N. L. Carothers' Real Analysis - A partial solutions manual

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Contents

Part I

Metric Spaces

Chapter 1

Calculus Review

1.1 The Real Numbers

Exercise 1
If A is a nonempty subset of $\mathbb{R}$ that is bounded below, show that A has a greatest lower bound. That is, show that there is a number $m \in \mathbb{R}$ satisfying: (i) m is a lower bound for A; and (ii) if x is a lower bound for A. then $x \leq m$. [Hint: Consider the set $-A = \{-a : a \in A\}$ and show that $m = -\sup(-A)$ works.]

Solution.
As pointed out in the hint, let $m = \sup(-A)$. Let also $B = \{-a : a \in A\}$ and b be any element of B. Since A is non-empty, B is non-empty too. By the definition of B, $b = -a$ for some $a \in A$, Because A is bounded below, it holds that there exists l such that $l \leq a$. This means that $-l \geq -a \implies -l \geq b$. Therefore, $-l$ is an upper bound for B. By the completeness of real numbers, B has a supremum, m. For any $a \in A$, $-a \in B$, therefore $-a \leq m \implies a \geq -m$. This means that $-m$ is a lower bound for A. For any lower bound m' of A, it holds that $m' \leq a$ for all $a \in A$. This means that $-m' \geq -a$ for all $-a \in B$. But then $-m'$ is an upper bound for B, thus by the definition of the supremum, $-m' \geq m \implies m' \leq -m$. We have thus shown that $-m$ is the greatest lower bound of A.

Exercise 3
Establish the following apparently different (but "fancier") characterization of the supremum. Let A be a non-empty set of $\mathbb{R}$ that is bounded above. Prove that $s = \sup A$ if and only if (i) s is an upper bound for A, and (ii) for every $\epsilon > 0$ there is an $a \in A$ such that $a > s - \epsilon$. State and prove the corresponding result for the infinum of an non-empty subset of $\mathbb{R}$ that is bounded below.

Solution.
$\implies$: Suppose first that s is the supremum of A. By the definition of the supremum, s is indeed an upper bound for A. Suppose now that there exists $\epsilon > 0$ such that for every $a \in A$ it is the case that $a \leq s - \epsilon$. Observe that this means that $s - \epsilon$ is an upper bound for A, while it also holds that $s - \epsilon < s$. But this contradicts the definition of s being the supremum of A.
We have arrived at a contradiction, and thus the negation of the assumed statement must hold. Namely, it must be the case that for every $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.
$\impliedby$: Suppose now that s is an upper bound for A and that for every $\epsilon > 0$ there is an $a \in A$ such that $a > s - \epsilon$. Suppose that s is not the supremum of A. By the completeness of the real numbers, A must have a supremum t. Because s is an upper bound for A, it must hold that $t < s$. Set $\epsilon = s - t > 0$. By the definition of s, there exists $a \in A$ such that $a > s - \epsilon = s - (s - t) = t$. This contradicts the fact that t is an upper bound for A. Therefore, s must be the supremum of A.
The corresponding result for the infinum is that a number s is the infinum of $A \subset \mathbb{R}$, with A non-empty and bounded below, if and only if (i) s is a lower bound for A and (ii) for every $\epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$. To prove this, we have that:

$\implies$: Suppose s is the infinum of A. Then s is a lower bound for A. Suppose that there exists $\epsilon > 0$ such that for every $a \in A, a \geq s + \epsilon$. But then $s + \epsilon > s$ is a lower bound for A, contradiction. The negation of our assumption leads to the desired property for s.
$\impliedby$: Now suppose s is a lower bound for A and for every $\epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$. Suppose s is not the infinum of A, and t is instead. Because s is a lower bound, $s < t$. Set $\epsilon = t - s$. Then there exists $a \in A$ such that $a < s + \epsilon = s + (t - s) = t$. But this contradicts the fact that t is a lower bound for A.
Note that this second proof can also be done by relating the infinum of A to the supremum of $-A$, but since we have not yet proved this (exercise 1) we do not use it here.

Exercise 4
Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that there is a sequence of elements x_n of A that converges to $\sup A$.

Solution.
Recall from exercise 3 that if $s = \sup A$, it holds that for every $\epsilon > 0$ there exists $a \in A$ such that $a > s - \epsilon$. Consider then the sequence that is formed by taking x_i be an element of A for which it holds that $x_i > s - \frac{1}{i}$. For any $\epsilon > 0$, take $M = \lceil \frac{1}{\epsilon} \rceil$. It then holds that $\frac{1}{M} \leq \epsilon$. Additionally, it is true by the definition of the sequence that $x_M > s - \lceil \frac{1}{M} \rceil \geq s - \epsilon \implies \epsilon > s - x_M$. Also, by the definition of the supremum, $s - x_M \geq 0$, thus $|s - x_M| < \epsilon$. Now, for any $j > M$ we have that $x_j > s - \frac{1}{j} \implies \frac{1}{j} > s - x_j$. By the definition of the supremum, $s - x_j \geq 0$, thus $|s - x_j| < \frac{1}{j} < \frac{1}{M} \leq \epsilon$.
We have thus precisely proved that the limit of the sequence x_i is the supremum s of A.

Exercise 5
Suppose that $a_n \leq b$ for all n and that $a = \lim_{n\to\infty} a_n$ exists. Show that $a \leq b$. Conclude that $a \leq \sup\{a_n : n \in \mathbb{N}\}$.

Solution.
Suppose that $a > b$, which means that $a - b > 0$. Set $\epsilon = a - b$. By the definition of the limit of a sequence, there exists $M > 0$ such that for all $n > M$, it holds that:

$$|a - a_n| < \epsilon \implies -\epsilon < a - a_n < \epsilon \implies a < a_n + \epsilon$$

It is the case that $a_n \leq b$ for all n, thus $a < b + \epsilon = b + (a - b) \implies a < a$, which is clearly a contradiction. Therefore $a \leq b$. Now because $s = \sup\{a_n : n \in \mathbb{N}\}$ is by definition a number for which $a_n \leq s$ for all n, the previous result applies, and thus $a \leq s = \sup\{a_n : n \in \mathbb{N}\}$.

Exercise 6
Prove that every convergent sequence of real numbers is bounded. Moreover, if a_n is convergent, show that $\inf a_n \leq \lim_{n\to\infty} a_n \leq \sup a_n$.

Solution.
Suppose a_n is a convergent sequence of real numbers, and suppose that it converges to $a \in \mathbb{R}$. By definition, for any $\epsilon > 0$, there exists $M > 0$ such that for every $n > M$ it holds that $|a_n - a| < \epsilon \implies a_n < a + \epsilon$. Pick any such ϵ, e.g. $\epsilon = 1$. We can thus see that there exist *at most* M elements of the sequence, chosen from $a_1, a_2, \ldots a_M$ such that $a_i \geq a + \epsilon$. Let then S be the —possibly empty— set of all such a_i. We can then see that the set $S \cup \{a + \epsilon\}$ has a finite number of elements. Thus its maximum element s is well defined. Observe then that $s \geq a_n$ for all elements of the sequence. Now recall from exercise 5 that the limit of the sequence is indeed at most equal to its supremum (which we showed is well defined, since at least one upper bound exists).
Again by the definition of the limit, for every $\epsilon > 0$ there exists $M > 0$ such that for $n > M$ it holds that $|a_n - a| < \epsilon \implies -\epsilon < a_n - a \implies a + \epsilon < a_n$. Observe that we can thus apply a completely symmetric argument: pick, say, $\epsilon = 1$ and then there are at most M elements such that $a_n \leq a + \epsilon$. By constructing S from these elements in the same way as above, the rest of the proof becomes completely symmetric, as is the part regarding $\inf a_n \leq \lim_{n\to\infty} a_n$.

Exercise 8
Given $a < b$, show that there are, in fact, infinitely many distinct rationals between a and b. The same goes for irrationals too.

Solution.
Suppose that there are only a finite number of distinct rationals between a and b, and call them $q_1, q_2, \ldots q_n$, listed in ascending order. This means $a < q_1 < q_2 < \ldots < q_n < b$. But then we have that q_n, b are real numbers with $q_n < b$, and therefore there has to exist a rational number q such that $q_n < r < b$. But this is a contradiction, because $q \neq q_i$ for all i and $a < q < b$. Therefore there exists an infinite number of distinct rationals between a and b.
Now suppose that there is only a finite number of irrationals $r_1, r_2, \ldots r_n$ between a, b, listed in ascending order. However, in exercise 7 we've seen that if $a < b$, there exists an irrational x such that $a < x < b$. This means that in our case there exists an irrational r such that $r_n < r < b$. Clearly, this is a contradiction since $r_i \neq r$ for all i. Therefore there exist infinitely many distinct irrationals between a, b.

Exercise 13
Let $a_n \geq 0$ for all n, and let $s_n = \sum_{i=1}^{n} a_i$. Show that (s_n) converges if and only if (s_n) is bounded.

Solution.
$\implies$: Assume that (s_n) converges to s. Then recall from exercise 6 that every convergent sequence is bounded, thus s_n is indeed bounded.
$\impliedby$: Suppose (s_n) is bounded. Because $a_n \geq 0$, the sequence of partial sums s_n increases monotonically. We know then that a monotone, bounded sequence always converges, thus (s_n) converges.

Exercise 14
Prove that a convergent sequence is Cauchy, and that any Cauchy sequence is bounded.

Solution.
A sequence of real numbers is Cauchy if, for every $\epsilon > 0$, there is an integer $N \geq 1$ such that $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$.
Let then a_n be a convergent sequence that converges to a, and select any $\epsilon > 0$. Then, by the definition of convergence, for $\epsilon' = \frac{\epsilon}{2}$ there exists $N > 0$ such that for all $n > N$ it holds that $|a_n - a| < \epsilon'$. Therefore, for any two $n, m > N$, it holds that:

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, which is the defining property of Cauchy sequences. Now let a_n be any Cauchy sequence. Suppose that a_n is not bounded. Because a_n is Cauchy, it must hold that for $\epsilon = 1$, there exists $N \geq 1$ such that for all $n, m > N$ it holds that $|a_n - a_m| < 1$. Let then a_k be the first element of the sequence for which this inequality holds when setting $n = k$ and m any other index $m > k$. This means that for any $m > k$, $|a_m - a_k| < 1$. In particular, this means that a_m can never be larger than $a_k + 1$, therefore for all elements of the sequence after that point, $a_k + 1$ is an upper bound. Then, because k is finite, $m = \max\{a_1, \ldots, a_k\}$ is well defined. If we then set $s = \max\{m, a_k + 1\}$, we see that s constitutes an upper bound for the entire sequence.
A symmetric argument can be applied to extract a lower bound for the sequence, thus any Cauchy sequence is bounded.

Exercise 15
Show that a Cauchy sequence with a convergent subsequence actually converges.

Solution.
Let a_n be a Cauchy sequence and $b_n = a_{f(n)}$ such that $f(j) > f(k)$ whenever $j > k$ be a convergent subsequence of it that converges to b. Pick any $\epsilon > 0$. Now it must be the case that there exists $N > 0$ such that whenever $n > N$, $|b_n - b| < \frac{\epsilon}{2}$. Now because a_n is Cauchy, there exists $M > 0$ such that whenever $n, m > 0$, $|a_n - a_m| < \frac{\epsilon}{2}$. Set $K = \max\{N, M\}$ and observe then that for any $n > K$, it holds that:

$$|a_n - b| < |a_n - b_{K+1} + b_{K+1} - b| \leq |a_n - b_{K+1}| + |b_{K+1} - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, where in the last step we used the fact that a_n is Cauchy (and b_{K+1} is of course an element of it) for the first term and that b_n converges to b for the second term. We have thus shown that a_n converges, and more specifically it converges to the limit of its convergent subsequence.

Exercise 17
Given real numbers a and b establish the following formulas: $|a + b| \leq |a| + |b|, \big||a| - |b|\big| \leq |a - b|, \max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

Solution.

- $|a+b| \leq |a| + |b|$: one way to do this is to observe that $\mathbb{R}$ is a vector space over the field $\mathbb{R}$ for which standard multiplication satisfies the properties of being an inner product. Thus we can use Axler's result of the triangle inequality for inner product spaces, whose proof does not assume that the triangle inequality holds for real numbers. Another way is to do a variant of this proof "manually":

$$(a + b)^2 = a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a| \cdot |b| = (|a| + |b|)^2$$

$$\implies |a + b| \leq |a| + |b|$$

  , where we used some properties of the absolute value which easily follow from its definition ($|a|^2 = |a|^2, x \leq |x|$)., and we took square roots at the last step, observing that $|a| + |b|$ is always non-negative.

- $\big||a| - |b|\big| \leq |a - b|$: Quite similar to the above, we have that:

$$(|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a| \cdot |b| \leq |a|^2 + |b|^2 - 2ab = (a - b)^2 \implies \big||a| - |b|\big| \leq |a - b|$$

  , where when taking square roots the absolute values are now necessary.

- $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$: It is either the case that $a \geq b$ or that $a \leq b$ (here we are using the fact that $\mathbb{R}$ is a totally ordered set). In the first case, we have that $\max\{a, b\} = a$ and also that $|a - b| = a - b$, which means $\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = a = \max\{a, b\}$. The second case is almost exactly the same, except for $|a - b|$ evaluating to $b - a$.

- $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$: This is done in the same manner as above, again by utilizing the fact that for any two a, b, it either holds that $a \leq b$ or that $a \geq b$.

Exercise 18
(a) Given $a > -1, a \neq 0$, use induction to show that $(1 + a)^n > 1 + na$ for any integer $n > 1$.
(b) Use (a) to show that, for any $x > 0$, the sequence $(1 + \frac{x}{n})^n$ increases.
(c) If $a > 0$, show that $(1 + a)^r > 1 + ra$ holds for any *rational* exponent $r > 1$.
[Hint: If $r = \frac{p}{q}$, then apply (a) with $n = q$ and (b) with $x = ap$.]
(d) Finally, show that (c) holds for any *real* exponent $r > 1$.

Solution.
(a) The base case of the induction is $n = 2$. We then have that:

$$(1 + a)^2 = 1 + 2a + a^2 > 1 + 2a$$

, since a is not zero, meaning that $a^2 > 0$. Suppose now that the inequality holds for $n = k > 1$. Then we have that:

$$(1 + a)^{k+1} = (1 + a)(1 + a)^k > (1 + a)(1 + ka) = 1 + ka + a + ka^2 = 1 + (k + 1)a + ka^2 > 1 + (k + 1)a$$

, where, crucially, we used the fact that $a > -1$, thus that $a + 1 > 0$, thus that we can multiply the inequality for $n = k$ with $(1 + a)$. In the last step we again use the fact that $a \neq 0$, thus that $ka^2 > 0$.

(b) We'll work through this the same way as the book's examples, i.e., compute the ratio between two successive terms of the sequence:

$$\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^n} = \left(1+\frac{x}{n}\right)\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^{n+1}} = \left(1+\frac{x}{n}\right)\left(\frac{\frac{n+1+x}{n+1}}{\frac{n+x}{n}}\right)^{n+1}$$

$$= \left(1+\frac{x}{n}\right)\left(\frac{n^2+n+nx}{n^2+x+n+nx}\right)^{n+1} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n^2+x+n+nx}\right)^{n+1} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{(x+n)(n+1)}\right)^{n+1}$$

Now, for any $x > 0$ and n positive integer we have that:

$$xn + n^2 + n > 0 \implies xn + n^2 + n + x > x \implies (x+n)(n+1) > x \implies -(x+n)(n+1) < -x$$

$$\implies -1 < -\frac{x}{(x+n)(n+1)}$$

Also, $x > 0$ thus this quantity is never zero. Therefore, using it as an a in the Bernoulli inequality (part (a)) we can obtain that:

$$\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^n} > \left(1+\frac{x}{n}\right)(1-(n+1)\frac{x}{(x+n)(n+1)}) = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{x+n}\right) = \frac{x+n}{n}\cdot\frac{n}{x+n} = 1$$

, which means that the sequence $\left(1+\frac{x}{n}\right)$ does indeed increase.
(c)Consider any rational $r > 1$. r can be written as $\frac{p}{q}$, where p, q are positive integers, and in fact $p > q$. Setting $x = ap$ (for this part, $a > 0$) and observing that $p > q$, from part (b) we have that:

$$\left(1+\frac{ap}{q}\right)^q < \left(1+\frac{ap}{p}\right)^p \implies \left(1+\frac{ap}{q}\right)^q < (1+a)^p$$

For $a > 0$, both quantities inside the parentheses are positive, and thus we can take q-th roots and obtain:

$$\left(1+\frac{ap}{q}\right) < (1+a)^{\frac{p}{q}}$$

, which, since $r = \frac{p}{q}$, is the equivalent of the Bernoulli inequality for a rational exponent $r > 1$ and $a > 0$.
(d) We know that we can approach any real number $r > 1$ with a sequence of rationals. Furthermore, the Bernoulli inequality will hold for all of those rationals that are larger than 1, so as we approach $r > 1$, the Bernoulli inequality will hold. Now, at the limit, this strict inequality holds as a non-strict inequality, thus we can conclude that:

$$(1+a)^r \geq 1 + ra$$

Now pick a rational q such that $1 < q < r$ (such a rational always exists). Observe that:

$$(1+a)^r = (1+a)^{\frac{rq}{q}} = ((1+a)^q)^{\frac{r}{q}} > (1+qa)^{\frac{r}{q}}$$

, where we applied the Bernoulli inequality for q. Now, however, $\frac{r}{q}$ is a real number greater than 1, and hence we can use our non-strict inequality above to arrive at the desired result:

$$(1+a)^r > (1+qa)^{\frac{r}{q}} \geq 1 + q\frac{r}{q}a = 1 + ra \implies (1+a)^r > 1 + ra$$

Exercise 21
Let $p \geq 2$ be a fixed integer, and let $0 < x < 1$. If x has a finite-length base p decimal expansion, that is, if $x = a_1/p + \ldots + a_n/p^n$ with $a_n \neq 0$, prove that x has precisely two base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique. Characterize the numbers $0 < x < 1$ that have repeating base p decimal expansions. How about eventually repeating?

Solution.
We consider first the decimal expansion $0.a_1a_2\ldots a_n00\ldots$, with an infinite number of trailing zeros after the n-th decimal, and call the i-th decimal here b_i. As per the book's definition, this corresponds to an infinite series defined as $\sum_{i=1}^{\infty} b_i/p^i$. It's clear that the limit of this sum is x, since x is precisely the sum of the first n terms, and all of the terms after n are zero. Thererefore, the expansion above indeed corresponds to x.
Now consider the decimal expansion corresponding to the series $\frac{a_1}{p} + \frac{a_2}{p^2} + \ldots + \frac{a_n - 1}{p^n} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^k}$. This would yield a series of decimals the first n of which would equal $a_1, \ldots a_n - 1$ (note that because $a_n \neq 0, a_n - 1$ causes no problems), while all decimals after that would equal $p-1$ (call the sequence of digits c_i). If we consider the limit of this series, we can see that the last term is a series summing to $\frac{1}{p^n}$, and that the first n terms sum to $x - \frac{1}{p^n}$. Thus, the series as a whole tends to x, which means that it *is* a decimal expansion for x.
We now have to show that there is no other decimal expansion for x. We will do this by selecting a decimal expansion $0.b_1b_2\ldots b_nb_{n+1}\ldots$ that is not equal to $0.a_1a_2\ldots a_n00\ldots$ and show that it must necessarily equal $0.a_1a_2\ldots(a_n-1)(p-1)(p-1)\ldots$. Let us then examine such a decimal expansion. Since it differs from $0.a_1a_2\ldots a_n00\ldots$, there must exist a first j such that $a_j \neq b_j$. Because both decimal expansions correspond to x, for the two series it must hold that:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, where we have erased from both sides the first $j-1$ digits that are equal (note that the set of these may be empty). Because we are manipulating convergent series, we can write:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i} = \sum_{i=j+1}^{n} \frac{b_i - a_i}{p^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{p^i}$$

, where we noted that after the n-th digit, all a_i are zero. Now, note that if $b_j > a_j$, the LHS here equals at most $-\frac{1}{p^j}$. At the same time, the RHS equals at least 0, which happens when $b_i = 0, i \geq n+1$ and all $b_i - a_i = 0, j+1 \leq i \leq n$. Clearly then, they can never be equal. On the other hand, if $b_j < a_j$ the LHS equals at least $\frac{1}{p^j}$. The RHS equals at most $\sum_{i=j+1}^{\infty} \frac{p-1}{p^i} = \frac{1}{p^j}$, and, crucially, this happens if all $b_i = p-1, i \geq n+1, b_i - a_i = p-1, j+1 \leq i \leq n$. Because the value is achieved when all of the digits fulfill these conditions, the RHS will be strictly smaller than the LHS in all other cases. The consequence of this is that all b_i starting at $i = n+1$ are equal to $p-1$. Additionally, all $a_i, j+1 \leq i \leq n$ have to be zero. But because $a_n \neq 0$, it must hold that $j+1 > n$. That is, the first digit where the two expansions differ must be at least at position n. If this was strictly larger than n, the LHS would not achieve its minimum value (a_j would be 0 and b_j would be $p-1$), thus it could not equal the RHS, contradiction. Thus $j = n$. By these observations, the only digit for which we have not yet determined a value is at position n. Recall that $a_j - b_j$ must equal 1, which means that $b_j = a_j - 1$. In other words, $b_n = a_n - 1$, all $b_i = a_i, i < n$ and all $b_i = p-1, i > n$, leading us to conclude that there exists no third possible representation.
Now we proceed to examine an x with no finite-length base p decimal expansion. Let $a_i, i = 1, 2, \ldots$ be one of its expansions, which is necessarily infinite in length. We will show that this is unique by taking another expansion b_i and showing that it must equal a_i at all digits. Indeed, suppose that they differ, and that the first digit at which this happens is at position j. As we did above, we can write:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, and, again because the series converge, we can rewrite this as:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i}$$

If $b_j > a_j$ we again observe that the LHS equals at most $-\frac{1}{p^j}$. The RHS equals at least $\sum_{i=j+1}^{\infty} \frac{-(p-1)}{p^i} = -\frac{1}{p^j}$, which happens if all $b_i - a_i = -(p-1), i > j$. This cannot otherwise be true because this minimum value is achieved when all $b_i - a_i$ are minimized. The consequence is each $b_i = 0, a_i = p-1, i > j$. However, this implies that x can then be written as $0.b_1 b_2 \ldots b_j 00 \ldots$, which is a finite-length expansion since all trailing digits after j are zeros. This is a contradiction. An exactly symmetrical argument applies when $b_j < a_j$, leading us to conclude that $b_j = a_j$ for *all* j, thus a_i is a unique expansion for x.
Now we examine $0 < x < 1$ that have an eventually repeating base p decimal expansion. This means that there exists a finite-length "unique" first part of the number, consisting of m digits, as well as a minimum "period" of repetition n (this can also be zero, in which case the part below does not apply but the number is trivially rational), such that the number can be written as:

$$0.a_1 a_2 \ldots a_m a_{m+1} a_{m+2} a_{m+3} \ldots a_{m+n} a_{m+1} \ldots a_{m+n} \ldots$$

, where we have defined n as the minimum integer for which this holds. Then observe that we can write:

$$\begin{aligned} x = (a_1 a_2 \ldots a_m) p^{-m} + (a_{m+1} a_{m+2} \ldots a_{m+n}) p^{-m-n} + (a_{m+1} a_{m+2} \ldots a_{m+n}) p^{-m-2n} \\ +(a_{m+1} a_{m+2} \ldots a_{m+n}) p^{-m-3n} + \ldots \\ = \frac{a_1 a_2 \ldots a_m}{p^m} + (a_{m+1} a_{m+2} \ldots a_{m+n}) (\sum_{i=1}^{\infty} p^{-m-in}) = \\ \frac{a_1 a_2 \ldots a_m}{p^m} + \frac{a_{m+1} a_{m+2} \ldots a_{m+n}}{p^m (p^n - 1)} \end{aligned}$$

, which is a sum of two quotients of integers, and therefore is a rational number. Note that for the case of repeating expansions everything above simplifies to $m = 0$. Now we need to examine whether *every* rational number can be written in this way, in other words, whether every rational number has an eventually repeating (which includes the trivially "eventually repeating" finite-length digit sequences) base p decimal expansion.
Recall that a rational number can be written as the quotient of two integers. Furthermore, because we are interested in numbers in $[0, 1]$, we can restrict ourselves to $x = \frac{a}{b}$ where a, b natural, non-zero numbers with $a < b$. We now recall that for finding a base p decimal expansion for x we can use the division algorithm iteratively. Namely, we first find the smallest power p_1^n such that $ap_1^n \geq b$, and then we use the division algorithm on ap^n, b. According to it, there will exist unique $q_1, r_1, 0 \leq r_1 < b \in \mathbb{N}$ such that $ap^{n_1} = bq_1 + r_1$. Notice that this implies that $\frac{a}{b} = \frac{q}{p^{n_1}} + \frac{r_1}{bp^{n_1}}$. The first $n_1 - 1$ digits of the decimal expansion of x will be zeros, since for all powers $p^k, k < n$ it has to be that $ap^k < b \implies x < \frac{1}{p^k}$. Now notice that $q_1 \leq \frac{ap^{n_1}}{b} < p$ because $ap^{n_1 - 1} < b$ due to the choice of n_1. Therefore q_1 can be thought of as the n_1-th digit in the decimal expansion of x. If $r_1 = 0$, the decimal expansion is finished, and finite in length. Otherwise, we can apply the same procedure to r_1, bp^{n_1} to obtain that $r_1 p^{n_2} = bp^{n_1} q_2 + r_2$, where n_2 has been chosen in the same manner as above, and signifies that the next $n_2 - 1$ digits of x will be zero, while the n_2-th will be q_2. Now we observe that if at any point r_k becomes zero, the expansion is finite in length. Otherwise, we record this sequence of remainders. Each of them is in the range $(0, b)$. Therefore, after at most $b - 1$ steps we will encounter a remainder that has been encountered before. The consequence is that after this point, the sequence of digits is fully known, because we are always performing a division with b. Therefore, from that point on the decimal expansion will indeed be infinitely repeating, thus completing the proof.

Exercise 23
If a_n is convergent, show that $\text{liminf}_{n\to\infty} a_n = \text{limsup}_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

Solution.
We begin by showing $\text{liminf}_{n\to\infty} a_n = \lim_{n\to\infty} a_n$. Let L be the limit of a_n. By definition, $\text{liminf}_{n\to\infty} a_n = \sup_{n\geq 1}\{\inf\{a_n, a_{n+1}, \ldots\}\}$. Suppose first that this supremum did not exist, i.e. that it "equals infinity". Then for every $M > 0$ there must exist $N > 0$ such that $\inf\{a_N, a_{N+1}, \ldots\} > M$. But this in turn would mean that a_n is not bounded and yet converges to L, a contradiction. Now suppose that the supremum

equals $L+\epsilon, \epsilon>0$. Then $L+\epsilon', \epsilon'<\epsilon$ cannot constitute an upper bound for the infimums. Thus there exists $N>0$ such that $\inf\{a_N, a_{N+1}, \ldots\} > L+\epsilon'$, which means that for $n \geq N, a_n > L+\epsilon'$. This is a clear contradiction of the definition of limit, since a_n cannot get more than ϵ'-close to L. Therefore, the liminf of a_n is at most L.

Now, by the definition of the limit, for any $\epsilon>0$ there exists $N>0$ such that for $n>N, |a_n - L| < \epsilon \implies a_n > L-\epsilon$. Notice that this means that $\inf\{a_N, a_{N+1}, \ldots\} \geq L-\epsilon$, which in turn means that the liminf of a_n, as an upper bound for these infimums, must equal at least $L-\epsilon$ for *any* $\epsilon>0$. At the same time, it equals at most L. The only possibility then is that $\text{liminf}_{n\to\infty} a_n = L$, which is what we are asked to prove.

The argument for showing that $\text{limsup}_{n\to\infty} a_n = L$ is exactly symmetrical.

Exercise 24

Show that $\lim\ \sup_{n\to\infty}(-a_n) = -\lim\ \inf_{n\to\infty} a_n$.

Solution.

We begin by the definition of lim sup:

$$\lim\ \sup_{n\to\infty}(-a_n) = \inf_{n\geq 1}\{\sup\{-a_n, -a_{n+1}, \ldots\}\}$$

We recall from exercise 1 that if $-A = \{-a, a \in A\}$ for some set A, then $\inf A = -\sup(-A)$. This means that for any of the sets $\{-a_n, -a_{n+1}, \ldots\}$ it holds that $\sup\{-a_n, -a_{n+1}, \ldots\} = -\inf\{a_n, a_{n+1}, \ldots\}$. Thus:

$$\lim\ \sup_{n\to\infty}(-a_n) = \inf_{n\geq 1}\{-\inf\{a_n, a_{n+1}, \ldots\}\}$$

Now observe that the "outer" infimum here is taken over the set $\{-\inf\{a_1, a_2, \ldots\}, -\inf\{a_2, a_3, \ldots\}, \ldots\}$. If we now set $A = \{-\inf\{a_1, a_2, \ldots\}, -\inf\{a_2, a_3, \ldots\}, \ldots\}$, then $-A = \{\inf\{a_1, a_2, \ldots\}, \inf\{a_2, a_3, \ldots\}, \ldots\}$. Therefore we can again apply the result of exercise 1 to get that:

$$\lim\ \sup_{n\to\infty}(-a_n) = \inf A = -\sup(-A) = -\sup_{n\geq 1}\{\inf\{a_1, a_2, \ldots\}, \inf\{a_2, a_3, \ldots\}, \ldots\} = -\lim\ \inf_{n\to\infty} a_n$$

Exercise 25

If $\lim\ \sup_{n\to\infty} a_n = -\infty$, show that a_n diverges to $-\infty$. If $\lim\ \sup_{n\to\infty} a_n = +\infty$, show that a_n has a *subsequence* that diverges to $+\infty$. What happens if $\lim\ \inf_{n\to\infty} = \pm\infty$?

Solution.

We have that $\lim\ \sup_{n\to\infty} a_n = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$. If this "equals negative infinity", this means that the set over which the infimum is computed is not bounded below. In other words, for any $M<0$, we can find $n \geq 1$ such that $\sup\{a_n, a_{n+1}, \ldots\} < M$. If the least upper bound of such a set is less than M, then clearly all elements of the set must also be less than M. This means that given an $M<0$ we can always find an n such that $a_k < M$ for all $k>n$, by appropriately picking the set as above. This means precisely that a_n diverges to $-\infty$.

Now let us examine what happens if lim sup "equals positive infinity". This means that the set over which the infimum is computed is empty. To see why this is the case, suppose that it was not. Then either it has an lower bound or it does not (law of excluded middle). In the first case, the (corollary of the) Least Upper Bound axiom tells us that the infimum would have to equal some real number. In the second case, by definition we would write that the infimum equals $-\infty$. The only case left is thus for the set to be empty, and in this case we have defined (page 3-4 for the supremum, symmetric for the infimum) that $\inf \emptyset = +\infty$, since every real number is a lower bound for the empty set. For the set to be empty, it must be the case that *all* $\sup\{a_n, a_{n+1}, \ldots\}$, i.e. for every n, are $+\infty$. This means that all of these sets are unbounded, i.e. that for every n, it is the case that $\{a_n, a_{n+1}, \ldots\}$ contains, for every $M>0$, an element a_{n_k} such that $a_{n_k} > M$.

Form then a subsequence b_j of a_n in the following way. Select b_1 to be the first element of $\{a_1, a_2, \ldots\}$ such that $b_1 > 1$, which is guaranteed to exist. Then select b_2 to be the first element of $\{a_2, a_3, \ldots\}$ such

that $b_2 > 2, b_2 \neq b_1$. This is also guaranteed to exist, because if it did not, then all elements of a_n after b_1 would be at most 2, which would mean that after that point a_n is bounded and then at least one of the suprema above would be a real number, in which the set over which lim sup is computed would not be empty, a contradiction. Continue this way to pick any b_j to never equal any of the previously selected elements, and then observe that this is a subsequence of a_n that indeed diverges to $+\infty$.
For lim inf the cases are symmetrical.

Exercise 26
Prove the characterization of lim sup given above. That is, given a bounded sequence (a_n), show that the number $M = \limsup_{n\to\infty} a_n$ satisfies (*) (page 12) and, conversely, that any number M satisfying (*) must equal $\limsup_{n\to\infty} a_n$. State and prove the corresponding result for $m = \liminf_{n\to\infty} a_n$.

Solution.
Suppose M is the limit supremum of a bounded sequence (a_n). This means that:

$$M = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$$

Now, pick any $\epsilon > 0$. Recall from exercise 3 that there must exist an element s of the set of suprema such that $s < M + \epsilon$. This means that there must exist N such that $s = \sup\{a_N, a_{N+1}, \ldots\} < M + \epsilon$. Now, by the definition of supremum, this means that for all $a_n, n \geq N$ it must hold that $a_n < M + \epsilon$. These are clearly infinitely many elements of the sequence, and thus the inequality stated here *may* not be true only for *finitely* many elements (up to and not including a_N).
Furthermore, suppose that $M - \epsilon < a_n$ is true for finitely many elements of the sequence, and call N' the largest integer for which $M - \epsilon < a_{N'}$ (well defined due to the set containing finitely many elements). Then for all $n > N'$ it must hold that $M - \epsilon \geq a_n$, which means that $M - \epsilon < M$ constitutes an upper bound for all subsequences $a_n, a_{n+1}, \ldots, n > N'$, and thus $\sup\{a_n, a_{n+1}, \ldots\} \leq M - \epsilon$. But by the definition of lim sup, M is the infimum of the set of suprema, yet is larger than any $\sup\{a_n, a_{n+1}, \ldots\}, n > N'$, a contradiction. Therefore $M - \epsilon < a_n$ has to hold for infinitely many n.
In the other direction, suppose M satisfies (*). We need to show that $M = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$. Suppose first that M is greater than the lim sup. By the definition of infimum, M cannot then constitute a lower bound for the set of suprema. Therefore there exists $N > 0$ such that $\sup\{a_N, a_{N+1}, \ldots\} < M$. Set then $\epsilon = M - \sup\{a_N, a_{N+1}, \ldots\}$, in which case by (*) there must exist infinitely many n such that $M - \epsilon < a_n \implies \sup\{a_N, a_{N+1}, \ldots\} < a_n$. But because n are infinitely many, some of them are greater than N, and thus this inequality contradicts the definition of supremum for the subsequence starting at N.
Now suppose M is smaller than the lim sup, in which case $\limsup_{n\to\infty} a_n > M$. Then $\limsup_{n\to\infty} a_n - M = \epsilon > 0$. This furthermore means that there exists $\epsilon' > 0, \epsilon' < \epsilon$. Now, there must exist infinite n such that $a_n < M + \epsilon' < M + \epsilon = \limsup_{n\to\infty} a_n$. Observe that this means the following. First, that for some $N > 0$, for all $n \geq N, M + \epsilon'$ constitutes an upper bound for all a_n, i.e. $M + \epsilon' \geq \sup\{a_N, a_{N+1}, \ldots\}$. Second, that $M + \epsilon' < \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$, which means that more specifically $M + \epsilon' < \sup\{a_N, a_{N+1}, \ldots\}$. We have arrived at a contradiction. Therefore M cannot be smaller than the lim sup, leading us to conclude that these must in fact be equal.
For lim inf, the corresponding result is as follows. m equals $\liminf_{n\to\infty} a_n$ if and only if for every $\epsilon > 0, a_n < m + \epsilon$ for infinitely many n and $a_n > m - \epsilon$ for all but finitely many n. The proof would use symmetrical arguments.

Exercise 27
Prove that every sequence of real numbers (a_n) has a subsequence (a_{n_k}) that converges to

$$\limsup_{n\to\infty} a_n$$

[Hint: If $M = \limsup_{n\to\infty} a_n = \pm\infty$, we must interpret the conclusion loosely; this case is handled in exercise 25. If $M \neq \pm\infty$, use (*) to choose (a_{n_k}) satisfying $|a_{n_k} - M| < 1/k$, for example. There is also a subsequence that converges to $\liminf_{n\to\infty} a_n$. Why?]

Solution.
Firstly, as mentioned in the hint let us categorize the cases of $\pm\infty$. One, if lim sup equals $+\infty$ then (a_n) has a subsequence diverging to $+\infty$. Two, if lim sup equals $-\infty$, then the sequence itself diverges to $-\infty$. Now let's examine the case where lim sup equals some real number M. We consider the following sequence: $\epsilon_k = \frac{1}{k}$. By the characterization of the supremum, there exist infinitely many n such that $M - \epsilon_1 < a_n$, and it also holds that *for all but* finitely many $n, a_n < M + \epsilon_1$. Crucially, the second observation means that there exists a maximum N for which the second inequality *does not* hold. Therefore, the second inequality holds for all $n > N$ for some N. Additionally, the first inequality holds for infinitely many n, and thus infinitely many of those have to be greater than N. Therefore, there exist infinitely many n such that $|a_n - M| < \epsilon_1$. Pick the first of those and call it a_1.
Precisely because these elements are always infinite in number, for any $k > 1$ we will always be able to pick an a_k as above with the additional constraint that a_k has not been picked before. Repeating this procedure will yield a subsequence (a_{n_k}) that clearly converges to M because $e_k \to 0$ (and also decrease monotonically). This concludes the proof.
Because a symmetric characterization exists for lim inf, and because a symmetric version of exercise 25 also exists for it, there will also be a subsequence that converges to lim inf, regardless of whether it equals $\pm\infty$ or some real number.

Exercise 29
If (a_{n_k}) is a convergent subsequence of (a_n), show that $\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$.

Solution.
By exercise 23, we know that $\liminf_{k\to\infty} a_{n_k} = \limsup_{k\to\infty} a_{n_k} = \lim_{k\to\infty} a_{n_k}$, since (a_{n_k}) converges. Recall the definition of lim sup for $(a_n), (a_{n_k})$:

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}, \limsup_{k\to\infty} a_{n_k} = \inf_{k\geq 1}\{\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\}\}$$

We claim that the second lim sup equals at most the first one. To see why this is the case, observe that any $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\}$ is the supremum of a number of elements of (a_n) starting at the n_k-th one and possibly excluding some elements after that. This means that $\{a_{n_k}, a_{n_{k+1}}, \ldots\} \subset \{a_{n_k}, a_{n_k+1}, \ldots\}$, and thus by exercise 2, $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\} \leq \sup\{a_{n_k}, a_{n_k+1}, \ldots\}$. Now suppose that the second lim sup was larger than the first. This would mean that at least one of $\sup\{a_n, a_{n+1}, \ldots\}$ is smaller than $\limsup_{k\to\infty} a_{n_k}$ (such that it cannot be the largest *lower* bound for the set of suprema of subsequences of (a_n)). Suppose that this happens for $n = N$.
But then because (a_{n_k}) has infinite terms, it has to be the case that for some $k, n_k > N$. Then $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\} \leq \sup\{a_{n_k}, a_{n_k+1}, \ldots\} \leq \sup\{a_N, a_{N+1}, \ldots\} < \limsup_{k\to\infty} a_{n_k}$. The first equality here was proved above, while the second is again based on exercise 3 applied on two subsequences, one starting at N and one starting at n_k. But then $\limsup_{k\to\infty} a_{n_k}$ is not a lower bound for the suprema of the subsequences of (a_{n_k}), which is a contradiction. Therefore, it has to be the case that $\limsup_{n\to\infty} a_n \leq \limsup_{k\to\infty} a_{n_k}$.
As stated in the beginning, by the convergence of (a_{n_k}) this also implies that $\lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$.
The proof for the inequality involving the infimum makes use of exercise 23 and a completely symmetric argument.

Exercise 31
If (a_n) is convergent and (b_n) is bounded, show that $\text{limsup}_{n\to\infty}(a_n+b_n) \leq \lim_{n\to\infty} a_n+\text{limsup}_{n\to\infty}b_n$.

Solution.
First of all, because a_n is convergent, it is also bounded. Since b_n is bounded as well, their sum is bounded. This means that a_n+b_n has a limit supremum that is indeed a real number. Then, by exercise 27, we know that the sequence a_n+b_n has a subsequence $a_{n_k}+b_{n_k}$ that converges to $\text{limsup}_{n\to\infty}(a_n+b_n)$. Because a_n converges, the subsequence a_{n_k} that corresponds to the subsequence of the sum also converges to the limit of a_n. Consequently, the subsequence b_{n_k} that corresponds to the subsequence of the sum also converges as a difference of convergent sequences. Observe that this means that:

$$\text{limsup}_{n\to\infty}(a_n+b_n) = \lim_{k\to\infty}(a_{n_k}+b_{n_k}) = \lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k} = \lim_{n\to\infty} a_n + \lim_{k\to\infty} b_{n_k}$$

Now, from exercise 29, because b_{n_k} is a convergent subsequnce of b_n we conclude that $\lim_{k\to\infty} b_{n_k} \leq \text{limsup}_{n\to\infty}b_n$. Combining these two observations we obtain that:

$$\text{limsup}_{n\to\infty}(a_n+b_n) \leq \lim_{n\to\infty} a_n + \text{limsup}_{n\to\infty}b_n$$

Exercise 33
Show that (x_n) converges to $x \in \mathbb{R}$ if and only if every subsequence (x_{n_k}) of (x_n) has a *further* subsequence $(x_{n_{k_l}})$ that converges to x.

Solution.
$\implies$: First suppose that (x_n) converges to $x \in \mathbb{R}$. Pick any subsequence (x_{n_k}), and pick any $\epsilon > 0$. There exists $N > 0$ such that for all $n > N$ it holds that $|x_n - x| < \epsilon$. Because (x_{n_k}) has infinite terms, there exist infinitely many $n_k > N$, and for all of these, by the definition of subsequences, the above inequality holds. Therefore, any subsequence of (x_n) converges to x. Any sequence is a trivial subsequence of itself, and thus we have completed the proof in this direction.
$\impliedby$: Conversely, suppose that every subsequence (x_{n_k}) of a sequence (x_n) has a subsequence $(x_{n_{k_l}})$ that converges to $x \in \mathbb{R}$. Suppose by way of contradiction that (x_n) does not converge to $x \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that for all $N > 0$ there exists $n > N$ for which $|x_n - x| \geq \epsilon$. Construct the following subsequence (x_{n_k}) of (x_n): the i-th term is the first element of (x_n) with $n > i$ such that $|x_n - x| \geq \epsilon$. By the observation above, this is always well defined.
By construction, all elements of this (x_{n_k}) are at least ϵ-away from x. Therefore, no subsequence $(x_{n_{k_l}})$ of (x_{n_k}) can ever converge to x. which directly contradicts our hypothesis, leading us to conclude that (x_n) converges to $x \in \mathbb{R}$.

Exercise 34
Suppose that $a_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n < \infty$.
(i) Show that $\liminf_{n\to\infty} na_n = 0$.
(ii) Give an example showing that $\limsup_{n\to\infty} na_n > 0$ is possible.

Solution.
(i) In order to show that $\liminf_{n\to\infty} na_n = 0$, we need to show two things. One, that for every $\epsilon > 0, na_n > 0 - \epsilon$ holds for all but finitely many n and two, that for every $\epsilon > 0, na_n < 0 + \epsilon$ holds for infinite n. The first inequality is obvious, since $a_n \geq 0$. Suppose then that the second inequality does not hold. This means that there exists $\epsilon > 0$ for which $na_n \geq \epsilon$ for only a finite number of n. Call these $n_1, n_2, \dots n_k$. Then, observe that for all $n > n_k$, it must be the case that $na_n \geq \epsilon \implies a_n \geq \frac{\epsilon}{n}$. Then:

$$\sum_{n>n_k} a_n \geq \sum_{n>n_k} \frac{\epsilon}{n} = \epsilon \sum_{n>n_k} \frac{1}{n}$$

We know, however, that this infinite sum diverges (since the infinite sum of all $\frac{1}{n}$ diverges), whereas our LHS here was assumed to converge. We have thus arrived at a contradiction, which means that the second inequality must indeed hold, and thus $\liminf_{n\to\infty} a_n = 0$.

(ii) Consider the sequence $a_n = \frac{1}{n}$ for $n = 2^k, k = 1, 2, \dots$ and $a_n = 0$ for every other n. Observe then that the corresponding series is a geometric series, and thus converges. At the same time, observe that $na_n = 1$ for $n = 2^k, k = 1, 2, \dots$ and $na_n = 0$ for every other n. Clearly then it is the case that $\limsup na_n = 1 > 0$.

Exercise 35
(The ratio test): Let $a_n \geq 0$.
(i) If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, show that $\sum_{n=1}^{\infty} a_n < \infty$.
(ii) If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, show that $\sum_{n=1}^{\infty} a_n$ diverges.
(iii) Find examples of both a convergent and a divergent series having $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

Solution.
(i) Observe, first of all, that $a_n \geq 0$. This means that the corresponding sequence of partial sums is non-decreasing. If we can show that it is also bounded, then we will have proved that it converges, which by definition means that $\sum_{n=1}^{\infty} a_n < \infty$.
By our hypothesis, we have that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. Call this quantity M. Then we can always find some $\epsilon > 0$ such that $M + \epsilon < 1$. For this ϵ, we apply the characterization of lim sup on the sequence $\frac{a_{n+1}}{a_n}$. This means that for all but finitely many n, it holds that $\frac{a_{n+1}}{a_n} < M + \epsilon$. Call N the largest n for which this does not hold, and then observe that we have that $a_{N+2} < a_{N+1}(M + \epsilon)$. This implies then that $a_{N+3} < a_{N+2}(M + \epsilon) < a_{N+1}(M + \epsilon)^2$. More generally, if $N' = N + 1$ for $k \geq 1$ we have that $a_{N'+k} < a_{N'}(M + \epsilon)^k$.
Then for the sequence of partial sums we have that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{N} a_i + \sum_{i=N'}^{n} a_i < \sum_{i=1}^{N} a_i + \sum_{k=1}^{n-N'} a_{N'}(M + \epsilon)^k$$

The first term here is clearly bounded as a finite sum. The second term can also be bounded by the *infinite* sum for k, precisely because it corresponds to a geometric series with ratio $M + \epsilon < 1$. But then this means that the sequence of partial sums is indeed bounded, and thus it must converge.
(ii) Call the lim inf m. Then $M > 1$ and we can always find $\epsilon > 0$ such that $m - \epsilon > 1$. Then, by the characterization of lim inf, for all but finitely many n it holds that $\frac{a_{n+1}}{a_n} > m - \epsilon$. Call N the first n after which (and including it) this holds. Then observe that, similarly to (i), for $k \geq 1$ we have that $a_{N+k} > (m - \epsilon)^k a_N$. But because $m - \epsilon > 1$ this implies that the terms of the sequence are not bounded, and hence the infinite series must necessarily diverge.
(iii) Consider the series corresponding to the sequence $a_n = 1$. Clearly, the ratio of two subsequent terms is 1, but the series obviously diverges.
Consider also the series corresponding to the sequence $a_n = \frac{1}{n^2}$, which we take as known that it converges to $\frac{\pi^2}{6}$. Observe that:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{n^2 + 2n + 1}$$

, which can easily be shown to converge to 1 as $n \to \infty$.

Exercise 37
If (E_n) is a sequence of subsets of a fixed set S, we define

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} (\bigcup_{k=n}^{\infty} E_k) \text{ and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} E_k)$$

Show that

$$\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n \text{ and that } \liminf_{n\to\infty} (E_n^c) = (\limsup_{n\to\infty} E_n)^c$$

Solution.
For the first part, we have the following. Suppose $x \in \liminf_{n\to\infty} E_n$. We need to show that $x \in \limsup_{n\to\infty} E_n$. By this definition of lim inf, it must be the case that $x \in \bigcap_{k=n}^{\infty} E_k$ for at least one $n \geq 1$.

This in turn means that there exists $n \geq 1$ such that $x \in E_n, E_{n+1}, \ldots$. Now observe that the following hold:

- For $l \leq n$, it is the case that $E_n \subset \bigcup_{k=l}^{\infty} E_k$. Therefore, $x \in \bigcup_{k=l}^{\infty} E_k, l \leq n$.
- For $i \geq 1$, it is the case that $E_{n+i} \subset \bigcup_{k=n+i}^{\infty} E_k$. Therefore, $x \in \bigcup_{k=n+i}^{\infty} E_k$. This can equivalently be written as $x \in \bigcup_{k=l}^{\infty} E_k, l > n$.

But then we have shown that $x \in \bigcup_{k=n}^{\infty} E_n$ for any $n \geq 1$, which means that, by the definition of lim sup used here, $x \in \limsup_{n\to\infty} E_n$ (since this is the intersection of all of these sets). Since x was selected as an arbitrary element of $\liminf_{n\to\infty} E_n$, we've shown that $\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n$.
For the second part, we will use De Morgan's laws, which we know hold for infinite unions and intersections as well (this is easy to prove using the same arguments as for finite unions and intersections). Namely, we have that:

$$\lim \inf_{n\to\infty}(E_n^c) = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} E_k^c) = \bigcup_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k)^c = (\bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k))^c = (\lim \sup_{n\to\infty} E_n)^c$$

1.2 Limits and Continuity

Exercise 40
Prove the following theorem (1.17):
Let f be a real-valued function defined in some punctured neighborhood of $a \in \mathbb{R}$. Then, the following are equivalent:
(i) There exists a number L such that $\lim_{x\to a} f(x) = L$ (by the $\epsilon - \delta$ definition).
(ii) There exists a number L such that $f(x_n) \to L$ whenever $x_n \to a$, where $x_n \neq a$ for all n.
(iii) $(f(x_n))$ converges (to something) whenever $x_n \to a$, where $x_n \neq a$ for all n.

Solution.
(i) $\implies$ (ii): Pick any sequence $x_n \to a$, with $x_n \neq a$ for all of its terms. For any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x_n - a| < \delta$ we have that $|f(x_n) - L| < \epsilon$. Since $x_n \to a$, for this $\delta > 0$ there exists $N > 0$ such that for $n > N$ we have that $0 < |x_n - a| < \delta$. But then it also holds that for $n > N$, $|f(x_n) - L| < \epsilon$, which means that for any $\epsilon > 0$ we are able to find $N > 0$ such that the definition of the limit for $(f(x_n))$ holds, with the limit value being L.
Therefore, whenever $x_n \to a, x_n \neq a$, it is also the case that $f(x_n) \to L$.
(ii) $\implies$ (iii): Pick any sequence $(f(x_n))$ such that the corresponding (x_n) converges to a and such that $x_n \neq a$. Then, applying (ii) one can obtain that $f(x_n) \to L$, which means indeed that $(f(x_n))$ converges to *something*.
(iii) $\implies$ (ii): Pick any two sequences $x_n \to a, y_n \to a$, such that $x_n \neq a, y_n \neq a$ for all n. We then have that $f(x_n) \to L_1, f(y_n) \to L_2$. As pointed out in the book, construct the sequence $x_1, y_1, x_2, y_2, \ldots$ by "interlacing" $(x_n), (y_n)$. It's easy to see that this sequence converges to a as well. Therefore, $f(z_n) \to L_3$. But now observe that both $f(x_n)$ and $f(y_n)$ are subsequences of $f(z_n)$, which means more specifically that all three of them must converge to the same limit, i.e. $L_1 = L_2 = L_3$.
We have thus shown that for any two sequences $x_n \to a, y_n \to a$, the corresponding $(f(x_n)), (f(y_n))$ always converge to the same limit, which is an equivalent way of stating (ii).
To complete the full equivalence of the theorem, we will now show that (ii) $\implies$ (i). To do this, suppose that (i) does not hold. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ it holds that for some x it is the case that both $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. Construct a sequence of $\delta_i = \frac{1}{i}$, and from that construct a corresponding sequence of $i \to x_i$, where each x_i fulfills the above mentioned conditions. Then clearly the sequence (x_i) converges to a, but the corresponding $(f(x_i))$ does not converge to L. This directly contradicts (ii), and we have thus shown that (ii) does indeed imply (i).

Exercise 45
Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(x) = 0$ whenever x is rational. Show that $f(x) = 0$ for every $x \in [a, b]$.

Solution.
Pick any $x \in [a, b]$. If $x \in \mathbb{Q}$, we immediately know that $f(x) = 0$. If $x \notin \mathbb{Q}$, then we know from a previous result that we can always approach the real number x with a sequence of rational numbers (x_n). In other words, $x_n \to x, x_n \in \mathbb{Q}$ for all x_n. Because f is continuous, it must then hold that $f(x_n) \to f(x)$. Observe that the sequence of $f(x_n)$ is a sequence of zeros, since all $x_n \in \mathbb{Q}$. Clearly then the limit has to be zero as well, which directly means that $f(x) = 0$, thus showing that $f(x)$ is indeed the zero function on $[a, b]$.

Exercise 46
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some open interval $(-a, a)$.
(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≥ 0" replaced by "> 0"? Explain.

Solution.
(a) Suppose for the sake of contradiction that this does not hold. This is equivalent to saying that for any $a \in \mathbb{R}$, there exist some $x_a \in (-a, a)$ such that $f(x_a) \leq 0$. Consider constructing the following sequence of a_i: $a_i = \frac{1}{n}$. Then, in every one of these intervals there has to exist x_i such that $f(x_i) \leq 0$. Observe that $a_i \to 0$, which means also that $x_i \to 0$. Since f is continuous, this means that it has to be the case that $f(x_i) \to f(0) > 0$. However, this is by construction a sequence of non-positive numbers, and because limits preserve non-strict inequalities, it cannot be the case that the limit of $(f(x_i))$ is positive. We arrive at a contradiction, and therefore it must be the case that for some a, $f(x) > 0$ for all $x \in (-a, a)$.
(b) Again, for an $x \in \mathbb{R} \setminus \mathbb{Q}$, we can approach it with a sequence of rationals (x_n). Then it is the case that $f(x_n) \geq 0$ for all n, and because f is continuous it also holds that $f(x_n) \to f(x)$. Again, limits preserve non-strict inequalities, and thus $f(x) \geq 0$ holds for all real x.
Now we consider whether this is true if we alter "$\geq$" to "$>$". Consider the function $f(x) = |x - \sqrt{2}|$. Clearly, this function is continuous, and also positive whenever $x \neq \sqrt{2}$ and zero for $x = \sqrt{2}$. More specifically, it's a function that is strictly positive for rational x. However, for the real $x = \sqrt{2}$, $f(\sqrt{2}) = 0$, thus providing a counterexample to the claim "if $f(x) > 0$ for every rational x, show that $f(x) > 0$ for all real x".

Exercise 50
Let D denote the set of rationals in $[0, 1]$ and suppose that $f : D \to \mathbb{R}$ is increasing. Show that there is an increasing function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ whenever x is rational. [Hint: For $x \in [0, 1]$, define $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}]$.

Solution.
Define g as in the provided hint. We first show that if $x \in \mathbb{Q}, f(x) = g(x)$. We have that $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}\}$. Recall that f is increasing, therefore, for all $0 \leq t \leq x, t \in \mathbb{Q}$ it has to hold that $f(t) \leq f(x)$. Therefore $f(x)$ is an upper bound for the set over which we are computing the supremum. For any $s < f(x)$, it is clear that s cannot be an upper bound for this set, because $f(x)$ belongs in it. We conclude that $f(x)$ is indeed the supremum, and thus that $g(x) = f(x)$.
Now to show that g is increasing, pick $x_1 < x_2, x_1, x_2 \in [0, 1]$ and call the sets over which $g(x_1), g(x_2)$ are computed S_1, S_2 respectively. If $y \in S_1$, we have that $y = f(t)$ for some $0 \leq t \leq x_1, t \in \mathbb{Q}$. Since $x_1 < x_2$, this means that $y \in S_2$ as well. Therefore $S_1 \subset S_2$, which means that $g(x_1) \leq g(x_2)$ by the properties of the supremum. Therefore g is indeed an increasing function.

Exercise 51
Let $f : [a, b] \to \mathbb{R}$ be increasing and define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x+)$ for $a \leq x < b$. and $g(b) = f(b)$. Prove that g is increasing and right-continuous.

Solution.
Before we begin we note that by preposition 1.19, $f(x+)$ is well-defined for all $x \in [a, b)$. First we will show that g is increasing. Pick $x_1, x_2 \in [a, b], x_1 < x_2$. We need to show that $g(x_1) \leq g(x_2)$, or, equivalently, that $f(x_1+) \leq f(x_2+)$. We will do this in two steps.
First, we will show that for every $x \in [a, b), f(x+) \geq f(x)$. Suppose this was not the case for some x, which would mean that $f(x+) < f(x) \implies 0 < f(x) - f(x+) = \epsilon$. Let $L = f(x+)$ to ease notation. Because L is well-defined, for this ϵ there exists $\delta > 0$ such that for all $y, x < y < x + \delta$ it holds that $|f(y) - L| < \epsilon$. More specifically, this implies that $f(y) < L + \epsilon = L + f(x) - L \implies f(y) < f(x)$. This contradicts the fact that f is increasing.
Secondly, we will show that for all $x \in [a, b), y \in [a, b], x < y$ it holds that $f(x+) \leq f(y)$. Suppose again that for some such x, y this does not hold, which means that $f(x+) > f(y)$, and again for ease of notation let $L = f(x+)$. Firstly, let $\delta_1 = y - x > 0$. Secondly, let $\epsilon = L - f(y) > 0$. Because L is well-defined, for this ϵ there exists $\delta_2 > 0$ such that for all $z, x < z < x + \delta_2$ it holds that:

$$|f(z) - L| < \epsilon \implies f(z) > L - \epsilon = L - L + f(y) \implies f(z) > f(y)$$

Set $\delta = \min\{\delta_1, \delta_2\}$, which means that $z < x + \delta \leq x + \delta_1 = y$ and also that, as shown above, $f(z) > f(y)$. Again, this directly contradicts the fact that f is increasing.
We can now complete the proof by observing that $g(x_1) = f(x_1+) \leq f(x_2)$ by using the second of the two proven lemmas, and then that $f(x_2) \leq f(x_2+) = g(x_2)$ by using the first of the two proven lemmas. Note that the last step is omitted if $x_2 = b$, in which case $f(x_2) = f(b) = g(b)$.
Now, because g is increasing in $[a, b]$, we know by proposition 1.19 that $g(x+)$ always exists for $x \in [a, b)$. We thus only need to show that $g(x+) = g(x)$. Because $f(x+)$ is well defined, for any given $\epsilon > 0$, there exists $\delta > 0$ such that for $x < y < x + \delta$ it holds that $|f(y) - f(x+)| < \epsilon$. Set $\delta' = \frac{\delta}{2} < \delta$. Then for all $y, x < y < x + \delta'$ it is the case that there exists $z \in (x, x + \delta)$ such that $y < z$. By the second of the two lemmas above, $g(y) = f(y+) \leq f(z) < f(x+) + \epsilon$. By the first of the two lemmas above, it is furthermore true that $g(y) = f(y+) \geq f(y) > f(x+) - \epsilon$.
Putting these together we obtain that $f(x+) - \epsilon < g(y) < f(x+) + \epsilon$, and since $f(x+) = g(x)$ this means that by picking $\delta' = \frac{\delta}{2}$ the definition of right continuity for g at x is satisfied.

Chapter 2

Countable and Uncountable Sets

2.1 Equivalence and Cardinality

Exercise 3
Given finitely many countable sets $A_1, \ldots, A_n$, show that $A_1 \cup \ldots \cup A_n$ and $A_1 \times \ldots \times A_n$ are countable sets.

Solution.
We will use induction on n:

- Base case, for $n = 2$: Consider two countable sets, A_1, A_2. If both of them are finite, then their union and Cartesian product are also finite, and thus trivially countable.

  If exactly one is finite, say $A_2 = \{a'_1, a'_2, \ldots, a'_n\}$, then observe that $A_1 \cup A_2$ contains at most all elements of A_1 and all elements of A_2, and possibly fewer if their intersection is not empty. In any case, suppose $|A_1 \cap A_2| = m$, and name f the bijection from A_1 to $\mathbb{N}$. Then let $S = A_2 \setminus A_1 = \{a'_{k_1}, \ldots a'_{k_{n-m}}\}$, and $f'(a'_{k_1}) = 1, \ldots f'(a'_{k_{n-m}}) = n - m$, in which case $A_1 \cup A_2 = A_1 \cup S$, and A_1, S have an empty intersection. Let $g : A_1 \cup S \to \mathbb{N}$ be such that:

  $$g(a) = \begin{cases} f(a) + (n-m) & , a \in A_1 \\ f'(a) & , a \in S \end{cases}$$

  Because f is a bijection, one can clearly see that g is surjective. Furthermore, g is one-to-one because f, f' are one-to-one and because the two "branches" of g have no overlapping values ($\min\{f(a) + (n-m)\} = n - m + 1 > n - m = \max\{f'(a)\}$). Therefore g is a bijection from $A_1 \cup A_2$ to $\mathbb{N}$, which means precisely that the union is countable.

  Now for the case where both A_1, A_2 are infinite, there exist bijections $f_1 : A_1 \to \mathbb{N}, f_2 : A_2 \to \mathbb{N}$. These impose orders $f_1(a_1) = 1, f_1(a_2) = 2, \ldots$ for $a'_i \in A_1$ and $f_2(a'_1) = 1, f_2(a'_2) = 2$ for $a_i \in A_2$. One can again set $S = A_2 \backslash A_1$, and f_2 can again be used to extract an order for the elements $s_1, s_2, \ldots$ of S. If S is finite, the problem reduces to the case above. If S is also infinite, we know that S is also countable, with $f'_2 : S \to \mathbb{N}$ the corresponding bijection. Additionally, $A_1 \cup A_2 = A_1 \cup S = \{a_1, s_1, a_2, s_2, \ldots\}$ ands $A_1 \cap S = \emptyset$. Use the orders imposed by f_1, f'_2 to sort the elements of A_1 in the order $a_1, a_2, \ldots$ and the elements of S in the order $s_1, s_2, \ldots$. Then define $g : A_1 \cup A_2 \to \mathbb{N}$ as:

  $$g(a) = \begin{cases} 2i, & a = a_i \in A_1 \\ 2i + 1, & a = s_i \in S \end{cases}$$

  For $n \in \mathbb{N}$, if n is even the equation $n = g(a)$ has a unique solution for $a = a_n$, due to the bijectivity of f_1. If n is odd, the equation $n = g(a)$ has a unique solution for $a = s_n$, due to the bijectivity of f'_2. In any case, $A_1 \cup A_2$ has been shown to be equivalent to $\mathbb{N}$.

  For the Cartesian product, the case where both A_1, A_2 are finite is again trivial. If A_1 is infinite and A_2 finite, then:

  $$A_1 = \{a_1, a_2, \ldots\}, A_2 = \{a'_1, a'_2, \ldots, a'_n\}$$

Let then $g : A_1 \times A_2 \to \mathbb{N}$:

$$g(a_i, a'_j) = n(i-1) + (j-1),\ j = 1, 2, \ldots, n$$

Observe that the fact that $A_1 \sim \mathbb{N}$ and j only takes a finite number of values makes g a surjection. Indeed, for $x = k \cdot n + l, l = 0, 1, \ldots, n-1$ (here we use the division algorithm for integers), one has but to set $i = 1+k$ (possible due to $A_1 \sim \mathbb{N}$ and $j = l+1$ (always possible due to the range of values l can achieve) to obtain $g(a_i, a'_j) = x$.

To show that g is injective, suppose $g(a_i, a'_j) = g(a_k, a'_l)$, and we then have that:

$$n(i-1) + (j-1) = n(k-1) + (l-1) \implies n(i-k) = l - j$$

This implies that the RHS is a multiple of n, which, because $1 \leq l, j \leq n$ is only possible if $l = j$. But then we also have that $i = k$, thus that $(a_i, a'_j) = (a_k, a'_l)$, i.e. that g is injective.

g is therefore a bijection, and thus $A_1 \times A_2 \sim \mathbb{N}$.

If both A_1, A_2 are infinite, then consider the function $g : A_1 \times A_2 \to \mathbb{N}$:

$$g(a_i, a'_j) = 2^i(2j-1)$$

, which, because of the fact that $A_1 \sim \mathbb{N}, A_2 \sim \mathbb{N}$ (thus i, j take all natural numbers as values), and by a proof completely analogous to $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ can be shown to be a bijection, thus proving that $A_1 \times A_2 \sim \mathbb{N}$.

- Inductive step: If this holds for $n = k \geq 2$, then observe that $A_1 \cup \ldots \cup A_{k+1} = (A_1 \cup \ldots \cup A_k) \cup A_{k+1}$, and we can therefore apply the inductive hypothesis to the sets $A_1, \ldots A_k$, and the base case to the union of those with A_k to obtain the statement for $n = k+1$ as well. The same argument applies to the Cartesian product as well, thus concluding the proof.

Exercise 5
Prove that a set is infinite if and only if it is equivalent to a proper subset of itself.
[Hint: If A is infinite and $x \in A$, show that A is equivalent to $A \setminus \{x\}$.]

Solution.
$\implies$: Suppose A is infinite. Then A contains an element x, and the set $S = A \setminus \{x\}$ must also be infinite. By exercise 4, the set S contains a countably infinite subset S'. We then have that:

$$S = S' \cup (S \setminus S'), A = S' \cup (S \setminus S') \cup \{x\}$$

, where all of the sets used in the unions are disjoint. By exercise 3, it holds that $S' \cup \{x\} \sim S'$, and this yields a corresponding bijection $g : S' \cup \{x\} \to S'$. Consider then the following function $f : A \to S$:

$$f(z) = \begin{cases} z, & z \in S \setminus S' \\ g(z), & z \in S' \cup \{x\} \end{cases}$$

, which, by the above observations regarding the bijectivity of g and the disjointness of the sets $S', S \setminus S', \{x\}$, means that f is a bijection as well, proving that $A \sim S$.
$\impliedby$: Now suppose that a set A is equivalent to S, S being a proper subset of A. By contradiction, suppose A is not infinite and that it contains n elements. Then S must contain at most $n-1$ elements. By the pigeonhole principle, there cannot exist a bijection from A to S, which means that $A \sim S$ cannot be true, a contradiction. Therefore A must be infinite.

Exercise 8
Show that $(0, 1)$ is equivalent to $[0, 1]$ and to $\mathbb{R}$.

Solution.
First, observe that $[0,1] = \{0\} \cup \{1\} \cup (0,1)$, and that $[0,1], [0,1), (0,1)$ are all infinite sets. By exercise 5, we have that $[0,1) \sim (0,1)$ (by explicitly picking $x = 0$ and $S = (0,1)$. Similarly, we also have that $[0,1] \sim [0,1)$ (by explicitly picking $x = 1$ and $S = [0,1)$). The transitivity property of the equivalence relation "is equivalent to" (exercise 1) yields then that $[0,1] \sim (0,1)$.
For $(0,1) \sim \mathbb{R}$, consider the function $f : \mathbb{R} \to (0,1), f(x) = \frac{1}{1+e^{-x}}$, which is known to be both injective and surjective, thus sufficing to show that $(0,1) \sim \mathbb{R}$.

Exercise 9
Show that $(0,1)$ is equivalent to the unit square $(0,1) \times (0,1)$.
[Hint: "Interlace" decimals, but carefully!]

Solution.
Pick any $x \in (0,1) \times (0,1)$. Then $x = (a,b), a, b \in (0,1)$. If a or b can be written as a decimal ending in infinite 9's (e.g. $0.3999\ldots$), write them in the equivalent form that features finite decimals (in this example, 0.4). Now consider the function $f : (0,1) \times (0,1) \to (0,1)$:

$$f(0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots) = 0.a_1b_1a_2b_2a_3b_3\ldots$$

Observe first that for any $x \in (0,1)$, x can be written as $x = 0.x_1x_2x_3\ldots$, and again if this can be written as ending in infinite 9's, "round it up". Then $f(0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots) = x$. Furthermore, suppose that $f(0.y_1y_2y_3\ldots, 0.z_1z_2z_3\ldots) = x = 0.x_1x_2x_3\ldots$. By definition, it is also the case that

$$f(0.y_1y_2y_3\ldots, 0.z_1z_2z_3\ldots) = 0.y_1z_1y_2z_2y_3z_3\ldots$$

Because we always choose to write decimals ending in infinite 9's as "rounded up", there can be no ambiguities here, in the sense that the equality of these two numbers directly implies that $y_1 = x_1, z_1 = x_2, y_2 = x_3, z_2 = x_4\ldots$ (i.e. any two decimals that do not end in infinite 9's can only be equal if all corresponding digits are equal). These two observations yields that f is bijective, thus proving the desired equivalence.

Exercise 10
Prove that $(0,1)$ can be put into one-to-one correspondence with the set of all *functions* $f : \mathbb{N} \to \{0,1\}$.

Solution.
To begin, we observe that a function from $\mathbb{N}$ to $\{0,1\}$ can be thought of a set of the form:

$$\{(0, v_0), (1, v_1), (2, v_2), \ldots\}$$

, where each v_i is the value of the function on i, and as such can be either 0 or 1. We now observe that the set of all such functions can be written as $S \cup \overline{S}$, where $S = \{f \in \{0,1\}^{\mathbb{N}} | \exists k \in \mathbb{N} \cup \{-1\}, f(i) = 1, i > k\}$ (in other words, a function either becomes 1 "forever" after some point, or it does not). Note that from now on we will use the binary system in this exercise.
Observe that all functions that belong in S are uniquely characterized by a "finite-length prefix" of values. This means that we can list all of them: start with a prefix of length 0, list all possible f such that in the above definition $k = -1$, then list all f with a prefix of length 1, etc. Since each length N yields only a finite number of functions (each corresponds to a binary number with N digits such that the last digit is not 1), we conclude that S is in fact countable, and that there exists a bijection $g : S \to \mathbb{N}$. Define now the following mapping $F : \{0,1\}^{\mathbb{N}} \to (0,1) \cup \mathbb{N}$:

$$F(f) = \begin{cases} 0.v_0v_1v_2\ldots, & f \in \overline{S} \\ g(f), & f \in S \end{cases}$$

By the bijectivity of g, F achieves all natural numbers. Furthermore, for any $x \in (0,1)$, we can always write $x = 0.b_0b_1\ldots$ such that it does not end in infinite ones, and then we can define an $f : \mathbb{N} \to \{0,1\}$

such that $f(i) = b_i$, which means that $F(f) = x$. We have thus shown that F is surjective. Additionally, we observe that due to our construction, if $f_1 \in \overline{S}, f_2 \in S, F(f_1), F(f_2)$ can never be equal: one is always in $(0,1)$ and the other is a non-zero natural number. For $f_1, f_2 \in \overline{S}, F(f_1) = F(f_2)$ either implies that all corresponding digits are equal, in which case clearly $f_1 = f_2$, or that one of them ends in infinite ones, which is impossible since $f_1, f_2 \in \overline{S}$. Finally, for $f_1, f_2 \in S$, the bijectivity of g implies that $f_1 = f_2$.
This shows that F is also injective, which means it is a bijection from $\{0,1\}^{\mathbb{N}}$ to $(0,1) \cup \mathbb{N}$. Now , $\mathbb{N}$ is clearly countable, and from exercise 6 we then have that $(0,1) \cup \mathbb{N} \sim (0,1)$, which by exercise 1 also gives us our desired result, $\{0,1\}^{\mathbb{N}} \sim (0,1)$.

Exercise 13
Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

Solution.
We know that there exist infinitely many primes, $p_1, p_2, \ldots$. Consider then forming the sets $S_i = \{p_i^k | k \in \mathbb{N} \setminus \{0\}\}$. These are clearly infinitely many, and each of them is infinite. Suppose that two of them, S_i, S_j had a non-empty intersection. Then there exists $x \in \mathbb{N}$ such that $x = p_i^k = p_j^l, k, l \geq 1$. But then this number has two different prime factorizations, which is a contradiction. Therefore all of these sets are pairwise disjoint.

Exercise 15
Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable. [Hint: Each one contains a rational!]

Solution.
Suppose that there exists an uncountable collection of pairwise disjoint and nonempty intervals of the form $(a, b), a, b \in \mathbb{R}$. As indicated in the hint, each such interval must contain a rational, regardless of whether a, b are rationals or not. Furthermore, because the intervals are all disjoint, no two such rationals can be equal. This means that there exist at least as many rationals as intervals, therefore an uncountable number of them, which we know is a contradiction.

2.2 The Cantor Set

Exercise 21
Show that any ternary decimal of the form $0.a_1a_2 \ldots a_n11$ (base 3), i.e., any finite-length decimal ending in two (or more) 1s is *not* an element of Δ.

Solution.
For this problem we will be using the following formalization of the Cantor set $\mathcal{C}$:

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \Big(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\Big)$$

We now consider a ternary decimal x of the form $x = 0.a_1 \ldots a_n11$, where n may be zero, in which case the "prefix" has zero length. Consider then what x equals:

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} = \frac{a_1 3^{n+1} + a_2 3^n + \ldots + a_n 3^2 + 3 + 1}{3^{n+2}}$$

$$= \frac{1}{3^{n+1}}(a_1 3^n + a_2 3^{n-1} + \ldots + a_n 3 + \frac{4}{3}) = \frac{1}{3^{n+1}}(3(a_1 3^{n-1} + a_2 3^{n-2} + \ldots + a_n) + \frac{4}{3})$$

Notice now that if $y = a_1 3^{n-1} + a_2 3^{n-2} + \ldots + a_n$, y is an integer that is at least 0 and at most $6\frac{3^n-1}{2} = 3(3^n - 1)$ (by using the geometric progression sum). But then this range in which y can move means that x is precisely equal to *one* of the left endpoints in the n-th union in the definition of $\mathcal{C}$ plus $\frac{4}{3}$, which, because $1 < \frac{4}{3} < 2$, means that x will always lie *inside* one of the intervals that comprise this union, which means of course that it lies outside of the Cantor set.

Exercise 22
Show that Δ contains no (nonempty) open intervals. In particular, show that if $x, y \in \Delta$, then there is some $z \in [0,1] \setminus \Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small".)

Solution.
Let $x, y \in \Delta$. Following the notation of the book, we name I_k the union of closed intervals that comprises the k-th "level" of the "tree" that creates the Cantor set. Then recall that $\Delta = \bigcap_{k=0}^{\infty} I_k$. Since $x, y \in \Delta$, it holds that $x, y \in I_k$ for all k. We now claim that there exists k such that x, y belong in two disjoint intervals that comprise the union I_k. This will of course imply that there exists a "removed middle third" interval between x, y at the k-th level, any of whose elements fulfill the condition $z \notin \Delta$. We do this by contradiction. Namely, suppose that x, y belong in the same closed interval at every level of the "tree", and call those intervals $I_1, I_2, \ldots$. By construction, these are nested intervals, and we know that I_n's length goes to zero as $n \to \infty$. But then the nested interval theorem dictates that the infinite intersection of $I_1, I_2, \ldots$ contains precisely one element, which means $x = y$, a contradiction.

Exercise 23
The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in Δ of the form $a/3^n$ for some integers n and $0 \le a \le 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as $0.a_1a_2\ldots a_{n+1}$ (base 3) where each a_k is 0 or 2, except a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2\ldots a_n1, 0.a_1a_2\ldots a_n2)$ where both entires are points of Δ written in base 3.

Solution.
We will apply induction on n, which corresponds to the "level" of the "tree" of the Cantor set. More specifically, we will prove the following.

- If $x \neq 0$ is a left endpoint of the Cantor set, that is, the the intersection forming the Cantor set contains it in the n-th "tree-level union " in some interval $[x, y]$, then it can be written in the form $0.a_1\ldots a_n2$ where $a_i \in \{0,2\}, i = 1, \ldots, n$.

- If $x \neq 1$ is a right endpoint of the Cantor set, that is, the intersection forming the Cantor set contains it in the n-th "tree-level union" in some interval $[y, x]$, then it can be written in the form $0.a_1\ldots a_n1$ where $a_i \in \{0,2\}, i = 1, \ldots, n$.

As stated, we will use induction on n starting at 0, which corresponds to the first level of the tree.

- **Base case**: if $n = 0$, then the corresponding union is $I_1 = [0, 0.1] \cup [0.2, 1]$ (written in ternary). Obviously, the only non-zero left endpoint is 0.2, which satisfies the claim stated above. Similarly, the only right endpoint not equal to 1 is 0.1, which also satisfies the claim.

- **Inductive hypothesis**: Suppose that for $n = k, k \ge 0$ it holds that $I_n = [0, x_1] \cup [x_2, x_3] \cup \ldots \cup [x_M, 1]$, where each left and right endpoint fulfill the conditions stated above (note that $M = 2^{n+1}$).

- **Inductive step**: Observe then that the intervals comprising I_{n+1} are formed by taking each interval $[x, y]$ comprising I_n and replacing it by $[x, x + 0.\underbrace{0\ldots0}_{n+1}1] \cup [y - \underbrace{0.0\ldots0}_{n+1}1, y]$. Therefore, new right endpoints are formed by taking either 0 or a previous left endpoint and adding $0.\underbrace{0\ldots0}_{n+1}1$. Clearly, adding such a number to 0 results in a right endpoint fulfilling the claim stated in the beginning. In the other case, we have an endpoint of the form $0.a_1\ldots a_n2 + 0.\underbrace{0\ldots0}_{n+1}1 = 0.a_1\ldots a_n21$, which based on the inductive hypothesis also fulfills the initial claim. Similarly, new left endpoints are formed by taking either 1 or previous right endpoints and subtracting $0.\underbrace{0\ldots0}_{n+1}1$. Subtracting this quantity from 1 results in $0.\underbrace{2\ldots 2}_{n+1}2$, which fulfills the claim for a left endpoint. On the other hand, subtracting it from a previous right endpoint results in an endpoint of the form $0.a_1 \ldots a_n 1 - 0.\underbrace{0\ldots 0}_{n+1}1 = 0.a_1 \ldots a_n 02$, which based on the inductive hypothesis also satisfies the claim for left endpoints.

This concludes the proof that endpoints of Δ have a ternary decimal expansion described in the exercise.

Exercise 26
Let $f : \Delta \to [0,1]$ be the Cantor function and let $x, y \in \Delta$ with $x < y$. Show that $f(x) \leq f(y)$. If $f(x) = f(y)$, show that x has two distinct ternary decimal representations. Finally, show that $f(x) = f(y)$ if and only if x, y are "consecutive" endpoints of the form $x = 0.a_1 a_1 \ldots a_n 1$ and $y = a_1 a_2 \ldots a_n 2$ (base 3).

Solution.
Let $x = 0.a_1 a_2 \ldots$ and $y = 0.b_1 b_2 \ldots$. Because $x, y \in \Delta$, we know that we can write x, y such that each of a_i, b_i is either 0 or 1. Now, because $x < y$ it has to be the case that for some $i \geq 1, a_i < b_i$ and $a_k = b_k$ for $k < i$. This can be seen by making a "digit-wise" comparison: the first decimal digit in which x, y differ determines which one is larger, since after that point the decimal digits get multiplied by smaller powers of 3 and the smaller number can never "catch up". The only possible exception would be if $x = 0.a_1 \ldots a_i 00 \ldots$ and $y = 0.a_1 \ldots (a_i - 1)22 \ldots$, but we know that this cannot be true since it would require $a_i = 1$ or $a_i - 1 = 1$, which is impossible due to the way we've written x, y. Now consider the decimal strings $x' = 0.(\frac{a_1}{2})(\frac{a_2}{2})\ldots, y' = 0.(\frac{b_1}{2})(\frac{b_2}{2})\ldots$. What the Cantor function does is interpret these as binary instead of ternary numbers, and set then $f(x) = x'_{\text{bin}}, f(y) = y'_{\text{bin}}$. But then $\frac{a_k}{2} = \frac{b_k}{2}$ for $k < i, \frac{a_i}{2} < \frac{b_i}{2}$. By this we have established that it is impossible that $f(x) > f(y)$, thus $f(x) \leq f(y)$. If it is the case that $f(x) = f(y)$, then for $k > i, \frac{a_k}{2} = 1, \frac{b_k}{2} = 0$ and $\frac{a_i}{2} = 0, \frac{b_i}{2} = 1$. Going back to the ternary numbers x, y, this implies that x is of the form:

$$x = 0.a_1 \ldots a_{i-1} 0222 \ldots,$$

which can also be written as $x = 0.a_1 \ldots a_{i-1} 1$. Furthermore, it implies that y is of the form:

$$y = 0.a_1 \ldots a_{i-1} 2000 \ldots,$$

which completes the proof that a) x has two distinct ternary decimal expansions, and that whenever $f(x) = f(y), x, y$ are endpoints of "discarded" middle third intervals. For the other direction of the last equivalence, if x, y are endpoints of "discarded" middle third intervals, write both of them such that they only contain 0s and 2s in ternary, apply f and observe that $f(x)$ ends in infinite 1s, $f(y)$ in 0s, and for some $i, f(x)_i 0, f(y)_i = 1$ and $f(x)_k = f(y)_k, k < i$, which of course means that $f(x) = f(y)$.

Exercise 29
Prove that the extended Cantor function $f : [0,1] \to [0,1]$ (as defined above) is increasing. [Hint: consider cases.]

Solution.
For any two $x, y \in [0,1]$, with $x < y$, we have the following:

- If $x, y \in \Delta$, we have shown in exercise 23 that $f(x) \leq f(y)$.
- If $x \in \Delta, y \notin \Delta$, then $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$. This means that $f(x) \in \{f(z), z \leq y, z \in \Delta\}$, and thus it must be the case that $f(x) \leq f(y)$.
- If $x, y \notin \Delta$, then $f(x) = \sup\{f(z), z \leq x, z \in \Delta\}$ and $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$. But then since $x < y$, the set over which the second supremum is computed is a superset of the set over which the first supremum is computed, meaning that $f(x) \leq f(y)$.

2.3 Monotone Functions

Exercise 34
Let $D = \{x_1, x_2, \ldots\}$, and let $\epsilon_n > 0$ with $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Define $f(x) = \sum_{x_n \leq x} \epsilon_n$ (as above). Check the following:
(i) f is discontinuous at the points of D
(ii) f is right-continuous everywhere
(iii) f is continuous at each point $x \in \mathbb{R} \setminus D$
How might this construction be modified to yield a *strictly* increasing function with these same properties?

Solution.
(i) Notice that we've already seen in the main text that for $x_k \in D, f(x_k-) = f(x_k) - \epsilon_k$, that is, the left limit of f at each point of D can never equal its value on the respective point, since $\epsilon_k > 0$. Therefore f is not continuous at the points of D.
(ii) Again, in the main text we've seen that f is right-continuous at the points of D. It suffices therefore to check that for $x \in \mathbb{R} \setminus D, f(x+) = f(x)$. Expanding this, we have:

$$f(x+) = \lim_{y \to x+} f(y)$$

Pick any $\epsilon > 0$. We need to show then that there exists $\delta > 0$ such that whenever $y - x < \delta$ it holds that $f(y) - f(x) < \epsilon$ (note that the fact that f is increasing and the fact that we are taking a right limit allows us to omit absolute values). We have that:

$$f(y) - f(x) = \sum_{\{n: x_n \leq y\}} \epsilon_n - \sum_{\{n: x_n \leq x\}} \epsilon_n = \sum_{\{n: x < x_n \leq y\}} \epsilon_n$$

As has already been observed in the main text, since the series corresponding to ϵ_n converges, it is the case that $\sum_{n=N}^{\infty} \epsilon_n = 0$ as $N \to \infty$. This means that given our chosen $\epsilon > 0$, we can find an $N > 0$ such that $\sum_{n=N}^{\infty} \epsilon_n < \epsilon$. Therefore, after excluding a *finite* number of terms from the series, namely $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\}$ we can make the corresponding remaining infinite sum arbitrarily small. Because the set is finite, we can then pick $\delta > 0$ such that none of these terms are in the interval $(x, y]$: simply find the closest term to x and pick y to be closer to x than that. Then by necessity:

$$\sum_{\{n: x < x_n \leq y\}} \epsilon_n \leq \sum_{n=N}^{\infty} \epsilon_n < \epsilon,$$

which means precisely that $f(y) - f(x) < \epsilon$, thus completing the proof that f is right-continuous everywhere.
(iii) Due to part (ii) we only need to prove that f is left-continuous at $x \in \mathbb{R} \setminus D$. That is, we need to prove that for $x \in \mathbb{R} \setminus D$:

$$\lim_{y \to x-} f(y) = f(x)$$

Pick any $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $x - y < \delta$:

$$f(x) - f(y) < \epsilon \implies \sum_{\{n: x_n \leq x\}} \epsilon_n - \sum_{\{n: x_n \leq y\}} \epsilon_n < \epsilon \implies \sum_{\{n: y < x_n \leq x\}} \epsilon_n < \epsilon$$

This means that we can now use the exact same observation as in (ii) to bring y sufficiently close to x such that the LHS sum is strictly smaller than ϵ, thus proving that f is indeed left-continuous at x. Note that the important difference from the case $x \in D$ examined in the book is that here $x_n \leq x$ does not introduce a fixed term in the sum, which would of course impose a lower bound on it. In contrast, when $x \in D$ it is not guaranteed that we can always exclude the finite set of terms obtained in (ii) since one of them *may be x itself.*
Now for the question posed in the exercise, consider the function $g(x) = f(x) + x$. Because the identity function is continuous everywhere, g will maintain all properties of f shown in the exercise: it will be right-continuous everywhere as a sum of right-continuous functions, it will be continuous at $\mathbb{R} \setminus D$ and at the points of D it will be discontinuous (since otherwise $f(x) = g(x) - x$ would be continuous as a difference of continuous functions). In addition, for any two $x_1 < x_2$, we have that $f(x_1) \leq f(x_2)$, and therefore clearly $g(x_1) < g(x_2)$, meaning that g is strictly increasing.

Exercise 35
Let $f : [a, b] \to \mathbb{R}$ be increasing, and let (x_n) be an enumeration of the discontinuities of f. For each n, let $a_n = f(x_n) - f(x_n-)$ and $b_n = f(x_n+) - f(x_n)$ be the left and right "jumps" in the graph of f, where $a_n = 0$ if $x_n = a$ and $b_n = 0$ if $x_n = b$. Show that $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$ and $\sum_{n=1}^{\infty} b_n \leq f(b) - f(a)$.

Solution.
We will first use exercise 33 (without the absolute values since we have an *increasing* function) to obtain an intermediate result about the first n terms of the sums that appear in this exercise. Namely, applying exercise 33 on the points (x_n) we have that:

$$\sum_{i=1}^{n} f(x_i+) - f(x_i-) \leq f(b) - f(a) \implies \sum_{i=1}^{n} f(x_i+) - f(x_i) + f(x_i) - f(x_i-) \leq f(b) - f(a)$$

$$\implies \sum_{i=1}^{n} f(x_i+) - f(x_i) + \sum_{i=1}^{n} f(x_i) - f(x_i-) \leq f(b) - f(a) \implies \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i \leq f(b) - f(a)$$

Since we are dealing with non-negative quantities from this we can conclude that

$$\sum_{i=1}^{n} b_i \leq f(b) - f(a), \sum_{i=1}^{n} a_i \leq f(b) - f(a)$$

Observe therefore that if the left or right discontinuities are finite in number, this result proves what is needed in the exercise (for a_i or b_i respectively). We now examine the case where the left discontinuities are infinite in number. Observe that each a_n is non-negative, and therefore the sequence of partial sums is non-decreasing, while we've already shown that it is bounded. We thus know that it converges, and since limits maintain non-strict inequalities, we obtain that $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$. The exact same reasoning can also be applied to right discontinuities.

Exercise 36
In the notation of Exercise 35, define $h(x) = \sum_{\{n: x_n \leq x\}} a_n + \sum_{\{n: x_n < x\}} b_n$. Show that h is increasing and that $g = f - h$ is both continuous and increasing. Thus, each increasing function f can be written as the sum of a continuous increasing function g and a "pure jump" function h.

Solution.
We begin by showing that h is increasing. Consider $y_1, y_2 \in [a, b], y_1 < y_2$. Then:

$$h(y_2) - h(y_1) = \sum_{\{n: x_n \leq y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \leq y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n$$
$$= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n \geq 0,$$

since each term of these sums is non-negative (recall that f is increasing). Therefore h is increasing. Now we examine the continuity of g, for which we'll have to examine the left and right limits at the points of discontinuity of f and at the points where f is continuous.
First let's consider $x_i \in \{x_1, x_2, \ldots\}$, i.e. a point of discontinuity of f:

- For $y \to x_i+$, we are interested in the following limit:

$$\lim_{y \to x_i+} g(y) = \lim_{y \to x_i+} (f(y) - h(y)) = \lim_{y \to x_i+} \Big(f(y) - \sum_{\{n: x_n \leq y\}} a_n - \sum_{\{n: x_n < y\}} b_n \Big)$$

  Now notice that in this case it is always true that $y > x_i$, leading us to decompose the sums as follows:

$$f(y) - \sum_{\{n: x_n \leq x_i\}} a_n - \sum_{\{n: x_i < x_n \leq y\}} a_n - \sum_{\{n: x_n < x_i\}} b_n - \sum_{\{n: x_i \leq x_n < y\}} b_n$$

$$= f(y) - h(x_i) - \Big(\sum_{\{n: x_i < x_n \le y\}} a_n + \sum_{\{n: x_i \le x_n < y\}} b_n \Big)$$

Now we observe that since we saw in exercise 35 that the series of the non-negative (a_n) converges, the sum of a_n inside the parenthesis can be made to go to zero as $y \to x_i+$: the argument here is the same as in exercise 34, where we show that we need but to exclude a finite number of terms of the series. The same applies to the sum of b_n, except that it tends to b_i instead (due to the "less than or equal to x_i" in its sum, where x_i does correspond to a term of the sequence). Consequently, the limit of this expression as $y \to x_i+$ must be:

$$f(x_i+) - h(x_i) - 0 - b_i = f(x_i+) - h(x_i) - f(x_i+) + f(x_i) = f(x_i) - h(x_i),$$

where the first term comes from the definition of the right limit of f at x_i, the second term is constant and the third and fourth were explained above. Therefore we conclude that $\lim_{y \to x_i+} g(y) = f(x_i) - h(x_i) = g(x_i)$.

- For $y \to x_i-$, the proof will be highly similar:

$$\lim_{y \to x_i-} g(y) = \lim_{y \to x_i-} \Big(f(y) - \sum_{\{n: x_n \le y\}} a_n - \sum_{\{n: x_n < y\}} b_n \Big)$$

The decomposition now looks like:

$$f(y) - \sum_{\{n: x_n \le x_i\}} a_n + \sum_{\{n: y < x_n \le x_i\}} a_n - \sum_{\{n: x_n < x_i\}} b_n + \sum_{\{n: y \le x_n < x_i\}} b_n$$

$$= f(y) - h(x_i) + \Big(\sum_{\{n: y < x_n \le x_i\}} a_n + \sum_{\{n: y \le x_n < x_i\}} b_n \Big)$$

A similar argument as above yields that the first sum here tends to a_i and the second to zero. Therefore the desired limit will equal:

$$f(x_i-) - h(x_i) + a_i = f(x_i-) - h(x_i) + f(x_i) - f(x_i-) = f(x_i) - h(x_i),$$

which is again the desired quantity.

Now we consider $x \notin \{x_1, x_2, \ldots\}$:

- For $y \to x+$, we are interested in:

$$\lim_{y \to x+} g(y) = \lim_{y \to x} (f(y) - h(y))$$

Carrying out the calculations will lead to an expression identical to the first bullet above, except that since $x \notin \{x_1, x_2, \ldots\}$ all sums that involve a_n, b_n will now go to zero (unlike before, they never contain a "fixed" term corresponding to x_i). Therefore we'll eventually arrive at $\lim_{y \to x+} g(y) = f(x+) - h(x) = f(x) - h(x)$, since f is now continuous at x.

- For $y \to x-$, everything is exactly symmetrical to the above, and therefore once more $\lim_{y \to x-} g(y) = f(x) - h(x)$.

This completes the proof that g is in fact continuous in all of $[a, b]$.
All that remains is to show that g is increasing as well. Take $y_1, y_2 \in [a, b]$ with $y_1 < y_2$. We want to show that $g(y_1) \le g(y_2)$. We have that:

$$g(y_2) - g(y_1) = f(y_2) - h(y_2) - f(y_1) + h(y_1) = f(y_2) - f(y_1) - (h(y_2) - h(y_1))$$

We examine $h(y_2) - h(y_1)$:

$$h(y_2) - h(y_1) = \sum_{\{n: x_n \le y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \le y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n$$

$$= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n$$

Notice that every term that appears in this sum corresponds to a $x_n \in [y_1, y_2]$. Recall also from exercise 35 that we had $\sum_{i=1}^n b_i + \sum_{i=1}^n a_i \leq f(b) - f(a)$ and that taking the limit of each of the two sums was well-defined, which means that this inequality as well holds at the limit. The only detail that may potentially be problematic here is that we may also be summing over the terms $f(y_1+) - f(y_1), f(y_2) - f(y_2-)$ in case one or both endpoints y_1, y_2 are also discontinuities. However, it can be shown fairly easily that with a small modification to 33 the inequality given there can also be shown to hold for n points in the closed interval [a, b] instead of just the open (one just has to take a finite number of $x_1, \ldots, x_n$ and consider $a < a' < x_1, x_n < b' < b$). Then 35 can be applied to generalize this to countably infinite points which may be the case here.
All of this results in the fact that the sum above is at most $f(y_2) - f(y_1)$ (since these are the endpoints of the relevant interval), and therefore also $g(y_2) \geq g(y_1)$, which shows that g is increasing.

Chapter 3

Metrics and Norms

3.1 Metric Spaces

Exercise 2
If d is a metric on M, show that $|d(x,z) - d(y,z)| \leq d(x,y)$ for any $x, y, z \in M$.

Solution.
First, we apply the triangle inequality as follows:

$$d(x,z) \leq d(x,y) + d(y,z) \implies d(x,z) - d(y,z) \leq d(x,y)$$

Another application of it yields:

$$d(y,z) \leq d(y,x) + d(x,z) \implies -d(y,x) \leq d(x,z) - d(y,z) \implies -d(x,y) \leq d(x,z) - d(y,z),$$

where we used the symmetry property of metric d. Putting the two inequalities together results in:

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

Exercise 3
As it happens, some of our requirements for a metric are redundant. To see why this is so, let M be a set and suppose $d : M \times M \to \mathbb{R}$ satisfies $d(x,y) = 0$ if and only if $x = y$, and $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in M$. Prove that d is a metric; that is, show that $d(x,y) \geq 0$ and $d(x,y) = d(y,x)$ hold for all x, y.

Solution.
Pick any two $x, y \in M$. We know then that for any $z \in M$ it holds that $d(x,z) \leq d(x,y) + d(z,y)$. More specifically, this holds for $z = x$, in which case we obtain:

$$d(x,x) \leq d(x,y) + d(x,y) \implies 0 \leq 2d(x,y),$$

which means that for any two $x, y \in M, d(x,y) \geq 0$.
For the symmetry propety, once again pick any two $x, y \in M$. Then $d(x,y) \leq d(x,z) + d(y,z)$ for any z. Pick then $z = x$, in which case we obtain $d(x,y) \leq d(x,x) + d(y,x) \implies d(x,y) \leq d(y,x)$. By exchanging the roles of x, y, we have that $d(y,x) \leq d(y,z) + d(x,z)$ for any z. Pick $z = y$ to observe that $d(y,x) \leq d(y,y) + d(x,y) \implies d(y,x) \leq d(x,y)$. But then $d(x,y) \leq d(y,x) \leq d(x,y)$, thus the only possibility is that $d(x,y) = d(y,x)$. Therefore d is indeed a metric.

Exercise 6
If d is any metric on M, show that $\rho(x,y) = \sqrt{d(x,y)}, \sigma(x,y) = d(x,y)/(1+d(x,y))$ and $\tau(x,y) = \min\{d(x,y),1\}$ are also metrics on M. [Hint: $\sigma(x,y) = F(d(x,y))$, where F is as in exercise 5.]

Solution.
We will solve this as a simple application of 7. If $F_1 : [0,\infty) \to [0,\infty), F_1(x) = \sqrt{x}$, then $\rho(x,y) = F_1(d(x,y))$. Then $F_1(x)/x = 1/\sqrt{x}$, which is clearly decreasing for $x > 0$, and therefore exercise 7 guarantees that ρ is a metric.
Similarly, if $F_2 : [0,\infty) \to [0,\infty), F_2(x) = \frac{x}{1+x}$, then $F_2'(x) = \frac{x+1-x}{(1+x)^2} = \frac{1}{(x+1)^2}$, which is clearly decreasing as for $x \geq 0$, and therefore σ is also a metric.
Lastly, if $F_3 : [0,\infty) \to [0,\infty), F_3(x) = \min\{x,1\}$, then $\tau(x,y) = F_3(d(x,y))$ and:

$$F_3(x)/x = \begin{cases} 1, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases},$$

which is of course a decreasing function for $x > 0$, and therefore τ is a metric.

Exercise 7
Here is a generalization of exercises 5 and 6. Let $f : [0,\infty) \to [0,\infty)$ be increasing and satisfy $f(0) = 0$, and $f(x) > 0$ for all $x > 0$. If f also satisfies $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric. Show that each of the following conditions is sufficient to ensure that $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$:
(a) f has a second derivative satisfying $f'' \leq 0$
(b) f has a decreasing first derivative
(c) $f(x)/x$ is decreasing for $x > 0$
[Hint: First show that (a) $\implies$ (b) $\implies$ (c).]

Solution.
As indicated in the hint, we first show that (a) $\implies$ (b) $\implies$ (c). If, after that, we can show that (c) implies the triangle inequality for f, we know also that any of (a), (b) imply it as well (since they imply (c)).
(a) $\implies$ (b): Since $f'' \leq 0$, f more specifically has a first derivative on every point of $[0,\infty)$. From calculus 1, it's known also that this implies that f' is decreasing (one would prove this via the Mean Value Theorem).
(b) $\implies$ (c): (b) implies that f is differentiable, thus $g(x) = f(x)/x, x > 0$ is also differentiable, with $g'(x) = \frac{f'(x)x - f(x)}{x^2}$. If we can show that the numerator here is non-positive for $x > 0$, we'll have shown that g is decreasing. Pick any $x > 0$, and apply the Mean Value Theorem on f in the interval $[0,x]$. This means that there exists $y \in (0,x)$ such that:

$$f'(y) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x},$$

where we used the fact that $f(0) = 0$. Therefore, we have that:

$$f'(x)x - f(x) = f'(x)x - xf'(y) = x(f'(x) - f'(y)) \leq 0,$$

since f' is decreasing and $y < x$. This completes the proof that (b) $\implies$ (c).
We now need to show that the triangle inequality for f follows from (c). Pick any two $x, y \geq 0$. If any of them, or both, are zero, the triangle inequality follows from the fact that $f(0) = 0$. If $x, y \neq 0$, then we know that $x \leq x+y, y \leq x+y$, and thus that $\frac{f(x)}{x} \geq \frac{f(x+y)}{x+y}, \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$ (by the fact that $f(x)/x$ is decreasing). We can rewrite these as:

$$(x+y)f(x) \geq xf(x+y), (x+y)f(y) \geq yf(x+y),$$

and now we can add them to obtain that:

$$(x+y)(f(x)+f(y)) \geq f(x+y)(x+y) \implies f(x)+f(y) \geq f(x+y),$$

which is of course what we wanted to show.

Exercise 10
The *Hilbert cube*, H^∞, is the collection of all real sequences $x = (x_n)$ with $|x_n| \leq 1, n = 1, 2, \ldots$.
(i) Show that $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞.
(ii) Given $x, y \in H^\infty$ and $k \in \mathbb{N}$, let $M_k = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. Show that $2^{-k}M_k \leq d(x, y) \leq M_k + 2^{-k+1}$.

Solution.
(i) We will make use of exercise 3, which allows us to conclude that d is a metric if the following three things hold:

- First, that $d(x, y)$ is a non-negative real number for any x, y, which, due to the infinite sum, is non-trivial here. Consider any two sequences x, y. We know that $|x_n| \leq 1, |y_n| \leq 1$, for all n, therefore the maximum value that $|x_n - y_n|$ achieves is 2. This means that $2^{-n}|x_n - y_n| \leq 2^{-n+1}$. The partial sums that correspond to d are therefore non-decreasing and upper bounded by a convergent geometric series, therefore the series that defines d also converges.

- Second, that $d(x, y) = 0$ iff $x = y$. If two sequences x, y are equal, then $x_n = y_n$ for all n, thus $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - x_n| = 0$. Conversely, if $d(x, y) = 0$, then we have a non-decreasing sequence of non-negative partial sums that converges to zero. The only way this can happen is if each term of the sequence is zero, i.e. if $|x_n - y_n| = 0$ for all n, that is, if $x_n = y_n$ for all n. But this means precisely that $x = y$.

- Third, that the triangle inequality holds. Pick any three sequences x, y, z, and $k \in \mathbb{N}^+$. Then:

$$\sum_{n=1}^{k} 2^{-n}|x_n - y_n| = \sum_{n=1}^{k} 2^{-n}|x_n - z_n + z_n - y_n| \leq \sum_{n=1}^{k} 2^{-n}|x_n - z_n| + \sum_{n=1}^{k} 2^{-n}|z_n - y_n|,$$

  by the triangle inequality. Since this holds for any k and all series converge (by the first item proven), we know that the inequality holds for the infinite series as well, i.e. that:

$$d(x, y) \leq d(x, z) + d(y, z)$$

Therefore d is a metric on the Hilbert cube.
(ii) Pick any two sequences x, y in the Hilbert cube and $k \in \mathbb{N}$. We know then by definition that $M_k \geq |x_n - y_n|$ for $i \leq k$. In addition, for $n > k$, the n-th term of the series that forms the metric d is $2^{-n}|x_n - y_n| \leq 2^{-n+1}$. Thus:

$$d(x, y) = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n| \leq M_k \sum_{n=1}^{k} 2^{-n} + \sum_{n=k+1}^{\infty} 2^{-n+1}$$

$$= M_k(1 - 2^{-k}) + 2(2 - 1 - (1 - 2^{-k})) = M_k - 2^{-k}M_k + 2^{-k+1} \leq M_k + 2^{-k+1},$$

where we used the fact that the geometric series with ratio $r = 1/2$ sums to 2, and our second sum here was thus 2 minus the sum of the first k terms minus 1 since the series starts at 1 instead of 0. In the last step we also use the fact that $M_k \geq 0$. For the other inequality, we have again that:

$$d(x, y) = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n|$$

Notice that in order for x, y to minimize this, it should be the case that $x_n - y_n = 0, n > k$, and that the maximum absolute difference M_k is multiplied with the smallest possible quantity up to k, that is, with 2^{-k}, while for all other $n < k, x_n - y_n = 0$. This would yield that $d(x, y) \geq M_k 2^{-k} + 0 + 0 = M_k 2^{-k}$.

Exercise 12
Check that $d(f,g) = \max_{a \le t \le b} |f(t) - g(t)|$ defines a metric on $C[a,b]$, the collection of all continuous, real valued functions defined on the closed interval $[a,b]$.

Solution.
Note that here continuity is important to ensure that the difference of any two functions is continuous, and as such achieves a maximum value in any closed interval. Using exercise 3, we need to check the following:

- One, that $d(f,g) = 0$ iff $f = g$. Suppose first that $d(f,g) = 0$, that is, $\max_{a \le b} |f(t) - g(t)| = 0$. If f, g were not equal, there would exist $x \in [a,b]$ such that $f(x) \neq g(x)$. Clearly then $\max_{a \le t \le b} |f(t) - g(t)| \ge |f(x) - g(x)| > 0$, which is a contradiction. Therefore $f = g$. In the other direction, if $f = g$, we equivalently have that $f - g$ is the zero function on $[a,b]$, and therefore $d(f,g) = \max_{a \le t \le b} |f(t) - g(t)| = 0$.

- Two, that for any three $f, g, h \in C[a,b], d(f,h) \le d(f,g) + d(g,h)$. We have first that $d(f,h) = \max_{a \le t \le b} |f(t) - h(t)|$. Suppose that this function achieves its maximum value on $t_M \in [a,b]$ (if there are more than one, pick any). Then $d(f,h) = |f(t_M) - h_(t_M)| = |f(t_M) - g(t_M) + g(t_M) - h(t_M)| \le |f(t_M) - g(t_M)| + |g(t_M) - h(t_M)|$. By definition, it holds that $|f(t_M) - g(t_M)| \le \max_{a \le t \le b} |f(t) - g(t)| = d(f,g)$, and similarly $|g(t_M) - h(t_M)| \le d(g,h)$. We therefore obtain that $d(f,h) \le d(f,g) + d(g,h)$.

Therefore d is indeed a metric on $C[a,b]$.

Exercise 14
We say that a subset A of a metric space is **bounded** if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \le C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.

Solution.
Suppose we have the finite union

$$U = A_1 \cup A_2 \cup \ldots \cup A_n$$

of n bounded subsets of a metric space M. Let then $x_1, x_2, \ldots, x_n, C_1, \ldots, C_n$ be such that $d(a, x_i) \le C$ for $a \in A_i$. Form the set $\{d(x_1, x_1), d(x_1, x_2), \ldots, d(x_n, x_1)\}$, which, crucially, has a finite number of elements, and therefore also has a maximum element, M. By the same reasoning, there exists C such that $C = \max\{C_1, \ldots, C_n\}$. Now pick any $a \in U$, which must belong in at least one A_i. Therefore:

$$d(a, x_1) \le d(a, x_i) + d(x_i, x_1) \le C_i + M \le C + M$$

If we thus set $x_U = x_1$ and $C_U = C + M$, we have shown precisely that U is bounded.

Exercise 15
We define the **diameter** of a nonempty subset A of M by $\text{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Show that A is bounded if and only if $\text{diam}(A)$ is finite.

Solution.
$\implies$: Suppose first that A is bounded. Then there exists $x_0 \in M, C \in \mathbb{R}$ such that $d(a, x_0) \le C$ for all $a \in A$. Pick any two $a, b \in A$. We then have that $d(a,b) \le d(a, x_0) + d(x_0, b) \le 2C$. Therefore the set $\{d(a,b) : a, b \in A\}$ has an upper bound, namely, $2C$, and therefore also a finite least upper bound, which means precisely that $\text{diam}(A)$ is finite.
$\impliedby$: Now suppose $\text{diam}(A) = C \in \mathbb{R}$. Fix an $a \in A$ (which exists since A is nonempty). Pick any $b \in A$, in which case we have that:

$$d(a,b) \in \{d(x,y) : x, y \in A\} \implies d(a,b) \le \text{diam}(A) = C$$

If thus set $x_0 = a$ and C the respective constant, we see that the definition of A being bounded is fulfilled.

3.2 Normed Vector Spaces

Exercise 16
Let V be a vector space, and let d be a metric on V satisfying $d(x, y) = d(x - y, 0)$ and $d(ax, ay) = |a|d(x, y)$ for every $x, y \in V$ and every scalar a. Show that $||x|| = d(x, 0)$ defines a norm on V (that has d as its "usual" metric). Give an example of a metric on the vector space $\mathbb{R}$ that fails to be associated with a norm in this way.

Solution. Let us examine the properties that would make $||\cdot||$ one by one:

- Suppose $x \in V$. Then $||x|| = d(x, 0) \geq 0$, since d is a metric (and of course $||x||$ is well-defined as a finite real number for the same reason).
- Suppose $||x|| = 0$. Then $d(x, 0) = 0$, which by the properties of metrics we know is true iff $x = 0$. Conversely, if $x = 0$, by the same argument $d(x, 0) = 0 \implies ||x|| = 0$. Thus $||x|| = 0 \iff x = 0$.
- For any scalar a, we have that $||ax|| = d(ax, 0) = d(ax, a0) = |a|d(x, 0) = |a|||x||$ by the exercise hypothesis.
- Pick any two $x, y \in V$. Then:

  $$||x+y|| = d(x+y, 0) \leq d(x+y, y) + d(y, 0) = d(x+y-y, 0) + d(y, 0) = d(x, 0) + d(y, 0) = ||x|| + ||y||,$$

  where we used the triangle inequality property for metrics and the fact that $d(x, y) = d(x - y, 0)$. Therefore the triangle inequality holds for the proposed norm.

We have thus shown that $||\cdot||$ is indeed a norm, and its usual metric will of course be $d'(x, y) = ||x - y|| = d(x - y, 0) = d(x, y)$, i.e., d.
For the requested example, consider the metric $\sigma(x, y) = |x - y|/(1 + |x - y|)$ from exercise 6 of section 3.1. Notice that the "proposed" norm would then be $||x|| = \sigma(x, 0) = |x|/(1 + |x|)$. Then for a scalar a:

$$||ax|| = |ax|/(1 + |ax|) = |a| \cdot |x|/(1 + |a| \cdot |x|),$$

which we can see does not necessarily equal $|a| \cdot ||x||$, and therefore the "proposed" norm is not really a norm. The cause for this is the fact that $\sigma(ax, ay) \neq |a|\sigma(x, y)$ in general, for similar reasons.

Exercise 18
Show that $||x||_\infty \leq ||x||_2 \leq ||x||_1$ for any $x \in \mathbb{R}^n$. Also check that $||x||_1 \leq n||x||_\infty$ and $||x||_1 \leq \sqrt{n}||x||_2$.

Solution.
We have that $||x||_\infty = \max_{1 \leq i \leq n}|x_i|$, while $||x||_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2}$. By definition, there must exist at least one $j \in \{1, 2, \dots, n\}$ such that $|x_j| = \max_{1 \leq i \leq n}|x_i|$. In addition, the square is an increasing function for $x \geq 0$, which would mean that $(\max_{1 \leq i \leq n}|x_i|)^2 = \max_{1 \leq i \leq n}|x_i|^2$. We therefore have that:

$$\sum_{i=1}^{n}||x_i||^2 = (\max_{1 \leq i \leq n}|x_i|)^2 + X,$$

for $X \geq 0$ (the sum of the absolute values of the remaining coordinates). Thus $(||x||_2)^2 \geq (||x||_\infty)^2$, and by taking square roots we have the first desired result.
We also have that:

$$(||x||_1)^2 = \left(\sum_{i=1}^{n}|x_i|\right)\left(\sum_{i=1}^{n}|x_i|\right) \geq \sum_{i=1}^{n}|x_i|^2 = (||x_2||)^2,$$

which we obtain by observing that the product results in a sum of non-negative terms over all pairs of indices $i, j \in \{1, 2, \dots, n\}$, which of course includes all terms for which $i = j$. Taking square roots yields the second desired result.
For the third result, observe that for each i, $|x_i| \leq \max_{1 \leq i \leq n}|x_i| = ||x||_\infty$. This means of course that:

$$||x||_1 = \sum_{i=1}^{n}|x_i| \leq n||x||_\infty$$

Lastly, we consider the following "trick" for the last part. For a given $x = (x_1, x_2, \dots, x_n)$, consider the vectors $y, z \in \mathbb{R}^{n^2}$:

$$y = \begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \\ \vdots \\ |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix}, z = \begin{pmatrix} |x_1| \\ |x_1| \\ \vdots \\ |x_1| \\ \vdots \\ |x_n| \\ |x_n| \\ \vdots \\ |x_n| \end{pmatrix}$$

We have that $||z||_2 = ||y||_2 = \sqrt{\sum_{i=1}^n n|x_i|^2} = \sqrt{n}||x||_2$, while $\langle y, z\rangle = \sum_{i=1}^n \sum_{j=1}^n |x_i| \cdot |x_j| = (\sum_{i=1}^n |x_i|)^2 = (||x||_1)^2$. Applying the Cauchy-Schwarz inequality on y, z, we then have that:

$$\langle y, z\rangle \leq ||y|| \cdot ||z|| \implies (||x||_1)^2 \leq \sqrt{n}||x||_2\sqrt{n}||x||_2,$$

which, once more by taking square roots, gives us the last desired result.

Exercise 19
Show that we have $\sum_{i=1}^n x_i y_i = ||x||_2||y||_2$ (equality in the Cauchy-Schwarz inequality) if and only if x, y are *proportional*, that is, if and only if $x = ay$ or $y = ax$ for some $a \geq 0$.

Solution.
$\impliedby$: Suppose $x = ay$ for some $a \geq 0$ (everything is symmetric for $y = ax$). Then:

$$||x||_2||y||_2 = ||ay||_2||y||_2 = |a| \cdot ||y||_2^2,$$

where we used the "scalar multiplication"/positive homogeneity property of the norm. Furthermore:

$$\sum_{i=1}^n x_i y_i = \sum_{i=1}^n (a y_i) y_i = a \sum_{i=1}^n y_i^2 = a||y||_2^2,$$

by the definition of the 2-norm on sequences, which yields the desired equality.
$\implies$: First, if $y = 0$ then this is trivially true. If $y \neq 0$, it's also the case that $||y||_2 \neq 0$ and we have the following. Consider the beginning of the proof of the Cauchy-Schwarz inequality:

$$0 \leq ||x + ty||_2^2 = ||x||_2^2 + 2t\langle x, y\rangle + t^2||y||_2^2,$$

for any $t \in \mathbb{R}$. As we saw, the discriminant here is $\Delta = (2\langle x, y\rangle)^2 - 4||x||_2^2||y||_2^2 = 0$, by our hypothesis that $\sum_{i=1}^n x_i y_i = ||x||_2||y||_2$ (since $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$). Then this means that the corresponding second-degree polynomial of t has a *unique* solution a:

$$a = \frac{-2\langle x, y\rangle}{2||y||_2^2} = -\frac{\langle x, y\rangle}{||y||_2^2}$$

By the above equation, it must also be the case that:

$$0 = ||x + ay||_2^2,$$

which, by the properties of the norm, means that by necessity $x = -ay$. Now we only need to show that $a \leq 0$. Observe that the denominator of a is clearly positive as the norm of y. The numerator equals, by definition, $\sum_{i=}^n x_i y_i = ||x||_2||y||_2$ (by the hypothesis), so it's clearly also non-negative. The minus sign in front of the fraction makes a non-positive, which completes the proof.

Exercise 21
Recall that we defined l_1 to be the collection of all absolutely summable sequences under the norm $||x||_1 = \sum_{n=1}^{\infty}|x_n|$, and we defined l_∞ to be the collection of all bounded sequences under the norm $||x||_\infty = \sup_{n\geq 1}|x_n|$. Fill in the details showing that each of these spaces is in fact a normed vector space.

Solution.
For each of the two proposed norms, we need to show that they are in fact norms, and also that the sequences which converge under them form a vector space. We begin with showing that l_1 defines a norm:

- It's obvious from the definition that $||x||_1 \geq 0$ whenever it exists, as a sum of non-negative terms.
- For any scalar a, we have that if for a sequence x the series $\sum_{n=1}^{\infty}|x_n|$ converges to $||x||_1$, then by well-known properties of limits, the series $\sum_{n=1}^{\infty}|ax_n|$ will converge to $|a| \cdot ||x||_1$.
- If $\sum_{i=0}^{\infty}|x_n|$ converges to zero, then since all terms are strictly non-negative, it must be the case that all of them are zero, thus that $||x||_1 = 0$ implies $x = 0$. The converse is obvious.
- We now need to prove the triangle inequality. Pick $x, y \in l_1$ and any $n > 0$. Then, by the triangle inequality for absolute values:

  $$\sum_{i=1}^{n}|x_i + y_i| \leq \sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}|y_i|$$

  Notice that as $n \to \infty$, the two sums on the RHS converge to $||x||_1, ||y||_1$ respectively. This imposes a bound on the series corresponding to the LHS, whose partial sums are also non-decreasing. Hence, the LHS also converges, and in fact the inequality holds at infinity, yielding $||x+y||_1 \leq ||x||_1 + ||y||_1$.

Therefore, $||x||_1$ is a norm. The third point above showed that $0 \in l_1$, while the triangle inequality proof showed that l_1 is closed under addition. Lastly, the second point above showed that l_1 is closed under scalar multiplication. Therefore, we've already shown that l_1 is a vector space, and thus $(l_1, ||\cdot||_1)$ is a normed vector space.
Now we examine $||\cdot||_\infty$ in a similar fashion.

- Again, from the definition it is obvious that whenever x is a bounded sequence, $||x||_\infty \geq 0$.
- For any scalar a, and any bounded sequence x, the sequence $(|ax_i|)$ will have as supremum $|a| \cdot ||x||_\infty$. If this were not the case, one would get a contradiction for x by dividing with $|a|$ (and if $a = 0$, then the supremum of $(|ax_i|)$ is obviously zero).
- It's clear that $||x||_\infty$ is zero iff x is the zero sequence, since the supremum of a set of non-negative numbers is zero iff all of them are zero.
- Lastly, the triangle inequality in this case arises as follows. Pick any $x, y \in l_\infty$. Then, for any i we have that $|x_i + y_i| \leq |x_i| + |y_i| \leq ||x||_\infty + ||y||_\infty$ by the definition of the infinity norm. But then this sum constitutes an upper bound for $|x_i + y_i|$ for all i, which means that by definition the supremum $\sup_i|x_i + y_i|$ both exists and is such that $||x+y||_\infty \leq ||x||_\infty + ||y||_\infty$.

The above shows both that $||\cdot||_\infty$ is a norm, and that l_∞ is a vector space, thus that $(l_\infty, ||\cdot||_\infty)$ is a normed vector space.

3.3 More Inequalities

Exercise 24
The conclusion of Lemma 3.7 (Hölder's inequality) also holds in the case $p = 1$ and $q = \infty$. Why?

Solution.
As a reminder, we say that a sequence y is in l_∞ if y is bounded above, and in that case we define $||y||_\infty = \sup_n |y_n|$. Let us first formally state what Hölder's inequality would assert in this case:
Given $x \in l_1$ and $y \in l_\infty$, we have $\sum_{i=1}^{\infty} |x_i y_i| \leq ||x||_1 ||y||_\infty$.
Inded, pick any $n \geq 1$. We then have that:

$$\sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i| \cdot |y_i| \leq \sum_{i=1}^{n} |x_i| \cdot \sup_k |y_k| = \sup_k |y_k| \cdot \sum_{i=1}^{n} |x_i| \leq ||y||_\infty \cdot ||x||_1,$$

where we used the definition of the supremum and the fact that $||x||_1$ is well-defined. Since the partial sums are non-decreasing and bounded, the LHS converges and the inequality (i.e., Hölder's) holds for the infinite series as well.

Exercise 25
The same techniques can be used to show that $||f||_p = (\int_0^1 |f(t)|^p dt)^{1/p}$ defines a norm on $C[0,1]$ for any $1 < p < \infty$. State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for $p = 1, q = \infty$?)

Solution.
We begin by noting that due to the absolute value, the functions being integrated are always non-negative. For any $t \in (0,1)$, we can thus apply Young's inequality on $a = \frac{|f(t)|}{||f||_p}, b = \frac{|g(t)|}{||g||_q}$:

$$\frac{|f(t)|}{||f||_p} \cdot \frac{|g(t)|}{||g||_q} \leq \frac{1}{p} \cdot \frac{|f(t)|^p}{||f||_p^p} + \frac{1}{q} \cdot \frac{|g(t)|^q}{||g||_q^q}$$

Since this is true for all $t \in (0,1)$, we can integrate both sides wrt. t to obtain:

$$\int_0^1 \frac{|f(t)|}{||f||_p} \cdot \frac{|g(t)|}{||g||_q} dt \leq \frac{1}{p} \cdot \int_0^1 \frac{|f(t)|^p}{||f||_p^p} dt + \frac{1}{q} \cdot \int_0^1 \frac{|g(t)|^q}{||g||_q^q} dt \implies$$

$$\frac{1}{||f||_p \cdot ||g||_q} \int_0^1 |f(t)g(t)| dt \leq \frac{1}{p} \cdot \frac{1}{||f||_p^p} \int_0^1 |f(t)|^p dt + \frac{1}{q} \cdot \frac{1}{||g||_q^q} \int_0^1 |g(t)|^q dt = \frac{1}{p} + \frac{1}{q} = 1$$

This yields $\int_0^1 |f(t)g(t)| dt \leq ||f||_p \cdot ||g||_q$, which would be the equivalent of Hölder's inequality.
Now, Minkowski's inequality would state firstly that for any $1 < p < \infty$, if the p-norm exists for $f, g \in C[0,1]$, it also exists for $f + g$, and secondly that $||f + g||_p \leq ||f||_p + ||g||_p$.
For the analogue of Minkowski's inequality ("if $f, g \in C[0,1]$ and $||f||_p, ||g||_p$ both exist, then $||f+g||_p$ exists and $||f+g||_p \leq ||f||_p + ||g||_p$"), we first prove an analogue of Lemma 3.5 that allows us to immediately obtain that $||f + g||_p$ exists. More specifically, for any $t \in (0,1)$, from Lemma 3.5 applied on $a = |f(t)|, b = |g(t)|$, we have that:

$$(|f(t)| + |g(t)|)^p \leq 2^p (|f(t)|^p + |g(t)|^p)$$

By the triangle inequality of the absolute value, we also have that $|f(t) + g(t)|^p \leq (|f(t)| + |g(t)|)^p$ since $p > 1$. Therefore, since these hold for any $t \in (0,1)$:

$$|f(t) + g(t)|^p \leq 2^p (|f(t)|^p + |g(t)|^p) \implies \int_0^1 |f(t) + g(t)|^p dt \leq 2^p \left(\int_0^1 |f(t)|^p dt + \int_0^1 |g(t)|^p dt \right)$$

$$\implies ||f + g||_p^p \leq 2^p (||f||_p^p + ||g||_p^p),$$

which imposes a bound on $||f + g||_p$, thus showing it exists.
Now, following the proof in the book, we observe the following regarding $||f||_p^{p-1}$, for $1/p + 1/q = 1$:

$$||f^{p-1}||_q = (\int_0^1 |f^{q(p-1)}(t)|dt)^{1/q} = (\int_0^1 |f(t)|^p dt)^{1/q} = (\int_0^1 |f(t)|^p dt)^{\frac{p-1}{p}} = ||f||_p^{p-1}$$

Now, for any $t \in (0,1)$:

$$|f(t)+g(t)|^p = |f(t)+g(t)| \cdot |f(t)+g(t)|^{p-1} \leq |f(t)| \cdot |f(t)+g(t)|^{p-1} + |g(t)| \cdot |f(t)+g(t)|^{p-1},$$

which means we can integrate to obtain that:

$$\int_0^1 |f(t)+g(t)|^p dt \leq \int_0^1 |f(t)| \cdot |f(t)+g(t)|^{p-1} dt + \int_0^1 |g(t)| \cdot |f(t)+g(t)|^{p-1} dt$$

Now, define q such that $1/p + 1/q = 1$, and by Hölder's inequality and the observation above:

$$\int_0^1 |f(t)+g(t)|^p dt \leq ||f||_p \cdot ||(f+g)^{p-1}||_q + ||g||_p \cdot ||(f+g)^{p-1}||_q \leq ||f||_p \cdot ||f+g||_p^{p-1} + ||g||_p \cdot ||f+g||_p^{p-1}$$

$$\implies ||f+g||^p \leq ||f+g||_p^{p-1}(||f||_p + ||g||_p),$$

from which Minkowski's inequality follows directly. For the case $p = 1, q = \infty$, we have that for any $t \in (0,1)$:

$$|f(t)g(t)| = |f(t)| \cdot |g(t)| \leq |f(t)| \cdot \max_{0 \leq t \leq 1} |g(t)| = |f(t)| \cdot ||g||_\infty$$

$$\implies \int_0^1 |f(t)g(t)| dt \leq ||g||_\infty \int_0^1 |f(t)| dt = ||g||_\infty \cdot ||f||_1,$$

showing that Hölder's inequality does indeed hold again.

Exercise 26
Given $a, b > 0$, show that $\lim_{p\to\infty}(a^p + b^p)^{1/p} = \max\{a, b\}$. [Hint: If $a < b$ and $r = a/b$ show that $(1/p)\log(1 + r^p) \to 0$ as $p \to \infty$.] What happens as $p \to 0$? as $p \to -1$? as $p \to -\infty$?

Solution.
First, if $a = b$, the statement is obvious: the quantity inside the limit simplifies to $(2a^p)^{1/p} = 2^{1/p}a$, and as $p \to \infty, 1/p \to 0$ thus $2^{1/p}a \to 1a = a = \max\{a, b\}$. Therefore we can assume from now on that $a < b$, and, as indicated in the hint, set $r = a/b < 1$. Since $r < 1$, we have that $\lim_p \to \infty r^p \to 0$ (an exponential with a base less than 1). Therefore, by standard limit rules, $\lim_{p\to\infty} \log(1 + r^p) = \log(1) = 0$. Furthermore $\lim_{p\to\infty} \frac{1}{p} = 0$, which means $\lim_{p\to\infty}(1/p)\log(1 + r^p) = 0$. Now we apply the following "trick":

$$(a^p + b^p)^{1/p} = e^{\log(a^p+b^p)^{1/p}} = e^{\frac{\log(a^p+b^p)}{p}} = e^{\frac{\log((rb)^p+b^p)}{p}} = e^{\frac{\log(b^p)+\log(1+r^p)}{p}} = e^{\log(b)+\frac{\log(1+r^p)}{p}}$$

Clearly, what we showed based on the hint allows us to take limits on both sides and easily obtain that:

$$\lim_{p\to\infty}(a^p + b^p)^{1/p} = e^{\log(b)} = b = \max\{a, b\}$$

Now, for the case of $p \to 0$, if $a = b, \lim_{p\to 0}(a^p + b^p)^{1/p} = \lim_{p\to 0}(2a^p)^{1/p} = \lim_{p\to 0}(2^{1/p}a)$, and as $p \to 0+$ this quantity will tend to positive infinity, whereas if $p \to 0-$ it will tend to zero, thus the limit does not exist. If $a < b$, based on the above observation what interests us is $\lim_{p\to 0} \frac{\log(1+r^p)}{p}$. In this case, $r^p \to 1$ as $p \to 0$, and thus the numerator tends to $\log(2)$, thus the entire fraction tends to positive infinity as $p \to 0+$ and negative infinity as $p \to 0-$, which means the limit does not exist.
The expression inside the limit is continuous as a function of p at -1, since $a, b > 0$. Therefore $\lim_{p\to -1}(a^p + b^p)^{1/p} = (\frac{1}{a} + \frac{1}{b})^{-1} = \frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{ab}{a+b}$.
Lastly, for $p \to -\infty$, for $a = b$ we are interested in $\lim_{p\to -\infty}(2^{1/p}a)$, which is easily seen to equal a. For $a < b$, we are interested in $\lim_{p\to -\infty} \frac{\log(1+r^p)}{p}$. Because $0 < r < 1$, the quantity inside the logarithm will

tend to positive infinity, whereas the denominator tends to negative infinity. Applying L'Hospital's rule and calling this limit L we have that:

$$L = \lim_{p \to -\infty} \frac{\log(r) r^p}{1 + r^p} = \lim_{p \to -\infty} \frac{\log(r)}{1 + \frac{1}{r^p}}$$

Here the numerator is constant, whereas since $0 < r < 1, r^p \to \infty$, and thus the denominator will tend to 1. This means that $L = \log(r) = \log(a/b) = \log(a) - \log(b)$. Going back to our original limit we would have that $\lim_{p \to -\infty}(a^p + b^p)^{1/p} = e^{\log(b)+\log(a)-\log(b)} = a = \min\{a, b\}$, which we can see was also true for $a = b$.

Exercise - Unlisted; Arose from a discussion of exercise 26
a) Prove that if $1 \leq p \leq q \leq \infty$, then $l^p \subset l^q$.
b) If $x \in l_p \subset l_q$ for $1 \leq p \leq q \leq \infty$, then $||x||_p \geq ||x||_q$.
c) If $x \in l^{p_0}$ for some p_0, prove that $\lim_{p \to \infty} ||x||_p = ||x||_\infty$.

Solution.
a) Let x be a sequence in l^p. If $p = q$, then the statement is obvious, so we continue the proof assuming that $p \neq q$. In the case where $q = \infty$, we know that if, for some $1 \leq p < \infty$, $||x||_p$ exists, then the sequence of partial sums corresponding to $(|x_i|^p)$ must be bounded and non-decreasing. But then the same must hold true for the sequence $(|x_i|)$, which leads us to conclude that x is in fact bounded, and thus has a supremum. This means then that $x \in l_q$, since $q = \infty$ and the infinity norm is defined as the supremum of absolute values.
Now, in the remaining case we have that $1 \leq p < q$ and both are real numbers. Note that, by using e.g. the Archimidean property of $\mathbb{R}$, we can write $q = np + \epsilon$, where n is a positive integer and $\epsilon > 0$. Consider now examining the first m terms of x in the following way:

$$\sum_{i=1}^{m} |x_i|^q = \sum_{i=1}^{m} |x_i|^{np+\epsilon} = \sum_{i=1}^{m} |x_i|^{np} \cdot |x_i|^\epsilon$$

Recall that we already showed that if $x \in l_p$, then $|x_i|$ are bounded above, and thus we can say that for every $i, |x_i|^\epsilon \leq S^\epsilon, S = \sup_j |x_j|$. Furthermore, since n is a positive integer, we have that:

$$\sum_{i=1}^{m} |x_i|^{np} \leq (\sum_{i=1}^{m} |x_i|^p)^n,$$

since if one expands the power of the RHS, we get at least all terms of the LHS plus possibly more, all of which are non-negative. Combining these two facts, we have that:

$$\sum_{i=1}^{m} |x_i|^q \leq (\sum_{i=1}^{m} |x_i|^p)^n S^\epsilon \leq ||x||_p^{pn} S^\epsilon,$$

which we can safely conclude since the p-norm exists. But then the partial sums are bounded above for each m and are non-decrasing, meaning that the LHS converges as $m \to \infty$, and this equals precisely $||x||_q^q$. Thus x is indeed also in l^q.
b) Consider a sequence $x = (x_1, x_2, \dots) \in l_p \subset l_q$. The statement is obvious if $p = q$, and also if x is the zero sequence. Therefore from now on we assume $p < q, x \neq 0$, which means also $||x||_p > 0, ||x||_q > 0$. In the case where $q = \infty$, we have that $||x||_q = \sup_i |x_i|$. For any $m > 0$, we have that:

$$\sum_{i=1}^{m} |x_i|^p \geq \max_{1 \leq i \leq m} |x|^p$$

If we take limits on both sides, and by thinking about the ϵ-based definition of the supremum, we can see that this yields $||x||_p^p \geq ||x||_\infty^p \implies ||x||_p \geq ||x||_\infty$.
In the case where $p < q < \infty$, consider the following. Let y be the sequence formed by $y_i = \frac{x_i}{||x||_p}$. Notice first that $||y||_p = 1$. Notice also that for any $i, |y_i|^p \leq 1$, since $||y||_p = 1 \implies \sum_{i=1}^{\infty} |y_i|^p = 1^p$, so any

individual term of the series must be at most 1. Since $q > p \geq 1$, we have that it must be the case that $q = rp$ for some $r > 1$. Then, by using properties of powers, for any i:

$$|y_i|^p \leq 1 \implies |y_i|^{rp} \leq |y_i|^p \implies |y_i|^q \leq |y_i|^p$$

This in turn implies that for any $m > 0$:

$$\sum_{i=1}^{m} |y_i|^q \leq \sum_{i=1}^{m} |y_i|^p \leq ||y||_p^p = 1$$

Since this holds for any m, it also holds at infinity, meaning that $||y||_q^q \leq 1 = ||y||_p^q$, and thus by taking q-roots we obtain that $||y||_q \leq ||y||_p$. But then by using the definition of y:

$$||y||_q \leq ||y||_p \implies \left|\left|\frac{x}{||x||_p}\right|\right|_q \leq \left|\left|\frac{x}{||x||_p}\right|\right|_p \implies ||x||_q \leq ||x||_p,$$

where we used the "multiplication by scalar" property of norms. This completes the proof.
c) First of all, we observe from part (a) that since $||x||_{p_0}$ exists for some p_0, it will also be the case that $x \in l^p$ for all $p \geq p_0$, as well as that $x \in l^\infty$. We now have the following:

$$\lim_{p\to\infty} \frac{||x||_p}{||x||_\infty} = \lim_{p\to\infty} e^{\log\left(\frac{||x||_p}{||x||_\infty}\right)}$$

We focus on the exponent:

$$\lim_{p\to\infty} \log\left(\frac{||x||_p}{||x||_\infty}\right) = \lim_{p\to\infty} \log\left(\frac{(\sum_{i=1}^{\infty}|x_i|^p)^{1/p}}{(||x||_\infty^p)^{1/p}}\right) = \lim_{p\to\infty} \frac{1}{p} \cdot \log\left(\sum_{i=1}^{\infty} \frac{|x_i|^p}{||x||_\infty^p}\right)$$

Now we make the observation that since the series corresponding to $||x||_p^p$ converges, the individual terms tend to zero. This means that must exist at least one j such that $|x_j|$ equals the supremum of x. This is because the only other possibility would be for $|x_i|$ to tend to their supremum without achieving it, but then the series would not converge. Therefore, the argument of the logarithm is at least 1 (since $\frac{|x_j|^p}{||x||_\infty^p} = 1$). Additionally, the existence of $||x||_p$ implies that the argument does not go to infinity. Thus, the numerator of the last limit above tends to $L > 1$ and the denominator to positive infinity, which means that the limit tends to zero. But then:

$$\lim_{p\to\infty} \frac{||x||_p}{||x||_\infty} = e^0 = 1,$$

which concludes the proof since $||x||_\infty$ is constant with respect to the limit variable.

Exercise - Unlisted; Arose from a discussion of exercise 26
Recall that for a continuous function $f \in C([0,1])$ we define

$$||f||_p = (\int_0^1 |f(x)|^p dx)^{1/p}, ||f||_\infty = \max_{x\in[0,1]} |f(x)|$$

Show that:
a) If $f \in C[0,1]$, then for $1 \leq p \leq q \leq \infty$, $||f||_1 \leq ||f||_p \leq ||f||_q \leq ||f||_\infty$.
b) If $f \in C[0,1]$, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Solution.
a) We begin by showing that for any real number $p \geq 1, ||f||_p \leq ||f||_\infty$. We do this as follows. First, we have that:

$$||f||_p^p = \int_0^1 |f(x)|^p dx$$

Since $p \geq 1$, the p-th power is a non-decreasing function, and we know that by definition, $|f^p(x)| \leq \max_{0\leq x\leq 1}|f^p(x)|$ for any $x \in [0,1]$. By integrating both sides we have that:

$$\int_0^1 |f^p(x)|dx \leq \max_{0\leq x\leq 1}|f^p(x)| \implies ||f||_p^p \leq ||f||_\infty^p$$

By taking p-roots, we have the desired inequality. Now consider any two real numbers $p, q \geq 1$ such that $p < q$. This means that there exists $r > 1$ such that $q = rp$. Let $g(x) = f^P(x)$ and set r' to be the number satisfying $1/r + 1/r' = 1$. We now apply Hölder's inequality for the functions g and $h(x) = 1$:

$$\int_0^1 |g(x) \cdot 1|dx \leq ||g||_r \cdot ||1||_{r'} \implies \int_0^1 |f^p(x)|dx \leq (\int_0^1 |f^{rp}(x)|)^{(1/r)} \implies (\int_0^1 |f^p(x)|dx)^r \leq \int_0^1 |f^{rp}(x)|dx,$$

where for the last step we used the fact that $r > 1$. Continuing:

$$(\int_0^1 |f^p(x)|dx)^r \leq \int_0^1 |f^{rp}(x)|dx \implies (\int_0^1 |f^p(x)|dx)^{q/p} \leq \int_0^1 |f^q(x)|dx$$

$$\implies (\int_0^1 |f^p(x)|dx)^{1/p} \leq (\int_0^1 |f^q(x)|dx)^{1/q} \implies ||f||_p \leq ||f||_q$$

This completes the proof of the remaining inequalities, thus establishing that for $1 \leq p \leq q \leq \infty, ||f||_1 \leq ||f||_p \leq ||f||_q \leq ||f||_\infty$.

b) From part (a) we already know that as $p \to \infty, ||f||_p$ always exists, and in fact is bounded above by $||f||_\infty$ and is non-decreasing (since for $p \leq q, ||f||_p \leq ||f||_q$). This means that $\lim_{p\to\infty}||f||_p$ exists, and will equal the supremum of the set $S = \{||f||_p, p \geq 1\}$. We thus only need to show that this supremum is in fact $||f||_\infty$. Since we already know that this constitutes an upper bound for S, assume that the supremum of S is $M < ||f||_\infty$. Then, by the definition of the infinity norm, there exists $x_0 \in [0,1]$ such that $|f(x_0)| = ||f||_\infty > M$. For the sake of simplicity, we shall assume that $x_0 \in (0,1)$: as will become clear, in the "edge cases" the only thing that changes is that some quantities lack a factor of 2. Set now $\epsilon = |f(x_0)| - M > 0$. Since f is continuous, there must exist $\delta > 0$ such that:

$$|x - x_0| < \delta \implies |f(x_0)| - |f(x)| < \epsilon,$$

where the absolute value in the second inequality can be omitted since $|f(x_0)|$ is the maximum value of $|f|$. Now, this can be rewritten as $|f(x)| > |f(x_0)| - \epsilon = |f(x_0)| - |f(x_0)| + M = M$, which means that in the interval $(x_0 - \delta, x_0 + \delta), |f(x)| > M$. Therefore, for all x in this interval we have that $|f(x)| > M \implies |f^p(x)| > M^p$. By integrating:

$$\int_{x_0-\delta}^{x_0+\delta} |f^p(x)|dx > 2M^p\delta \implies \left(\int_{x_0-\delta}^{x_0+\delta} |f^p(x)|dx\right)^{1/p} > (2\delta)^{1/p}M$$

We now have that $||f||_p$ is greater than or equal to the LHS here, since $(x_0-\delta, x_0+\delta) \subset (0,1)$. Furthermore, as $p \to \infty$, the RHS tends to M, since δ is constant. By taking limits, we can thus obtain that:

$$\lim_{p\to\infty}||f||_p \geq M$$

Now we observe that we can repeat the entirety of the argument above for some N with $M < N < ||f||_\infty$. But then this means also that $\lim_{p\to\infty}||f||_p \geq N$, and then the ϵ-based limit definition would allow us to find p such that $||f||_p > M$, which contradicts the defining property of M as the supremum of all $||f||_p$. Therefore, we have arrived at a contradiction, and thus it must be the case that $\lim_{p\to\infty}||f||_p = ||f||_\infty$.

3.4 Limits in Metric Spaces

Exercise 30
If $A \subset B$, show that $\text{diam}(A) \leq \text{diam}(B)$.

Solution.
Consider the definition of $\text{diam}(A)$: $\text{diam}(A) = \{\sup\{d(a, b) : a, b \in A\}$. Because $A \subset B$, we have that any two $a, b \in A$ also belong in B. Therefore, the set S_1 over which the diameter is computed for B is a superset of the set S_2 over which the diameter is computed for A. But then exercise 2 of Chapter 1 guarantees that $\sup S_1 \leq \sup S_2$, which means precisely that $\text{diam}(A) \leq \text{diam}(B)$.

Exercise 32
In a normed vector space $(V, ||\cdot||)$ show that $B_r(x) = x + B_r(0) = \{x + y : ||y|| < r\}$ and that $B_r(0) = rB_1(0) = \{rx : ||x|| < 1\}$.

Solution.
Let us call $S = \{x+y : ||y|| < r\}$, in which case we asked to show that $B_r(x) = S$. First, let $z \in B_r(x)$. By definition, this means that $d(x, z) < r$. Recall that in a normed vector space we have that $||z-x|| = d(x, y)$ (unless a different metric is explicitly specified). Observe then that we can write $z = x + (z - x)$, where $||z - x|| = d(z, x) = d(x, z) < r$. By setting $y = z - x$, we obtain that $z \in S$. Therefore, $B_r(x) \subset S$.
In the other direction, suppose $z \in S$, which means that there exists $y, ||y|| < r$ such that $z = x + y$. Then we observe that $d(x, z) = ||z - x|| = x + y - x|| = ||y|| < r$. By definition, this means that $z \in B_r(x)$, and thus that $S \subset B_r(x)$, meaning that in fact $S = B_r(x)$.
For the second part of the exercise, set $S = \{rx : ||x|| < 1\}$. First, suppose $x \in B_r(0)$, which means $||x|| < r$. By using the scalar multiplication properties of vector spaces, we can then write $x = r \cdot \frac{x}{r}$. Set $y = \frac{x}{r}$, in which case $||y|| = ||\frac{x}{r}|| = \frac{1}{|r|}||x|| < 1$. This means that x can be written in the form $ry, ||y|| < 1$, thus that $x \in S$, thus that $B_r(0) \subset S$. Conversely, assume $x \in S$, which means $x = ry, ||y|| < 1$. But then $||x|| = ||ry|| = |r| \cdot ||y|| < r$, i.e. that $x \in B_r(0)$, and thus that $S \subset B_r(0)$, completing the proof that $B_r(0) = S$.

Exercise 33
Limits are unique. [Hint: $d(x, y) \leq d(x, x_n) + d(x_n, y)$.]

Solution.
Suppose that a sequence (x_n) in a metric space M converges to two $a, b \in M$ such that $a \neq b$. By the definition of metrics, we know then that it must be the case that $d(a, b) > 0$. Set $\epsilon = d(a, b) > 0$, in which case by the definition of the limit in a metric space there must exist N_1, N_2 such that $d(x_n, a) < \epsilon/4, d(x_n, b) < \epsilon/4$ for $n \geq N_1, n \geq N_2$ respectively. If we then pick $n > \max\{N_1, N_2\}$ we have that both of these inequalities hold for x_n. Recall the triangle inequality for metrics:

$$d(a, b) \leq d(a, x_n) + d(x_n, b) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon = d(a, b)$$

This is a clear contradiction, which means that if a sequence in a metric space has a limit, the limit has to be unique.

Exercise 34
If $x_n \to x$ in (M, d), show that $d(x_n, y) \to d(x, y)$ for any $y \in M$. More generally, if $x_n \to x, y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$.

Solution.
Pick $\epsilon > 0$. Because $x_n r \to x$, we know that there exists $N > 0$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. By the triangle inequality, for $n \geq N$ we have that:

$$d(x_n, y) \leq d(x, y) + d(x_n, x) < \epsilon + d(x, y) \implies d(x_n, y) - d(x, y) < \epsilon$$

By exercise 2 of Chapter 3, we also have that:

$$|d(x_n, x) - d(y, x)| \leq d(x_n, y) \implies -d(x_n, y) \leq d(x_n, x) - d(x, y) < \epsilon - d(x, y)$$

$$\implies d(x_n, y) - d(x, y) > -\epsilon$$

Combining these two inequalities we obtain that $|d(x_n, y) - d(x, y)| < \epsilon$ for all $n > N$, which is precisely the definition of $d(x_n, y) \to d(x, y)$.
For the second, more general statement, pick again $\epsilon > 0$. Then there exist N_1, N_2 such that $d(x_n, x) < \frac{\epsilon}{2}, d(y_n, y) < \frac{\epsilon}{2}$ whenever $n > N_1, n > N_2$ respectively. Set then $N = \max\{N_1, N_2\}$. By the triangle inequality for $n > N$ we have that:

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$\implies d(x_n, y_n) - d(x, y) < \epsilon$$

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + d(x_n, y_n)$$
$$\implies d(x, y) - d(x_n, y_n) < \epsilon \implies -\epsilon < d(x_n, y_n) - d(x, y)$$

Putting together these two inequalities results in $|d(x_n, y_n) - d(x, y)| < \epsilon$ for $n > N$, which is precisely the definition of $d(x_n, y_n) \to d(x, y)$.

Exercise 35
If $x_n \to x$, then $x_{n_k} \to x$ for any subsequence (x_{n_k}) of (x_n).

Solution.
Since $x_n \to x$, for any given $\epsilon > 0$ we can always find $N > 0$ such that for all $n \geq N, d(x_n, x) < \epsilon$. Because (x_{n_k}) contains infinite terms of (x_n) selected in an order which maintains the order of indices, it must be the case that for $n_k \geq N$, $d(x_{n_k}, x) < \epsilon$. We conclude that $x_{n_k} \to x$.

Exercise 36
A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set $\{x_n : n \geq 1\}$ is bounded).

Solution.
We begin by showing that a convergent sequence is Cauchy. Pick any $\epsilon > 0$. Because the sequence converges, say to x, there exists $N > 0$ such that whenever $n \geq N$ it is the case that $d(x_n, x) < \epsilon/2$. For any two such $n_1, n_2 \geq N$ we then have:

$$d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x) + d(x, x_{n_2}) < \epsilon,$$

which is precisely the definition of x being Cauchy.
Now assume (x_n) is Cauchy. Pick e.g. $\epsilon = 1$, and find $N > 0$ such that for $n_1, n_2 \geq N$ it holds that $d(x_{n_1}, x_{n_2}) < \epsilon$. Consider the set $\{d(x_n, x_1), 1 \leq n \leq N\}$. This has a finite number of non-negative elements, and as such a well-defined non-negative maximum M. For any $n > N$ we then have that:

$$d(x_n, x_1) \leq d(x_n, x_N) + d(x_N, x_1) < M + \epsilon,$$

and therefore all elements of the sequence satisfy $d(x_n, x_1) < M + \epsilon$, i.e. they are contained in the open ball $B_{M+\epsilon}(x_1)$, which is to say that the sequence is bounded.

Exercise 37
A Cauchy sequence with a convergent subsequence converges.

Solution.
Let (x_n) be a Cauchy sequence with a convergent subsequence $(x_{n_k}) \to x$. Pick any $\epsilon > 0$. Because $(x_{n_k}) \to x$, there exists $N_1 > 0$ such that for $n_k \geq N_1$ it holds that $d(x_{n_k}, x) < \epsilon/2$. Furthermore, because (x_n) is Cauchy, there exists $N_2 > 0$ such that for $i, j \geq N_2$ it holds that $d(x_i, x_j) < \epsilon/2$. Set $N = \max\{N_1, N_2\}$ and pick any $n > N$. Then, by the triangle inequality:

$$d(x_n, x) \leq d(x_n, x_{N_1}) + d(x_{N_1}, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

But then this means that (x_n) converges to x, which completes the proof.

Exercise 39
If every subsequence of (x_n) has a *further* subsequence that converges to x, then (x_n) converges to x.

Solution.
By way of contradiction, assume (x_n) does not converge to x. Then there exists $\epsilon > 0$ such that for all $N > 0$ there exists $n \geq N$ such that $d(x_n, x) \geq \epsilon$. Consider constructing the following subsequence of (x_n): for any $i > 0$, the i-th element of the subsequence equals the first x_n such that $d(x_n, x) \geq \epsilon, n \geq i$ and x_n has not been selected before. By the hypothesis above, this is always well-defined.
As a subsequence of (x_n), by the hypothesis of the exercise this subsequence will have a further subsequence converging to x. Notice, however, that by construction all elements of this "further subsequence" are such that $d(x_{n_{k_l}}, x) \geq \epsilon$, which contradicts convergence to x. Therefore, (x_n) itself must converge to x.

Exercise 40
Here is a positive result about l_1 that may restore your faith in intuition. Given any fixed element $x \in l_1$, show that the sequence $x^{(k)} = (x_1, \ldots, x_k, 0, \ldots) \in l_1$ (i.e. the first k terms of x followed by all 0s) converges to x in l_1-norm. Show that the same holds true in l_2, but give an example showing that it fails (in general) in l_∞.

Solution.
For any given k, we make the following observation:

$$||x - x^{(k)}||_1 = \sum_{i=1}^{k} |x_i - x_i| + \sum_{i=k+1}^{\infty} |x_i| = \sum_{i=k+1}^{\infty} |x_i|$$

Notice now that, since $x \in l_1$, the RHS here equals $||x||_1 - \sum_{i=1}^{k} |x_i|$. Furthermore, because the $l-1$ norm of x viewed as an infinite series converges, it must be the case that by selecting k large enough, the term $\sum_{i=1}^{k} |x_i|$ can be made to be arbitrarily close to it, which in other words means that the RHS above can be made arbitrarily small. Of course, this has the immediate consequence that for any $\epsilon > 0$ we can find $K > 0$ such that for $n \geq K, ||x - x^{(n)}||_1 < \epsilon$, which means that the sequence $x^{(k)}$ converges to x under the l_1-norm.
To show that the same holds for the l_2-norm, we compute:

$$||x - x^{(k)}||_2^2 = \sum_{i=1}^{k} |x_i - x_i|^2 + \sum_{i=k+1}^{\infty} |x_i|^2 = \sum_{i=k+1}^{\infty} |x_i|^2$$

The RHS here equals $||x||_2^2 - \sum_{i=1}^{k} |x_i|^2$. Because the square of the l_2-norm of x viewed as an infinite series converges, it must be the case that $\sum_{i=1}^{k} |x_i|^2$ gets arbitrarily close to it by picking an appropriate k. The remaining argument is also the same as for l_1, which means that $x^{(k)} \to x$ under l_2 as well.
For the case of l_∞, we consider the sequence formed by $x_n = 1 - \frac{1}{n}$. This can be easily shown to have $||x||_\infty = 1$. However, for $x^{(k)}$ as defined in the exercise:

$$||x - x^{(k)}||_\infty = \sup\{0, x_{k+1}, x_{k+2}, \ldots\} = 1$$

The reason this holds is because $x_n \to 1$, and it is thus made clear that $x^{(k)}$ cannot converge to x under the l_∞ norm.

Exercise 41
Given $x, y \in l_2$, recall that $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$. Show that if $x^{(k)} \to x$ and $y^{(k)} \to y$ in l_2, then $\langle x^{(k)}, y^{(k)} \rangle \to \langle x, y \rangle$.

Solution.
We begin first by verifying that if $a, b, c \in l_2$, then $\langle a + b, c \rangle = \langle a, b \rangle + \langle a, c \rangle$. We have that:

$$\langle a + b, c \rangle = \sum_{i=1}^{\infty} (a_i + b_i) c_i = \sum_{i=1}^{\infty} (a_i c_i + b_i c_i)$$

Notice that because a, b, c are all in l_2, we can safely split this series into a sum of the two series corresponding $\langle a, c\rangle, \langle b, c\rangle$. Now we move on to the main part of the proof by picking any $\epsilon > 0$. To begin with, for any $k, x^{(k)}, y^{(k)} \in l_2$ so any inner products involving them are well-defined:

$$|\langle x, y\rangle - \langle x^{(k)}, y^{(k)}\rangle| = |\langle x, y\rangle - \langle x, y^{(k)}\rangle + \langle x, y^{(k)}\rangle - \langle x^{(k)}, y^{(k)}\rangle| = |\langle x, y - y^{(k)}\rangle + \langle x - x^{(k)}, y^{(k)}\rangle|$$

$$\leq |\langle x, y - y^{(k)}\rangle| + |\langle x - x^{(k)}, y^{(k)}\rangle| \leq ||x||_2 \cdot ||y - y^{(k)}||_2 + ||x - x^{(k)}||_2 \cdot ||y^{(k)}||_2$$

Notice now that because $y^{(k)} \to y, x^{(k)} \to x$, we can, for this ϵ, find $K > 0$ such that for all $k \geq K, ||x - x^{(k)}||_2 < \epsilon, ||y - y^{(k)}||_2 < \epsilon$. By the extension of the triangle inequality, it will then also hold that $\big|||y^{(k)}||_2 - ||y||_2\big| < \epsilon$. Note that here we are using the association of the l_2 norm to the corresponding metric in l_2. Coupled with these observations, the above inequality implies:

$$|\langle x, y\rangle - \langle x^{(k)}, y^{(k)}\rangle| < ||x||_2 \cdot \epsilon + \epsilon \cdot (||y||_2 + \epsilon)$$

Now note that the RHS is a function of ϵ that tends to zero as ϵ tends to zero, which means (by the "elegance is not required" theorem from Hubbard & Hubbard) that indeed $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$.

Exercise 42
Two metrics d, ρ on a set M are said to be **equivalent** if they generate the same convergent sequences; that is, $d(x_n, x) \to 0$ iff $\rho(x_n, x) \to 0$. If d is any metric on M, show that the metrics ρ, σ, τ defined in Exercise 6 are all equivalent to d.

Solution.
We begin with $\rho(x, y) = \sqrt{d(x, y)}$. Suppose that some sequence x_n converges to x under d. Then, pick any $\epsilon > 0$ and find $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon^2$. We have then that $\rho(x_n, x) = \sqrt{d(x_n, x)} < \sqrt{\epsilon^2} = \epsilon$, which of course shows that $\rho(x_n, x) \to 0$, i.e. that x_n converges to x under ρ. Conversely, if $\rho(x_n, x) \to 0$ for some $(x_n), x$, then for any given $\epsilon > 0$, pick $N > 0$ such that for $n \geq N, \rho(x_n, x) < \sqrt{\epsilon} \implies \sqrt{d(x_n, x)} < \sqrt{\epsilon} \implies d(x_n, x) < \epsilon$, which means that $d(x_n, x) \to 0$.
Now, for $\sigma(x, y) = d(x, y)/(1 + d(x, y))$, suppose again $d(x_n, x) \to 0$ and pick first $\epsilon < 1$. Then there exists $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon/(1 - \epsilon) \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x) < \epsilon(1 + d(x_n, x)) \implies \sigma(x_n, x) < \epsilon$. Notice that for $\epsilon \geq 1$, it suffices to pick, say, the $N > 0$ for which the above holds when e.g. $\epsilon' = 1/2$. This shows that $\sigma(x_n, x) \to 0$. Conversely, for $\sigma(x_n, x) \to 0$, for any given $\epsilon > 0$ there exists $N > 0$ such that for $n \geq N, \sigma(x_n, x) < \epsilon \implies d(x_n, x)/(1 + d(x_n, x)) < \epsilon \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x)(1 - \epsilon) < \epsilon$. Notice that for $\epsilon < 1$, we obtain $d(x_n, x) < \epsilon/(1 - \epsilon)$, which means that $d(x_n, x)$ is bound by a function of $\epsilon > 0$ which tends to zero as $\epsilon \to 0$. It furthermore holds that this function bounds $d(x_n, x)$ when $\epsilon < 1$. Also, for $\epsilon \geq 1$, it holds trivially that $d(x_n, x)/(1 + d(x_n, x)) < \epsilon$. By defining $u(\epsilon) = \epsilon/(1 - \epsilon), \epsilon < 1$ and $u(\epsilon) = 1$ for $\epsilon \geq 1$, we have a function that fulfills the conditions of the "elegance is not required" theorem for limits, thus showing that $d(x_n, x) \to 0$.
Lastly, for $\tau(x, y) = \min\{d(x, y), 1\}$, if $d(x_n, x) \to 0$, then for $\epsilon > 1$ it trivially holds that $\tau(x_n, x) < \epsilon$ for $n \geq 1$, while for $\epsilon \leq 1$ we have that for sufficiently large $N > 0, d(x_n, x) < \epsilon$ and thus $\tau(x_n, x) = d(x_n, x) < \epsilon$, thus showing that $\tau(x_n, x) \to 0$. Conversely, if $\tau(x_n, x) \to 0$ then for any given $\epsilon > 0$, there exists $N > 0$ such that for $n \geq N, \tau(x_n, x) < \epsilon \implies \min\{d(x_n, x), 1\} < \epsilon$, thus for $\epsilon < 1$ it has to be that $d(x_n, x) < \epsilon$, which is enough to show that $d(x_n, x) \to 0$ (since for larger ϵ, N can trivially be found by using $\epsilon' < 1$).

Exercise 43
Show that the usual metric on $\mathbb{N}$ is equivalent to the discrete metric. Show that any metric on a *finite* set is equivalent to the discrete metric.

Solution.
As a reminder, for $x, y \in \mathbb{N}$ the discrete metric is $d(x, y) = 1$ iff $x \neq y$ and $d(x, y) = 0$ otherwise. The usual metric on $\mathbb{N}$ is $\rho(x, y) = |x - y|$. To show that the two metrics are equivalent, suppose first that $x_n \to x$ under d. In order for this to happen, notice that for any given $\epsilon > 0$ we must be able to find $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon$. The definition of d then dictates that in fact after some N, x_n

are always equal to x. Then it is clear that $\rho(x_n, x) = 0$ for $n \geq N$, meaning of course that $\rho(x_n, x) \to 0$. Conversely, assume $x_n \to x$ under ρ. Once again, for any given $\epsilon > 0$ we can find $N > 0$ such that for $n \geq N, \rho(x_n, x) < \epsilon$. Choose $\epsilon = 1/2$. Then $\rho(x_n, x) < 1/2 \implies |x_n - x| < 1/2$ means that in fact $x_n = x$, since these are natural numbers. Then it is of course true that $d(x_n, x) \to 0$ as well.
For the second part of the exercise, let $M = \{x_1, x_2, \dots, x_N\}$ be a finite set, d be the discrete metric and ρ be any metric on it. Suppose the sequence (y_n) is such that $d(y_n, x_i) \to$ for some $x_i \in M$. By the same observation as above, for $n \geq K$ for some K it must hold that $y_n = x_i$. Then it must be the case that *any* metric ρ, purely by being a metric, must satisfy $\rho(y_n, x_i) = 0$. This shows that $\rho(y_n, x_i) \to 0$. Conversely, if $\rho(y_n, x_i) \to 0$, then for any $\epsilon > 0$ there exists $K > 0$ such that for $n \geq K, \rho(y_n, x_i) < \epsilon$. Pick then $\epsilon = \min\{\rho(x_j, x_i), j = 1, 2, \dots, N\, j \neq i\}$, which is well-defined due to M being finite. Then the fact that $\rho(y_n, x_i)$ eventually becomes less than ϵ implies that it must be the case that after some $N, y_n = x_i$, such that the metric becomes zero. Consequently, it also holds that $d(y_n, x_i) \to 0$.

Exercise 44
Show that the metrics induced by $||\cdot||_1, ||\cdot||_2, ||\cdot||_\infty$ on $\mathbb{R}^n$ are all equivalent. [Hint: see exercise 18.]

Solution.
Call the corresponding metrics d_1, d_2, d_∞ respectively. Let (x_n) be a sequence in $\mathbb{R}^n$ that converges to x under d_2. Pick any $\epsilon > 0$ and find N such that $d_2(x_n, x) < \epsilon$ for $n \geq N$. By exercise 18, we have that:

$$d_\infty(x_n, x) = ||x_n - x||_\infty \leq ||x_n - x||_2 = d_2(x_n, x) < \epsilon,$$

which shows that $d_\infty(x_n, x) \to 0$. Also by exercise 18, we have that:

$$d_1(x_n, x) = ||x_n - x||_1 \leq \sqrt{n}||x_n - x||_2 = \sqrt{n} d_2(x_n, x) < \sqrt{n}\epsilon,$$

and we once again refer to the "elegance is not required" theorem which guarantees that $d_1(x_n, x) \to 0$. Now suppose (x_n) converges to x under d_1. The bound $||x||_2 \leq ||x||_1$ from exercise 18 and a process almost identical to the above show that it converges under d_2 as well, and then of course under d_∞. Lastly, the bound $||x||_1 \leq n||x||_\infty$ shows that convergence under d_∞ implies convergence under d_1, thus completing the "implication circle" that indicates that all three metrics are equivalent.

Exercise 45
We say that two norms on the same vector space X are equivalent if the metrics they induce are equivalent. Show that $||\cdot||$ and $|||\cdot|||$ are equivalent on X iff they generate the same sequences tending to 0; that is, $||x_n|| \to 0$ iff $|||x_n||| \to 0$.

Solution.
$\implies$: We call the induced metrics of $||\cdot||, |||\cdot|||, d_1$ and d_2 respectively. Suppose $||\cdot||$ is equivalent to $|||\cdot|||$. Pick any (x_n) such that it converges to 0 under d_1. This means that $d(x_n, 0) \to 0$, and thus also that $d_2(x_n, 0) \to 0$ (by the equivalency of the metrics). Obviously, the argument is symmetric for exchanging the roles of the two norms, which means that $||x_n|| \to 0$ iff $|||x_n||| \to 0$.
$\impliedby$: Now suppose that $||x_n|| \to 0$ iff $|||x_n||| \to 0$, and let (y_n) be a sequence that converges to y under d_1. This means that $||y_n - y|| \to 0$. In other words, the sequence $(y_n) - y$ tends 0 under $||\cdot||$, and thus by our hypothesis also under $|||\cdot|||$, which directly translates to $d_2(y_n, y) \to 0$, i.e., (y_n) converges to y under d_2. Once again, the argument is exactly symmetric for exchanging the roles of the two norms, and this leads us to conclude that the metrics they induce are equivalent, and thus that the norms are also equivalent.

Exercise 46

Given two metric spaces (M, d) and (N, ρ), we can define a metric on the product $M \times N$ in a variety of ways. Our only requirement is that a sequence of pairs (a_n, x_n) in $M \times N$ should converge precisely when both coordinate sequences $(a_n), (x_n)$ converge in $(M, d), (N, \rho)$ respectively. Show that each of the following define metrics on $M \times n$ that enjoy this property and that all three are equivalent:

$$d_1((a,x),(b,y)) = d(a,b) + \rho(x,y)$$
$$d_2((a,x),(b,y)) = (d(a,b)^2 + \rho(x,y)^2)^{1/2}$$
$$d_\infty((a,x),(b,y)) = \max\{d(a,b), \rho(x,y)\}$$

Henceforth, any implict reference to "the" metric on $M \times N$, sometimes called the **product metric**, will mean one of d_1, d_2, d_∞. Any one of them will serve equally well; use whichever looks most convenient for the argument at hand.

Solution.

We begin by showing that d_1 has the property mentioned. First, suppose $(x_n) \in M, (y_n) \in N$ converge to x, y respectively under d, ρ. Pick $\epsilon > 0$ and $N, M > 0$ such that for $n \geq N, m \geq M, d(x_n, x) < \epsilon/2, \rho(y_n, y) < \epsilon/2$. Then, for the sequence (x_n, y_n), for $n \geq \max\{N, M\}$ we have that:

$$d_1((x_n, y_n), (x, y)) = d(x_n, x) + \rho(y_n, y) < \epsilon,$$

which of course shows that $(x_n, y_n) \to (x, y)$ under d_1.
Conversely, let's assume that $((x_n, y_n))$ is a sequence of ordered pairs in $M \times N$ that converges to (x, y) under d_1. Then, for any given $\epsilon > 0$ we can find $N > 0$ such that for $n \geq N$, $d_1((x_n, y_n), (x, y)) < \epsilon \implies d(x_n, x) + \rho(y_n, y) < \epsilon$. Because a metric always yields non-negative values, each of $d(x_n, x) < \epsilon, \rho(y_n, y) < \epsilon$ must hold for $n \geq N$, which shows that $d(x_n, x) \to 0, \rho(y_n, y) \to 0$.
For d_2, observe that not much will change in the proof: in one direction, we simply need to adjust the chosen ϵ for x_n, y_n based on the squares and square roots, and in the other we will merely have to take squares of both sides of an inequality.
For d_∞, in one direction it suffices to make both $d(x_n, x), \rho(y_n, y)$ less than ϵ, and then their max also satisfies this. In the other, for a "target" ϵ, start from d_∞ with ϵ and observe that the max enforces both $d(x_n, x), \rho(y_n, y)$ to satisfy this.
Hence we only need to show the equivalency of the the three. Suppose (x_n, y_n) converges to (x, y) under any of them. Then we already showed that $x_n \to x, y_n \to y$ under d, ρ respectively. But we also showed that for any of the other two product metrics, this is enough to guarantee that (x_n, y_n) converges to (x, y) under them, thus showing that they are in fact equivalent.

Chapter 4

Open Sets and Closed Sets

4.1 Closed sets

Exercise 1
Show that an "open rectangle" $(a, b) \times (c, d)$ is an open set in $\mathbb{R}^2$. More generally, if A, B are open in $\mathbb{R}$, show that $A \times B$ is open in $\mathbb{R}^2$. If A, B are closed in $\mathbb{R}$, show that $A \times B$ is closed in $\mathbb{R}^2$.

Solution.
Call the open rectangle S, in which case its defining property is $S = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$. Let $p = (x, y) \in S$, and set $r = \min\{x - a, b - x, c - y, d - y\}$. Now let $p' = (x', y')$ be any point in $B_r(p)$. This means that $||p' - p||_2 < r$. More specifically, this translates to the following inequalities:

$$\begin{aligned}
|x' - x| < x - a &\implies x' - x > a - x \implies a < x' \\
|x' - x| < b - x &\implies x' - x < b - x \implies x' < b \\
|y' - y| < y - c &\implies y' - y > c - y \implies y' > c \\
|y' - y| < d - y &\implies y' - y > d - y \implies y' < d
\end{aligned}$$

These four properties mean that $p' \in S$, and thus we have proved that S is indeed open.
For the sake of generality, for the remaining exercise we will work with A, B being subsets of two general metric spaces $(M, d), (N, \rho)$, respectively, and then the general result will imply what is asked for here. First of all, we select d_∞ as the metric of $M \times N$, with the definition given in exercise 46 of Chapter 3. Suppose first that A, B are open, and pick $(a, b) \in A \times B$. There exist then open balls $B^d_{\epsilon_1}(a) \subset A, B^\rho_{\epsilon_2}(b) \subset B$. Pick $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, and consider any $(x, y) \in B^{d_\infty}_\epsilon((a, b))$. We have that:

$$d_\infty((x, y), (a, b)) < \epsilon \implies \max\{d(x, a), \rho(y, b)\} < \min\{\epsilon_1, \epsilon_2\}$$

Notice that this implies that, more specifically, $d(x, a) < \epsilon_1, \rho(y, b) < \epsilon_2$, which means $x \in B^d_{\epsilon_1}(a) \subset A, y \in B^\rho_{\epsilon_2}(b) \subset B$. In other words, $x \in A, b \in B$, and so $(x, y) \in A \times B$, which shows that $B^{d_\infty}_\epsilon(a, b) \subset A \times B$, and thus the Cartesian product is also open.
Now suppose that A, B are closed, which implies $M \setminus A, N \setminus B$ are open. Consider rewriting $A \times B$ as follows:

$$\begin{aligned}
A \times B = \{(x, y) \in M \times N | x \in A, y \in B\} &= (M \times N) \setminus \{(x, y) \in M \times N | x \notin A \text{ or } y \notin B\} \\
&= (M \times N) \setminus ((A^c \times N) \cup (M \times B^c))
\end{aligned}$$

Now, from the immediately preceding proof, $A^c \times N, M \times B^c$ are open as Cartesian products of open sets (a metric space is always open). Their union is thus also open, and then the complement of this union, which we showed equals $A \times B$, is thus closed.

Exercise 3
Some authors say that two metrics d, ρ on a set M are equivalent if they generate the same open sets. Prove this. (Recall that we defined equivalence to mean that d, ρ generate the same convergent sequences.)

Solution.
First suppose that two metrics d, ρ generate the same open sets. Pick any sequence $(x_n) \in M$ such that $d(x_n, x) \to 0$, and pick $\epsilon > 0$. The ball $B^\rho_\epsilon(x)$ is open under ρ, and must thus also be open (although not necessarily a ball) under d. By convergence of $x_n \to x$ under d, we have that (x_n) is eventually in $B^\rho_\epsilon(x)$, which of course means that we have showed that $\rho(x_n, x) \to 0$ as well. Exchanging the roles of d, ρ completes the proof that they are equivalent under the sequence-based definition.
Conversely, assume that d, ρ generate the same convergent sequences. Suppose S is an open set under d. If S is the empty set, it is also clearly open under ρ. Otherwise, suppose for the sake of contradiction that S is not open under ρ, which means that there exists $x \in S$ such that for every $\epsilon > 0$, $B^\rho_\epsilon(x) \not\subset S$. More specifically, this means that for a sequence of $\epsilon_n = \frac{1}{n}$, we can always find $x_n \notin S$ such that $x_n \in B^\rho_{\epsilon_n}(x)$. But then it becomes clear that $\rho(x_n, x) \to 0$, which means that $d(x_n, x) \to 0$ as well. Therefore we have constructed a sequence that has an infinite number of terms that do not belong in S, and at the same time converges to $x \in S$ under d. Because S is open under d, this is a contradiction, and we therefore conclude that S must also be open under ρ. Exchanging the roles of the two metrics completes the proof in this direction, thus showing that the two definitions are equivalent.

Exercise 5
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$, and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

Solution.
Let $S = \{x : f(x) > 0\}$. Pick any $x \in S$, in which case $f(x) > 0$. Let $\epsilon = f(x)$, and then because f is continous, there must exist $\delta > 0$ such that:

$$|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

The second inequality implies $-\epsilon < f(y) - f(x) \implies f(y) > f(x) - f(x) = 0$. Notice then that for every $y \in B_\delta(x)$ we have $f(y) > 0$, and thus that $y \in S$, which shows that S is open.
Now let $T = \{x : f(x) = 0\}$. Notice that $T = \mathbb{R} \setminus (\{x : f(x) > 0\} \cup \{x : f(x) < 0\})$. It is very easy to show that the above also holds for "less than zero", thus rendering the union here a union of open sets, and then T a closed set as a complement of an open set.

Exercise 7
Show that every open set in $\mathbb{R}$ is the union of countably many open intervals with *rational* endpoints. Use this to show that the collection $\mathcal{U}$ of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself.

Solution.
We know already that an open set $S \subset \mathbb{R}$ can be written as $S = \cup_{n=1}^\infty (a_n, b_n)$, where the intervals are all disjoint, and perhaps unbounded. Consider the following procedure. We take the interval (a_n, b_n), where in general the endpoints are real numbers. We know that between them there will always be a rational number q_n. In fact, b_n can be approached by a (increasing) sequence of rational numbers starting at q_n and getting infinitely close to it, and the same holds for a_n (only difference being the fact that this will be a decreasing sequence). These are *countably infinite* in number: indeed, each sequence is equivalent to the natural numbers, since these directly form its indices, and of course $\mathbb{N} \cup \mathbb{N} \sim \mathbb{N}$ ("two sequences are equivalent to one sequence").
We can thus replace (a_n, b_n) with a union of the form $\cup_{k=1}^\infty (q_k, r_k)$ where the endpoints are rationals. But then we have that S is a countable union of countable sets, which we know (Theorem 2.6) is also countable, and thus S is indeed a countable union of intervals with rational endpoints.
Now let $f : \mathbb{R} \setminus \{0\} \to \mathcal{U}, f(x) = (0, x)$ if $x > 0$, and $f(x) = (x, 0)$ otherwise. This function is clearly an injection from the reals except zero to the set of open sets. Recall also (exercise 17, ch. 2) that

$\mathbb{R} \sim \mathbb{R} \setminus \{0\}$, and thus we can easily see that there exists an injection h_1 from $\mathbb{R}$ to $\mathcal{U}$. Additionally, let $g((q_n)) = \cup_{n=1}^{\infty}(q_n, q_{n+1})$ be a function mapping sequences of rationals to open sets. We've shown that *every* open set can be written in this form, meaning that g is a surjection. But then for every open interval we can find at least one sequence of rationals that via g (since it's a function) corresponds to it and to no other open interval. This shows the existence of an injection $h_2 : \mathcal{U} \to \mathbb{Q}^{\mathbb{N}}$. Putting together the existence of h_1, h_2 and the fact that $\mathbb{Q}^{\mathbb{N}} \sim \mathbb{R}$, we apply the Cantor-Schröder-Bernstein theorem to see that $\mathcal{U}$ and $\mathbb{R}$ are in fact equivalent.

Exercise 8
Show that every open interval (and hence every open set) in $\mathbb{R}$ is a countable union of closed intervals and that every closed interval in $\mathbb{R}$ is a countable intersection of open intervals.

Solution.
Consider any interval of the form (a, b), first where $a, b \in \mathbb{R}$, and consider the union:

$$U = \bigcup_{n=1}^{\infty}\left[\frac{a+b}{2} - \sum_{i=2}^{n}(b-a)\frac{1}{2^i}, \frac{a+b}{2} + \sum_{i=2}^{n}(b-a)\frac{1}{2^i}\right]$$

For the sake of brevity, we will omit some details, but one can see that as $n \to \infty$, the terms inside the sums converge to $(b-a)/2$. But this means that for any given $x \in (a, b)$, it suffices to check whether $x < (a+b)/2$ or $x > (a+b)/2$, and then pick n accordingly to make the left or right endpoint of the n-th term of the union become smaller or greater than x respectively. This is always possible due to the sums converging to $(b-a)/2$, which in turn makes the endpoints converge to a, b. The union thus includes all points in (a, b), and furthermore it is clearly countable.
If it is the case that $a = -\infty$ and $b = \infty$, we propose the union $U = \cup_{n=1}^{\infty}[-2^n, 2^n]$, and we can thus clearly always find n big enough such that any $x \in \mathbb{R}$ belongs in one such interval. The "intermediate" case of one of a, b being infinity and the other a real number would be handled as a mixture of the above.
Now consider instead an interval of the form $[a, b], a, b \in \mathbb{R}$. Consider the intersection:

$$I = \bigcap_{n=1}^{\infty}\left[a - \frac{1}{2^n}, b + \frac{1}{2^n}\right]$$

We claim that this (countable) intersection equals $[a, b]$. First, it is easy to see that any $x \in [a, b]$ is contained in every interval, since $b + \frac{1}{2^n} > b, a - \frac{1}{2^n} < a$ for all n. Second, we claim that no $y \notin [a, b]$ can be contained in I. Indeed, if $y < a$, because $a - \frac{1}{2^n}$ gets arbitrarily close to a from the left, we can always find n sufficiently large such that $y < a - \frac{1}{2^n}$, and a similar argument works for $y > b$, showing that y will not be contained in the intervals corresponding to $k \geq n$, and thus neither will it be contained in I. This concludes the proof.

Exercise 10
Given $y = (y_n) \in H^{\infty}, N \in \mathbb{N}, \epsilon > 0$, show that $\{x = (x_n) \in H^{\infty} : |x_k - y_k| < \epsilon, k = 1, \ldots, N\}$ is open in H^{∞} (see Exercise 3.10).

Solution.
Let $S = \{x = (x_n) \in H^{\infty} : |x_k - y_k| < \epsilon, k = 1, \ldots, N\}$, and suppose $x \in S$. Suppose also, by way of contradiction, that there exists a sequence $z^{(k)} \to x$ with an infinite number of its elements not in S. This means that there must exist infinite k such that $|z_n^{(k)} - y_n| \geq \epsilon$ for at least one $n = 1, 2, \ldots, N$. We first make the observation that $z^{(k)} \to x$ implies coordinate-wise convergence for each n, in other words that $z^{(k)} \to x_n$ for all n. Indeed, given a particular n, pick any $\epsilon > 0$ and set $\epsilon' = 2^{-n}\epsilon$. Then there must exist $K > 0$ such that for $k \geq K$ (recall the metric for H^{∞} from exercise 3.10):

$$d(z^{(k)}, x) < \epsilon' \implies \sum_{m=1}^{\infty} 2^{-m}|z_m^{(k)} - x_m| < \epsilon'$$

But then it is the case that $2^{-n}|z_n^{(k)} - x_n| < 2^{-n}\epsilon \implies |z_n^{(k)} - x_n| < \epsilon$ for all $k \geq K$, which is the definition of convergence for the n-th coordinate. Now pick an arbitrary n, and consider the following:

$$|z_n^{(k)} - y_n| \leq |z_n^{(k)} - x_n| + |x_n - y_n|$$

The second term here is independent of z, so let it equal $c_n < \epsilon$ (by the definition of S). Set $\epsilon_n = \epsilon - c_n > 0$, and notice that there must exist $K > 0$ such that for $k \geq K, |z_n^k - x_n| < \epsilon_n$. Then:

$$|z_n^{(k)} - y_n| < \epsilon_n + c_n = \epsilon - c_n + c_n = \epsilon$$

The implication here is that after some K, this n can no longer "play the part" of keeping $z^{(k)} \notin S$ by satisfying $|z_n^{(k)} - y_n| \geq \epsilon$. This is true for any $n = 1, 2, \ldots, N$, so we can find a maximum such K such that for $k \geq K$ the above holds for *all* n. However, the existence of infinite k such that $z^{(k)} \notin S$ directly contradicts this, and thus S must indeed be open.

Exercise 11
Let $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where the k-th entry is 1 and the rest are 0s. Show that $\{e^{(k)} : k \geq 1\}$ is closed as a subset of l_1.

Solution.
Let $x = (x_1, x_2, \ldots)$ be any element of $S = \{e^{(k)} : k \geq 1\}$. We will show that if, for every $\epsilon > 0$, it is the case that $B_\epsilon(x) \cap S \neq \emptyset$ (under l_1), then $x \in S$, which we know is one of the three equivalent ways of defining closed sets. What this statement means is that for every $\epsilon > 0$, there must exist a sequence $y = (0, 0, \ldots, 1, 0, \ldots)$ with precisely one (say, the i-th) element equal to 1 and the rest equal to 0, and at the same time $||(x_1, x_2, \ldots, x_i, \ldots) - (0, 0, \ldots, 1, 0, \ldots)||_1 < \epsilon \implies ||(x_1, x_2, \ldots 1 - x_i, x_{i+1}, \ldots)||_1 < \epsilon$. Suppose then that $x \notin S$. One possibility is that x has at least two non-zero elements, x_j, x_k. Notice then that no matter the position of i, it holds that $||(x_1, x_2, \ldots 1 - x_i, x_{i+1}, \ldots)||_1 \geq \min\{|x_j|, |x_k|\}$: indeed, y can at most zero out one of the two non-zero elements of x. But this means that for sufficiently small ϵ the statement above will not hold, a contradiction. Therefore x has at most one non-zero element. If x is the zero sequence, it is clear that the LHS above will always have a norm of 1, and thus we again arrive at a contradiction. The only possibility is thus that $x = (0, \ldots, a, 0, \ldots), a \neq 0$. Suppose that $a \neq 1$, and observe that for $\epsilon < 1$ it must be the case that the y we seek has $y_i = 1$, otherwise $||(0, \ldots, 1, \ldots, a, \ldots)||_1 \geq 1$. Therefore, it must be the case that $||(0, \ldots 1 - a, 0, \ldots)||_1 < \epsilon$ for all $\epsilon < 1$, which enforces $|1 - a| < \epsilon$, and of course this can only be true for $a = 1$, thus showing that $x \in S$.

Exercise 13
Show that c_0 is a closed subset of l_∞. [Hint: If $(x^{(n)})$ is a sequence of sequences in c_0 converging to $x \in l_\infty$, note that $|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}|$ and now choose n so that $|x_k - x_k^{(n)}|$ is small *independent* of k.]

Solution.
Suppose that $(x^{(n)}) \in c_0$ is a sequence of sequences converging to x. We need to show that $x \in c_0$, in other words that $x_k \to 0$. We follow the hint by observing that, for any given term $x^{(n)}$ of the sequence of sequences, via the triangle inequality we have that:

$$|x_k| = |x_k - x_k^{(n)} + x_k^{(n)}| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}|$$

Pick any $\epsilon > 0$. Because $x^{(n)} \to x$, there exists $N > 0$ such that for $n \geq N$ we have that $d_\infty(x^{(n)}, x) < \epsilon \implies \sup_k |x_k^{(n)} - x_k| < \epsilon/2$. This guarantees that $|x_k^{(n)} - x_k| < \epsilon/2$ for *any* k. In addition, we know that for any $n, x^{(n)} \in c_0 \implies x_k^n \to 0$, and so there must exist $K \geq 0$ such that $|x_k^n| < \epsilon/2$ for $k \geq K$. By using any $n \geq N$ in the inequality stated above, we see that for $k \geq K$ we have that $|x_k| < \epsilon$, which means that $x \to 0$, and thus that $x \in c_0$.

Exercise 15
The set $A = \{y \in M : d(x, y) \leq r\}$ is sometimes called the *closed ball* about x of radius r. Show that A is a closed set, but give an example showing that A need not equal the closure of the open ball $B_r(x)$.

Solution.
To show that A is closed, pick any point y and suppose that the ball $B_\epsilon(y)$ intersects A for any $\epsilon > 0$. We need to show that $y \in A$. Indeed, if we assume $y \notin A$, then $d(x, y) > r$. If we pick $\epsilon = d(x, y) - r > 0$, then there must exist $z \in B_\epsilon(y) \cap A$. This means that $d(z, x) \leq r$. Then:

$$d(y, x) \leq d(y, z) + d(z, x) < \epsilon + d(x, z) = d(x, y) - r + d(x, z) \leq d(x, y) - r + r = d(x, y)$$

We have arrived at the contradiction $d(y, x) < d(x, y)$, and so $y \in A$, which means that A is closed.
For the second part of the exercise, consider the discrete metric d. Then for some $x \in M, B_1^d(x) = \{x\}$, and the closure of this open ball is also the set $\{x\}$: this contains precisely all the limit points of $\{x\}$. However, the corresponding closed ball would be the set $\{y \in M : d(x, y) \leq 1\} = M$, by definition of the discrete metric. Clearly, the closure of the open ball around x does not equal the closed ball around x.

Exercise 17
Show that A is open if and only if $A^\circ = A$ and that A is closed if and only if $\overline{A} = A$.

Solution.
We begin with the first statement. Note that it always holds that $A^\circ \subset A$, so in both directions of the equivalence we'll only need to show or assume that $A \subset A^\circ$.
$\implies$: Suppose A is open and pick any $x \in A$. We need to show that $x \in A^\circ = \cup\{B_\epsilon(y) \subset A,$ for some $y \in A, \epsilon > 0\}$. Because A is open, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset A$. This clearly implies that $B_\epsilon(x)$ is included in the union that forms A°, and so $x \in A^\circ$, which means $A \subset A^\circ$.
$\impliedby$: Suppose now that $A \subset A^\circ$, and pick any $x \in A$. To show that A is open, we need to find $\epsilon > 0$ such that $B_\epsilon(x) \subset A$. By our assumption, it must be the case that there exist $y \in A, \rho > 0$ such that $x \in B_\rho(y) \subset A$. In other words, it is the case that $d(x, y) < \rho$. Set $\epsilon = \rho - d(x, y)$, and consider the ball $B_\epsilon(x)$. For any z in this ball, we have that:

$$d(z, x) < \epsilon \implies d(z, x) < \rho - d(x, y) \implies d(x, y) + (z, x) < \rho \implies d(y, z) < \rho,$$

where we used the triangle inequality in the last step. Thus $z \in B_\rho(y) \subset A$, which means $B_\epsilon(x) \subset A$, and so we've shown that A is indeed open. Now, for the second statement we have that $A \subset \overline{A}$, so we'll only need to show or assume that $\overline{A} \subset A$:
$\implies$: Suppose A is closed, and pick any $x \in \overline{A}$. Then, for every $\epsilon > 0$ we have that $B_\epsilon(x) \cap A \neq \emptyset$. Because A is closed, recall that this must imply that $x \in A$.
$\impliedby$: Conversely, assume $\overline{A} \subset A$, and pick any x such that for every $\epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$. If we can show that this implies $x \in A$, we will have shown that A is closed. By the definition of the closure, we have that it implies that $x \in \overline{A}$, and by our assumption also that $x \in A$, thus completing the proof that A is closed.

Exercise 18
Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E, \inf E$ are elements of $\overline{E}$. Thus $\sup E, \inf E$ are elements of E whenever E is closed.

Solution.
Recall from exercise 3 of Chapter 1 that for every $\epsilon > 0$, there exists $a \in E$ such that $a > \sup E - \epsilon$ and $b \in E$ such that $b < \inf E + \epsilon$. But then observe that in $\mathbb{R}$, $B_\epsilon(\sup E) = \{x \in \mathbb{R} : |x - \sup E| < \epsilon\}$, and the same holds for the infimum, which by the above means that $B_\epsilon(\sup E) \cap E \neq \emptyset, B_\epsilon(\inf E) \cap E \neq \emptyset$ since these intersections include a, b respectively. Since this holds for any $\epsilon > 0$, it must be the case that $\sup E, \inf E \in \overline{E}$. By exercise 17, we have that $\overline{E} = E$ whenever E is closed, and thus in this case it also holds that $\sup E, \inf E$ are elements of E.

Exercise 19
Show that $\text{diam}(A) = \text{diam}(\overline{A})$.

Solution.
By definition, we have that $\text{diam}(A) = \sup\{d(a,b), a, b \in A\}, \text{diam}(\overline{A}) = \sup\{d(a,b), a, b \in \overline{A}\}$. Notice that because $A \subset \overline{A}$, it must be the case that $\text{diam}(A) \leq \text{diam}(\overline{A})$. Suppose then that the inequality is strict, in which case it must hold that for at least two $a, b \in \overline{A}, d(a,b) > \text{diam}(A)$. By the definition of the closure, there must exist sequences $(x_n), (y_n) \subset A$ such that $x_n \to a, y_n \to b$. Then, by exercise 34 of 3.4 we know that $d(x,y_n) \to d(a,b)$. Thus, for any given $\epsilon > 0$, there exists $N > 0$ such that for $n \geq N, |d(x_n, y_n) - d(a,b)| < \epsilon \implies d(x_n, y_n) > d(a,b) - \epsilon$. But, because $d(a,b)$ is an upper bound for $\{d(x,y), x, y \in\}(A \subset \overline{A})$ again by exercise 3, Chapter 1), this means precisely that $\sup\{d(x_n, y_n), n = 1, 2, \ldots\} = d(a,b)$ and thus $\text{diam}(A) \geq d(a,b) = \text{diam}(\overline{A})$, which is a contradiction.

Exercise 20
If $A \subset B$, show that $\overline{A} \subset \overline{B}$. Does $\overline{A} \subset \overline{B}$ imply $A \subset B$? Explain.

Solution.
Suppose $x \in \overline{A}$. Then there exists a sequence $(x_n) \subset A$ such that $x_n \to x$. Because $A \subset B$, it must also be the case that $(x_n) \subset B$. Then, by the definition of the closure of a set, we have that the limit of (x_n), i.e., x, must also belong in $\overline{B}$, thus showing that $\overline{A} \subset \overline{B}$.
The statement posed as a question is false. We consider the following counterexample: $B = B_1(0) = (-1,1), A = \{1\}$ in $\mathbb{R}$. Here we easily obtain that $\overline{B} = [-1,1]$ and $\overline{A} = \{1\}$, and so we have that $\overline{A} \subset \overline{B}$. However, it clearly does not hold that $A \subset B$.

Exercise 22
True or false? $(A \cup B)^\circ = A^\circ \cup B^\circ$.

Solution.
This is not, in general, true. Set $A = (-1,1], B = [1,2)$, in which case $A \cup B = (-1,2)$, and so $(A \cup B)^\circ = (-1,2)$. Additionally, $A^\circ = (-1,1), B^\circ = (1,2) \implies A^\circ \cup B^\circ = (-1,1) \cup (1,2)$. Notice then that $1 \in (A \cup B)^\circ$ but $1 \notin A^\circ \cup B^\circ$, so the two sets cannot be equal.

Exercise 27
Show that $|d(x,A) - d(y,A)| \leq d(x,y)$ and conclude that the map $x \to d(x,A)$ is continuous. In the context of this problem, "continuity" refers to $x_n \to x \implies d(x_n, A) \to d(x,A)$.

Solution.
From exercise 26, we recall that $d(x,A) = \inf\{d(x,a), a \in A\}$. We begin by observing that for any given $a \in A$, it holds that:

$$d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a) \implies d(x,A) - d(x,y) \leq d(y,a)$$

Since this holds for any $a \in A$, the quantity on the LHS constitutes a lower bound for $\{d(y,a), a \in A\}$. By the definition of the infimum, it must then hold that $d(x,A) - d(x,y) \leq d(y,A) \implies d(x,A) - d(y,A) \leq d(x,y)$. By exchanging the roles of x, y above, we can arrive also at the inequality $d(y,A) - d(x,y) \leq d(x,a)$ for any $a \in A$, which leads to $d(y,A) - d(x,y) \leq d(x,A) \implies d(y,A) - d(x,A) \leq d(x,y)$. Putting these two together we obtain that $|d(x,A) - d(y,A)| \leq d(x,y)$. For the second part of the exercise, suppose that $x_n \to x$ under d, that is, that for any $\epsilon > 0$ there exists $N > 0$ such that $d(x_n, x) < \epsilon, n \geq N$. By using the inequality we just proved, we have that:

$$|d(x_n, A) - d(x,A)| \leq d(x_n, x) < \epsilon,$$

and so we can indeed safely conclude that $d(x_n, A) \to d(x,A)$, which is the definition of continuity we use here.

Exercise 28
Given a set A in M and $\epsilon > 0$, show that $\{x \in M : d(x,A) < \epsilon\}$ is an open set and that $\{x \in M : d(x,A) \leq \epsilon\}$ is a closed set (and each contains A).

Solution.
The fact that each of these sets contains A is easily seen: for any $x \in A$, we have that $d(x,A) = \inf\{d(x,a) : a \in A\} = d(x,x) = 0$, and thus of course $d(x,A) \leq \epsilon$ for any $\epsilon > 0$. Now let $S = \{x \in M : d(x,A) < \epsilon\}$, and suppose that S is not open. Then there must exist $x \in S$ such that there exists a sequence $x_i \to x$ with an infinite number of terms that are not in S. Take those terms $(x_{i_k}) \subset S^c$ and form a sequence converging to x. Observe that $d(x_{i_k}, A) \geq \epsilon$, and that by exercise 27, the function $x \to d(x,A)$ is continuous. This then means that the inequality must hold at the limit, i.e. that $d(x,A) \geq \epsilon$, which is a contradiction. Therefore S is open. Now set instead $S = \{x \in M : d(x,A) \leq \epsilon\}$ and pick any convergent sequence $(x_i) \subset S$ converging to $x \in M$. It holds that $d(x_i,A) \geq \epsilon$, and again by exercise 27 we obtain $d(x,A) \geq \epsilon$, which means $x \in S$ and so S is closed.

Exercise 29
Show that every closed set in M is the intersection of countably many open sets and that every open set in M is the union of countably many closed sets. [Hint: What is $\bigcap_{n=1}^{\infty}\{x \in M : d(x,A) < 1/n\}$?]

Solution.
We begin by following the hint and setting $S = \bigcap_{n=1}^{\infty}\{x \in M : d(x,A) < 1/n\}$, where A is a closed set. By exercise 28, we know that each of the sets comprising the infinite intersection is open. It suffices therefore to show that $A = S$. First, let $y \in A$. Then, it of course holds that $d(y,A) = 0 < 1/n$ for every $n > 0$, and so $y \in S$. Therefore, $A \subset S$. In the other direction, suppose $y \in S$. It must then be the case that $d(y,A) < 1/n$ for every $n > 0$. The only way this can hold is if $d(y,A) = 0$, which by exercise 26 we know is equivalent to $y \in \overline{A}$. Here A is closed, so $A = \overline{A}$, which means $y \in A$, and so $S \subset A$. We've therefore shown that $A = S$, with S being a countable intersection of open sets.
Now suppose instead that A is open. Then A^c must be closed. By the first part, we can write $A^c = \bigcap_{n=1}^{\infty}\{x \in M : d(x,A^c) < 1/n\}$. Consider what it means for an element to *not* belong in A^c (therefore, to belong in A): for it to not belong in this infinite intersection, it is both sufficient and necessary for it to not belong in at least one of the sets comprising it, i.e. we have that $A = \bigcup_{n=1}^{\infty}\{x \in M : d(x,A^c) < 1/n\}^c = \bigcup_{n=1}^{\infty}\{x \in M : d(x,A^c) \geq 1/n\}$. This is a countable union, so we must only show that the sets comprising it are closed. Let $S_n = \{x \in M : d(x,A^c) \geq 1/n\}$ and let $(x_i) \in S_n$ be a convergent sequence with $x_i \to x$. It is then the case that $d(x_i,A^c) \geq 1/n$ for all i, and recall from exercise 27 that the function $f(x) = d(x,A)$ is continuous, and so because $x_i \to x$, it holds that $f(x_i) \to f(x) = d(x,A)$. Because $d(x_i,A^c) \geq 1/n$ for all i, this also holds at the limit: $d(x,A^c) \geq 1/n$, and so $x \in S_n$ and S_n is closed.

Exercise 33
Let A be a subset of M. A point $x \in M$ is called a **limit point** of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ for every $\epsilon > 0$. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

Solution.
Suppose that this is not true, i.e. that x is a limit point of A and a neighborhood S of x contains only finitely many points of A. Call these $x_1, x_2, \dots x_n$ and define $\epsilon = \min\{d(x_1,x), d(x_2,x), \dots, d(x_n,x)\}$. Then consider the ball $B_\epsilon(x) \subset S$: it becomes clear that no point of A in S, except for x, is inside this ball. But this contradicts the definition of limit point as given in the exercise, and thus every neighborhood of x must contain infinitely many points of A.

Exercise 34
Show that x is a limit point of A if and only if there is a sequence $x(x_n)$ in A such that $x_n \to x$ and $x_n \neq x$ for all n.

Solution.

$\Longrightarrow$: If x is a limit point of A, for any given $\epsilon > 0$, find $x_n \in A, x_n \neq x$ such that $x_n \in B_\epsilon(x)$. Constructing a sequence of such x_n by taking successive $\epsilon_n = 1/n, n = 1, 2, \dots$ yields the desired result: $x_n \to x, x_n \neq x$ for all n.
$\Longleftarrow$: Now suppose that there exists a sequence (x_n) in A with $x_n \to x, x_n \neq x$. Then for any given $\epsilon > 0$, we can find $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon$, which means that the ball $B_\epsilon(x)$ contains a point (in fact, infinitely many) of A that does not equal x (since $x_n \neq x$ for all n).

Exercise 36
Suppose that $x_n \to x \in M$ under d, and let $A = \{x\} \cup \{x_n : n \geq 1\}$. Prove that A is closed.

Solution.
If we can show that for any given $y \in M$ for which $B_\epsilon(y) \cap A \neq \emptyset$ holds for all $\epsilon > 0$, it is the case that $y \in A$, we will have shown that A is closed. Pick a $y \in M$ that fulfills the stated condition. If $y = x$ or $y = x_n$ for some x_n, it is of course true that $y \in A$. If $y \neq x, y \neq x_n$ for all n, then for any given $\epsilon > 0$, we can find an element $z \in A$ such that $d(z, y) < \epsilon$. It is easily seen that by choosing $\epsilon_n = 1/n$, eventually these z must no longer equal x: the distance $d(x, y)$ is of course constant and cannot be smaller than all $\epsilon > 0$ if $y \neq x$. Therefore, after some N, the z corresponding to ϵ_n must be elements of the sequence (x_n), and furthermore these elements cannot all originate from the first K terms of (x_n) for any K, since then we would be able to find a minimum distance $d(z_n, y)$. We conclude that, by picking z_n with appropriate increasing indices, we can then form a subsequence of (x_n) that converges to $y \neq x$, which by the uniqueness of the limit is a contradiction. Therefore y is either equal to x or some x_n and A is closed.

Exercise 38
A set P is called **perfect** if it is empty or if it is a closed set and every point of P is a limit point of P. Show that Δ is perfect. Show that $\mathbb{R}$ is perfect when considered as a subset of $\mathbb{R}^2$.

Solution.
We need to show that Δ is closed (which is stated in example 4.12 but not proved). Suppose $(x_n) \subset \Delta$ converges to x, and suppose that $x \notin \Delta$, which means that it must be outside at least one of the "levels" of intervals comprising the infinite intersection that is Δ. It is rather onerous to completely formalize everything here, but this would mean that $d(x, e) > 0$ for every endpoint e of the intervals of that level. Since all x_n are in Δ, each must belong in some interval of that level as well, and so for each $x_n, d(x_n, x) \geq d(x, e_{k_n}) > 0$ for *some* endpoint e_{k_n} out of a finite number of endpoints of that level. Therefore, for each $n, d(x_n, x) \geq \min_e d(x, e) > 0$ which contradicts convergence of (x_n) to x, and so Δ must be closed.
Now, to show that P is perfect, we must, for any given $x \in P$, prove that some sequence $(x_n) \subset P$ converges to x. Recall that $x \in \Delta$ iff $x = \sum_{n=1}^{\infty} a_n/3^n$ where each a_n is either 0 or 2. If x does not end in an infinite length of zeros, consider the sequence (x_n) where $x_n = \sum_{k=1}^{n} a_k/3^k + \sum_{k=n+1}^{\infty} 0/3^k$, that is, x_n equals x up to the n-th digit and then is zero. If x does end in an infinite string of zeros, modify x_n to have 2s instead of 0s as the digits after n. It is fairly easy to see that $d(x_n, x) \to 0$ as $n \to \infty$ (since the difference $|x_n - x|$ will always be upper bounded by $2/3^n$), while we also observe that x_n can never equal x. Therefore, x is indeed a limit point of a sequence in P.
For the second part of the exercise, if we identify $\mathbb{R}$ with $S = \{(x, 0), x \in \mathbb{R}\}$, then this is quite obviously a closed set. Additionally, if $x \in S$, consider the sequence $p_n = (x + 1/n, 0) \in \mathbb{R}^2$. Again, it is fairly easy to show that it converges coordinate-wise to $(x, 0)$, without ever resulting in $p_n = (x, 0)$, and so every point of S is a limit point of S.

Exercise 39
Show that a nonempty perfect subset P of $\mathbb{R}$ is uncountable. This gives yet another proof that the Cantor set is uncountable. [Hint: First convince yourself that P is infinite, and assume that P is countable, say $P = \{x_1, x_2, \dots\}$. Construct a decreasing sequence of nested closed intervals $[a_n, b_n]$ such that $(a_n, b_n) \cap P \neq \emptyset$ but $x_n \notin [a_n, b_n]$. Use the nest interval theorem to get a contradiction.]

Solution.
We follow the thought process indicated in the hint. First of all, if we assume P to be finite and have, say, N elements, it becomes clear that none of its points can be limit points: there is a well-defined minimum

distance d_i of each point x_i from the other $N-1$, and for any point x_i, no other elements of P lie in $B_\epsilon(x_i), \epsilon < d_i$. Now assume that P is countable, such that $P = \{x_1, x_2, \ldots\}$. We can thus find an interval $[a_1, b_1]$ such that $x_1 \notin [a_1, b_1]$ but $[a_1, b_1] \cap P \neq \emptyset$. One easy way to do this is to pick it such that it contains x_2.
Now we can repeat this procedure to find an interval $[a_2, b_2] \subset [a_1, b_1], b_2 - a_2 < (b_1 - a_1)/2$ (we "cut" the interval in half) that again is such that $[a_2, b_2] \cap P \neq \emptyset$ but $x_2 \notin [a_2, b_2]$. The key observation is that if we were unable to do this, it would mean that the interval $[a_1, b_1]$ does not, in fact, contain infinite points, and so x_2 cannot be the limit of a sequence of points in P, contradicting P's property of being perfect. Notice that it does not have to be the case that $x_3 \in [a_2, b_2]$ (e.g. if $x_3 < x_1$), but *some* element of P must be in this interval, and it cannot equal either x_1 or x_2.
The process can be repeated ad infinitum, with the following observations:

- The interval length goes to zero, and the intervals are nested.
- The n-th interval contains an element of P that cannot equal any of $x_1, x_2, \ldots, x_n$.

These observations lead us to conclude that we can form a sequence of x_{k_i}, one corresponding to each selected interval, and that this sequence must converge to some x, which will be the single point contained in the infinite intersection of intervals (by using the nested interval theorem). Furthermore, we have that $x_{k_i} \to x$. If $x \in P$, we arrive at a contradiction, since it cannot equal any of the elements $x_1, x_2, \ldots$: indeed, if it was equal to one of them, say the i-th, the above process would not have continued after the $(i-1)$-th step (by the definition of these intervals). If $x \notin P$, then this contradicts the fact that P is closed as a perfect set. In both cases, we have arrived at a contradiction, which means that P cannot be countable.

Exercise 40
If $x \in A$ and x is *not* a limit point of A, then x is called an **isolated point** of A. Show that a point $x \in A$ is an isolated point of A if and only if $(B_\epsilon(x) \setminus \{x\}) \cap A = \emptyset$ for some $\epsilon > 0$. Prove that a subset of $\mathbb{R}$ can have at most countably many isolated points, thus showing that every uncountable subset of $\mathbb{R}$ has a limit point.

Solution.
The first part of the exercise arises simply by negating the definition of limit point in exercise 33. For the second part, we note that if we can find a one-to-one mapping of isolated points of a subset $A \subset \mathbb{R}$ to the rational numbers, we will have shown that these are at most countable. We describe a process by which such a mapping can be created.
Let p_1 be any isolated point of A, which means that there exists ϵ_1 such that $(p_1 - \epsilon_1, p_1 + \epsilon_1)$ contains no other points of A. We know that must always exist a rational $q_1 \in (p_1, p_1 + \epsilon_1)$. If there exists an isolated point $p_2 > p_1$ such that its *corresponding* ϵ_2 is such that $q_1 \in (p_2 - \epsilon, p_2)$, we select a new rational $q_1' \in (p_1, q_1)$. We claim that there can now be no isolated point $p_3 > p_1$ such that $q_1' \in (p_3 - \epsilon_3, p_3)$ (again, ϵ_3 is such that it satisfies the definition of isolated point). Indeed, if this was true, there would be two cases:

- $p_3 > p_2$: Observe that $p_2 > q_1' > p_3 - \epsilon_3$, which would mean $p_2 \in (p_3 - \epsilon_3, p_3)$, which contradicts the fact that p_3 is an isolated point with a corresponding ϵ_3.
- $p_3 < p_2$: Then it must necessarily be that $p_3 > q_1$, otherwise p_3 would contradict the definitionf of isolated point for p_1. However, if this was the case we would have that $p_3 \in (p_2 - \epsilon_2, p_2)$, which contradicts the definition of isolated point for p_2.

Notice, also, that q_1' cannot be in any $(p_3 - \epsilon_3, p_3 + \epsilon_3)$ for $p_3 < p_1$, because then p_1 would also be in this interval, a contradiction. Therefore, the procedure that we describe will always match an interval of the form $(p_i - \epsilon_i, p_i + \epsilon_i)$ to a rational number that lies *in no other* interval of the form $(p_j - \epsilon_j, p_j + \epsilon_j)$, and thus the mapping of isolated points to rationals must necessarily be injective, which completes the proof.

Exercise 41
Related to the notion of limit points and isolated points are boundary points. A point $x \in M$ is said to be a **boundary point** of A if each neighborhood of x hits both A and A^c. In symbols, x is a boundary point of A if and only if $B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap A^c \neq \emptyset$ for every $\epsilon > 0$. Verify each of the following formulas, where ∂A denotes the set of boundary points of A:
(a) $\partial A = \partial A^c$
(b) $\overline{A} = \partial A \cup A^\circ$
(c) $M = A^\circ \cup \partial A \cup (A^c)^\circ$
Notice that the first and last equations tell us that each set A partitions M into three regions: the points "well inside" A, the points "well outside" A, and the points on the common boundary of A, A^c.

Solution.
(a) Suppose first that $x \in \partial A$. Then for every $\epsilon > 0$, it holds that $B_\epsilon(x) \cap A \neq \emptyset \implies B_\epsilon(x) \cap (A^c)^c \neq set$ and also that $B_\epsilon(x) \cap A^c \neq \emptyset$. But this means of course that $x \in \partial A^c$, and thus that $\partial A \subset \partial A^c$. Exchanging the roles of A, A^c yields the inclusion in the other direction as well.
(b) Now suppose $x \in \overline{A}$. Then for every $\epsilon > 0$, the ball $B_\epsilon(x)$ has a non-empty intersection with A. In the case where, for some such $\epsilon, B_\epsilon(x) \subset A$, by definition of the interior it holds that $x \in A^\circ$. In the case where for *every* $\epsilon, B_\epsilon(x) \cap A^c \neq \emptyset$, we conclude that $x \in \partial A$. Therefore $x \in \partial A \cup A^\circ$. Conversely, assume $x \in \partial A \cup A^\circ$. If $x \in \partial A$, then more specifically every ball $B_\epsilon(x)$ intersects A, and thus $x \in \overline{A}$ by definition. If $x \in A^\circ$, then it is trivially true that $x \in A \subset \overline{A}$, thus proving that $\partial A \cup A^\circ \subset \overline{A}$.
(c) We begin by observing that $M = A^\circ \cup (A^\circ)^c = A^\circ \cup \overline{A^c}$, where we made use of exercise 24, observing that $A^\circ = (\overline{A^c})^c$ and taking the complement of both sides. Now by part (b), $\overline{A^c} = \partial A^c \cup (A^c)^\circ$, and by part (a), $\partial A = \partial A^c$, so $M = A^\circ \cup \partial A \cup (A^c)^\circ$.

Exercise 42
If E is a nonempty bounded subset of $\mathbb{R}$, show that both $\sup E$, inf are both boundary points of E. Hence, if E is also closed, then $\sup E, \inf E$ are elements of E.

Solution.
By exercise 18, we already know that $\sup E, \inf E \in \overline{E}$. By exercise 40, we thus know that $\sup E, \inf E \in \partial E \cup E^\circ$. Suppose $\sup E \in E^\circ$. Then there exists $\epsilon > 0$ such that $B_\epsilon(\sup E) \subset E \implies \{x \in \mathbb{R} : |x - \sup E| < \epsilon\} \subset E$. But then it holds that $x = \sup E + \epsilon/2 \in B_\epsilon(\sup E) \subset E$, which contradicts the definition of the supremum, since $x > \sup E$. Then it must necessarily hold that $x \in \partial E$. The proof for the infimum is symmetrical. Notice that the second part of the exercise follows immediately from exercise 18. Suppose (d) holds. Recall from exercise 41 that $M = A^\circ \cup \partial A \cup (A^c)^\circ$, and so in this case it must hold that $M = A^\circ \cup \partial A$. But, also from exercise 41, $\overline{A} = A^\circ \cup \partial A$, which shows that $M = \overline{A}$, i.e. that A is dense in M.

Exercise 43
Show that ∂A is always a closed set; in fact, $\partial A = \overline{A} \setminus A^\circ$.

Solution.
Pick any $x \in M$ and any $\epsilon > 0$. We will show that if it always holds that $B_\epsilon(x) \cap \partial A \neq \emptyset$, then $x \in \partial$A, which is one of the definitions of ∂A being a closed set. By using $\epsilon' = \epsilon/2$ in our hypothesis, there exists $y \in \partial A$ such that $d(x, y) < \epsilon/2$. It then holds that $B_{\epsilon/2}(y) \cap A \neq \emptyset, B_{\epsilon/2}(y) \cap A^c \neq \emptyset$, by definition of the boundary. Therefore there must exist $z_1 \in A$ with $d(z_1, y) < \epsilon/2$ and $z_2 \in A^c$ with $d(z_2, y) < \epsilon/2$. Then:

$$d(x, z_1) \leq d(x, y) + d(y, z_2) < \epsilon, d(x, z_2) \leq d(x, y) + d(y, z_2) < \epsilon$$

This means that $B_\epsilon(x) \cap A \neq \emptyset, B_\epsilon(x) \cap A^c \neq \emptyset$, since z_1, z_2 belong in these two intersections respectively. Because this holds for any $\epsilon > 0, x$ satisfies the defining property of ∂A, and as such $x \in \partial A$, which means that ∂A is closed. Now recall from exercise 41 that $\overline{A} = \partial A \cup A^\circ$, and notice that the two sets on the RHS must always be disjoint: if $x \in \partial A$, then the intersection $B_\epsilon(x) \cap A^c$ is always non-empty, which means that there can be no $\epsilon > 0$ such that $B_\epsilon(c) \subset A$, and thus $x \notin A^\circ$. Therefore, we can safely conclude that $\partial A = \overline{A} \setminus A^\circ$.

Exercise 46
A set A is said to be **dense** in M (or, as some authors say, *everywhere dense*) if $\overline{A} = M$. For example, both $\mathbb{Q}, \mathbb{R} \setminus Q$ are dense in $\mathbb{R}$. Show that A is dense in M if and only if any of the following hold:
(a) Every point in M is the limit of a sequence from A.
(b) $B_\epsilon(x) \cap A \neq \emptyset$ for every $x \in M$ and every $\epsilon > 0$.
(c) $U \cap A \neq \emptyset$ for every nonempty open set U.
(d) A^c has empty interior.

Solution.
Suppose first that $(A^c)^\circ = \emptyset$. Then from exercise 41 we have that $M = A^\circ \cup \partial A \cup (A^c)^\circ, \overline{A} = A^\circ \cup \partial A$, and so we conclude that $M = \overline{A}$. Conversely, suppose $(A^c)^\circ = \emptyset$, in which case by the exact same properties as above, $M = \overline{A} \cup (A^c)^\circ \implies M = \overline{A}$, which means that A is indeed dense, and that we are finished with condition (d).
For (a) recall Corollary 4.11: $x \in \overline{A}$ iff $x_n \to x$ for some sequence $(x_n) \subset A$, i.e. $\overline{A}$ is precisely the set of limits of sequences in A, and so condition (a) immediately reduces to the definition of A being dense.
For (b), observe first that if $\overline{A} = M$, then by the definition of the closure, for every $x \in A$ and every $\epsilon > 0$ it must be the case that $B_\epsilon(x) \cap A \neq \emptyset$. Conversely, if (b) holds then every point $x \in M$ fulfills the defining property of $\overline{A}$, and so $M \subset \overline{A}$, which of course means $\overline{A} = M$.
For (c) we'll make use of the equivalence of (d) to A being dense. Namely, assume first that (d) does not hold, i.e. that there exists $x \in (A^c)^\circ$, which means that for some $\epsilon > 0, B_\epsilon(x) \subset A^c$, and so $B_\epsilon(x) \cap A = \emptyset$. But this ball is then an open set that renders (c) false, which means we've shown that $\neg(d) \implies \neg(c)$, or, equivalently, that $(c) \implies (d)$. Conversely, if (c) does not hold, then there exists a nonempty open set U such that $U \cap A = \emptyset$, which means $U \subset A^c$. Then, by the definition of the interior, it must be the case that $U \subset (A^c)^\circ$, and therefore that A^c does *not* have an empty interior. This means that (d) does not hold, i.e. we've shown that $\neg(c) \implies \neg(d)$, which is equivalent to $(d) \implies (c)$, which completes the proof that (c), (d) are equivalent, and therefore that (c) is also equivalent to M being dense.

Exercise 48
A metric space is called **separable** if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n$.

Solution.
For $\mathbb{R}$, we know that $\mathbb{Q}$ is countable and that $\overline{Q} = \mathbb{R}$, so it's a countable dense set of $\mathbb{R}$. For $\mathbb{R}^2$, consider the set $\mathbb{Q} \times \mathbb{Q}$: by exercise 3 of chapter 2, we have that this is a countable set. Also, any element $(x, y) \in \mathbb{R}^2$ can be approximated coordinate-wise by two sequences $q_n \to x, q_n' \to y$ each of which is in $\mathbb{Q}$. Then, exercise 46 of chapter 3 guarantees that $(q_n, q_n') \to (x, y)$, and so we've shown condition (a) of exercise 46 above, which means that $\mathbb{Q} \times \mathbb{Q}$ is indeed dense and countable. For $\mathbb{R}^n$, by a similar reasoning as above (extending ex. 46 to $n > 2$) we can use the n-times Cartesian product of $\mathbb{Q}$ as an example of a dense countable set.

Exercise 49
Prove that l_2, H^∞ are separable. [Hint: Consider finitely nonzero sequences of the form $(r_1, \ldots, r_n, 0, 0, \ldots)$, where each r_k is rational.]

Solution.
We begin with l_2, and follow the reasoning of the hint. Namely, consider the set $S \subset l_2$ such that :

$$S = \{(r_1, r_2, \ldots, r_n, 0, 0 \ldots) | r_i \in \mathbb{Q} \ \forall i, n \geq 0\}$$

Notice that this a countable set: it is equivalent to $\mathbb{Q} \cup (\mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}) \cup \ldots$, i.e. a countable union of countable sets. We need to show that for any given $x \in l_2$ there exists a sequence (of sequences) $(r)^{(k)} \subset S$ such that $(r)^{(k)} \to x$. To do this, we first pick any $\epsilon > 0$. Because $x \in l_2$, we know that $\sum_{i=1}^\infty x_i^2$ converges. A consequence of this is that there exists $N > 0$ such that $\sum_{i=N}^\infty x_i^2 < \epsilon/2$ (the choice of $\epsilon/2$ was made by "working backwards"). We furthermore know that any real element x_i of x can be approximated arbitrarily well by rational numbers. Thus, we can always find r_i such that $|x_i - r_i| < \frac{\epsilon}{\sqrt{N}}$, for the above choice of N. For this particular ϵ, we thus define $r_\epsilon = (r_1, r_2, \ldots, r_N, 0, 0, \ldots)$, and we observe that:

$$\sum_{i=1}^{\infty}(x_i - r_i)^2 = \sum_{i=1}^{N}(x_i - r_i)^2 + \sum_{i=N+1}^{\infty}(x_i - r_i)^2 < \sum_{i=1}^{N}\Big(\frac{\epsilon}{\sqrt{N}}\Big)^2 + \sum_{i=N+1}^{\infty}(x_i - 0)^2 < N\frac{\epsilon^2}{N} + \frac{\epsilon^2}{2} = \epsilon^2$$

$$\implies ||x - r_\epsilon||_2 < \epsilon$$

So far we have been working with a single sequence at a time. The final step is thus to define $(r)^k = (r_1, r_{1/2}, r_{1/3}, \ldots)$, i.e. a sequence whose k-th term is the sequence described above for $\epsilon = 1/k$. Notice then that for any given $\epsilon > 0$, this construction guarantees that for any $N > 1/\epsilon$ we have that $||x - r^{(N)}|| = ||x - r_{1/N}|| < 1/N < \epsilon$, which completes the proof that $(r)^{(k)} \to x$.
The proof for H^∞ will be quite similar. We recall that the corresponding metric is:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$$

Consider then any $x \in H^\infty, x = (x_1, x_2, \ldots)$, and pick any $\epsilon > 0$. Because $x \in H^\infty$, it must be the case that $|x_n| < 1$ for all n. Therefore, we have that for any given $K > 0, \sum_{n=K}^{\infty} \frac{|x_n|}{2^n} \leq \sum_{n=K}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^{K-1}}$. We select N such that $\frac{1}{2^N} < \frac{\epsilon}{2} \implies \frac{1}{\epsilon} < 2^{N-1} \implies N > 1 + \log(1/\epsilon)$, and we consider a sequence of rationals $r = (r_1, r_2, \ldots, r_N, 0, 0, \ldots)$. Then:

$$d(x, r) = \sum_{n=1}^{\infty} 2^{-n}|x_n - r_n| = \sum_{n=1}^{N}|x_n - r_n| + \sum_{n=N+1}^{\infty} 2^{-n}|x_n| < \sum_{n=1}^{\infty} 2^{-n}|x_n - r_n| = \sum_{n=1}^{N}|x_n - r_n| + \frac{\epsilon}{2}$$

Using the exact same observation as above regarding rationals, we can make the first term less than $\epsilon/2$ as well. Finally, we form $r^{(k)} = (r_1, r_{1/2}, r_{1/3}, \ldots)$ as above, to conclude that $r^{(k)} \to x$, and thus that H^∞ is separable.

Exercise 51
Show that a separable metric space has at most countably many isolated points.

Solution.
We claim that if a metric space M is separable with S being a countable dense subset, then every one of M's isolated points must be in S. Indeed, suppose $x \in M$ is an isolated point and $x \notin S$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \emptyset$. Therefore, no sequence $(x_n) \subset S$ can ever have x as a limit point. Furthermore, $x \notin S$ means that the trivial constant sequence with $x_n = x$ does not belong in S either, and thus there is no sequence in S that converges to x. But then by the denseness of $S, x \notin \overline{S} = M$, a contradiction. We conclude that all isolated points of M must be contained in the at most countably infinite set S, and so they must also be at most countable.

Exercise 53
Can you find a countable dense subset of $C[0, 1]$?

Solution.
We will give an example of such a set of continuous functions $F = \{f_1, f_2, \ldots\}$. In order to do this, let us call $P_j = \{0, \frac{1}{2^j}, \frac{2}{2^j}, \ldots, \frac{2^j-1}{2^j}, 1\}$ for $j = 1, 2, 3, \ldots$. We can think of P_j as partitioning $[0, 1]$ in 2^j disjoint intervals $([0, 1/2^j), [1/2^j, 2/2^j), \ldots, [(2^j - 1)/2^j, 1])$. We now define a set of functions F_j that achieve a rational value on all the endpoints of these intervals, and are linear in the intervals between successive endpoints. Namely, for a choice of $2^j + 1$ rational numbers $q_1, q_2, \ldots, q_{2^j+1}$ we can define:

$$g(x) = \begin{cases} q_j & , x = p_j \text{ for some } p_j \in P_j \\ \frac{q_{j+1} - q_j}{p_{j+1} - p_j}(x - p_j) + q_j & , x \in (p_j, p_{j+1}) \end{cases}$$

We define F_j as the set of all possible such functions. We can see that any such function is continuous, and also that for a particular j there are "as many" such functions as the elements of $\mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}$,

where the Cartesian product is taken $j+1$ times. This means that each F_j is countable. If we now define $F = \cup_{i=1}^{\infty} F_j$, we have a countable union of countable sets, and so this is countable as well. We can think of increasing j as computing piecewise linear functions where the "pieces" get smaller and smaller. What remains now is to show that this is actually a dense subset of $C[0,1]$, i.e. that any function $f \in C[0,1]$ can be approximated by a sequence $(f_n) \subset F$. In order to do this we must, for any given f, be able to find a sequence (f_n) such that for any $\epsilon > 0$ there exists $N > 0$ such that for $n \geq N, d(f, f_n) < \epsilon$.
We first recall the definition of d (the Frechet metric): $d(f, f_n) = \max_{0 \leq t \leq 1} |f(t) - f_n(t)|$. Given then a particular f, we pick the n-th term of the sequence to be the function $f_i \in F_n$ for which $|f_i(p_n) - f(p_n)| < 1/n$ for all $p_n \in P_n$. Note that such a function can always be found: we can always find rationals $f_i(p_n)$ arbitrarily close to the corresponding real $f(p_n)$. We recall, also, (e.g. from Hubbard & Hubbard) that any continuous function on a compact set is actually uniformly continuous. Thus, pick any $\epsilon > 0$, and:

- Select first a $N_1 > 4/\epsilon$.
- From uniform continuity, obtain $\delta > 0$ such that whenever $|x-y| < \delta$ it holds that $|f(x)-f(y)| < \epsilon/4$, and pick $N_2 > \log_2(1/\delta)$. Notice then that for $n > N_2$, the intervals formed by the endpoints of P_n have a length of less than δ (since their length is $1/2^n$).
- Set $N = \max\{N_1, N_2\}$.

For $n > N$, we have the following. By definition, on all $p_k \in P_n$ we will have that $|f_n(p_k) - f(p_k)| < 1/n \leq 1/N_1 < \epsilon/4$. On all other points $x \notin P_n$, it is the case that $x \in (p_j, p_{j+1})$ for some j. Then:

$$|f(x) - f_n(x)| = \left| f(x) - \Big(\frac{f_n(p_{j+1}) - f_n(p_j)}{p_{j+1} - p_j}(x - p_j) + f(p_j) \Big) \right|$$

$$\leq |f(x) - f(p_j)| + \left| \frac{f_n(p_{j+1}) - f_n(p_j)}{p_{j+1} - p_j}(x - p_j) \right| < \epsilon/4 + |f_n(p_{j+1}) - f_n(p_j)|$$

$$= \epsilon/4 + |f_n(p_{j+1}) - f(p_j) + f(p_j) - f_n(p_j)| \leq \epsilon/4 + |f_n(p_{j+1}) - f(p_j)| + |f(p_j) - f_n(p_j)|$$

$$\leq \epsilon/4 + \epsilon/4 + |f_n(p_{j+1}) - f(p_{j+1}) + f(p_{j+1}) - f(p_j)| \leq \epsilon/2 + |f_n(p_{j+1}) - f(p_{j+1})| + |f(p_{j+1}) - f(p_j)| = \epsilon$$

Notice that here we used the uniform continuity of f as well as the defining property of N_1. We have shown that $|f(x) - f_n(x)| < \epsilon$ for all $x \in (0,1)$, which means of course that $d(f_n, f) < \epsilon$, and thus that $f_n \to f$. Therefore, the countable set F is dense in $C[0,1]$.

4.2 The relative metric

Exercise 1
Complete the proof of Proposition 4.13.

Solution.
We first restate parts (ii) and (iii) of 4.13 in some more detail:
(ii) We say that a set $F \subset A \subset M$ is closed in (A, d) iff $A \setminus F$ is open in A. We then have that F is closed in (A, d) if and only if $F = A \cap C$, where C is closed in (M, d).
(iii) Given any $E \subset A \subset M$, we define the closure of E in A, $\mathrm{cl}_A(E)$, as the smallest set that is closed in A and contains E. Then, for any $E \subset A$, $\mathrm{cl}_A(E) = A \cap \mathrm{cl}_M(E)$.
We now begin with (ii).
$\Longrightarrow$: Suppose F is closed in (A, d). Then $A \setminus F$ is open in A, which by part (i) is equivalent to the existence of $U \subset M$ such that $A \setminus F = U \cap A$, with U open in M. Therefore $F = A \setminus (A \setminus F) = A \setminus (U \cap A) = A \cap U^c$. By definition, U^c is closed in M, thus completing the proof in this direction.
$\Longleftarrow$: Suppose now that $F = A \cap C$ for some C closed in (M, d). Then C^c is open in M, and $A \setminus F = A \setminus (A \cap C) = A \cap C^c$, which means that $A \setminus F$ is open in A, and so by definition F is closed in A.
Continuing, for part (iii) we have the following.

Pick any $E \subset A$, and suppose $x \in \mathrm{cl}_A(E)$. It is easy to see that $x \in A$, by the definition of this closure. In addition, for any closed set $C \subset M$ such that $E \subset C$, we have that the set $C \cap A$ is closed in A, and contains E, therefore $C \cap A \supset \mathrm{cl}_A(E)$, and so $x \in C \cap A$. Since this holds for any closed C, it holds for $\mathrm{cl}_M(E)$, and therefore $x \in A \cap \mathrm{cl}_M(E)$, which implies $\mathrm{cl}_A(E) \subset A \cap \mathrm{cl}_M(E)$.
In the other direction, suppose $x \in A \cap \mathrm{cl}_M(E)$, and let F be any set closed in A containing E. By part (ii), $F = A \cap C$ for some C closed in M, for which it holds that $\mathrm{cl}_M(E) \subset C \implies x \in C$. But then because $x \in A$ also holds, we obtain $x \in A \cap C = F$. Because this holds for *any* F closed in A, it holds for $\mathrm{cl}_A(E)$, which means $x \in \mathrm{cl}_A(E)$, thus proving the inclusion in the other direction, and thus showing that in fact $\mathrm{cl}_A(E) = A \cap \mathrm{cl}_M(E)$.

Exercise 62
Suppose that A is open in (M, d) and that $G \subset A$. Show that G is open in A if and only if G is open in M. Is the result still true if "open" is replaced everywhere by "closed"? Explain.

Solution.
$\implies$: Suppose $G \subset A$ is open in A. Then there exists $U \subset M$ open such that $G = A \cap U$. Therefore, G is open as the intersection of two open sets.
$\impliedby$: Suppose $G \subset A$ is open in M. Then, by writing $G = A \cap G$, we trivially fulfill the definition of G being open in A as well.
Now suppose that we replace "open" with "closed" everywhere, to obtain the statement "if A is closed in (M, d), then any $F \subset A$ is closed in A if and only if F is closed in M".
$\implies$: Suppose F is closed in A, in which case $F = A \cap C$ for some C closed in M. Since A is also closed, we have that F is closed in M as the intersection of two closed sets.
$\impliedby$: Suppose now that $F \subset A$ is closed in M. Then we can write $F = A \cap F$, where F is closed in M, and so we also have that F is closed in A, which means that the statement we obtained is also true.

Exercise 63
Is there a nonempty subset of $\mathbb{R}$ that is open when considered as a subset of $\mathbb{R}^2$? Closed?

Solution.
Identify $\mathbb{R}$ with the set $\{(x, 0) | x \in \mathbb{R}\} \subset \mathbb{R}^2$ and consider any $S \subset \mathbb{R}, S \neq \emptyset$ that is open in $\mathbb{R}$ as well as in $\mathbb{R}^2$. Then for any $p \in S$ (there exists at least one since S is not empty), there must exist $\epsilon > 0$ such that $B_\epsilon^{\mathbb{R}^2}(p) \subset \mathbb{R}$. p is of the form $p = (x, 0)$, so the point $(x, \epsilon/2)$ is contained in this open ball, and would therefore have to belong in $\mathbb{R}$, which is by definition impossible. Therefore, no nonempty open subset of $\mathbb{R}$ can be empty as a subset of $\mathbb{R}^2$. Turning the discussion to closed sets instead, consider the set $[-1, 1] \subset \mathbb{R}$. It is easy to see that this is a closed set in $\mathbb{R}^2$ (by e.g. using the sequence-based definition of closed-ness). Furthermore, if C is the (closed) unit disk in $\mathbb{R}^2$ we can see that $C \cap \mathbb{R} = [-1, 1]$, and thus $[-1, 1]$ is indeed closed in $\mathbb{R}$.

Exercise 68
If A is a separable subset of M (that is, if A has a countable dense subset of its own), show that $\overline{A}$ is also separable.

Solution.
Let $S = \{s_1, s_2, s_3, \ldots\}$ be a countable dense subset of A. Recall that $\overline{A}$ consists precisely of elements y such that some sequence $(x_n) \subset A$ converges to y (this sequence may be constant). For any given y, we have two cases:

- $x_n \to y$ is a constant sequence, which means $y \in A$. Then by the separability of A, there exists $(s_k) \subset S$ such that $s_k \to y$.
- $x_n \to y$ is not constant, and therefore $y \notin A$ (in this case, y lies on the boundary of A). We describe a process for forming a sequence $(s_k) \subset S$ such that $s_k \to y$. Let $\epsilon_i = 1/i$, and find $N_i > 0$ such that for $n \geq N_i, d(x_n, y) < \epsilon_i/2$. Find also $K_i > 0$ such that for $k \geq K_i, d(s_k, x_{N_i}) < \epsilon_i/2$ (this is possible due to A being separable). By the triangle inequality we obtain that $d(s_{K_i}, y) \leq d(s_{K_i}, x_{N_i}) + d(x_{N_i}, y) < \epsilon_i/2 + \epsilon_i/2 = \epsilon_i$. Therefore, we set the i-th term of the constructed sequence to be s_{K_i}. The fact that ϵ_i decrease monotonically means that the sequence formed does indeed converge to y.

Since we were able to show that for any $y \in \overline{A}$ there exists $(s_k) \subset S \subset A \subset \overline{A}$ such that $(s_k) \to y$ and S is countable, we have that $\overline{A}$ is indeed separable.

Chapter 5

Open Sets and Closed Sets

5.1 Continuous Functions

Exercise 4
Show that $\mathcal{X}_\Delta : \mathbb{R} \to \mathbb{R}$, the characteristic function of the Cantor set, is discontinuous at each point of Δ.

Solution.
We will do this by contradiction. Suppose $\epsilon = 1/2$ and $x \in \Delta$. If $\mathcal{X}_\Delta$ is continuous at x, then there exists $\delta > 0$ such that whenever $|y - x| < \delta$ it holds that $|\mathcal{X}_\Delta(y) - \mathcal{X}_\Delta(x)| < \epsilon$. Pick any such y. By definition of the characteristic function, this implies that $\mathcal{X}_\Delta(y) = 1$, and so that $y \in \Delta$. But then by exercise 22 of Chapter 2 (page 29 on the book), we know that for our $x, y \in \Delta$, where either $x < y$ or $y < x$, there exists $x < z < y$ or $y < z < x$ respectively such that $z \notin \Delta$. The consequence of this is that $|z - x| < \delta$ but $\mathcal{X}_\Delta(z) = 0$, and so $|\mathcal{X}_\Delta(z) - \mathcal{X}_\Delta(x)| = 1 > 1/2$, a contradiction. Therefore $\mathcal{X}_\Delta$ is not continuous at x.

Exercise 7
(a) If $f : M \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$, show that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open subsets of M.
(b) Conversely, if the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every $a \in \mathbb{R}$, show that f is continuous.
(c) Show that f is continuous even if we assume only that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every *rational* a.

Solution.
(a) For any $a \in \mathbb{R}$, consider the set $S = \{y \in \mathbb{R} \mid y > a\}$. This is clearly seen to be an open set. Then, theorem 5.1 (equivalent definitions of continuity) guarantees that $f^{-1}(S)$ is open in M. This set equals $f^{-1}(S) = \{x \in M \mid f(x) \in S\} = \{x \in M \mid f(x) > a\}$. The same holds if "$>$" is replaced by "$<$", thus completing the proof.
(b) Suppose N is any open subset of $\mathbb{R}$. If we can show that $f^{-1}(N)$ is open in M, then we will have equivalently shown that f is continuous. We know (theorem 4.6) that N can be written as a countable union of disjoint open intervals:

$$N = \bigcup_{n=1}^{\infty} I_n, I_n = (a_n, b_n),$$

where $I_n \cap I_m = \emptyset$ for $n \neq m$. Then we have that $f^{-1}(N) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$. Note that in general the boundaries of these intervals may equal the two infinities. First, in the special case where this union is simply the interval $(-\infty, \infty)$ (i.e., the chosen N equals $\mathbb{R}$), we can immediately conclude that $f^{-1}(N) = M$: if this was not the case, then there would exist some $x \in M$ such that $f(x) \notin N = \mathbb{R}$, which is clearly impossible. Note that, trivially, $f^{-1}(N)$ is then open (example 4.1 (a), page 51). In every other case, no interval I_n is of this form, and N is a union of *at least one* interval (a_n, b_n) where not both of the endpoints are infinities. We are now interested in $f^{-1}(I_n)$. If both a_n, b_n are real numbers, write

$I_n = (-\infty, b_n) \cap (a_n, \infty)$. If not, proceed with $I_n = (-\infty, b_n)$ or $I_n = (a_n, \infty)$. In the last two cases, one easily sees that $f^{-1}(N) = \{x \in M : f(x) < b_n\}, f^{-1}(N) = \{x \in M : f(x) > a_n\}$ respectively, both of which are open by our hypothesis. In the first case, $f^{-1}(N)$ equals a finite intersection of open sets (by utilizing what we proved just now), and as such is again open. Then we conclude that $f^{-1}(N) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$ is also open as an arbitrary union of open sets, thus also obtaining that f is continuous.
(c) The proof here relies on exercise 7 of chapter 4 (page 55), which states that every open set in $\mathbb{R}$ can be written as the union of countably many open intervals with rational endpoints. Since disjointness was never necessary in (b), everything else follows in exactly the same way, because we know that $\{x : f(x) > a\}, \{x : f(x) < a\}$ are open for rational a, and every interval we will encounter here will be of this form. Therefore, f is again continuous.

Exercise 8
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some interval $(-a, a)$.
(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≥ 0" replaced by "> 0"? Explain.

Solution.
(a) Suppose $f(x) = c > 0$, and set $\epsilon = c/2 > 0$. Due to the continuity of f, there exists $\delta > 0$ such that whenever $|x - 0| < \delta$, it holds that $|f(x) - f(0)| < \epsilon$. This implies that $-c/2 < f(x) - c < c/2 \implies c/2 < f(x) < 3c/2$. But then setting $a = \delta$ means that for $x \in (-a, a)$ it holds that $f(x) > 0$.
(b) Pick any $y \in \mathbb{R}$. We know that we can always find a sequence $(x_n) \subset \mathbb{Q}$ such that $x_n \to y$. By the continuity of f, it must hold that $f(x_n) \to f(y)$. Suppose now that $f(y) < 0$, and set $\epsilon = -f(y)/2$. Then we can find $N > 0$ such that whenever $n \geq N$ it holds that:

$$|f(x_n) - f(y)| < \epsilon \implies f(y)/2 < f(x_n) - f(y) < -f(y)/2 \implies f(x_n) < f(y)/2 < 0,$$

which is a contradiction. Therefore it must hold that $f(y) \geq 0$ for every $y \in \mathbb{R}$. Now for the second part, consider the function $f(x) = (x - \sqrt{2})^2$. Clearly, f is continuous and strictly positive for all $x \in \mathbb{R} \setminus \{\sqrt{2}\}$, which means that it is more specifically continuous for all $x \in \mathbb{Q}$. However, $f(\sqrt{2}) = 0$, which is not strictly positive, and thus the assertion that "$f(x) > 0$ for every rational x implies that $f(x) > 0$ for every real x" fails.

Exercise 9
Let $A \subset M$. Show that $f : (A, d) \to (N, \rho)$ is continuous at $a \in A$ if and only if, given $\epsilon > 0$, there is a $\delta > 0$ such that $\rho(f(x), f(a)) < \epsilon$ whenever $d(x, a) < \delta$ *and* $x \in A$. We paraphrase this statement by saying that "f has a point of continuity relative to A".

Solution.
Recall the definition of continuity in a metric space M: $f : (M, d) \to (N, \rho)$ is continuous at $a \in M$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $d(x, y) < \delta$ it holds that $\rho(f(x), f(y)) < \epsilon$. We know that if M is a metric space, $A \subset M$ is also a metric space. Therefore, the definition of continuity for functions $f : (A, d) \to (N, \rho)$ states that f is continuous at $a \in A$ precisely when, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(f(x), f(a)) < \epsilon$ whenever $d(x, a) < \delta$ for x inside the metric space, which in this case is A.

Exercise 10
Let $A = (0, 1] \cup \{2\}$, considered as a subset of $\mathbb{R}$. Show that every function $f : A \to \mathbb{R}$ is continuous, relative to A, at 2.

Solution.
Pick any $\epsilon > 0$, and set e.g. $\delta = 1/3$. Observe then that $B^A_{1/3}(2) = \{2\}$. Therefore, for every $x \in A$ such that $|x - 2| < 1/3$, it trivially holds that $|f(x) - f(2)| < \epsilon$, and so f is continuous at 2 relative to A.

Exercise 14
A continuous function on $\mathbb{R}$ is completely determined by its values on $\mathbb{Q}$. Use this to "count" the continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Solution.
First of all, to see that a continuous function is indeed determined by its values on $\mathbb{Q}$, suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Then if $x \in \mathbb{R} \setminus \mathbb{Q}$, we know that there exists a sequence $(x_n) \subset \mathbb{Q}$ such that $x_n \to x$. It is the case that $f(x_n) = g(x_n)$ for all n, and by the continuity of f, g we have that $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$, therefore $f = g$.
We begin our proof by noticing that the above observation implies that there exists an injection $F : C^0 \to \mathbb{R}^{\mathbb{Q}}$.
Furthermore, there exists a trivial injection G from $\mathbb{R}$ to C^0: map any real number a to the clearly continuous $f(x) = a$.
Therefore, if we can now show that $\mathbb{R}^{\mathbb{Q}} \sim \mathbb{R}$, the existence of F guarantees the existence of an injection from the set of continuous functions to $\mathbb{R}$ as well. This injection coupled with G will allow us to apply the Cantor-Schröder-Bernstein theorem to obtain that in fact $C^0 \sim \mathbb{R}$. In showing $\mathbb{R}^{\mathbb{Q}} \sim \mathbb{R}$, the following lemma will be useful:
If $A \sim B$, and C is countable, then $A^C \sim B^C$. Indeed, we can construct a bijection $F : A^C \to B^C$ in the following manner. First, let h be a bijection from A to B, which we know exists. If $f \in A^C$, we have $f(c_1) = a_1, f(c_2) = a_2, \ldots$. We obtain $b_1, b_2, \ldots$ by $h(a_i) = b_i$, and we define $g \in B^C$ as $g(c_1) = b_1, g(c_2) = b_2, \ldots$. We then define $F(f) = g$. To prove injectivity, suppose $F(f_1) = F(f_2)$ for $f_1 \neq f_2$. This means that there exists $c_n \in C$ such that $f_1(c_n) \neq f_2(c_n)$, and so $h(f_1(c_n)) = F(f_1)(c_n) \neq h(f_2(c_n)) = F(f_2)(c_n)$, due to the injectivity of h. But then $F(f_1) \neq F(f_2)$, a contradiction. For surjectivity, pick $g \in B^C$, which is completely determined by the values $g(c_1), g(c_2), \ldots$. Let then $a_n \in A, a_n = h^{-1}(g(c_n))$, which are well-defined since h is a bijection. Then by defining $f \in A^C$ such that $f(c_n) = a_n$ we have $F(f) = g$, and thus F is also a surjection, and therefore a bijection.
Notice now the following. We know $\mathbb{R} \sim [0,1]$, and by the lemma above we then have $\mathbb{R}^{\mathbb{Q}} \sim [0,1]^{\mathbb{Q}}$ since $\mathbb{Q}$ is countable. For the remaining proof, we can thus work with $[0,1]^{\mathbb{Q}}$.
Pick any $f \in [0,1]^{\mathbb{Q}}$, and let $f(q_n) = a_n \in [0,1]$, where we write a_n in binary form, and if it can be written as both ending in infinite 1s and in infinite 0s, we choose the latter. In a manner similar to Cantor's diagonal argument, we construct the following list:

$$f(q_1) = a_{11}.a_{12}a_{13}\ldots$$

$$f(q_2) = a_{21}.a_{22}a_{23}\ldots$$

$$\vdots$$

Define then $b_f = 0.a_{11}a_{21}a_{12}a_{31}a_{22}a_{13}\ldots \in [0,1]$. We claim that the function $F : [0,1]^{\mathbb{Q}} \to [0,1]$ defined by $F(f) = b_f$ is an injection. Indeed, if $b_f = b_g$ for two such f, g, if b_f, b_g are equal on all corresponding digits, then $f = g$ trivially. If not, then one of them, say b_f, ends in infinite ones, and the other in infinite zeros. However, this would imply that the construction above yields a b_f ending in infinite ones, and thus that $f(q_i)$ end in infinite ones, which is impossible due to how we chose to construct these b. Therefore F is injective. We now also find an injection $g : [0,1] \to [0,1]^{\mathbb{Q}}$. For $a \in [0,1]$, write $a = a_1.a_2a_3\ldots$ in binary form, again choosing infinite zeros over infinite ones as a postfix whenever necessary. We then define $g(a) = f$ such that $f(q_i) = 0.0\ldots0a_i$, where the prefix is of length $i-1$ (for $f(q_1)$, we have $f(q_1) = a_1$). Notice again that this is easily seen to be an injection, since $g(a) = g(b)$ implies that all corresponding digits of a, b are equal. By the Cantor-Schröder-Bernstein theorem, we then have that $[0,1]^{\mathbb{Q}} \sim [0,1]$, and thus that $\mathbb{R}^{\mathbb{Q}} \sim \mathbb{R}$, i.e. there are "as many" continuous functions as real numbers.

Exercise 17
Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

Solution.
Suppose that there exists $x \in M \setminus D$ such that $f(x) \neq g(x)$, which means that $\rho(f(x), g(x)) = \epsilon > 0$. By the denseness of D, there exists a sequence $(x_n) \subset D, x_n \to x$. Because f, g are both continuous, it holds that $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. Therefore, there exist $N_1, N_2 > 0$ such that for $n \geq \max\{N_1, N_2\}$ it holds that $\rho(f(x_n), f(x)) < \frac{\epsilon}{2}, \rho(g(x_n), g(x)) < \frac{\epsilon}{2}$. Notice, furthermore, that $f(x_n) = g(x_n)$ since the two functions are equal on D. Then, by the triangle inequality we obtain that:

$$\epsilon = \rho(f(x), g(x)) \leq \rho(f(x), f(x_n)) + \rho(f(x_n), g(x)) = \rho(f(x), f(x_n)) + \rho(g(x_n), g(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$\implies \epsilon < \epsilon,$$

which is a contradiction. Therefore $f(x) = g(x)$ on all of M.
Now suppose that f is onto, and pick any $y \in N$. Then there exists $x \in M$ such that $f(x) = y$. Furthermore, by the denseness of D there exists $(x_n) \subset D$ such that $x_n \to x$. The continuity of f then implies $f(x_n) \to f(x) = y$. But $f(x_n)$ is a sequence entirely in $f(D)$, which means that $f(D)$ is dense in N, since y was picked arbitrarily in N.

Exercise 18
Let $f : (M, d) \to (N, \rho)$ be continuous, and let A be a separable subset of M. Prove that $f(A)$ is separable.

Solution.
We need to find a countable dense subset of $f(A)$. Since A is separable, it countains a countable and dense subset $S \subset A, S = \{a_1, a_2, \ldots\}$. Pick any element $y \in f(A)$, which means there exists $a \in A$ such that $y = f(a)$. Since S is dense, there exists a sequence $(a_n) \subset A$ such that $a_n \to a$, and because f is continuous, $f(a_n) \to f(a)$. Notice that this holds for any $y \in f(A)$. The consequence is that if we form the set $S' = \{f(a_1), f(a_2), \ldots\}$, which is clearly countable because S is countable, we are always able to find a sequence in s' converging to any element of A. Therefore $f(A)$ is separable.

Exercise 19
A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a **Lipschitz condition** if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. More economically, we may say that f is Lipschitz (or Lipschitz with constant K if a particular constant seems to matter). Show that $\sin x$ is Lipschitz with constant $K = 1$. Prove that a Lipschitz function is (uniformly) continuous.

Solution.
Pick any two $x, y \in \mathbb{R}$, and let $h = x - y$. Then we have that:

$$|f(x) - f(y)| = |\sin(x) - \sin(y)| = |\sin(y + h) - \sin(y)| = \left|2\sin(\frac{h}{2})\cos(\frac{y + y + h}{2})\right|$$
$$\leq \left|2\sin(\frac{h}{2})\right| \leq 2\left|\frac{h}{2}\right| \leq |h| = |x - y|,$$

where we used the identity $\sin(a) - \sin(b) = 2\sin((a - b)/2)\cos((a + b)/2)$, the fact that $|\cos(a)| \leq 1$ for any a, and the inequality $|\sin(a)| \leq |a|$ for any a. We've thus shown that sin is Lipschitz continuous with $K = 1$. Now we want to show that any Lipschitz continuous function (with constant K) is uniformly continuous. Pick any ϵ, and set $\delta = \frac{\epsilon}{K}$. Then observe that, whenever $|x - y| < \delta$:

$$|f(x) - f(y)| \leq K|x - y| < K\delta = K\frac{\epsilon}{K} = \epsilon,$$

which is precisely the definition of uniform continuity for f.

Exercise 20
If d is a metric on M, show that $|d(x, z) - d(y, z)| \leq d(x, y)$ and conclude that the function $f(x) = d(x, z)$ is continuous on M for any fixed $z \in M$. This says that $d(x, y)$ is *separately continuous* - continuous in each variable separately.

Solution.
We have already shown this inequality in exercise 2 of chapter 3. What we are then asked to show is that for any *fixed* $z \in M$, and for any $x \in M$, the function $f(x) = d(x, z)$ is continuous at x. To do this, pick any $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $d(x, y) < \delta$ it holds that $|f(y) - f(x)| < \epsilon$ (note that the metric space containing the codomain is simply $\mathbb{R}$). Observe that:

$$|f(y) - f(x)| = |d(y, z) - d(x, z)| \leq d(x, y)$$

Therefore, if we set $\delta = \epsilon/2$, we obtain that $|f(y) - f(x)| \leq \epsilon/2 < \epsilon$, and thus that f is continuous on M.

Exercise 23
Define $S : c_0 \to c_0$ by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. That is, S shifts the entries forward and puts 0 in the empty slot. Show that S is an isometry (into).

Solution.
We need to show that for any two sequences $(x_n), (y_n) \in c_0, d(S((x_n)), S((y_n))) = d((x_n), (y_n))$, where $d((x_n), (y_n)) = \sup_n |x_n - y_n|$. We notice that:

$$d(S((x_n)), S((y_n))) = \sup_n \{|0 - 0|, |x_1 - y_1|, |x_2 - y_2|, \ldots\} = \sup_n \{|x_1 - y_1|, |x_2 - y_2|, \ldots\},$$

since every term here is non-negative. This last quantity equals precisely $d((x_n), (y_n))$, which completes the proofthat S is an isometry.

Exercise 24
Let V be a normed vector space. If $y \in V$ is fixed, show that the maps $a \mapsto ay$ from $\mathbb{R}$ to V and $x \mapsto x + y$ from V to V are continuous.

Solution.
Let $g : \mathbb{R} \to V, g(a) = ay$ for a fixed $y \in V$. Note that g is obviously continuous if $y = 0$, so from now on assume $y \neq 0$. We need to show that for any $a \in \mathbb{R}$, it holds that for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|b - a| < \delta$, it holds that $||g(b) - g(a)|| < \epsilon$ (for the usual metric on V). By the properties of normed vector spaces, we have that:

$$||g(b) - g(a)|| = ||by - ba|| = |b - a| \cdot ||y||$$

For a given $\epsilon > 0$, set $\delta = \frac{\epsilon}{||y||}$. Then, whenever $|b - a| < \delta$, the equality above shows that $||g(b) - g(a)|| < \delta ||y|| = \epsilon$, i.e., that g is indeed continuous.
Now let instead $g : V \to V$ be $g(x) = x + y$ for a fixed $y \in V$. For any $x, z \in V$ we have that:

$$||g(z) - g(x)|| = ||(z + y) - (x + y)|| = ||z - x||$$

Therefore, for any given $\epsilon > 0$, if we set $\delta = \epsilon$, whenever $||z - x|| < \delta$, the equality above tells us that $||g(z) - g(x)|| < \delta = \epsilon$, and so that g is continuous on any $x \in V$.

Exercise 25
A function $f : (M, d) \to (N, \rho)$ is called **Lipschitz** if there is a constant K such that $\rho(f(x), f(y)) \leq K d(x, y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

Solution.
Suppose first that $K = 0$. Then for any two $x, y \in M$ it holds that $\rho(f(x), f(y)) \leq 0$. By the properties of metrics, we know that this means that $\rho(f(x), f(y)) = 0$, and this in turn means that $f(x) = f(y)$ for all x, y, thus f is constant, and trivially continuous.
If $K \neq 0$, pick any $x \in M$ and any $\epsilon > 0$. Set $\delta = \frac{\epsilon}{K}$, and observe that whenever $d(x, y) < \delta$, we have that:

$$\rho(f(x), f(y)) \leq K d(x, y) < K\delta = K\frac{\epsilon}{K} = \epsilon,$$

which is precisely the definition of continuity for f.

Exercise 26
Provide the answer to a question raised in Chapter Three by showing that integration is continuous. specifically, show that the map $L(f) = \int_a^b f(t)dt$ is Lipschitz with constant $K = b - a$ for $f \in C[a, b]$.

Solution.
We recall first the Frechet metric for continuous functions on $[a, b] : d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$. The question raised in the beginning of Chapter Three was whether integration is continuous, which, in other words, would mean that the quantities $L(f), L(g)$ are "close" whenever f, g are close. To answer this, we observe that:

$$|L(f) - L(g)| = \left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| = \left| \int_a^b (f(t) - g(t))dt \right|$$

Now we call the mean value theorem for integration: there exists $c \in (a, b)$ such that $\int_a^b (f(t) - g(t))dt = (f(c) - g(c))(b - a)$. Therefore:

$$|L(f) - L(g)| = |(f(c) - g(c))(b - a)| = (b - a)|f(c) - g(c)| \leq (b - a) \max_{a \leq t \leq b} |f(t) - g(t)| = (b - a)d(f, g)$$

We've therefore shown that $|L(f) - L(g)| \leq (b - a)d(f, g)$, which means precisely that L is Lipschitz with $K = b - a$.

Exercise 28
Define $g : l_2 \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} x_n/n$. Is g continuous?

Solution.
Let $y = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, i.e. the sequence defined by $y_k = \frac{1}{k}$, which as we know is in l_2. We observe then that for a given sequence of sequences $(x^{(n)})$ that converges to a sequence x (all sequences involved are in l_2), we have that:

$$|g(x^{(n)}) - g(x)| = \left| \sum_{k=1}^{\infty} (x_k^{(n)} - x_k)/k \right| \leq \sum_{k=1}^{\infty} |(x_k^{(n)} - x_k)/k|,$$

where we note that while at this point we are not sure that the sums converge, we will see that the RHS is in fact bounded, and so we would obtain the above result by initially examining finite sums only. Specifically, we observe that we can apply the Cauchy-Schwarz inequality to the RHS, since both $(x^{(n)} - x)$ and y are in l_2. From this we obtain that:

$$|g(x^{(n)}) - g(x)| \leq ||x^{(n)} - x||_2 \cdot ||y||_2$$

We've thus shown that g is Lipschitz with constant $||y||_2$, and so also continuous.

Exercise 30
Let $f : (M, d) \to (N, \rho)$. Prove that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset M$ if and only if $f^{-1}(\mathring{B}) \subset (f^{-1}(B))^{\circ}$ for every $B \subset N$. Give an example of a continuous f such that $f(\overline{A}) \neq \overline{f(A)}$ for some $A \subset M$.

Solution.
For ease of notation, name the three propositions:
(a): f is continuous
(b): $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset M$
(c): $f^{-1}(\mathring{B}) \subset (f^{-1}(B))^{\circ}$ for every $B \subset N$
(a) $\implies$ (b): Pick any $A \in M$ and any $y \in f(\overline{A})$. We need to show that $y \in \overline{f(A)}$. Because $y \in f(\overline{A})$, we have that there exists $x \in \overline{A}$ such that $f(x) = y$. We recall that $x \in \overline{A}$ is equivalent to the existence of a sequence $(x_n) \subset A$ such that $x_n \to x$. Because f is continuous, we must then have that $f(x_n) \to f(x) = y$.

By definition of the image of a function, $f(x_n) \in f(A)$ for every n. Once again, $f(x_n) \to y$ with $(f(x_n)) \subset f(A)$ is equivalent to $y \in \overline{f(A)}$.
(b) $\implies$ (a): For the converse, pick any $x \in M$. If no sequences in M converge to x, then there exists $\delta > 0$ such that $d(x, y) > \delta$ for every $y \in M, y \neq x$. Now pick any $\epsilon > 0$ and observe that $d(x, y) < \delta$ is only satisfied when $y = x$, in which case of course $\rho(f(x), f(y)) = 0 < \epsilon$, so f is continuous at x. In the other case, we have that there exists at least one sequence in M that converges to x. By way of contradiction, assume that there f is not continuous, and thus that for some $x_n \to x$ it does not hold that $f(x_n) \to f(x)$. By negating the definition of the limit, we obtain that there exists $\epsilon > 0$ such that for every $N > 0$ there exists $n \geq N$ such that $\rho(f(x_n), f(x)) \geq \epsilon$. Gather then an infinite sequence $(f(x_i))$ for which $\rho(f(x_i), f(x)) \geq \epsilon$, and notice that $x_i \to x$ since the corresponding x_i form a subsequence of (x_n). We observe now that if $A = (x_i), \overline{A} = \{x_1, x_2, \ldots, \} \cup \{x\}$: this is due to the uniqueness of the limit, as well as the fact that $x_i \to x$. Then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$, which in turn means that there exists a sequence $(f(y_j)) \subset f(A)$ such that $f(y_j) \to f(x)$ (by the definition of the closure). Since $f(A) = \{f(x_1), f(x_2), \ldots\}$, we conclude that each $f(y_j)$ must equal some $f(x_i)$. But then $\rho(f(y_j), f(x)) \geq \epsilon$ for every j, and thus $f(y_j)$ cannot convert to x, a contradiction. Therefore f must be continuous.
(a) $\implies$ (c): Pick any $B \subset N$, and any $x \in f^{-1}(\mathring{B})$. This means that there exists $\epsilon > 0$ such that, if $y = f(x)$, then $B^{\rho}_{\epsilon}(y) \subset B$. Since f is continuous, there exists $\delta > 0$ such that whenever $d(x', x) < \delta$ it holds that $\rho(f(x'), y) < \epsilon$. Therefore, for every such $x', f(x') \in B^{\rho}_{\epsilon}(y)$, which implies $f(x') \in B$. We thus obtain that $x' \in f^{-1}(B)$, and since this holds for every x' with $d(x', x) < \delta$, we have that $B^d_{\delta}(x) \subset f^{-1}(B)$, which means precisely that $x \in (f^{-1}(B))^{\circ}$.
(c) $\implies$ (a): Suppose $B \subset N$ is open. Then $B = \mathring{B}$ and we have that $f^{-1}(B) = f^{-1}(\mathring{B}) \subset (f^{-1}(B))^{\circ}$. We know of course that $f^{-1}(B)^{\circ} \subset f^{-1}(B)$, which means $f^{-1}(B) = f^{-1}(B)^{\circ}$, which is equivalent to $f^{-1}(B)$ being open. Since this holds for any open B, we have one of the conditions that are known to be equivalent to f being continuous: namely, that the inverse image of any open set under f is also open.
For the requested example, consider $M = \mathbb{N}$ equipped with the discrete metric, $N = \mathbb{R}$ and $f : M \to N, f(i) = \frac{1}{i}, i \neq 0, f(0) = 1$, which we easily see is continuous. Then if $A = \mathbb{N}^+$, we notice that $\overline{A} = A$ and $f(\overline{A}) = \{1, 1/2, 1/3, \ldots\}$. However, $\overline{f(A)} = \{0, 1, 1/2, 1/3, \ldots\}$, and so $f(\overline{A}) \neq \overline{f(A)}$.

Exercise 31
Let $f : (M, d) \to (N, \rho)$.
(a) If $M = \bigcup_{n=1}^{\infty} U_n$, where each U_n is an open set in M, and if f is continuous on each U_n relative to that U_n, show that f is continuous on M.
(b) If $M = U_{n=1}^{N} E_n$, where each E_n is a closed set in M, and if f is continuous on each E_n relative to that E_n, show that f is continuous on M.
(c) Give an example showing that f can fail to be continuous on all of M if, instead, we use a countably infinite union of closed sets $M = \bigcup_{n=1}^{\infty} E_n$ in (b).

Solution.
(a) Pick any $x \in M$. Then $x \in U_i$ for at least one i. Because U_i is open, there exists δ_1 such that $B^d_{\delta_1}(x) \subset U_i$. In addition, because f is continuous on x relative to U_i, for every $\epsilon > 0$ there exists $\delta_2 > 0$ such that whenever $d(x', x) < \delta_2$ and $x' \in U_i$ it holds that $\rho(f(x'), f(x)) < \epsilon$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $d(x', x) < \delta$ we have $x' \in U_i$, and also $d(x', x) < \delta_2$, thus $\rho(f(x'), f(x)) < \epsilon$. Therefore f is indeed continuous on x in the general sense.
(b) Again, pick any $x \in M$. Then $x \in E_i$ for at least one i. If $x \in \mathring{E}_i$, an argument like the one in (a) serves to show that f is continuous at x. The complication here is it may be the case that $x \in \partial E_i$, in which case for any open ball $B^d_{\delta}(x), B^d_{\delta}(x) \cap E_i^c \neq \emptyset$. Suppose then that $x \in \partial E_i$, pick any $\epsilon > 0$ and obtain a $\delta_i > 0$ from the definition of relative continuity to E_i. Then set $S = B^d_{\delta_i}(x) \cap E_i^c$. Any point $x' \in S$ must belong in at least one $E_j, j \neq i$. For all such E_j, we examine whether their intersection with S is non-empty (call the corresponding set of indices I), and if it isn't, we know that it must contain at least x. It must thus hold that f is continuous on x relative to each $E_j, j \in I$. From this we can obtain *at most* $N - 1$ quantities $\delta_j > 0$ that satisfy the definition of relative continuity of f on x (with respect to each of these sets). The minimum of these quantities is thus well defined, and we call it δ'. Notice then that $\delta = \min\{\delta_1, \delta'\}$ is such that whenever $d(x', x) < \delta$, it holds that x' is in *some* E_i, and relative continuity with respect to that E_i

guarantees that $\rho(f(x'), f(x)) < \epsilon$. Therefore f is indeed continuous on x in the general sense.
(c) Consider first the following construction of subsets of $\mathbb{R}^2$. We define l_i to be the line passing through the origin at an angle of $0 + \sum_{j=1}^{i-1} \frac{\pi}{2^j}$ with the $x'x$ axis. Geometrically, this forms lines that start from an angle of 0 radians with the $x'x$ axis and tend towards an angle of π, but without ever attaining it. Let now U_i be the subset of $\mathbb{R}^2$ contained inside l_i, l_{i+1}, the lines included, i.e. increasingly small "pie slices". Notice that each of these sets is closed. Now discard U_i for i even, which yields a countably infinite collection of sets $\mathcal{U}$ that are closed, and every pair of which only has the origin in its intersection. Define M to be the union of all members of U, which is a metric space as a subset of $\mathbb{R}^2$, and let $N = \mathbb{R}$. Then define $f : M \to N$ as:

$$f(x, y) = ix + iy, (x, y) \in U_i$$

This function is clearly continuous on each U_i relative to U_i, since this restriction of it is linear (furthermore, it is well-defined since it is zero at the origin no matter which U_i we examine in the formula). However, it fails to be continuous at (0, 0): notice that these are increasingly "steep" planes, each of which requires an increasingly small $\delta > 0$ in the definition of continuity for a given $\epsilon > 0$, and as such the argument used in (b) regarding a well-defined minimum δ fails.

Exercise 34
Show that d is continuous on $M \times M$, where $M \times M$ is supplied with "the" product metric (see Exercise 3.46). This says that d is jointly continuous, that is, continuous as a function of two variables. [Hint: If $x_n \to x, y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$.]

Solution.
In order to discuss continuity of d as a function defined on $M \times M$, we first need to equip $M \times M$ with an appropriate metric. By exercise 3.46, we are given three choices all of which define equivalent metrics, from which we choose $e : M \times M \to \mathbb{R}$:

$$e((x, y), (a, b)) = d(x, a) + d(y, b)$$

Now we need to show that whenever $(x_n, y_n) \to^e (x, y)$, it holds that $d(x_n, y_n) \to d(x, y)$. Select any $\epsilon > 0$, in which case we can find $N > 0$ such that whenever $n \geq N$ it holds that:

$$e((x_n, y_n), (x, y)) < \epsilon \implies d(x_n, x) + d(y_n, y) < \epsilon$$

By the triangle inequality for d we have that:

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \implies d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n)$$

$$\implies d(x_n, y_n) - d(x, y) < \epsilon$$

And also:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \implies d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y, y_n)$$

$$\implies d(x, y) - d(x_n, y_n) < \epsilon$$

Combining these two inequalities yields $|d(x, y) - d(x_n, y_n)| < \epsilon$, so by setting $\delta = \epsilon$ we obtain the desired definition of continuity for d.

Exercise 35
Show that a set U is open in M if and only if $U = f^{-1}(V)$ for some continuous function $f : M \to \mathbb{R}$ and some open set V in $\mathbb{R}$.

Solution.
$\Longrightarrow$: If U is open, we have that U^c is closed. Define then the function $f : M \to \mathbb{R}, f(x) = d(x, U^c)$. Notice first that this is continuous, as discussed in preposition 5.4. Furthermore, $f(x) > 0$ iff $x \notin \overline{U^c} = U^c \iff x \in U$, where we used the fact that U^c is closed. Consequently, we have that $f^{-1}(\mathbb{R} \setminus \{0\}) = U$, and of course the set $\mathbb{R} \setminus \{0\}$ is open, thus completing the proof in this direction.
$\Longleftarrow$: If we think of f as a continuous function between (M, d) and $(\mathbb{R}, \rho)$ where ρ is the usual metric on $\mathbb{R}$, then theorem 5.1 guarantees that $f^{-1}(V) = U$ is open in M.

Exercise 36
Suppose that we are given a point x and a sequence (x_n) in a metric space M, and suppose that $f(x_n) \to f(x)$ for every continuous, real-valued function f on M. Does it follow that $x_n \to x$ in M? Explain.

Solution.
Suppose that it is not the case that $x_n \to x$. This means that there must exist $\epsilon > 0$ such that for every $N > 0$ there exists $n \geq N$ such that $d(x_n, x) \geq \epsilon$. Select the subsequence (x_{n_k}) that is formed by picking a different such x_n for every value of N (these can always be found since the statement holds for arbitrarily large N). Let then $U = B_{\epsilon/2}(x)$, and $f(y) = d(y, U)$, which we know (prep. 5.4) is continuous. Note now that $x \in U$, so $f(x) = 0$. Note also that for any $z \in U$ and any x_{n_k} we have that:

$$d(z, x_{n_k}) \geq |d(z, x) - d(x_{n_k}, x)|,$$

and we know that $d(z, x) \leq \epsilon/2, d(x_{n_k}, x) \geq \epsilon$, so we can rewrite this as $d(z, x_n) \geq d(x_{n_k}, x) - d(z, x)$. Continuing:

$$d(z, x_{n_k}) \geq \epsilon - d(z, x) > \epsilon - \epsilon/2 = \epsilon/2,$$

and since this holds for any $z \in U$ by taking the infimum over them we can conclude that $d(x_{n_k}, U) \geq \epsilon/2$. Therefore $f(x_{n_k}) \geq \epsilon/2$ for all x_{n_k}. What this now means is that we can never satisfy the definition of $f(x_n) \to f(x) = 0$: we can never find N such that $f(x_n) < \epsilon/2$, since at least one such x_{n_k} will violate this no matter how large N is. Because we are given $f(x_n) \to f(x)$, we arrive at a contradiction. This means that it must indeed be the case that $x_n \to x$.

Exercise 41
Let C be a closed set in $\mathbb{R}$ and let $f : C \to \mathbb{R}$ be continuous. Show that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = f(x)$ for every $x \in C$. We say that g is a continuous *extension* of f to all of $\mathbb{R}$. In particular, every continuous function on the Cantor set Δ extends continuously to all of $\mathbb{R}$. [Hint: The complement of C is the countable union of disjoint open intervals. Define g by "connecting the dots" across each of these open intervals.]

Solution.
Since C is closed, we know that $B = C^c$ is open. In particular, as stated in the hint, this means that we can write B as a countable union of disjoint open intervals, $B = \cup_{j=1}^{\infty} I_j, I_j \cap I_k = \emptyset$ whenever $j \neq k$. We are therefore tasked with finding values for g on each $I_j = (a_j, b_j)$ such that it is continuous everywhere. For $x \in I_j$, consider defining:

$$g(x) = f(a_j) + (f(b_j) - f(a_j))\frac{x - a_j}{b_j - a_j},$$

and for $x \in C, g(x) = f(x)$. We then know that g is continuous on all of C except for the endpoints of all I_j due to f's continuity. Furthermore, g is continuous inside each I_j as a linear function. Lastly, by an easy computation of left and right limits on each a_j, b_j we conclude that g is also continuous on each of them as well. Therefore g is continuous everywhere. An immediate consequence of this is the fact that continous functions on Δ can be continuously extended to all of $\mathbb{R}$, given that Δ is closed (Example 4.12).

5.2 Homeomorphisms

Exercise 43
If you are not already convinced, prove that two metrics d, ρ on a set M are equivalent if and only if the identity map on M is a homeomorphism from (M, d) to (M, ρ).

Solution.
Let i be the identity map from (M, d) to (M, ρ).
$\implies$: It is obvious that i is a bijection. In addition, if $x_n \to^d x$, we have that $x_n \to^\rho x$ by the equivalence of the two metrics. But this then means precisely that $i(x_n) \to^\rho i(x)$ in (M, ρ), so i is continuous. With a symmetrical argument we can show that i^{-1} is continuous, thus completing the proof that i is a homeomorphism from (M, d) to (M, ρ).
$\impliedby$: If i is a homeomorphism from (M, d) to (M, ρ), then $x_n \to^d x$ iff $x_n \to^\rho x$, which is of course the definition of d, ρ being equivalent.

Exercise 44
Check that the relation "is homeomorphic to" is an equivalence relation on pairs of metric spaces.

Solution.
Denote the relation "is homeomorphic to" with $\sim$.
Reflexivity: For any metric space (M, d), it is clear that $(M, d) \sim (M, d)$, since the identity function is a trivial homeomorphism.
Symmetry: If $(M, d) \sim (N, \rho)$, then there exists a homeomorphism f from (M, d) to (N, ρ), and so by the defining properties of homeomorphisms, f^{-1} is also a homeomorphism from (N, ρ) to (M, d).
Transitivity: If $(M, d) \sim (N, \rho), (N, \rho) \sim (O, \sigma)$, let f, g be the corresponding homeomorphisms. Then the function $h : M \to O, h(x) = g(f(x))$ is a bijection as a composition of bijections. Furthermore, if $x_n \to^d x$, we have that $f(x_n) \to^\rho f(x)$ by the continuity of f, and then that $g(f(x_n)) \to^\sigma g(f(x))$ by the continuity of g, which means $h(x_n) \to^\sigma h(x)$, which means h is continuous. A symmetrical argument shows that h^{-1} is also continuous, thus completing the proof that h is a homeomorphism from (M, d) to (O, σ), thus showing that $(M, d) \sim (O, \sigma)$. We've thus shown that $\sim$ is an equivalence relation.

Exercise 46
Show that every metric space is homeomorphic to one of finite diameter. [Hint: Every metric is equivalent to a bounded metric.]

Solution.
As stated in the hint, recall that in Ex. 42, Ch. 3 we showed that any metric d is equivalent to the *bounded* metric $\tau(x, y) = \min\{d(x, y), 1\}$. For any metric space (M, d), consider then the metric space (M, τ). Clearly, for any two elements x, y of (M, τ) we have that $\tau(x, y) \leq 1$, and so the diameter of (M, τ) is at most 1. Furthermore, Exercise 43 guarantees that (M, d) is homeomorphic to (M, τ), thus completing the proof.

Exercise 47
Define $E : \mathbb{N} \to l_1$ by $E(n) = (1, \ldots, 1, 0, \ldots)$, where the first n entries are 1 and the rest are 0. Show that E is an isometry (into).

Solution.
We need to show that $l_1(E(n), E(m)) = |n - m|$ for any two $n, m \in \mathbb{N}$. Suppose WLOG that $n < m$, which means that:

$$l_1(E(n), E(m)) = \sum_{i=1}^{\infty} |E(n)_i - E(m)_i| = \sum_{i=n+1}^{m} |0 - 1| = m - (n + 1) + 1 = m - n = |n - m|$$

Exercise 48
Prove that $\mathbb{R}$ is homeomorphic to $(0,1)$ and that $(0,1)$ is homeomorphic to $(0,\infty)$. Is $\mathbb{R}$ *isometric* to $(0,1)$? To $(0,\infty)$? Explain.

Solution.
Let $f:\mathbb{R}\to(0,1), f(x)=\frac{1}{1+e^{-x}}$, which is easily seen to be continuous and bijective, with a continuous inverse ($f^{-1}(x)=\log(\frac{x}{1-x})$). Therefore $\mathbb{R}$ is homeomorphic to $(0,1)$. Let also $g:(0,1)\to(0,\infty), g(x)=\log(\frac{1}{1-x})$, which in $(0, 1)$ assumes strictly positive and unbounded values, is continuous and bijective, and thus is a homeomorphism between $(0,1)$ and $(0,\infty)$. To examine whether $\mathbb{R}$ is isometric to $(0,1)$, we consider the following. Suppose that this is indeed true, which means that there exists a homeomorphism f between them with the property $|f(x)-f(y)|=|x-y|$ for all $x,y\in\mathbb{R}$. Fix $x=0$ and let $y\to\infty$. Then the RHS here is unbounded. However, since $f(x),f(y)\in(0,1)$ the LHS is clearly bounded, and so we arrive at a contradiction, which means that the two spaces cannot be isometric. Similarly, suppose f is now a homeomorphism between $\mathbb{R}$ and $(0,1)$ with the property $|f(x)-f(y)|=|x-y|, x,y\in\mathbb{R}$. For any $x\in\mathbb{R}$, it has to hold that:

$$|f(x)-f(0)|=|x-0|=|-x|=|f(-x)-f(0)|$$

This means that for any given $x\in\mathbb{R}$, either $f(x)-f(0)=f(-x)-f(0)\implies f(x)=f(-x)$ or $f(x)-f(0)=-f(-x)+f(0)\implies f(x)+f(-x)=2f(0)$. The first case is impossible due to f being one-to-one as a homeomorphism. Consequently, the second equality has to hold for all x. Notice, however, that the RHS is fixed and positive, whereas the LHS is a sum of positive quantities, which means that $f(x)$ must be bounded. But then $|f(x)-f(0)|=|x|$ leads to a contradiction as $x\to\infty$. Therefore, the two spaces are not isometric.

Exercise 50
Let (M,d) denote the set $\{0\}\cup\{(1/n):n\geq 1\}$ under its usual metric. Define a second metric ρ on M by setting $\rho(1/n,1/m)=|1/n-1/m|$ for $m,n\geq 2, \rho(1/n,1)=1/n$ for $n\geq 2, \rho(1/n,0)=1-1/n$ for $n\geq 2$, and $\rho(0,1)=1$. Show that (M,d) and (M,ρ) are homeomorphic but that the identity map from (M,d) to (M,ρ) is *not* continuous.

Solution.
Consider the following map $f:(M,d)\to(M,\rho)$:

$$f(x)=\begin{cases}1, & x=0\\ 0, & x=1\\ x, & \text{otherwise}\end{cases}$$

It is easy to see that this is a bijection, and that $f^{-1}(x)=f(x)$. Suppose now that $x_n\to^d x$ for $(x_n)\subset(M,d)$. We observe first that for any sequence that does not eventually become constant, this means that $x=0$ and that after some N, all x_n must be of the form $\frac{1}{m_n}$ for $m_n>1$. For any $\epsilon>0$, set $\delta=\epsilon$, in which case whenever $d(x_n,x)<\epsilon$, we have that:

$$\rho(f(x_n),f(x))=\rho\Big(\frac{1}{m_n},f(0)\Big)=\Big|\frac{1}{m_n}\Big|=d(x_n,x)<\epsilon,$$

which shows that f is continuous, since this holds for an arbitrary sequence (x_n) (note that it trivially holds for sequences that eventually becomes constant). One can show that the converse holds as well: $f(x_n)\to^\rho f(x)\implies x_n\to^d x$. This is done by observing that under ρ, the only non-trivial way that this can happen is if $f(x)=1$, and if after some N, $f(x_n)$ eventually equal $\frac{1}{m_n}$. The remaining proof is then the same as above. From this we conclude that f is a homeomorphism between $(M,d),(M,\rho)$.
Define the sequence $(x_n)\subset M$ by $x_n=\frac{1}{n}$. Notice that with the standard metric d on M, we have $d(x_n,0)=|0-\frac{1}{n}|=\frac{1}{n}$, which goes to zero as $n\to\infty$. However, with the metric ρ, we have that $\rho(x_n,0)=1-\frac{1}{n}$ for $n\geq 2$, which *does not* go to zero as $n\to\infty$. Therefore, d,ρ do not generate the same convergent sequences, i.e. they are not equivalent. Exercise 43 Ch. 5 then tells us that the identity map on M cannot be a homeomorphism from (M,d) to (M,ρ).

Exercise 51
Let (M, ρ) be a separable metric space and assume that $\rho(x, y) \leq 1$ for every $x, y \in M$. Given a countable dense set $\{x_n : n \geq 1\} \subset M$, define a map $f : M \to H^\infty$, from M into the Hilbert cube (Exercise 3.10), by $f(x) = (\rho(x, x_n))_{n=1}^\infty$.
(i) Prove that f is one-to-one and continuous. In fact, f satisfies $d(f(x), f(y)) \leq \rho(x, y)$, where d is the metric on H^∞.
(ii) Fix $\epsilon > 0$ and $x \in M$. Find $\delta > 0$ such that $\rho(x, y) < \epsilon$ whenever $d(f(x), f(y)) < \delta$. Conclude that f is a homeomorphism into H^∞.

Solution.
(i) Suppose $f(x) = f(y)$ for $x, y \in M$. Then the sequences $(\rho(x, x_n))_{n=1}^\infty, (\rho(y, x_n))_{n=1}^\infty$ are equal, which means each of their respective terms is equal, i.e. $\rho(x, x_n) = \rho(y, x_n)$ for every $n = 1, 2, \ldots$. Due to the separability of M, there exists a sequence $(x_{n_k}) \subset \{x_n : n \geq 1\}$ that converges to x. Since we have that $\rho(x_{n_k}, x) = \rho(x_{n_k}, y)$, we conclude that it is also the case that $x_{n_k} \to y$, and from the uniqueness of the limit we have $x = y$, which means f is one-to-one. The metric used for the Hilbert cube is:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

For any two $x, y \in M$, we therefore have that:

$$d(f(x), f(y)) = \sum_{n=1}^{\infty} 2^{-n} |\rho(x, x_n) - \rho(y, x_n)| \leq \sum_{n=1}^{\infty} 2^{-n} \rho(x, y) = \rho(x, y)$$

Notice, now, that the continuity of f follows immediately from this, where for each ϵ the required δ equals ϵ.
(ii) We fix $\epsilon > 0$ and $x \in M$. Notice, first that since the set of x_n is dense, we can find one such x_i such that $\rho(x, x_i) < \epsilon/4$. Now, based on this i, suppose that $\delta = \epsilon/2^{i+2}$ and that y is such that we have $d(f(x), f(y)) < \delta$. Then:

$$d(f(x), f(y)) < \delta \implies \sum_{n=1}^{\infty} 2^{-n} |\rho(x, x_n) - \rho(y, x_n)| < \delta \implies 2^{-i} |\rho(x, x_i) - \rho(y, x_i)| < \delta,$$

where we separated the term corresponding to the x_i we picked above. Thus:

$$|\rho(x, x_i) - \rho(y, x_i)| < \delta 2^i = \epsilon/4 \implies \rho(y, x_i) < \rho(x, x_i) + \epsilon/4 = \epsilon/2$$

Now we have that $\rho(x, y) < \rho(x, x_i) + \rho(y, x_i) = \epsilon/4 + \epsilon/2 = \epsilon \cdot 3/4 < \epsilon$, and so we have effectively shown that f^{-1} (which is well-defined when thinking about it as a function inside the range of f and *not* all of H^∞), thus showing that f is a homeomorphism into H^∞.

Exercise 52
Prove Theorem 5.5:
Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Then the following are equivalent:
(i) f is a homeomorphism
(ii) $x_n \to^d x \iff f(x_n) \to^\rho f(x)$
(iii) G is open in $M \iff f(G)$ is open in N
(iv) E is closed in $M \iff f(E)$ is closed in N
(v) $\hat{d}(x, y) = \rho(f(x), f(y))$ defines a metric on M equivalent to d

Solution.
(i) $\implies$ (ii): As a homeomorphism, f is continuous, and so is f^{-1}. Then Theorem 5.1 (ii) guarantees that whenever $x_n \to^d x$, it holds that $f(x_n) \to^\rho f(x)$. Also, whenever $f(x_n) \to^\rho f(x)$ it holds that $f^{-1}(f(x_n)) \to^d f^{-1}(f(x))$, i.e. that $x_n \to^d x$.
(ii) $\implies$ (i): This utilizes Theorem 5.1 (ii) in the other direction, and yields that f, f^{-1} are both continuous, thus that f is a homeomorphism.

(i) $\implies$ (iii): Let $G \subset M$ be open. Then $(f^{-1})^{-1}(G)$ is open in N by the continuity of f^{-1} and Theorem 5.1 (iv). We have that $f(G) = (f^{-1})^{-1}(G)$ (relatively simple to prove via set inclusions), therefore $f(G)$ is open in N. Conversely, if $f(G)$ is open in N, then a direct application of Theorem 5.1 (iv) yields that $f^{-1}(f(G)) = G$ is open in M. Therefore we've shown that (iii) holds.
(iii) $\implies$ (i): Notice that each of the two directions of the given equivalence can be used to apply Theorem 5.1 (iv) to obtain that both f, f^{-1} are continuous.
(i) $\implies$ (iv), (iv) $\implies$ (i): This is done in the same way as above, except that we use Theorem 5.1 (iii) instead of (iv).
(ii) $\implies$ (v): Whenever $x_n \to^d x$, we have that $f(x_n) \to^\rho f(x)$, which means precisely that $\rho(f(x_n), f(x)) \to 0$, and so $\hat{d}(x_n, x) \to 0$, which means $x_n \to^{\hat{d}} x$. Since we also have that whenever $f(x_n) \to^\rho f(x)$ it holds that $x_n \to^d x$, we obtain that whenever $\rho(f(x_n), f(x)) = \hat{d}(x_n, x) \to 0$, it also holds that $d(x_n, x) \to 0$. We've thus shown that $d, \hat{d}$ are equivalent.
(v) $\implies$ (ii): This is a simple reversal of the immediately preceding argument.

Exercise 53
Suppose that we are given a point x and a sequence (x_n) in a metric space M, and suppose that $f(x_n) \to f(x)$ for every continuous real-valued function f on M. Prove that $x_n \to x$ in M.

Solution.
We have already shown this in Exercise 36, Ch. 5.

Exercise 54
Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Prove that the following are equivalent: (i) f is a homeomorphism and (ii) $g : N \to \mathbb{R}$ is continuous if and only if $g \circ f : M \to \mathbb{R}$ is continuous. [Hint: Use the characterization given in Theorem 5.5 (ii).]

Solution.
(i) $\implies$ (ii): Suppose that f is a homeomorphism. First, suppose that g is continuous. Then, for any $x_n \to^d x$ we have that $f(x_n) \to^\rho f(x)$ due to the continuity of f, and by the continuity of $g, g(f(x_n)) \to g(f(x))$. Therefore $g \circ f$ is clearly continuous. Conversely, suppose $g \circ f$ is continuous and pick any $(y_n) \in N$ that converges to $y \in N$. To show that g is continuous, we need to show that $g(y_n) \to g(y)$. Because f is a homeomorphism, it is the case that $f^{-1}(y_n) \to f^{-1}(y)$. Now recall that $g \circ f$ is continuous, and thus $(g \circ f)(f^{-1}(y_n)) \to (g \circ f)(f^{-1}(y))$, which simplifies to $g(y_n) \to g(y)$. Therefore, g is indeed continuous.
(ii) $\implies$ (i): Pick any $x_n \to^d x$, and define $g(y) = \rho(y, f(x))$, which is always well-defined since f is one-to-one. This is a continuous mapping (Exercise 20 Ch. 5), and therefore from our assumption we have that $g \circ f$ is also continuous. This means that $\rho(f(y), f(x))$ is continuous on all $y \in M$, and so that whenever $x_n \to^d x$, it must be the case that $\rho(f(x_n), f(x)) \to \rho(f(x), f(x)) = 0$, which means precisely that $f(x_n) \to^\rho f(x)$, i.e. that f is continuous. We now need to show that f^{-1} is continuous as well. Suppose thus that $y_n \to y, (y_n) \subset N, y \in N$. Because f is one-to-one and onto, there exist unique $(x_n), x$ such that $f(x_n) = y_n, f(x) = y$. Keeping x fixed, consider the function $g(y) = d(f^{-1}(y), x)$, which is once again continuous. Therefore, $g \circ f$ is also continuous, and because $f(x_n) \to^\rho f(x)$, it must be the case that:

$$(g \circ f)(f(x_n)) \to (g \circ f)(f(x)) \implies d(f^{-1}(f(x_n)), x) \to d(f^{-1}(f(x)), x) \implies d(x_n, x) \to 0,$$

which means $x_n \to^d x$, which means f^{-1} is continuous. We've therefore established that f is a homeomorphism.

Exercise 55
Let $f : (M, d) \to (N, \rho)$ be a homeomorphism. Prove that M is separable iff N is separable.

Solution.
M separable $\implies$ N separable: Pick any element $y \in N$. Then there exists a unique $x \in M$ such that $f(x) = y$ (since f is a bijection). Because M is separable, there exists a countable dense subset A, which means that there exists a sequence $(x_n) \subset A$ such that $x_n \to x$. Due to f being a homeomorphism, we obtain $f(x_n) \to f(x) = y$. Furthermore, because A is countable and f is a bijection, $f(A)$ is also countable,

and we have $(f(x_n)) \subset f(A)$. Since y was picked arbitrarily, we conclude that $\overline{f(A)} = N$, i.e. $f(A)$ is dense in N and N is thus separable.
N separable $\implies$ M separable: The proof in this direction relies on exactly the same argument except for using f^{-1} instead of f.

Exercise 56
Let $f : (M, d) \to (N, \rho)$.
(i) We say that f is an **open** map if $f(U)$ is open in N whenever U is open in M; that is, f maps open sets to open sets. Give examples of a continuous open map that is not open and open map that is not continuous. [Hint: Please note that the definition depends on the target space N.]
(ii) Similarly, f is **closed** if it maps closed sets to closed sets. Give examples of a continuous map that is not closed and a closed map that is not continuous.

Solution.
(i) Consider $M = N = \mathbb{R}$ and the function $f(x) = 1$. This is clearly a continuous map, but it is not open: $\mathbb{R}$ is open and yet f maps it to $\{1\}$, a closed set. Consider also $M = \mathbb{R}, N = \{0, 1\}$ and $f(x) = 0, x < 0, f(x) = 1, x \geq 0$. Notice that f is not continuous, but that by this definition of N, all possible subsets of N are open and thus f is trivially an open map.
(ii) Consider $M = N = \mathbb{R}$ and the clearly continuous $f(x) = \frac{1}{x}$. Here $f(M) = (0, \infty)$, and M is closed, whereas $f(M)$ thus isn't. Therefore f is continuous but not closed. For a map that is closed but not continuous, observe that our second example for (i) works here as well: every possible subset of N is closed, so f is closed but it is not continuous.

Exercise 57
Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Show that the following are equivalent: (i) f is open (ii) f is closed and (iii) f^{-1} is continuous. Consequently, f is a homeomorphism if and only if both f, f^{-1} are open (closed).

Solution.
(i) $\implies$ (ii): Let $S \subset M$ be closed. Then S^c is by definition open, and $f(S^c)$ is open because f is open. We thus have that $f(S^c)^c = N \setminus f(S^c)$ is closed. This set consists of all elements $y \in N$ such that they do not equal any $f(x)$ for $x \in S^c$. Because f is one-to-one and onto, each must necessarily equal $f(x)$ for *some unique* $x \in M$, and thus the only possibility is $x \in M \setminus S^c = S$, therefore $N \setminus f(S^c) = f(S)$, and so $f(S)$ is closed, which means f is closed.
(ii) $\implies$ (i): Let $S \subset M$ be open, which means S^c is closed and $f(S^c)$ is also closed since f is closed. Therefore $f(S^c)^c = N \setminus f(S^c)$ is open, and the remaining argument is as above, showing that this set equals $f(S)$, which means f is open.
(i) $\implies$ (iii): Suppose f is open, and pick any open $S \subset M$. If we can show that $(f^{-1})^{-1}(S)$ is open in N, by Theorem 5.1 we will also have that f^{-1} is continuous. We have that $(f^{-1})^{-1}(S) = f(S)$ (easily shown using the definitions and bijectivity of f), and so by our assumption this set is indeed open and f^{-1} is continuous.
(ii) $\implies$ (iii): This is shown in much the same way, changing "open" to "closed" and again using Theorem 5.1.
(iii) $\implies$ (i): Suppose now f^{-1} is continuous and $S \subset M$ is open. Using again the fact that $(f^{-1})^{-1}(S) = f(S)$ and Theorem 5.1 in the other direction, we obtain that $f(S)$ is open, and so f is open.
(iii) $\implies$ (ii): Once again it suffices to change "open" to "closed" above.

Exercise 60
Let (M, d) be a metric space, and let τ be the discrete metric on M. Then, (M, d) and (M, τ) are homeomorphic if and only if every subset of M is open in (M, d) if and only if every function $f : (M, d) \to \mathbb{R}$ is continuous.

Solution.
For convenience, we refer to the three statements as (i), (ii), (iii) in the order they are given. Then:

(i) $\implies$ (ii): Suppose f is a homeomorphism between $(M, d), (M, \tau)$. We have seen (Example 4.1 (c)) that in a discrete space every subset is open. Due to the bijectivity of f, every subset $S \subset (M, d)$ is mapped uniquely to a subset $S' = f(S) \subset (M, \tau)$, and from Theorem 5.5 each of them is open iff the other one is too. Therefore since every $f(S)$ is open in (M, τ), every $S \subset (M, d)$ is open too.
(ii) $\implies$ (i): The argument above also works here, since Theorem 5.5 presents equivalent conditions: if every subset of (M, d) is open, then of course every subset of (M, d) is open iff its image in (M, τ) is open, and so the identity function is a homeomorphism between them.
(i) $\implies$ (iii): Let g be a homeomorphism between $(M, d), (M, \tau)$ and $x_n \to^d x$. Then by Theorem 5.5 we have that $g(x_n) \to^\tau g(x)$, which means for $\epsilon = 1/2$ there exists $N > 0$ such that for $n \geq N, \tau(g(x_n), g(x)) < \epsilon \implies g(x_n) = g(x)$ due to the definition of the discrete metric. But since g is a bijection, this yields $x_n = x$, so *every convergent sequence in* (M, d) *is eventually constant*. Clearly then $f(x_n) \to f(x)$ will also hold for any function $f : (M, d) \to \mathbb{R}$, and so any such f is continuous.
(iii) $\implies$ (ii): For any set $S \subset (M, d)$, consider the indicator function $f_S(x) = 1, x \in S, f_S(x) = 0$ otherwise. By the hypothesis, this must be continuous. Now pick any $S \subset (M, d)$, and any $x \in S$, and any sequence $(x_n) \subset M, x_n \to^d x$. Due to the continuity of f_S, it must hold that $f_S(x_n) \to f_S(x)$, which means that for $\epsilon > 1/2$ there must exist $N > 0$ such that for $n \geq N, |f_S(x_n) - f_S(x)| < 1/2 \implies |f_S(x_n) - 1| < 1/2$. By the definition of f_S this is only possible if all such $f_S(x_n)$ equal 1, which happens precisely when $x_n \in S$, and so the sequence (x_n) contains only finitely many points not in S. Therefore by Theorem 4.7, S is open, and since it was picked arbitrarily we have shown (ii).

5.3 The Space of Continuous Functions

Exercise 62
If $f, g \in C[a, b]$, show that $||fg||_\infty \leq ||f||_\infty ||g||_\infty$. Also show that $||\max\{f, g\}||_\infty \leq \max\{||f||_\infty, ||g||_\infty\}$, and that $||f||_\infty \leq ||g||_\infty$ whenever $|f| \leq |g|$.

Solution.
For any $x \in [a, b], |f(x)| \leq \max_{a \leq t \leq b}|f(t)| = ||f||_\infty$, and similarly $|g(x)| < ||g||_\infty$. We thus obtain $|f(x)g(x)| \leq ||f||_\infty \cdot ||g||_\infty$ and:

$$||fg||_\infty = \max_{a \leq t \leq b} |f(t)g(t)| \leq ||f||_\infty \cdot ||g||_\infty$$

We also have that:

$$||\max\{f, g\}||_\infty = \max_{a \leq t \leq b} |\max\{f, g\}(t)| = \max_{a \leq t \leq b} |\max\{f(t), g(t)\}| \leq \max\{||f||_\infty, ||g||_\infty\},$$

since every value of f, g is smaller than the corresponding maximum value of the function.
Lastly, suppose $|f| \leq |g|$. This means that for all $t \in [a, b], |f(t)| \leq g(t)|$. Consequently, the same inequality must hold for their maximum values, which means $||f||_\infty \leq ||g||_\infty$.

Exercise 64
Given $n \in \mathbb{N}$ and $f, g \in C(\mathbb{R})$, let $d_n(f, g) = \max_{|t| \leq n}|f(t) - g(t)|$. Then d_n defines a pseudometric on $C(\mathbb{R})$ (Why?) Show that $d(f, g) = \sum_{n=1}^\infty 2^{-n} d_n(f, g)/(1 + d_n(f, g))$ defines a metric on $C(\mathbb{R})$.

Solution.
To show that d_n is a pseudometric on $C(\mathbb{R})$, we have the following:

- From its definition, it is obvious that $d_n(f, g) \geq 0$ for any two $f, g \in C(\mathbb{R})$.
- Again from the definition, $d_n(f, g) = d_n(g, f)$ for any two $f, g \in C(\mathbb{R})$.
- If $f = g$, then clearly $d_n(f, g) = 0$. *However*, what makes d_n a pseudometric instead of a metric is the fact that for e.g. $f(t) = t, t \leq n, f(t) = 2(t - n) + n$ and $g(t) = t, t \leq n, g(t) = 3(t - n) + n$ (both clearly continuous, and not equal functions), $d_n(f, g) = 0$.

- For $f, g, h \in C(\mathbb{R})$, we have that:

$$d_n(f,h) = \max_{|t|\leq n} |f(t) - h(t)| = \max_{|t|\leq n} |f(t) - g(t) + g(t) - h(t)|$$

Since it is the case that $|f(t) - g(t) + g(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$, the inequality must also hold when applying $\max_{|t|\leq n}$ to both sides. Furthermore, the maximum of a sum equals at most the sum of the two maxima, and so we obtain:

$$d_n(f,h) \leq \max_{t\leq n} |f(t) - g(t)| + \max_{t\leq n} |g(t) - h(t)| = d_n(f,g) + d_n(g,h),$$

which is the triangle inequality for d_n.

Now we examine d in the same manner:

- Due to the definition of this metric as an infinite sum, we need to first determine whether it is always well-defined, i.e. whether the sum always converges. To see this, observe that $d_n(f,g)/(1+d_n(f,g)) \leq 1$, and so each partial sum of d is bounded above by the geometric series $\sum_{n=1}^{\infty} 2^{-n}$, which we know converges. Since the partial sums are also non-decreasing, $d(f,g)$ does in fact always converge.

- By the non-negativity of each d_n, it is clear that $d(f,g) \geq 0$ always holds as well.

- Similarly, by the symmetry of each d_n, $d(f,g) = d(g,f)$.

- Clearly, $d(f,f) = 0$ for any $f \in C(\mathbb{R})$. Now pick any two $f, g \in C(\mathbb{R})$, $f \neq g$. Then there exists $x \in \mathbb{R}$ such that $f(x) \neq g(x)$. This means that for $n = \lceil x \rceil, d_n(f,g) = \max_{|t|\leq n} |f(t) - g(t)| > 0$, and so $d(f,g) > 0$ as well, thus showing that $d(f,g) = 0$ iff $f = g$.

- For the triangle inequality, we use Exercise 7 of Chapter 3, by observing that each term of the sum of d equals $2^{-n}\sigma(d_n(f,g))$ (for the σ of Exercise 6 of that Chapter). Although d_n is not a metric, the proof that for some metric d, $\sigma(d(x,y))$ satisfies the triangle inequality if $\sigma(x+y) \leq \sigma(x) + \sigma(y)$ only depended on d satisfying the triangle inequality, which is the case for d_n. Since the triangle inequality is satisfied for each term of the convergent sum d, it also holds for d itself, thus completing the proof that d is a metric.

Chapter 6

Connectedness

6.1 Connected Sets

Exercise 1
Supply the missing details in the proof of Lemma 6.3:
Let E be a subset of a metric space M. If U and V are disjoint open sets in E, then there are disjoint open sets A and B in M such that $U = A \cap E, V = B \cap E$.

Solution.
The proof provided in the book arrives in full detail at the observation that for any two $x \in U, y \in V$ it is the case that $E \cap B_{\epsilon_x}(x) \cap B_{\epsilon_y}(y) = \emptyset$. Notice that this shows that if these two open balls intersect, then their intersection must be entirely contained in $M \setminus E$. If it was the case that they didn't intersect at all, then simply writing A as the union of all $B_{\epsilon_x}(x)$ and B as the union of all $B_{\delta_y}(y)$ would make these open as unions of open sets, and of course disjoint. The claim that is given in the proof is that one can in fact achieve this by substituting ϵ_x with $\epsilon_x/2$ and δ_y with $\delta_y/2$, i.e., it holds that $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$ for any two $x \in U, y \in V$. Suppose then that there exists $z \in M \setminus E$ such that $z \in B_{\epsilon_x/2}(x), z \in B_{\delta_y/2}$. We then have:

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{\epsilon_x}{2} + \frac{\delta_y}{2}$$

WLOG, assume $\delta_y \leq \epsilon_x$, in which case we conclude $d(x, y) < 2\frac{\epsilon_x}{2} = \epsilon_x$. But then this means $y \in B_{\epsilon_x}(x)$, and of course $y \in E$. We originally had that $E \cap B_{\epsilon_x}(x) = U$, and so $y \in U$, which is a contradiction since $y \in V$ and $U \cap V = \emptyset$. Therefore $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$, and so indeed the open sets $A = \cup\{B_{\epsilon_x/2}(x), x \in U\}, B = \bigcup\{B_{\delta_y/2}(y), y \in V\}$ are disjoint.

Exercise 2
Show that the only nonempty connected subsets of Δ are singletons. (We would say that Δ is *totally disconnected.*)

Solution.
Suppose that there exists a connected subset of Δ that contains at least two points $x, y, x < y$. By Theorem 6.4, it must also contain the entire interval $[x, y]$, and so it must hold that $[x, y] \subset \Delta$. But this directly contradicts Exercise 22 of 2.2: Δ contains no nonempty open intervals, but here it must contain (x, y), a contradiction. Therefore, the only nonempty connected subsets of Δ are singletons.

Exercise 5
If E and F are connected subsets of M with $E \cap F \neq \emptyset$, show that $E \cup F$ is connected.

Solution.
Suppose for the sake of contradiction that $E \cup F$ is disconnected. Then there exist open sets $A, B, A \cap B = \emptyset$ such that $E \cup F \subset A \cup B, (E \cup F) \cap A \neq \emptyset, (E \cup F) \cap B \neq \emptyset$. By the first of these conditions, we obtain $E \subset A \cup B$ and $F \subset A \cup B$. By the second of these conditions, we obtain that at least one of $E \cap A \neq \emptyset$

and $F \cap A \neq \emptyset$ must be true. By the third of these conditions, we obtain that at least one of $E \cap B \neq \emptyset$ and $F \cap B \neq \emptyset$ must be true. Then we have the following possibilities:

- If $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$ both hold, because we also have $E \subset A \cup B$, we obtain the contradiction that E is disconnected.
- If $E \cap A \neq \emptyset$ and $E \cap B = \emptyset$, we have necessarily that $F \cap B \neq \emptyset$. Furthermore, because $E \subset A \cup B$, it must be the case that $E \subset A$. Now, if it holds that $F \cap A \neq \emptyset$, by the same reasoning as above we arrive at the contradiction that F is disconnected. Therefore it must be the case that $F \cap A = \emptyset$, which means $F \subset B$. Notice then that $E \subset A, F \subset B, A \cap B = \emptyset$, but $E \cap F \neq \emptyset$ by the hypothesis. This is clearly a contradiction as well.
- The case $E \cap A = \emptyset$ and $E \cap B \neq \emptyset$ is symmetric to the above.

Therefore, $E \cup F$ must be connected.

Exercise 6
More generally, if $\mathcal{C}$ is a collection of connected subsets of M, all having a point in common, prove that $\bigcup \mathcal{C}$ is connected. Use this to give another proof that $\mathbb{R}$ is connected.

Solution.
Suppose for the sake of contradiction that $\bigcup \mathcal{C}$ is disconnected, which means that there exist open sets $A, B, A \cap B = \emptyset$ such that $\bigcup \mathcal{C} \subset A \cup B, \bigcup \mathcal{C} \cap A \neq \emptyset, \bigcup \mathcal{C} \cap B \neq \emptyset$. Now, this means that there must exist $C_i, C_j \in \mathcal{C}$, such that $C_i \cap A \neq \emptyset$ and $C_j \cap B \neq \emptyset$. Furthermore, it must hold that $C_i \cap C_j \neq \emptyset$, since all sets forming $\bigcup \mathcal{C}$ have a common point. These now imply that $(C_i \cup C_j) \cap A \neq \emptyset$ and $(C_i \cup C_j) \cap B \neq \emptyset$, while it is also clear that $(C_i \cup C_j) \subset (A \cup B)$, thus yielding that $C_i \cup C_j$ is disconnected. However, in Exercise 5 we showed that since C_i, C_j are connected and $C_i \cap C_j \neq \emptyset$, their union is also connected, which means that we arrive at a contradiction. Therefore $\bigcup \mathcal{C}$ is also connected. As an application, consider the fact that $\mathbb{R} = \bigcup_{a>0}[-a, a]$, and that each of these intervals is connected, which means that since all of the intervals have 0 as a common point, $\mathbb{R}$ is also connected.

Exercise 7
If every pair of points in M is contained in some connected set, show that M is itself connected.

Solution.
Fix a point $x \in M$, and observe that $M = \cup_{y \in M}(\{x\} \cup \{y\})$. We know that each pair x, y is contained in some connected set A_y. Furthermore, because M is the entire metric space, it must also hold that $M = \cup_{y \in M} A_y$ (i.e., it cannot be that some point in some A_y is not contained in the metric space). We have thus written M as a union of connected sets that have a point in common, namely, x. By exercise 6, M is therefore also connected.

Exercise 9
If $A \subset B \subset \overline{A} \subset M$, and if A is connected, show that B is connected. In particular, $\overline{A}$ is connected.

Solution.
Suppose B is disconnected. Observe first that obviously it cannot be that $B = A$. Because $B \subset \overline{A}$, it must then be that B contains at least one $x \in \overline{A} \setminus A$, and possibly more. For any such x, there exists some sequence $(x_n) \subset A$ such that $x_n \to x$ (by the definition of the closure). Because B is disconnected, by Lemma 6.5 there exists $f : B \to \{0, 1\}$ continuous and onto. Note that by the corollary noted immediately after Lemma 6.5, the restriction of f to A must be constant because A is connected. This means that, WLOG, $f(a) = 0$ for each $a \in A$. More specifically, $f(x_n) = 0$ for all n. However, because f is onto, it must be the case that $f(x) = 1$ for at least one of the x described above. But this is a contradiction due to the fact that continuous functions preserve limits on convergent sequences. Therefore, B must be connected.

Exercise 13
If $f : [a, b] \to [a, b]$ is continuous, show that f has a fixed point; that is, show that there is some point $x \in [a, b]$ with $f(x) = x$.

Solution.
If $f(b) = b, b$ serves as the fixed point. If $f(a) = a, a$ serves as the fixed point. If neither of these hold, then due to the range of f we have that $f(b) < b \implies f(b) - b < 0, f(a) > a \implies f(a) - a > 0$. Let then $g : [a, b] \to \mathbb{R}, g(x) = f(x) - x$, which is a continuous function such that $g(a) > 0, g(b) < 0$. Therefore, by the Intermediate Value Theorem (Corollary 6.7), we obtain that g assumes every value between $g(b)$ and $g(a)$, so more specifically it achieves the value of 0 for some $c \in (a, b)$, meaning that $g(c) = 0 \implies f(c) = c$.

Exercise 15
If $f : \mathbb{R} \to \mathbb{R}$ is continuous and open, show that f is strictly monotone.

Solution.
If f is constant in any interval (a, b), we trivially see that it cannot be open (the image of (a, b) under f is a single point, which is a closed set). From now on we therefore assume that it is not constant in any interval. Suppose that it is not strictly monotone, which means that there exist $a, b, c \in \mathbb{R}, a < b < c$ such that (assuming WLOG a direction for the inequalities) $f(a) < f(b)$ but $f(b) > f(c)$. The important observation here is that f, as a continuous function, achieves a minimum value m and a maximum value M in the interval $[a, c]$, and that M cannot be achieved at either a or c (since $f(b) > f(a), f(b) > f(c)$). Therefore, when we consider the open interval (a, b), it must be the case that $f((a, b))$ is an interval that is either of the form $[m, M]$, or of the form $(m, M]$. In both cases, this is not an open set, which contradicts the assumption that f is open. Therefore, f is strictly monotone.

Exercise 26
Let $f : [0, 1] \to \mathbb{R}$ be defined as $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that although f is not continuous, the graph of f is a connected subset of $\mathbb{R}^2$. [Hint: Use Exercise 9.]

Solution.
First we show that f is not continuous at 0. Indeed, pick $\epsilon = 1/2$ and observe that it should be the case that for some $\delta > 0$, whenever $|x| < \delta, |f(x)| < \epsilon$. However, for any δ, if we select a sufficiently large $k \in \mathbb{N}$, we obtain that $\frac{2}{4k\pi+\pi} < \delta$ but $\sin(1/x) = \sin(\frac{4k\pi+\pi}{2}) = \sin(2k\pi + \pi/2) = 1 > \epsilon$, and so f cannot be continuous at 0 (this is one way to formalize the increasingly fast oscillations of this function's graph). Now we show that its graph is connected in $\mathbb{R}^2$. We have that the set $\{x > 0\}$ is connected (as an interval). Furthermore, the map $x \mapsto (x, f(x)), x > 0$ is continuous (Lemma 5.8), and so the set $\{(x, f(x)), x > 0\}$ is connected as the image of a connected set under a continuous function. Now, the closure of this set of course contains $\{(0, 0)\} \cup \{(x, f(x)), x > 0\}$ (i.e., the graph of f): the sequence $((\frac{1}{n\pi}, f(\frac{1}{n\pi}))), n > 0$ converges to $(0, 0)$. By exercise 9, we thus have that the graph of f is connected.

Exercise 27
Let V be a normed vector space, and let $x \neq y \in V$. Show that the map $f(t) = x + t(y - x)$ is a homeomorphism from $[0, 1]$ into V. The range of f is the line segment joining x and y, and is often written $[x, y]$ (since f is a homeomorphism, the interval notation is justified). [Hint: That f is continuous and one-to-one is easy; next, show that if $f(t_n) \to z$, then (t_n) converges to some t in $[0, 1]$ with $z = f(t)$.]

Solution.
To show that f is one-to-one, we have that for $t_1 \neq t_2 \in (0, 1), f(t_1) = x + t_1(y - x), f(t_2) = x + t_2(y - x)$

Chapter 7

Completeness

7.1 Completeness

> **Exercise 1**
> If $A \subset B \subset M$ and B is totally bounded, then A is totally bounded.

Solution.
Pick any $\epsilon > 0$. By Lemma 7.1, because B is totally bounded, there exist finitely many sets $B_1, \ldots, B_n \subset B$ such that $\text{diam}(B_i) < \epsilon$ for all i and $B \subset \bigcup_{i=1}^n B_i$. Let then $A_i = B_i \cap A$, each of which is clearly a subset of A. It holds of course that $\text{diam}(A_i) \leq \text{diam}(B_i) < \epsilon$. Furthermore, because $A \subset B \subset \bigcup_{i=1}^n B_i$, any $x \in A$ must belong in at least one B_i, and hence also in the corresponding A_i, which means that $A \subset \bigcup_{i=1}^n A_i$. By all of the above and Lemma 7.1, we conclude that A is totally bounded.

> **Exercise 2**
> Show that a subset $A \subset \mathbb{R}$ is totally bounded if and only if it is bounded. In particular, if I is a closed, bounded, interval in $\mathbb{R}$ and $\epsilon > 0$, show that I can be covered by finitely many closed subintervals $J_1, \ldots, J_n$, each of length at most ϵ.

Solution.
We recall example 7.2 (a): a totally bounded set A is necessarily bounded. To show this, pick $\epsilon = 1$ and obtain $x_1, \ldots, x_n$ such that $A \subset \bigcup_{i=1}^n B_1(x_i)$. Set $M = \max_{i,j} d(x_i, x_j)$ and pick any two $x, y \in A$, for which it must hold that $x \in B_i, y \in B_j$ for some i, j. By the triangle inequality:

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 1 + M + 1 = M + 2$$

This of course shows that A is bounded.
Conversely, if $A \subset \mathbb{R}$ is bounded, then $\text{diam} A < R$, which in $\mathbb{R}$ is equivalent to $A \subset [-R, R]$. For any $\epsilon > 0$, set $N = \lceil \frac{2R}{\epsilon} \rceil$ and subdivide $[-R, R]$ into N closed subintervals of length ϵ, $[-R, a_1], [a_1, a_2], \ldots [a_{N-1}, R]$. Then observe that any element of A must belong in one of these intervals, and thus is at most ϵ-far from some a_i. Since there are finitely many a_i, we conclude that A is totally bounded.

> **Exercise 4**
> Show that A is totally bounded if and only if A can be covered by finitely many *closed* sets of diameter at most ϵ for every $\epsilon > 0$.

Solution.
$\implies$: Suppose first that A is totally bounded and pick any $\epsilon > 0$. Then there exist finitely many $x_1, \ldots, x_n$ such that $A \subset \bigcup_{i=1}^n B_{\epsilon/4}(x_i)$. As in the book, we may assume that $A \cap B_{\epsilon/4}(x_i) \neq \emptyset$ for all i, and thus for each x_i find $a_i \in A, a_i \in B_{\epsilon/4}(x_i)$. Now, pick any $y \in B_{\epsilon/4}(x_i)$, and by the triangle inequality:

$$d(y, a_i) \leq d(y, x_i) + d(x_i, a_i) \leq \epsilon/4 + \epsilon/4 \leq \epsilon/2$$

This shows that all elements of $B_{\epsilon/4}(x_i)$ belong in the *closed* ball of radius $\epsilon/2$ around a_i. More specifically then, all elements of A that were covered by $B_{\epsilon/4}(x_i)$ are also covered by this closed ball. We conclude that the union of these n closed balls —each of which of course has diameter ϵ— covers A.

$\Longleftarrow$: Conversely, suppose that for any $\epsilon > 0$, A can be covered by finitely many closed sets of diameter at most ϵ. For any $\epsilon > 0$, use this hypothesis on $\epsilon/2$ to obtain finitely many closed sets $S_1, \ldots, S_n$ such that $A \subset \bigcup_{i=1}^{n} S_i$ and $\text{diam}(S_i) \leq \epsilon/2$ for all i. Once again, we may safely assume that $A \cap S_i \neq \emptyset$ for all i, and thus for every i find an $a_i \in A, a_i \in S_i$. For any $y \in S_i$, it holds that $d(y, a_i) \leq \text{diam}(S_i) \leq \epsilon/2 < \epsilon$, and so $y \in B_\epsilon(a_i)$. This means that all elements of A that were covered by S_i are also covered by $B_\epsilon(a_i)$, meaning that $A \subset \bigcup_{i=1}^{n} B_\epsilon(a_i)$, which is of course the definition of total boundedness for A.