same closed interval at every level of the “tree”, and call those intervals  $I_1, I_2, \dots$ . By construction, these are nested intervals, and we know that  $I_n$ ’s length goes to zero as  $n \rightarrow \infty$ . But then the nested interval theorem dictates that the infinite intersection of  $I_1, I_2, \dots$  contains precisely one element, which means  $x = y$ , a contradiction.

**Exercise 23**

The endpoints of  $\Delta$  are those points in  $\Delta$  having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in  $\Delta$  of the form  $a/3^n$  for some integers  $n$  and  $0 \leq a \leq 3^n$ . Show that the endpoints of  $\Delta$  other than 0 and 1 can be written as  $0.a_1a_2\dots a_{n+1}$  (base 3) where each  $a_k$  is 0 or 2, except  $a_{n+1}$ , which is either 1 or 2. That is, the discarded “middle third” intervals are of the form  $(0.a_1a_2\dots a_n1, 0.a_1a_2\dots a_n2)$  where both entires are points of  $\Delta$  written in base 3.

*Solution.*

We will apply induction on  $n$ , which corresponds to the “level” of the “tree” of the Cantor set. More specifically, we will prove the following.

- If  $x \neq 0$  is a left endpoint of the Cantor set, that is, the the intersection forming the Cantor set contains it in the  $n$ -th “tree-level union ” in some interval  $[x, y]$ , then it can be written in the form  $0.a_1\dots a_n2$  where  $a_i \in \{0, 2\}, i = 1, \dots, n$ .
- If  $x \neq 1$  is a right endpoint of the Cantor set, that is, the intersection forming the Cantor set contains it in the  $n$ -th “tree-level union” in some interval  $[y, x]$ , then it can be written in the form  $0.a_1\dots a_n1$  where  $a_i \in \{0, 2\}, i = 1, \dots, n$ .

As stated, we will use induction on  $n$  starting at 0, which corresponds to the first level of the tree.

- **Base case:** if  $n = 0$ , then the corresponding union is  $I_1 = [0, 0.1] \cup [0.2, 1]$  (written in ternary). Obviously, the only non-zero left endpoint is 0.2, which satisfies the claim stated above. Similarly, the only right endpoint not equal to 1 is 0.1, which also satisfies the claim.
- **Inductive hypothesis:** Suppose that for  $n = k, k \geq 0$  it holds that  $I_n = [0, x_1] \cup [x_2, x_3] \cup \dots \cup [x_M, 1]$ , where each left and right endpoint fulfill the conditions stated above (note that  $M = 2^{n+1}$ ).
- **Inductive step:** Observe then that the intervals comprising  $I_{n+1}$  are formed by taking each interval  $[x, y]$  comprising  $I_n$  and replacing it by  $[x, x + \underbrace{0.0\dots 0}_{n+1}1] \cup [y - \underbrace{0.0\dots 0}_{n+1}1, y]$ . Therefore, new right endpoints are formed by taking either 0 or a previous left endpoint and adding  $\underbrace{0.0\dots 0}_{n+1}1$ . Clearly, adding such a number to 0 results in a right endpoint fulfilling the claim stated in the beginning. In the other case, we have an endpoint of the form  $0.a_1\dots a_n2 + \underbrace{0.0\dots 0}_{n+1}1 = 0.a_1\dots a_n21$ , which based on the inductive hypothesis also fulfills the initial claim. Similarly, new left endpoints are formed by taking either 1 or previous right endpoints and subtracting  $\underbrace{0.0\dots 0}_{n+1}1$ . Subtracting this quantity from 1

results in  $0.\underbrace{2\dots 2}_{n+1}2$ , which fulfills the claim for a left endpoint. On the other hand, subtracting it from a previous right endpoint results in an endpoint of the form  $0.a_1\dots a_n1 - 0.\underbrace{0\dots 0}_{n+1}1 = 0.a_1\dots a_n02$ , which based on the inductive hypothesis also satisfies the claim for left endpoints.

This concludes the proof that endpoints of  $\Delta$  have a ternary decimal expansion described in the exercise.

### Exercise 26

Let  $f : \Delta \rightarrow [0, 1]$  be the Cantor function and let  $x, y \in \Delta$  with  $x < y$ . Show that  $f(x) \leq f(y)$ . If  $f(x) = f(y)$ , show that  $x$  has two distinct ternary decimal representations. Finally, show that  $f(x) = f(y)$  if and only if  $x, y$  are “consecutive” endpoints of the form  $x = 0.a_1a_1\dots a_n1$  and  $y = a_1a_2\dots a_n2$  (base 3).

*Solution.*

Let  $x = 0.a_1a_2\dots$  and  $y = 0.b_1b_2\dots$ . Because  $x, y \in \Delta$ , we know that we can write  $x, y$  such that each of  $a_i, b_i$  is either 0 or 1. Now, because  $x < y$  it has to be the case that for some  $i \geq 1$ ,  $a_i < b_i$  and  $a_k = b_k$  for  $k < i$ . This can be seen by making a “digit-wise” comparison: the first decimal digit in which  $x, y$  differ determines which one is larger, since after that point the decimal digits get multiplied by smaller powers of 3 and the smaller number can never “catch up”. The only possible exception would be if  $x = 0.a_1\dots a_i00\dots$  and  $y = 0.a_1\dots (a_i-1)22\dots$ , but we know that this cannot be true since it would require  $a_i = 1$  or  $a_i - 1 = 1$ , which is impossible due to the way we’ve written  $x, y$ . Now consider the decimal strings  $x' = 0.(\frac{a_1}{2})(\frac{a_2}{2})\dots, y' = 0.(\frac{b_1}{2})(\frac{b_2}{2})\dots$ . What the Cantor function does is interpret these as binary instead of ternary numbers, and set then  $f(x) = x'_{\text{bin}}, f(y) = y'_{\text{bin}}$ . But then  $\frac{a_k}{2} = \frac{b_k}{2}$  for  $k < i$ ,  $\frac{a_i}{2} < \frac{b_i}{2}$ . By this we have established that it is impossible that  $f(x) > f(y)$ , thus  $f(x) \leq f(y)$ . If it is the case that  $f(x) = f(y)$ , then for  $k > i$ ,  $\frac{a_k}{2} = 1, \frac{b_k}{2} = 0$  and  $\frac{a_i}{2} = 0, \frac{b_i}{2} = 1$ . Going back to the ternary numbers  $x, y$ , this implies that  $x$  is of the form:

$$x = 0.a_1\dots a_{i-1}0222\dots,$$

which can also be written as  $x = 0.a_1\dots a_{i-1}1$ . Furthermore, it implies that  $y$  is of the form:

$$y = 0.a_1\dots a_{i-1}2000\dots,$$

which completes the proof that a)  $x$  has two distinct ternary decimal expansions, and that whenever  $f(x) = f(y)$ ,  $x, y$  are endpoints of “discarded” middle third intervals. For the other direction of the last equivalence, if  $x, y$  are endpoints of “discarded” middle third intervals, write both of them such that they only contain 0s and 2s in ternary, apply  $f$  and observe that  $f(x)$  ends in infinite 1s,  $f(y)$  in 0s, and for some  $i$ ,  $f(x)_i0, f(y)_i1$  and  $f(x)_k = f(y)_k, k < i$ , which of course means that  $f(x) = f(y)$ .

### Exercise 29

Prove that the extended Cantor function  $f : [0, 1] \rightarrow [0, 1]$  (as defined above) is increasing. [Hint: consider cases.]

*Solution.*

For any two  $x, y \in [0, 1]$ , with  $x < y$ , we have the following:

- If  $x, y \in \Delta$ , we have shown in exercise 23 that  $f(x) \leq f(y)$ .
- If  $x \in \Delta, y \notin \Delta$ , then  $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$ . This means that  $f(x) \in \{f(z), z \leq y, z \in \Delta\}$ , and thus it must be the case that  $f(x) \leq f(y)$ .
- If  $x, y \notin \Delta$ , then  $f(x) = \sup\{f(z), z \leq x, z \in \Delta\}$  and  $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$ . But then since  $x < y$ , the set over which the second supremum is computed is a superset of the set over which the first supremum is computed, meaning that  $f(x) \leq f(y)$ .

## 2.3 Monotone Functions

### Exercise 34

Let  $D = \{x_1, x_2, \dots\}$ , and let  $\epsilon_n > 0$  with  $\sum_{n=1}^{\infty} \epsilon_n < \infty$ . Define  $f(x) = \sum_{x_n \leq x} \epsilon_n$  (as above). Check the following:

- (i)  $f$  is discontinuous at the points of  $D$
- (ii)  $f$  is right-continuous everywhere
- (iii)  $f$  is continuous at each point  $x \in \mathbb{R} \setminus D$

How might this construction be modified to yield a *strictly* increasing function with these same properties?

*Solution.*

(i) Notice that we've already seen in the main text that for  $x_k \in D$ ,  $f(x_k-) = f(x_k) - \epsilon_k$ , that is, the left limit of  $f$  at each point of  $D$  can never equal its value on the respective point, since  $\epsilon_k > 0$ . Therefore  $f$  is not continuous at the points of  $D$ .

(ii) Again, in the main text we've seen that  $f$  is right-continuous at the points of  $D$ . It suffices therefore to check that for  $x \in \mathbb{R} \setminus D$ ,  $f(x+) = f(x)$ . Expanding this, we have:

$$f(x+) = \lim_{y \rightarrow x+} f(y)$$

Pick any  $\epsilon > 0$ . We need to show then that there exists  $\delta > 0$  such that whenever  $y - x < \delta$  it holds that  $f(y) - f(x) < \epsilon$  (note that the fact that  $f$  is increasing and the fact that we are taking a right limit allows us to omit absolute values). We have that:

$$f(y) - f(x) = \sum_{\{n: x_n \leq y\}} \epsilon_n - \sum_{\{n: x_n \leq x\}} \epsilon_n = \sum_{\{n: x < x_n \leq y\}} \epsilon_n$$

As has already been observed in the main text, since the series corresponding to  $\epsilon_n$  converges, it is the case that  $\sum_{n=N}^{\infty} \epsilon_n = 0$  as  $N \rightarrow \infty$ . This means that given our chosen  $\epsilon > 0$ , we can find an  $N > 0$  such that  $\sum_{n=N}^{\infty} \epsilon_n < \epsilon$ . Therefore, after excluding a *finite* number of terms from the series, namely  $\{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}\}$  we can make the corresponding remaining infinite sum arbitrarily small. Because the set is finite, we can then pick  $\delta > 0$  such that none of these terms are in the interval  $(x, y]$ : simply find the closest term to  $x$  and pick  $y$  to be closer to  $x$  than that. Then by necessity:

$$\sum_{\{n: x < x_n \leq y\}} \epsilon_n \leq \sum_{n=N}^{\infty} \epsilon_n < \epsilon,$$

which means precisely that  $f(y) - f(x) < \epsilon$ , thus completing the proof that  $f$  is right-continuous everywhere.

(iii) Due to part (ii) we only need to prove that  $f$  is left-continuous at  $x \in \mathbb{R} \setminus D$ . That is, we need to prove that for  $x \in \mathbb{R} \setminus D$ :

$$\lim_{y \rightarrow x-} f(y) = f(x)$$

Pick any  $\epsilon > 0$ . We need to find  $\delta > 0$  such that whenever  $x - y < \delta$ :

$$f(x) - f(y) < \epsilon \implies \sum_{\{n: x_n \leq x\}} \epsilon_n - \sum_{\{n: x_n \leq y\}} \epsilon_n < \epsilon \implies \sum_{\{n: y < x_n \leq x\}} \epsilon_n < \epsilon$$

This means that we can now use the exact same observation as in (ii) to bring  $y$  sufficiently close to  $x$  such that the LHS sum is strictly smaller than  $\epsilon$ , thus proving that  $f$  is indeed left-continuous at  $x$ . Note that the important difference from the case  $x \in D$  examined in the book is that here  $x_n \leq x$  does not introduce a fixed term in the sum, which would of course impose a lower bound on it. In contrast, when  $x \in D$  it is not guaranteed that we can always exclude the finite set of terms obtained in (ii) since one of them *may be*  $x$  *itself*.

Now for the question posed in the exercise, consider the function  $g(x) = f(x) + x$ . Because the identity function is continuous everywhere,  $g$  will maintain all properties of  $f$  shown in the exercise: it will be right-continuous everywhere as a sum of right-continuous functions, it will be continuous at  $\mathbb{R} \setminus D$  and at the points of  $D$  it will be discontinuous (since otherwise  $f(x) = g(x) - x$  would be continuous as a difference of continuous functions). In addition, for any two  $x_1 < x_2$ , we have that  $f(x_1) \leq f(x_2)$ , and therefore clearly  $g(x_1) < g(x_2)$ , meaning that  $g$  is strictly increasing.

### Exercise 35

Let  $f : [a, b] \rightarrow \mathbb{R}$  be increasing, and let  $(x_n)$  be an enumeration of the discontinuities of  $f$ . For each  $n$ , let  $a_n = f(x_n) - f(x_{n-})$  and  $b_n = f(x_{n+}) - f(x_n)$  be the left and right “jumps” in the graph of  $f$ , where  $a_n = 0$  if  $x_n = a$  and  $b_n = 0$  if  $x_n = b$ . Show that  $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$  and  $\sum_{n=1}^{\infty} b_n \leq f(b) - f(a)$ .

*Solution.*

We will first use exercise 33 (without the absolute values since we have an *increasing* function) to obtain an intermediate result about the first  $n$  terms of the sums that appear in this exercise. Namely, applying exercise 33 on the points  $(x_n)$  we have that:

$$\begin{aligned} \sum_{i=1}^n f(x_i+) - f(x_i-) &\leq f(b) - f(a) \implies \sum_{i=1}^n f(x_i+) - f(x_i) + f(x_i) - f(x_i-) \leq f(b) - f(a) \\ \implies \sum_{i=1}^n f(x_i+) - f(x_i) + \sum_{i=1}^n f(x_i) - f(x_i-) &\leq f(b) - f(a) \implies \sum_{i=1}^n b_i + \sum_{i=1}^n a_i \leq f(b) - f(a) \end{aligned}$$

Since we are dealing with non-negative quantities from this we can conclude that

$$\sum_{i=1}^n b_i \leq f(b) - f(a), \quad \sum_{i=1}^n a_i \leq f(b) - f(a)$$

Observe therefore that if the left or right discontinuities are finite in number, this result proves what is needed in the exercise (for  $a_i$  or  $b_i$  respectively). We now examine the case where the left discontinuities are infinite in number. Observe that each  $a_n$  is non-negative, and therefore the sequence of partial sums is non-decreasing, while we’ve already shown that it is bounded. We thus know that it converges, and since limits maintain non-strict inequalities, we obtain that  $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$ . The exact same reasoning can also be applied to right discontinuities.

### Exercise 36

In the notation of Exercise 35, define  $h(x) = \sum_{\{n: x_n \leq x\}} a_n + \sum_{\{n: x_n < x\}} b_n$ . Show that  $h$  is increasing and that  $g = f - h$  is both continuous and increasing. Thus, each increasing function  $f$  can be written as the sum of a continuous increasing function  $g$  and a “pure jump” function  $h$ .

*Solution.*

We begin by showing that  $h$  is increasing. Consider  $y_1, y_2 \in [a, b], y_1 < y_2$ . Then:

$$\begin{aligned} h(y_2) - h(y_1) &= \sum_{\{n: x_n \leq y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \leq y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n \\ &= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n \geq 0, \end{aligned}$$

since each term of these sums is non-negative (recall that  $f$  is increasing). Therefore  $h$  is increasing. Now we examine the continuity of  $g$ , for which we’ll have to examine the left and right limits at the points of discontinuity of  $f$  and at the points where  $f$  is continuous.

First let’s consider  $x_i \in \{x_1, x_2, \dots\}$ , i.e. a point of discontinuity of  $f$ :

- For  $y \rightarrow x_i+$ , we are interested in the following limit:

$$\lim_{y \rightarrow x_i+} g(y) = \lim_{y \rightarrow x_i+} (f(y) - h(y)) = \lim_{y \rightarrow x_i+} \left( f(y) - \sum_{\{n: x_n \leq y\}} a_n - \sum_{\{n: x_n < y\}} b_n \right)$$

Now notice that in this case it is always true that  $y > x_i$ , leading us to decompose the sums as follows:

$$f(y) - \sum_{\{n: x_n \leq x_i\}} a_n - \sum_{\{n: x_i < x_n \leq y\}} a_n - \sum_{\{n: x_n < x_i\}} b_n - \sum_{\{n: x_i \leq x_n < y\}} b_n$$

$$= f(y) - h(x_i) - \left( \sum_{\{n: x_n < x_i \leq y\}} a_n + \sum_{\{n: x_i \leq x_n < y\}} b_n \right)$$

Now we observe that since we saw in exercise 35 that the series of the non-negative ( $a_n$ ) converges, the sum of  $a_n$  inside the parenthesis can be made to go to zero as  $y \rightarrow x_i+$ : the argument here is the same as in exercise 34, where we show that we need but to exclude a finite number of terms of the series. The same applies to the sum of  $b_n$ , except that it tends to  $b_i$  instead (due to the “less than or equal to  $x_i$ ” in its sum, where  $x_i$  does correspond to a term of the sequence). Consequently, the limit of this expression as  $y \rightarrow x_i+$  must be:

$$f(x_i+) - h(x_i) - 0 - b_i = f(x_i+) - h(x_i) - f(x_i+) + f(x_i) = f(x_i) - h(x_i),$$

where the first term comes from the definition of the right limit of  $f$  at  $x_i$ , the second term is constant and the third and fourth were explained above. Therefore we conclude that  $\lim_{y \rightarrow x_i+} g(y) = f(x_i) - h(x_i) = g(x_i)$ .

- For  $y \rightarrow x_i-$ , the proof will be highly similar:

$$\lim_{y \rightarrow x_i-} g(y) = \lim_{y \rightarrow x_i-} \left( f(y) - \sum_{\{n: x_n \leq y\}} a_n - \sum_{\{n: x_n < y\}} b_n \right)$$

The decomposition now looks like:

$$\begin{aligned} f(y) - \sum_{\{n: x_n \leq x_i\}} a_n + \sum_{\{n: y < x_n \leq x_i\}} a_n - \sum_{\{n: x_n < x_i\}} b_n + \sum_{\{n: y \leq x_n < x_i\}} b_n \\ = f(y) - h(x_i) + \left( \sum_{\{n: y < x_n \leq x_i\}} a_n + \sum_{\{n: y \leq x_n < x_i\}} b_n \right) \end{aligned}$$

A similar argument as above yields that the first sum here tends to  $a_i$  and the second to zero. Therefore the desired limit will equal:

$$f(x_i-) - h(x_i) + a_i = f(x_i-) - h(x_i) + f(x_i) - f(x_i-) = f(x_i) - h(x_i),$$

which is again the desired quantity.

Now we consider  $x \notin \{x_1, x_2, \dots\}$ :

- For  $y \rightarrow x+$ , we are interested in:

$$\lim_{y \rightarrow x+} g(y) = \lim_{y \rightarrow x} (f(y) - h(y))$$

Carrying out the calculations will lead to an expression identical to the first bullet above, except that since  $x \notin \{x_1, x_2, \dots\}$  all sums that involve  $a_n, b_n$  will now go to zero (unlike before, they never contain a “fixed” term corresponding to  $x_i$ ). Therefore we’ll eventually arrive at  $\lim_{y \rightarrow x+} g(y) = f(x+) - h(x) = f(x) - h(x)$ , since  $f$  is now continuous at  $x$ .

- For  $y \rightarrow x-$ , everything is exactly symmetrical to the above, and therefore once more  $\lim_{y \rightarrow x-} g(y) = f(x) - h(x)$ .

This completes the proof that  $g$  is in fact continuous in all of  $[a, b]$ .

All that remains is to show that  $g$  is increasing as well. Take  $y_1, y_2 \in [a, b]$  with  $y_1 < y_2$ . We want to show that  $g(y_1) \leq g(y_2)$ . We have that:

$$g(y_2) - g(y_1) = f(y_2) - h(y_2) - f(y_1) + h(y_1) = f(y_2) - f(y_1) - (h(y_2) - h(y_1))$$

We examine  $h(y_2) - h(y_1)$ :

$$h(y_2) - h(y_1) = \sum_{\{n: x_n \leq y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \leq y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n$$

$$= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n$$

Notice that every term that appears in this sum corresponds to a  $x_n \in [y_1, y_2]$ . Recall also from exercise 35 that we had  $\sum_{i=1}^n b_i + \sum_{i=1}^n a_i \leq f(b) - f(a)$  and that taking the limit of each of the two sums was well-defined, which means that this inequality as well holds at the limit. The only detail that may potentially be problematic here is that we may also be summing over the terms  $f(y_1+) - f(y_1)$ ,  $f(y_2) - f(y_2-)$  in case one or both endpoints  $y_1, y_2$  are also discontinuities. However, it can be shown fairly easily that with a small modification to 33 the inequality given there can also be shown to hold for  $n$  points in the closed interval  $[a, b]$  instead of just the open (one just has to take a finite number of  $x_1, \dots, x_n$  and consider  $a < a' < x_1, x_n < b' < b$ ). Then 35 can be applied to generalize this to countably infinite points which may be the case here.

All of this results in the fact that the sum above is at most  $f(y_2) - f(y_1)$  (since these are the endpoints of the relevant interval), and therefore also  $g(y_2) \geq g(y_1)$ , which shows that  $g$  is increasing.

# Chapter 3

## Metrics and Norms

### 3.1 Metric Spaces

**Exercise 2**

If  $d$  is a metric on  $M$ , show that  $|d(x, z) - d(y, z)| \leq d(x, y)$  for any  $x, y, z \in M$ .

*Solution.*

First, we apply the triangle inequality as follows:

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y)$$

Another application of it yields:

$$d(y, z) \leq d(y, x) + d(x, z) \implies -d(y, x) \leq d(x, z) - d(y, z) \implies -d(x, y) \leq d(x, z) - d(y, z),$$

where we used the symmetry property of metric  $d$ . Putting the two inequalities together results in:

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

**Exercise 3**

As it happens, some of our requirements for a metric are redundant. To see why this is so, let  $M$  be a set and suppose  $d : M \times M \rightarrow \mathbb{R}$  satisfies  $d(x, y) = 0$  if and only if  $x = y$ , and  $d(x, y) \leq d(x, z) + d(y, z)$  for all  $x, y, z \in M$ . Prove that  $d$  is a metric; that is, show that  $d(x, y) \geq 0$  and  $d(x, y) = d(y, x)$  hold for all  $x, y$ .

*Solution.*

Pick any two  $x, y \in M$ . We know then that for any  $z \in M$  it holds that  $d(x, z) \leq d(x, y) + d(z, y)$ . More specifically, this holds for  $z = x$ , in which case we obtain:

$$d(x, x) \leq d(x, y) + d(x, y) \implies 0 \leq 2d(x, y),$$

which means that for any two  $x, y \in M$ ,  $d(x, y) \geq 0$ .

For the symmetry property, once again pick any two  $x, y \in M$ . Then  $d(x, y) \leq d(x, z) + d(y, z)$  for any  $z$ . Pick then  $z = x$ , in which case we obtain  $d(x, y) \leq d(x, x) + d(y, x) \implies d(x, y) \leq d(y, x)$ . By exchanging the roles of  $x, y$ , we have that  $d(y, x) \leq d(y, z) + d(x, z)$  for any  $z$ . Pick  $z = y$  to observe that  $d(y, x) \leq d(y, y) + d(x, y) \implies d(y, x) \leq d(x, y)$ . But then  $d(x, y) \leq d(y, x) \leq d(x, y)$ , thus the only possibility is that  $d(x, y) = d(y, x)$ . Therefore  $d$  is indeed a metric.

**Exercise 6**

If  $d$  is any metric on  $M$ , show that  $\rho(x, y) = \sqrt{d(x, y)}$ ,  $\sigma(x, y) = d(x, y)/(1 + d(x, y))$  and  $\tau(x, y) = \min\{d(x, y), 1\}$  are also metrics on  $M$ . [Hint:  $\sigma(x, y) = F(d(x, y))$ , where  $F$  is as in exercise 5.]

*Solution.*

We will solve this as a simple application of 7. If  $F_1 : [0, \infty) \rightarrow [0, \infty)$ ,  $F_1(x) = \sqrt{x}$ , then  $\rho(x, y) = F_1(d(x, y))$ . Then  $F_1(x)/x = 1/\sqrt{x}$ , which is clearly decreasing for  $x > 0$ , and therefore exercise 7 guarantees that  $\rho$  is a metric.

Similarly, if  $F_2 : [0, \infty) \rightarrow [0, \infty)$ ,  $F_2(x) = \frac{x}{1+x}$ , then  $F'_2(x) = \frac{x+1-x}{(1+x)^2} = \frac{1}{(x+1)^2}$ , which is clearly decreasing as for  $x \geq 0$ , and therefore  $\sigma$  is also a metric.

Lastly, if  $F_3 : [0, \infty) \rightarrow [0, \infty)$ ,  $F_3(x) = \min\{x, 1\}$ , then  $\tau(x, y) = F_3(d(x, y))$  and:

$$F_3(x)/x = \begin{cases} 1, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases},$$

which is of course a decreasing function for  $x > 0$ , and therefore  $\tau$  is a metric.

**Exercise 7**

Here is a generalization of exercises 5 and 6. Let  $f : [0, \infty) \rightarrow [0, \infty)$  be increasing and satisfy  $f(0) = 0$ , and  $f(x) > 0$  for all  $x > 0$ . If  $f$  also satisfies  $f(x+y) \leq f(x) + f(y)$  for all  $x, y \geq 0$ , then  $f \circ d$  is a metric whenever  $d$  is a metric. Show that each of the following conditions is sufficient to ensure that  $f(x+y) \leq f(x) + f(y)$  for all  $x, y \geq 0$ :

- (a)  $f$  has a second derivative satisfying  $f'' \leq 0$
- (b)  $f$  has a decreasing first derivative
- (c)  $f(x)/x$  is decreasing for  $x > 0$

[Hint: First show that (a)  $\implies$  (b)  $\implies$  (c).]

*Solution.*

As indicated in the hint, we first show that (a)  $\implies$  (b)  $\implies$  (c). If, after that, we can show that (c) implies the triangle inequality for  $f$ , we know also that any of (a), (b) imply it as well (since they imply (c)).

(a)  $\implies$  (b): Since  $f'' \leq 0$ ,  $f$  more specifically has a first derivative on every point of  $[0, \infty)$ . From calculus 1, it's known also that this implies that  $f'$  is decreasing (one would prove this via the Mean Value Theorem).

(b)  $\implies$  (c): (b) implies that  $f$  is differentiable, thus  $g(x) = f(x)/x, x > 0$  is also differentiable, with  $g'(x) = \frac{f'(x)x - f(x)}{x^2}$ . If we can show that the numerator here is non-positive for  $x > 0$ , we'll have shown that  $g$  is decreasing. Pick any  $x > 0$ , and apply the Mean Value Theorem on  $f$  in the interval  $[0, x]$ . This means that there exists  $y \in (0, x)$  such that:

$$f'(y) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x},$$

where we used the fact that  $f(0) = 0$ . Therefore, we have that:

$$f'(x)x - f(x) = f'(x)x - xf'(y) = x(f'(x) - f'(y)) \leq 0,$$

since  $f'$  is decreasing and  $y < x$ . This completes the proof that (b)  $\implies$  (c).

We now need to show that the triangle inequality for  $f$  follows from (c). Pick any two  $x, y \geq 0$ . If any of them, or both, are zero, the triangle inequality follows from the fact that  $f(0) = 0$ . If  $x, y \neq 0$ , then we know that  $x \leq x+y, y \leq x+y$ , and thus that  $\frac{f(x)}{x} \geq \frac{f(x+y)}{x+y}, \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$  (by the fact that  $f(x)/x$  is decreasing). We can rewrite these as:

$$(x+y)f(x) \geq xf(x+y), (x+y)f(y) \geq yf(x+y),$$

and now we can add them to obtain that:

$$(x+y)(f(x) + f(y)) \geq f(x+y)(x+y) \implies f(x) + f(y) \geq f(x+y),$$

which is of course what we wanted to show.

### Exercise 10

The *Hilbert cube*,  $H^\infty$ , is the collection of all real sequences  $x = (x_n)$  with  $|x_n| \leq 1, n = 1, 2, \dots$

- (i) Show that  $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$  defines a metric on  $H^\infty$ .
- (ii) Given  $x, y \in H^\infty$  and  $k \in \mathbb{N}$ , let  $M_k = \max\{|x_1 - y_1|, \dots, |x_k - y_k|\}$ . Show that  $2^{-k}M_k \leq d(x, y) \leq M_k + 2^{-k+1}$ .

*Solution.*

(i) We will make use of exercise 3, which allows us to conclude that  $d$  is a metric if the following three things hold:

- First, that  $d(x, y)$  is a non-negative real number for any  $x, y$ , which, due to the infinite sum, is non-trivial here. Consider any two sequences  $x, y$ . We know that  $|x_n| \leq 1, |y_n| \leq 1$ , for all  $n$ , therefore the maximum value that  $|x_n - y_n|$  achieves is 2. This means that  $2^{-n}|x_n - y_n| \leq 2^{-n+1}$ . The partial sums that correspond to  $d$  are therefore non-decreasing and upper bounded by a convergent geometric series, therefore the series that defines  $d$  also converges.
- Second, that  $d(x, y) = 0$  iff  $x = y$ . If two sequences  $x, y$  are equal, then  $x_n = y_n$  for all  $n$ , thus  $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - x_n| = 0$ . Conversely, if  $d(x, y) = 0$ , then we have a non-decreasing sequence of non-negative partial sums that converges to zero. The only way this can happen is if each term of the sequence is zero, i.e. if  $|x_n - y_n| = 0$  for all  $n$ , that is, if  $x_n = y_n$  for all  $n$ . But this means precisely that  $x = y$ .
- Third, that the triangle inequality holds. Pick any three sequences  $x, y, z$ , and  $k \in \mathbb{N}^+$ . Then:

$$\sum_{n=1}^k 2^{-n}|x_n - y_n| = \sum_{n=1}^k 2^{-n}|x_n - z_n + z_n - y_n| \leq \sum_{n=1}^k 2^{-n}|x_n - z_n| + \sum_{n=1}^k 2^{-n}|z_n - y_n|,$$

by the triangle inequality. Since this holds for any  $k$  and all series converge (by the first item proven), we know that the inequality holds for the infinite series as well, i.e. that:

$$d(x, y) \leq d(x, z) + d(y, z)$$

Therefore  $d$  is a metric on the Hilbert cube.

(ii) Pick any two sequences  $x, y$  in the Hilbert cube and  $k \in \mathbb{N}$ . We know then by definition that  $M_k \geq |x_i - y_i|$  for  $i \leq k$ . In addition, for  $n > k$ , the  $n$ -th term of the series that forms the metric  $d$  is  $2^{-n}|x_n - y_n| \leq 2^{-n+1}$ . Thus:

$$\begin{aligned} d(x, y) &= \sum_{n=1}^k 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n| \leq M_k \sum_{n=1}^k 2^{-n} + \sum_{n=k+1}^{\infty} 2^{-n+1} \\ &= M_k(1 - 2^{-k}) + 2(2 - 1 - (1 - 2^{-k})) = M_k - 2^{-k}M_k + 2^{-k+1} \leq M_k + 2^{-k+1}, \end{aligned}$$

where we used the fact that the geometric series with ratio  $r = 1/2$  sums to 2, and our second sum here was thus 2 minus the sum of the first  $k$  terms minus 1 since the series starts at 1 instead of 0. In the last step we also use the fact that  $M_k \geq 0$ . For the other inequality, we have again that:

$$d(x, y) = \sum_{n=1}^k 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n|$$

Notice that in order for  $x, y$  to minimize this, it should be the case that  $x_n - y_n = 0, n > k$ , and that the maximum absolute difference  $M_k$  is multiplied with the smallest possible quantity up to  $k$ , that is, with  $2^{-k}$ , while for all other  $n < k, x_n - y_n = 0$ . This would yield that  $d(x, y) \geq M_k 2^{-k} + 0 + 0 = M_k 2^{-k}$ .

**Exercise 12**

Check that  $d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$  defines a metric on  $C[a, b]$ , the collection of all continuous, real valued functions defined on the closed interval  $[a, b]$ .

*Solution.*

Note that here continuity is important to ensure that the difference of any two functions is continuous, and as such achieves a maximum value in any closed interval. Using exercise 3, we need to check the following:

- One, that  $d(f, g) = 0$  iff  $f = g$ . Suppose first that  $d(f, g) = 0$ , that is,  $\max_{a \leq t \leq b} |f(t) - g(t)| = 0$ . If  $f, g$  were not equal, there would exist  $x \in [a, b]$  such that  $f(x) \neq g(x)$ . Clearly then  $\max_{a \leq t \leq b} |f(t) - g(t)| \geq |f(x) - g(x)| > 0$ , which is a contradiction. Therefore  $f = g$ . In the other direction, if  $f = g$ , we equivalently have that  $f - g$  is the zero function on  $[a, b]$ , and therefore  $d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)| = 0$ .
- Two, that for any three  $f, g, h \in C[a, b]$ ,  $d(f, h) \leq d(f, g) + d(g, h)$ . We have first that  $d(f, h) = \max_{a \leq t \leq b} |f(t) - h(t)|$ . Suppose that this function achieves its maximum value on  $t_M \in [a, b]$  (if there are more than one, pick any). Then  $d(f, h) = |f(t_M) - h(t_M)| = |f(t_M) - g(t_M) + g(t_M) - h(t_M)| \leq |f(t_M) - g(t_M)| + |g(t_M) - h(t_M)|$ . By definition, it holds that  $|f(t_M) - g(t_M)| \leq \max_{a \leq t \leq b} |f(t) - g(t)| = d(f, g)$ , and similarly  $|g(t_M) - h(t_M)| \leq d(g, h)$ . We therefore obtain that  $d(f, h) \leq d(f, g) + d(g, h)$ .

Therefore  $d$  is indeed a metric on  $C[a, b]$ .

**Exercise 14**

We say that a subset  $A$  of a metric space is **bounded** if there is some  $x_0 \in M$  and some constant  $C < \infty$  such that  $d(a, x_0) \leq C$  for all  $a \in A$ . Show that a finite union of bounded sets is again bounded.

*Solution.*

Suppose we have the finite union

$$U = A_1 \cup A_2 \cup \dots \cup A_n$$

of  $n$  bounded subsets of a metric space  $M$ . Let then  $x_1, x_2, \dots, x_n, C_1, \dots, C_n$  be such that  $d(a, x_i) \leq C$  for  $a \in A_i$ . Form the set  $\{d(x_1, x_1), d(x_1, x_2), \dots, d(x_n, x_1)\}$ , which, crucially, has a finite number of elements, and therefore also has a maximum element,  $M$ . By the same reasoning, there exists  $C$  such that  $C = \max\{C_1, \dots, C_n\}$ . Now pick any  $a \in U$ , which must belong in at least one  $A_i$ . Therefore:

$$d(a, x_1) \leq d(a, x_i) + d(x_i, x_1) \leq C_i + M \leq C + M$$

If we thus set  $x_U = x_1$  and  $C_U = C + M$ , we have shown precisely that  $U$  is bounded.

**Exercise 15**

We define the **diameter** of a nonempty subset  $A$  of  $M$  by  $\text{diam}(A) = \sup\{d(a, b) : a, b \in A\}$ . Show that  $A$  is bounded if and only if  $\text{diam}(A)$  is finite.

*Solution.*

$\implies$  : Suppose first that  $A$  is bounded. Then there exists  $x_0 \in M, C \in \mathbb{R}$  such that  $d(a, x_0) \leq C$  for all  $a \in A$ . Pick any two  $a, b \in A$ . We then have that  $d(a, b) \leq d(a, x_0) + d(x_0, b) \leq 2C$ . Therefore the set  $\{d(a, b) : a, b \in A\}$  has an upper bound, namely,  $2C$ , and therefore also a finite least upper bound, which means precisely that  $\text{diam}(A)$  is finite.

$\impliedby$  : Now suppose  $\text{diam}(A) = C \in \mathbb{R}$ . Fix an  $a \in A$  (which exists since  $A$  is nonempty). Pick any  $b \in A$ , in which case we have that:

$$d(a, b) \in \{d(x, y) : x, y \in A\} \implies d(a, b) \leq \text{diam}(A) = C$$

If thus set  $x_0 = a$  and  $C$  the respective constant, we see that the definition of  $A$  being bounded is fulfilled.

## 3.2 Normed Vector Spaces

**Exercise 16**

Let  $V$  be a vector space, and let  $d$  be a metric on  $V$  satisfying  $d(x, y) = d(x - y, 0)$  and  $d(ax, ay) = |a|d(x, y)$  for every  $x, y \in V$  and every scalar  $a$ . Show that  $\|x\| = d(x, 0)$  defines a norm on  $V$  (that has  $d$  as its “usual” metric). Give an example of a metric on the vector space  $\mathbb{R}$  that fails to be associated with a norm in this way.

*Solution.* Let us examine the properties that would make  $\|\cdot\|$  one by one:

- Suppose  $x \in V$ . Then  $\|x\| = d(x, 0) \geq 0$ , since  $d$  is a metric (and of course  $\|x\|$  is well-defined as a finite real number for the same reason).
- Suppose  $\|x\| = 0$ . Then  $d(x, 0) = 0$ , which by the properties of metrics we know is true iff  $x = 0$ . Conversely, if  $x = 0$ , by the same argument  $d(x, 0) = 0 \implies \|x\| = 0$ . Thus  $\|x\| = 0 \iff x = 0$ .
- For any scalar  $a$ , we have that  $\|ax\| = d(ax, 0) = d(ax, a0) = |a|d(x, 0) = |a|\|x\|$  by the exercise hypothesis.
- Pick any two  $x, y \in V$ . Then:

$$\|x + y\| = d(x + y, 0) \leq d(x + y, y) + d(y, 0) = d(x + y - y, 0) + d(y, 0) = d(x, 0) + d(y, 0) = \|x\| + \|y\|,$$

where we used the triangle inequality property for metrics and the fact that  $d(x, y) = d(x - y, 0)$ . Therefore the triangle inequality holds for the proposed norm.

We have thus shown that  $\|\cdot\|$  is indeed a norm, and its usual metric will of course be  $d'(x, y) = \|x - y\| = d(x - y, 0) = d(x, y)$ , i.e.,  $d$ .

For the requested example, consider the metric  $\sigma(x, y) = |x - y|/(1 + |x - y|)$  from exercise 6 of section 3.1. Notice that the “proposed” norm would then be  $\|x\| = \sigma(x, 0) = |x|/(1 + |x|)$ . Then for a scalar  $a$ :

$$\|ax\| = |ax|/(1 + |ax|) = |a| \cdot |x|/(1 + |a| \cdot |x|),$$

which we can see does not necessarily equal  $|a| \cdot \|x\|$ , and therefore the “proposed” norm is not really a norm. The cause for this is the fact that  $\sigma(ax, ay) \neq |a|\sigma(x, y)$  in general, for similar reasons.

**Exercise 18**

Show that  $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$  for any  $x \in \mathbb{R}^n$ . Also check that  $\|x\|_1 \leq n\|x\|_\infty$  and  $\|x\|_1 \leq \sqrt{n}\|x\|_2$ .

*Solution.*

We have that  $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ , while  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ . By definition, there must exist at least one  $j \in \{1, 2, \dots, n\}$  such that  $|x_j| = \max_{1 \leq i \leq n} |x_i|$ . In addition, the square is an increasing function for  $x \geq 0$ , which would mean that  $(\max_{1 \leq i \leq n} |x_i|)^2 = \max_{1 \leq i \leq n} |x_i|^2$ . We therefore have that:

$$\sum_{i=1}^n \|x_i\|^2 = (\max_{1 \leq i \leq n} |x_i|)^2 + X,$$

for  $X \geq 0$  (the sum of the absolute values of the remaining coordinates). Thus  $(\|x\|_2)^2 \geq (\|x\|_\infty)^2$ , and by taking square roots we have the first desired result.

We also have that:

$$(\|x\|_1)^2 = \left( \sum_{i=1}^n |x_i| \right) \left( \sum_{i=1}^n |x_i| \right) \geq \sum_{i=1}^n |x_i|^2 = (\|x\|_2)^2,$$

which we obtain by observing that the product results in a sum of non-negative terms over all pairs of indices  $i, j \in \{1, 2, \dots, n\}$ , which of course includes all terms for which  $i = j$ . Taking square roots yields the second desired result.

For the third result, observe that for each  $i$ ,  $|x_i| \leq \max_{1 \leq i \leq n} |x_i| = \|x\|_\infty$ . This means of course that:

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leq n\|x\|_\infty$$

Lastly, we consider the following “trick” for the last part. For a given  $x = (x_1, x_2, \dots, x_n)$ , consider the vectors  $y, z \in \mathbb{R}^{n^2}$ :

$$y = \begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \\ \vdots \\ |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix}, z = \begin{pmatrix} |x_1| \\ |x_1| \\ \vdots \\ |x_1| \\ |x_n| \\ |x_n| \\ \vdots \\ |x_n| \end{pmatrix}$$

We have that  $\|z\|_2 = \|y\|_2 = \sqrt{\sum_{i=1}^n n|x_i|^2} = \sqrt{n}\|x\|_2$ , while  $\langle y, z \rangle = \sum_{i=1}^n \sum_{j=1}^n |x_i| \cdot |x_j| = (\sum_{i=1}^n |x_i|)^2 = (\|x\|_1)^2$ . Applying the Cauchy-Schwarz inequality on  $y, z$ , we then have that:

$$\langle y, z \rangle \leq \|y\| \cdot \|z\| \implies (\|x\|_1)^2 \leq \sqrt{n}\|x\|_2 \sqrt{n}\|x\|_2,$$

which, once more by taking square roots, gives us the last desired result.

### Exercise 19

Show that we have  $\sum_{i=1}^n x_i y_i = \|x\|_2 \|y\|_2$  (equality in the Cauchy-Schwarz inequality) if and only if  $x, y$  are proportional, that is, if and only if  $x = ay$  or  $y = ax$  for some  $a \geq 0$ .

*Solution.*

$\Leftarrow$  : Suppose  $x = ay$  for some  $a \geq 0$  (everything is symmetric for  $y = ax$ ). Then:

$$\|x\|_2 \|y\|_2 = \|ay\|_2 \|y\|_2 = |a| \cdot \|y\|_2^2,$$

where we used the “scalar multiplication”/positive homogeneity property of the norm. Furthermore:

$$\sum_{i=1}^n x_i y_i = \sum_{i=1}^n (ay_i) y_i = a \sum_{i=1}^n y_i^2 = a \|y\|_2^2,$$

by the definition of the 2-norm on sequences, which yields the desired equality.

$\Rightarrow$  : First, if  $y = 0$  then this is trivially true. If  $y \neq 0$ , it’s also the case that  $\|y\|_2 \neq 0$  and we have the following. Consider the beginning of the proof of the Cauchy-Schwarz inequality:

$$0 \leq \|x + ty\|_2^2 = \|x\|_2^2 + 2t\langle x, y \rangle + t^2\|y\|_2^2,$$

for any  $t \in \mathbb{R}$ . As we saw, the discriminant here is  $\Delta = (2\langle x, y \rangle)^2 - 4\|x\|_2^2\|y\|_2^2 = 0$ , by our hypothesis that  $\sum_{i=1}^n x_i y_i = \|x\|_2 \|y\|_2$  (since  $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ ). Then this means that the corresponding second-degree polynomial of  $t$  has a unique solution  $a$ :

$$a = \frac{-2\langle x, y \rangle}{2\|y\|_2^2} = -\frac{\langle x, y \rangle}{\|y\|_2^2}$$

By the above equation, it must also be the case that:

$$0 = \|x + ay\|_2^2,$$

which, by the properties of the norm, means that by necessity  $x = -ay$ . Now we only need to show that  $a \leq 0$ . Observe that the denominator of  $a$  is clearly positive as the norm of  $y$ . The numerator equals, by definition,  $\sum_{i=1}^n x_i y_i = \|x\|_2 \|y\|_2$  (by the hypothesis), so it’s clearly also non-negative. The minus sign in front of the fraction makes  $a$  non-positive, which completes the proof.

**Exercise 21**

Recall that we defined  $l_1$  to be the collection of all absolutely summable sequences under the norm  $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$ , and we defined  $l_{\infty}$  to be the collection of all bounded sequences under the norm  $\|x\|_{\infty} = \sup_{n \geq 1} |x_n|$ . Fill in the details showing that each of these spaces is in fact a normed vector space.

*Solution.*

For each of the two proposed norms, we need to show that they are in fact norms, and also that the sequences which converge under them form a vector space. We begin with showing that  $l_1$  defines a norm:

- It's obvious from the definition that  $\|x\|_1 \geq 0$  whenever it exists, as a sum of non-negative terms.
- For any scalar  $a$ , we have that if for a sequence  $x$  the series  $\sum_{n=1}^{\infty} |x_n|$  converges to  $\|x\|_1$ , then by well-known properties of limits, the series  $\sum_{n=1}^{\infty} |ax_n|$  will converge to  $|a| \cdot \|x\|_1$ .
- If  $\sum_{i=0}^{\infty} |x_n|$  converges to zero, then since all terms are strictly non-negative, it must be the case that all of them are zero, thus that  $\|x\|_1 = 0$  implies  $x = 0$ . The converse is obvious.
- We now need to prove the triangle inequality. Pick  $x, y \in l_1$  and any  $n > 0$ . Then, by the triangle inequality for absolute values:

$$\sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|$$

Notice that as  $n \rightarrow \infty$ , the two sums on the RHS converge to  $\|x\|_1, \|y\|_1$  respectively. This imposes a bound on the series corresponding to the LHS, whose partial sums are also non-decreasing. Hence, the LHS also converges, and in fact the inequality holds at infinity, yielding  $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ .

Therefore,  $\|x\|_1$  is a norm. The third point above showed that  $0 \in l_1$ , while the triangle inequality proof showed that  $l_1$  is closed under addition. Lastly, the second point above showed that  $l_1$  is closed under scalar multiplication. Therefore, we've already shown that  $l_1$  is a vector space, and thus  $(l_1, \|\cdot\|_1)$  is a normed vector space.

Now we examine  $\|\cdot\|_{\infty}$  in a similar fashion.

- Again, from the definition it is obvious that whenever  $x$  is a bounded sequence,  $\|x\|_{\infty} \geq 0$ .
- For any scalar  $a$ , and any bounded sequence  $x$ , the sequence  $(|ax_i|)$  will have as supremum  $|a| \cdot \|x\|_{\infty}$ . If this were not the case, one would get a contradiction for  $x$  by dividing with  $|a|$  (and if  $a = 0$ , then the supremum of  $(|ax_i|)$  is obviously zero).
- It's clear that  $\|x\|_{\infty}$  is zero iff  $x$  is the zero sequence, since the supremum of a set of non-negative numbers is zero iff all of them are zero.
- Lastly, the triangle inequality in this case arises as follows. Pick any  $x, y \in l_{\infty}$ . Then, for any  $i$  we have that  $|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_{\infty} + \|y\|_{\infty}$  by the definition of the infinity norm. But then this sum constitutes an upper bound for  $|x_i + y_i|$  for all  $i$ , which means that by definition the supremum  $\sup_i |x_i + y_i|$  both exists and is such that  $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$ .

The above shows both that  $\|\cdot\|_{\infty}$  is a norm, and that  $l_{\infty}$  is a vector space, thus that  $(l_{\infty}, \|\cdot\|_{\infty})$  is a normed vector space.

### 3.3 More Inequalities

**Exercise 24**

The conclusion of Lemma 3.7 (Hölder's inequality) also holds in the case  $p = 1$  and  $q = \infty$ . Why?

*Solution.*

As a reminder, we say that a sequence  $y$  is in  $l_\infty$  if  $y$  is bounded above, and in that case we define  $\|y\|_\infty = \sup_n |y_n|$ . Let us first formally state what Hölder's inequality would assert in this case:

Given  $x \in l_1$  and  $y \in l_\infty$ , we have  $\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_1 \|y\|_\infty$ .

Indeed, pick any  $n \geq 1$ . We then have that:

$$\sum_{i=1}^n |x_i y_i| = \sum_{i=1}^n |x_i| \cdot |y_i| \leq \sum_{i=1}^n |x_i| \cdot \sup_k |y_k| = \sup_k |y_k| \cdot \sum_{i=1}^n |x_i| \leq \|y\|_\infty \cdot \|x\|_1,$$

where we used the definition of the supremum and the fact that  $\|x\|_1$  is well-defined. Since the partial sums are non-decreasing and bounded, the LHS converges and the inequality (i.e., Hölder's) holds for the infinite series as well.

**Exercise 25**

The same techniques can be used to show that  $\|f\|_p = (\int_0^1 |f(t)|^p dt)^{1/p}$  defines a norm on  $C[0, 1]$  for any  $1 < p < \infty$ . State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for  $p = 1, q = \infty$ ?)

*Solution.*

We begin by noting that due to the absolute value, the functions being integrated are always non-negative. For any  $t \in (0, 1)$ , we can thus apply Young's inequality on  $a = \frac{|f(t)|}{\|f\|_p}, b = \frac{|g(t)|}{\|g\|_q}$ :

$$\frac{|f(t)|}{\|f\|_p} \cdot \frac{|g(t)|}{\|g\|_q} \leq \frac{1}{p} \cdot \frac{|f(t)|^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{|g(t)|^q}{\|g\|_q^q}$$

Since this is true for all  $t \in (0, 1)$ , we can integrate both sides wrt.  $t$  to obtain:

$$\begin{aligned} \int_0^1 \frac{|f(t)|}{\|f\|_p} \cdot \frac{|g(t)|}{\|g\|_q} dt &\leq \frac{1}{p} \cdot \int_0^1 \frac{|f(t)|^p}{\|f\|_p^p} dt + \frac{1}{q} \cdot \int_0^1 \frac{|g(t)|^q}{\|g\|_q^q} dt \implies \\ \frac{1}{\|f\|_p \cdot \|g\|_q} \int_0^1 |f(t)g(t)| dt &\leq \frac{1}{p} \cdot \frac{1}{\|f\|_p^p} \int_0^1 |f(t)|^p dt + \frac{1}{q} \cdot \frac{1}{\|g\|_q^q} \int_0^1 |g(t)|^q dt = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

This yields  $\int_0^1 |f(t)g(t)| dt \leq \|f\|_p \cdot \|g\|_q$ , which would be the equivalent of Hölder's inequality.

Now, Minkowski's inequality would state firstly that for any  $1 < p < \infty$ , if the  $p$ -norm exists for  $f, g \in C[0, 1]$ , it also exists for  $f + g$ , and secondly that  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

For the analogue of Minkowski's inequality ("if  $f, g \in C[0, 1]$  and  $\|f\|_p, \|g\|_p$  both exist, then  $\|f + g\|_p$  exists and  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ "), we first prove an analogue of Lemma 3.5 that allows us to immediately obtain that  $\|f + g\|_p$  exists. More specifically, for any  $t \in (0, 1)$ , from Lemma 3.5 applied on  $a = |f(t)|, b = |g(t)|$ , we have that:

$$(|f(t)| + |g(t)|)^p \leq 2^p (|f(t)|^p + |g(t)|^p)$$

By the triangle inequality of the absolute value, we also have that  $|f(t) + g(t)|^p \leq (|f(t)| + |g(t)|)^p$  since  $p > 1$ . Therefore, since these hold for any  $t \in (0, 1)$ :

$$\begin{aligned} |f(t) + g(t)|^p \leq 2^p (|f(t)|^p + |g(t)|^p) &\implies \int_0^1 |f(t) + g(t)|^p dt \leq 2^p \left( \int_0^1 |f(t)|^p dt + \int_0^1 |g(t)|^p dt \right) \\ &\implies \|f + g\|_p^p \leq 2^p (\|f\|_p^p + \|g\|_p^p), \end{aligned}$$

which imposes a bound on  $\|f + g\|_p$ , thus showing it exists.

Now, following the proof in the book, we observe the following regarding  $\|f\|_p^{p-1}$ , for  $1/p + 1/q = 1$ :

$$\|f^{p-1}\|_q = \left( \int_0^1 |f^{q(p-1)}(t)| dt \right)^{1/q} = \left( \int_0^1 |f(t)|^p dt \right)^{1/q} = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{p-1}{p}} = \|f\|_p^{p-1}$$

Now, for any  $t \in (0, 1)$ :

$$|f(t) + g(t)|^p = |f(t) + g(t)| \cdot |f(t) + g(t)|^{p-1} \leq |f(t)| \cdot |f(t) + g(t)|^{p-1} + |g(t)| \cdot |f(t) + g(t)|^{p-1},$$

which means we can integrate to obtain that:

$$\int_0^1 |f(t) + g(t)|^p dt \leq \int_0^1 |f(t)| \cdot |f(t) + g(t)|^{p-1} dt + \int_0^1 |g(t)| \cdot |f(t) + g(t)|^{p-1} dt$$

Now, define  $q$  such that  $1/p + 1/q = 1$ , and by Hölder's inequality and the observation above:

$$\begin{aligned} \int_0^1 |f(t) + g(t)|^p dt &\leq \|f\|_p \cdot \|(f + g)^{p-1}\|_q + \|g\|_p \cdot \|(f + g)^{p-1}\|_q \leq \|f\|_p \cdot \|f + g\|_p^{p-1} + \|g\|_p \cdot \|f + g\|_p^{p-1} \\ &\implies \|f + g\|^p \leq \|f + g\|_p^{p-1} (\|f\|_p + \|g\|_p), \end{aligned}$$

from which Minkowski's inequality follows directly. For the case  $p = 1, q = \infty$ , we have that for any  $t \in (0, 1)$ :

$$\begin{aligned} |f(t)g(t)| &= |f(t)| \cdot |g(t)| \leq |f(t)| \cdot \max_{0 \leq t \leq 1} |g(t)| = |f(t)| \cdot \|g\|_\infty \\ &\implies \int_0^1 |f(t)g(t)| dt \leq \|g\|_\infty \int_0^1 |f(t)| dt = \|g\|_\infty \cdot \|f\|_1, \end{aligned}$$

showing that Hölder's inequality does indeed hold again.

### Exercise 26

Given  $a, b > 0$ , show that  $\lim_{p \rightarrow \infty} (a^p + b^p)^{1/p} = \max\{a, b\}$ . [Hint: If  $a < b$  and  $r = a/b$  show that  $(1/p) \log(1 + r^p) \rightarrow 0$  as  $p \rightarrow \infty$ .] What happens as  $p \rightarrow 0$ ? as  $p \rightarrow -1$ ? as  $p \rightarrow -\infty$ ?

*Solution.*

First, if  $a = b$ , the statement is obvious: the quantity inside the limit simplifies to  $(2a^p)^{1/p} = 2^{1/p}a$ , and as  $p \rightarrow \infty, 1/p \rightarrow 0$  thus  $2^{1/p}a \rightarrow 1a = a = \max\{a, b\}$ . Therefore we can assume from now on that  $a < b$ , and, as indicated in the hint, set  $r = a/b < 1$ . Since  $r < 1$ , we have that  $\lim_{p \rightarrow \infty} r^p \rightarrow 0$  (an exponential with a base less than 1). Therefore, by standard limit rules,  $\lim_{p \rightarrow \infty} \log(1 + r^p) = \log(1) = 0$ . Furthermore  $\lim_{p \rightarrow \infty} \frac{1}{p} = 0$ , which means  $\lim_{p \rightarrow \infty} (1/p) \log(1 + r^p) = 0$ . Now we apply the following “trick”:

$$(a^p + b^p)^{1/p} = e^{\log(a^p + b^p)^{1/p}} = e^{\frac{\log(a^p + b^p)}{p}} = e^{\frac{\log((rb)^p + b^p)}{p}} = e^{\frac{\log(b^p) + \log(1 + r^p)}{p}} = e^{\log(b) + \frac{\log(1 + r^p)}{p}}$$

Clearly, what we showed based on the hint allows us to take limits on both sides and easily obtain that:

$$\lim_{p \rightarrow \infty} (a^p + b^p)^{1/p} = e^{\log(b)} = b = \max\{a, b\}$$

Now, for the case of  $p \rightarrow 0$ , if  $a = b$ ,  $\lim_{p \rightarrow 0} (a^p + b^p)^{1/p} = \lim_{p \rightarrow 0} (2a^p)^{1/p} = \lim_{p \rightarrow 0} (2^{1/p}a)$ , and as  $p \rightarrow 0+$  this quantity will tend to positive infinity, whereas if  $p \rightarrow 0-$  it will tend to zero, thus the limit does not exist. If  $a < b$ , based on the above observation what interests us is  $\lim_{p \rightarrow 0} \frac{\log(1 + r^p)}{p}$ . In this case,  $r^p \rightarrow 1$  as  $p \rightarrow 0$ , and thus the numerator tends to  $\log(2)$ , thus the entire fraction tends to positive infinity as  $p \rightarrow 0+$  and negative infinity as  $p \rightarrow 0-$ , which means the limit does not exist.

The expression inside the limit is continuous as a function of  $p$  at  $-1$ , since  $a, b > 0$ . Therefore  $\lim_{p \rightarrow -1} (a^p + b^p)^{1/p} = (\frac{1}{a} + \frac{1}{b})^{-1} = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}$ .

Lastly, for  $p \rightarrow -\infty$ , for  $a = b$  we are interested in  $\lim_{p \rightarrow -\infty} (2^{1/p}a)$ , which is easily seen to equal  $a$ . For  $a < b$ , we are interested in  $\lim_{p \rightarrow -\infty} \frac{\log(1 + r^p)}{p}$ . Because  $0 < r < 1$ , the quantity inside the logarithm will

tend to positive infinity, whereas the denominator tends to negative infinity. Applying L'Hospital's rule and calling this limit  $L$  we have that:

$$L = \lim_{p \rightarrow -\infty} \frac{\log(r)r^p}{1+r^p} = \lim_{p \rightarrow -\infty} \frac{\log(r)}{1+\frac{1}{r^p}}$$

Here the numerator is constant, whereas since  $0 < r < 1, r^p \rightarrow \infty$ , and thus the denominator will tend to 1. This means that  $L = \log(r) = \log(a/b) = \log(a) - \log(b)$ . Going back to our original limit we would have that  $\lim_{p \rightarrow -\infty} (a^p + b^p)^{1/p} = e^{\log(b) + \log(a) - \log(b)} = a = \min\{a, b\}$ , which we can see was also true for  $a = b$ .

**Exercise - Unlisted; Arose from a discussion of exercise 26**

- a) Prove that if  $1 \leq p \leq q \leq \infty$ , then  $l^p \subset l^q$ .
- b) If  $x \in l_p \subset l_q$  for  $1 \leq p \leq q \leq \infty$ , then  $\|x\|_p \geq \|x\|_q$ .
- c) If  $x \in l^{p_0}$  for some  $p_0$ , prove that  $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$ .

*Solution.*

a) Let  $x$  be a sequence in  $l^p$ . If  $p = q$ , then the statement is obvious, so we continue the proof assuming that  $p \neq q$ . In the case where  $q = \infty$ , we know that if, for some  $1 \leq p < \infty$ ,  $\|x\|_p$  exists, then the sequence of partial sums corresponding to  $(|x_i|^p)$  must be bounded and non-decreasing. But then the same must hold true for the sequence  $(|x_i|)$ , which leads us to conclude that  $x$  is in fact bounded, and thus has a supremum. This means then that  $x \in l_q$ , since  $q = \infty$  and the infinity norm is defined as the supremum of absolute values.

Now, in the remaining case we have that  $1 \leq p < q$  and both are real numbers. Note that, by using e.g. the Archimedean property of  $\mathbb{R}$ , we can write  $q = np + \epsilon$ , where  $n$  is a positive integer and  $\epsilon > 0$ . Consider now examining the first  $m$  terms of  $x$  in the following way:

$$\sum_{i=1}^m |x_i|^q = \sum_{i=1}^m |x_i|^{np+\epsilon} = \sum_{i=1}^m |x_i|^{np} \cdot |x_i|^\epsilon$$

Recall that we already showed that if  $x \in l_p$ , then  $|x_i|$  are bounded above, and thus we can say that for every  $i, |x_i|^\epsilon \leq S^\epsilon, S = \sup_j |x_j|$ . Furthermore, since  $n$  is a positive integer, we have that:

$$\sum_{i=1}^m |x_i|^{np} \leq \left( \sum_{i=1}^m |x_i|^p \right)^n,$$

since if one expands the power of the RHS, we get at least all terms of the LHS plus possibly more, all of which are non-negative. Combining these two facts, we have that:

$$\sum_{i=1}^m |x_i|^q \leq \left( \sum_{i=1}^m |x_i|^p \right)^n S^\epsilon \leq \|x\|_p^{pn} S^\epsilon,$$

which we can safely conclude since the  $p$ -norm exists. But then the partial sums are bounded above for each  $m$  and are non-decreasing, meaning that the LHS converges as  $m \rightarrow \infty$ , and this equals precisely  $\|x\|_q^q$ . Thus  $x$  is indeed also in  $l^q$ .

b) Consider a sequence  $x = (x_1, x_2, \dots) \in l_p \subset l_q$ . The statement is obvious if  $p = q$ , and also if  $x$  is the zero sequence. Therefore from now on we assume  $p < q, x \neq 0$ , which means also  $\|x\|_p > 0, \|x\|_q > 0$ . In the case where  $q = \infty$ , we have that  $\|x\|_q = \sup_i |x_i|$ . For any  $m > 0$ , we have that:

$$\sum_{i=1}^m |x_i|^p \geq \max_{1 \leq i \leq m} |x_i|^p$$

If we take limits on both sides, and by thinking about the  $\epsilon$ -based definition of the supremum, we can see that this yields  $\|x\|_p^p \geq \|x\|_\infty^p \implies \|x\|_p \geq \|x\|_\infty$ .

In the case where  $p < q < \infty$ , consider the following. Let  $y$  be the sequence formed by  $y_i = \frac{x_i}{\|x\|_p}$ . Notice first that  $\|y\|_p = 1$ . Notice also that for any  $i, |y_i|^p \leq 1$ , since  $\|y\|_p = 1 \implies \sum_{i=1}^\infty |y_i|^p = 1^p$ , so any

individual term of the series must be at most 1. Since  $q > p \geq 1$ , we have that it must be the case that  $q = rp$  for some  $r > 1$ . Then, by using properties of powers, for any  $i$ :

$$|y_i|^p \leq 1 \implies |y_i|^{rp} \leq |y_i|^p \implies |y_i|^q \leq |y_i|^p$$

This in turn implies that for any  $m > 0$ :

$$\sum_{i=1}^m |y_i|^q \leq \sum_{i=1}^m |y_i|^p \leq \|y\|_p^p = 1$$

Since this holds for any  $m$ , it also holds at infinity, meaning that  $\|y\|_q^q \leq 1 = \|y\|_p^q$ , and thus by taking  $q$ -roots we obtain that  $\|y\|_q \leq \|y\|_p$ . But then by using the definition of  $y$ :

$$\|y\|_q \leq \|y\|_p \implies \left\| \frac{x}{\|x\|_p} \right\|_q \leq \left\| \frac{x}{\|x\|_p} \right\|_p \implies \|x\|_q \leq \|x\|_p,$$

where we used the “multiplication by scalar” property of norms. This completes the proof.

c) First of all, we observe from part (a) that since  $\|x\|_{p_0}$  exists for some  $p_0$ , it will also be the case that  $x \in l^p$  for all  $p \geq p_0$ , as well as that  $x \in l^\infty$ . We now have the following:

$$\lim_{p \rightarrow \infty} \frac{\|x\|_p}{\|x\|_\infty} = \lim_{p \rightarrow \infty} e^{\log\left(\frac{\|x\|_p}{\|x\|_\infty}\right)}$$

We focus on the exponent:

$$\lim_{p \rightarrow \infty} \log\left(\frac{\|x\|_p}{\|x\|_\infty}\right) = \lim_{p \rightarrow \infty} \log\left(\frac{(\sum_{i=1}^{\infty} |x_i|^p)^{1/p}}{(\|x\|_\infty^p)^{1/p}}\right) = \lim_{p \rightarrow \infty} \frac{1}{p} \cdot \log\left(\sum_{i=1}^{\infty} \frac{|x_i|^p}{\|x\|_\infty^p}\right)$$

Now we make the observation that since the series corresponding to  $\|x\|_p^p$  converges, the individual terms tend to zero. This means that there must exist at least one  $j$  such that  $|x_j|$  equals the supremum of  $x$ . This is because the only other possibility would be for  $|x_i|$  to tend to their supremum without achieving it, but then the series would not converge. Therefore, the argument of the logarithm is at least 1 (since  $\frac{|x_j|^p}{\|x\|_\infty^p} = 1$ ). Additionally, the existence of  $\|x\|_p$  implies that the argument does not go to infinity. Thus, the numerator of the last limit above tends to  $L > 1$  and the denominator to positive infinity, which means that the limit tends to zero. But then:

$$\lim_{p \rightarrow \infty} \frac{\|x\|_p}{\|x\|_\infty} = e^0 = 1,$$

which concludes the proof since  $\|x\|_\infty$  is constant with respect to the limit variable.

### Exercise - Unlisted; Arose from a discussion of exercise 26

Recall that for a continuous function  $f \in C([0, 1])$  we define

$$\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p}, \|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$$

Show that:

- a) If  $f \in C[0, 1]$ , then for  $1 \leq p \leq q \leq \infty$ ,  $\|f\|_1 \leq \|f\|_p \leq \|f\|_q \leq \|f\|_\infty$ .
- b) If  $f \in C[0, 1]$ , then  $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$ .

*Solution.*

a) We begin by showing that for any real number  $p \geq 1$ ,  $\|f\|_p \leq \|f\|_\infty$ . We do this as follows. First, we have that:

$$\|f\|_p^p = \int_0^1 |f(x)|^p dx$$

Since  $p \geq 1$ , the  $p$ -th power is a non-decreasing function, and we know that by definition,  $|f^p(x)| \leq \max_{0 \leq x \leq 1} |f^p(x)|$  for any  $x \in [0, 1]$ . By integrating both sides we have that:

$$\int_0^1 |f^p(x)| dx \leq \max_{0 \leq x \leq 1} |f^p(x)| \implies \|f\|_p^p \leq \|f\|_\infty^p$$

By taking  $p$ -roots, we have the desired inequality. Now consider any two real numbers  $p, q \geq 1$  such that  $p < q$ . This means that there exists  $r > 1$  such that  $q = rp$ . Let  $g(x) = f^p(x)$  and set  $r'$  to be the number satisfying  $1/r + 1/r' = 1$ . We now apply Hölder's inequality for the functions  $g$  and  $h(x) = 1$ :

$$\int_0^1 |g(x) \cdot 1| dx \leq \|g\|_r \cdot \|1\|_{r'} \implies \int_0^1 |f^p(x)| dx \leq \left( \int_0^1 |f^{rp}(x)| \right)^{(1/r)} \implies \left( \int_0^1 |f^p(x)| dx \right)^r \leq \int_0^1 |f^{rp}(x)| dx,$$

where for the last step we used the fact that  $r > 1$ . Continuing:

$$\begin{aligned} \left( \int_0^1 |f^p(x)| dx \right)^r &\leq \int_0^1 |f^{rp}(x)| dx \implies \left( \int_0^1 |f^p(x)| dx \right)^{q/p} \leq \int_0^1 |f^q(x)| dx \\ &\implies \left( \int_0^1 |f^p(x)| dx \right)^{1/p} \leq \left( \int_0^1 |f^q(x)| dx \right)^{1/q} \implies \|f\|_p \leq \|f\|_q \end{aligned}$$

This completes the proof of the remaining inequalities, thus establishing that for  $1 \leq p \leq q \leq \infty$ ,  $\|f\|_1 \leq \|f\|_p \leq \|f\|_q \leq \|f\|_\infty$ .

b) From part (a) we already know that as  $p \rightarrow \infty$ ,  $\|f\|_p$  always exists, and in fact is bounded above by  $\|f\|_\infty$  and is non-decreasing (since for  $p \leq q$ ,  $\|f\|_p \leq \|f\|_q$ ). This means that  $\lim_{p \rightarrow \infty} \|f\|_p$  exists, and will equal the supremum of the set  $S = \{\|f\|_p, p \geq 1\}$ . We thus only need to show that this supremum is in fact  $\|f\|_\infty$ . Since we already know that this constitutes an upper bound for  $S$ , assume that the supremum of  $S$  is  $M < \|f\|_\infty$ . Then, by the definition of the infinity norm, there exists  $x_0 \in [0, 1]$  such that  $|f(x_0)| = \|f\|_\infty > M$ . For the sake of simplicity, we shall assume that  $x_0 \in (0, 1)$ : as will become clear, in the “edge cases” the only thing that changes is that some quantities lack a factor of 2. Set now  $\epsilon = |f(x_0)| - M > 0$ . Since  $f$  is continuous, there must exist  $\delta > 0$  such that:

$$|x - x_0| < \delta \implies |f(x_0)| - |f(x)| < \epsilon,$$

where the absolute value in the second inequality can be omitted since  $|f(x_0)|$  is the maximum value of  $|f|$ . Now, this can be rewritten as  $|f(x)| > |f(x_0)| - \epsilon = |f(x_0)| - |f(x_0)| + M = M$ , which means that in the interval  $(x_0 - \delta, x_0 + \delta)$ ,  $|f(x)| > M$ . Therefore, for all  $x$  in this interval we have that  $|f(x)| > M \implies |f^p(x)| > M^p$ . By integrating:

$$\int_{x_0-\delta}^{x_0+\delta} |f^p(x)| dx > 2M^p \delta \implies \left( \int_{x_0-\delta}^{x_0+\delta} |f^p(x)| dx \right)^{1/p} > (2\delta)^{1/p} M$$

We now have that  $\|f\|_p$  is greater than or equal to the LHS here, since  $(x_0 - \delta, x_0 + \delta) \subset (0, 1)$ . Furthermore, as  $p \rightarrow \infty$ , the RHS tends to  $M$ , since  $\delta$  is constant. By taking limits, we can thus obtain that:

$$\lim_{p \rightarrow \infty} \|f\|_p \geq M$$

Now we observe that we can repeat the entirety of the argument above for some  $N$  with  $M < N < \|f\|_\infty$ . But then this means also that  $\lim_{p \rightarrow \infty} \|f\|_p \geq N$ , and then the  $\epsilon$ -based limit definition would allow us to find  $p$  such that  $\|f\|_p > M$ , which contradicts the defining property of  $M$  as the supremum of all  $\|f\|_p$ . Therefore, we have arrived at a contradiction, and thus it must be the case that  $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$ .

### 3.4 Limits in Metric Spaces

**Exercise 30**

If  $A \subset B$ , show that  $\text{diam}(A) \leq \text{diam}(B)$ .

*Solution.*

Consider the definition of  $\text{diam}(A) : \text{diam}(A) = \{\sup\{d(a, b) : a, b \in A\}\}$ . Because  $A \subset B$ , we have that any two  $a, b \in A$  also belong in  $B$ . Therefore, the set  $S_1$  over which the diameter is computed for  $B$  is a superset of the set  $S_2$  over which the diameter is computed for  $A$ . But then exercise 2 of Chapter 1 guarantees that  $\sup S_1 \leq \sup S_2$ , which means precisely that  $\text{diam}(A) \leq \text{diam}(B)$ .

**Exercise 32**

In a normed vector space  $(V, \|\cdot\|)$  show that  $B_r(x) = x + B_r(0) = \{x + y : \|y\| < r\}$  and that  $B_r(0) = rB_1(0) = \{rx : \|x\| < 1\}$ .

*Solution.*

Let us call  $S = \{x + y : \|y\| < r\}$ , in which case we asked to show that  $B_r(x) = S$ . First, let  $z \in B_r(x)$ . By definition, this means that  $d(x, z) < r$ . Recall that in a normed vector space we have that  $\|z - x\| = d(x, y)$  (unless a different metric is explicitly specified). Observe then that we can write  $z = x + (z - x)$ , where  $\|z - x\| = d(z, x) = d(x, z) < r$ . By setting  $y = z - x$ , we obtain that  $z \in S$ . Therefore,  $B_r(x) \subset S$ .

In the other direction, suppose  $z \in S$ , which means that there exists  $y, \|y\| < r$  such that  $z = x + y$ . Then we observe that  $d(x, z) = \|z - x\| = \|x + y - x\| = \|y\| < r$ . By definition, this means that  $z \in B_r(x)$ , and thus that  $S \subset B_r(x)$ , meaning that in fact  $S = B_r(x)$ .

For the second part of the exercise, set  $S = \{rx : \|x\| < 1\}$ . First, suppose  $x \in B_r(0)$ , which means  $\|x\| < r$ . By using the scalar multiplication properties of vector spaces, we can then write  $x = r \cdot \frac{x}{r}$ . Set  $y = \frac{x}{r}$ , in which case  $\|y\| = \left\| \frac{x}{r} \right\| = \frac{1}{|r|} \|x\| < 1$ . This means that  $x$  can be written in the form  $ry, \|y\| < 1$ , thus that  $x \in S$ , thus that  $B_r(0) \subset S$ . Conversely, assume  $x \in S$ , which means  $x = ry, \|y\| < 1$ . But then  $\|x\| = \|ry\| = |r| \cdot \|y\| < r$ , i.e. that  $x \in B_r(0)$ , and thus that  $S \subset B_r(0)$ , completing the proof that  $B_r(0) = S$ .

**Exercise 33**

Limits are unique. [Hint:  $d(x, y) \leq d(x, x_n) + d(x_n, y)$ .]

*Solution.*

Suppose that a sequence  $(x_n)$  in a metric space  $M$  converges to two  $a, b \in M$  such that  $a \neq b$ . By the definition of metrics, we know then that it must be the case that  $d(a, b) > 0$ . Set  $\epsilon = d(a, b) > 0$ , in which case by the definition of the limit in a metric space there must exist  $N_1, N_2$  such that  $d(x_n, a) < \epsilon/4, d(x_n, b) < \epsilon/4$  for  $n \geq N_1, n \geq N_2$  respectively. If we then pick  $n > \max\{N_1, N_2\}$  we have that both of these inequalities hold for  $x_n$ . Recall the triangle inequality for metrics:

$$d(a, b) \leq d(a, x_n) + d(x_n, b) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon = d(a, b)$$

This is a clear contradiction, which means that if a sequence in a metric space has a limit, the limit has to be unique.

**Exercise 34**

If  $x_n \rightarrow x$  in  $(M, d)$ , show that  $d(x_n, y) \rightarrow d(x, y)$  for any  $y \in M$ . More generally, if  $x_n \rightarrow x, y_n \rightarrow y$ , show that  $d(x_n, y_n) \rightarrow d(x, y)$ .

*Solution.*

Pick  $\epsilon > 0$ . Because  $x_n \rightarrow x$ , we know that there exists  $N > 0$  such that  $d(x_n, x) < \epsilon$  for all  $n \geq N$ . By the triangle inequality, for  $n \geq N$  we have that:

$$d(x_n, y) \leq d(x, y) + d(x_n, x) < \epsilon + d(x, y) \implies d(x_n, y) - d(x, y) < \epsilon$$

By exercise 2 of Chapter 3, we also have that:

$$|d(x_n, x) - d(y, x)| \leq d(x_n, y) \implies -d(x_n, y) \leq d(x_n, x) - d(x, y) < \epsilon - d(x, y)$$

$$\implies d(x_n, y) - d(x, y) > -\epsilon$$

Combining these two inequalities we obtain that  $|d(x_n, y) - d(x, y)| < \epsilon$  for all  $n > N$ , which is precisely the definition of  $d(x_n, y) \rightarrow d(x, y)$ .

For the second, more general statement, pick again  $\epsilon > 0$ . Then there exist  $N_1, N_2$  such that  $d(x_n, x) < \frac{\epsilon}{2}, d(y_n, y) < \frac{\epsilon}{2}$  whenever  $n > N_1, n > N_2$  respectively. Set then  $N = \max\{N_1, N_2\}$ . By the triangle inequality for  $n > N$  we have that:

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\implies d(x_n, y_n) - d(x, y) < \epsilon \end{aligned}$$

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + d(x_n, y_n) \\ &\implies d(x, y) - d(x_n, y_n) < \epsilon \implies -\epsilon < d(x_n, y_n) - d(x, y) \end{aligned}$$

Putting together these two inequalities results in  $|d(x_n, y_n) - d(x, y)| < \epsilon$  for  $n > N$ , which is precisely the definition of  $d(x_n, y_n) \rightarrow d(x, y)$ .

### Exercise 35

If  $x_n \rightarrow x$ , then  $x_{n_k} \rightarrow x$  for any subsequence  $(x_{n_k})$  of  $(x_n)$ .

*Solution.*

Since  $x_n \rightarrow x$ , for any given  $\epsilon > 0$  we can always find  $N > 0$  such that for all  $n \geq N, d(x_n, x) < \epsilon$ . Because  $(x_{n_k})$  contains infinite terms of  $(x_n)$  selected in an order which maintains the order of indices, it must be the case that for  $n_k \geq N, d(x_{n_k}, x) < \epsilon$ . We conclude that  $x_{n_k} \rightarrow x$ .

### Exercise 36

A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set  $\{x_n : n \geq 1\}$  is bounded).

*Solution.*

We begin by showing that a convergent sequence is Cauchy. Pick any  $\epsilon > 0$ . Because the sequence converges, say to  $x$ , there exists  $N > 0$  such that whenever  $n \geq N$  it is the case that  $d(x_n, x) < \epsilon/2$ . For any two such  $n_1, n_2 \geq N$  we then have:

$$d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x) + d(x, x_{n_2}) < \epsilon,$$

which is precisely the definition of  $x$  being Cauchy.

Now assume  $(x_n)$  is Cauchy. Pick e.g.  $\epsilon = 1$ , and find  $N > 0$  such that for  $n_1, n_2 \geq N$  it holds that  $d(x_{n_1}, x_{n_2}) < \epsilon$ . Consider the set  $\{d(x_n, x_1), 1 \leq n \leq N\}$ . This has a finite number of non-negative elements, and as such a well-defined non-negative maximum  $M$ . For any  $n > N$  we then have that:

$$d(x_n, x_1) \leq d(x_n, x_N) + d(x_N, x_1) < M + \epsilon,$$

and therefore all elements of the sequence satisfy  $d(x_n, x_1) < M + \epsilon$ , i.e. they are contained in the open ball  $B_{M+\epsilon}(x_1)$ , which is to say that the sequence is bounded.

### Exercise 37

A Cauchy sequence with a convergent subsequence converges.

*Solution.*

Let  $(x_n)$  be a Cauchy sequence with a convergent subsequence  $(x_{n_k}) \rightarrow x$ . Pick any  $\epsilon > 0$ . Because  $(x_{n_k}) \rightarrow x$ , there exists  $N_1 > 0$  such that for  $n_k \geq N_1$  it holds that  $d(x_{n_k}, x) < \epsilon/2$ . Furthermore, because  $(x_n)$  is Cauchy, there exists  $N_2 > 0$  such that for  $i, j \geq N_2$  it holds that  $d(x_i, x_j) < \epsilon/2$ . Set  $N = \max\{N_1, N_2\}$  and pick any  $n > N$ . Then, by the triangle inequality:

$$d(x_n, x) \leq d(x_n, x_{N_1}) + d(x_{N_1}, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

But then this means that  $(x_n)$  converges to  $x$ , which completes the proof.

**Exercise 39**

If every subsequence of  $(x_n)$  has a *further* subsequence that converges to  $x$ , then  $(x_n)$  converges to  $x$ .

*Solution.*

By way of contradiction, assume  $(x_n)$  does not converge to  $x$ . Then there exists  $\epsilon > 0$  such that for all  $N > 0$  there exists  $n \geq N$  such that  $d(x_n, x) \geq \epsilon$ . Consider constructing the following subsequence of  $(x_n)$ : for any  $i > 0$ , the  $i$ -th element of the subsequence equals the first  $x_n$  such that  $d(x_n, x) \geq \epsilon$ ,  $n \geq i$  and  $x_n$  has not been selected before. By the hypothesis above, this is always well-defined.

As a subsequence of  $(x_n)$ , by the hypothesis of the exercise this subsequence will have a further subsequence converging to  $x$ . Notice, however, that by construction all elements of this “further subsequence” are such that  $d(x_{n_{k_l}}, x) \geq \epsilon$ , which contradicts convergence to  $x$ . Therefore,  $(x_n)$  itself must converge to  $x$ .

**Exercise 40**

Here is a positive result about  $l_1$  that may restore your faith in intuition. Given any fixed element  $x \in l_1$ , show that the sequence  $x^{(k)} = (x_1, \dots, x_k, 0, \dots) \in l_1$  (i.e. the first  $k$  terms of  $x$  followed by all 0s) converges to  $x$  in  $l_1$ -norm. Show that the same holds true in  $l_2$ , but give an example showing that it fails (in general) in  $l_\infty$ .

*Solution.*

For any given  $k$ , we make the following observation:

$$\|x - x^{(k)}\|_1 = \sum_{i=1}^k |x_i - x_i| + \sum_{i=k+1}^{\infty} |x_i| = \sum_{i=k+1}^{\infty} |x_i|$$

Notice now that, since  $x \in l_1$ , the RHS here equals  $\|x\|_1 - \sum_{i=1}^k |x_i|$ . Furthermore, because the  $l-1$  norm of  $x$  viewed as an infinite series converges, it must be the case that by selecting  $k$  large enough, the term  $\sum_{i=1}^k |x_i|$  can be made to be arbitrarily close to it, which in other words means that the RHS above can be made arbitrarily small. Of course, this has the immediate consequence that for any  $\epsilon > 0$  we can find  $K > 0$  such that for  $n \geq K$ ,  $\|x - x^{(n)}\|_1 < \epsilon$ , which means that the sequence  $x^{(k)}$  converges to  $x$  under the  $l_1$ -norm.

To show that the same holds for the  $l_2$ -norm, we compute:

$$\|x - x^{(k)}\|_2^2 = \sum_{i=1}^k |x_i - x_i|^2 + \sum_{i=k+1}^{\infty} |x_i|^2 = \sum_{i=k+1}^{\infty} |x_i|^2$$

The RHS here equals  $\|x\|_2^2 - \sum_{i=1}^k |x_i|^2$ . Because the square of the  $l_2$ -norm of  $x$  viewed as an infinite series converges, it must be the case that  $\sum_{i=1}^k |x_i|^2$  gets arbitrarily close to it by picking an appropriate  $k$ . The remaining argument is also the same as for  $l_1$ , which means that  $x^{(k)} \rightarrow x$  under  $l_2$  as well.

For the case of  $l_\infty$ , we consider the sequence formed by  $x_n = 1 - \frac{1}{n}$ . This can be easily shown to have  $\|x\|_\infty = 1$ . However, for  $x^{(k)}$  as defined in the exercise:

$$\|x - x^{(k)}\|_\infty = \sup\{0, x_{k+1}, x_{k+2}, \dots\} = 1$$

The reason this holds is because  $x_n \rightarrow 1$ , and it is thus made clear that  $x^{(k)}$  cannot converge to  $x$  under the  $l_\infty$  norm.

**Exercise 41**

Given  $x, y \in l_2$ , recall that  $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ . Show that if  $x^{(k)} \rightarrow x$  and  $y^{(k)} \rightarrow y$  in  $l_2$ , then  $\langle x^{(k)}, y^{(k)} \rangle \rightarrow \langle x, y \rangle$ .

*Solution.*

We begin first by verifying that if  $a, b, c \in l_2$ , then  $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$ . We have that:

$$\langle a + b, c \rangle = \sum_{i=1}^{\infty} (a_i + b_i) c_i = \sum_{i=1}^{\infty} (a_i c_i + b_i c_i)$$

Notice that because  $a, b, c$  are all in  $l_2$ , we can safely split this series into a sum of the two series corresponding  $\langle a, c \rangle, \langle b, c \rangle$ . Now we move on to the main part of the proof by picking any  $\epsilon > 0$ . To begin with, for any  $k, x^{(k)}, y^{(k)} \in l_2$  so any inner products involving them are well-defined:

$$\begin{aligned} |\langle x, y \rangle - \langle x^{(k)}, y^{(k)} \rangle| &= |\langle x, y \rangle - \langle x, y^{(k)} \rangle + \langle x, y^{(k)} \rangle - \langle x^{(k)}, y^{(k)} \rangle| = |\langle x, y - y^{(k)} \rangle + \langle x - x^{(k)}, y^{(k)} \rangle| \\ &\leq |\langle x, y - y^{(k)} \rangle| + |\langle x - x^{(k)}, y^{(k)} \rangle| \leq \|x\|_2 \cdot \|y - y^{(k)}\|_2 + \|x - x^{(k)}\|_2 \cdot \|y^{(k)}\|_2 \end{aligned}$$

Notice now that because  $y^{(k)} \rightarrow y, x^{(k)} \rightarrow x$ , we can, for this  $\epsilon$ , find  $K > 0$  such that for all  $k \geq K, \|x - x^{(k)}\|_2 < \epsilon, \|y - y^{(k)}\|_2 < \epsilon$ . By the extension of the triangle inequality, it will then also hold that  $\|y^{(k)}\|_2 - \|y\|_2 < \epsilon$ . Note that here we are using the association of the  $l_2$  norm to the corresponding metric in  $l_2$ . Coupled with these observations, the above inequality implies:

$$|\langle x, y \rangle - \langle x^{(k)}, y^{(k)} \rangle| < \|x\|_2 \cdot \epsilon + \epsilon \cdot (\|y\|_2 + \epsilon)$$

Now note that the RHS is a function of  $\epsilon$  that tends to zero as  $\epsilon$  tends to zero, which means (by the “elegance is not required” theorem from Hubbard & Hubbard) that indeed  $\langle x^{(k)}, y^{(k)} \rangle \rightarrow \langle x, y \rangle$ .

### Exercise 42

Two metrics  $d, \rho$  on a set  $M$  are said to be **equivalent** if they generate the same convergent sequences; that is,  $d(x_n, x) \rightarrow 0$  iff  $\rho(x_n, x) \rightarrow 0$ . If  $d$  is any metric on  $M$ , show that the metrics  $\rho, \sigma, \tau$  defined in Exercise 6 are all equivalent to  $d$ .

*Solution.*

We begin with  $\rho(x, y) = \sqrt{d(x, y)}$ . Suppose that some sequence  $x_n$  converges to  $x$  under  $d$ . Then, pick any  $\epsilon > 0$  and find  $N > 0$  such that for  $n \geq N, d(x_n, x) < \epsilon^2$ . We have then that  $\rho(x_n, x) = \sqrt{d(x_n, x)} < \sqrt{\epsilon^2} = \epsilon$ , which of course shows that  $\rho(x_n, x) \rightarrow 0$ , i.e. that  $x_n$  converges to  $x$  under  $\rho$ . Conversely, if  $\rho(x_n, x) \rightarrow 0$  for some  $(x_n), x$ , then for any given  $\epsilon > 0$ , pick  $N > 0$  such that for  $n \geq N, \rho(x_n, x) < \sqrt{\epsilon} \implies \sqrt{d(x_n, x)} < \sqrt{\epsilon} \implies d(x_n, x) < \epsilon$ , which means that  $d(x_n, x) \rightarrow 0$ .

Now, for  $\sigma(x, y) = d(x, y)/(1 + d(x, y))$ , suppose again  $d(x_n, x) \rightarrow 0$  and pick first  $\epsilon < 1$ . Then there exists  $N > 0$  such that for  $n \geq N, d(x_n, x) < \epsilon/(1 - \epsilon) \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x) < \epsilon(1 + d(x_n, x)) \implies \sigma(x_n, x) < \epsilon$ . Notice that for  $\epsilon \geq 1$ , it suffices to pick, say, the  $N > 0$  for which the above holds when e.g.  $\epsilon' = 1/2$ . This shows that  $\sigma(x_n, x) \rightarrow 0$ . Conversely, for  $\sigma(x_n, x) \rightarrow 0$ , for any given  $\epsilon > 0$  there exists  $N > 0$  such that for  $n \geq N, \sigma(x_n, x) < \epsilon \implies d(x_n, x)/(1 + d(x_n, x)) < \epsilon \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x)(1 - \epsilon) < \epsilon$ . Notice that for  $\epsilon < 1$ , we obtain  $d(x_n, x) < \epsilon/(1 - \epsilon)$ , which means that  $d(x_n, x)$  is bound by a function of  $\epsilon > 0$  which tends to zero as  $\epsilon \rightarrow 0$ . It furthermore holds that this function bounds  $d(x_n, x)$  when  $\epsilon < 1$ . Also, for  $\epsilon \geq 1$ , it holds trivially that  $d(x_n, x)/(1 + d(x_n, x)) < \epsilon$ . By defining  $u(\epsilon) = \epsilon/(1 - \epsilon), \epsilon < 1$  and  $u(\epsilon) = 1$  for  $\epsilon \geq 1$ , we have a function that fulfills the conditions of the “elegance is not required” theorem for limits, thus showing that  $d(x_n, x) \rightarrow 0$ .

Lastly, for  $\tau(x, y) = \min\{d(x, y), 1\}$ , if  $d(x_n, x) \rightarrow 0$ , then for  $\epsilon > 1$  it trivially holds that  $\tau(x_n, x) < \epsilon$  for  $n \geq 1$ , while for  $\epsilon \leq 1$  we have that for sufficiently large  $N > 0, d(x_n, x) < \epsilon$  and thus  $\tau(x_n, x) = d(x_n, x) < \epsilon$ , thus showing that  $\tau(x_n, x) \rightarrow 0$ . Conversely, if  $\tau(x_n, x) \rightarrow 0$  then for any given  $\epsilon > 0$ , there exists  $N > 0$  such that for  $n \geq N, \tau(x_n, x) < \epsilon \implies \min\{d(x_n, x), 1\} < \epsilon$ , thus for  $\epsilon < 1$  it has to be that  $d(x_n, x) < \epsilon$ , which is enough to show that  $d(x_n, x) \rightarrow 0$  (since for larger  $\epsilon, N$  can trivially be found by using  $\epsilon' < 1$ ).

### Exercise 43

Show that the usual metric on  $\mathbb{N}$  is equivalent to the discrete metric. Show that any metric on a *finite* set is equivalent to the discrete metric.

*Solution.*

As a reminder, for  $x, y \in \mathbb{N}$  the discrete metric is  $d(x, y) = 1$  iff  $x \neq y$  and  $d(x, y) = 0$  otherwise. The usual metric on  $\mathbb{N}$  is  $\rho(x, y) = |x - y|$ . To show that the two metrics are equivalent, suppose first that  $x_n \rightarrow x$  under  $d$ . In order for this to happen, notice that for any given  $\epsilon > 0$  we must be able to find  $N > 0$  such that for  $n \geq N, d(x_n, x) < \epsilon$ . The definition of  $d$  then dictates that in fact after some  $N, x_n$

are always equal to  $x$ . Then it is clear that  $\rho(x_n, x) = 0$  for  $n \geq N$ , meaning of course that  $\rho(x_n, x) \rightarrow 0$ . Conversely, assume  $x_n \rightarrow x$  under  $\rho$ . Once again, for any given  $\epsilon > 0$  we can find  $N > 0$  such that for  $n \geq N$ ,  $\rho(x_n, x) < \epsilon$ . Choose  $\epsilon = 1/2$ . Then  $\rho(x_n, x) < 1/2 \implies |x_n - x| < 1/2$  means that in fact  $x_n = x$ , since these are natural numbers. Then it is of course true that  $d(x_n, x) \rightarrow 0$  as well.

For the second part of the exercise, let  $M = \{x_1, x_2, \dots, x_N\}$  be a finite set,  $d$  be the discrete metric and  $\rho$  be any metric on it. Suppose the sequence  $(y_n)$  is such that  $d(y_n, x_i) \rightarrow 0$  for some  $x_i \in M$ . By the same observation as above, for  $n \geq K$  for some  $K$  it must hold that  $y_n = x_i$ . Then it must be the case that *any* metric  $\rho$ , purely by being a metric, must satisfy  $\rho(y_n, x_i) = 0$ . This shows that  $\rho(y_n, x_i) \rightarrow 0$ . Conversely, if  $\rho(y_n, x_i) \rightarrow 0$ , then for any  $\epsilon > 0$  there exists  $K > 0$  such that for  $n \geq K$ ,  $\rho(y_n, x_i) < \epsilon$ . Pick then  $\epsilon = \min\{\rho(x_j, x_i), j = 1, 2, \dots, N, j \neq i\}$ , which is well-defined due to  $M$  being finite. Then the fact that  $\rho(y_n, x_i)$  eventually becomes less than  $\epsilon$  implies that it must be the case that after some  $N$ ,  $y_n = x_i$ , such that the metric becomes zero. Consequently, it also holds that  $d(y_n, x_i) \rightarrow 0$ .

#### Exercise 44

Show that the metrics induced by  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$  on  $\mathbb{R}^n$  are all equivalent. [Hint: see exercise 18.]

*Solution.*

Call the corresponding metrics  $d_1, d_2, d_\infty$  respectively. Let  $(x_n)$  be a sequence in  $\mathbb{R}^n$  that converges to  $x$  under  $d_2$ . Pick any  $\epsilon > 0$  and find  $N$  such that  $d_2(x_n, x) < \epsilon$  for  $n \geq N$ . By exercise 18, we have that:

$$d_\infty(x_n, x) = \|x_n - x\|_\infty \leq \|x_n - x\|_2 = d_2(x_n, x) < \epsilon,$$

which shows that  $d_\infty(x_n, x) \rightarrow 0$ . Also by exercise 18, we have that:

$$d_1(x_n, x) = \|x_n - x\|_1 \leq \sqrt{n} \|x_n - x\|_2 = \sqrt{n} d_2(x_n, x) < \sqrt{n} \epsilon,$$

and we once again refer to the “elegance is not required” theorem which guarantees that  $d_1(x_n, x) \rightarrow 0$ . Now suppose  $(x_n)$  converges to  $x$  under  $d_1$ . The bound  $\|x\|_2 \leq \|x\|_1$  from exercise 18 and a process almost identical to the above show that it converges under  $d_2$  as well, and then of course under  $d_\infty$ . Lastly, the bound  $\|x\|_1 \leq n \|x\|_\infty$  shows that convergence under  $d_\infty$  implies convergence under  $d_1$ , thus completing the “implication circle” that indicates that all three metrics are equivalent.

#### Exercise 45

We say that two norms on the same vector space  $X$  are equivalent if the metrics they induce are equivalent. Show that  $\|\cdot\|$  and  $\|\|\cdot\|\|$  are equivalent on  $X$  iff they generate the same sequences tending to 0; that is,  $\|x_n\| \rightarrow 0$  iff  $\|\|x_n\|\| \rightarrow 0$ .

*Solution.*

$\implies$  : We call the induced metrics of  $\|\cdot\|, \|\|\cdot\|\|, d_1$  and  $d_2$  respectively. Suppose  $\|\cdot\|$  is equivalent to  $\|\|\cdot\|\|$ . Pick any  $(x_n)$  such that it converges to 0 under  $d_1$ . This means that  $d(x_n, 0) \rightarrow 0$ , and thus also that  $d_2(x_n, 0) \rightarrow 0$  (by the equivalency of the metrics). Obviously, the argument is symmetric for exchanging the roles of the two norms, which means that  $\|x_n\| \rightarrow 0$  iff  $\|\|x_n\|\| \rightarrow 0$ .

$\impliedby$  : Now suppose that  $\|x_n\| \rightarrow 0$  iff  $\|\|x_n\|\| \rightarrow 0$ , and let  $(y_n)$  be a sequence that converges to  $y$  under  $d_1$ . This means that  $\|y_n - y\| \rightarrow 0$ . In other words, the sequence  $(y_n) - y$  tends 0 under  $\|\cdot\|$ , and thus by our hypothesis also under  $\|\|\cdot\|\|$ , which directly translates to  $d_2(y_n, y) \rightarrow 0$ , i.e.,  $(y_n)$  converges to  $y$  under  $d_2$ . Once again, the argument is exactly symmetric for exchanging the roles of the two norms, and this leads us to conclude that the metrics they induce are equivalent, and thus that the norms are also equivalent.

### Exercise 46

Given two metric spaces  $(M, d)$  and  $(N, \rho)$ , we can define a metric on the product  $M \times N$  in a variety of ways. Our only requirement is that a sequence of pairs  $(a_n, x_n)$  in  $M \times N$  should converge precisely when both coordinate sequences  $(a_n), (x_n)$  converge in  $(M, d), (N, \rho)$  respectively. Show that each of the following define metrics on  $M \times N$  that enjoy this property and that all three are equivalent:

$$\begin{aligned}d_1((a, x), (b, y)) &= d(a, b) + \rho(x, y) \\d_2((a, x), (b, y)) &= (d(a, b)^2 + \rho(x, y)^2)^{1/2} \\d_\infty((a, x), (b, y)) &= \max\{d(a, b), \rho(x, y)\}\end{aligned}$$

Henceforth, any implicit reference to “the” metric on  $M \times N$ , sometimes called the **product metric**, will mean one of  $d_1, d_2, d_\infty$ . Any one of them will serve equally well; use whichever looks most convenient for the argument at hand.

*Solution.*

We begin by showing that  $d_1$  has the property mentioned. First, suppose  $(x_n) \in M, (y_n) \in N$  converge to  $x, y$  respectively under  $d, \rho$ . Pick  $\epsilon > 0$  and  $N, M > 0$  such that for  $n \geq N, m \geq M, d(x_n, x) < \epsilon/2, \rho(y_n, y) < \epsilon/2$ . Then, for the sequence  $(x_n, y_n)$ , for  $n \geq \max\{N, M\}$  we have that:

$$d_1((x_n, y_n), (x, y)) = d(x_n, x) + \rho(y_n, y) < \epsilon,$$

which of course shows that  $(x_n, y_n) \rightarrow (x, y)$  under  $d_1$ .

Conversely, let's assume that  $((x_n, y_n))$  is a sequence of ordered pairs in  $M \times N$  that converges to  $(x, y)$  under  $d_1$ . Then, for any given  $\epsilon > 0$  we can find  $N > 0$  such that for  $n \geq N, d_1((x_n, y_n), (x, y)) < \epsilon \implies d(x_n, x) + \rho(y_n, y) < \epsilon$ . Because a metric always yields non-negative values, each of  $d(x_n, x) < \epsilon, \rho(y_n, y) < \epsilon$  must hold for  $n \geq N$ , which shows that  $d(x_n, x) \rightarrow 0, \rho(y_n, y) \rightarrow 0$ .

For  $d_2$ , observe that not much will change in the proof: in one direction, we simply need to adjust the chosen  $\epsilon$  for  $x_n, y_n$  based on the squares and square roots, and in the other we will merely have to take squares of both sides of an inequality.

For  $d_\infty$ , in one direction it suffices to make both  $d(x_n, x), \rho(y_n, y)$  less than  $\epsilon$ , and then their max also satisfies this. In the other, for a “target”  $\epsilon$ , start from  $d_\infty$  with  $\epsilon$  and observe that the max enforces both  $d(x_n, x), \rho(y_n, y)$  to satisfy this.

Hence we only need to show the equivalency of the the three. Suppose  $(x_n, y_n)$  converges to  $(x, y)$  under any of them. Then we already showed that  $x_n \rightarrow x, y_n \rightarrow y$  under  $d, \rho$  respectively. But we also showed that for any of the other two product metrics, this is enough to guarantee that  $(x_n, y_n)$  converges to  $(x, y)$  under them, thus showing that they are in fact equivalent.

## Chapter 4

# Open Sets and Closed Sets

### 4.1 Closed sets

#### Exercise 1

Show that an “open rectangle”  $(a, b) \times (c, d)$  is an open set in  $\mathbb{R}^2$ . More generally, if  $A, B$  are open in  $\mathbb{R}$ , show that  $A \times B$  is open in  $\mathbb{R}^2$ . If  $A, B$  are closed in  $\mathbb{R}$ , show that  $A \times B$  is closed in  $\mathbb{R}^2$ .

*Solution.*

Call the open rectangle  $S$ , in which case its defining property is  $S = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$ . Let  $p = (x, y) \in S$ , and set  $r = \min\{x - a, b - x, c - y, d - y\}$ . Now let  $p' = (x', y')$  be any point in  $B_r(p)$ . This means that  $\|p' - p\|_2 < r$ . More specifically, this translates to the following inequalities:

$$\begin{aligned}|x' - x| &< x - a \implies x' - x > a - x \implies a < x' \\|x' - x| &< b - x \implies x' - x < b - x \implies x' < b \\|y' - y| &< y - c \implies y' - y > c - y \implies y' > c \\|y' - y| &< d - y \implies y' - y > d - y \implies y' < d\end{aligned}$$

These four properties mean that  $p' \in S$ , and thus we have proved that  $S$  is indeed open.

For the sake of generality, for the remaining exercise we will work with  $A, B$  being subsets of two general metric spaces  $(M, d), (N, \rho)$ , respectively, and then the general result will imply what is asked for here. First of all, we select  $d_\infty$  as the metric of  $M \times N$ , with the definition given in exercise 46 of Chapter 3. Suppose first that  $A, B$  are open, and pick  $(a, b) \in A \times B$ . There exist then open balls  $B_{\epsilon_1}^d(a) \subset A, B_{\epsilon_2}^\rho(b) \subset B$ . Pick  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ , and consider any  $(x, y) \in B_\epsilon^{d_\infty}((a, b))$ . We have that:

$$d_\infty((x, y), (a, b)) < \epsilon \implies \max\{d(x, a), \rho(y, b)\} < \min\{\epsilon_1, \epsilon_2\}$$

Notice that this implies that, more specifically,  $d(x, a) < \epsilon_1, \rho(y, b) < \epsilon_2$ , which means  $x \in B_{\epsilon_1}^d(a) \subset A, y \in B_{\epsilon_2}^\rho(b) \subset B$ . In other words,  $x \in A, b \in B$ , and so  $(x, y) \in A \times B$ , which shows that  $B_\epsilon^{d_\infty}(a, b) \subset A \times B$ , and thus the Cartesian product is also open.

Now suppose that  $A, B$  are closed, which implies  $M \setminus A, N \setminus B$  are open. Consider rewriting  $A \times B$  as follows:

$$\begin{aligned}A \times B &= \{(x, y) \in M \times N | x \in A, y \in B\} = (M \times N) \setminus \{(x, y) \in M \times N | x \notin A \text{ or } y \notin B\} \\&= (M \times N) \setminus ((A^c \times N) \cup (M \times B^c))\end{aligned}$$

Now, from the immediately preceding proof,  $A^c \times N, M \times B^c$  are open as Cartesian products of open sets (a metric space is always open). Their union is thus also open, and then the complement of this union, which we showed equals  $A \times B$ , is thus closed.

**Exercise 3**

Some authors say that two metrics  $d, \rho$  on a set  $M$  are equivalent if they generate the same open sets. Prove this. (Recall that we defined equivalence to mean that  $d, \rho$  generate the same convergent sequences.)

*Solution.*

First suppose that two metrics  $d, \rho$  generate the same open sets. Pick any sequence  $(x_n) \in M$  such that  $d(x_n, x) \rightarrow 0$ , and pick  $\epsilon > 0$ . The ball  $B_\epsilon^\rho(x)$  is open under  $\rho$ , and must thus also be open (although not necessarily a ball) under  $d$ . By convergence of  $x_n \rightarrow x$  under  $d$ , we have that  $(x_n)$  is eventually in  $B_\epsilon^\rho(x)$ , which of course means that we have showed that  $\rho(x_n, x) \rightarrow 0$  as well. Exchanging the roles of  $d, \rho$  completes the proof that they are equivalent under the sequence-based definition.

Conversely, assume that  $d, \rho$  generate the same convergent sequences. Suppose  $S$  is an open set under  $d$ . If  $S$  is the empty set, it is also clearly open under  $\rho$ . Otherwise, suppose for the sake of contradiction that  $S$  is not open under  $\rho$ , which means that there exists  $x \in S$  such that for every  $\epsilon > 0$ ,  $B_\epsilon^\rho(x) \not\subset S$ . More specifically, this means that for a sequence of  $\epsilon_n = \frac{1}{n}$ , we can always find  $x_n \notin S$  such that  $x_n \in B_{\epsilon_n}^\rho(x)$ . But then it becomes clear that  $\rho(x_n, x) \rightarrow 0$ , which means that  $d(x_n, x) \rightarrow 0$  as well. Therefore we have constructed a sequence that has an infinite number of terms that do not belong in  $S$ , and at the same time converges to  $x \in S$  under  $d$ . Because  $S$  is open under  $d$ , this is a contradiction, and we therefore conclude that  $S$  must also be open under  $\rho$ . Exchanging the roles of the two metrics completes the proof in this direction, thus showing that the two definitions are equivalent.

**Exercise 5**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Show that  $\{x : f(x) > 0\}$  is an open subset of  $\mathbb{R}$ , and that  $\{x : f(x) = 0\}$  is a closed subset of  $\mathbb{R}$ .

*Solution.*

Let  $S = \{x : f(x) > 0\}$ . Pick any  $x \in S$ , in which case  $f(x) > 0$ . Let  $\epsilon = f(x)$ , and then because  $f$  is continuous, there must exist  $\delta > 0$  such that:

$$|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

The second inequality implies  $-\epsilon < f(y) - f(x) \implies f(y) > f(x) - \epsilon = 0$ . Notice then that for every  $y \in B_\delta(x)$  we have  $f(y) > 0$ , and thus that  $y \in S$ , which shows that  $S$  is open.

Now let  $T = \{x : f(x) = 0\}$ . Notice that  $T = \mathbb{R} \setminus (\{x : f(x) > 0\} \cup \{x : f(x) < 0\})$ . It is very easy to show that the above also holds for “less than zero”, thus rendering the union here a union of open sets, and then  $T$  a closed set as a complement of an open set.

**Exercise 7**

Show that every open set in  $\mathbb{R}$  is the union of countably many open intervals with *rational* endpoints. Use this to show that the collection  $\mathcal{U}$  of all open subsets of  $\mathbb{R}$  has the same cardinality as  $\mathbb{R}$  itself.

*Solution.*

We know already that an open set  $S \subset \mathbb{R}$  can be written as  $S = \bigcup_{n=1}^{\infty} (a_n, b_n)$ , where the intervals are all disjoint, and perhaps unbounded. Consider the following procedure. We take the interval  $(a_n, b_n)$ , where in general the endpoints are real numbers. We know that between them there will always be a rational number  $q_n$ . In fact,  $b_n$  can be approached by a (increasing) sequence of rational numbers starting at  $q_n$  and getting infinitely close to it, and the same holds for  $a_n$  (only difference being the fact that this will be a decreasing sequence). These are *countably infinite* in number: indeed, each sequence is equivalent to the natural numbers, since these directly form its indices, and of course  $\mathbb{N} \cup \mathbb{N} \sim \mathbb{N}$  (“two sequences are equivalent to one sequence”).

We can thus replace  $(a_n, b_n)$  with a union of the form  $\bigcup_{k=1}^{\infty} (q_k, r_k)$  where the endpoints are rationals. But then we have that  $S$  is a countable union of countable sets, which we know (Theorem 2.6) is also countable, and thus  $S$  is indeed a countable union of intervals with rational endpoints.

Now let  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathcal{U}$ ,  $f(x) = (0, x)$  if  $x > 0$ , and  $f(x) = (x, 0)$  otherwise. This function is clearly an injection from the reals except zero to the set of open sets. Recall also (exercise 17, ch. 2) that

$\mathbb{R} \sim \mathbb{R} \setminus \{0\}$ , and thus we can easily see that there exists an injection  $h_1$  from  $\mathbb{R}$  to  $\mathcal{U}$ . Additionally, let  $g((q_n)) = \cup_{n=1}^{\infty} (q_n, q_{n+1})$  be a function mapping sequences of rationals to open sets. We've shown that every open set can be written in this form, meaning that  $g$  is a surjection. But then for every open interval we can find at least one sequence of rationals that via  $g$  (since it's a function) corresponds to it and to no other open interval. This shows the existence of an injection  $h_2 : \mathcal{U} \rightarrow \mathbb{Q}^{\mathbb{N}}$ . Putting together the existence of  $h_1, h_2$  and the fact that  $\mathbb{Q}^{\mathbb{N}} \sim \mathbb{R}$ , we apply the Cantor-Schröder-Bernstein theorem to see that  $\mathcal{U}$  and  $\mathbb{R}$  are in fact equivalent.

### Exercise 8

Show that every open interval (and hence every open set) in  $\mathbb{R}$  is a countable union of closed intervals and that every closed interval in  $\mathbb{R}$  is a countable intersection of open intervals.

*Solution.*

Consider any interval of the form  $(a, b)$ , first where  $a, b \in \mathbb{R}$ , and consider the union:

$$U = \bigcup_{n=1}^{\infty} \left[ \frac{a+b}{2} - \sum_{i=2}^n (b-a) \frac{1}{2^i}, \frac{a+b}{2} + \sum_{i=2}^n (b-a) \frac{1}{2^i} \right]$$

For the sake of brevity, we will omit some details, but one can see that as  $n \rightarrow \infty$ , the terms inside the sums converge to  $(b-a)/2$ . But this means that for any given  $x \in (a, b)$ , it suffices to check whether  $x < (a+b)/2$  or  $x > (a+b)/2$ , and then pick  $n$  accordingly to make the left or right endpoint of the  $n$ -th term of the union become smaller or greater than  $x$  respectively. This is always possible due to the sums converging to  $(b-a)/2$ , which in turn makes the endpoints converge to  $a, b$ . The union thus includes all points in  $(a, b)$ , and furthermore it is clearly countable.

If it is the case that  $a = -\infty$  and  $b = \infty$ , we propose the union  $U = \cup_{n=1}^{\infty} [-2^n, 2^n]$ , and we can thus clearly always find  $n$  big enough such that any  $x \in \mathbb{R}$  belongs in one such interval. The “intermediate” case of one of  $a, b$  being infinity and the other a real number would be handled as a mixture of the above.

Now consider instead an interval of the form  $[a, b], a, b \in \mathbb{R}$ . Consider the intersection:

$$I = \bigcap_{n=1}^{\infty} \left[ a - \frac{1}{2^n}, b + \frac{1}{2^n} \right]$$

We claim that this (countable) intersection equals  $[a, b]$ . First, it is easy to see that any  $x \in [a, b]$  is contained in every interval, since  $b + \frac{1}{2^n} > b, a - \frac{1}{2^n} < a$  for all  $n$ . Second, we claim that no  $y \notin [a, b]$  can be contained in  $I$ . Indeed, if  $y < a$ , because  $a - \frac{1}{2^n}$  gets arbitrarily close to  $a$  from the left, we can always find  $n$  sufficiently large such that  $y < a - \frac{1}{2^n}$ , and a similar argument works for  $y > b$ , showing that  $y$  will not be contained in the intervals corresponding to  $k \geq n$ , and thus neither will it be contained in  $I$ . This concludes the proof.

### Exercise 10

Given  $y = (y_n) \in H^{\infty}, N \in \mathbb{N}, \epsilon > 0$ , show that  $\{x = (x_n) \in H^{\infty} : |x_k - y_k| < \epsilon, k = 1, \dots, N\}$  is open in  $H^{\infty}$  (see Exercise 3.10).

*Solution.*

Let  $S = \{x = (x_n) \in H^{\infty} : |x_k - y_k| < \epsilon, k = 1, \dots, N\}$ , and suppose  $x \in S$ . Suppose also, by way of contradiction, that there exists a sequence  $z^{(k)} \rightarrow x$  with an infinite number of its elements not in  $S$ . This means that there must exist infinite  $k$  such that  $|z_n^{(k)} - y_n| \geq \epsilon$  for at least one  $n = 1, 2, \dots, N$ . We first make the observation that  $z^{(k)} \rightarrow x$  implies coordinate-wise convergence for each  $n$ , in other words that  $z^{(k)} \rightarrow x_n$  for all  $n$ . Indeed, given a particular  $n$ , pick any  $\epsilon' > 0$  and set  $\epsilon' = 2^{-n}\epsilon$ . Then there must exist  $K > 0$  such that for  $k \geq K$  (recall the metric for  $H^{\infty}$  from exercise 3.10):

$$d(z^{(k)}, x) < \epsilon' \implies \sum_{m=1}^{\infty} 2^{-m} |z_m^{(k)} - x_m| < \epsilon'$$

But then it is the case that  $2^{-n}|z_n^{(k)} - x_n| < 2^{-n}\epsilon \implies |z_n^{(k)} - x_n| < \epsilon$  for all  $k \geq K$ , which is the definition of convergence for the  $n$ -th coordinate. Now pick an arbitrary  $n$ , and consider the following:

$$|z_n^{(k)} - y_n| \leq |z_n^{(k)} - x_n| + |x_n - y_n|$$

The second term here is independent of  $z$ , so let it equal  $c_n < \epsilon$  (by the definition of  $S$ ). Set  $\epsilon_n = \epsilon - c_n > 0$ , and notice that there must exist  $K > 0$  such that for  $k \geq K$ ,  $|z_n^{(k)} - x_n| < \epsilon_n$ . Then:

$$|z_n^{(k)} - y_n| < \epsilon_n + c_n = \epsilon - c_n + c_n = \epsilon$$

The implication here is that after some  $K$ , this  $n$  can no longer “play the part” of keeping  $z^{(k)} \notin S$  by satisfying  $|z_n^{(k)} - y_n| \geq \epsilon$ . This is true for any  $n = 1, 2, \dots, N$ , so we can find a maximum such  $K$  such that for  $k \geq K$  the above holds for *all*  $n$ . However, the existence of infinite  $k$  such that  $z^{(k)} \notin S$  directly contradicts this, and thus  $S$  must indeed be open.

### Exercise 11

Let  $e^{(k)} = (0, \dots, 0, 1, 0, \dots)$ , where the  $k$ -th entry is 1 and the rest are 0s. Show that  $\{e^{(k)} : k \geq 1\}$  is closed as a subset of  $l_1$ .

*Solution.*

Let  $x = (x_1, x_2, \dots)$  be any element of  $S = \{e^{(k)} : k \geq 1\}$ . We will show that if, for every  $\epsilon > 0$ , it is the case that  $B_\epsilon(x) \cap S \neq \emptyset$  (under  $l_1$ ), then  $x \in S$ , which we know is one of the three equivalent ways of defining closed sets. What this statement means is that for every  $\epsilon > 0$ , there must exist a sequence  $y = (0, 0, \dots, 1, 0, \dots)$  with precisely one (say, the  $i$ -th) element equal to 1 and the rest equal to 0, and at the same time  $\|(x_1, x_2, \dots, x_i, \dots) - (0, 0, \dots, 1, 0, \dots)\|_1 < \epsilon \implies \|(x_1, x_2, \dots, 1 - x_i, x_{i+1}, \dots)\|_1 < \epsilon$ . Suppose then that  $x \notin S$ . One possibility is that  $x$  has at least two non-zero elements,  $x_j, x_k$ . Notice then that no matter the position of  $i$ , it holds that  $\|(x_1, x_2, \dots, 1 - x_i, x_{i+1}, \dots)\|_1 \geq \min\{|x_j|, |x_k|\}$ : indeed,  $y$  can at most zero out one of the two non-zero elements of  $x$ . But this means that for sufficiently small  $\epsilon$  the statement above will not hold, a contradiction. Therefore  $x$  has at most one non-zero element. If  $x$  is the zero sequence, it is clear that the LHS above will always have a norm of 1, and thus we again arrive at a contradiction. The only possibility is thus that  $x = (0, \dots, a, 0, \dots)$ ,  $a \neq 0$ . Suppose that  $a \neq 1$ , and observe that for  $\epsilon < 1$  it must be the case that the  $y$  we seek has  $y_i = 1$ , otherwise  $\|(0, \dots, 1, \dots, a, \dots)\|_1 \geq 1$ . Therefore, it must be the case that  $\|(0, \dots, 1 - a, 0, \dots)\|_1 < \epsilon$  for all  $\epsilon < 1$ , which enforces  $|1 - a| < \epsilon$ , and of course this can only be true for  $a = 1$ , thus showing that  $x \in S$ .

### Exercise 13

Show that  $c_0$  is a closed subset of  $l_\infty$ . [Hint: If  $(x^{(n)})$  is a sequence of sequences in  $c_0$  converging to  $x \in l_\infty$ , note that  $|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}|$  and now choose  $n$  so that  $|x_k - x_k^{(n)}|$  is small *independent* of  $k$ .]

*Solution.*

Suppose that  $(x^{(n)}) \in c_0$  is a sequence of sequences converging to  $x$ . We need to show that  $x \in c_0$ , in other words that  $x_k \rightarrow 0$ . We follow the hint by observing that, for any given term  $x^{(n)}$  of the sequence of sequences, via the triangle inequality we have that:

$$|x_k| = |x_k - x_k^{(n)} + x_k^{(n)}| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}|$$

Pick any  $\epsilon > 0$ . Because  $x^{(n)} \rightarrow x$ , there exists  $N > 0$  such that for  $n \geq N$  we have that  $d_\infty(x^{(n)}, x) < \epsilon \implies \sup_k |x_k^{(n)} - x_k| < \epsilon/2$ . This guarantees that  $|x_k^{(n)} - x_k| < \epsilon/2$  for *any*  $k$ . In addition, we know that for any  $n$ ,  $x^{(n)} \in c_0 \implies x_k^n \rightarrow 0$ , and so there must exist  $K \geq 0$  such that  $|x_k^n| < \epsilon/2$  for  $k \geq K$ . By using any  $n \geq N$  in the inequality stated above, we see that for  $k \geq K$  we have that  $|x_k| < \epsilon$ , which means that  $x \rightarrow 0$ , and thus that  $x \in c_0$ .

**Exercise 15**

The set  $A = \{y \in M : d(x, y) \leq r\}$  is sometimes called the *closed ball* about  $x$  of radius  $r$ . Show that  $A$  is a closed set, but give an example showing that  $A$  need not equal the closure of the open ball  $B_r(x)$ .

*Solution.*

To show that  $A$  is closed, pick any point  $y$  and suppose that the ball  $B_\epsilon(y)$  intersects  $A$  for any  $\epsilon > 0$ . We need to show that  $y \in A$ . Indeed, if we assume  $y \notin A$ , then  $d(x, y) > r$ . If we pick  $\epsilon = d(x, y) - r > 0$ , then there must exist  $z \in B_\epsilon(y) \cap A$ . This means that  $d(z, x) \leq r$ . Then:

$$d(y, x) \leq d(y, z) + d(z, x) < \epsilon + d(x, z) = d(x, y) - r + d(x, z) \leq d(x, y) - r + r = d(x, y)$$

We have arrived at the contradiction  $d(y, x) < d(x, y)$ , and so  $y \in A$ , which means that  $A$  is closed.

For the second part of the exercise, consider the discrete metric  $d$ . Then for some  $x \in M$ ,  $B_1^d(x) = \{x\}$ , and the closure of this open ball is also the set  $\{x\}$ : this contains precisely all the limit points of  $\{x\}$ . However, the corresponding closed ball would be the set  $\{y \in M : d(x, y) \leq 1\} = M$ , by definition of the discrete metric. Clearly, the closure of the open ball around  $x$  does not equal the closed ball around  $x$ .

**Exercise 17**

Show that  $A$  is open if and only if  $A^\circ = A$  and that  $A$  is closed if and only if  $\overline{A} = A$ .

*Solution.*

We begin with the first statement. Note that it always holds that  $A^\circ \subset A$ , so in both directions of the equivalence we'll only need to show or assume that  $A \subset A^\circ$ .

$\implies$  : Suppose  $A$  is open and pick any  $x \in A$ . We need to show that  $x \in A^\circ = \cup\{B_\epsilon(y) \subset A, \text{ for some } y \in A, \epsilon > 0\}$ . Because  $A$  is open, there exists some  $\epsilon > 0$  such that  $B_\epsilon(x) \subset A$ . This clearly implies that  $B_\epsilon(x)$  is included in the union that forms  $A^\circ$ , and so  $x \in A^\circ$ , which means  $A \subset A^\circ$ .

$\impliedby$  : Suppose now that  $A \subset A^\circ$ , and pick any  $x \in A$ . To show that  $A$  is open, we need to find  $\epsilon > 0$  such that  $B_\epsilon(x) \subset A$ . By our assumption, it must be the case that there exist  $y \in A, \rho > 0$  such that  $x \in B_\rho(y) \subset A$ . In other words, it is the case that  $d(x, y) < \rho$ . Set  $\epsilon = \rho - d(x, y)$ , and consider the ball  $B_\epsilon(x)$ . For any  $z$  in this ball, we have that:

$$d(z, x) < \epsilon \implies d(z, x) < \rho - d(x, y) \implies d(x, y) + (z, x) < \rho \implies d(y, z) < \rho,$$

where we used the triangle inequality in the last step. Thus  $z \in B_\rho(y) \subset A$ , which means  $B_\epsilon(x) \subset A$ , and so we've shown that  $A$  is indeed open. Now, for the second statement we have that  $A \subset \overline{A}$ , so we'll only need to show or assume that  $\overline{A} \subset A$ :

$\implies$  : Suppose  $A$  is closed, and pick any  $x \in \overline{A}$ . Then, for every  $\epsilon > 0$  we have that  $B_\epsilon(x) \cap A \neq \emptyset$ . Because  $A$  is closed, recall that this must imply that  $x \in A$ .

$\impliedby$  : Conversely, assume  $\overline{A} \subset A$ , and pick any  $x$  such that for every  $\epsilon > 0$ ,  $B_\epsilon(x) \cap A \neq \emptyset$ . If we can show that this implies  $x \in A$ , we will have shown that  $A$  is closed. By the definition of the closure, we have that it implies that  $x \in \overline{A}$ , and by our assumption also that  $x \in A$ , thus completing the proof that  $A$  is closed.

**Exercise 18**

Given a nonempty bounded subset  $E$  of  $\mathbb{R}$ , show that  $\sup E, \inf E$  are elements of  $\overline{E}$ . Thus  $\sup E, \inf E$  are elements of  $E$  whenever  $E$  is closed.

*Solution.*

Recall from exercise 3 of Chapter 1 that for every  $\epsilon > 0$ , there exists  $a \in E$  such that  $a > \sup E - \epsilon$  and  $b \in E$  such that  $b < \inf E + \epsilon$ . But then observe that in  $\mathbb{R}$ ,  $B_\epsilon(\sup E) = \{x \in \mathbb{R} : |x - \sup E| < \epsilon\}$ , and the same holds for the infimum, which by the above means that  $B_\epsilon(\sup E) \cap E \neq \emptyset, B_\epsilon(\inf E) \cap E \neq \emptyset$  since these intersections include  $a, b$  respectively. Since this holds for any  $\epsilon > 0$ , it must be the case that  $\sup E, \inf E \in \overline{E}$ . By exercise 17, we have that  $\overline{E} = E$  whenever  $E$  is closed, and thus in this case it also holds that  $\sup E, \inf E$  are elements of  $E$ .

**Exercise 19**

Show that  $\text{diam}(A) = \text{diam}(\overline{A})$ .

*Solution.*

By definition, we have that  $\text{diam}(A) = \sup\{d(a, b), a, b \in A\}$ ,  $\text{diam}(\overline{A}) = \sup\{d(a, b), a, b \in \overline{A}\}$ . Notice that because  $A \subset \overline{A}$ , it must be the case that  $\text{diam}(A) \leq \text{diam}(\overline{A})$ . Suppose then that the inequality is strict, in which case it must hold that for at least two  $a, b \in \overline{A}$ ,  $d(a, b) > \text{diam}(A)$ . By the definition of the closure, there must exist sequences  $(x_n), (y_n) \subset A$  such that  $x_n \rightarrow a, y_n \rightarrow b$ . Then, by exercise 34 of 3.4 we know that  $d(x_n, y_n) \rightarrow d(a, b)$ . Thus, for any given  $\epsilon > 0$ , there exists  $N > 0$  such that for  $n \geq N$ ,  $|d(x_n, y_n) - d(a, b)| < \epsilon \implies d(x_n, y_n) > d(a, b) - \epsilon$ . But, because  $d(a, b)$  is an upper bound for  $\{d(x, y), x, y \in A \subset \overline{A}\}$  again by exercise 3, Chapter 1, this means precisely that  $\sup\{d(x_n, y_n), n = 1, 2, \dots\} = d(a, b)$  and thus  $\text{diam}(A) \geq d(a, b) = \text{diam}(\overline{A})$ , which is a contradiction.

**Exercise 20**

If  $A \subset B$ , show that  $\overline{A} \subset \overline{B}$ . Does  $\overline{A} \subset \overline{B}$  imply  $A \subset B$ ? Explain.

*Solution.*

Suppose  $x \in \overline{A}$ . Then there exists a sequence  $(x_n) \subset A$  such that  $x_n \rightarrow x$ . Because  $A \subset B$ , it must also be the case that  $(x_n) \subset B$ . Then, by the definition of the closure of a set, we have that the limit of  $(x_n)$ , i.e.,  $x$ , must also belong in  $\overline{B}$ , thus showing that  $\overline{A} \subset \overline{B}$ .

The statement posed as a question is false. We consider the following counterexample:  $B = B_1(0) = (-1, 1)$ ,  $A = \{1\}$  in  $\mathbb{R}$ . Here we easily obtain that  $\overline{B} = [-1, 1]$  and  $\overline{A} = \{1\}$ , and so we have that  $\overline{A} \subset \overline{B}$ . However, it clearly does not hold that  $A \subset B$ .

**Exercise 22**

True or false?  $(A \cup B)^\circ = A^\circ \cup B^\circ$ .

*Solution.*

This is not, in general, true. Set  $A = (-1, 1)$ ,  $B = [1, 2)$ , in which case  $A \cup B = (-1, 2)$ , and so  $(A \cup B)^\circ = (-1, 2)$ . Additionally,  $A^\circ = (-1, 1)$ ,  $B^\circ = (1, 2)$   $\implies A^\circ \cup B^\circ = (-1, 1) \cup (1, 2)$ . Notice then that  $1 \in (A \cup B)^\circ$  but  $1 \notin A^\circ \cup B^\circ$ , so the two sets cannot be equal.

**Exercise 27**

Show that  $|d(x, A) - d(y, A)| \leq d(x, y)$  and conclude that the map  $x \rightarrow d(x, A)$  is continuous. In the context of this problem, “continuity” refers to  $x_n \rightarrow x \implies d(x_n, A) \rightarrow d(x, A)$ .

*Solution.*

From exercise 26, we recall that  $d(x, A) = \inf\{d(x, a), a \in A\}$ . We begin by observing that for any given  $a \in A$ , it holds that:

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \implies d(x, A) - d(x, y) \leq d(y, a)$$

Since this holds for any  $a \in A$ , the quantity on the LHS constitutes a lower bound for  $\{d(y, a), a \in A\}$ . By the definition of the infimum, it must then hold that  $d(x, A) - d(x, y) \leq d(y, A) \implies d(x, A) - d(y, A) \leq d(x, y)$ . By exchanging the roles of  $x, y$  above, we can arrive also at the inequality  $d(y, A) - d(x, y) \leq d(x, a)$  for any  $a \in A$ , which leads to  $d(y, A) - d(x, y) \leq d(x, A) \implies d(y, A) - d(x, A) \leq d(x, y)$ . Putting these two together we obtain that  $|d(x, A) - d(y, A)| \leq d(x, y)$ . For the second part of the exercise, suppose that  $x_n \rightarrow x$  under  $d$ , that is, that for any  $\epsilon > 0$  there exists  $N > 0$  such that  $d(x_n, x) < \epsilon, n \geq N$ . By using the inequality we just proved, we have that:

$$|d(x_n, A) - d(x, A)| \leq d(x_n, x) < \epsilon,$$

and so we can indeed safely conclude that  $d(x_n, A) \rightarrow d(x, A)$ , which is the definition of continuity we use here.

**Exercise 28**

Given a set  $A$  in  $M$  and  $\epsilon > 0$ , show that  $\{x \in M : d(x, A) < \epsilon\}$  is an open set and that  $\{x \in M : d(x, A) \leq \epsilon\}$  is a closed set (and each contains  $A$ ).

*Solution.*

The fact that each of these sets contains  $A$  is easily seen: for any  $x \in A$ , we have that  $d(x, A) = \inf\{d(x, a) : a \in A\} = d(x, x) = 0$ , and thus of course  $d(x, A) \leq \epsilon$  for any  $\epsilon > 0$ . Now let  $S = \{x \in M : d(x, A) < \epsilon\}$ , and suppose that  $S$  is not open. Then there must exist  $x \in S$  such that there exists a sequence  $x_i \rightarrow x$  with an infinite number of terms that are not in  $S$ . Take those terms  $(x_{i_k}) \subset S^c$  and form a sequence converging to  $x$ . Observe that  $d(x_{i_k}, A) \geq \epsilon$ , and that by exercise 27, the function  $x \rightarrow d(x, A)$  is continuous. This then means that the inequality must hold at the limit, i.e. that  $d(x, A) \geq \epsilon$ , which is a contradiction. Therefore  $S$  is open. Now set instead  $S = \{x \in M : d(x, A) \leq \epsilon\}$  and pick any convergent sequence  $(x_i) \subset S$  converging to  $x \in M$ . It holds that  $d(x_i, A) \geq \epsilon$ , and again by exercise 27 we obtain  $d(x, A) \geq \epsilon$ , which means  $x \in S$  and so  $S$  is closed.

**Exercise 29**

Show that every closed set in  $M$  is the intersection of countably many open sets and that every open set in  $M$  is the union of countably many closed sets. [Hint: What is  $\cap_{n=1}^{\infty} \{x \in M : d(x, A) < 1/n\}?$ ]

*Solution.*

We begin by following the hint and setting  $S = \cap_{n=1}^{\infty} \{x \in M : d(x, A) < 1/n\}$ , where  $A$  is a closed set. By exercise 28, we know that each of the sets comprising the infinite intersection is open. It suffices therefore to show that  $A = S$ . First, let  $y \in A$ . Then, it of course holds that  $d(y, A) = 0 < 1/n$  for every  $n > 0$ , and so  $y \in S$ . Therefore,  $A \subset S$ . In the other direction, suppose  $y \in S$ . It must then be the case that  $d(y, A) < 1/n$  for every  $n > 0$ . The only way this can hold is if  $d(y, A) = 0$ , which by exercise 26 we know is equivalent to  $y \in \bar{A}$ . Here  $A$  is closed, so  $A = \bar{A}$ , which means  $y \in A$ , and so  $S \subset A$ . We've therefore shown that  $A = S$ , with  $S$  being a countable intersection of open sets.

Now suppose instead that  $A$  is open. Then  $A^c$  must be closed. By the first part, we can write  $A^c = \cap_{n=1}^{\infty} \{x \in M : d(x, A^c) < 1/n\}$ . Consider what it means for an element to *not* belong in  $A^c$  (therefore, to belong in  $A$ ): for it to not belong in this infinite intersection, it is both sufficient and necessary for it to not belong in at least one of the sets comprising it, i.e. we have that  $A = \cup_{n=1}^{\infty} \{x \in M : d(x, A^c) \geq 1/n\} = \cup_{n=1}^{\infty} \{x \in M : d(x, A^c) \geq 1/n\}$ . This is a countable union, so we must only show that the sets comprising it are closed. Let  $S_n = \{x \in M : d(x, A^c) \geq 1/n\}$  and let  $(x_i) \in S_n$  be a convergent sequence with  $x_i \rightarrow x$ . It is then the case that  $d(x_i, A^c) \geq 1/n$  for all  $i$ , and recall from exercise 27 that the function  $f(x) = d(x, A)$  is continuous, and so because  $x_i \rightarrow x$ , it holds that  $f(x_i) \rightarrow f(x) = d(x, A)$ . Because  $d(x_i, A^c) \geq 1/n$  for all  $i$ , this also holds at the limit:  $d(x, A^c) \geq 1/n$ , and so  $x \in S_n$  and  $S_n$  is closed.

**Exercise 33**

Let  $A$  be a subset of  $M$ . A point  $x \in M$  is called a **limit point** of  $A$  if every neighborhood of  $x$  contains a point of  $A$  that is different from  $x$  itself, that is, if  $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$  for every  $\epsilon > 0$ . If  $x$  is a limit point of  $A$ , show that every neighborhood of  $x$  contains infinitely many points of  $A$ .

*Solution.*

Suppose that this is not true, i.e. that  $x$  is a limit point of  $A$  and a neighborhood  $S$  of  $x$  contains only finitely many points of  $A$ . Call these  $x_1, x_2, \dots, x_n$  and define  $\epsilon = \min\{d(x_1, x), d(x_2, x), \dots, d(x_n, x)\}$ . Then consider the ball  $B_\epsilon(x) \subset S$ : it becomes clear that no point of  $A$  in  $S$ , except for  $x$ , is inside this ball. But this contradicts the definition of limit point as given in the exercise, and thus every neighborhood of  $x$  must contain infinitely many points of  $A$ .

**Exercise 34**

Show that  $x$  is a limit point of  $A$  if and only if there is a sequence  $x(x_n)$  in  $A$  such that  $x_n \rightarrow x$  and  $x_n \neq x$  for all  $n$ .

*Solution.*

$\implies$  : If  $x$  is a limit point of  $A$ , for any given  $\epsilon > 0$ , find  $x_n \in A, x_n \neq x$  such that  $x_n \in B_\epsilon(x)$ . Constructing a sequence of such  $x_n$  by taking successive  $\epsilon_n = 1/n, n = 1, 2, \dots$  yields the desired result:  $x_n \rightarrow x, x_n \neq x$  for all  $n$ .

$\impliedby$  : Now suppose that there exists a sequence  $(x_n)$  in  $A$  with  $x_n \rightarrow x, x_n \neq x$ . Then for any given  $\epsilon > 0$ , we can find  $N > 0$  such that for  $n \geq N, d(x_n, x) < \epsilon$ , which means that the ball  $B_\epsilon(x)$  contains a point (in fact, infinitely many) of  $A$  that does not equal  $x$  (since  $x_n \neq x$  for all  $n$ ).

### Exercise 36

Suppose that  $x_n \rightarrow x \in M$  under  $d$ , and let  $A = \{x\} \cup \{x_n : n \geq 1\}$ . Prove that  $A$  is closed.

*Solution.*

If we can show that for any given  $y \in M$  for which  $B_\epsilon(y) \cap A \neq \emptyset$  holds for all  $\epsilon > 0$ , it is the case that  $y \in A$ , we will have shown that  $A$  is closed. Pick a  $y \in M$  that fulfills the stated condition. If  $y = x$  or  $y = x_n$  for some  $x_n$ , it is of course true that  $y \in A$ . If  $y \neq x, y \neq x_n$  for all  $n$ , then for any given  $\epsilon > 0$ , we can find an element  $z \in A$  such that  $d(z, y) < \epsilon$ . It is easily seen that by choosing  $\epsilon_n = 1/n$ , eventually these  $z$  must no longer equal  $x$ : the distance  $d(x, y)$  is of course constant and cannot be smaller than all  $\epsilon > 0$  if  $y \neq x$ . Therefore, after some  $N$ , the  $z$  corresponding to  $\epsilon_n$  must be elements of the sequence  $(x_n)$ , and furthermore these elements cannot all originate from the first  $K$  terms of  $(x_n)$  for any  $K$ , since then we would be able to find a minimum distance  $d(z_n, y)$ . We conclude that, by picking  $z_n$  with appropriate increasing indices, we can then form a subsequence of  $(x_n)$  that converges to  $y \neq x$ , which by the uniqueness of the limit is a contradiction. Therefore  $y$  is either equal to  $x$  or some  $x_n$  and  $A$  is closed.

### Exercise 38

A set  $P$  is called **perfect** if it is empty or if it is a closed set and every point of  $P$  is a limit point of  $P$ . Show that  $\Delta$  is perfect. Show that  $\mathbb{R}$  is perfect when considered as a subset of  $\mathbb{R}^2$ .

*Solution.*

We need to show that  $\Delta$  is closed (which is stated in example 4.12 but not proved). Suppose  $(x_n) \subset \Delta$  converges to  $x$ , and suppose that  $x \notin \Delta$ , which means that it must be outside at least one of the “levels” of intervals comprising the infinite intersection that is  $\Delta$ . It is rather onerous to completely formalize everything here, but this would mean that  $d(x, e) > 0$  for every endpoint  $e$  of the intervals of that level. Since all  $x_n$  are in  $\Delta$ , each must belong in some interval of that level as well, and so for each  $x_n, d(x_n, x) \geq d(x, e_{k_n}) > 0$  for *some* endpoint  $e_{k_n}$  out of a finite number of endpoints of that level. Therefore, for each  $n, d(x_n, x) \geq \min_e d(x, e) > 0$  which contradicts convergence of  $(x_n)$  to  $x$ , and so  $\Delta$  must be closed.

Now, to show that  $P$  is perfect, we must, for any given  $x \in P$ , prove that some sequence  $(x_n) \subset P$  converges to  $x$ . Recall that  $x \in \Delta$  iff  $x = \sum_{n=1}^{\infty} a_n / 3^n$  where each  $a_n$  is either 0 or 2. If  $x$  does not end in an infinite length of zeros, consider the sequence  $(x_n)$  where  $x_n = \sum_{k=1}^n a_k / 3^k + \sum_{k=n+1}^{\infty} 0 / 3^k$ , that is,  $x_n$  equals  $x$  up to the  $n$ -th digit and then is zero. If  $x$  does end in an infinite string of zeros, modify  $x_n$  to have 2s instead of 0s as the digits after  $n$ . It is fairly easy to see that  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$  (since the difference  $|x_n - x|$  will always be upper bounded by  $2/3^n$ ), while we also observe that  $x_n$  can never equal  $x$ . Therefore,  $x$  is indeed a limit point of a sequence in  $P$ .

For the second part of the exercise, if we identify  $\mathbb{R}$  with  $S = \{(x, 0), x \in \mathbb{R}\}$ , then this is quite obviously a closed set. Additionally, if  $x \in S$ , consider the sequence  $p_n = (x + 1/n, 0) \in \mathbb{R}^2$ . Again, it is fairly easy to show that it converges coordinate-wise to  $(x, 0)$ , without ever resulting in  $p_n = (x, 0)$ , and so every point of  $S$  is a limit point of  $S$ .

### Exercise 39

Show that a nonempty perfect subset  $P$  of  $\mathbb{R}$  is uncountable. This gives yet another proof that the Cantor set is uncountable. [Hint: First convince yourself that  $P$  is infinite, and assume that  $P$  is countable, say  $P = \{x_1, x_2, \dots\}$ . Construct a decreasing sequence of nested closed intervals  $[a_n, b_n]$  such that  $(a_n, b_n) \cap P \neq \emptyset$  but  $x_n \notin [a_n, b_n]$ . Use the nest interval theorem to get a contradiction.]

*Solution.*

We follow the thought process indicated in the hint. First of all, if we assume  $P$  to be finite and have, say,  $N$  elements, it becomes clear that none of its points can be limit points: there is a well-defined minimum

distance  $d_i$  of each point  $x_i$  from the other  $N - 1$ , and for any point  $x_i$ , no other elements of  $P$  lie in  $B_\epsilon(x_i), \epsilon < d_i$ . Now assume that  $P$  is countable, such that  $P = \{x_1, x_2, \dots\}$ . We can thus find an interval  $[a_1, b_1]$  such that  $x_1 \notin [a_1, b_1]$  but  $[a_1, b_1] \cap P \neq \emptyset$ . One easy way to do this is to pick it such that it contains  $x_2$ .

Now we can repeat this procedure to find an interval  $[a_2, b_2] \subset [a_1, b_1], b_2 - a_2 < (b_1 - a_1)/2$  (we “cut” the interval in half) that again is such that  $[a_2, b_2] \cap P \neq \emptyset$  but  $x_2 \notin [a_2, b_2]$ . The key observation is that if we were unable to do this, it would mean that the interval  $[a_1, b_1]$  does not, in fact, contain infinite points, and so  $x_2$  cannot be the limit of a sequence of points in  $P$ , contradicting  $P$ ’s property of being perfect. Notice that it does not have to be the case that  $x_3 \in [a_2, b_2]$  (e.g. if  $x_3 < x_1$ ), but *some* element of  $P$  must be in this interval, and it cannot equal either  $x_1$  or  $x_2$ .

The process can be repeated ad infinitum, with the following observations:

- The interval length goes to zero, and the intervals are nested.
- The  $n$ -th interval contains an element of  $P$  that cannot equal any of  $x_1, x_2, \dots, x_n$ .

These observations lead us to conclude that we can form a sequence of  $x_{k_i}$ , one corresponding to each selected interval, and that this sequence must converge to some  $x$ , which will be the single point contained in the infinite intersection of intervals (by using the nested interval theorem). Furthermore, we have that  $x_{k_i} \rightarrow x$ . If  $x \in P$ , we arrive at a contradiction, since it cannot equal any of the elements  $x_1, x_2, \dots$ : indeed, if it was equal to one of them, say the  $i$ -th, the above process would not have continued after the  $(i - 1)$ -th step (by the definition of these intervals). If  $x \notin P$ , then this contradicts the fact that  $P$  is closed as a perfect set. In both cases, we have arrived at a contradiction, which means that  $P$  cannot be countable.

#### Exercise 40

If  $x \in A$  and  $x$  is *not* a limit point of  $A$ , then  $x$  is called an **isolated point** of  $A$ . Show that a point  $x \in A$  is an isolated point of  $A$  if and only if  $(B_\epsilon(x) \setminus \{x\}) \cap A = \emptyset$  for some  $\epsilon > 0$ . Prove that a subset of  $\mathbb{R}$  can have at most countably many isolated points, thus showing that every uncountable subset of  $\mathbb{R}$  has a limit point.

*Solution.*

The first part of the exercise arises simply by negating the definition of limit point in exercise 33. For the second part, we note that if we can find a one-to-one mapping of isolated points of a subset  $A \subset \mathbb{R}$  to the rational numbers, we will have shown that these are at most countable. We describe a process by which such a mapping can be created.

Let  $p_1$  be any isolated point of  $A$ , which means that there exists  $\epsilon_1$  such that  $(p_1 - \epsilon_1, p_1 + \epsilon_1)$  contains no other points of  $A$ . We know that must always exist a rational  $q_1 \in (p_1, p_1 + \epsilon_1)$ . If there exists an isolated point  $p_2 > p_1$  such that its *corresponding*  $\epsilon_2$  is such that  $q_1 \in (p_2 - \epsilon_2, p_2)$ , we select a new rational  $q'_1 \in (p_1, q_1)$ . We claim that there can now be no isolated point  $p_3 > p_1$  such that  $q'_1 \in (p_3 - \epsilon_3, p_3)$  (again,  $\epsilon_3$  is such that it satisfies the definition of isolated point). Indeed, if this was true, there would be two cases:

- $p_3 > p_2$ : Observe that  $p_2 > q'_1 > p_3 - \epsilon_3$ , which would mean  $p_2 \in (p_3 - \epsilon_3, p_3)$ , which contradicts the fact that  $p_3$  is an isolated point with a corresponding  $\epsilon_3$ .
- $p_3 < p_2$ : Then it must necessarily be that  $p_3 > q_1$ , otherwise  $p_3$  would contradict the definition of isolated point for  $p_1$ . However, if this was the case we would have that  $p_3 \in (p_2 - \epsilon_2, p_2)$ , which contradicts the definition of isolated point for  $p_2$ .

Notice, also, that  $q'_1$  cannot be in any  $(p_3 - \epsilon_3, p_3 + \epsilon_3)$  for  $p_3 < p_1$ , because then  $p_1$  would also be in this interval, a contradiction. Therefore, the procedure that we describe will always match an interval of the form  $(p_i - \epsilon_i, p_i + \epsilon_i)$  to a rational number that lies *in no other* interval of the form  $(p_j - \epsilon_j, p_j + \epsilon_j)$ , and thus the mapping of isolated points to rationals must necessarily be injective, which completes the proof.

**Exercise 41**

Related to the notion of limit points and isolated points are boundary points. A point  $x \in M$  is said to be a **boundary point** of  $A$  if each neighborhood of  $x$  hits both  $A$  and  $A^c$ . In symbols,  $x$  is a boundary point of  $A$  if and only if  $B_\epsilon(x) \cap A \neq \emptyset$  and  $B_\epsilon(x) \cap A^c \neq \emptyset$  for every  $\epsilon > 0$ . Verify each of the following formulas, where  $\partial A$  denotes the set of boundary points of  $A$ :

- (a)  $\partial A = \partial A^c$
- (b)  $\overline{A} = \partial A \cup A^\circ$
- (c)  $M = A^\circ \cup \partial A \cup (A^c)^\circ$

Notice that the first and last equations tell us that each set  $A$  partitions  $M$  into three regions: the points “well inside”  $A$ , the points “well outside”  $A$ , and the points on the common boundary of  $A, A^c$ .

*Solution.*

- (a) Suppose first that  $x \in \partial A$ . Then for every  $\epsilon > 0$ , it holds that  $B_\epsilon(x) \cap A \neq \emptyset \implies B_\epsilon(x) \cap (A^c)^c \neq \emptyset$  and also that  $B_\epsilon(x) \cap A^c \neq \emptyset$ . But this means of course that  $x \in \partial A^c$ , and thus that  $\partial A \subset \partial A^c$ . Exchanging the roles of  $A, A^c$  yields the inclusion in the other direction as well.
- (b) Now suppose  $x \in \overline{A}$ . Then for every  $\epsilon > 0$ , the ball  $B_\epsilon(x)$  has a non-empty intersection with  $A$ . In the case where, for some such  $\epsilon$ ,  $B_\epsilon(x) \subset A$ , by definition of the interior it holds that  $x \in A^\circ$ . In the case where for *every*  $\epsilon$ ,  $B_\epsilon(x) \cap A^c \neq \emptyset$ , we conclude that  $x \in \partial A$ . Therefore  $x \in \partial A \cup A^\circ$ . Conversely, assume  $x \in \partial A \cup A^\circ$ . If  $x \in \partial A$ , then more specifically every ball  $B_\epsilon(x)$  intersects  $A$ , and thus  $x \in \overline{A}$  by definition. If  $x \in A^\circ$ , then it is trivially true that  $x \in A \subset \overline{A}$ , thus proving that  $\partial A \cup A^\circ \subset \overline{A}$ .
- (c) We begin by observing that  $M = A^\circ \cup (A^c)^\circ = A^\circ \cup \overline{A^c}$ , where we made use of exercise 24, observing that  $A^\circ = (\overline{A^c})^c$  and taking the complement of both sides. Now by part (b),  $\overline{A^c} = \partial A^c \cup (A^c)^\circ$ , and by part (a),  $\partial A = \partial A^c$ , so  $M = A^\circ \cup \partial A \cup (A^c)^\circ$ .

**Exercise 42**

If  $E$  is a nonempty bounded subset of  $\mathbb{R}$ , show that both  $\sup E, \inf E$  are both boundary points of  $E$ . Hence, if  $E$  is also closed, then  $\sup E, \inf E$  are elements of  $E$ .

*Solution.*

By exercise 18, we already know that  $\sup E, \inf E \in \overline{E}$ . By exercise 40, we thus know that  $\sup E, \inf E \in \partial E \cup E^\circ$ . Suppose  $\sup E \in E^\circ$ . Then there exists  $\epsilon > 0$  such that  $B_\epsilon(\sup E) \subset E \implies \{x \in \mathbb{R} : |x - \sup E| < \epsilon\} \subset E$ . But then it holds that  $x = \sup E + \epsilon/2 \in B_\epsilon(\sup E) \subset E$ , which contradicts the definition of the supremum, since  $x > \sup E$ . Then it must necessarily hold that  $\sup E \in \partial E$ . The proof for the infimum is symmetrical. Notice that the second part of the exercise follows immediately from exercise 18. Suppose (d) holds. Recall from exercise 41 that  $M = A^\circ \cup \partial A \cup (A^c)^\circ$ , and so in this case it must hold that  $M = A^\circ \cup \partial A$ . But, also from exercise 41,  $\overline{A} = A^\circ \cup \partial A$ , which shows that  $M = \overline{A}$ , i.e. that  $A$  is dense in  $M$ .

**Exercise 43**

Show that  $\partial A$  is always a closed set; in fact,  $\partial A = \overline{A} \setminus A^\circ$ .

*Solution.*

Pick any  $x \in M$  and any  $\epsilon > 0$ . We will show that if it always holds that  $B_\epsilon(x) \cap \partial A \neq \emptyset$ , then  $x \in \partial A$ , which is one of the definitions of  $\partial A$  being a closed set. By using  $\epsilon' = \epsilon/2$  in our hypothesis, there exists  $y \in \partial A$  such that  $d(x, y) < \epsilon/2$ . It then holds that  $B_{\epsilon/2}(y) \cap A \neq \emptyset, B_{\epsilon/2}(y) \cap A^c \neq \emptyset$ , by definition of the boundary. Therefore there must exist  $z_1 \in A$  with  $d(z_1, y) < \epsilon/2$  and  $z_2 \in A^c$  with  $d(z_2, y) < \epsilon/2$ . Then:

$$d(x, z_1) \leq d(x, y) + d(y, z_1) < \epsilon, d(x, z_2) \leq d(x, y) + d(y, z_2) < \epsilon$$

This means that  $B_\epsilon(x) \cap A \neq \emptyset, B_\epsilon(x) \cap A^c \neq \emptyset$ , since  $z_1, z_2$  belong in these two intersections respectively. Because this holds for any  $\epsilon > 0$ ,  $x$  satisfies the defining property of  $\partial A$ , and as such  $x \in \partial A$ , which means that  $\partial A$  is closed. Now recall from exercise 41 that  $\overline{A} = \partial A \cup A^\circ$ , and notice that the two sets on the RHS must always be disjoint: if  $x \in \partial A$ , then the intersection  $B_\epsilon(x) \cap A^c$  is always non-empty, which means that there can be no  $\epsilon > 0$  such that  $B_\epsilon(x) \subset A$ , and thus  $x \notin A^\circ$ . Therefore, we can safely conclude that  $\partial A = \overline{A} \setminus A^\circ$ .

**Exercise 46**

A set  $A$  is said to be **dense** in  $M$  (or, as some authors say, *everywhere dense*) if  $\overline{A} = M$ . For example, both  $\mathbb{Q}, \mathbb{R} \setminus Q$  are dense in  $\mathbb{R}$ . Show that  $A$  is dense in  $M$  if and only if any of the following hold:

- (a) Every point in  $M$  is the limit of a sequence from  $A$ .
- (b)  $B_\epsilon(x) \cap A \neq \emptyset$  for every  $x \in M$  and every  $\epsilon > 0$ .
- (c)  $U \cap A \neq \emptyset$  for every nonempty open set  $U$ .
- (d)  $A^c$  has empty interior.

*Solution.*

Suppose first that  $(A^c)^\circ = \emptyset$ . Then from exercise 41 we have that  $M = A^\circ \cup \partial A \cup (A^c)^\circ$ ,  $\overline{A} = A^\circ \cup \partial A$ , and so we conclude that  $M = \overline{A}$ . Conversely, suppose  $(A^c)^\circ = \emptyset$ , in which case by the exact same properties as above,  $M = \overline{A} \cup (A^c)^\circ \implies M = \overline{A}$ , which means that  $A$  is indeed dense, and that we are finished with condition (d).

For (a) recall Corollary 4.11:  $x \in \overline{A}$  iff  $x_n \rightarrow x$  for some sequence  $(x_n) \subset A$ , i.e.  $\overline{A}$  is precisely the set of limits of sequences in  $A$ , and so condition (a) immediately reduces to the definition of  $A$  being dense.

For (b), observe first that if  $\overline{A} = M$ , then by the definition of the closure, for every  $x \in A$  and every  $\epsilon > 0$  it must be the case that  $B_\epsilon(x) \cap A \neq \emptyset$ . Conversely, if (b) holds then every point  $x \in M$  fulfills the defining property of  $\overline{A}$ , and so  $M \subset \overline{A}$ , which of course means  $\overline{A} = M$ .

For (c) we'll make use of the equivalence of (d) to  $A$  being dense. Namely, assume first that (d) does not hold, i.e. that there exists  $x \in (A^c)^\circ$ , which means that for some  $\epsilon > 0$ ,  $B_\epsilon(x) \subset A^c$ , and so  $B_\epsilon(x) \cap A = \emptyset$ . But this ball is then an open set that renders (c) false, which means we've shown that  $\neg(d) \implies \neg(c)$ , or, equivalently, that  $(c) \implies (d)$ . Conversely, if (c) does not hold, then there exists a nonempty open set  $U$  such that  $U \cap A = \emptyset$ , which means  $U \subset A^c$ . Then, by the definition of the interior, it must be the case that  $U \subset (A^c)^\circ$ , and therefore that  $A^c$  does *not* have an empty interior. This means that (d) does not hold, i.e. we've shown that  $\neg(c) \implies \neg(d)$ , which is equivalent to  $(d) \implies (c)$ , which completes the proof that (c), (d) are equivalent, and therefore that (c) is also equivalent to  $M$  being dense.

**Exercise 48**

A metric space is called **separable** if it contains a countable dense subset. Find examples of countable dense sets in  $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n$ .

*Solution.*

For  $\mathbb{R}$ , we know that  $\mathbb{Q}$  is countable and that  $\overline{\mathbb{Q}} = \mathbb{R}$ , so it's a countable dense set of  $\mathbb{R}$ . For  $\mathbb{R}^2$ , consider the set  $\mathbb{Q} \times \mathbb{Q}$ : by exercise 3 of chapter 2, we have that this is a countable set. Also, any element  $(x, y) \in \mathbb{R}^2$  can be approximated coordinate-wise by two sequences  $q_n \rightarrow x, q'_n \rightarrow y$  each of which is in  $\mathbb{Q}$ . Then, exercise 46 of chapter 3 guarantees that  $(q_n, q'_n) \rightarrow (x, y)$ , and so we've shown condition (a) of exercise 46 above, which means that  $\mathbb{Q} \times \mathbb{Q}$  is indeed dense and countable. For  $\mathbb{R}^n$ , by a similar reasoning as above (extending ex. 46 to  $n > 2$ ) we can use the  $n$ -times Cartesian product of  $\mathbb{Q}$  as an example of a dense countable set.

**Exercise 49**

Prove that  $l_2, H^\infty$  are separable. [Hint: Consider finitely nonzero sequences of the form  $(r_1, \dots, r_n, 0, 0, \dots)$ , where each  $r_k$  is rational.]

*Solution.*

We begin with  $l_2$ , and follow the reasoning of the hint. Namely, consider the set  $S \subset l_2$  such that :

$$S = \{(r_1, r_2, \dots, r_n, 0, 0, \dots) | r_i \in \mathbb{Q} \ \forall i, n \geq 0\}$$

Notice that this a countable set: it is equivalent to  $\mathbb{Q} \cup (\mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}) \cup \dots$ , i.e. a countable union of countable sets. We need to show that for any given  $x \in l_2$  there exists a sequence (of sequences)  $(r)^{(k)} \subset S$  such that  $(r)^{(k)} \rightarrow x$ . To do this, we first pick any  $\epsilon > 0$ . Because  $x \in l_2$ , we know that  $\sum_{i=1}^{\infty} x_i^2$  converges. A consequence of this is that there exists  $N > 0$  such that  $\sum_{i=N}^{\infty} x_i^2 < \epsilon/2$  (the choice of  $\epsilon/2$  was made by “working backwards”). We furthermore know that any real element  $x_i$  of  $x$  can be approximated arbitrarily well by rational numbers. Thus, we can always find  $r_i$  such that  $|x_i - r_i| < \frac{\epsilon}{\sqrt{N}}$ , for the above choice of  $N$ . For this particular  $\epsilon$ , we thus define  $r_\epsilon = (r_1, r_2, \dots, r_N, 0, 0, \dots)$ , and we observe that:

$$\begin{aligned} \sum_{i=1}^{\infty} (x_i - r_i)^2 &= \sum_{i=1}^N (x_i - r_i)^2 + \sum_{i=N+1}^{\infty} (x_i - r_i)^2 < \sum_{i=1}^N \left( \frac{\epsilon}{\sqrt{N}} \right)^2 + \sum_{i=N+1}^{\infty} (x_i - 0)^2 < N \frac{\epsilon^2}{N} + \frac{\epsilon^2}{2} = \epsilon^2 \\ \implies \|x - r_\epsilon\|_2 &< \epsilon \end{aligned}$$

So far we have been working with a single sequence at a time. The final step is thus to define  $(r)^k = (r_1, r_{1/2}, r_{1/3}, \dots)$ , i.e. a sequence whose  $k$ -th term is the sequence described above for  $\epsilon = 1/k$ . Notice then that for any given  $\epsilon > 0$ , this construction guarantees that for any  $N > 1/\epsilon$  we have that  $\|x - r^{(N)}\| = \|x - r_{1/N}\| < 1/N < \epsilon$ , which completes the proof that  $(r)^{(k)} \rightarrow x$ .

The proof for  $H^\infty$  will be quite similar. We recall that the corresponding metric is:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

Consider then any  $x \in H^\infty$ ,  $x = (x_1, x_2, \dots)$ , and pick any  $\epsilon > 0$ . Because  $x \in H^\infty$ , it must be the case that  $|x_n| < 1$  for all  $n$ . Therefore, we have that for any given  $K > 0$ ,  $\sum_{n=K}^{\infty} \frac{|x_n|}{2^n} \leq \sum_{n=K}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^{K-1}}$ . We select  $N$  such that  $\frac{1}{2^N} < \frac{\epsilon}{2} \implies \frac{1}{\epsilon} < 2^{N-1} \implies N > 1 + \log(1/\epsilon)$ , and we consider a sequence of rationals  $r = (r_1, r_2, \dots, r_N, 0, 0, \dots)$ . Then:

$$d(x, r) = \sum_{n=1}^{\infty} 2^{-n} |x_n - r_n| = \sum_{n=1}^N |x_n - r_n| + \sum_{n=N+1}^{\infty} 2^{-n} |x_n| < \sum_{n=1}^N 2^{-n} |x_n - r_n| = \sum_{n=1}^N |x_n - r_n| + \frac{\epsilon}{2}$$

Using the exact same observation as above regarding rationals, we can make the first term less than  $\epsilon/2$  as well. Finally, we form  $r^{(k)} = (r_1, r_{1/2}, r_{1/3}, \dots)$  as above, to conclude that  $r^{(k)} \rightarrow x$ , and thus that  $H^\infty$  is separable.

### Exercise 51

Show that a separable metric space has at most countably many isolated points.

*Solution.*

We claim that if a metric space  $M$  is separable with  $S$  being a countable dense subset, then every one of  $M$ 's isolated points must be in  $S$ . Indeed, suppose  $x \in M$  is an isolated point and  $x \notin S$ . Then there exists  $\epsilon > 0$  such that  $B_\epsilon(x) \cap S = \emptyset$ . Therefore, no sequence  $(x_n) \subset S$  can ever have  $x$  as a limit point. Furthermore,  $x \notin S$  means that the trivial constant sequence with  $x_n = x$  does not belong in  $S$  either, and thus there is no sequence in  $S$  that converges to  $x$ . But then by the denseness of  $S$ ,  $x \notin \overline{S} = M$ , a contradiction. We conclude that all isolated points of  $M$  must be contained in the at most countably infinite set  $S$ , and so they must also be at most countable.

### Exercise 53

Can you find a countable dense subset of  $C[0, 1]$ ?

*Solution.*

We will give an example of such a set of continuous functions  $F = \{f_1, f_2, \dots\}$ . In order to do this, let us call  $P_j = \{0, \frac{1}{2^j}, \frac{2}{2^j}, \dots, \frac{2^j-1}{2^j}, 1\}$  for  $j = 1, 2, 3, \dots$ . We can think of  $P_j$  as partitioning  $[0, 1]$  in  $2^j$  disjoint intervals  $([0, 1/2^j], [1/2^j, 2/2^j], \dots, [(2^j-1)/2^j, 1])$ . We now define a set of functions  $F_j$  that achieve a rational value on all the endpoints of these intervals, and are linear in the intervals between successive endpoints. Namely, for a choice of  $2^j + 1$  rational numbers  $q_1, q_2, \dots, q_{2^j+1}$  we can define:

$$g(x) = \begin{cases} q_j & , x = p_j \text{ for some } p_j \in P_j \\ \frac{q_{j+1}-q_j}{p_{j+1}-p_j}(x - p_j) + q_j & , x \in (p_j, p_{j+1}) \end{cases}$$

We define  $F_j$  as the set of all possible such functions. We can see that any such function is continuous, and also that for a particular  $j$  there are “as many” such functions as the elements of  $\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}$ ,

where the Cartesian product is taken  $j + 1$  times. This means that each  $F_j$  is countable. If we now define  $F = \cup_{i=1}^{\infty} F_j$ , we have a countable union of countable sets, and so this is countable as well. We can think of increasing  $j$  as computing piecewise linear functions where the “pieces” get smaller and smaller. What remains now is to show that this is actually a dense subset of  $C[0, 1]$ , i.e. that any function  $f \in C[0, 1]$  can be approximated by a sequence  $(f_n) \subset F$ . In order to do this we must, for any given  $f$ , be able to find a sequence  $(f_n)$  such that for any  $\epsilon > 0$  there exists  $N > 0$  such that for  $n \geq N$ ,  $d(f, f_n) < \epsilon$ .

We first recall the definition of  $d$  (the Frechet metric):  $d(f, f_n) = \max_{0 \leq t \leq 1} |f(t) - f_n(t)|$ . Given then a particular  $f$ , we pick the  $n$ -th term of the sequence to be the function  $f_i \in F_n$  for which  $|f_i(p_n) - f(p_n)| < 1/n$  for all  $p_n \in P_n$ . Note that such a function can always be found: we can always find rationals  $f_i(p_n)$  arbitrarily close to the corresponding real  $f(p_n)$ . We recall, also, (e.g. from Hubbard & Hubbard) that any continuous function on a compact set is actually uniformly continuous. Thus, pick any  $\epsilon > 0$ , and:

- Select first a  $N_1 > 4/\epsilon$ .
- From uniform continuity, obtain  $\delta > 0$  such that whenever  $|x - y| < \delta$  it holds that  $|f(x) - f(y)| < \epsilon/4$ , and pick  $N_2 > \log_2(1/\delta)$ . Notice then that for  $n > N_2$ , the intervals formed by the endpoints of  $P_n$  have a length of less than  $\delta$  (since their length is  $1/2^n$ ).
- Set  $N = \max\{N_1, N_2\}$ .

For  $n > N$ , we have the following. By definition, on all  $p_k \in P_n$  we will have that  $|f_n(p_k) - f(p_k)| < 1/n \leq 1/N_1 < \epsilon/4$ . On all other points  $x \notin P_n$ , it is the case that  $x \in (p_j, p_{j+1})$  for some  $j$ . Then:

$$\begin{aligned} |f(x) - f_n(x)| &= \left| f(x) - \left( \frac{f_n(p_{j+1}) - f_n(p_j)}{p_{j+1} - p_j} (x - p_j) + f(p_j) \right) \right| \\ &\leq |f(x) - f(p_j)| + \left| \frac{f_n(p_{j+1}) - f_n(p_j)}{p_{j+1} - p_j} (x - p_j) \right| < \epsilon/4 + |f_n(p_{j+1}) - f_n(p_j)| \\ &= \epsilon/4 + |f_n(p_{j+1}) - f(p_j) + f(p_j) - f_n(p_j)| \leq \epsilon/4 + |f_n(p_{j+1}) - f(p_j)| + |f(p_j) - f_n(p_j)| \\ &\leq \epsilon/4 + \epsilon/4 + |f_n(p_{j+1}) - f(p_{j+1}) + f(p_{j+1}) - f(p_j)| \leq \epsilon/2 + |f_n(p_{j+1}) - f(p_{j+1})| + |f(p_{j+1}) - f(p_j)| = \epsilon \end{aligned}$$

Notice that here we used the uniform continuity of  $f$  as well as the defining property of  $N_1$ . We have shown that  $|f(x) - f_n(x)| < \epsilon$  for all  $x \in (0, 1)$ , which means of course that  $d(f_n, f) < \epsilon$ , and thus that  $f_n \rightarrow f$ . Therefore, the countable set  $F$  is dense in  $C[0, 1]$ .

## 4.2 The relative metric

### Exercise 1

Complete the proof of Proposition 4.13.

*Solution.*

We first restate parts (ii) and (iii) of 4.13 in some more detail:

(ii) We say that a set  $F \subset A \subset M$  is closed in  $(A, d)$  iff  $A \setminus F$  is open in  $A$ . We then have that  $F$  is closed in  $(A, d)$  if and only if  $F = A \cap C$ , where  $C$  is closed in  $(M, d)$ .

(iii) Given any  $E \subset A \subset M$ , we define the closure of  $E$  in  $A$ ,  $\text{cl}_A(E)$ , as the smallest set that is closed in  $A$  and contains  $E$ . Then, for any  $E \subset A$ ,  $\text{cl}_A(E) = A \cap \text{cl}_M(E)$ .

We now begin with (ii).

$\implies$ : Suppose  $F$  is closed in  $(A, d)$ . Then  $A \setminus F$  is open in  $A$ , which by part (i) is equivalent to the existence of  $U \subset M$  such that  $A \setminus F = U \cap A$ , with  $U$  open in  $M$ . Therefore  $F = A \setminus (A \setminus F) = A \setminus (U \cap A) = A \cap U^c$ . By definition,  $U^c$  is closed in  $M$ , thus completing the proof in this direction.

$\impliedby$ : Suppose now that  $F = A \cap C$  for some  $C$  closed in  $(M, d)$ . Then  $C^c$  is open in  $M$ , and  $A \setminus F = A \setminus (A \cap C) = A \cap C^c$ , which means that  $A \setminus F$  is open in  $A$ , and so by definition  $F$  is closed in  $A$ .

Continuing, for part (iii) we have the following.

Pick any  $E \subset A$ , and suppose  $x \in \text{cl}_A(E)$ . It is easy to see that  $x \in A$ , by the definition of this closure. In addition, for any closed set  $C \subset M$  such that  $E \subset C$ , we have that the set  $C \cap A$  is closed in  $A$ , and contains  $E$ , therefore  $C \cap A \supset \text{cl}_A(E)$ , and so  $x \in C \cap A$ . Since this holds for any closed  $C$ , it holds for  $\text{cl}_M(E)$ , and therefore  $x \in A \cap \text{cl}_M(E)$ , which implies  $\text{cl}_A(E) \subset A \cap \text{cl}_M(E)$ .

In the other direction, suppose  $x \in A \cap \text{cl}_M(E)$ , and let  $F$  be any set closed in  $A$  containing  $E$ . By part (ii),  $F = A \cap C$  for some  $C$  closed in  $M$ , for which it holds that  $\text{cl}_M(E) \subset C \implies x \in C$ . But then because  $x \in A$  also holds, we obtain  $x \in A \cap C = F$ . Because this holds for *any*  $F$  closed in  $A$ , it holds for  $\text{cl}_A(E)$ , which means  $x \in \text{cl}_A(E)$ , thus proving the inclusion in the other direction, and thus showing that in fact  $\text{cl}_A(E) = A \cap \text{cl}_M(E)$ .

### Exercise 62

Suppose that  $A$  is open in  $(M, d)$  and that  $G \subset A$ . Show that  $G$  is open in  $A$  if and only if  $G$  is open in  $M$ . Is the result still true if “open” is replaced everywhere by “closed”? Explain.

*Solution.*

$\implies$  : Suppose  $G \subset A$  is open in  $A$ . Then there exists  $U \subset M$  open such that  $G = A \cap U$ . Therefore,  $G$  is open as the intersection of two open sets.

$\impliedby$  : Suppose  $G \subset A$  is open in  $M$ . Then, by writing  $G = A \cap G$ , we trivially fulfill the definition of  $G$  being open in  $A$  as well.

Now suppose that we replace “open” with “closed” everywhere, to obtain the statement “if  $A$  is closed in  $(M, d)$ , then any  $F \subset A$  is closed in  $A$  if and only if  $F$  is closed in  $M$ ”.

$\implies$  : Suppose  $F$  is closed in  $A$ , in which case  $F = A \cap C$  for some  $C$  closed in  $M$ . Since  $A$  is also closed, we have that  $F$  is closed in  $M$  as the intersection of two closed sets.

$\impliedby$  : Suppose now that  $F \subset A$  is closed in  $M$ . Then we can write  $F = A \cap F$ , where  $F$  is closed in  $M$ , and so we also have that  $F$  is closed in  $A$ , which means that the statement we obtained is also true.

### Exercise 63

Is there a nonempty subset of  $\mathbb{R}$  that is open when considered as a subset of  $\mathbb{R}^2$ ? Closed?

*Solution.*

Identify  $\mathbb{R}$  with the set  $\{(x, 0) | x \in \mathbb{R}\} \subset \mathbb{R}^2$  and consider any  $S \subset \mathbb{R}$ ,  $S \neq \emptyset$  that is open in  $\mathbb{R}$  as well as in  $\mathbb{R}^2$ . Then for any  $p \in S$  (there exists at least one since  $S$  is not empty), there must exist  $\epsilon > 0$  such that  $B_\epsilon^{\mathbb{R}^2}(p) \subset \mathbb{R}$ .  $p$  is of the form  $p = (x, 0)$ , so the point  $(x, \epsilon/2)$  is contained in this open ball, and would therefore have to belong in  $\mathbb{R}$ , which is by definition impossible. Therefore, no nonempty open subset of  $\mathbb{R}$  can be empty as a subset of  $\mathbb{R}^2$ . Turning the discussion to closed sets instead, consider the set  $[-1, 1] \subset \mathbb{R}$ . It is easy to see that this is a closed set in  $\mathbb{R}^2$  (by e.g. using the sequence-based definition of closedness). Furthermore, if  $C$  is the (closed) unit disk in  $\mathbb{R}^2$  we can see that  $C \cap \mathbb{R} = [-1, 1]$ , and thus  $[-1, 1]$  is indeed closed in  $\mathbb{R}$ .

### Exercise 68

If  $A$  is a separable subset of  $M$  (that is, if  $A$  has a countable dense subset of its own), show that  $\overline{A}$  is also separable.

*Solution.*

Let  $S = \{s_1, s_2, s_3, \dots\}$  be a countable dense subset of  $A$ . Recall that  $\overline{A}$  consists precisely of elements  $y$  such that some sequence  $(x_n) \subset A$  converges to  $y$  (this sequence may be constant). For any given  $y$ , we have two cases:

- $x_n \rightarrow y$  is a constant sequence, which means  $y \in A$ . Then by the separability of  $A$ , there exists  $(s_k) \subset S$  such that  $s_k \rightarrow y$ .
- $x_n \rightarrow y$  is not constant, and therefore  $y \notin A$  (in this case,  $y$  lies on the boundary of  $A$ ). We describe a process for forming a sequence  $(s_k) \subset S$  such that  $s_k \rightarrow y$ . Let  $\epsilon_i = 1/i$ , and find  $N_i > 0$  such that for  $n \geq N_i$ ,  $d(x_n, y) < \epsilon_i/2$ . Find also  $K_i > 0$  such that for  $k \geq K_i$ ,  $d(s_k, x_{N_i}) < \epsilon_i/2$  (this is possible due to  $A$  being separable). By the triangle inequality we obtain that  $d(s_{K_i}, y) \leq d(s_{K_i}, x_{N_i}) + d(x_{N_i}, y) < \epsilon_i/2 + \epsilon_i/2 = \epsilon_i$ . Therefore, we set the  $i$ -th term of the constructed sequence to be  $s_{K_i}$ . The fact that  $\epsilon_i$  decrease monotonically means that the sequence formed does indeed converge to  $y$ .

Since we were able to show that for any  $y \in \overline{A}$  there exists  $(s_k) \subset S \subset A \subset \overline{A}$  such that  $(s_k) \rightarrow y$  and  $S$  is countable, we have that  $\overline{A}$  is indeed separable.



# Chapter 5

## Open Sets and Closed Sets

### 5.1 Continuous Functions

#### Exercise 4

Show that  $\chi_{\Delta} : \mathbb{R} \rightarrow \mathbb{R}$ , the characteristic function of the Cantor set, is discontinuous at each point of  $\Delta$ .

*Solution.*

We will do this by contradiction. Suppose  $\epsilon = 1/2$  and  $x \in \Delta$ . If  $\chi_{\Delta}$  is continuous at  $x$ , then there exists  $\delta > 0$  such that whenever  $|y - x| < \delta$  it holds that  $|\chi_{\Delta}(y) - \chi_{\Delta}(x)| < \epsilon$ . Pick any such  $y$ . By definition of the characteristic function, this implies that  $\chi_{\Delta}(y) = 1$ , and so that  $y \in \Delta$ . But then by exercise 22 of Chapter 2 (page 29 on the book), we know that for our  $x, y \in \Delta$ , where either  $x < y$  or  $y < x$ , there exists  $z$  such that  $x < z < y$  or  $y < z < x$  respectively such that  $z \notin \Delta$ . The consequence of this is that  $|z - x| < \delta$  but  $\chi_{\Delta}(z) = 0$ , and so  $|\chi_{\Delta}(z) - \chi_{\Delta}(x)| = 1 > 1/2$ , a contradiction. Therefore  $\chi_{\Delta}$  is not continuous at  $x$ .

#### Exercise 7

- If  $f : M \rightarrow \mathbb{R}$  is continuous and  $a \in \mathbb{R}$ , show that the sets  $\{x : f(x) > a\}$  and  $\{x : f(x) < a\}$  are open subsets of  $M$ .
- Conversely, if the sets  $\{x : f(x) > a\}$  and  $\{x : f(x) < a\}$  are open for every  $a \in \mathbb{R}$ , show that  $f$  is continuous.
- Show that  $f$  is continuous even if we assume only that the sets  $\{x : f(x) > a\}$  and  $\{x : f(x) < a\}$  are open for every *rational*  $a$ .

*Solution.*

(a) For any  $a \in \mathbb{R}$ , consider the set  $S = \{y \in \mathbb{R} \mid y > a\}$ . This is clearly seen to be an open set. Then, theorem 5.1 (equivalent definitions of continuity) guarantees that  $f^{-1}(S)$  is open in  $M$ . This set equals  $f^{-1}(S) = \{x \in M \mid f(x) \in S\} = \{x \in M \mid f(x) > a\}$ . The same holds if “ $>$ ” is replaced by “ $<$ ”, thus completing the proof.

(b) Suppose  $N$  is any open subset of  $\mathbb{R}$ . If we can show that  $f^{-1}(N)$  is open in  $M$ , then we will have equivalently shown that  $f$  is continuous. We know (theorem 4.6) that  $N$  can be written as a countable union of disjoint open intervals:

$$N = \bigcup_{n=1}^{\infty} I_n, I_n = (a_n, b_n),$$

where  $I_n \cap I_m = \emptyset$  for  $n \neq m$ . Then we have that  $f^{-1}(N) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$ . Note that in general the boundaries of these intervals may equal the two infinities. First, in the special case where this union is simply the interval  $(-\infty, \infty)$  (i.e., the chosen  $N$  equals  $\mathbb{R}$ ), we can immediately conclude that  $f^{-1}(N) = M$ : if this was not the case, then there would exist some  $x \in M$  such that  $f(x) \notin N = \mathbb{R}$ , which is clearly impossible. Note that, trivially,  $f^{-1}(N)$  is then open (example 4.1 (a), page 51). In every other case, no interval  $I_n$  is of this form, and  $N$  is a union of *at least one* interval  $(a_n, b_n)$  where not both of the endpoints are infinities. We are now interested in  $f^{-1}(I_n)$ . If both  $a_n, b_n$  are real numbers, write

$I_n = (-\infty, b_n) \cap (a_n, \infty)$ . If not, proceed with  $I_n = (-\infty, b_n)$  or  $I_n = (a_n, \infty)$ . In the last two cases, one easily sees that  $f^{-1}(N) = \{x \in M : f(x) < b_n\}$ ,  $f^{-1}(N) = \{x \in M : f(x) > a_n\}$  respectively, both of which are open by our hypothesis. In the first case,  $f^{-1}(N)$  equals a finite intersection of open sets (by utilizing what we proved just now), and as such is again open. Then we conclude that  $f^{-1}(N) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$  is also open as an arbitrary union of open sets, thus also obtaining that  $f$  is continuous.

(c) The proof here relies on exercise 7 of chapter 4 (page 55), which states that every open set in  $\mathbb{R}$  can be written as the union of countably many open intervals with rational endpoints. Since disjointness was never necessary in (b), everything else follows in exactly the same way, because we know that  $\{x : f(x) > a\}$ ,  $\{x : f(x) < a\}$  are open for rational  $a$ , and every interval we will encounter here will be of this form. Therefore,  $f$  is again continuous.

### Exercise 8

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous.

(a) If  $f(0) > 0$ , show that  $f(x) > 0$  for all  $x$  in some interval  $(-a, a)$ .

(b) If  $f(x) \geq 0$  for every rational  $x$ , show that  $f(x) \geq 0$  for all real  $x$ . Will this result hold with “ $\geq 0$ ” replaced by “ $> 0$ ”? Explain.

*Solution.*

(a) Suppose  $f(x) = c > 0$ , and set  $\epsilon = c/2 > 0$ . Due to the continuity of  $f$ , there exists  $\delta > 0$  such that whenever  $|x - 0| < \delta$ , it holds that  $|f(x) - f(0)| < \epsilon$ . This implies that  $-c/2 < f(x) - c < c/2 \implies c/2 < f(x) < 3c/2$ . But then setting  $a = \delta$  means that for  $x \in (-a, a)$  it holds that  $f(x) > 0$ .

(b) Pick any  $y \in \mathbb{R}$ . We know that we can always find a sequence  $(x_n) \subset \mathbb{Q}$  such that  $x_n \rightarrow y$ . By the continuity of  $f$ , it must hold that  $f(x_n) \rightarrow f(y)$ . Suppose now that  $f(y) < 0$ , and set  $\epsilon = -f(y)/2$ . Then we can find  $N > 0$  such that whenever  $n \geq N$  it holds that:

$$|f(x_n) - f(y)| < \epsilon \implies f(y)/2 < f(x_n) - f(y) < -f(y)/2 \implies f(x_n) < f(y)/2 < 0,$$

which is a contradiction. Therefore it must hold that  $f(y) \geq 0$  for every  $y \in \mathbb{R}$ . Now for the second part, consider the function  $f(x) = (x - \sqrt{2})^2$ . Clearly,  $f$  is continuous and strictly positive for all  $x \in \mathbb{R} \setminus \{\sqrt{2}\}$ , which means that it is more specifically continuous for all  $x \in \mathbb{Q}$ . However,  $f(\sqrt{2}) = 0$ , which is not strictly positive, and thus the assertion that “ $f(x) > 0$  for every rational  $x$  implies that  $f(x) > 0$  for every real  $x$ ” fails.

### Exercise 9

Let  $A \subset M$ . Show that  $f : (A, d) \rightarrow (N, \rho)$  is continuous at  $a \in A$  if and only if, given  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $\rho(f(x), f(a)) < \epsilon$  whenever  $d(x, a) < \delta$  and  $x \in A$ . We paraphrase this statement by saying that “ $f$  has a point of continuity relative to  $A$ ”.

*Solution.*

Recall the definition of continuity in a metric space  $M$ :  $f : (M, d) \rightarrow (N, \rho)$  is continuous at  $a \in M$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that whenever  $d(x, y) < \delta$  it holds that  $\rho(f(x), f(y)) < \epsilon$ . We know that if  $M$  is a metric space,  $A \subset M$  is also a metric space. Therefore, the definition of continuity for functions  $f : (A, d) \rightarrow (N, \rho)$  states that  $f$  is continuous at  $a \in A$  precisely when, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\rho(f(x), f(a)) < \epsilon$  whenever  $d(x, a) < \delta$  for  $x$  inside the metric space, which in this case is  $A$ .

### Exercise 10

Let  $A = (0, 1] \cup \{2\}$ , considered as a subset of  $\mathbb{R}$ . Show that every function  $f : A \rightarrow \mathbb{R}$  is continuous, relative to  $A$ , at 2.

*Solution.*

Pick any  $\epsilon > 0$ , and set e.g.  $\delta = 1/3$ . Observe then that  $B_{1/3}^A(2) = \{2\}$ . Therefore, for every  $x \in A$  such that  $|x - 2| < 1/3$ , it trivially holds that  $|f(x) - f(2)| < \epsilon$ , and so  $f$  is continuous at 2 relative to  $A$ .

**Exercise 14**

A continuous function on  $\mathbb{R}$  is completely determined by its values on  $\mathbb{Q}$ . Use this to “count” the continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

*Solution.*

First of all, to see that a continuous function is indeed determined by its values on  $\mathbb{Q}$ , suppose that  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $f(x) = g(x)$  for all  $x \in \mathbb{Q}$ . Then if  $x \in \mathbb{R} \setminus \mathbb{Q}$ , we know that there exists a sequence  $(x_n) \subset \mathbb{Q}$  such that  $x_n \rightarrow x$ . It is the case that  $f(x_n) = g(x_n)$  for all  $n$ , and by the continuity of  $f, g$  we have that  $f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = g(x)$ , therefore  $f = g$ .

We begin our proof by noticing that the above observation implies that there exists an injection  $F : C^0 \rightarrow \mathbb{R}^\mathbb{Q}$ .

Furthermore, there exists a trivial injection  $G$  from  $\mathbb{R}$  to  $C^0$ : map any real number  $a$  to the clearly continuous  $f(x) = a$ .

Therefore, if we can now show that  $\mathbb{R}^\mathbb{Q} \sim \mathbb{R}$ , the existence of  $F$  guarantees the existence of an injection from the set of continuous functions to  $\mathbb{R}$  as well. This injection coupled with  $G$  will allow us to apply the Cantor-Schröder-Bernstein theorem to obtain that in fact  $C^0 \sim \mathbb{R}$ . In showing  $\mathbb{R}^\mathbb{Q} \sim \mathbb{R}$ , the following lemma will be useful:

If  $A \sim B$ , and  $C$  is countable, then  $A^C \sim B^C$ . Indeed, we can construct a bijection  $F : A^C \rightarrow B^C$  in the following manner. First, let  $h$  be a bijection from  $A$  to  $B$ , which we know exists. If  $f \in A^C$ , we have  $f(c_1) = a_1, f(c_2) = a_2, \dots$ . We obtain  $b_1, b_2, \dots$  by  $h(a_i) = b_i$ , and we define  $g \in B^C$  as  $g(c_1) = b_1, g(c_2) = b_2, \dots$ . We then define  $F(f) = g$ . To prove injectivity, suppose  $F(f_1) = F(f_2)$  for  $f_1 \neq f_2$ . This means that there exists  $c_n \in C$  such that  $f_1(c_n) \neq f_2(c_n)$ , and so  $h(f_1(c_n)) = F(f_1)(c_n) \neq h(f_2(c_n)) = F(f_2)(c_n)$ , due to the injectivity of  $h$ . But then  $F(f_1) \neq F(f_2)$ , a contradiction. For surjectivity, pick  $g \in B^C$ , which is completely determined by the values  $g(c_1), g(c_2), \dots$ . Let then  $a_n \in A, a_n = h^{-1}(g(c_n))$ , which are well-defined since  $h$  is a bijection. Then by defining  $f \in A^C$  such that  $f(c_n) = a_n$  we have  $F(f) = g$ , and thus  $F$  is also a surjection, and therefore a bijection.

Notice now the following. We know  $\mathbb{R} \sim [0, 1]$ , and by the lemma above we then have  $\mathbb{R}^\mathbb{Q} \sim [0, 1]^\mathbb{Q}$  since  $\mathbb{Q}$  is countable. For the remaining proof, we can thus work with  $[0, 1]^\mathbb{Q}$ .

Pick any  $f \in [0, 1]^\mathbb{Q}$ , and let  $f(q_n) = a_n \in [0, 1]$ , where we write  $a_n$  in binary form, and if it can be written as both ending in infinite 1s and in infinite 0s, we choose the latter. In a manner similar to Cantor’s diagonal argument, we construct the following list:

$$f(q_1) = a_{11}.a_{12}a_{13}\dots$$

$$f(q_2) = a_{21}.a_{22}a_{23}\dots$$

⋮

Define then  $b_f = 0.a_{11}a_{21}a_{12}a_{31}a_{22}a_{13}\dots \in [0, 1]$ . We claim that the function  $F : [0, 1]^\mathbb{Q} \rightarrow [0, 1]$  defined by  $F(f) = b_f$  is an injection. Indeed, if  $b_f = b_g$  for two such  $f, g$ , if  $b_f, b_g$  are equal on all corresponding digits, then  $f = g$  trivially. If not, then one of them, say  $b_f$ , ends in infinite ones, and the other in infinite zeros. However, this would imply that the construction above yields a  $b_f$  ending in infinite ones, and thus that  $f(q_i)$  end in infinite ones, which is impossible due to how we chose to construct these  $b$ . Therefore  $F$  is injective. We now also find an injection  $g : [0, 1] \rightarrow [0, 1]^\mathbb{Q}$ . For  $a \in [0, 1]$ , write  $a = a_1.a_2a_3\dots$  in binary form, again choosing infinite zeros over infinite ones as a postfix whenever necessary. We then define  $g(a) = f$  such that  $f(q_i) = 0.0\dots 0a_i$ , where the prefix is of length  $i - 1$  (for  $f(q_1)$ , we have  $f(q_1) = a_1$ ). Notice again that this is easily seen to be an injection, since  $g(a) = g(b)$  implies that all corresponding digits of  $a, b$  are equal. By the Cantor-Schröder-Bernstein theorem, we then have that  $[0, 1]^\mathbb{Q} \sim [0, 1]$ , and thus that  $\mathbb{R}^\mathbb{Q} \sim \mathbb{R}$ , i.e. there are “as many” continuous functions as real numbers.

**Exercise 17**

Let  $f, g : (M, d) \rightarrow (N, \rho)$  be continuous, and let  $D$  be a dense subset of  $M$ . If  $f(x) = g(x)$  for all  $x \in D$ , show that  $f(x) = g(x)$  for all  $x \in M$ . If  $f$  is onto, show that  $f(D)$  is dense in  $N$ .

*Solution.*

Suppose that there exists  $x \in M \setminus D$  such that  $f(x) \neq g(x)$ , which means that  $\rho(f(x), g(x)) = \epsilon > 0$ . By the denseness of  $D$ , there exists a sequence  $(x_n) \subset D, x_n \rightarrow x$ . Because  $f, g$  are both continuous, it holds that  $f(x_n) \rightarrow f(x)$  and  $g(x_n) \rightarrow g(x)$ . Therefore, there exist  $N_1, N_2 > 0$  such that for  $n \geq \max\{N_1, N_2\}$  it holds that  $\rho(f(x_n), f(x)) < \frac{\epsilon}{2}, \rho(g(x_n), g(x)) < \frac{\epsilon}{2}$ . Notice, furthermore, that  $f(x_n) = g(x_n)$  since the two functions are equal on  $D$ . Then, by the triangle inequality we obtain that:

$$\begin{aligned}\epsilon &= \rho(f(x), g(x)) \leq \rho(f(x), f(x_n)) + \rho(f(x_n), g(x)) = \rho(f(x), f(x_n)) + \rho(g(x_n), g(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\implies \epsilon < \epsilon,\end{aligned}$$

which is a contradiction. Therefore  $f(x) = g(x)$  on all of  $M$ .

Now suppose that  $f$  is onto, and pick any  $y \in N$ . Then there exists  $x \in M$  such that  $f(x) = y$ . Furthermore, by the denseness of  $D$  there exists  $(x_n) \subset D$  such that  $x_n \rightarrow x$ . The continuity of  $f$  then implies  $f(x_n) \rightarrow f(x) = y$ . But  $f(x_n)$  is a sequence entirely in  $f(D)$ , which means that  $f(D)$  is dense in  $N$ , since  $y$  was picked arbitrarily in  $N$ .

**Exercise 18**

Let  $f : (M, d) \rightarrow (N, \rho)$  be continuous, and let  $A$  be a separable subset of  $M$ . Prove that  $f(A)$  is separable.

*Solution.*

We need to find a countable dense subset of  $f(A)$ . Since  $A$  is separable, it contains a countable and dense subset  $S \subset A, S = \{a_1, a_2, \dots\}$ . Pick any element  $y \in f(A)$ , which means there exists  $a \in A$  such that  $y = f(a)$ . Since  $S$  is dense, there exists a sequence  $(a_n) \subset A$  such that  $a_n \rightarrow a$ , and because  $f$  is continuous,  $f(a_n) \rightarrow f(a)$ . Notice that this holds for any  $y \in f(A)$ . The consequence is that if we form the set  $S' = \{f(a_1), f(a_2), \dots\}$ , which is clearly countable because  $S$  is countable, we are always able to find a sequence in  $S'$  converging to any element of  $A$ . Therefore  $f(A)$  is separable.

**Exercise 19**

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to satisfy a **Lipschitz condition** if there is a constant  $K < \infty$  such that  $|f(x) - f(y)| \leq K|x - y|$  for all  $x, y \in \mathbb{R}$ . More economically, we may say that  $f$  is Lipschitz (or Lipschitz with constant  $K$  if a particular constant seems to matter). Show that  $\sin x$  is Lipschitz with constant  $K = 1$ . Prove that a Lipschitz function is (uniformly) continuous.

*Solution.*

Pick any two  $x, y \in \mathbb{R}$ , and let  $h = x - y$ . Then we have that:

$$\begin{aligned}|f(x) - f(y)| &= |\sin(x) - \sin(y)| = |\sin(y + h) - \sin(y)| = \left| 2 \sin\left(\frac{h}{2}\right) \cos\left(\frac{y+y+h}{2}\right) \right| \\ &\leq \left| 2 \sin\left(\frac{h}{2}\right) \right| \leq 2 \left| \frac{h}{2} \right| \leq |h| = |x - y|,\end{aligned}$$

where we used the identity  $\sin(a) - \sin(b) = 2 \sin((a - b)/2) \cos((a + b)/2)$ , the fact that  $|\cos(a)| \leq 1$  for any  $a$ , and the inequality  $|\sin(a)| \leq |a|$  for any  $a$ . We've thus shown that  $\sin$  is Lipschitz continuous with  $K = 1$ . Now we want to show that any Lipschitz continuous function (with constant  $K$ ) is uniformly continuous. Pick any  $\epsilon$ , and set  $\delta = \frac{\epsilon}{K}$ . Then observe that, whenever  $|x - y| < \delta$ :

$$|f(x) - f(y)| \leq K|x - y| < K\delta = K \frac{\epsilon}{K} = \epsilon,$$

which is precisely the definition of uniform continuity for  $f$ .

**Exercise 20**

If  $d$  is a metric on  $M$ , show that  $|d(x, z) - d(y, z)| \leq d(x, y)$  and conclude that the function  $f(x) = d(x, z)$  is continuous on  $M$  for any fixed  $z \in M$ . This says that  $d(x, y)$  is *separately continuous* - continuous in each variable separately.

*Solution.*

We have already shown this inequality in exercise 2 of chapter 3. What we are then asked to show is that for any *fixed*  $z \in M$ , and for any  $x \in M$ , the function  $f(x) = d(x, z)$  is continuous at  $x$ . To do this, pick any  $\epsilon > 0$ . We need to find  $\delta > 0$  such that whenever  $d(x, y) < \delta$  it holds that  $|f(y) - f(x)| < \epsilon$  (note that the metric space containing the codomain is simply  $\mathbb{R}$ ). Observe that:

$$|f(y) - f(x)| = |d(y, z) - d(x, z)| \leq d(x, y)$$

Therefore, if we set  $\delta = \epsilon/2$ , we obtain that  $|f(y) - f(x)| \leq \epsilon/2 < \epsilon$ , and thus that  $f$  is continuous on  $M$ .

**Exercise 23**

Define  $S : c_0 \rightarrow c_0$  by  $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ . That is,  $S$  shifts the entries forward and puts 0 in the empty slot. Show that  $S$  is an isometry (into).

*Solution.*

We need to show that for any two sequences  $(x_n), (y_n) \in c_0$ ,  $d(S((x_n)), S((y_n))) = d((x_n), (y_n))$ , where  $d((x_n), (y_n)) = \sup_n |x_n - y_n|$ . We notice that:

$$d(S((x_n)), S((y_n))) = \sup_n \{|0 - 0|, |x_1 - y_1|, |x_2 - y_2|, \dots\} = \sup_n \{|x_1 - y_1|, |x_2 - y_2|, \dots\},$$

since every term here is non-negative. This last quantity equals precisely  $d((x_n), (y_n))$ , which completes the proof that  $S$  is an isometry.

**Exercise 24**

Let  $V$  be a normed vector space. If  $y \in V$  is fixed, show that the maps  $a \mapsto ay$  from  $\mathbb{R}$  to  $V$  and  $x \mapsto x + y$  from  $V$  to  $V$  are continuous.

*Solution.*

Let  $g : \mathbb{R} \rightarrow V$ ,  $g(a) = ay$  for a fixed  $y \in V$ . Note that  $g$  is obviously continuous if  $y = 0$ , so from now on assume  $y \neq 0$ . We need to show that for any  $a \in \mathbb{R}$ , it holds that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $|b - a| < \delta$ , it holds that  $\|g(b) - g(a)\| < \epsilon$  (for the usual metric on  $V$ ). By the properties of normed vector spaces, we have that:

$$\|g(b) - g(a)\| = \|by - ba\| = |b - a| \cdot \|y\|$$

For a given  $\epsilon > 0$ , set  $\delta = \frac{\epsilon}{\|y\|}$ . Then, whenever  $|b - a| < \delta$ , the equality above shows that  $\|g(b) - g(a)\| < \delta \|y\| = \epsilon$ , i.e., that  $g$  is indeed continuous.

Now let instead  $g : V \rightarrow V$  be  $g(x) = x + y$  for a fixed  $y \in V$ . For any  $x, z \in V$  we have that:

$$\|g(z) - g(x)\| = \|(z + y) - (x + y)\| = \|z - x\|$$

Therefore, for any given  $\epsilon > 0$ , if we set  $\delta = \epsilon$ , whenever  $\|z - x\| < \delta$ , the equality above tells us that  $\|g(z) - g(x)\| < \delta = \epsilon$ , and so that  $g$  is continuous on any  $x \in V$ .

**Exercise 25**

A function  $f : (M, d) \rightarrow (N, \rho)$  is called **Lipschitz** if there is a constant  $K$  such that  $\rho(f(x), f(y)) \leq Kd(x, y)$  for all  $x, y \in M$ . Prove that a Lipschitz mapping is continuous.

*Solution.*

Suppose first that  $K = 0$ . Then for any two  $x, y \in M$  it holds that  $\rho(f(x), f(y)) \leq 0$ . By the properties of metrics, we know that this means that  $\rho(f(x), f(y)) = 0$ , and this in turn means that  $f(x) = f(y)$  for all  $x, y$ , thus  $f$  is constant, and trivially continuous.

If  $K \neq 0$ , pick any  $x \in M$  and any  $\epsilon > 0$ . Set  $\delta = \frac{\epsilon}{K}$ , and observe that whenever  $d(x, y) < \delta$ , we have that:

$$\rho(f(x), f(y)) \leq Kd(x, y) < K\delta = K \frac{\epsilon}{K} = \epsilon,$$

which is precisely the definition of continuity for  $f$ .

**Exercise 26**

Provide the answer to a question raised in Chapter Three by showing that integration is continuous. Specifically, show that the map  $L(f) = \int_a^b f(t)dt$  is Lipschitz with constant  $K = b - a$  for  $f \in C[a, b]$ .

*Solution.*

We recall first the Frechet metric for continuous functions on  $[a, b] : d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$ . The question raised in the beginning of Chapter Three was whether integration is continuous, which, in other words, would mean that the quantities  $L(f), L(g)$  are “close” whenever  $f, g$  are close. To answer this, we observe that:

$$|L(f) - L(g)| = \left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| = \left| \int_a^b (f(t) - g(t))dt \right|$$

Now we call the mean value theorem for integration: there exists  $c \in (a, b)$  such that  $\int_a^b (f(t) - g(t))dt = (f(c) - g(c))(b - a)$ . Therefore:

$$|L(f) - L(g)| = |(f(c) - g(c))(b - a)| = (b - a)|f(c) - g(c)| \leq (b - a) \max_{a \leq t \leq b} |f(t) - g(t)| = (b - a)d(f, g)$$

We've therefore shown that  $|L(f) - L(g)| \leq (b - a)d(f, g)$ , which means precisely that  $L$  is Lipschitz with  $K = b - a$ .

**Exercise 28**

Define  $g : l_2 \rightarrow \mathbb{R}$  by  $g(x) = \sum_{n=1}^{\infty} x_n/n$ . Is  $g$  continuous?

*Solution.*

Let  $y = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ , i.e. the sequence defined by  $y_k = \frac{1}{k}$ , which as we know is in  $l_2$ . We observe then that for a given sequence of sequences  $(x^{(n)})$  that converges to a sequence  $x$  (all sequences involved are in  $l_2$ ), we have that:

$$|g(x^{(n)}) - g(x)| = \left| \sum_{k=1}^{\infty} (x_k^{(n)} - x_k)/k \right| \leq \sum_{k=1}^{\infty} |(x_k^{(n)} - x_k)/k|,$$

where we note that while at this point we are not sure that the sums converge, we will see that the RHS is in fact bounded, and so we would obtain the above result by initially examining finite sums only. Specifically, we observe that we can apply the Cauchy-Schwarz inequality to the RHS, since both  $(x^{(n)} - x)$  and  $y$  are in  $l_2$ . From this we obtain that:

$$|g(x^{(n)}) - g(x)| \leq \|x^{(n)} - x\|_2 \cdot \|y\|_2$$

We've thus shown that  $g$  is Lipschitz with constant  $\|y\|_2$ , and so also continuous.

**Exercise 30**

Let  $f : (M, d) \rightarrow (N, \rho)$ . Prove that  $f$  is continuous if and only if  $f(\overline{A}) \subset \overline{f(A)}$  for every  $A \subset M$  if and only if  $f^{-1}(\overset{\circ}{B}) \subset (f^{-1}(B))^\circ$  for every  $B \subset N$ . Give an example of a continuous  $f$  such that  $f(\overline{A}) \neq \overline{f(A)}$  for some  $A \subset M$ .

*Solution.*

For ease of notation, name the three propositions:

(a):  $f$  is continuous

(b):  $f(\overline{A}) \subset \overline{f(A)}$  for every  $A \subset M$

(c):  $f^{-1}(\overset{\circ}{B}) \subset (f^{-1}(B))^\circ$  for every  $B \subset N$

(a)  $\implies$  (b): Pick any  $A \subset M$  and any  $y \in f(\overline{A})$ . We need to show that  $y \in \overline{f(A)}$ . Because  $y \in f(\overline{A})$ , we have that there exists  $x \in \overline{A}$  such that  $f(x) = y$ . We recall that  $x \in \overline{A}$  is equivalent to the existence of a sequence  $(x_n) \subset A$  such that  $x_n \rightarrow x$ . Because  $f$  is continuous, we must then have that  $f(x_n) \rightarrow f(x) = y$ .

By definition of the image of a function,  $f(x_n) \in f(A)$  for every  $n$ . Once again,  $f(x_n) \rightarrow y$  with  $(f(x_n)) \subset f(A)$  is equivalent to  $y \in \overline{f(A)}$ .

(b)  $\implies$  (a): For the converse, pick any  $x \in M$ . If no sequences in  $M$  converge to  $x$ , then there exists  $\delta > 0$  such that  $d(x, y) > \delta$  for every  $y \in M, y \neq x$ . Now pick any  $\epsilon > 0$  and observe that  $d(x, y) < \delta$  is only satisfied when  $y = x$ , in which case of course  $\rho(f(x), f(y)) = 0 < \epsilon$ , so  $f$  is continuous at  $x$ . In the other case, we have that there exists at least one sequence in  $M$  that converges to  $x$ . By way of contradiction, assume that there  $f$  is not continuous, and thus that for some  $x_n \rightarrow x$  it does not hold that  $f(x_n) \rightarrow f(x)$ . By negating the definition of the limit, we obtain that there exists  $\epsilon > 0$  such that for every  $N > 0$  there exists  $n \geq N$  such that  $\rho(f(x_n), f(x)) \geq \epsilon$ . Gather then an infinite sequence  $(f(x_i))$  for which  $\rho(f(x_i), f(x)) \geq \epsilon$ , and notice that  $x_i \rightarrow x$  since the corresponding  $x_i$  form a subsequence of  $(x_n)$ . We observe now that if  $A = (x_i), \overline{A} = \{x_1, x_2, \dots\} \cup \{x\}$ : this is due to the uniqueness of the limit, as well as the fact that  $x_i \rightarrow x$ . Then  $f(x) \in \overline{f(A)} \subset \overline{f(A)}$ , which in turn means that there exists a sequence  $(f(y_j)) \subset f(A)$  such that  $f(y_j) \rightarrow f(x)$  (by the definition of the closure). Since  $f(A) = \{f(x_1), f(x_2), \dots\}$ , we conclude that each  $f(y_j)$  must equal some  $f(x_i)$ . But then  $\rho(f(y_j), f(x)) \geq \epsilon$  for every  $j$ , and thus  $f(y_j)$  cannot converge to  $x$ , a contradiction. Therefore  $f$  must be continuous.

(a)  $\implies$  (c): Pick any  $B \subset N$ , and any  $x \in f^{-1}(\overset{\circ}{B})$ . This means that there exists  $\epsilon > 0$  such that, if  $y = f(x)$ , then  $B_\epsilon^\rho(y) \subset B$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that whenever  $d(x', x) < \delta$  it holds that  $\rho(f(x'), y) < \epsilon$ . Therefore, for every such  $x', f(x') \in B_\epsilon^\rho(y)$ , which implies  $f(x') \in B$ . We thus obtain that  $x' \in f^{-1}(B)$ , and since this holds for every  $x'$  with  $d(x', x) < \delta$ , we have that  $B_\delta^d(x) \subset f^{-1}(B)$ , which means precisely that  $x \in (f^{-1}(B))$ .

(c)  $\implies$  (a): Suppose  $B \subset N$  is open. Then  $B = \overset{\circ}{B}$  and we have that  $f^{-1}(B) = f^{-1}(\overset{\circ}{B}) \subset (f^{-1}(\overset{\circ}{B}))$ . We know of course that  $f^{-1}(\overset{\circ}{B}) \subset f^{-1}(B)$ , which means  $f^{-1}(B) = f^{-1}(\overset{\circ}{B})$ , which is equivalent to  $f^{-1}(B)$  being open. Since this holds for any open  $B$ , we have one of the conditions that are known to be equivalent to  $f$  being continuous: namely, that the inverse image of any open set under  $f$  is also open.

For the requested example, consider  $M = \mathbb{N}$  equipped with the discrete metric,  $N = \mathbb{R}$  and  $f : M \rightarrow N, f(i) = \frac{1}{i}, i \neq 0, f(0) = 1$ , which we easily see is continuous. Then if  $A = \mathbb{N}^+$ , we notice that  $\overline{A} = A$  and  $f(\overline{A}) = \{1, 1/2, 1/3, \dots\}$ . However,  $\overline{f(A)} = \{0, 1, 1/2, 1/3, \dots\}$ , and so  $f(\overline{A}) \neq f(\overline{A})$ .

### Exercise 31

Let  $f : (M, d) \rightarrow (N, \rho)$ .

(a) If  $M = \bigcup_{n=1}^{\infty} U_n$ , where each  $U_n$  is an open set in  $M$ , and if  $f$  is continuous on each  $U_n$  relative to that  $U_n$ , show that  $f$  is continuous on  $M$ .

(b) If  $M = \bigcup_{n=1}^{\infty} E_n$ , where each  $E_n$  is a closed set in  $M$ , and if  $f$  is continuous on each  $E_n$  relative to that  $E_n$ , show that  $f$  is continuous on  $M$ .

(c) Give an example showing that  $f$  can fail to be continuous on all of  $M$  if, instead, we use a countably infinite union of closed sets  $M = \bigcup_{n=1}^{\infty} E_n$  in (b).

*Solution.*

(a) Pick any  $x \in M$ . Then  $x \in U_i$  for at least one  $i$ . Because  $U_i$  is open, there exists  $\delta_1$  such that  $B_{\delta_1}^d(x) \subset U_i$ . In addition, because  $f$  is continuous on  $x$  relative to  $U_i$ , for every  $\epsilon > 0$  there exists  $\delta_2 > 0$  such that whenever  $d(x', x) < \delta_2$  and  $x' \in U_i$  it holds that  $\rho(f(x'), f(x)) < \epsilon$ . Set  $\delta = \min\{\delta_1, \delta_2\}$ . Then, whenever  $d(x', x) < \delta$  we have  $x' \in U_i$ , and also  $d(x', x) < \delta_2$ , thus  $\rho(f(x'), f(x)) < \epsilon$ . Therefore  $f$  is indeed continuous on  $x$  in the general sense.

(b) Again, pick any  $x \in M$ . Then  $x \in E_i$  for at least one  $i$ . If  $x \in \overset{\circ}{E}_i$ , an argument like the one in (a) serves to show that  $f$  is continuous at  $x$ . The complication here is it may be the case that  $x \in \partial E_i$ , in which case for any open ball  $B_\delta^d(x), B_\delta^d(x) \cap E_i^c \neq \emptyset$ . Suppose then that  $x \in \partial E_i$ , pick any  $\epsilon > 0$  and obtain a  $\delta_i > 0$  from the definition of relative continuity to  $E_i$ . Then set  $S = B_{\delta_i}^d(x) \cap E_i^c$ . Any point  $x' \in S$  must belong in at least one  $E_j, j \neq i$ . For all such  $E_j$ , we examine whether their intersection with  $S$  is non-empty (call the corresponding set of indices  $I$ ), and if it isn't, we know that it must contain at least  $x$ . It must thus hold that  $f$  is continuous on  $x$  relative to each  $E_j, j \in I$ . From this we can obtain *at most*  $N - 1$  quantities  $\delta_j > 0$  that satisfy the definition of relative continuity of  $f$  on  $x$  (with respect to each of these sets). The minimum of these quantities is thus well defined, and we call it  $\delta'$ . Notice then that  $\delta = \min\{\delta_1, \delta'\}$  is such that whenever  $d(x', x) < \delta$ , it holds that  $x'$  is in *some*  $E_i$ , and relative continuity with respect to that  $E_i$

guarantees that  $\rho(f(x'), f(x)) < \epsilon$ . Therefore  $f$  is indeed continuous on  $x$  in the general sense.

(c) Consider first the following construction of subsets of  $\mathbb{R}^2$ . We define  $l_i$  to be the line passing through the origin at an angle of  $0 + \sum_{j=1}^{i-1} \frac{\pi}{2^j}$  with the  $x'x$  axis. Geometrically, this forms lines that start from an angle of 0 radians with the  $x'x$  axis and tend towards an angle of  $\pi$ , but without ever attaining it. Let now  $U_i$  be the subset of  $\mathbb{R}^2$  contained inside  $l_i, l_{i+1}$ , the lines included, i.e. increasingly small “pie slices”. Notice that each of these sets is closed. Now discard  $U_i$  for  $i$  even, which yields a countably infinite collection of sets  $\mathcal{U}$  that are closed, and every pair of which only has the origin in its intersection. Define  $M$  to be the union of all members of  $\mathcal{U}$ , which is a metric space as a subset of  $\mathbb{R}^2$ , and let  $N = \mathbb{R}$ . Then define  $f : M \rightarrow N$  as:

$$f(x, y) = ix + iy, (x, y) \in U_i$$

This function is clearly continuous on each  $U_i$  relative to  $U_i$ , since this restriction of it is linear (furthermore, it is well-defined since it is zero at the origin no matter which  $U_i$  we examine in the formula). However, it fails to be continuous at  $(0, 0)$ : notice that these are increasingly “steep” planes, each of which requires an increasingly small  $\delta > 0$  in the definition of continuity for a given  $\epsilon > 0$ , and as such the argument used in (b) regarding a well-defined minimum  $\delta$  fails.

**Exercise 34**

Show that  $d$  is continuous on  $M \times M$ , where  $M \times M$  is supplied with “the” product metric (see Exercise 3.46). This says that  $d$  is jointly continuous, that is, continuous as a function of two variables. [Hint: If  $x_n \rightarrow x, y_n \rightarrow y$ , show that  $d(x_n, y_n) \rightarrow d(x, y)$ .]

*Solution.*

In order to discuss continuity of  $d$  as a function defined on  $M \times M$ , we first need to equip  $M \times M$  with an appropriate metric. By exercise 3.46, we are given three choices all of which define equivalent metrics, from which we choose  $e : M \times M \rightarrow \mathbb{R}$ :

$$e((x, y), (a, b)) = d(x, a) + d(y, b)$$

Now we need to show that whenever  $(x_n, y_n) \rightarrow^e (x, y)$ , it holds that  $d(x_n, y_n) \rightarrow d(x, y)$ . Select any  $\epsilon > 0$ , in which case we can find  $N > 0$  such that whenever  $n \geq N$  it holds that:

$$e((x_n, y_n), (x, y)) < \epsilon \implies d(x_n, x) + d(y_n, y) < \epsilon$$

By the triangle inequality for  $d$  we have that:

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \implies d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n) \\ &\implies d(x_n, y_n) - d(x, y) < \epsilon \end{aligned}$$

And also:

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \implies d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y, y_n) \\ &\implies d(x, y) - d(x_n, y_n) < \epsilon \end{aligned}$$

Combining these two inequalities yields  $|d(x, y) - d(x_n, y_n)| < \epsilon$ , so by setting  $\delta = \epsilon$  we obtain the desired definition of continuity for  $d$ .

**Exercise 35**

Show that a set  $U$  is open in  $M$  if and only if  $U = f^{-1}(V)$  for some continuous function  $f : M \rightarrow \mathbb{R}$  and some open set  $V$  in  $\mathbb{R}$ .

*Solution.*

$\implies$  : If  $U$  is open, we have that  $U^c$  is closed. Define then the function  $f : M \rightarrow \mathbb{R}$ ,  $f(x) = d(x, U^c)$ . Notice first that this is continuous, as discussed in preposition 5.4. Furthermore,  $f(x) > 0$  iff  $x \notin \overline{U^c} = U^c \iff x \in U$ , where we used the fact that  $U^c$  is closed. Consequently, we have that  $f^{-1}(\mathbb{R} \setminus \{0\}) = U$ , and of course the set  $\mathbb{R} \setminus \{0\}$  is open, thus completing the proof in this direction.

$\impliedby$  : If we think of  $f$  as a continuous function between  $(M, d)$  and  $(\mathbb{R}, \rho)$  where  $\rho$  is the usual metric on  $\mathbb{R}$ , then theorem 5.1 guarantees that  $f^{-1}(V) = U$  is open in  $M$ .

**Exercise 36**

Suppose that we are given a point  $x$  and a sequence  $(x_n)$  in a metric space  $M$ , and suppose that  $f(x_n) \rightarrow f(x)$  for every continuous, real-valued function  $f$  on  $M$ . Does it follow that  $x_n \rightarrow x$  in  $M$ ? Explain.

*Solution.*

Suppose that it is not the case that  $x_n \rightarrow x$ . This means that there must exist  $\epsilon > 0$  such that for every  $N > 0$  there exists  $n \geq N$  such that  $d(x_n, x) \geq \epsilon$ . Select the subsequence  $(x_{n_k})$  that is formed by picking a different such  $x_n$  for every value of  $N$  (these can always be found since the statement holds for arbitrarily large  $N$ ). Let then  $U = B_{\epsilon/2}(x)$ , and  $f(y) = d(y, U)$ , which we know (prep. 5.4) is continuous. Note now that  $x \in U$ , so  $f(x) = 0$ . Note also that for any  $z \in U$  and any  $x_{n_k}$  we have that:

$$d(z, x_{n_k}) \geq |d(z, x) - d(x_{n_k}, x)|,$$

and we know that  $d(z, x) \leq \epsilon/2$ ,  $d(x_{n_k}, x) \geq \epsilon$ , so we can rewrite this as  $d(z, x_n) \geq d(x_{n_k}, x) - d(z, x)$ . Continuing:

$$d(z, x_{n_k}) \geq \epsilon - d(z, x) > \epsilon - \epsilon/2 = \epsilon/2,$$

and since this holds for any  $z \in U$  by taking the infimum over them we can conclude that  $d(x_{n_k}, U) \geq \epsilon/2$ . Therefore  $f(x_{n_k}) \geq \epsilon/2$  for all  $x_{n_k}$ . What this now means is that we can never satisfy the definition of  $f(x_n) \rightarrow f(x) = 0$ : we can never find  $N$  such that  $f(x_n) < \epsilon/2$ , since at least one such  $x_{n_k}$  will violate this no matter how large  $N$  is. Because we are given  $f(x_n) \rightarrow f(x)$ , we arrive at a contradiction. This means that it must indeed be the case that  $x_n \rightarrow x$ .

**Exercise 41**

Let  $C$  be a closed set in  $\mathbb{R}$  and let  $f : C \rightarrow \mathbb{R}$  be continuous. Show that there is a continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g(x) = f(x)$  for every  $x \in C$ . We say that  $g$  is a continuous *extension* of  $f$  to all of  $\mathbb{R}$ . In particular, every continuous function on the Cantor set  $\Delta$  extends continuously to all of  $\mathbb{R}$ . [Hint: The complement of  $C$  is the countable union of disjoint open intervals. Define  $g$  by “connecting the dots” across each of these open intervals.]

*Solution.*

Since  $C$  is closed, we know that  $B = C^c$  is open. In particular, as stated in the hint, this means that we can write  $B$  as a countable union of disjoint open intervals,  $B = \bigcup_{j=1}^{\infty} I_j$ ,  $I_j \cap I_k = \emptyset$  whenever  $j \neq k$ . We are therefore tasked with finding values for  $g$  on each  $I_j = (a_j, b_j)$  such that it is continuous everywhere. For  $x \in I_j$ , consider defining:

$$g(x) = f(a_j) + (f(b_j) - f(a_j)) \frac{x - a_j}{b_j - a_j},$$

and for  $x \in C$ ,  $g(x) = f(x)$ . We then know that  $g$  is continuous on all of  $C$  except for the endpoints of all  $I_j$  due to  $f$ 's continuity. Furthermore,  $g$  is continuous inside each  $I_j$  as a linear function. Lastly, by an easy computation of left and right limits on each  $a_j, b_j$  we conclude that  $g$  is also continuous on each of them as well. Therefore  $g$  is continuous everywhere. An immediate consequence of this is the fact that continuous functions on  $\Delta$  can be continuously extended to all of  $\mathbb{R}$ , given that  $\Delta$  is closed (Example 4.12).

## 5.2 Homeomorphisms

### Exercise 43

If you are not already convinced, prove that two metrics  $d, \rho$  on a set  $M$  are equivalent if and only if the identity map on  $M$  is a homeomorphism from  $(M, d)$  to  $(M, \rho)$ .

*Solution.*

Let  $i$  be the identity map from  $(M, d)$  to  $(M, \rho)$ .

$\implies$ : It is obvious that  $i$  is a bijection. In addition, if  $x_n \rightarrow^d x$ , we have that  $x_n \rightarrow^\rho x$  by the equivalence of the two metrics. But this then means precisely that  $i(x_n) \rightarrow^\rho i(x)$  in  $(M, \rho)$ , so  $i$  is continuous. With a symmetrical argument we can show that  $i^{-1}$  is continuous, thus completing the proof that  $i$  is a homeomorphism from  $(M, d)$  to  $(M, \rho)$ .

$\impliedby$ : If  $i$  is a homeomorphism from  $(M, d)$  to  $(M, \rho)$ , then  $x_n \rightarrow^d x$  iff  $x_n \rightarrow^\rho x$ , which is of course the definition of  $d, \rho$  being equivalent.

### Exercise 44

Check that the relation “is homeomorphic to” is an equivalence relation on pairs of metric spaces.

*Solution.*

Denote the relation “is homeomorphic to” with  $\sim$ .

**Reflexivity:** For any metric space  $(M, d)$ , it is clear that  $(M, d) \sim (M, d)$ , since the identity function is a trivial homeomorphism.

**Symmetry:** If  $(M, d) \sim (N, \rho)$ , then there exists a homeomorphism  $f$  from  $(M, d)$  to  $(N, \rho)$ , and so by the defining properties of homeomorphisms,  $f^{-1}$  is also a homeomorphism from  $(N, \rho)$  to  $(M, d)$ .

**Transitivity:** If  $(M, d) \sim (N, \rho), (N, \rho) \sim (O, \sigma)$ , let  $f, g$  be the corresponding homeomorphisms. Then the function  $h : M \rightarrow O, h(x) = g(f(x))$  is a bijection as a composition of bijections. Furthermore, if  $x_n \rightarrow^d x$ , we have that  $f(x_n) \rightarrow^\rho f(x)$  by the continuity of  $f$ , and then that  $g(f(x_n)) \rightarrow^\sigma g(f(x))$  by the continuity of  $g$ , which means  $h(x_n) \rightarrow^\sigma h(x)$ , which means  $h$  is continuous. A symmetrical argument shows that  $h^{-1}$  is also continuous, thus completing the proof that  $h$  is a homeomorphism from  $(M, d)$  to  $(O, \sigma)$ , thus showing that  $(M, d) \sim (O, \sigma)$ . We've thus shown that  $\sim$  is an equivalence relation.

### Exercise 46

Show that every metric space is homeomorphic to one of finite diameter. [Hint: Every metric is equivalent to a bounded metric.]

*Solution.*

As stated in the hint, recall that in Ex. 42, Ch. 3 we showed that any metric  $d$  is equivalent to the *bounded* metric  $\tau(x, y) = \min\{d(x, y), 1\}$ . For any metric space  $(M, d)$ , consider then the metric space  $(M, \tau)$ . Clearly, for any two elements  $x, y$  of  $(M, \tau)$  we have that  $\tau(x, y) \leq 1$ , and so the diameter of  $(M, \tau)$  is at most 1. Furthermore, Exercise 43 guarantees that  $(M, d)$  is homeomorphic to  $(M, \tau)$ , thus completing the proof.

### Exercise 47

Define  $E : \mathbb{N} \rightarrow l_1$  by  $E(n) = (1, \dots, 1, 0, \dots)$ , where the first  $n$  entries are 1 and the rest are 0. Show that  $E$  is an isometry (into).

*Solution.*

We need to show that  $l_1(E(n), E(m)) = |n - m|$  for any two  $n, m \in \mathbb{N}$ . Suppose WLOG that  $n < m$ , which means that:

$$l_1(E(n), E(m)) = \sum_{i=1}^{\infty} |E(n)_i - E(m)_i| = \sum_{i=n+1}^m |0 - 1| = m - (n + 1) + 1 = m - n = |n - m|$$

**Exercise 48**

Prove that  $\mathbb{R}$  is homeomorphic to  $(0, 1)$  and that  $(0, 1)$  is homeomorphic to  $(0, \infty)$ . Is  $\mathbb{R}$  *isometric* to  $(0, 1)$ ? To  $(0, \infty)$ ? Explain.

*Solution.*

Let  $f : \mathbb{R} \rightarrow (0, 1)$ ,  $f(x) = \frac{1}{1+e^{-x}}$ , which is easily seen to be continuous and bijective, with a continuous inverse ( $f^{-1}(x) = \log(\frac{x}{1-x})$ ). Therefore  $\mathbb{R}$  is homeomorphic to  $(0, 1)$ . Let also  $g : (0, 1) \rightarrow (0, \infty)$ ,  $g(x) = \log(\frac{1}{1-x})$ , which in  $(0, 1)$  assumes strictly positive and unbounded values, is continuous and bijective, and thus is a homeomorphism between  $(0, 1)$  and  $(0, \infty)$ . To examine whether  $\mathbb{R}$  is isometric to  $(0, 1)$ , we consider the following. Suppose that this is indeed true, which means that there exists a homeomorphism  $f$  between them with the property  $|f(x) - f(y)| = |x - y|$  for all  $x, y \in \mathbb{R}$ . Fix  $x = 0$  and let  $y \rightarrow \infty$ . Then the RHS here is unbounded. However, since  $f(x), f(y) \in (0, 1)$  the LHS is clearly bounded, and so we arrive at a contradiction, which means that the two spaces cannot be isometric. Similarly, suppose  $f$  is now a homeomorphism between  $\mathbb{R}$  and  $(0, 1)$  with the property  $|f(x) - f(y)| = |x - y|$ ,  $x, y \in \mathbb{R}$ . For any  $x \in \mathbb{R}$ , it has to hold that:

$$|f(x) - f(0)| = |x - 0| = |-x| = |f(-x) - f(0)|$$

This means that for any given  $x \in \mathbb{R}$ , either  $f(x) - f(0) = f(-x) - f(0) \implies f(x) = f(-x)$  or  $f(x) - f(0) = -f(-x) + f(0) \implies f(x) + f(-x) = 2f(0)$ . The first case is impossible due to  $f$  being one-to-one as a homeomorphism. Consequently, the second equality has to hold for all  $x$ . Notice, however, that the RHS is fixed and positive, whereas the LHS is a sum of positive quantities, which means that  $f(x)$  must be bounded. But then  $|f(x) - f(0)| = |x|$  leads to a contradiction as  $x \rightarrow \infty$ . Therefore, the two spaces are not isometric.

**Exercise 50**

Let  $(M, d)$  denote the set  $\{0\} \cup \{(1/n) : n \geq 1\}$  under its usual metric. Define a second metric  $\rho$  on  $M$  by setting  $\rho(1/n, 1/m) = |1/n - 1/m|$  for  $m, n \geq 2$ ,  $\rho(1/n, 1) = 1/n$  for  $n \geq 2$ ,  $\rho(1/n, 0) = 1 - 1/n$  for  $n \geq 2$ , and  $\rho(0, 1) = 1$ . Show that  $(M, d)$  and  $(M, \rho)$  are homeomorphic but that the identity map from  $(M, d)$  to  $(M, \rho)$  is *not* continuous.

*Solution.*

Consider the following map  $f : (M, d) \rightarrow (M, \rho)$ :

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x = 1 \\ x, & \text{otherwise} \end{cases}$$

It is easy to see that this is a bijection, and that  $f^{-1}(x) = f(x)$ . Suppose now that  $x_n \xrightarrow{d} x$  for  $(x_n) \subset (M, d)$ . We observe first that for any sequence that does not eventually become constant, this means that  $x = 0$  and that after some  $N$ , all  $x_n$  must be of the form  $\frac{1}{m_n}$  for  $m_n > 1$ . For any  $\epsilon > 0$ , set  $\delta = \epsilon$ , in which case whenever  $d(x_n, x) < \epsilon$ , we have that:

$$\rho(f(x_n), f(x)) = \rho\left(\frac{1}{m_n}, f(0)\right) = \left|\frac{1}{m_n}\right| = d(x_n, x) < \epsilon,$$

which shows that  $f$  is continuous, since this holds for an arbitrary sequence  $(x_n)$  (note that it trivially holds for sequences that eventually becomes constant). One can show that the converse holds as well:  $f(x_n) \xrightarrow{\rho} f(x) \implies x_n \xrightarrow{d} x$ . This is done by observing that under  $\rho$ , the only non-trivial way that this can happen is if  $f(x) = 1$ , and if after some  $N$ ,  $f(x_n)$  eventually equal  $\frac{1}{m_n}$ . The remaining proof is then the same as above. From this we conclude that  $f$  is a homeomorphism between  $(M, d), (M, \rho)$ .

Define the sequence  $(x_n) \subset M$  by  $x_n = \frac{1}{n}$ . Notice that with the standard metric  $d$  on  $M$ , we have  $d(x_n, 0) = |0 - \frac{1}{n}| = \frac{1}{n}$ , which goes to zero as  $n \rightarrow \infty$ . However, with the metric  $\rho$ , we have that  $\rho(x_n, 0) = 1 - \frac{1}{n}$  for  $n \geq 2$ , which *does not* go to zero as  $n \rightarrow \infty$ . Therefore,  $d, \rho$  do not generate the same convergent sequences, i.e. they are not equivalent. Exercise 43 Ch. 5 then tells us that the identity map on  $M$  cannot be a homeomorphism from  $(M, d)$  to  $(M, \rho)$ .

### Exercise 51

Let  $(M, \rho)$  be a separable metric space and assume that  $\rho(x, y) \leq 1$  for every  $x, y \in M$ . Given a countable dense set  $\{x_n : n \geq 1\} \subset M$ , define a map  $f : M \rightarrow H^\infty$ , from  $M$  into the Hilbert cube (Exercise 3.10), by  $f(x) = (\rho(x, x_n))_{n=1}^\infty$ .

- (i) Prove that  $f$  is one-to-one and continuous. In fact,  $f$  satisfies  $d(f(x), f(y)) \leq \rho(x, y)$ , where  $d$  is the metric on  $H^\infty$ .
- (ii) Fix  $\epsilon > 0$  and  $x \in M$ . Find  $\delta > 0$  such that  $\rho(x, y) < \epsilon$  whenever  $d(f(x), f(y)) < \delta$ . Conclude that  $f$  is a homeomorphism into  $H^\infty$ .

*Solution.*

(i) Suppose  $f(x) = f(y)$  for  $x, y \in M$ . Then the sequences  $(\rho(x, x_n))_{n=1}^\infty, (\rho(y, x_n))_{n=1}^\infty$  are equal, which means each of their respective terms is equal, i.e.  $\rho(x, x_n) = \rho(y, x_n)$  for every  $n = 1, 2, \dots$ . Due to the separability of  $M$ , there exists a sequence  $(x_{n_k}) \subset \{x_n : n \geq 1\}$  that converges to  $x$ . Since we have that  $\rho(x_{n_k}, x) = \rho(x_{n_k}, y)$ , we conclude that it is also the case that  $x_{n_k} \rightarrow y$ , and from the uniqueness of the limit we have  $x = y$ , which means  $f$  is one-to-one. The metric used for the Hilbert cube is:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

For any two  $x, y \in M$ , we therefore have that:

$$d(f(x), f(y)) = \sum_{n=1}^{\infty} 2^{-n} |\rho(x, x_n) - \rho(y, x_n)| \leq \sum_{n=1}^{\infty} 2^{-n} \rho(x, y) = \rho(x, y)$$

Notice, now, that the continuity of  $f$  follows immediately from this, where for each  $\epsilon$  the required  $\delta$  equals  $\epsilon$ .

(ii) We fix  $\epsilon > 0$  and  $x \in M$ . Notice, first that since the set of  $x_n$  is dense, we can find one such  $x_i$  such that  $\rho(x, x_i) < \epsilon/4$ . Now, based on this  $i$ , suppose that  $\delta = \epsilon/2^{i+2}$  and that  $y$  is such that we have  $d(f(x), f(y)) < \delta$ . Then:

$$d(f(x), f(y)) < \delta \implies \sum_{n=1}^{\infty} 2^{-n} |\rho(x, x_n) - \rho(y, x_n)| < \delta \implies 2^{-i} |\rho(x, x_i) - \rho(y, x_i)| < \delta,$$

where we separated the term corresponding to the  $x_i$  we picked above. Thus:

$$|\rho(x, x_i) - \rho(y, x_i)| < \delta 2^i = \epsilon/4 \implies \rho(y, x_i) < \rho(x, x_i) + \epsilon/4 = \epsilon/2$$

Now we have that  $\rho(x, y) < \rho(x, x_i) + \rho(y, x_i) = \epsilon/4 + \epsilon/2 = \epsilon \cdot 3/4 < \epsilon$ , and so we have effectively shown that  $f^{-1}$  (which is well-defined when thinking about it as a function inside the range of  $f$  and *not* all of  $H^\infty$ ), thus showing that  $f$  is a homeomorphism into  $H^\infty$ .

### Exercise 52

Prove Theorem 5.5:

Let  $f : (M, d) \rightarrow (N, \rho)$  be one-to-one and onto. Then the following are equivalent:

- (i)  $f$  is a homeomorphism
- (ii)  $x_n \rightarrow^d x \iff f(x_n) \rightarrow^\rho f(x)$
- (iii)  $G$  is open in  $M \iff f(G)$  is open in  $N$
- (iv)  $E$  is closed in  $M \iff f(E)$  is closed in  $N$
- (v)  $\hat{d}(x, y) = \rho(f(x), f(y))$  defines a metric on  $M$  equivalent to  $d$

*Solution.*

(i)  $\implies$  (ii): As a homeomorphism,  $f$  is continuous, and so is  $f^{-1}$ . Then Theorem 5.1 (ii) guarantees that whenever  $x_n \rightarrow^d x$ , it holds that  $f(x_n) \rightarrow^\rho f(x)$ . Also, whenever  $f(x_n) \rightarrow^\rho f(x)$  it holds that  $f^{-1}(f(x_n)) \rightarrow^d f^{-1}(f(x))$ , i.e. that  $x_n \rightarrow^d x$ .

(ii)  $\implies$  (i): This utilizes Theorem 5.1 (ii) in the other direction, and yields that  $f, f^{-1}$  are both continuous, thus that  $f$  is a homeomorphism.

(i)  $\implies$  (iii): Let  $G \subset M$  be open. Then  $(f^{-1})^{-1}(G)$  is open in  $N$  by the continuity of  $f^{-1}$  and Theorem 5.1 (iv). We have that  $f(G) = (f^{-1})^{-1}(G)$  (relatively simple to prove via set inclusions), therefore  $f(G)$  is open in  $N$ . Conversely, if  $f(G)$  is open in  $N$ , then a direct application of Theorem 5.1 (iv) yields that  $f^{-1}(f(G)) = G$  is open in  $M$ . Therefore we've shown that (iii) holds.

(iii)  $\implies$  (i): Notice that each of the two directions of the given equivalence can be used to apply Theorem 5.1 (iv) to obtain that both  $f, f^{-1}$  are continuous.

(i)  $\implies$  (iv), (iv)  $\implies$  (i): This is done in the same way as above, except that we use Theorem 5.1 (iii) instead of (iv).

(ii)  $\implies$  (v): Whenever  $x_n \rightarrow^d x$ , we have that  $f(x_n) \rightarrow^\rho f(x)$ , which means precisely that  $\rho(f(x_n), f(x)) \rightarrow 0$ , and so  $\hat{d}(x_n, x) \rightarrow 0$ , which means  $x_n \rightarrow^{\hat{d}} x$ . Since we also have that whenever  $f(x_n) \rightarrow^\rho f(x)$  it holds that  $x_n \rightarrow^d x$ , we obtain that whenever  $\rho(f(x_n), f(x)) = \hat{d}(x_n, x) \rightarrow 0$ , it also holds that  $d(x_n, x) \rightarrow 0$ . We've thus shown that  $d, \hat{d}$  are equivalent.

(v)  $\implies$  (ii): This is a simple reversal of the immediately preceding argument.

### Exercise 53

Suppose that we are given a point  $x$  and a sequence  $(x_n)$  in a metric space  $M$ , and suppose that  $f(x_n) \rightarrow f(x)$  for every continuous real-valued function  $f$  on  $M$ . Prove that  $x_n \rightarrow x$  in  $M$ .

*Solution.*

We have already shown this in Exercise 36, Ch. 5.

### Exercise 54

Let  $f : (M, d) \rightarrow (N, \rho)$  be one-to-one and onto. Prove that the following are equivalent: (i)  $f$  is a homeomorphism and (ii)  $g : N \rightarrow \mathbb{R}$  is continuous if and only if  $g \circ f : M \rightarrow \mathbb{R}$  is continuous. [Hint: Use the characterization given in Theorem 5.5 (ii).]

*Solution.*

(i)  $\implies$  (ii): Suppose that  $f$  is a homeomorphism. First, suppose that  $g$  is continuous. Then, for any  $x_n \rightarrow^d x$  we have that  $f(x_n) \rightarrow^\rho f(x)$  due to the continuity of  $f$ , and by the continuity of  $g, g(f(x_n)) \rightarrow g(f(x))$ . Therefore  $g \circ f$  is clearly continuous. Conversely, suppose  $g \circ f$  is continuous and pick any  $(y_n) \in N$  that converges to  $y \in N$ . To show that  $g$  is continuous, we need to show that  $g(y_n) \rightarrow g(y)$ . Because  $f$  is a homeomorphism, it is the case that  $f^{-1}(y_n) \rightarrow f^{-1}(y)$ . Now recall that  $g \circ f$  is continuous, and thus  $(g \circ f)(f^{-1}(y_n)) \rightarrow (g \circ f)(f^{-1}(y))$ , which simplifies to  $g(y_n) \rightarrow g(y)$ . Therefore,  $g$  is indeed continuous.

(ii)  $\implies$  (i): Pick any  $x_n \rightarrow^d x$ , and define  $g(y) = \rho(y, f(x))$ , which is always well-defined since  $f$  is one-to-one. This is a continuous mapping (Exercise 20 Ch. 5), and therefore from our assumption we have that  $g \circ f$  is also continuous. This means that  $\rho(f(y), f(x))$  is continuous on all  $y \in M$ , and so that whenever  $x_n \rightarrow^d x$ , it must be the case that  $\rho(f(x_n), f(x)) \rightarrow \rho(f(x), f(x)) = 0$ , which means precisely that  $f(x_n) \rightarrow^\rho f(x)$ , i.e. that  $f$  is continuous. We now need to show that  $f^{-1}$  is continuous as well. Suppose thus that  $y_n \rightarrow y, (y_n) \subset N, y \in N$ . Because  $f$  is one-to-one and onto, there exist unique  $(x_n), x$  such that  $f(x_n) = y_n, f(x) = y$ . Keeping  $x$  fixed, consider the function  $g(y) = d(f^{-1}(y), x)$ , which is once again continuous. Therefore,  $g \circ f$  is also continuous, and because  $f(x_n) \rightarrow^\rho f(x)$ , it must be the case that:

$$(g \circ f)(f(x_n)) \rightarrow (g \circ f)(f(x)) \implies d(f^{-1}(f(x_n)), x) \rightarrow d(f^{-1}(f(x)), x) \implies d(x_n, x) \rightarrow 0,$$

which means  $x_n \rightarrow^d x$ , which means  $f^{-1}$  is continuous. We've therefore established that  $f$  is a homeomorphism.

### Exercise 55

Let  $f : (M, d) \rightarrow (N, \rho)$  be a homeomorphism. Prove that  $M$  is separable iff  $N$  is separable.

*Solution.*

$M$  separable  $\implies$   $N$  separable: Pick any element  $y \in N$ . Then there exists a unique  $x \in M$  such that  $f(x) = y$  (since  $f$  is a bijection). Because  $M$  is separable, there exists a countable dense subset  $A$ , which means that there exists a sequence  $(x_n) \subset A$  such that  $x_n \rightarrow x$ . Due to  $f$  being a homeomorphism, we obtain  $f(x_n) \rightarrow f(x) = y$ . Furthermore, because  $A$  is countable and  $f$  is a bijection,  $f(A)$  is also countable,

and we have  $(f(x_n)) \subset f(A)$ . Since  $y$  was picked arbitrarily, we conclude that  $\overline{f(A)} = N$ , i.e.  $f(A)$  is dense in  $N$  and  $N$  is thus separable.

$N$  separable  $\implies M$  separable: The proof in this direction relies on exactly the same argument except for using  $f^{-1}$  instead of  $f$ .

### Exercise 56

Let  $f : (M, d) \rightarrow (N, \rho)$ .

(i) We say that  $f$  is an **open** map if  $f(U)$  is open in  $N$  whenever  $U$  is open in  $M$ ; that is,  $f$  maps open sets to open sets. Give examples of a continuous open map that is not open and open map that is not continuous. [Hint: Please note that the definition depends on the target space  $N$ .]

(ii) Similarly,  $f$  is **closed** if it maps closed sets to closed sets. Give examples of a continuous map that is not closed and a closed map that is not continuous.

*Solution.*

(i) Consider  $M = N = \mathbb{R}$  and the function  $f(x) = 1$ . This is clearly a continuous map, but it is not open:  $\mathbb{R}$  is open and yet  $f$  maps it to  $\{1\}$ , a closed set. Consider also  $M = \mathbb{R}, N = \{0, 1\}$  and  $f(x) = 0, x < 0, f(x) = 1, x \geq 0$ . Notice that  $f$  is not continuous, but that by this definition of  $N$ , all possible subsets of  $N$  are open and thus  $f$  is trivially an open map.

(ii) Consider  $M = N = \mathbb{R}$  and the clearly continuous  $f(x) = \frac{1}{x}$ . Here  $f(M) = (0, \infty)$ , and  $M$  is closed, whereas  $f(M)$  thus isn't. Therefore  $f$  is continuous but not closed. For a map that is closed but not continuous, observe that our second example for (i) works here as well: every possible subset of  $N$  is closed, so  $f$  is closed but it is not continuous.

### Exercise 57

Let  $f : (M, d) \rightarrow (N, \rho)$  be one-to-one and onto. Show that the following are equivalent: (i)  $f$  is open (ii)  $f$  is closed and (iii)  $f^{-1}$  is continuous. Consequently,  $f$  is a homeomorphism if and only if both  $f, f^{-1}$  are open (closed).

*Solution.*

(i)  $\implies$  (ii): Let  $S \subset M$  be closed. Then  $S^c$  is by definition open, and  $f(S^c)$  is open because  $f$  is open. We thus have that  $f(S^c)^c = N \setminus f(S^c)$  is closed. This set consists of all elements  $y \in N$  such that they do not equal any  $f(x)$  for  $x \in S^c$ . Because  $f$  is one-to-one and onto, each must necessarily equal  $f(x)$  for some unique  $x \in M$ , and thus the only possibility is  $x \in M \setminus S^c = S$ , therefore  $N \setminus f(S^c) = f(S)$ , and so  $f(S)$  is closed, which means  $f$  is closed.

(ii)  $\implies$  (i): Let  $S \subset M$  be open, which means  $S^c$  is closed and  $f(S^c)$  is also closed since  $f$  is closed. Therefore  $f(S^c)^c = N \setminus f(S^c)$  is open, and the remaining argument is as above, showing that this set equals  $f(S)$ , which means  $f$  is open.

(i)  $\implies$  (iii): Suppose  $f$  is open, and pick any open  $S \subset M$ . If we can show that  $(f^{-1})^{-1}(S)$  is open in  $N$ , by Theorem 5.1 we will also have that  $f^{-1}$  is continuous. We have that  $(f^{-1})^{-1}(S) = f(S)$  (easily shown using the definitions and bijectivity of  $f$ ), and so by our assumption this set is indeed open and  $f^{-1}$  is continuous.

(ii)  $\implies$  (iii): This is shown in much the same way, changing “open” to “closed” and again using Theorem 5.1.

(iii)  $\implies$  (i): Suppose now  $f^{-1}$  is continuous and  $S \subset M$  is open. Using again the fact that  $(f^{-1})^{-1}(S) = f(S)$  and Theorem 5.1 in the other direction, we obtain that  $f(S)$  is open, and so  $f$  is open.

(iii)  $\implies$  (ii): Once again it suffices to change “open” to “closed” above.

### Exercise 60

Let  $(M, d)$  be a metric space, and let  $\tau$  be the discrete metric on  $M$ . Then,  $(M, d)$  and  $(M, \tau)$  are homeomorphic if and only if every subset of  $M$  is open in  $(M, d)$  if and only if every function  $f : (M, d) \rightarrow \mathbb{R}$  is continuous.

*Solution.*

For convenience, we refer to the three statements as (i), (ii), (iii) in the order they are given. Then:

(i)  $\implies$  (ii): Suppose  $f$  is a homeomorphism between  $(M, d), (M, \tau)$ . We have seen (Example 4.1 (c)) that in a discrete space every subset is open. Due to the bijectivity of  $f$ , every subset  $S \subset (M, d)$  is mapped uniquely to a subset  $S' = f(S) \subset (M, \tau)$ , and from Theorem 5.5 each of them is open iff the other one is too. Therefore since every  $f(S)$  is open in  $(M, \tau)$ , every  $S \subset (M, d)$  is open too.

(ii)  $\implies$  (i): The argument above also works here, since Theorem 5.5 presents equivalent conditions: if every subset of  $(M, d)$  is open, then of course every subset of  $(M, d)$  is open iff its image in  $(M, \tau)$  is open, and so the identity function is a homeomorphism between them.

(i)  $\implies$  (iii): Let  $g$  be a homeomorphism between  $(M, d), (M, \tau)$  and  $x_n \rightarrow^d x$ . Then by Theorem 5.5 we have that  $g(x_n) \rightarrow^\tau g(x)$ , which means for  $\epsilon = 1/2$  there exists  $N > 0$  such that for  $n \geq N, \tau(g(x_n), g(x)) < \epsilon \implies g(x_n) = g(x)$  due to the definition of the discrete metric. But since  $g$  is a bijection, this yields  $x_n = x$ , so *every convergent sequence in  $(M, d)$  is eventually constant*. Clearly then  $f(x_n) \rightarrow f(x)$  will also hold for any function  $f : (M, d) \rightarrow \mathbb{R}$ , and so any such  $f$  is continuous.

(iii)  $\implies$  (ii): For any set  $S \subset (M, d)$ , consider the indicator function  $f_S(x) = 1, x \in S, f_S(x) = 0$  otherwise. By the hypothesis, this must be continuous. Now pick any  $S \subset (M, d)$ , and any  $x \in S$ , and any sequence  $(x_n) \subset M, x_n \rightarrow^d x$ . Due to the continuity of  $f_S$ , it must hold that  $f_S(x_n) \rightarrow f_S(x)$ , which means that for  $\epsilon > 1/2$  there must exist  $N > 0$  such that for  $n \geq N, |f_S(x_n) - f_S(x)| < 1/2 \implies |f_S(x_n) - 1| < 1/2$ . By the definition of  $f_S$  this is only possible if all such  $f_S(x_n)$  equal 1, which happens precisely when  $x_n \in S$ , and so the sequence  $(x_n)$  contains only finitely many points not in  $S$ . Therefore by Theorem 4.7,  $S$  is open, and since it was picked arbitrarily we have shown (ii).

### 5.3 The Space of Continuous Functions

#### Exercise 62

If  $f, g \in C[a, b]$ , show that  $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ . Also show that  $\|\max\{f, g\}\|_\infty \leq \max\{\|f\|_\infty, \|g\|_\infty\}$ , and that  $\|f\|_\infty \leq \|g\|_\infty$  whenever  $|f| \leq |g|$ .

*Solution.*

For any  $x \in [a, b], |f(x)| \leq \max_{a \leq t \leq b} |f(t)| = \|f\|_\infty$ , and similarly  $|g(x)| < \|g\|_\infty$ . We thus obtain  $|f(x)g(x)| \leq \|f\|_\infty \cdot \|g\|_\infty$  and:

$$\|fg\|_\infty = \max_{a \leq t \leq b} |f(t)g(t)| \leq \|f\|_\infty \cdot \|g\|_\infty$$

We also have that:

$$\|\max\{f, g\}\|_\infty = \max_{a \leq t \leq b} |\max\{f, g\}(t)| = \max_{a \leq t \leq b} |\max\{f(t), g(t)\}| \leq \max\{\|f\|_\infty, \|g\|_\infty\},$$

since every value of  $f, g$  is smaller than the corresponding maximum value of the function.

Lastly, suppose  $|f| \leq |g|$ . This means that for all  $t \in [a, b], |f(t)| \leq g(t)|$ . Consequently, the same inequality must hold for their maximum values, which means  $\|f\|_\infty \leq \|g\|_\infty$ .

#### Exercise 64

Given  $n \in \mathbb{N}$  and  $f, g \in C(\mathbb{R})$ , let  $d_n(f, g) = \max_{|t| \leq n} |f(t) - g(t)|$ . Then  $d_n$  defines a pseudometric on  $C(\mathbb{R})$  (Why?) Show that  $d(f, g) = \sum_{n=1}^{\infty} 2^{-n} d_n(f, g) / (1 + d_n(f, g))$  defines a metric on  $C(\mathbb{R})$ .

*Solution.*

To show that  $d_n$  is a pseudometric on  $C(\mathbb{R})$ , we have the following:

- From its definition, it is obvious that  $d_n(f, g) \geq 0$  for any two  $f, g \in C(\mathbb{R})$ .
- Again from the definition,  $d_n(f, g) = d_n(g, f)$  for any two  $f, g \in C(\mathbb{R})$ .
- If  $f = g$ , then clearly  $d_n(f, g) = 0$ . However, what makes  $d_n$  a pseudometric instead of a metric is the fact that for e.g.  $f(t) = t, t \leq n, f(t) = 2(t-n) + n$  and  $g(t) = t, t \leq n, g(t) = 3(t-n) + n$  (both clearly continuous, and not equal functions),  $d_n(f, g) = 0$ .

- For  $f, g, h \in C(\mathbb{R})$ , we have that:

$$d_n(f, h) = \max_{|t| \leq n} |f(t) - h(t)| = \max_{|t| \leq n} |f(t) - g(t) + g(t) - h(t)|$$

Since it is the case that  $|f(t) - g(t) + g(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$ , the inequality must also hold when applying  $\max_{|t| \leq n}$  to both sides. Furthermore, the maximum of a sum equals at most the sum of the two maxima, and so we obtain:

$$d_n(f, h) \leq \max_{t \leq n} |f(t) - g(t)| + \max_{t \leq n} |g(t) - h(t)| = d_n(f, g) + d_n(g, h),$$

which is the triangle inequality for  $d_n$ .

Now we examine  $d$  in the same manner:

- Due to the definition of this metric as an infinite sum, we need to first determine whether it is always well-defined, i.e. whether the sum always converges. To see this, observe that  $d_n(f, g)/(1+d_n(f, g)) \leq 1$ , and so each partial sum of  $d$  is bounded above by the geometric series  $\sum_{n=1}^{\infty} 2^{-n}$ , which we know converges. Since the partial sums are also non-decreasing,  $d(f, g)$  does in fact always converge.
- By the non-negativity of each  $d_n$ , it is clear that  $d(f, g) \geq 0$  always holds as well.
- Similarly, by the symmetry of each  $d_n$ ,  $d(f, g) = d(g, f)$ .
- Clearly,  $d(f, f) = 0$  for any  $f \in C(\mathbb{R})$ . Now pick any two  $f, g \in C(\mathbb{R}), f \neq g$ . Then there exists  $x \in \mathbb{R}$  such that  $f(x) \neq g(x)$ . This means that for  $n = \lceil x \rceil$ ,  $d_n(f, g) = \max_{|t| \leq n} |f(t) - g(t)| > 0$ , and so  $d(f, g) > 0$  as well, thus showing that  $d(f, g) = 0$  iff  $f = g$ .
- For the triangle inequality, we use Exercise 7 of Chapter 3, by observing that each term of the sum of  $d$  equals  $2^{-n}\sigma(d_n(f, g))$  (for the  $\sigma$  of Exercise 6 of that Chapter). Although  $d_n$  is not a metric, the proof that for some metric  $d$ ,  $\sigma(d(x, y))$  satisfies the triangle inequality if  $\sigma(x+y) \leq \sigma(x) + \sigma(y)$  only depended on  $d$  satisfying the triangle inequality, which is the case for  $d_n$ . Since the triangle inequality is satisfied for each term of the convergent sum  $d$ , it also holds for  $d$  itself, thus completing the proof that  $d$  is a metric.

# Chapter 6

## Connectedness

### 6.1 Connected Sets

#### Exercise 1

Supply the missing details in the proof of Lemma 6.3:

Let  $E$  be a subset of a metric space  $M$ . If  $U$  and  $V$  are disjoint open sets in  $E$ , then there are disjoint open sets  $A$  and  $B$  in  $M$  such that  $U = A \cap E, V = B \cap E$ .

*Solution.*

The proof provided in the book arrives in full detail at the observation that for any two  $x \in U, y \in V$  it is the case that  $E \cap B_{\epsilon_x}(x) \cap B_{\epsilon_y}(y) = \emptyset$ . Notice that this shows that if these two open balls intersect, then their intersection must be entirely contained in  $M \setminus E$ . If it was the case that they didn't intersect at all, then simply writing  $A$  as the union of all  $B_{\epsilon_x}(x)$  and  $B$  as the union of all  $B_{\delta_y}(y)$  would make these open as unions of open sets, and of course disjoint. The claim that is given in the proof is that one can in fact achieve this by substituting  $\epsilon_x$  with  $\epsilon_x/2$  and  $\delta_y$  with  $\delta_y/2$ , i.e., it holds that  $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$  for any two  $x \in U, y \in V$ . Suppose then that there exists  $z \in M \setminus E$  such that  $z \in B_{\epsilon_x/2}(x), z \in B_{\delta_y/2}(y)$ . We then have:

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{\epsilon_x}{2} + \frac{\delta_y}{2}$$

WLOG, assume  $\delta_y \leq \epsilon_x$ , in which case we conclude  $d(x, y) < 2\frac{\epsilon_x}{2} = \epsilon_x$ . But then this means  $y \in B_{\epsilon_x}(x)$ , and of course  $y \in E$ . We originally had that  $E \cap B_{\epsilon_x}(x) = U$ , and so  $y \in U$ , which is a contradiction since  $y \in V$  and  $U \cap V = \emptyset$ . Therefore  $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$ , and so indeed the open sets  $A = \cup\{B_{\epsilon_x/2}(x), x \in U\}, B = \cup\{B_{\delta_y/2}(y), y \in V\}$  are disjoint.

#### Exercise 2

Show that the only nonempty connected subsets of  $\Delta$  are singletons. (We would say that  $\Delta$  is *totally disconnected*.)

*Solution.*

Suppose that there exists a connected subset of  $\Delta$  that contains at least two points  $x, y, x < y$ . By Theorem 6.4, it must also contain the entire interval  $[x, y]$ , and so it must hold that  $[x, y] \subset \Delta$ . But this directly contradicts Exercise 22 of 2.2:  $\Delta$  contains no nonempty open intervals, but here it must contain  $(x, y)$ , a contradiction. Therefore, the only nonempty connected subsets of  $\Delta$  are singletons.

#### Exercise 5

If  $E$  and  $F$  are connected subsets of  $M$  with  $E \cap F \neq \emptyset$ , show that  $E \cup F$  is connected.

*Solution.*

Suppose for the sake of contradiction that  $E \cup F$  is disconnected. Then there exist open sets  $A, B, A \cap B = \emptyset$  such that  $E \cup F \subset A \cup B, (E \cup F) \cap A \neq \emptyset, (E \cup F) \cap B \neq \emptyset$ . By the first of these conditions, we obtain  $E \subset A \cup B$  and  $F \subset A \cup B$ . By the second of these conditions, we obtain that at least one of  $E \cap A \neq \emptyset$

and  $F \cap A \neq \emptyset$  must be true. By the third of these conditions, we obtain that at least one of  $E \cap B \neq \emptyset$  and  $F \cap B \neq \emptyset$  must be true. Then we have the following possibilities:

- If  $E \cap A \neq \emptyset$  and  $E \cap B \neq \emptyset$  both hold, because we also have  $E \subset A \cup B$ , we obtain the contradiction that  $E$  is disconnected.
- If  $E \cap A \neq \emptyset$  and  $E \cap B = \emptyset$ , we have necessarily that  $F \cap B \neq \emptyset$ . Furthermore, because  $E \subset A \cup B$ , it must be the case that  $E \subset A$ . Now, if it holds that  $F \cap A \neq \emptyset$ , by the same reasoning as above we arrive at the contradiction that  $F$  is disconnected. Therefore it must be the case that  $F \cap A = \emptyset$ , which means  $F \subset B$ . Notice then that  $E \subset A, F \subset B, A \cap B = \emptyset$ , but  $E \cap F \neq \emptyset$  by the hypothesis. This is clearly a contradiction as well.
- The case  $E \cap A = \emptyset$  and  $E \cap B \neq \emptyset$  is symmetric to the above.

Therefore,  $E \cup F$  must be connected.

### Exercise 6

More generally, if  $\mathcal{C}$  is a collection of connected subsets of  $M$ , all having a point in common, prove that  $\bigcup \mathcal{C}$  is connected. Use this to give another proof that  $\mathbb{R}$  is connected.

*Solution.*

Suppose for the sake of contradiction that  $\bigcup \mathcal{C}$  is disconnected, which means that there exist open sets  $A, B, A \cap B = \emptyset$  such that  $\bigcup \mathcal{C} \subset A \cup B, \bigcup \mathcal{C} \cap A \neq \emptyset, \bigcup \mathcal{C} \cap B \neq \emptyset$ . Now, this means that there must exist  $C_i, C_j \in \mathcal{C}$ , such that  $C_i \cap A \neq \emptyset$  and  $C_j \cap B \neq \emptyset$ . Furthermore, it must hold that  $C_i \cap C_j \neq \emptyset$ , since all sets forming  $\bigcup \mathcal{C}$  have a common point. These now imply that  $(C_i \cup C_j) \cap A \neq \emptyset$  and  $(C_i \cup C_j) \cap B \neq \emptyset$ , while it is also clear that  $(C_i \cup C_j) \subset (A \cup B)$ , thus yielding that  $C_i \cup C_j$  is disconnected. However, in Exercise 5 we showed that since  $C_i, C_j$  are connected and  $C_i \cap C_j \neq \emptyset$ , their union is also connected, which means that we arrive at a contradiction. Therefore  $\bigcup \mathcal{C}$  is also connected. As an application, consider the fact that  $\mathbb{R} = \bigcup_{a>0} [-a, a]$ , and that each of these intervals is connected, which means that since all of the intervals have 0 as a common point,  $\mathbb{R}$  is also connected.

### Exercise 7

If every pair of points in  $M$  is contained in some connected set, show that  $M$  is itself connected.

*Solution.*

Fix a point  $x \in M$ , and observe that  $M = \bigcup_{y \in M} (\{x\} \cup \{y\})$ . We know that each pair  $x, y$  is contained in some connected set  $A_y$ . Furthermore, because  $M$  is the entire metric space, it must also hold that  $M = \bigcup_{y \in M} A_y$  (i.e., it cannot be that some point in some  $A_y$  is not contained in the metric space). We have thus written  $M$  as a union of connected sets that have a point in common, namely,  $x$ . By exercise 6,  $M$  is therefore also connected.

### Exercise 9

If  $A \subset B \subset \overline{A} \subset M$ , and if  $A$  is connected, show that  $B$  is connected. In particular,  $\overline{A}$  is connected.

*Solution.*

Suppose  $B$  is disconnected. Observe first that obviously it cannot be that  $B = A$ . Because  $B \subset \overline{A}$ , it must then be that  $B$  contains at least one  $x \in \overline{A} \setminus A$ , and possibly more. For any such  $x$ , there exists some sequence  $(x_n) \subset A$  such that  $x_n \rightarrow x$  (by the definition of the closure). Because  $B$  is disconnected, by Lemma 6.5 there exists  $f : B \rightarrow \{0, 1\}$  continuous and onto. Note that by the corollary noted immediately after Lemma 6.5, the restriction of  $f$  to  $A$  must be constant because  $A$  is connected. This means that, WLOG,  $f(a) = 0$  for each  $a \in A$ . More specifically,  $f(x_n) = 0$  for all  $n$ . However, because  $f$  is onto, it must be the case that  $f(x) = 1$  for at least one of the  $x$  described above. But this is a contradiction due to the fact that continuous functions preserve limits on convergent sequences. Therefore,  $B$  must be connected.

**Exercise 13**

If  $f : [a, b] \rightarrow [a, b]$  is continuous, show that  $f$  has a fixed point; that is, show that there is some point  $x \in [a, b]$  with  $f(x) = x$ .

*Solution.*

If  $f(b) = b$ ,  $b$  serves as the fixed point. If  $f(a) = a$ ,  $a$  serves as the fixed point. If neither of these hold, then due to the range of  $f$  we have that  $f(b) < b \implies f(b) - b < 0$ ,  $f(a) > a \implies f(a) - a > 0$ . Let then  $g : [a, b] \rightarrow \mathbb{R}$ ,  $g(x) = f(x) - x$ , which is a continuous function such that  $g(a) > 0$ ,  $g(b) < 0$ . Therefore, by the Intermediate Value Theorem (Corollary 6.7), we obtain that  $g$  assumes every value between  $g(b)$  and  $g(a)$ , so more specifically it achieves the value of 0 for some  $c \in (a, b)$ , meaning that  $g(c) = 0 \implies f(c) = c$ .

**Exercise 15**

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and open, show that  $f$  is strictly monotone.

*Solution.*

If  $f$  is constant in any interval  $(a, b)$ , we trivially see that it cannot be open (the image of  $(a, b)$  under  $f$  is a single point, which is a closed set). From now on we therefore assume that it is not constant in any interval. Suppose that it is not strictly monotone, which means that there exist  $a, b, c \in \mathbb{R}$ ,  $a < b < c$  such that (assuming WLOG a direction for the inequalities)  $f(a) < f(b)$  but  $f(b) > f(c)$ . The important observation here is that  $f$ , as a continuous function, achieves a minimum value  $m$  and a maximum value  $M$  in the interval  $[a, c]$ , and that  $M$  cannot be achieved at either  $a$  or  $c$  (since  $f(b) > f(a), f(b) > f(c)$ ). Therefore, when we consider the open interval  $(a, b)$ , it must be the case that  $f((a, b))$  is an interval that is either of the form  $[m, M]$ , or of the form  $(m, M]$ . In both cases, this is not an open set, which contradicts the assumption that  $f$  is open. Therefore,  $f$  is strictly monotone.

**Exercise 26**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined as  $f(x) = \sin(1/x)$  for  $x \neq 0$  and  $f(0) = 0$ . Show that although  $f$  is not continuous, the graph of  $f$  is a connected subset of  $\mathbb{R}^2$ . [Hint: Use Exercise 9.]

*Solution.*

First we show that  $f$  is not continuous at 0. Indeed, pick  $\epsilon = 1/2$  and observe that it should be the case that for some  $\delta > 0$ , whenever  $|x| < \delta$ ,  $|f(x)| < \epsilon$ . However, for any  $\delta$ , if we select a sufficiently large  $k \in \mathbb{N}$ , we obtain that  $\frac{2}{4k\pi+\pi} < \delta$  but  $\sin(1/x) = \sin(\frac{4k\pi+\pi}{2}) = \sin(2k\pi + \pi/2) = 1 > \epsilon$ , and so  $f$  cannot be continuous at 0 (this is one way to formalize the increasingly fast oscillations of this function's graph). Now we show that its graph is connected in  $\mathbb{R}^2$ . We have that the set  $\{x > 0\}$  is connected (as an interval). Furthermore, the map  $x \mapsto (x, f(x))$ ,  $x > 0$  is continuous (Lemma 5.8), and so the set  $\{(x, f(x)), x > 0\}$  is connected as the image of a connected set under a continuous function. Now, the closure of this set of course contains  $\{(0, 0)\} \cup \{(x, f(x)), x > 0\}$  (i.e., the graph of  $f$ ): the sequence  $((\frac{1}{n\pi}, f(\frac{1}{n\pi}))), n > 0$  converges to  $(0, 0)$ . By exercise 9, we thus have that the graph of  $f$  is connected.

**Exercise 27**

Let  $V$  be a normed vector space, and let  $x \neq y \in V$ . Show that the map  $f(t) = x + t(y - x)$  is a homeomorphism from  $[0, 1]$  into  $V$ . The range of  $f$  is the line segment joining  $x$  and  $y$ , and is often written  $[x, y]$  (since  $f$  is a homeomorphism, the interval notation is justified). [Hint: That  $f$  is continuous and one-to-one is easy; next, show that if  $f(t_n) \rightarrow z$ , then  $(t_n)$  converges to some  $t$  in  $[0, 1]$  with  $z = f(t)$ .]

*Solution.*

To show that  $f$  is one-to-one, we have that if  $f(t_1) = f(t_2)$ :

$$f(t_1) = f(t_2) \implies x + t_1(y - x) = x + t_2(y - x) \implies (t_1 - t_2)(y - x) = 0,$$

by the properties of vector spaces. Because  $y \neq x$ ,  $y - x$  cannot equal the zero vector, and so it must be the case that  $t_1 = t_2$ , thus showing that  $f$  is one-to-one. The *into* part of the statement implies that we do not need to show that  $f$  is onto, since we only care about its range. For continuity, notice that:

$$\|f(t_1) - f(t_2)\| = \|x + t_1(y-x) - x - t_2(y-x)\| = \|(t_1 - t_2)(y-x)\| = |t_1 - t_2| \cdot \|y-x\|,$$

and so  $f$  is in fact Lipschitz with constant  $\|y-x\|$ , and therefore continuous. The only thing that remains is showing that  $f^{-1}$  (defined in  $f$ 's range) is also continuous. To do this, we follow the hint: suppose  $f(t_n) \rightarrow z = f(t)$ . Then, for any given  $\epsilon > 0$ , set  $\delta = \epsilon/\|y-x\|$  and we have that for  $n \geq N$  for some  $N$ :

$$\|f(t_n) - f(t)\| < \delta \implies |t_n - t| \cdot \|y-x\| < \epsilon \|y-x\| \implies |t_n - t| < \epsilon,$$

which of course shows that  $t_n \rightarrow t$ , which completes the proof that  $f^{-1}$  is continuous, and thus  $f$  is a homeomorphism into  $V$ .

**Exercise 28**

Deduce from Exercises 7 and 27 that any normed space  $V$  is connected.

*Solution.*

If we can show that any pair of vectors  $x, y \in V$  is contained in some connected set, we will have shown that  $V$  is connected (Exercise 7). The observation here is that for any two such vectors, consider the set  $\{x + t(y-x), t \in [0, 1]\} \subset V$ . This is the image  $f([0, 1])$  if  $f$  is defined as in Exercise 27. Because  $[0, 1]$  is connected as an interval and because  $f$  is continuous, we conclude that  $f([0, 1])$  is also connected. This set contains  $x, y$ , and is connected, so by our initial observation we conclude that the normed vector space  $V$  is connected.

# Chapter 7

## Completeness

### 7.1 Totally Bounded Sets

#### Exercise 1

If  $A \subset B \subset M$  and  $B$  is totally bounded, then  $A$  is totally bounded.

*Solution.*

Pick any  $\epsilon > 0$ . By Lemma 7.1, because  $B$  is totally bounded, there exist finitely many sets  $B_1, \dots, B_n \subset B$  such that  $\text{diam}(B_i) < \epsilon$  for all  $i$  and  $B \subset \bigcup_{i=1}^n B_i$ . Let then  $A_i = B_i \cap A$ , each of which is clearly a subset of  $A$ . It holds of course that  $\text{diam}(A_i) \leq \text{diam}(B_i) < \epsilon$ . Furthermore, because  $A \subset B \subset \bigcup_{i=1}^n B_i$ , any  $x \in A$  must belong in at least one  $B_i$ , and hence also in the corresponding  $A_i$ , which means that  $A \subset \bigcup_{i=1}^n A_i$ . By all of the above and Lemma 7.1, we conclude that  $A$  is totally bounded.

#### Exercise 2

Show that a subset  $A \subset \mathbb{R}$  is totally bounded if and only if it is bounded. In particular, if  $I$  is a closed, bounded, interval in  $\mathbb{R}$  and  $\epsilon > 0$ , show that  $I$  can be covered by finitely many closed subintervals  $J_1, \dots, J_n$ , each of length at most  $\epsilon$ .

*Solution.*

We recall example 7.2 (a): a totally bounded set  $A$  is necessarily bounded. To show this, pick  $\epsilon = 1$  and obtain  $x_1, \dots, x_n$  such that  $A \subset \bigcup_{i=1}^n B_1(x_i)$ . Set  $M = \max_{i,j} d(x_i, x_j)$  and pick any two  $x, y \in A$ , for which it must hold that  $x \in B_i, y \in B_j$  for some  $i, j$ . By the triangle inequality:

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 1 + M + 1 = M + 2$$

This of course shows that  $A$  is bounded.

Conversely, if  $A \subset \mathbb{R}$  is bounded, then  $\text{diam } A < R$ , which in  $\mathbb{R}$  is equivalent to  $A \subset [-R, R]$ . For any  $\epsilon > 0$ , set  $N = \lceil \frac{2R}{\epsilon} \rceil$  and subdivide  $[-R, R]$  into  $N$  closed subintervals of length  $\epsilon$ ,  $[-R, a_1], [a_1, a_2], \dots, [a_{N-1}, R]$ . Then observe that any element of  $A$  must belong in one of these intervals, and thus is at most  $\epsilon$ -far from some  $a_i$ . Since there are finitely many  $a_i$ , we conclude that  $A$  is totally bounded.

#### Exercise 4

Show that  $A$  is totally bounded if and only if  $A$  can be covered by finitely many *closed* sets of diameter at most  $\epsilon$  for every  $\epsilon > 0$ .

*Solution.*

$\implies$ : Suppose first that  $A$  is totally bounded and pick any  $\epsilon > 0$ . Then there exist finitely many  $x_1, \dots, x_n$  such that  $A \subset \bigcup_{i=1}^n B_{\epsilon/4}(x_i)$ . As in the book, we may assume that  $A \cap B_{\epsilon/4}(x_i) \neq \emptyset$  for all  $i$ , and thus for each  $x_i$  find  $a_i \in A, a_i \in B_{\epsilon/4}(x_i)$ . Now, pick any  $y \in B_{\epsilon/4}(x_i)$ , and by the triangle inequality:

$$d(y, a_i) \leq d(y, x_i) + d(x_i, a_i) \leq \epsilon/4 + \epsilon/4 \leq \epsilon/2$$

This shows that all elements of  $B_{\epsilon/4}(x_i)$  belong in the *closed* ball of radius  $\epsilon/2$  around  $a_i$ . More specifically then, all elements of  $A$  that were covered by  $B_{\epsilon/4}(x_i)$  are also covered by this closed ball. We conclude that the union of these  $n$  closed balls —each of which of course has diameter  $\epsilon$ — covers  $A$ .

$\Leftarrow$  : Conversely, suppose that for any  $\epsilon > 0$ ,  $A$  can be covered by finitely many closed sets of diameter at most  $\epsilon$ . For any  $\epsilon > 0$ , use this hypothesis on  $\epsilon/2$  to obtain finitely many closed sets  $S_1, \dots, S_n$  such that  $A \subset \bigcup_{i=1}^n S_i$  and  $\text{diam}(S_i) \leq \epsilon/2$  for all  $i$ . Once again, we may safely assume that  $A \cap S_i \neq \emptyset$  for all  $i$ , and thus for every  $i$  find an  $a_i \in A, a_i \in S_i$ . For any  $y \in S_i$ , it holds that  $d(y, a_i) \leq \text{diam}(S_i) \leq \epsilon/2 < \epsilon$ , and so  $y \in B_\epsilon(a_i)$ . This means that all elements of  $A$  that were covered by  $S_i$  are also covered by  $B_\epsilon(a_i)$ , meaning that  $A \subset \bigcup_{i=1}^n B_\epsilon(a_i)$ , which is of course the definition of total boundedness for  $A$ .

### Exercise 5

Prove that  $A$  is totally bounded if and only if  $\overline{A}$  is totally bounded.

*Solution.*

$\Rightarrow$  : Suppose  $A$  is totally bounded and pick any  $\epsilon > 0$ . Find  $x_1, \dots, x_n$  such that  $A \subset \bigcup_{i=1}^n B_{\epsilon/2}(x_i)$ , which is always possible due to  $A$  being totally bounded. For any  $z \in \overline{A}$ , we have the following. If  $z \in A$ , we have that  $z$  is inside at least one of these  $n$  balls of radius  $\epsilon/2$ , and thus also inside the corresponding ball of radius  $\epsilon$ . If  $z \notin A$ , by the definition of the closure it must be that  $y_i \rightarrow z$  for some  $(y_i) \subset A$ . Then there exists  $N > 0$  such that for all  $i > N, d(y_i, z) < \epsilon/2$ . Pick any of these  $y_i$ , and find a corresponding  $x_j$  such that  $y_i \in B_{\epsilon/2}(x_j)$ . By the triangle inequality,  $d(y_i, z) \leq d(y_i, x_j) + d(x_j, z) < \epsilon/2 + \epsilon/2 = \epsilon$ , which means that  $y_i \in B_\epsilon(z)$ . We've therefore shown total boundedness for  $\overline{A}$ .

$\Leftarrow$  : In the other direction, recall that  $A \subset \overline{A}$ , so by Exercise 1 if  $\overline{A}$  is totally bounded,  $A$  is as well.

### Exercise 9

Give an example of a closed bounded subset of  $l_\infty$  that is not totally bounded.

*Solution.*

Consider the set corresponding to the sequence examined in Example 7.2 (d):

$$S = \{(1, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\} \subset l_\infty$$

It's easy to see that the set is bounded, since all of its elements have a  $l_\infty$  norm of 1. At the same time, it is closed, since the only convergent sequences it contains are eventually constant ones. However, the observation given in that example shows that it is not totally bounded: any two of its elements are at a distance of 1, and hence no finite number of balls of radius less than 1 can cover it.

### Exercise 10

Prove that a totally bounded metric space  $M$  is separable. [Hint: For each  $n$ , let  $D_n$  be a finite  $(1/n)$ -net for  $M$ . Show that  $D = \bigcup_{n=1}^\infty D_n$  is a countable dense set.]

*Solution.*

Suppose  $M$  is totally bounded, and form the construction indicated in the hint. Namely, for each  $n$ , collect the *finite*, due to total boundedness,  $x_1, x_2, \dots, x_{k_n}$  needed to cover  $M$  with balls of radius  $1/n$ . Call this set of  $k_n$  elements  $D_n$ , and let  $D = \bigcup_{n=1}^\infty D_n$ , which is countable as a countable union of countable sets. Now, select any  $y \in M$  and for each  $i$ , select an element  $x_i \in D_i$  such that  $d(y, x_i) < 1/i$ , which is guaranteed to exist. Notice that this sequence gets arbitrarily close to  $y$ , and hence  $D$  is also dense, showing that  $M$  is separable.

### Exercise 11

Prove that  $H^\infty$  is totally bounded (see Exercises 3.10 and 4.48).

*Solution.*

Recall the methodology we followed in Exercise 4.48. In particular, recall the observation that given any  $\epsilon > 0$ , we can find  $N$  such that for any  $x \in H^\infty$  and any sequence  $r = (r_1, \dots, r_N, 0, \dots)$  it holds that under the metric of  $H^\infty$ :

$$d(x, r) < \sum_{n=1}^N |x_n - r_n| + \frac{\epsilon}{2}$$

If we can show that for any  $\epsilon > 0$  we can select *finitely* many sequences of the form  $r = (r_1, \dots, r_N, 0, 0, \dots)$  such that for any  $x \in H^\infty$ , it is the case that for at least one of them it holds that  $\sum_{n=1}^N |x_n - r_{i,n}| < \epsilon/2$ , we will have shown that  $d(x, r_i) < \epsilon$ , and thus that  $H^\infty$  is totally bounded. Let then  $K = \lceil \frac{4N}{\epsilon} \rceil$ , and consider subdividing the interval  $[-1, 1]$  in  $K$  equal subintervals, each of length  $\frac{2}{K}$ . Let  $r_1, r_2, \dots, r_K$  be the endpoints of these subintervals. Then, for any given sequence  $(x_n)$ , and for any of its terms  $x_i$ , it must hold that for some  $r_j$ :

$$|x_i - r_j| < \frac{2}{K} < \frac{2}{\frac{4N}{\epsilon}} < \frac{\epsilon}{2N}$$

We form  $r$  by selecting such an  $r_j$  for each  $i = 1, 2, \dots, N$ , and then we have that:

$$d(x, r) < \sum_{n=1}^N \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon$$

To summarize, for any  $\epsilon > 0$ , pick first  $N$  as described previously, then select  $r_1, \dots, r_K$  as above (where  $K$  is a function of  $N, \epsilon$ ), and then set  $R = \{(\rho_1, \rho_2, \dots, \rho_N, 0, 0, \dots), \rho_i \in \{r_1, r_2, \dots, r_K\}\}$ , which is a finite set such that for any  $x \in H^\infty$ ,  $d(x, r) < \epsilon$  for some  $r \in R$ . Therefore,  $H^\infty$  is totally bounded.

## 7.2 Complete Metric Spaces

### Exercise 12

Let  $(A, d)$  be a subset of an arbitrary metric space  $(M, d)$ . If  $(A, d)$  is complete, show that  $A$  is closed in  $M$ .

*Solution.*

Suppose  $(x_n) \subset A$  is a sequence that converges to  $x$ . By Exercise 36 of Ch. 3, we know that  $(x_n)$  is Cauchy as a convergent sequence. Because  $A$  is complete, this means that  $(x_n)$  must converge to a point inside  $A$ , which of course shows that  $x \in A$ , meaning that  $A$  is indeed closed in  $M$ .

### Exercise 16

Prove that  $\mathbb{R}^n$  is complete under any of the norms  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ . [This is interesting because completeness is not usually preserved by the mere equivalence of *metrics*. Here we use the fact that all of the metrics involved are generated by *norms*. Specifically, we need the norms in question to be equivalent as functions:  $\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \|\cdot\|_1 \leq n\|\cdot\|_\infty$ . As we will see later, any two norms on  $\mathbb{R}^n$  are comparable in this way.]

*Solution.*

Suppose  $(x_i)$  is a Cauchy sequence in  $\mathbb{R}^n$  under  $\|\cdot\|_1$ . Then for every  $\epsilon > 0$ , there exists  $N > 0$  such that for every  $i, j \geq N$  it holds that:

$$\|x_i - x_j\|_1 < \epsilon \implies \sum_{k=1}^n |x_{i,k} - x_{j,k}| < \epsilon$$

This of course means that each of the coordinate sequences is Cauchy in  $\mathbb{R}$ , and so it converges to some limit, which we call  $l_k$  for the  $k$ -th coordinate. Then we already know that  $(x_i)$  converges to  $(l_1, l_2, \dots, l_n)$ , which is clearly in  $\mathbb{R}^n$  and so  $\mathbb{R}^n$  is complete under  $\|\cdot\|_1$ . Note also that the equivalence of the metrics induced by these norms shows that  $x_i \rightarrow_\infty (l_1, l_2, \dots, l_n)$ ,  $x_i \rightarrow_2 (l_1, l_2, \dots, l_n)$  as well. What we furthermore have is that, as stated in the hint,  $\|x_i - x_j\|_\infty \leq \|x_i - x_j\|_1 < \epsilon$ , so any sequence that is Cauchy under  $\|\cdot\|_1$  is also Cauchy under  $\|\cdot\|_\infty$ . Similarly, because  $\|x_i - x_j\|_1 \leq n\|x_i - x_j\|_\infty$ , any Cauchy sequence under  $\|\cdot\|_\infty$  is also Cauchy under  $\|\cdot\|_1$ , and so the two sets of Cauchy sequences are the identical, and the argument presented above for  $\|\cdot\|_1$  suffices to show that  $\mathbb{R}^n$  is complete under  $\|\cdot\|_\infty$  as well. By a similar argument we can see that  $\mathbb{R}^n$  is complete under  $\|\cdot\|_2$  as well.

**Exercise 17**

Given metric spaces  $M, N$ , show that  $M \times N$  is complete if and only if both  $M, N$  are complete.

*Solution.*

We will use  $d_1((x_1, y_1), (x_2, y_2)) = d_M(x_1, x_2) + d_N(y_1, y_2)$  as the product metric on  $M \times N$  (from Exercise 46 of Ch. 3 we know that we have three choices that define equivalent metrics). Suppose first that  $M, N$  are complete. Pick any Cauchy sequence  $((x_i, y_i)) \in M \times N$ . Then for any  $\epsilon > 0$  there exists  $K > 0$  such that for  $k, l \geq K$  it holds that:

$$d_1((x_k, y_k), (x_l, y_l)) < \epsilon \implies d_M(x_k, x_l) + d_N(y_k, y_l) < \epsilon$$

This of course shows that  $d_M(x_k, x_l) < \epsilon, d_N(y_k, y_l) < \epsilon$  for  $k, l \geq K$ , i.e., that  $(x_i), (y_i)$  are Cauchy in  $M, N$  respectively. By the completeness of  $M, N$ , we have that  $x_i \rightarrow x \in M, y_i \rightarrow y \in N$ , and then Exercise 46 Ch. 3 guarantees that  $(x_i, y_i) \rightarrow (x, y)$ , which is of course an element of  $M \times N$ , and so we've shown completeness for  $M \times N$ .

Conversely, assume  $M \times N$  is complete, and pick any Cauchy sequence  $(x_i) \in M$ . For a non-trivial problem, we assume that  $N \neq \emptyset$ , and so there exists at least one  $y \in N$ . Form the sequence  $(x_i, y) \subset M \times N$ , and pick any  $\epsilon > 0$ . We then have that there exists  $K > 0$  such that for  $k, l \geq K$ :

$$d_M(x_k, x_l) < \epsilon \implies d_M(x_k, x_l) + 0 < \epsilon \implies d_M(x_k, x_l) + d_N(y, y) < \epsilon \implies d_1((x_k, y), (x_l, y)) < \epsilon$$

This shows that  $(x_i, y)$  is Cauchy in  $M \times N$ , and by the completeness of  $M \times N$  we obtain that  $(x_i, y) \rightarrow (x, y')$  for some  $x \in M, y' \in N$ . Because convergence in the product space implies convergence in each of  $M, N$ , we have that  $x_i \rightarrow x, y \rightarrow y'$  ( $y' = y$  due to the fact that this sequence is constant). We've therefore shown that any Cauchy sequence in  $M$  converges to an element of  $M$ , which means that  $M$  is complete. Exchanging the roles of  $M, N$  shows that the same holds for  $N$ .

**Exercise 18**

Fill in the details of the proofs that  $l_1, l_\infty$  are complete.

*Solution.*

The two proofs are done in much the same way as the proof that  $l_2$  is complete. We begin with  $l_1$ . Consider a Cauchy sequence (of sequences)  $(f_n) \subset l_1$ , where we write  $f_n = (f_n(k))_{k=1}^\infty$ . Fix any  $k \in \mathbb{N}$  and consider the sequence  $(f_n(k))$ . Select any  $\epsilon > 0$  and observe that because  $(f_n)$  is Cauchy, there exists  $N > 0$  such that for any  $n, m \geq N$  we have that  $\|f_n - f_m\|_1 < \epsilon$ . Thus, it also holds that:

$$|f_n(k) - f_m(k)| \leq \|f_n - f_m\|_1 < \epsilon,$$

which shows that the sequence  $(f_1(k), f_2(k), \dots)$  is Cauchy in  $\mathbb{R}$  and thus converges. Call its limit  $f(k)$  and set  $f = (f(1), f(2), \dots)$ . We now show that  $f \in l_1$ . Fix any  $N \in \mathbb{N}$ , and consider the “prefix”  $(f(1), f(2), \dots, f(N))$ . We have that:

$$\sum_{n=1}^N |f(n)| = \sum_{n=1}^N |\lim_{m \rightarrow \infty} f_m(n)| = \lim_{m \rightarrow \infty} \sum_{n=1}^N |f_m(n)|$$

Notice now that the RHS here is bounded, since  $f_m \in l_1$  for any  $m$ . Therefore, for any  $N, \sum_{n=1}^N |f(n)| \leq B$ , and since limits preserve non-strict inequalities,  $\|f\|_1 \leq B \implies f \in l_1$ . Finally, we need to show that  $f_n \rightarrow f$  in  $l_1$ .

Pick any  $\epsilon > 0$  and select  $m_0$  such that for  $m, l \geq m_0, \|f_m - f_l\|_1 < \epsilon$ . Again, fix  $N$  and observe that if  $m \geq m_0$ :

$$\sum_{n=1}^N |f_m(n) - f(n)| = \sum_{n=1}^N |\lim_{k \rightarrow \infty} f_k(n) - f_m(n)| = \lim_{k \rightarrow \infty} \sum_{n=1}^N |f_k(n) - f_m(n)|$$

Notice also that for any  $m, k \geq m_0$ :

$$\sum_{n=1}^N |f_k(n) - f_m(n)| \leq \|f_k - f_m\|_1 < \epsilon$$

Taking the limit wrt.  $k$  we obtain that:

$$\lim_{k \rightarrow \infty} \sum_{n=1}^N |f_k(n) - f_m(n)| \leq \epsilon \implies \sum_{n=1}^N |f_m(n) - f(n)| \leq \epsilon$$

Finally, since this holds for any  $N$ , we have that  $\|f_m - f\| \leq \epsilon$  for  $m \geq m_0$  and so  $f_n \rightarrow f$ , and we've thus shown that  $l_1$  is complete (any Cauchy sequence in it converges to a point in it).

Now we do the same for  $l_\infty$ . Let  $(f_n) \subset l_\infty$  be Cauchy. First, fix  $k \in \mathbb{N}$  and examine  $(f_n(k))$  (sequence of reals). We have that, for any  $\epsilon > 0$  there exists  $N > 0$  such that for  $n, m \geq N$  it holds that  $\|f_n - f_m\|_\infty < \epsilon$ . From this we obtain:

$$|f_n(k) - f_m(k)| \leq \sup_l |f_n(l) - f_m(l)| = \|f_n - f_m\| < \epsilon,$$

showing that  $(f_n(k))$  is Cauchy in  $\mathbb{R}$  and thus converges, say to  $f(k)$ . We again define  $f$  as  $f = (f(1), f(2), \dots)$ , and we need to show that  $f \in l_\infty$ . For any fixed  $N$ :

$$\max_{n \leq N} |f(n)| = \max_{n \leq N} \left| \lim_{m \rightarrow \infty} f_m(n) \right| = \lim_{m \rightarrow \infty} \max_{n \leq N} |f_m(n)|$$

Because all  $f_m \in l_\infty$ , the RHS here is bounded, and since this holds for any  $N$ , we have also that  $f \in l_\infty$ . Now we need to show that  $f_n \rightarrow f$ . Pick any  $\epsilon > 0$ , and select  $m_0$  such that for  $m, l \geq m_0$ ,  $\|f_m - f_l\|_\infty < \epsilon$ . To begin, fix  $N$  and observe that if  $m \geq m_0$ :

$$\max_{n \leq N} |f_m(n) - f(n)| = \max_{n \leq N} \left| \lim_{k \rightarrow \infty} (f_k(n) - f_m(n)) \right| = \lim_{k \rightarrow \infty} \max_{n \leq N} |f_k(n) - f_m(n)|$$

Now, for any  $k \geq m_0$  we have that:

$$\max_{n \leq N} |f_k(n) - f_m(n)| \leq \sup_n |f_k(n) - f_m(n)| = \|f_k - f_m\|_\infty < \epsilon,$$

and thus taking the limit wrt.  $k$  yields that:

$$\lim_{k \rightarrow \infty} \max_{n \leq N} |f_k(n) - f_m(n)| \leq \epsilon$$

Therefore  $\max_{n \leq N} |f_m(n) - f(n)| \leq \epsilon$ , and since this holds for any  $N$ , we conclude that  $|f_m - f| \leq \epsilon$ , showing that  $f_n \rightarrow f \in l_\infty$  and hence that  $l_\infty$  is complete.

### Exercise 20

If  $(x_n), (y_n)$  are Cauchy in  $(M, d)$  show that  $(d(x_n, y_n))_{n=1}^\infty$  is Cauchy in  $\mathbb{R}$ .

*Solution.*

Pick any  $\epsilon > 0$ . We have first that:

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_m, y_n) + d(x_m, y_n) - d(x_m, y_m)| \leq \\ &|d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m), \end{aligned}$$

where we've used the triangle inequality in  $\mathbb{R}$  and Exercise 2 of Ch. 3 (reverse triangle inequality in a metric space). Due to  $(x_n), (y_n)$  being Cauchy, we can find  $N, M > 0$  such that whenever  $n, m \geq N$ ,  $d(x_n, x_m) < \epsilon/2$  and whenever  $n, m \geq M$ ,  $d(y_n, y_m) < \epsilon/2$ . Set  $K = \max\{N, M\}$  and then we conclude that for  $n, m \geq K$ ,  $d(x_n, x_m) < \epsilon/2$ ,  $d(y_n, y_m) < \epsilon/2$ , which by the above inequality shows that  $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$ , i.e. that  $(d(x_n, y_n))_{n=1}^\infty$  is Cauchy in  $\mathbb{R}$ .

**Exercise 21**

If  $(M, d)$  is complete, prove that two Cauchy sequences  $(x_n), (y_n)$  have the same limit if and only if  $d(x_n, y_n) \rightarrow 0$ .

*Solution.*

Suppose first that  $d(x_n, y_n) \rightarrow 0$ . Because  $(M, d)$  is complete, we have that  $(x_n)$  converges to some  $x$  as a Cauchy sequence. Pick any  $\epsilon > 0$ , and obtain  $N$  such that for  $n \geq N$ ,  $d(x_n, y_n) < \epsilon/2$ . By the reverse triangle inequality:

$$|d(y_n, x) - d(x_n, x)| \leq d(y_n, x_n) < \epsilon/2 \implies d(x_n, x) - \epsilon/2 < d(y_n, x) < \epsilon/2 + d(x_n, x)$$

There exists  $M > 0$  such that for  $n \geq M$ ,  $d(x_n, x) < \epsilon/2$ . For  $n \geq \max\{N, M\}$ , we then have that:

$$d(y_n, x) < \epsilon/2 + d(x_n, x) < \epsilon/2 + \epsilon/2 \implies d(y_n, x) < \epsilon,$$

which shows that  $y_n \rightarrow x$ . Conversely, if  $x_n \rightarrow x, y_n \rightarrow x$ , we have that:

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$$

Since the RHS can be made arbitrarily small, we obtain that  $d(x_n, y_n) \rightarrow 0$ .

**Exercise 22**

Let  $D$  be a dense subset of a metric space  $M$ , and suppose that every Cauchy sequence from  $D$  converges to some point of  $M$ . Prove that  $M$  is complete.

*Solution.*

Pick any Cauchy sequence  $(x_n)$  in  $M$ . Now, construct a sequence  $(y_n) \subset D$  in the following manner. For any  $n$ , pick  $y_n \in D$  such that  $d(x_n, y_n) < \frac{1}{2^n}$ . This is always possible due to  $D$  being dense in  $M$ . We show now that  $(y_n)$  is Cauchy. Pick any  $\epsilon > 0$  and find  $N > 0$  such that  $\frac{1}{2^N} < \frac{\epsilon}{3}$ . By the construction of  $(y_n)$ , for any  $n, m \geq N$ , it holds that  $d(x_n, y_n) < \epsilon/3, d(x_m, y_m) < \epsilon/3$ . Furthermore, because  $(x_n)$  is Cauchy we can find  $M$  such that for  $n, m \geq M$ ,  $d(x_n, x_m) < \epsilon/3$ .

Setting  $K = \max\{N, M\}$ , we observe that for  $n, m \geq K$ :

$$d(y_n, y_m) \leq d(x_n, y_n) + d(x_n, x_m) + d(x_m, y_m) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

This shows that  $(y_n)$  is Cauchy in  $D$ , and thus, by the hypothesis, converges to a point in  $M$ . Now, since  $d(x_n, y_n) \rightarrow 0$ , we can apply Exercise 21 (Ch. 7) to conclude that  $(x_n), (y_n)$  have the same limit, and hence that  $(x_n)$  converges to a point in  $M$ , showing that  $M$  is complete.

**Exercise 23**

Prove that  $M$  is complete if and only if every sequence  $(x_n)$  in  $M$  satisfying  $d(x_n, x_{n+1}) < 2^{-n}$ , for all  $n$ , converges to a point in  $M$ .

*Solution.*

First, we observe that every sequence that satisfies this constraint is Cauchy. Indeed, if  $(x_n)$  is one such sequence, pick any  $\epsilon > 0$  and set  $N = \lceil \log(\frac{1}{\epsilon}) \rceil$ , in which case  $2^N > \frac{1}{\epsilon} \implies \epsilon > 2^{-N}$ . For any  $n, m > N$  (suppose  $m > n$ ) it holds that:

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \leq \dots \leq \\ &\sum_{i=n}^{m-1} d(x_i, x_{i+1}) = \sum_{j=0}^{m-n-1} d(x_{n+j}, x_{n+j+1}) \end{aligned}$$

Now notice that for each such term it holds that  $d(x_{n+j}, x_{n+j+1}) < 2^{-n-j}$ . Substituting:

$$d(x_n, x_m) < \sum_{j=0}^{m-n-1} \frac{1}{2^{n+j}} = \frac{1}{2^n} \sum_{j=0}^{m-n-1} \frac{1}{2^j} < \frac{2}{2^n} = \frac{1}{2^{n-1}} \leq \frac{1}{2^N} < \epsilon,$$

completing the proof that the sequence is indeed Cauchy. The immediate consequence is that if  $M$  is complete, any such sequence in  $M$  converges to a point in  $M$  (by the definition of completeness).

Conversely, suppose that any such sequence in  $M$  converges to a point in  $M$ , and let  $(y_n) \subset M$  be Cauchy. Construct the following subsequence  $(y_{n_k})$ . For any  $k = 1, 2, \dots$ , set  $\epsilon_k = 2^{-k}$ , and obtain  $N_k$  such that for  $i, j \geq N_k$ ,  $d(x_i, x_j) < \epsilon_k = 2^{-k}$ . Set  $y_{n_k}$  to be equal to the first  $x_i, i \geq N_k$  that has not yet been previously selected (this is always possible since  $x_n$  has infinite terms). Now notice that by its definition,  $d(y_{n_k}, y_{n_{k+1}}) < \epsilon_k = 2^{-k} \leq 2^{-n_k}$  (since  $n_k \leq k$  always holds due to the fact that we select each term to be different from all previous ones). By our hypothesis,  $(y_{n_k})$  converges to a point in  $M$ , and then Exercise 15 Ch. 1 guarantees that  $(y_n)$  converges to the same point as a Cauchy sequence with a convergent subsequence.

### Exercise 24

Prove that the Hilbert cube  $H^\infty$  is complete.

*Solution.*

Let  $(x_n) \subset H^\infty$  be a Cauchy sequence, and consider, for any  $k$ , the sequence formed by taking the  $k$ -term of each of these, i.e.  $(x_{11}, x_{21}, x_{31}, \dots)$ , and call it  $y_k$ . For any  $\epsilon > 0$ , set  $\epsilon' = \epsilon/2^k$ . Since  $(x_n)$  is Cauchy, we can find  $N$  such that for  $n, m \geq N$ ,  $d(x_n, x_m) < \epsilon$ , which means:

$$\begin{aligned} d(x_n, x_m) < \epsilon &\implies \sum_{i=1}^{\infty} 2^{-i} |x_{ni} - x_{mi}| < \epsilon' \implies 2^{-k} |x_{nk} - x_{mk}| < \epsilon/2^k \implies |x_{nk} - x_{mk}| < \epsilon \\ &\implies |y_{k_n} - y_{k_m}| < \epsilon \end{aligned}$$

We have thus shown that  $y_k$  is Cauchy in  $\mathbb{R}$ , and thus converges, say to  $l_k$ . This holds for any  $k$ , so let  $l = (l_1, l_2, \dots)$ . Because it holds that  $|x_{nk}| \leq 1$ , it is also the case that  $|l_k| \leq 1$  for any  $k$ , and so  $l \in H^\infty$ . We now only need to show that  $x_n \rightarrow l$ . Take any  $\epsilon > 0$ , and fix any  $N > 0$ . We examine the quantity  $\sum_{k=1}^N 2^{-k} |x_{nk} - l_k|$ . We have that:

$$\sum_{k=1}^N 2^{-k} |x_{nk} - l_k| = \sum_{k=1}^N 2^{-k} |x_{nk} - \lim_{m \rightarrow \infty} x_{mk}| = \lim_{m \rightarrow \infty} \sum_{k=1}^N 2^{-k} |x_{nk} - x_{mk}|$$

We now pick  $M > 0$  such that  $d(x_n, x_m) < \epsilon$  (possible since  $(x_n)$  is Cauchy), and examine the quantity inside the limit for  $n, m \geq M$ :

$$\sum_{k=1}^N 2^{-k} |x_{nk} - x_{mk}| \leq \sum_{k=1}^{\infty} 2^{-k} |x_{nk} - x_{mk}| = d(x_n, x_m) < \epsilon$$

Since this holds for any  $m \geq M$ , we can take limits to obtain that  $\lim_{m \rightarrow \infty} \sum_{k=1}^N 2^{-k} |x_{nk} - x_{mk}| \leq \epsilon$ . Finally, since this holds for any  $N$ , we observe that we can thus also take the limit wrt. to  $N$  in this inequality to obtain that:

$$\sum_{k=1}^{\infty} 2^{-k} |x_{nk} - l_k| \leq \epsilon \implies d(x_n, l) \leq \epsilon,$$

as long as  $n \geq M$ , where  $M$  is defined as above. Since this holds for any  $\epsilon > 0$ , we conclude that  $x_n \rightarrow l \in H^\infty$ , hence that  $H^\infty$  is complete.

### Exercise 25

True or false? If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and if  $(x_n)$  is Cauchy, then  $(f(x_n))$  is Cauchy. Examples? How about if we insist that  $f$  be strictly increasing? Show that the answer is “true” if  $f$  is Lipschitz.

*Solution.*

We know that every Cauchy sequence in  $\mathbb{R}$  converges, which means that  $x_n \rightarrow x$  for some  $x$ . Then the continuity of  $f$  implies that  $f(x_n) \rightarrow f(x)$ . Now, select any  $\epsilon > 0$  and pick  $N$  such that for any  $n \geq N$ ,  $|f(x_n) - f(x)| < \epsilon/2$ . For any  $n, m \geq N$  we have that:

$$|f(x_n) - f(x_m)| \leq |f(x_n) - f(x) + f(x) - f(x_m)| \leq |f(x_n) - f(x)| + |f(x_m) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which shows that  $f(x_n)$  is Cauchy. The remaining questions in the exercise are obvious.

### Exercise 26

Just as with the nested interval theorem, it is essential that the sets  $F_n$  used in the nested set theorem be both closed and bounded. Why? Is the condition  $\text{diam}(F_n) \rightarrow 0$  really necessary? Explain.

*Solution.*

One simple reason is the fact that the counterexamples presented for the case of intervals are of course counterexamples for the case of sets too. Differently, we can extend the reasoning of these counterexamples in any metric space  $M$ . For example, if  $x \in M$  and we set  $F_n = \{y \in M : d(y, x) \leq \frac{1}{n}\} \setminus \{x\}$  (bounded but not closed), we observe that even though  $F_1 \supset F_2 \supset \dots$ , the infinite intersection must be the empty set: if it were to contain any point  $z$ , then  $d(z, x) \leq \frac{1}{n}$  for all  $n$ , which would mean  $z = x$ . Similarly, if we set  $F_n = \{y \in M : d(y, x) \geq n\}$  in a metric space where these sets are not bounded, we again have  $F_1 \supset F_2 \supset \dots$ , but again the infinite intersection must be the empty set, otherwise there would have to exist a  $z$  such that  $d(z, x) \geq n$  for all  $n$ , which is of course impossible (metrics are finite-valued).

Regarding the condition  $\text{diam}(F_n) \rightarrow 0$ , suppose  $x, y \in M$  with  $d(x, y) < 1$  and let  $F_n = \{z \in M : d(z, x) \leq 1 + \frac{1}{n}\}$ . Here the sets are nested but the diameter clearly does not go to zero: each  $F_n$  will always contain both  $x, y$ , since  $d(x, y) < 1 \leq 1 + \frac{1}{n}$  for every  $n$ . Therefore the diameter will always be at least  $d(x, y)$ , and the consequence is that we can no longer conclude that the infinite intersection contains only one point. Other than that, the nested set theorem still holds in the sense that the infinite intersection is not empty.

### Exercise 27

Note that the version of the Bolzano-Weierstrass theorem given in Theorem 7.11 replaces boundedness with total boundedness. Is this really necessary? Explain.

*Solution.*

We know that the metric space  $l_1$  is complete (Exercise 18 Ch. 7). Consider the set  $\{e_n\}$ , where  $e_n = (0, 0, \dots, 1, 0, \dots)$ , i.e.,  $e_n$  has 1 in its  $n$ -th coordinate and 0 elsewhere (example 7.2 (d)). As discussed in the example, this is a bounded but not totally bounded set in  $l_1$ . Suppose that it contains some convergent sequence  $x_i \rightarrow x \in l_1$ , as would have to be the case if Theorem 7.3 (i)  $\implies$  (iii) held for *bounded* sets. Note that this sequence cannot be eventually constant, otherwise we would not be able to obtain a limit point as defined in Exercise 4.33. Then, for  $\epsilon = 1$  there must exist  $N > 0$  such that for  $i \geq N$ :

$$\|x_i - x\|_1 < \epsilon \implies \sum_{j=1}^{\infty} |x_{ij} - x_j| < \epsilon$$

Because all  $x_n$  are drawn from the set of  $e_n$ , if we consider the infinitely many terms  $x_i, i \geq N$ , an infinite number of them must be drawn from the set  $\{e_m, e_{m+1}, \dots\}$  for *any*  $m$ . These must then have  $x_{ij} = 0$  for  $i < m$ , and hence from the inequality we have also  $x_j = 0$  for  $j < m$ . Since this holds for any  $m$ ,  $x = (0, 0, 0, \dots)$ . But then by definition it must be that  $\|x_i - x\| = 1 = \epsilon$ , since these differ in precisely one coordinate. This makes it impossible for the inequality above to hold, and thus the sequence fails to converge, showing that boundedness of a set alone is not enough in the nested set theorem.

### Exercise 30

If  $(M, d)$  is complete, prove that every open subset  $G \subset M$  is homeomorphic to a complete metric space. [Hint: Let  $F = M \setminus G$  and consider the metric  $\rho(x, y) = d(x, y) + |d(x, F)^{-1} - d(y, F)^{-1}|$  on  $G$ .]

*Solution.*

Based on the hint, we will show that the function  $i : (G, d) \rightarrow (G, \rho)$ ,  $i(x) = x$  is an homeomorphism. We know (Theorem 5.5) that this is equivalent to showing that  $x_n \xrightarrow{d} x \iff x_n \xrightarrow{\rho} x$ . Suppose

therefore that  $x_n \rightarrow^d x$ . Recall that by preposition 5.4 we know that  $x \rightarrow d(x, F)$  is a continuous function. This means that  $d(x_n, F) \rightarrow d(x, F)$ . We observe also that  $d(x, F) > 0, d(x_n, F) > 0$  must always hold: indeed,  $x_n, x \in G$ , and because  $G \cap F = \emptyset, \overline{F} = F$  ( $F$  is closed), it can never be that  $d(x_n, F) = 0, d(x, F) = 0$  since this would mean that these are in  $\overline{F}$ . Therefore the function  $x \mapsto d(x, F)^{-1}$  defined for  $x \in G$  is also well-defined and continuous. Pick now any  $\epsilon > 0$ . There exists  $N_1 > 0$  such that for  $n \geq N_1, |d(x_n, F)^{-1} - d(x, F)^{-1}| < \epsilon/2$ , and  $N_2 > 0$  such that for  $n \geq N_2, d(x_n, x) < \epsilon/2$ . Therefore for  $n \geq \max\{N_1, N_2\}$ :

$$\rho(x_n, x) = d(x_n, x) + |d(x_n, F)^{-1} - d(x, F)^{-1}| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which shows that  $x_n \rightarrow^\rho x$ . Conversely, suppose  $x_n \rightarrow^\rho x$ , which means that for any  $\epsilon > 0$ , there exists  $N > 0$  s.t. for  $n \geq N, \rho(x_n, x) < \epsilon \implies d(x_n, x) < \epsilon$ , which shows that  $x_n \rightarrow^d x$ . This completes the proof that  $i$  is a homeomorphism between  $(G, d), (G, \rho)$ . We also need to show that  $(G, \rho)$  is complete. Suppose therefore  $(x_n) \subset G$  is Cauchy under  $\rho$ . Then for any  $\epsilon > 0$  there exists  $N > 0$  such that for  $n, m \geq N$ :

$$\rho(x_n, x_m) < \epsilon \implies d(x_n, x_m) + |d(x_n, F)^{-1} - d(x_m, F)^{-1}| < \epsilon \implies d(x_n, x_m) < \epsilon,$$

which shows that  $(x_n)$  is Cauchy in  $M$  under  $d$ . This means that  $x_n \rightarrow^d x$ , since  $M$  is complete. If we can show that  $x \in G$ , then we can use the homeomorphism  $i$  to immediately obtain that  $x_n \rightarrow^\rho x$ , which would complete the proof that  $G$  is complete under  $\rho$ . Suppose  $x \notin G$ , which means  $x \in F$  and hence  $d(x, F) = 0$ , which implies that  $d(x_n, F) \rightarrow 0$ . Fix  $\epsilon > 0$  and obtain  $N > 0$  such that for  $n, m \geq N$  it holds that  $\rho(x_n, x_m) < \epsilon$ . In particular, fix then any such  $n \geq N$ , and select  $\epsilon' = \frac{1}{d(x_n, F)^{-1} + \epsilon}$ . There exists  $M > 0$  such that for  $m \geq M$ :

$$\begin{aligned} d(x_m, F) < \epsilon' &\implies d(x_m, F) < \frac{1}{d(x_n, F)^{-1} + \epsilon} \implies d(x_n, F)^{-1} + \epsilon < \frac{1}{d(x_m, F)} \implies \\ &\epsilon < d(x_m, F)^{-1} - d(x_n, F)^{-1} \end{aligned}$$

Since the LHS is positive, we have  $\epsilon < |d(x_m, F)^{-1} - d(x_n, F)^{-1}| \implies \rho(x_n, x_m) > \epsilon$ . But then if  $m \geq \max\{N, M\}$ , this directly contradicts the fact that  $(x_n)$  is Cauchy under  $\rho$ . Therefore  $x \notin F \implies x \in G$ , which completes the proof.

### Exercise 31

If  $\sum_{n=1}^{\infty} x_n$  is a convergent series in a normed vector space  $X$ , show that  $\|\sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|$ .

*Solution.*

If the RHS of the inequality doesn't converge, then the statement holds trivially. If it does converge, then consider the following for any  $N$ :

$$\left\| \sum_{n=1}^N x_n \right\| \leq \sum_{n=1}^N \|x_n\|$$

Because norms are continuous, we know that the LHS converges as  $N \rightarrow \infty$ , and in fact it converges to  $\|\sum_{n=1}^{\infty} x_n\|$ . We thus have that both sides converge, and since limits preserve non-strict inequalities we obtain that  $\|\sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|$ .

### Exercise 36

The function  $f(x) = x^2$  has two obvious fixed points:  $p_0 = 0, p_1 = 1$ . Show that there is a  $0 < \delta < 1$  such that  $|f(x) - p_0| < |x - p_0|$  whenever  $|x - p_0| < \delta, x \neq p_0$ . Conclude that  $f^{(n)}(x) \rightarrow p_0$  whenever  $|x - p_0| < \delta, x \neq p_0$ . This means that  $p_0$  is an *attracting* fixed point for  $f$ ; every orbit that starts out near 0 converges to 0. In contrast, find a  $\delta > 0$  such that if  $|x - p_1| < \delta, x \neq p_1$ , then  $|f(x) - p_1| > |x - p_1|$ . This means that  $p_1$  is a *repelling* fixed point for  $f$ ; orbits that start out near 1 are pushed away from 1. In fact, given any  $x \neq 1$ , we have  $f^{(n)}(x) \not\rightarrow 1$ .

*Solution.*

For the first case, we are interested in the inequality  $|f(x) - 0| < |x - 0| \iff |x^2| < |x|$ . For  $x \neq 0$ , this simplifies to  $|x| < 1$ . Therefore, select, for example,  $\delta = 1/2$ . For any  $x$  such that  $|x - 0| < 1/2$ , it holds that  $|f(x) - 0| < |x - 0|$ . Since  $f(x)$  will also be in  $(-1/2, 1/2)$ , this will hold upon re-application of  $f$  too. This shows that  $f^{(n)}(x) \rightarrow 0$ . Now consider the second case, where we are interested in  $|f(x) - 1| > |x - 1| \iff |x^2 - 1| > |x - 1|$ . For  $x \neq 1$ , this simplifies to  $|x + 1| > 1$ . Therefore, pick, for example,  $\delta = 1/2$ , in which case whenever  $|x - 1| < 1/2$  we have more specifically that  $x > 0$ , and so  $|x + 1| > 1$ . Therefore for all such  $x$  with  $x \neq 1$ , we get  $|f(x) - 1| > |x - 1|$ . Observe that this suffices to see why it can never be the case that  $f^{(n)}(x) \rightarrow 1$  for  $x \neq 1$ : if at any point a term of this sequence ends up in  $(1 - 1/2, 1 + 1/2)$  (which it must, in order to converge to 1), a re-application of  $f$  will push the next term away again, and thus we can never obtain convergence.

**Exercise 37**

Suppose that  $f : (a, b) \rightarrow (a, b)$  has a fixed point  $p$  in  $(a, b)$  and that  $f$  is differentiable at  $p$ . If  $|f'(p)| < 1$ , prove that  $p$  is an attracting fixed point for  $f$ . If  $|f'(p)| > 1$ , prove that  $f$  is a repelling fixed point for  $f$ .

*Solution.*

Suppose first that  $|f'(p)| < 1$ . We recall the definition of the derivative:  $f'(p) = \lim_{h \rightarrow 0} \frac{f(p+h)-f(p)}{h}$ , meaning that for every  $\epsilon > 0$  there exists  $\delta > 0$  such that whenever  $|h| < \delta$  it holds that:

$$\left| \frac{f(p+h) - f(p)}{h} - f'(p) \right| < \epsilon \implies \left| \left| \frac{f(p+h) - f(p)}{h} \right| - |f'(p)| \right| < \epsilon \implies \left| \frac{f(p+h) - f(p)}{h} \right| < \epsilon + |f'(p)|$$

Since  $|f'(p)| < 1$ , let  $\epsilon = 1 - |f'(p)|$ , obtain the corresponding  $\delta$  and rewrite the above as:

$$|f(p+h) - f(p)| < |h|(1 - |f'(p)| + |f'(p)|) = |h|$$

This holds for  $|h| < \delta$ , or, by substituting  $x = p + h$ , for  $|x - p| < \delta$ , which means  $|f(x) - f(p)| < |x - p|$ . Recall that  $f(p) = p$  since  $p$  is a fixed point for  $f$ , and observe that we have arrived at the same conclusion as in the previous exercise: if we choose  $x_0 \in (p - \delta, p + \delta)$ , it will always be the case that  $|f(x_0) - p| < |x_0 - p|$ , and this distance will monotonically shrink as we re-apply  $f$ , leading us to conclude that  $f^{(n)}(x_0) \rightarrow p$ , which means  $p$  is indeed an attracting fixed point for  $f$ . Now suppose instead that  $|f'(p)| > 1$ . Again, for every  $\epsilon > 0$  there exists  $\delta > 0$  such that for  $|h| < \delta$ :

$$\left| \frac{f(p+h) - f(p)}{h} - f'(p) \right| < \epsilon \implies \left| \left| \frac{f(p+h) - f(p)}{h} \right| - |f'(p)| \right| < \epsilon \implies |f'(p)| - \epsilon < \left| \frac{f(p+h) - f(p)}{h} \right|$$

Now set  $\epsilon = |f'(p)| - 1$ , and obtain the  $\delta$  for which:

$$|f'(p)| - |f'(p)| + 1 < \left| \frac{f(p+h) - f(p)}{h} \right| \implies |h| < |f(p+h) - f(p)|$$

By substituting  $x = p + h$  we obtain  $|x - p| < |f(x) - p|$  for all  $x \in (p - \delta, p + \delta)$ , which, just like the previous exercise, shows that if we start at any  $x_0$  and compute  $(f^{(n)}(x_0))$ , whenever a term is in  $(p - \delta, +\delta)$  the next one will *not* be, and thus it can never be that  $f^{(n)}(x_0) \rightarrow p$ .

**Exercise 39**

The cubic  $x^3 - x - 1$  has a unique real root  $x_0$  with  $1 < x_0 < 2$ . Find it! [Hint: Iterating the function  $f(x) = x^3 - 1$  won't work! Why?]

*Solution.*

First, because  $1^3 - 1 - 1 = -1 < 0$  and  $2^3 - 2 - 1 = 5 > 0$ , by the intermediate value theorem it is indeed true that there exists a real root  $x_0 \in (1, 2)$ . Furthermore,  $(x^3 - x - 1)' = 3x^2 - 1$ , which is positive in  $(1, 2)$ , and

so the polynomial is strictly increasing in  $(1, 2)$ , meaning that this root is unique. Let now  $f(x) = x^3 - 1$ . We observe that  $f'(x) = 3x^2$ , for which it holds that  $3 < |f'(x)| < 12$  for  $x \in [1, 2]$  (notice we use the closed interval such that our metric space is complete —subsets of  $\mathbb{R}$  are complete iff they are closed—). By exercise 37, this means that the fixed point  $x_0 \in [1, 2]$  is a repelling one, and so we cannot approximate it by iterating  $f^{(n)}(x_0)$  for any  $x_0$ . To deal with this, we consider the function  $g(x) = \lambda(f(x) - x) + x$ . If  $\lambda \neq 0$  and  $y$  is a fixed point for  $g$ , we have that  $y = g(y) = \lambda(f(y) - y) + y \implies \lambda(f(y) - y) = 0 \implies f(y) = y$ , and so we can find fixed points of  $f$  by finding fixed points of  $g$ . We begin by computing:

$$g'(x) = \lambda f'(x) - \lambda + 1 = 3\lambda x^2 - \lambda + 1$$

In order for the fixed point of  $g$  to be an attracting one, we want  $|g'(x)| < 1$  in  $(1, 2)$ . We have  $g''(x) = 6\lambda x$ , and so for  $\lambda < 0$ ,  $g'(x)$  is strictly decreasing in  $(1, 2)$ . This means that for  $x \in (1, 2)$ ,  $g'(1) > g'(x) > g'(2)$ . We have  $g'(1) = 3\lambda - \lambda + 1 = 2\lambda + 1$ ,  $g'(2) = 12\lambda - \lambda + 1 = 11\lambda + 1$ . We want these values to satisfy:

$$g'(1) < 1 \implies 2\lambda + 1 < 1 \implies \lambda < 0, g'(2) > -1 \implies 11\lambda + 1 > -1 \implies \lambda > -2/11$$

In order for  $g$  to be usable it must also be the case that  $g([1, 2]) \subset [1, 2]$ . We observed that  $g'(x)$  is strictly decreasing, and so its minimum value in  $[1, 2]$  is  $g'(2) = 12\lambda - \lambda + 1 = 11\lambda + 1$ , which is positive for  $\lambda > -1/11$ . As a consequence,  $g$  would then be strictly increasing, and so  $g([1, 2]) = [g(1), g(2)] = [\lambda(f(1) - 1) + 1, \lambda(f(2) - 2) + 2] = [\lambda(0 - 1) + 1, \lambda(7 - 2) + 2] = [1 - \lambda, 5\lambda + 2]$ . The constraints this yields are:

$$1 - \lambda \geq 1 \implies 0 \geq \lambda, 5\lambda + 2 \leq 2 \implies \lambda \leq 0$$

All in all, it suffices to pick  $\lambda < 0, \lambda > -1/11$ , so for example  $\lambda = -1/10$  does the job, and we can find the fixed point of  $g$  (and hence of  $f$ , and hence the root of  $x^3 - x - 1$ ) by iterating over  $g$ .

### Exercise 40

Extend the result in Example 7.15 as follows: Suppose that  $F : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$ , differentiable in  $(a, b)$ , and satisfies  $F(a) < 0, F(b) > 0$ , and  $0 < K_1 \leq F'(x) \leq K_2$ . Show that there is a unique solution to the equation  $F(x) = 0$ , and describe a methodology that is guaranteed to approximate it with arbitrary accuracy. [Hint: Consider the equation  $f(x) = x$ , where  $f(x) = x - \lambda F(x)$  for some suitably chosen  $\lambda$ .]

*Solution.*

First, the fact that  $F(a) < 0, F(b) > 0$  and that  $F$  is continuous on  $[a, b]$  allows us to use the intermediate value theorem to conclude that there exists at least one solution to  $F(x) = 0, x \in (a, b)$ . Secondly,  $F'(x) > 0$  means that the function is monotonically increasing in  $[a, b]$ , and hence this solution is unique. In a generalization of the previous problem (39), we would like to solve this by finding a fixed point of  $F(x) - x$ . However, the same issues that were described above arise, since we don't know whether  $|F'(x)| < 1$  in  $(a, b)$ . Consider the function  $f(x) = x - \lambda F(x)$ , and observe that for  $\lambda \neq 0$ , it holds that if  $f(x) = x$ , then  $\lambda F(x) = 0 \implies F(x) = 0$ . Therefore we can find roots of  $F$  by finding fixed points of  $f$ . We have that:

$$f'(x) = 1 - \lambda F'(x)$$

We want:

$$|f'(x)| < 1 \implies |1 - \lambda F'(x)| < 1 \implies -1 < 1 - \lambda F'(x) < 1 \implies \lambda F'(x) > 0, \lambda F'(x) < 2$$

Since  $F'(x) > 0$  for all  $x \in (a, b)$ , the first condition simplifies to  $\lambda > 0$ . In addition, from the second condition we obtain  $\lambda < \frac{2}{F'(x)} \leq \frac{2}{K_1}$ , since  $K_1 \leq F'(x)$  for  $x \in (a, b)$ . In order for any of this to be applicable, it must be the case that  $f([a, b]) \subset [a, b]$ . The analysis here would be simplified if  $f$  was strictly increasing, so let's examine what this would require:

$$f'(x) > 0 \implies 1 > \lambda F'(x) \implies \lambda < \frac{1}{F'(x)} \leq \frac{1}{K_1}$$

This is consistent with our constraints so far, since  $0 < \frac{1}{K_1} < \frac{2}{K_1}$ . We can thus assume that  $\lambda$  is such that  $f$  is strictly increasing, in which case:

$$f([a, b]) = [f(a), f(b)] = [a - \lambda F(a), b - \lambda F(b)]$$

We want  $b - \lambda F(b) \leq b \implies 0 \leq \lambda F(b)$ , which, because  $F(b) > 0$ , is always true for  $\lambda > 0$ . We also want  $a - \lambda F(a) \geq a \implies \lambda F(a) \leq 0$ , which, because  $F(a) < 0$ , is always true for  $\lambda > 0$ . Therefore, picking any  $\lambda \in (0, \frac{1}{K_1})$  results in  $f$  having a unique and attracting fixed point in  $(a, b)$ , and so iterative applications of  $f$  will always converge to it starting from any initial “guess” in  $(a, b)$ . As we observed, this fixed point  $p$  will satisfy  $F(p) = 0$ .

### Exercise 41

Let  $M$  be complete and let  $f : M \rightarrow M$  be continuous. If  $f^{(k)}$  is a strict contraction for some integer  $k > 1$ , show that  $f$  has a unique fixed point.

*Solution.*

By Theorem 7.13 we know that  $f^{(k)}$  has a unique fixed point  $p$ . We first show that if  $f$  has a fixed point, it must be unique. Indeed, suppose  $p' \in M$  is such that  $f(p') = p'$ . This means of course that  $f^{(k)}(p') = p'$ , and so  $p'$  is a fixed point of  $f^{(k)}$ , which has a unique fixed point. We've therefore more specifically shown that if  $f$  has a fixed point, this must be  $p$ . Furthermore, we know that for any  $x_0 \in M$ , it is the case that  $\lim_{n \rightarrow \infty} (f^{(k)})^{(n)}(x_0) = p$ . This can be thought of as a sequence of the form:

$$f^{(k)}(x_0), f^{(2k)}(x_0), f^{(3k)}(x_0), \dots$$

Consider the sequence  $f^{(n)}(x_0)$ . For any  $n$ , by the division theorem for integers we know that there exist unique  $q, r \in \mathbb{N}$  such that  $n = qk + r, r \in \{0, 1, \dots, k-1\}$ . Therefore, we have that:

$$\begin{aligned} d(f^{(n)}(x_0), p) &= d(f^{(qk+r)}(x_0), p) \leq d(f^{(qk)}(f^{(r)}(x_0)), f^{(qk)}(x_0)) + d(f^{(qk)}(x_0), p) \\ &\leq a^q d(f^{(r)}(x_0), x_0) + d(f^{(qk)}(x_0), p), \end{aligned}$$

where  $a$  is the contraction factor for  $f^{(k)}$ . Notice that the second term of the RHS here can be made arbitrarily small after some  $N_1$  due to  $\lim_{n \rightarrow \infty} (f^{(k)})^{(n)}(x_0) = p$ . Furthermore, notice that  $r$  can only take  $k$  different values, and so we can define:

$$M = \max_{r \in \{0, 1, \dots, k-1\}} \{d(f^{(r)}(x_0), x_0)\},$$

leading to  $d(f^{(n)}(x_0), p) \leq a^q M + d(f^{(qk)}(x_0), p)$ . Notice then that because  $a < 1$ , by selecting  $n \geq N_2$  large enough so that  $q$  also grows large enough, the first term of the RHS above can *also* be made arbitrarily small. This leads us to conclude that for any given  $\epsilon > 0$ , after some sufficiently large  $N = \max\{N_1, N_2\}$ , for all  $n \geq N$  it holds that  $d(f^{(n)}(x_0), p) < \epsilon$ . Therefore:

$$\lim_{n \rightarrow \infty} f^{(n)}(x_0) = p \implies f(p) = \lim_{n \rightarrow \infty} f^{(n+1)}(x_0) = \lim_{n \rightarrow \infty} f^{(n)}(x_0) = p,$$

where we have used the continuity of  $f$ . We thus conclude that  $p$  is indeed a fixed point of  $f$ .

### Exercise 43

Show that each of the hypotheses of the contraction mapping principle is necessary by finding examples of a space  $M$  and a map  $f : M \rightarrow M$  having no fixed point where:

- (a)  $M$  is incomplete (but  $f$  is still a strict contraction)
- (b)  $f$  satisfies only  $d(f(x), f(y)) < d(x, y)$  for all  $x \neq y$  (but  $M$  is still complete)

*Solution.*

(a) Let  $M = (0, 1]$ , which is incomplete due to the fact that we can find Cauchy sequences inside  $M$  that converge to  $0 \notin M$ . Let also  $f(x) = \frac{x^2}{4}$ , for which  $f'(x) = \frac{x}{2}$ . It's clear that  $f((0, 1]) = (0, 1/4] \subset (0, 1]$ . Furthermore, since  $|f'(x)| < 1$  in  $(0, 1)$ ,  $f$  is a strict contraction. To find fixed points of  $f$  we have that:

$$f(x) = x \implies \frac{x^2}{4} = x \implies x^2 - 4x = 0 \implies x(x-4) = 0 \implies x = 0 \text{ or } x = 4$$

Neither of these points are in  $M$ , and thus  $f$  has no fixed points when considered as a map from  $M$  to  $M$ .  
(b) Consider the metric space  $M = [0, \infty)$ , which is complete as a closed subset of  $\mathbb{R}$ . Let  $f : M \rightarrow \mathbb{R}$ ,  $f(x) = x + \frac{1}{x+1}$ , which has  $f(x) > 0$  whenever  $x \geq 0$ , and so we can think of it as  $f : M \rightarrow M$ . Notice that  $f(x) = x \implies \frac{1}{x+1} = 0$ , which has no solutions, and hence  $f$  has no fixed points. Pick any two  $x, y$  with  $x < y$ . We have that  $d(x, y) = y - x$ . Furthermore:

$$\begin{aligned} d(f(x), f(y)) &= \left| x + \frac{1}{x+1} - y - \frac{1}{y+1} \right| = \left| x - y + \frac{y+1-x-1}{(x+1)(y+1)} \right| = \left| (y-x)\left(-1 + \frac{1}{(x+1)(y+1)}\right) \right| \\ &= d(x, y) \left| 1 - \frac{1}{(x+1)(y+1)} \right| \end{aligned}$$

Notice that the factor that multiplies  $d(x, y)$  here is always strictly less than 1, since  $x, y \geq 0$ , and so indeed  $d(f(x), f(y)) < d(x, y)$ . Nevertheless, by making  $x, y$  increasingly large, we can make this factor arbitrarily close to 1, meaning that no constant  $a < 1$  would satisfy  $d(f(x), f(y)) \leq ad(x, y)$  for all  $x, y, x \neq y$ .

### 7.3 Completions

#### Exercise 46

If  $A$  is a dense subset of a metric space  $(M, d)$ , show that  $(A, d)$  and  $(M, d)$  have the same completion (isometrically). [Hint: If  $\hat{M}$  is the completion for  $M$ , then  $A$  is dense in  $\hat{M}$ . Why?]

*Solution.*

Since so far in the book we haven't worked with the strict definition (i.e., we've assumed for simplicity that a set is a subset of its completion), we will do this here for the sake of working through all the details at least once. Suppose then that  $\hat{M}$  is a/the completion for  $M$ . This means that  $M$  is isometric to a dense subset of  $\hat{M}$ , in other words that there exists a function  $F : M \rightarrow \hat{M}$  that is an *into* isometry, and is such that  $F(M)$  is dense in  $\hat{M}$ . Clearly, the function  $F|_A : A \rightarrow \hat{M}, F|_A(x) = F(x)$  is also an isometry. The question is whether  $F|_A(A)$  is dense in  $\hat{M}$ . Let therefore  $y \in \hat{M}$ . Because  $F(M)$  is dense in  $\hat{M}$ , there exists  $(y_n) \subset F(M)$  such that  $y_n \rightarrow y$ . For any  $\epsilon > 0$ , we can thus find  $N$  such that for  $n \geq N$ ,  $\hat{d}(y_n, y) < \epsilon/2$ . Now, there exist  $x_n \in M$  such that  $F(x_n) = y_n$ . Because  $A$  is dense in  $M$ , there exists  $(z_n) \subset A$  such that  $z_n \rightarrow x_N$ . Pick  $K > 0$  such that  $d(z_n, x_N) < \epsilon/2$  for all  $n \geq K$ . We observe that, for any  $n \geq K$ :

$$\hat{d}(F(z_n), y) \leq \hat{d}(F(z_n), y_N) + \hat{d}(y_N, y) = d(z_n, x_N) + \hat{d}(y_N, y) < \epsilon/2 + \epsilon/2 = \epsilon$$

If, then, for increasingly small  $\epsilon$  (e.g.,  $1/n$ ) we pick  $z_n \in A$  as described above, we know that the sequence  $F(z_n) \subset F|_A(A)$  converges to  $y$ , which means precisely that  $F|_A(A)$  is dense in  $\hat{M}$ , and so  $A, M$  indeed have the same completion isometrically.



# Chapter 8

## Compactness

### 8.1 Compact Metric Spaces

#### Exercise 1

If  $K$  is a nonempty compact subset of  $\mathbb{R}$ , show that  $\sup K, \inf K$  are elements of  $K$ .

*Solution.*

First, observe that  $\sup K$  is a well-defined real number since  $K$  is (totally) bounded. Second, recall the  $\epsilon$ -based characterization of the supremum: for every  $\epsilon > 0$ , there exists  $x \in K$  such that  $x > \sup K - \epsilon$ . Form then a sequence  $(x_n) \subset K$  such that  $x_n > \sup K - \frac{1}{n}$ , and notice that this converges to the supremum. Because  $K$  is also closed, it must be that  $\sup K \in K$  as a limit point. The proof for the infimum is exactly symmetrical.

#### Exercise 2

Let  $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ , considered as a subset of  $\mathbb{Q}$  (with its usual metric). Show that  $E$  is closed and bounded but *not* compact.

*Solution.*

First, we notice that any element  $x \in E$  must be such that  $\sqrt{2} < |x| < \sqrt{3}$ . This means that for any two  $x, y \in E$ :

$$\begin{aligned}\sqrt{2} < |x| < \sqrt{3}, \sqrt{2} < |y| < \sqrt{3} &\implies -\sqrt{2} < x < \sqrt{3}, -\sqrt{3} < -y < \sqrt{2} \implies \\ -(\sqrt{3} + \sqrt{2}) < x - y < \sqrt{3} + \sqrt{2} &\implies |x - y| < \sqrt{3} + \sqrt{2}\end{aligned}$$

This shows that  $E$  is indeed bounded (has bounded diameter). To show that it is closed, suppose  $(x_n) \subset E$  converges to some  $x \in \mathbb{Q}$ . Suppose that it was the case that  $x^2 > 3$ . By the continuity of the squaring function, we have that  $x_n^2 \rightarrow x^2$ , and this then means that the terms  $x_n^2$  get infinitely close to  $3 + \epsilon$ , and thus more specifically eventually become larger than 3, a contradiction. Similarly, it cannot be that  $x^2 < 2$ . The only possibilities that remain for  $x \notin E$  are  $x^2 = 2$  or  $x^2 = 3$ . Suppose  $x^2 = 2$ . This would mean that  $x_n^2 \rightarrow x^2$ , and by the continuity of the square root function we would have that  $x_n \rightarrow \sqrt{2}$ . However,  $\sqrt{2}$  is *not* a rational, which is a contradiction since we began with the assumption that  $x_n$  converges to some limit in the metric space. The same holds for  $x^2 = 3$ . Therefore  $x \in E$  always, and thus  $E$  is closed. However, consider the sequence constructed as follows. Start with  $x_1 = 3/2 \in E$ . Between any two real numbers we can always find a rational number, and so pick  $x_2$  to be between  $\sqrt{2}, (\sqrt{2} + x_1)/2$ . Continue this way to form a sequence that is Cauchy: the distance between subsequent terms is iteratively divided by two. Nevertheless, as observed above this sequence does *not* converge in  $\mathbb{Q}$ . In fact all its subsequences converge to  $\sqrt{2}$ , so none of them can converge in  $\mathbb{Q}$ , and thus more specifically in  $E$ , and thus by theorem 8.2,  $E$  is not compact.

**Exercise 3**

If  $A$  is compact in  $M$ , prove that  $\text{diam}(A)$  is finite. Moreover, if  $A$  is nonempty, show that there exist points  $x, y \in A$  such that  $\text{diam}(A) = d(x, y)$ .

*Solution.*

Because  $A$  is compact, it must be totally bounded, and hence also bounded. We've seen this (totally bounded  $\implies$  bounded) in Example 7.2 (a) and provided a proof in Ex. 2 of Ch. 7. Therefore,  $A$  has finite diameter. Now suppose  $A$  is nonempty. We know that  $\text{diam}(A) = \sup\{d(x, y), x, y \in A\}$  is finite. By the definition of the supremum, for every  $\epsilon > 0$ , there exist  $x, y \in A$  such that  $d(x, y) > \text{diam}(A) - \epsilon$ . For successive  $\epsilon_i = \frac{1}{i}$ , find  $x_i, y_i$  such that  $d(x_i, y_i) > \text{diam}(A) - \epsilon_i$ . Form two sequences,  $(x_i), (y_i) \subset A$ . Because  $A$  is compact, by Theorem 8.2, the sequence  $(x_i)$  contains a convergent subsequence  $x_{i_k} \rightarrow x \in A$ . Select the corresponding subsequence  $(y_{i_k})$  that satisfies the above inequality when paired with  $(x_{i_k})$ . Notice that again by Theorem 8.2, we can select a sub-subsequence  $(y_{i_{k_l}})$  such that  $y_{i_{k_l}} \rightarrow y \in A$ . Of course, a subsequence of a convergent subsequence converges, so take also the corresponding  $(x_{i_{k_l}})$ , which converges to  $x$ . By our initial inequality above we have  $d(x_{i_{k_l}}, y_{i_{k_l}}) \rightarrow \text{diam}(A)$ . By Exercise 34, Ch. 3, we know that  $d(x_{i_{k_l}}, y_{i_{k_l}}) \rightarrow d(x, y)$ , and thus by the uniqueness of the limit we conclude that  $d(x, y) = \text{diam}(A)$ .

**Exercise 4**

If  $A, B$  are compact sets in  $M$ , show that  $A \cup B$  is compact.

*Solution.*

Let  $(c_i) \subset A \cup B$  be a non-trivial sequence (i.e., not eventually constant). If  $(c_i)$  contains an infinite number of terms  $a_i \in A$ , then this sequence of  $a_i$  has a subsequence that converges to a point in  $A$ , since  $A$  is compact. A subsequence of a subsequence is also a subsequence of the sequence itself, and thus  $(c_i)$  contains a sequence converging to a point in  $A \cup B$ . If  $(c_i)$  contains only a finite number of terms from  $A$ , then it contains an infinite number of terms from  $B$ , and we use the same argument to show that it contains a subsequence converging to a point in  $B$ . Therefore,  $A \cup B$  is compact, by Theorem 8.2.

**Exercise 6**

If  $A$  is compact in  $M$  and  $B$  is compact in  $N$ , show that  $A \times B$  is compact in  $M \times N$  (see Exercise 3.46).

*Solution.*

Let  $((x_i, y_i)) \subset A \times B$  be a sequence. By Theorem 8.2,  $(x_i) \subset A$  has a convergent subsequence  $(x_{i_k}) \subset A$ , such that  $x_{i_k} \rightarrow x \in A$ , since  $A$  is compact in  $M$ . Take the subsequence  $(y_{i_k}) \subset B$ . Again, because  $B$  is compact in  $N$ , there exists  $(y_{i_{k_l}}) \subset B$  such that  $y_{i_{k_l}} \rightarrow y \in B$ . Any subsequence of  $(x_{i_k})$  converges to  $x$ , so take the corresponding  $x_{i_{k_l}} \rightarrow x$ . By Exercise 3.46, the sequence  $((x_{i_{k_l}}, y_{i_{k_l}})) \subset A \times B$  converges to  $(x, y) \in A \times B$ . Since this is a subsequence of  $((x_i, y_i))$ , we've shown that  $A \times B$  is compact in  $M \times N$ .

**Exercise 10**

Show that the Heine-Borel theorem (closed, bounded sets in  $\mathbb{R}$  are compact) implies the Bolzano-Weierstrass theorem. Conclude that the Heine-Borel theorem is equivalent to the completeness of  $\mathbb{R}$ .

*Solution.*

The Bolzano-Weierstrass theorem states that every bounded infinite subset of  $\mathbb{R}$  has a limit point in  $\mathbb{R}$ . Suppose  $A$  is such a subset, and consider its closure,  $\bar{A}$ , which must also be bounded (if not, due to the definition of the closure, there would be points of  $A$  infinitely close to points of  $\partial A$  that form an unbounded sequence, a contradiction). Because  $\bar{A}$  is also closed, then by our assumption of the Heine-Borel theorem, it is compact. Then Theorem 8.2 guarantees that every sequence in  $A \subset \bar{A}$  has a subsequence converging to a point in  $\bar{A}$ , which means we've shown the Bolzano-Weierstrass theorem. Observe, also, that what we've shown implies the completeness of the reals: any Cauchy sequence in  $\mathbb{R}$  has a bounded infinite range, and hence has a convergent subsequence, which for a Cauchy sequence means that it itself converges.

For a stronger version of this exercise, we work with the hypothesis that any closed interval  $[a, b]$  is compact. For a bounded, infinite subset of  $\mathbb{R}$  we know that it has a lower bound  $m$ , and an upper bound  $M$ , which

means  $A \subset [m, M]$ . Importantly, we do not assume that least upper bounds or greatest lower bounds exist, since this is the axiom of completeness. This interval is compact by our hypothesis. Thus, for any sequence in  $A \subset [m, M]$ , we can use theorem 8.2 to obtain a sequence with a subsequence converging to a point in  $[m, M] \subset \mathbb{R}$ , thus showing the Bolzano-Weierstrass theorem again.

### Exercise 11

Prove that compactness is not a relative property. That is, if  $K$  is compact in  $M$ , show that  $K$  is compact in *any* metric space that contains it (isometrically).

*Solution.*

Let  $(N, \rho)$  be any metric space such that for a subset  $L \subset N$  it holds that  $i : K \rightarrow L$  is an isometry onto  $L$ . We need to show that  $L$  is compact in  $N$ . Pick any sequence  $(y_i) \subset L$ . Because  $K, L$  are isometric, there exists a unique sequence  $(x_i) \subset K$  such that  $i(x_i) = y_i$ . Now, since  $K$  is compact in  $M$ , by Theorem 8.2,  $(x_i)$  contains a convergent subsequence  $(x_{i_k})$  such that  $x_{i_k} \rightarrow^d x \in K$ . Notice now that  $i$  is a homeomorphism, and thus  $i(x_{i_k}) \rightarrow^\rho i(x)$ , which means  $y_{i_k} \rightarrow^\rho i(x)$ . Because  $i$  is onto,  $i(x) \in L$ . We've therefore shown that  $(y_i)$  contains a convergent subsequence  $y_{i_k} \rightarrow^\rho i(x) \in L$ , which by Theorem 8.2 means that  $L$  is compact in  $(N, \rho)$ .

### Exercise 12

Show that the set  $A = \{x \in l_2 : |x_n| \leq 1/n, n = 1, 2, \dots\}$  is compact in  $l_2$ . [Hint: First show that  $A$  is closed. Next, use the fact that  $\sum_{n=1}^{\infty} 1/n^2 < \infty$  to show that  $A$  is “within  $\epsilon$ ” of the set  $A \cap \{x \in l_2 : |x_n| = 0, n \geq N\}$ .]

*Solution.*

We know (Example 7.7) that  $l_2$  is complete. Therefore, if we show that  $A$  is closed, we have that it will also be complete (Theorem 7.9). Suppose  $(x_k)_n \subset A$  is a sequence of sequences that converges to  $x \in l_2$ . Then, for any  $\epsilon > 0$ , there exists  $N > 0$  such that for all  $n \geq N$  we have that:

$$\left( \sum_{k=1}^{\infty} |x_{nk} - x_k|^2 \right)^{1/2} < \epsilon \implies \sum_{k=1}^{\infty} |x_{nk} - x_k|^2 < \epsilon^2$$

Therefore, for any  $k$  we have that:

$$|x_{nk} - x_k|^2 < \epsilon^2 \implies |x_{nk} - x_k| < \epsilon \implies x_{nk} - \epsilon < x_k < x_{nk} + \epsilon$$

By the definition of  $A$ , we have that  $|x_{nk}| \leq 1/k$ . Therefore,  $-1/k - \epsilon < x_k < 1/k + \epsilon \implies |x_k| < 1/k + \epsilon$ . Because this holds for *any*  $\epsilon > 0$ , we conclude that it must be the case that  $|x_k| \leq 1/k$  for all  $k$ . But this is precisely the defining property of  $A$ , and so  $x \in A$ , showing that  $A$  is indeed closed. Now we need to show that  $A$  is totally bounded. We begin by selecting an  $N > 0$  and examining sequences of the form  $(x_1, x_2, \dots, x_{N-1}, 0, 0, \dots)$ . In particular, consider the following construction. We subdivide each of the intervals  $[-1, 1], [-1/2, 1/2], \dots, [-1/(N-1), 1/(N-1)]$  into  $N$  intervals of equal length. For the  $i$ -th of these intervals, this yields  $N+1$  endpoints,  $x_{i,1}, x_{i,2}, \dots, x_{i,N+1}$ . We then form the set:

$$B =$$

$$\{(x_{i,1}, x_{i,2}, \dots, x_{i,N-1}, 0, \dots), i_1 \in \{1, 2, \dots, N+1\}, i_2 \in \{1, 2, \dots, N+1\}, \dots, i_{N-1} \in \{1, 2, \dots, N+1\}\}$$

Notice that  $B$  contains a finite number of points  $((N+1)^{N-1})$ . Furthermore, suppose  $(y_k) \in A$  is a sequence. Take the element  $z$  of  $B$  formed by selecting, for each of the  $N-1$  first coordinates, the nearest element  $z_i = x_{i,j} \in B$  to  $y_i$ . Then observe that  $|y_i - z_i| < 2/(Ni)$ , because we subdivided the interval  $[-1/i, 1/i]$  into  $N$  equally long subintervals. Then:

$$\sum_{k=1}^{\infty} |y_k - z_k|^2 = \sum_{k=1}^{N-1} |y_k - z_k|^2 + \sum_{k=N}^{\infty} |y_k - 0|^2 \leq \sum_{k=1}^{N-1} 4/(Nk)^2 + \sum_{k=N}^{\infty} |y_k|^2 = \frac{4}{N^2} \sum_{k=1}^{N-1} \frac{1}{k^2} + \sum_{k=N}^{\infty} |y_k|^2$$

$$\leq \frac{4(N-1)}{N^2} + \sum_{k=N}^{\infty} |y_k|^2$$

Now notice that because  $(y_k) \in l_2$ , the second term of the sum can be made arbitrarily small by selecting  $N$  sufficiently large (the sequence of partial sums must converge). Secondly, the first part of the sum can also be easily seen to tend to zero as  $N$  grows large. In particular, one can then select  $N$  large enough so that each of these is upper bound by  $\epsilon/2$ , which by the above means  $d(y, z) < \epsilon$ , i.e., that  $B$  constitutes a finite  $\epsilon$ -net for  $A$ , which shows that  $A$  is totally bounded. Since it was also shown to be complete in  $l_2$ , it is thus compact in  $l_2$ .

### Exercise 13

Given  $c_n \geq 0$  for all  $n$ , prove that the set  $\{x \in l_2 : |x_n| \leq c_n, n \geq 1\}$  is compact in  $l_2$  if and only if  $\sum_{n=1}^{\infty} c_n^2 < \infty$ .

*Solution.*

$\Rightarrow$  : Suppose first that  $S = \{x \in l_2 : |x_n| \leq c_n, n \geq 1\}$  is compact in  $l_2$ . By Corollary 8.3, it must then be closed in  $l_2$ . Consider now forming the sequence of sequences  $(c_n)_k = (c_1, c_2, \dots, c_k, 0, 0, \dots)$ , where all coordinates after the  $k$ -th one are zero. Importantly,  $(c_n)_k \subset S$ , and because  $S$  is compact, by Theorem 8.2,  $(c_n)_k$  contains a subsequence that converges to an element  $L = (L_1, L_2, \dots)$  of  $S$ . Our claim is now that  $L = (c_1, c_2, \dots)$ . Notice that if we can prove this, then we immediately obtain that  $\sum_{n=1}^{\infty} c_n^2 < \infty$ : because  $S$  is closed,  $L \in S \subset l_2$ , and so  $(c_1, c_2, \dots) \in l_2$ . Suppose then that for some  $i, L_i \neq c_i \Rightarrow |L_i - c_i| = \epsilon > 0$ . Any subsequence  $(c_n)_{k_j}$  of  $(c_n)_k$  must eventually contain terms of which the  $i$ -th coordinate equals  $c_i$ . Consequently, for sufficiently large  $k_j$ :

$$\sum_{n=1}^{\infty} |c_{k_j, n} - L_i|^2 \geq |c_i - L_i|^2 > \epsilon^2$$

This directly contradicts the fact that a subsequence of  $(c_n)_k$  converges to  $L$ . By the argument presented above, this also shows that  $(c_1, c_2, \dots) \in l_2$ .

Because  $S$  is closed, it must be the case that  $(c_1, c_2, \dots) \in S \subset l_2$ , which means of course that  $\sum_{n=1}^{\infty} c_n^2 < \infty$ .

$\Leftarrow$  : We need to show that  $S$  is both complete and totally bounded. We will do this in a similar manner as in Exercise 8.12. First, recall that  $l_2$  is complete, and so for completeness it suffices to show that  $S$  is closed. Suppose then  $(x_n)_k \rightarrow (l_n)$  is a sequence of sequences in  $S$  converging to  $(l_n)$ . To show that  $(l_n) \in S$ , we need to show that  $|l_n| \leq c_n$  for all  $n$ . Suppose that this is not the case, and that for some  $N, |l_N| > c_N$ . Then let  $\epsilon = |l_N| - c_N$ . For any term of the sequence of sequences  $(x_n)_k$  (i.e., for a fixed  $k$ ), we obtain:

$$\begin{aligned} |x_{kN}| \leq c_N &\Rightarrow |x_{kN} - l_N + l_N| \leq c_N \Rightarrow |x_{kN} - l_N| - |l_N| \leq c_N \\ &\Rightarrow -c_N \leq |x_{kN} - l_N| - |l_N| \Rightarrow |x_{kN} - l_N| \geq \epsilon \end{aligned}$$

Notice that imposes a lower bound on  $\|(x_n)_k - (l_n)\|_2$ , which means that  $(x_n)_k$  does *not* converge to  $(l_n)$ , a contradiction. Therefore,  $|l_n| < c_n$  for all  $n$ , and so  $(l_n) \in S$ , which means  $S$  is closed in  $l_2$  and thus complete. We now examine total boundedness. Once again, as in Ex. 8.12, we examine sequences of the form  $(x_1, x_2, \dots, x_{N-1}, 0, 0, \dots)$  for a fixed  $N > 0$ . In particular, we take the intervals  $[-c_1, c_1], [-c_2, c_2], \dots, [-c_{N-1}, c_{N-1}]$ , and subdivide each into  $N$  subintervals of equal length. Then the  $i$ -th interval is subdivided into  $N$  subintervals, each of which has length  $\frac{2c_i}{N}$  (note that if  $c_i = 0$ , everything collapses to a single point). We gather the endpoints of all intervals ( $N+1$  endpoints for the  $i$ -th interval,  $x_{i,1}, \dots, x_{i,N+1}$ ) and form the set:

$$B = \{(x_{i_1,1}, x_{i_2,2}, \dots, x_{i_{N-1},N-1}, 0, 0, \dots) | i_j \in \{1, 2, \dots, N+1\}, j = 1, 2, \dots, N-1\},$$

which contains a finite number of points. Suppose now  $(y_n) \in S$ , and form the sequence  $(z_n) \in B$  that has  $z_i$  equal to the  $x_{i,j}$  that is closest to  $y_i$  for each  $i = 1, \dots, N-1$ , and  $z_i = 0$  for all other  $i$ . Notice that  $|y_i - z_i| < 2c_i/N$  by our construction above. Then:

$$\sum_{k=1}^{\infty} |y_k - z_k|^2 = \sum_{k=1}^{N-1} |y_k - z_k|^2 + \sum_{k=N}^{\infty} |y_k - 0|^2 \leq \sum_{k=1}^{N-1} 4c_i^2/N^2 + \sum_{k=N}^{\infty} |y_k|^2 \leq \frac{4}{N^2} \sum_{k=1}^{N-1} c_i^2 + \sum_{k=N}^{\infty} |y_k|^2$$

Because  $y_k \in l_2$ , the second term can be made arbitrarily small by selecting a sufficiently large  $N$ . Notice, also, that  $\sum_{n=1}^{\infty} c_n^2 < \infty$ , therefore the term  $\sum_{k=1}^{N-1} c_i^2$  is upper bound, and so again the first term of the inequality can be made arbitrarily small by selecting sufficiently large  $N$  (so that  $\frac{4}{N^2}$  becomes sufficiently small). We conclude that  $\sum_{k=1}^{\infty} |y_k - z_k|^2$  can be upper bound by any arbitrarily small  $\epsilon$ , and so a set  $B$  containing a sufficiently large but finite number of elements  $z_k$  constitutes an  $\epsilon$ -net for  $S$ , showing that  $S$  is totally bounded, and hence compact.

#### Exercise 14

Show that the Hilbert cube  $H^\infty$  is compact. [Hint: First show that  $H^\infty$  is complete (Ex. 7.24). Now, given  $\epsilon > 0$ , choose  $N$  so that  $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$  and argue that  $H^\infty$  is “within  $\epsilon$ ” of the set  $\{x \in H^\infty : |x_n| = 0\}$  for  $n \geq N$ .]

*Solution.*

In Ex. 7.24, we showed that the Hilbert cube is complete. We now need to examine total boundedness as indicated in the hint. Pick any  $\epsilon > 0$ , and observe that we can find  $N > 0$  such that  $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$ : this is possible due to the series  $1, 1 + 1/2, 1 + 1/2 + 1/4, \dots$  converging. Consider also partitioning the interval  $[-1, 1]$  into  $N$  subintervals of equal length, as in Exercises 8.12, 8.13 above, collecting all endpoints  $x_1, x_2, \dots, x_{N+1}$  and forming the set:

$$B = \{(x_{i,1}, x_{i,2}, \dots, x_{i,N-1}, 0, 0, \dots) : i_j \in \{1, 2, \dots, N+1\}, j = 1, \dots, N-1\} \subset H^\infty$$

Let then  $(y_n) \in H^\infty$ , and let  $(z_n) \in B$  be the element of  $B$  such that  $|y_i - z_i|$  is minimized for each  $i = 1, \dots, N-1$  (i.e., for each  $i$  we select the subinterval endpoint that is closest to  $y_i$ ). Then  $|y_i - z_i| < \frac{2}{N}$ . Using the metric of  $H^\infty$ , we have that:

$$\begin{aligned} d((y_n), (z_n)) &= \sum_{n=1}^{\infty} 2^{-n} |y_n - z_n| = \sum_{n=1}^{N-1} 2^{-n} |y_n - z_n| + \sum_{n=N}^{\infty} 2^{-n} |y_n| \leq \sum_{n=1}^{N-1} 2^{-n} \frac{2}{N} + \sum_{n=N}^{\infty} 2^{-n} \\ &\leq \frac{4}{N} + \sum_{n=N}^{\infty} 2^{-n} \end{aligned}$$

where we used the fact that  $|y_n| < 1$ , and we upper-bounded the first term by the geometric series sum  $1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$ . Notice now that, as observed above, we can pick a sufficiently large  $N$  such that the second term is made to be smaller than any  $\epsilon > 0$ . Furthermore, the first term again becomes smaller than any  $\epsilon > 0$  for sufficiently large  $N$ . This shows that  $d((y_n), (z_n))$  can in turn be made arbitrarily small, and thus that  $B$  —which contains a finite number of elements— constitutes an  $\epsilon$ -net for  $H^\infty$ , thus showing that  $H^\infty$  is totally bounded, and thus compact.

#### Exercise 15

If  $A$  is a totally bounded subset of a complete metric space  $M$ , show that  $\overline{A}$  is compact in  $M$ . For this reason, totally bounded sets are sometimes called *precompact* or *conditionally compact*. In fact, any set with compact closure might be labeled precompact.

*Solution.*

By definition, we know that  $\overline{A}$  is closed in the complete metric space  $M$ . Therefore, it is complete (Theorem 7.9). Furthermore, by Exercise 7.5, we know that  $A$  is totally bounded iff  $\overline{A}$  is totally bounded, and thus  $\overline{A}$  is totally bounded, from which we can conclude that it is compact.

#### Exercise 16

Show that a metric space  $M$  is totally bounded if and only if its completion  $\hat{M}$  is compact.

*Solution.*

$\implies$  : By definition,  $\hat{M}$  is complete, as a completion. To show that it is totally bounded, we have the following. For any  $\epsilon > 0$ , let  $x_1, \dots, x_n \in M$  be such that  $M \subset \bigcup_{i=1}^n B_\epsilon^d(x_i)$ , which is always feasible due to  $M$  being totally bounded. Now, because  $i : M \rightarrow \hat{M}$  is an isometry into  $\hat{M}$ , it is easy to see from the above that  $i(M)$  is totally bounded. Take then any  $\epsilon > 0$  and find  $y_1, \dots, y_n \in i(M)$  such that  $i(M) \subset \bigcup_{i=1}^n B_{\epsilon/2}^{\hat{d}}(y_i)$ . Because  $i(M)$  is dense in  $\hat{M}$ , for any  $y \in \hat{M}$  there exists  $z \in i(M)$  such that  $\hat{d}(y, z) < \epsilon/2$ . But then since  $z \in \bigcup_{i=1}^n B_{\epsilon/2}^{\hat{d}}(y_i)$ , it must be the case that  $d(z, y_i) < \epsilon/2$  for at least one  $y_i$ . Consequently:

$$d(y, y_i) \leq d(y, z) + d(z, y_i) < \epsilon/2 + \epsilon/2 = \epsilon$$

We conclude then that in fact  $\hat{M} \subset \bigcup_{i=1}^n B_\epsilon^{\hat{d}}(y_i)$ , which means that  $\hat{M}$  is totally bounded and thus also compact.

$\Leftarrow$  : Conversely, assume  $\hat{M}$  is compact. Then it is totally bounded, and thus any subset of it, more specifically  $i(M)$ , is totally bounded as well. By using the same argument as in the forward direction, we can show that  $M$  is then also totally bounded.

**Exercise 17**

If  $M$  is compact, show that  $M$  is also separable.

*Solution.*

By Exercise 7.10, we know that a totally bounded metric space is separable. Compact metric spaces are by definition totally bounded, and hence also separable.

**Exercise 19**

Prove that  $M$  is separable if and only if  $M$  is homeomorphic to a totally bounded metric space (specifically, a subset of the Hilbert cube). [Hint: See Exercise 4.49.]

*Solution.*

$\implies$  : Suppose  $(M, \rho)$  is a separable metric space, and suppose, WLOG, that  $\rho(x, y) \leq 1$ . This is possible due to Exercise 3.42, where we proved that any metric is equivalent to a bounded metric. In Exercise 5.51 we showed that if  $\{x_n : n \geq 1\}$  is a countable dense subset of  $M$ , the function  $f(x) = (\rho(x, x_n))_{n=1}^\infty$  acts as a homeomorphism from  $M$  into  $H^\infty$ , meaning that the image  $f(M)$  is totally bounded as a subset of a totally bounded set ( $H^\infty$ ).

$\Leftarrow$  : Now suppose that  $(M, \rho)$  is homeomorphic to a subset  $S$  of the Hilbert cube.  $S$  is totally bounded as a subset of a totally bounded metric space, and hence by Exercise 7.10 is also separable. Let then  $T = \{(x_n)_1, (x_n)_2, \dots\}$  be a countable dense subset of  $S$ , and suppose that  $f$  is a homeomorphism between  $M$  and  $S$ . Find the unique  $a_1, a_2, \dots \in M$  such that  $f(a_i) = (x_n)_i$ . The set  $T_M = \{a_1, a_2, \dots\} \subset M$  is of course countable. Furthermore, let  $b \in M$ , and obtain  $(y_n) = f(b) \in S$ . By the denseness of  $T$  in  $S$ , there exists a sequence  $(x_n)_k$  such that  $(x_n)_k \rightarrow^d (y_n)$ , where  $d$  is the metric of  $H^\infty$ . By the properties of homeomorphisms, this also means that the corresponding sequence  $a_k \rightarrow^\rho b$ . This shows that  $T_M$  is dense in  $M$ , and thus that  $M$  is separable.

**Exercise 20**

Let  $E$  be a noncompact subset of  $\mathbb{R}$ . Find a continuous function  $f : E \rightarrow \mathbb{R}$  that is (i) not bounded; (ii) bounded but has no maximum value.

*Solution.*

By the Heine-Borel theorem, any closed and bounded subset of  $\mathbb{R}$  is compact. Therefore, any noncompact set  $E \subset \mathbb{R}$  must be either not closed or unbounded. We have that:

- If  $E$  is unbounded, then let  $(x_n) \subset E$  be a sequence formed by selecting  $x_i$  such that  $|x_i| > i$ . In fact, WLOG, suppose  $x_i > i$ , since the case for a subset not bounded below is symmetric. Then the function  $f : E \rightarrow \mathbb{R}$ ,  $f(x) = x$  is continuous but clearly unbounded, as evidenced by examining  $(f(x_n))$ . Again on  $E$ , the function  $f(x) = 1 - \frac{1}{1+x^2}$  is continuous, bounded by 0 and 1, but achieves no maximum value, since  $(f(x_n))$  approaches 1 arbitrarily.

- If  $E$  is bounded but not closed (the only case not covered above), then it must contain at least one convergent sequence  $(x_n)$  that converges to a point  $y \notin E$ . Then let  $f : E \rightarrow \mathbb{R}$  be such that  $f(x) = \frac{1}{x-y}$ , which is well defined since  $y \notin E$ . This is continuous, but unbounded on  $(x_n)$ . For our other counterexample, suppose, WLOG, that  $y \geq 0$ , since the other case will be symmetric. Then the function  $f : E \rightarrow \mathbb{R}$ ,  $f(x) = x, x < y, f(x) = -x + 2y, x > y$  is bounded (because  $E$  is bounded), but achieves no maximum value: indeed, as  $x_n \rightarrow y$ , the function's value approaches arbitrarily close to  $y$ , and for any  $x \in E$  it is the case that  $f(x) < y$ .

### Exercise 21

Prove Corollary 8.6: If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then the range of  $f$  is a compact interval  $[c, d]$  for some  $c, d \in \mathbb{R}$ .

*Solution.*

The statement is obvious in the degenerate case  $a = b$ , which implies that  $f([a, b]) = [f(a), f(a)]$ . It is also obvious for the case where  $[a, b]$  is a non-degenerate interval but  $f$  is constant. If neither of these hold, we first apply Corollary 6.7 to obtain that  $f([a, b])$  is an interval. We also apply Corollary 8.5 to obtain that  $f$  is bounded on  $[a, b]$  and attains its maximum and minimum values,  $M, m$  respectively. The only way that these can both hold is if  $f([a, b]) = [m, M]$ : if it was an open or half-open interval, then  $f$  would not achieve at least one of  $m, M$ .

### Exercise 22

If  $M$  is compact and  $f : M \rightarrow N$  is continuous, prove that  $f$  is a closed map.

*Solution.*

We need to show that  $f$  maps closed sets to closed sets. Suppose then that  $A \subset M$  is closed. Because  $M$  is compact, by Corollary 8.3 we have that  $A$  is compact. Because  $f$  is continuous, by Theorem 8.4 we have that  $f(A)$  is compact in  $N$ , and Corollary 8.3 then implies that  $f(A)$  is closed in  $N$ .

### Exercise 23

Suppose that  $M$  is compact and that  $f : M \rightarrow N$  is continuous, one-to-one, and onto. Prove that  $f$  is a homeomorphism.

*Solution.*

Suppose  $d, \rho$  are the metrics on  $M, N$  respectively. Pick any  $(x_n) \subset M$  such that  $x_n \rightarrow^d x \in M$ . By the continuity of  $f$ , we have that  $f(x_n) \rightarrow^\rho f(x)$ . Now suppose that  $y_n \rightarrow^\rho y, (y_n) \subset N, y \in N$ . Since  $f$  is one-to-one and onto, there exists a unique sequence  $(x_n) \subset M$  and a unique  $x \in M$  such that  $y_n = f(x_n), y = f(x)$ . To show that  $f$  is a homeomorphism, it suffices to show that  $x_n \rightarrow^d x$ . Suppose then that this is not the case, which means that there exists  $\epsilon > 0$  such that for every  $N > 0$  there exists  $n \geq N$  such that  $d(x_n, x) \geq \epsilon$ . Then, form a subsequence  $x_{n_k}$  by selecting as the  $k$ -term the first of the above  $x_n$  for which  $n \geq k$ , and which has not yet been selected. Notice that this is always possible, since the condition “for every  $N > 0$ ” implies that there exist an infinite number of  $x_n$  that satisfy the inequality. In this way we have formed a subsequence  $x_{n_k}$  of the original  $(x_n)$  for which  $d(x_{n_k}, x) \geq \epsilon$  for all  $k$ . Because  $M$  is compact, we can apply Theorem 8.2 to obtain a subsequence  $(x_{n_{k_l}})$  (that is, a sub-subsequence of the original  $(x_n)$ ) that converges to some  $x'$ . Because  $f$  is continuous, we then obtain that  $(f(x_{n_{k_l}}))$  converges to  $f(x')$ . But  $(f(x_{n_{k_l}}))$  is a subsequence of  $(y_n)$ , and thus it must converge to the same limit,  $y$ , which means  $f(x) = y = f(x')$ . Consequently,  $x' = x$  since  $f$  is one-to-one, and so  $x_{n_{k_l}} \rightarrow x$ . This contradicts the definition of  $(x_{n_{k_l}})$ , and so it must indeed be that  $x_n \rightarrow^d x$ , which means  $f$  is a homeomorphism.

**Exercise 24**

Let  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$  be continuous and one-to-one. Show that  $f$  cannot be onto. Moreover, show that the range of  $f$  is *nowhere dense* in  $[0, 1] \times [0, 1]$ . [Hint: The range of  $f$  is closed (why?); if it has nonempty interior, then it contains a closed rectangle. Argue that this rectangle is the image of some subinterval of  $[0, 1]$ .]

*Solution.*

We note first that if we can prove that the range of  $f$  is nowhere dense in  $[0, 1] \times [0, 1]$  (that is, that it cannot have a non-empty interior), we will have also shown that  $f$  cannot be onto. Indeed, if it were, its range would equal the entire unit square, whose interior equals itself. Suppose then, as stated in the hint, that  $f$ 's range has a non-empty interior  $I$ . Note that  $f$  is continuous and defined on the compact metric space  $[0, 1]$ , so by Exercise 8.22 it is a closed map. Furthermore,  $f$  is a homeomorphism *into* the unit square (Exercise 8.23). Therefore its range is also closed, since  $[0, 1]$  is closed. Because  $I$  is an open set, it contains a point  $x$  such that there exists  $\epsilon > 0$  for which  $B_\epsilon(x) \subset I$ .

Then, by taking a sufficiently small  $\epsilon'$ , we can always find a closed square of side  $\epsilon'$  that is a proper subset of  $B_\epsilon(x)$ . Notice that this holds even if  $x$  lies on the “boundary” of the unit square, because a disk of radius  $\epsilon$  around it always has a non-empty intersection with the “open” square  $(0, 1) \times (0, 1)$  (this means that the square does not degenerate to a segment). Call this closed square  $S$ , and since it is closed and  $f$  is a homeomorphism,  $f^{-1}(S)$  must be closed as well (Theorem 5.5). Not only that, but  $S$  is also connected, and so  $f^{-1}(S)$  is connected (Theorem 6.6 applied on  $f^{-1}$ ). The only possibility is thus that  $S$  is the image of a closed subinterval of  $[0, 1]$ .

Call this subinterval  $[a, b]$ , and notice that  $[0, 1] \setminus [a, b]$  must be of one of the forms  $[0, a)$ ,  $[0, a) \cup (b, 1]$ ,  $(b, 1]$ . In any one of these cases, we examine the behavior of  $f$  on the closure of these intervals,  $[0, a]$ ,  $[0, a] \cup [b, 1]$  and  $[b, 1]$  respectively, and we call this closure  $C$  (regardless of the specific case). These are all closed subsets of a compact space, and so are compact themselves (Corollary 8.3), while furthermore their images under  $f$  are also compact (Theorem 8.4). We argue that this is a contradiction. Indeed, it must hold that  $B_\epsilon(x) \setminus S \subset f(C)$ , and because  $f$  is one-to-one, *at most two* points of  $S$ ,  $f(a)$  and/or  $f(b)$  belong in  $f(C)$ . However, there are infinitely many Cauchy sequences that can be defined inside  $B_\epsilon(x)$  that converge to points on the boundary of  $S$  which do not belong in  $f(C)$ , which means that  $f(C)$  is not complete, which is a contradiction.

Therefore, the range of  $f$  must be nowhere dense in  $[0, 1] \times [0, 1]$ , and as discussed above, this means that  $f$  can also never be onto.

**Exercise 25**

Let  $V$  be a normed vector space and let  $x \neq y \in V$ . Show that the map  $f(t) = x + t(y - x)$  is a homeomorphism from  $[0, 1]$  into  $V$ . The range of  $f$  is the line segment joining  $x$  and  $y$ ; it is often written  $[x, y]$ .

*Solution.*

In Exercise 6.27, we showed that  $f$  is continuous and one-to-one. It is furthermore *onto*, as long as we think of it as a function from  $[0, 1]$  to  $f([0, 1]) \subset V$ . Furthermore,  $[0, 1]$  is compact (Heine-Borel), and so we can apply Exercise 8.23 to conclude that  $f$  is a homeomorphism *into*  $V$ .

**Exercise 29**

Let  $M$  be a compact metric space and suppose that  $f : M \rightarrow M$  satisfies  $d(f(x), f(y)) < d(x, y)$  whenever  $x \neq y$ . Show that  $f$  has a fixed point. [Hint: First note that  $f$  is continuous; next, consider  $g(x) = d(x, f(x))$ .]

*Solution.*

As indicated in the hint, we first show that  $f$  is continuous by observing that it is in fact Lipschitz continuous with constant  $K = 1$ . Now consider the function  $g(x) = d(x, f(x))$ . This is easily seen to be continuous as well: if  $x_n \rightarrow x$ , then  $f(x_n) \rightarrow f(x)$ , and hence  $d(x_n, f(x_n)) \rightarrow d(x, f(x))$  (a simple application of the triangle inequality shows this). By Corollary 8.5,  $g$  is bounded and attains its minimum

value  $m$ , which means that for some  $x_0 \in M$ ,  $d(x_0, f(x_0)) = m$ . We claim that  $f(x_0)$  is a fixed point for  $f$ . Indeed, if it is not it must be that  $f(x_0) \neq f(f(x_0))$  and then it also holds that:

$$d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = m,$$

which is impossible by the definition of  $m$ .

### Exercise 30

Prove Lemma 8.8: In a metric space  $M$ , the following are equivalent:

- (a) If  $\mathcal{G}$  is any collection of open sets in  $M$  with  $\cup\{G : G \in \mathcal{G}\} \supset M$ , then there are finitely many sets  $G_1, \dots, G_n \in \mathcal{G}$  with  $\cup_{i=1}^n G_i \supset M$ .
- (b) If  $\mathcal{F}$  is any collection of closed sets in  $M$  such that  $\cap_{i=1}^n F_i \neq \emptyset$  for all choices of finitely many sets  $F_1, \dots, F_n \in \mathcal{F}$ , then  $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$ .

*Solution.*

(a)  $\implies$  (b): Let  $\mathcal{F}$  be any collection of closed sets with the property that any choice of finitely many  $F_1, \dots, F_n \in \mathcal{F}$  have a nonempty intersection. Form the collection  $\mathcal{G} = \{\overline{F} : F \in \mathcal{F}\}$ , i.e., gather all complements of the sets in  $\mathcal{F}$ . Each of these is by definition open. The statement we would like to prove can be rewritten via DeMorgan's laws as:

$$\cap\{F : F \in \mathcal{F}\} \neq \emptyset \iff \overline{\cup\{\overline{F} : F \in \mathcal{F}\}} \neq \emptyset \iff \cup\{G : G \in \mathcal{G}\} \neq M$$

Suppose that this does not hold, i.e., that  $\cup\{G : G \in \mathcal{G}\} = M$ . Then by our hypothesis (a), there exist finitely many  $G_1, \dots, G_n$  such that:

$$\overline{\bigcup_{i=1}^n G_i} = M \implies \overline{\bigcup_{i=1}^n \overline{F_i}} = M \implies \overline{\bigcap_{i=1}^n F_i} = M \implies \overline{\bigcap_{i=1}^n F_i} = \emptyset$$

But this directly contradicts the defining property of  $\mathcal{F}$ , since any choice of finitely many  $F_i \in \mathcal{F}$  must have a nonempty intersection. We've therefore shown that  $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$ , i.e., that the implication stated by (b) holds.

(b)  $\implies$  (a): Suppose  $\mathcal{G}$  is a collection of open sets with  $\cup\{G : G \in \mathcal{G}\} \supset M$ , and suppose that no choice of finitely many  $G_1, \dots, G_n \in \mathcal{G}$  is such that  $\cup_{i=1}^n G_i \supset M$ . Form the collection of the complements  $\mathcal{F} = \{\overline{G} : G \in \mathcal{G}\}$ , and notice that each of these sets is closed. For any finitely many  $G_1, \dots, G_n \in \mathcal{G}$ , it must hold that:

$$\cup_{i=1}^n \{G : G \in \mathcal{G}\} \neq M \iff \overline{\cap_{i=1}^n \{\overline{G} : G \in \mathcal{G}\}} \neq M \iff \cap_{i=1}^n \{F : F \in \mathcal{F}\} \neq \emptyset$$

We have shown that the hypothesis of (b) holds for  $\mathcal{F}$ , therefore the conclusion holds as well. This means that:

$$\cap\{F : F \in \mathcal{F}\} \neq \emptyset \implies \overline{\cup\{\overline{F} : F \in \mathcal{F}\}} \neq \emptyset \implies \cup\{G : G \in \mathcal{G}\} \neq M$$

But this directly contradicts the defining property of  $\mathcal{G}$ , and so it must be the case that there exists a choice of finitely many  $G_1, \dots, G_n \in \mathcal{G}$  such that  $\cup_{i=1}^n G_i \supset M$ , and the implication stated by (a) holds.

### Exercise 31

Given an arbitrary metric space  $M$ , show that a decreasing sequence of nonempty compact sets in  $M$  has nonempty intersection.

*Solution.*

Consider such a sequence of sets  $S_1 \supset S_2 \supset \dots$ . Think of  $S_1$  as a compact metric space. Then, inside  $S_1$ , any number of finitely many sets from  $S_2, S_3, \dots$  have nonempty intersections: indeed, the intersection equals the smallest of these finitely many sets (i.e., the one with the largest subscript). Because  $S_1$  is compact, we can apply Theorem 8.9 to conclude that statement (b) from Lemma 8.8 holds, and so that  $\cap_{i \geq 2} S_i \neq \emptyset$ . Obviously, since  $S_2 \subset S_1$  the elements of this intersection belong in  $S_1$  as well, and so we conclude that the entire decreasing sequence of sets has nonempty intersection.

**Exercise 32**

Prove Corollary 8.11 ( $M$  is compact iff every countable open cover admits a finite subcover), by showing that the following two statements are equivalent:

- (i) Every decreasing sequence of nonempty closed sets in  $M$  has nonempty intersection.
- (ii) Every countable open cover of  $M$  admits a finite subcover; that is, if  $(G_n)$  is a sequence of open sets in  $M$  satisfying  $\cup_{n=1}^{\infty} G_n \supset M$ , then  $\cup_{n=1}^N G_n \supset M$  for some (finite)  $N$ .

*Solution.*

Suppose first that every decreasing sequence of nonempty closed sets in  $M$  has nonempty intersection, and suppose that  $G_1, G_2, \dots$  is a countable open cover of  $M$ . Form the sequence of sets:

$$\overline{G_1}, \overline{G_1} \cap \overline{G_2}, \overline{G_1} \cap \overline{G_2} \cap \overline{G_3}, \dots$$

Each of these sets is closed as a finite intersection of closed sets (complements of open sets). Furthermore, suppose that any of them is empty:

$$G_1 \cap G_2 \cap \dots \cap G_n = \emptyset \implies \overline{G_1 \cup \dots \cup G_n} = \emptyset \implies \cup_{i=1}^n G_i = M$$

This means that the countable open cover admits a finite subcover. If *none* of them are empty, then observe that they form a decreasing subsequence, and so we can use the hypothesis to obtain that the infinite sequence has nonempty intersection:

$$\cap_{i=1}^{\infty} \overline{G_i} \neq \emptyset \implies \overline{\cup_{i=1}^{\infty} G_i} \neq \emptyset \implies \cup_{i=1}^{\infty} G_i \neq M,$$

which contradicts the fact that  $G_i$  form a cover of  $M$ . Therefore, it is eventually the case that  $G_1 \cap \dots \cap G_i$  becomes empty, and so the countable open cover must admit a finite subcover.

Now suppose that every countable open cover admits a finite subcover, and pick a decreasing sequence  $F_1 \supset F_2 \supset \dots$  of closed, nonempty sets in  $M$ . Form the sequence of sets:

$$\overline{F_1}, \overline{F_2}, \overline{F_3}, \dots$$

All of these sets are open as complements of closed sets. Suppose now that the infinite union of  $\overline{F_i}$  does not cover the metric space. Then there exists  $x \notin \overline{F_i}$  for all  $i$ , which means  $x \in \cap_{i=1}^{\infty} F_i$ , and so the infinite intersection of all  $F_i$  is nonempty. If it does cover the metric space, then as a countable open cover it admits, by our hypothesis, a finite subcover  $\overline{F_1} \cup \dots \cup \overline{F_n} = M$ . This means that  $\overline{F_1 \cap \dots \cap F_n} = M$ , and consequently that  $F_1 \cap \dots \cap F_n = \emptyset$ . Because these sets are nested, the only way that this is possible is if  $F_j = \emptyset$  for all  $j$  after some  $i \leq n$ , which contradicts our hypothesis. Therefore, the decreasing sequence  $F_1 \supset F_2 \supset \dots$  has nonempty intersection. Having shown this equivalence, we recall from Corollary 8.10 that  $M$  is compact iff (i) holds, and so indeed  $M$  is compact iff (ii) holds (Corollary 8.11).

**Exercise 36**

Let  $F$  and  $K$  be disjoint, nonempty subsets of a metric space  $M$  with  $F$  closed and  $K$  compact. Show that  $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$ . Show that this may fail if we assume only that  $F$  and  $K$  are disjoint closed sets.

*Solution.*

Notice first that this quantity is indeed a real number, since  $K$  is compact, which means that it is bounded. Suppose then that it is not the case that  $d(F, K) > 0$ , which means  $d(F, K) = 0$ . This means that for any  $\epsilon > 0$ , there exist  $x \in F, y \in K$  such that  $d(x, y) < \epsilon$ . For the sequence of epsilons  $\epsilon_i = \frac{1}{i}$ , select  $x_i, y_i$  such that  $d(x_i, y_i) < \epsilon_i$ . Because the sequence  $(y_n)$  is in the compact set  $K$ , it contains a convergent subsequence  $(y_{n_k})$ , which converges to  $y \in K$  (Theorem 8.2). Form the corresponding subsequence  $(x_{n_k})$ . For any given  $\epsilon > 0$ , find  $N$  such that for  $k \geq N$ ,  $d(y_{n_k}, y) < \epsilon/2$ . Find also  $M > 0$  such that for  $k \geq M$ ,  $d(x_{n_k}, y_{n_k}) < \epsilon/2$ , which is always possible from the definition of  $(x_n), (y_n)$ . Notice that for any  $k \geq \max\{N, M\}$ ,  $d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) < \epsilon/2 + \epsilon/2 = \epsilon$ . This means that  $(x_{n_k}) \subset F$  converges to  $y \notin F$ , which contradicts the fact that  $F$  is closed. Therefore  $d(F, K) > 0$ . To show that it does not suffice for  $K$  to be closed only, consider  $M = \mathbb{R}^2, F = \{(x, 0) : x \in \mathbb{R}\}, K = \{(x, \frac{1}{x}) : x > 0\}$ . It's easy

enough to see that these are both closed by examining their complements, and they are also clearly disjoint. However, notice that for any  $\epsilon > 0$ ,  $d((\frac{2}{\epsilon}, 0), (\frac{2}{\epsilon}, \frac{1}{2\epsilon})) = \epsilon/2$ , which means that  $d(F, K)$  must be zero.

### Exercise 40

Let  $M$  be compact and let  $f : M \rightarrow M$  satisfy  $d(f(x), f(y)) = d(x, y)$  for all  $x, y \in M$ . Show that  $f$  is onto. [Hint: If  $B_\epsilon(x) \cap f(M) = \emptyset$ , consider the sequence  $(f^{(n)}(x))$ .]

*Solution.*

Notice that  $f$  is an isometry, and thus continuous. Therefore, the set  $f(M)$  must be compact in  $M$  (Theorem 8.4). Suppose then that it is the case that there exists  $x \notin f(M)$ , i.e., that  $f$  is not onto. We claim that there exists  $\epsilon > 0$  such that the entire open ball  $B_\epsilon(x)$  lies outside  $f(M)$ . If this was not the case, then by taking successive  $x_n$  such that  $d(x_n, x) < \frac{1}{n}$ , we could easily construct a Cauchy sequence  $(x_n) \subset f(M)$  that converges to  $x \notin f(M)$ , which would be a contradiction because  $f(M)$  is complete. Now consider what happens if, as indicated in the hint, we examine the sequence  $(f^{(n)}(x))$ . Since this belongs entirely in the compact set  $f(M)$ , it must contain a convergent subsequence (Theorem 8.2)  $(f^{(n_k)}(x))$ , converging to some  $y \in f(M)$ . Because this subsequence converges, it must be Cauchy. Therefore, there exists some  $N$  such that for  $k, l \geq N$  (WLOG,  $k < l$ ) it holds that:

$$d(f^{(n_k)}(x), f^{(n_l)}(x)) < \epsilon \implies d(x, f^{(n_l-n_k)}(x)) < \epsilon$$

What this implies, of course, is that  $f^{(n_k-n_l)}(x)$  lies inside the ball  $B_\epsilon(x)$ , which as we saw above is impossible. Therefore, it cannot be the case that  $x \notin f(M)$  for some  $x$ , and so  $f$  is onto.

### Exercise 41

Is compactness necessary in Exercise 40? That is, is it possible for a metric space to be isometric to a proper subset of itself? Explain.

*Solution.*

Consider the clearly non-compact metric space  $M = [1, +\infty)$ , and the function  $f : M \rightarrow M$ ,  $f(x) = x + 1$ . The range of  $f$  is  $f(M) = [2, +\infty)$ , which is a proper subset of  $M$ . We can easily see that  $f$  is an isometry, and so  $[1, +\infty)$  is isometric to  $[2, +\infty)$ .

### Exercise 42

Let  $M$  be compact and let  $f : M \rightarrow M$  satisfy  $d(f(x), f(y)) \geq d(x, y)$  for all  $x, y \in M$ . Prove that  $f$  is an isometry of  $M$  onto itself.

*Solution.*

Notice that if we can prove that the given inequality is in fact always an equality, then we can apply Exercise 8.40 to directly obtain that  $f$  is an onto isometry. Suppose, therefore, that there exist  $x, y \in m$  such that  $d(x, y) < d(f(x), f(y))$ . Form the sequences  $f^{(n)}(x), f^{(n)}(y)$ , and notice that for  $n \geq 1$ , it holds that  $d(f^{(n)}(x), f^{(n)}(y)) \geq d(f(x), f(y))$ , due to the given property of  $f$ . Now,  $M$  is compact, and hence by Theorem 8.2, there exists a subsequence  $(f^{(n_k)}(x))$  that converges to a point in  $M$ . If we then take these indices  $n_k$  and form the subsequence  $(f^{(n_k)}(y))$ , we can again apply Theorem 8.2 to obtain a further subsequence  $(f^{(n_{k_l})}(y))$  that converges to a point in  $M$ . Because any subsequence of a convergent sequence also converges, we can safely work with both  $(f^{(n_{k_l})}(x)), (f^{(n_{k_l})}(y))$  as convergent subsequences. Since these are convergent, they are also Cauchy. This means that for any given  $\epsilon > 0$ , there exists  $N_1 > 0$  such that for  $l_1, l_2 \geq N_1$  (WLOG,  $l_1 < l_2$ ), it holds that:

$$d(f^{(n_{k_{l_1}})}(x), f^{(n_{k_{l_2}})}(x)) < \epsilon \implies d(x, f^{(n_{k_{l_2}}-n_{k_{l_1}})}(x)) < \epsilon,$$

where we have again (repeatedly) used the defining property of  $f$ . Notice that what this says is that a subsequence of  $f^{(n_{k_l})}(x)$  converges to  $x$ , which means that this must hold for the sequence  $f^{(n_k)}(x)$  itself (since it's a convergent one). The exact same argument can be applied to show that  $f^{(n_k)}(y)$  converges to  $y$ . Now, for  $\epsilon = \frac{d(f(x), f(y))-d(x,y)}{2}$ , obtain  $N_1 > 0, N_2 > 0$  such that for  $l_1 \geq N_1, l_2 \geq N_2$ :

$$d(x, f^{(n_{k_l})}(x)) < \epsilon, d(y, f^{(n_{k_l})}(y)) < \epsilon$$

Consequently, for  $l \geq \max\{N_1, N_2\}$ :

$$\begin{aligned} d(f^{(n_{k_l})}(x), f^{(n_{k_l})}(y)) &\leq d(x, f^{(n_{k_l})}(x)) + d(x, f^{(n_{k_l})}(y)) \leq d(x, f^{(n_{k_l})}(x)) + d(x, y) + d(y, f^{(n_{k_l})}(y)) \\ &\leq 2\epsilon + d(x, y) = d(f(x), f(y)) - d(x, y) + d(x, y) = d(f(x), f(y)) \end{aligned}$$

But this directly contradicts our initial observation that for  $n \geq 1$ ,  $d(f^{(n)}(x), f^{(n)}(y)) \geq d(f(x), f(y))$ , and therefore it must always be the case that  $d(f(x), f(y)) = d(x, y)$ , which as discussed above means that  $f$  is an isometry onto  $M$ .

#### Exercise 43

Let  $M$  be compact and suppose that  $f : M \rightarrow M$  is one-to-one, onto, and satisfies  $d(f(x), f(y)) \leq d(x, y)$  for all  $x, y \in M$ . Prove that  $f$  is an isometry of  $M$  onto itself. [Hint: Exercise 42.]

*Solution.*

Since  $f$  is both one-to-one and onto, it is invertible. Then, for any two  $x, y \in M$ :

$$d(f(x), f(y)) \leq d(x, y) \implies d(x, y) \leq d(f^{-1}(x), f^{-1}(y))$$

We can therefore directly apply Exercise 8.42 to conclude that  $f^{-1}$  is an isometry of  $M$  onto itself. It's easy to see that this implies that  $f$  is then also an isometry of  $M$  onto itself.

## 8.2 Uniform Continuity

#### Exercise 44

Show that any Lipschitz map  $f : (M, d) \rightarrow (N, \rho)$  is uniformly continuous. In particular, any isometry is uniformly continuous.

*Solution.*

If  $f : (M, d) \rightarrow (N, \rho)$  is Lipschitz continuous, there exists  $K \geq 0$  such that  $\rho(f(x), f(y)) \leq Kd(x, y)$  for any two  $x, y \in M$ . If  $K = 0$ , then  $f$  is trivially uniformly continuous, since  $f(x) = f(y)$  for any two  $x, y \in M$ . If  $K > 0$ , then for any  $\epsilon > 0$ , let  $\delta = \frac{\epsilon}{K}$ . Then, whenever  $d(x, y) < \delta$ , we have:

$$\rho(f(x), f(y)) \leq Kd(x, y) < K \frac{\epsilon}{K} = \epsilon$$

The selection of  $\delta$  is independent of  $x, y$ , and therefore  $f$  is indeed uniformly continuous. Now, an isometry is Lipschitz continuous with  $K = 1$ , and therefore it follows immediately that it is also uniformly continuous.

#### Exercise 45

Prove that *every* map  $f : \mathbb{N} \rightarrow \mathbb{R}$  is uniformly continuous.

*Solution.*

Take any  $\epsilon$  and take  $\delta = 1/2$ . Then for any two  $x, y \in \mathbb{N}$  such that  $|m - n| < \delta$ , it vacuously holds that  $|f(m) - f(n)| < \epsilon$ , since no two natural numbers satisfy  $|m - n| < 1/2$  except if  $m = n$  (in which case clearly  $|f(m) - f(n)| = 0$ ).

**Exercise 46**

Show that  $|d(x, z) - d(y, z)| \leq d(x, y)$  and conclude that the map  $x \mapsto d(x, z)$  is uniformly continuous on  $M$  for each fixed  $z \in M$ .

*Solution.*

For the inequality, we already know it to hold from Exercise 3.2. Now, consider any  $\epsilon > 0$  and set  $\delta = \epsilon$ . Then, based on the “reverse triangle inequality”, whenever  $d(x, y) < \delta$ , we have that:

$$|d(x, z) - d(y, z)| \leq d(x, y) < \delta = \epsilon,$$

which shows that the map  $x \mapsto d(x, z)$  is uniformly continuous for any fixed  $z \in M$ .

**Exercise 47**

Given a nonempty subset  $A \subset M$ , show that  $|d(x, A) - d(y, A)| \leq d(x, y)$ , and conclude that the map  $x \mapsto d(x, A)$  is uniformly continuous on  $M$ .

*Solution.*

We know that this inequality holds from Proposition 5.4. Then, for any  $\epsilon > 0$ , set  $\delta = \epsilon$  and observe that whenever  $d(x, y) < \delta$ , we have that:

$$|d(x, A) - d(y, A)| \leq d(x, y) < \delta = \epsilon,$$

and so the map from  $x$  to  $d(x, A)$  is uniformly continuous on  $M$  for any fixed  $A$ .

**Exercise 48**

Prove that a uniformly continuous map sends Cauchy sequences into Cauchy sequences.

*Solution.*

Let  $f : (M, d) \rightarrow (N, \rho)$  be uniformly continuous, and  $(x_n) \subset M$  be Cauchy. Pick any  $\epsilon > 0$ , and obtain  $\delta > 0$  for which it holds that  $d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$  for any two  $x, y \in M$ , which exists because  $f$  is uniformly continuous. Furthermore, pick  $N > 0$  such that  $d(x_n, x_m) < \delta$  for any  $n, m \geq N$ , which exists because  $(x_n)$  is Cauchy. Then, for any  $n, m \geq N$  it must hold that  $d(x_n, x_m) < \delta$ , and thus also that  $\rho(f(x_n), f(x_m)) < \epsilon$ , which shows precisely that  $(f(x_n))$  is Cauchy in  $N$ .