N. L. Carothers' Real Analysis - A partial solutions manual

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Contents

Part I

Metric Spaces

Chapter 1

Calculus Review

1.1 The Real Numbers

Exercise 1
If A is a nonempty subset of $\mathbb{R}$ that is bounded below, show that A has a greatest lower bound. That is, show that there is a number $m \in \mathbb{R}$ satisfying: (i) m is a lower bound for A; and (ii) if x is a lower bound for A. then $x \leq m$. [Hint: Consider the set $-A = \{-a : a \in A\}$ and show that $m = -\sup(-A)$ works.]

Solution.
As pointed out in the hint, let $m = \sup(-A)$. Let also $B = \{-a : a \in A\}$ and b be any element of B. Since A is non-empty, B is non-empty too. By the definition of B, $b = -a$ for some $a \in A$, Because A is bounded below, it holds that there exists l such that $l \leq a$. This means that $-l \geq -a \implies -l \geq b$. Therefore, $-l$ is an upper bound for B. By the completeness of real numbers, B has a supremum, m. For any $a \in A$, $-a \in B$, therefore $-a \leq m \implies a \geq -m$. This means that $-m$ is a lower bound for A. For any lower bound m' of A, it holds that $m' \leq a$ for all $a \in A$. This means that $-m' \geq -a$ for all $-a \in B$. But then $-m'$ is an upper bound for B, thus by the definition of the supremum, $-m' \geq m \implies m' \leq -m$. We have thus shown that $-m$ is the greatest lower bound of A.

Exercise 3
Establish the following apparently different (but "fancier") characterization of the supremum. Let A be a non-empty set of $\mathbb{R}$ that is bounded above. Prove that $s = \sup A$ if and only if (i) s is an upper bound for A, and (ii) for every $\epsilon > 0$ there is an $a \in A$ such that $a > s - \epsilon$. State and prove the corresponding result for the infinum of an non-empty subset of $\mathbb{R}$ that is bounded below.

Solution.
$\implies$: Suppose first that s is the supremum of A. By the definition of the supremum, s is indeed an upper bound for A. Suppose now that there exists $\epsilon > 0$ such that for every $a \in A$ it is the case that $a \leq s - \epsilon$. Observe that this means that $s - \epsilon$ is an upper bound for A, while it also holds that $s - \epsilon < s$. But this contradicts the definition of s being the supremum of A.
We have arrived at a contradiction, and thus the negation of the assumed statement must hold. Namely, it must be the case that for every $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.
$\impliedby$: Suppose now that s is an upper bound for A and that for every $\epsilon > 0$ there is an $a \in A$ such that $a > s - \epsilon$. Suppose that s is not the supremum of A. By the completeness of the real numbers, A must have a supremum t. Because s is an upper bound for A, it must hold that $t < s$. Set $\epsilon = s - t > 0$. By the definition of s, there exists $a \in A$ such that $a > s - \epsilon = s - (s - t) = t$. This contradicts the fact that t is an upper bound for A. Therefore, s must be the supremum of A.
The corresponding result for the infinum is that a number s is the infinum of $A \subset \mathbb{R}$, with A non-empty and bounded below, if and only if (i) s is a lower bound for A and (ii) for every $\epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$. To prove this, we have that:

$\implies$: Suppose s is the infinum of A. Then s is a lower bound for A. Suppose that there exists $\epsilon > 0$ such that for every $a \in A, a \geq s + \epsilon$. But then $s + \epsilon > s$ is a lower bound for A, contradiction. The negation of our assumption leads to the desired property for s.
$\impliedby$: Now suppose s is a lower bound for A and for every $\epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$. Suppose s is not the infinum of A, and t is instead. Because s is a lower bound, $s < t$. Set $\epsilon = t - s$. Then there exists $a \in A$ such that $a < s + \epsilon = s + (t - s) = t$. But this contradicts the fact that t is a lower bound for A.
Note that this second proof can also be done by relating the infinum of A to the supremum of $-A$, but since we have not yet proved this (exercise 1) we do not use it here.

Exercise 4
Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that there is a sequence of elements x_n of A that converges to $\sup A$.

Solution.
Recall from exercise 3 that if $s = \sup A$, it holds that for every $\epsilon > 0$ there exists $a \in A$ such that $a > s - \epsilon$. Consider then the sequence that is formed by taking x_i be an element of A for which it holds that $x_i > s - \frac{1}{i}$. For any $\epsilon > 0$, take $M = \lceil \frac{1}{\epsilon} \rceil$. It then holds that $\frac{1}{M} \leq \epsilon$. Additionally, it is true by the definition of the sequence that $x_M > s - \lceil \frac{1}{M} \rceil \geq s - \epsilon \implies \epsilon > s - x_M$. Also, by the definition of the supremum, $s - x_M \geq 0$, thus $|s - x_M| < \epsilon$. Now, for any $j > M$ we have that $x_j > s - \frac{1}{j} \implies \frac{1}{j} > s - x_j$. By the definition of the supremum, $s - x_j \geq 0$, thus $|s - x_j| < \frac{1}{j} < \frac{1}{M} \leq \epsilon$.
We have thus precisely proved that the limit of the sequence x_i is the supremum s of A.

Exercise 5
Suppose that $a_n \leq b$ for all n and that $a = \lim_{n\to\infty} a_n$ exists. Show that $a \leq b$. Conclude that $a \leq \sup\{a_n : n \in \mathbb{N}\}$.

Solution.
Suppose that $a > b$, which means that $a - b > 0$. Set $\epsilon = a - b$. By the definition of the limit of a sequence, there exists $M > 0$ such that for all $n > M$, it holds that:

$$|a - a_n| < \epsilon \implies -\epsilon < a - a_n < \epsilon \implies a < a_n + \epsilon$$

It is the case that $a_n \leq b$ for all n, thus $a < b + \epsilon = b + (a - b) \implies a < a$, which is clearly a contradiction. Therefore $a \leq b$. Now because $s = \sup\{a_n : n \in \mathbb{N}\}$ is by definition a number for which $a_n \leq s$ for all n, the previous result applies, and thus $a \leq s = \sup\{a_n : n \in \mathbb{N}\}$.

Exercise 6
Prove that every convergent sequence of real numbers is bounded. Moreover, if a_n is convergent, show that $\inf a_n \leq \lim_{n\to\infty} a_n \leq \sup a_n$.

Solution.
Suppose a_n is a convergent sequence of real numbers, and suppose that it converges to $a \in \mathbb{R}$. By definition, for any $\epsilon > 0$, there exists $M > 0$ such that for every $n > M$ it holds that $|a_n - a| < \epsilon \implies a_n < a + \epsilon$. Pick any such ϵ, e.g. $\epsilon = 1$. We can thus see that there exist *at most* M elements of the sequence, chosen from $a_1, a_2, \ldots a_M$ such that $a_i \geq a + \epsilon$. Let then S be the —possibly empty— set of all such a_i. We can then see that the set $S \cup \{a + \epsilon\}$ has a finite number of elements. Thus its maximum element s is well defined. Observe then that $s \geq a_n$ for all elements of the sequence. Now recall from exercise 5 that the limit of the sequence is indeed at most equal to its supremum (which we showed is well defined, since at least one upper bound exists).
Again by the definition of the limit, for every $\epsilon > 0$ there exists $M > 0$ such that for $n > M$ it holds that $|a_n - a| < \epsilon \implies -\epsilon < a_n - a \implies a + \epsilon < a_n$. Observe that we can thus apply a completely symmetric argument: pick, say, $\epsilon = 1$ and then there are at most M elements such that $a_n \leq a + \epsilon$. By constructing S from these elements in the same way as above, the rest of the proof becomes completely symmetric, as is the part regarding $\inf a_n \leq \lim_{n\to\infty} a_n$.

Exercise 8
Given $a < b$, show that there are, in fact, infinitely many distinct rationals between a and b. The same goes for irrationals too.

Solution.
Suppose that there are only a finite number of distinct rationals between a and b, and call them $q_1, q_2, \dots q_n$, listed in ascending order. This means $a < q_1 < q_2 < \dots < q_n < b$. But then we have that q_n, b are real numbers with $q_n < b$, and therefore there has to exist a rational number q such that $q_n < r < b$. But this is a contradiction, because $q \neq q_i$ for all i and $a < q < b$. Therefore there exists an infinite number of distinct rationals between a and b.
Now suppose that there is only a finite number of irrationals $r_1, r_2, \dots r_n$ between a, b, listed in ascending order. However, in exercise 7 we've seen that if $a < b$, there exists an irrational x such that $a < x < b$. This means that in our case there exists an irrational r such that $r_n < r < b$. Clearly, this is a contradiction since $r_i \neq r$ for all i. Therefore there exist infinitely many distinct irrationals between a, b.

Exercise 13
Let $a_n \geq 0$ for all n, and let $s_n = \sum_{i=1}^{n} a_i$. Show that (s_n) converges if and only if (s_n) is bounded.

Solution.
$\implies$: Assume that (s_n) converges to s. Then recall from exercise 6 that every convergent sequence is bounded, thus s_n is indeed bounded.
$\impliedby$: Suppose (s_n) is bounded. Because $a_n \geq 0$, the sequence of partial sums s_n increases monotonically. We know then that a monotone, bounded sequence always converges, thus (s_n) converges.

Exercise 14
Prove that a convergent sequence is Cauchy, and that any Cauchy sequence is bounded.

Solution.
A sequence of real numbers is Cauchy if, for every $\epsilon > 0$, there is an integer $N \geq 1$ such that $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$.
Let then a_n be a convergent sequence that converges to a, and select any $\epsilon > 0$. Then, by the definition of convergence, for $\epsilon' = \frac{\epsilon}{2}$ there exists $N > 0$ such that for all $n > N$ it holds that $|a_n - a| < \epsilon'$. Therefore, for any two $n, m > N$, it holds that:

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, which is the defining property of Cauchy sequences. Now let a_n be any Cauchy sequence. Suppose that a_n is not bounded. Because a_n is Cauchy, it must hold that for $\epsilon = 1$, there exists $N \geq 1$ such that for all $n, m > N$ it holds that $|a_n - a_m| < 1$. Let then a_k be the first element of the sequence for which this inequality holds when setting $n = k$ and m any other index $m > k$. This means that for any $m > k$, $|a_m - a_k| < 1$. In particular, this means that a_m can never be larger than $a_k + 1$, therefore for all elements of the sequence after that point, $a_k + 1$ is an upper bound. Then, because k is finite, $m = \max\{a_1, \dots, a_k\}$ is well defined. If we then set $s = \max\{m, a_k + 1\}$, we see that s constitutes an upper bound for the entire sequence.
A symmetric argument can be applied to extract a lower bound for the sequence, thus any Cauchy sequence is bounded.

Exercise 15
Show that a Cauchy sequence with a convergent subsequence actually converges.

Solution.
Let a_n be a Cauchy sequence and $b_n = a_{f(n)}$ such that $f(j) > f(k)$ whenever $j > k$ be a convergent subsequence of it that converges to b. Pick any $\epsilon > 0$. Now it must be the case that there exists $N > 0$ such that whenever $n > N$, $|b_n - b| < \frac{\epsilon}{2}$. Now because a_n is Cauchy, there exists $M > 0$ such that whenever $n, m > 0$, $|a_n - a_m| < \frac{\epsilon}{2}$. Set $K = \max\{N, M\}$ and observe then that for any $n > K$, it holds that:

$$|a_n - b| < |a_n - b_{K+1} + b_{K+1} - b| \leq |a_n - b_{K+1}| + |b_{K+1} - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

, where in the last step we used the fact that a_n is Cauchy (and b_{K+1} is of course an element of it) for the first term and that b_n converges to b for the second term. We have thus shown that a_n converges, and more specifically it converges to the limit of its convergent subsequence.

Exercise 17
Given real numbers a and b establish the following formulas: $|a + b| \leq |a| + |b|, \big||a| - |b|\big| \leq |a - b|, \max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

Solution.

- $|a+b| \leq |a| + |b|$: one way to do this is to observe that $\mathbb{R}$ is a vector space over the field $\mathbb{R}$ for which standard multiplication satisfies the properties of being an inner product. Thus we can use Axler's result of the triangle inequality for inner product spaces, whose proof does not assume that the triangle inequality holds for real numbers. Another way is to do a variant of this proof "manually":

$$(a + b)^2 = a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a| \cdot |b| = (|a| + |b|)^2$$

$$\implies |a + b| \leq |a| + |b|$$

, where we used some properties of the absolute value which easily follow from its definition ($|a|^2 = |a|^2, x \leq |x|$)., and we took square roots at the last step, observing that $|a| + |b|$ is always non-negative.

- $\big||a| - |b|\big| \leq |a - b|$: Quite similar to the above, we have that:

$$(|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a| \cdot |b| \leq |a|^2 + |b|^2 - 2ab = (a - b)^2 \implies \big||a| - |b|\big| \leq |a - b|$$

, where when taking square roots the absolute values are now necessary.

- $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$: It is either the case that $a \geq b$ or that $a \leq b$ (here we are using the fact that $\mathbb{R}$ is a totally ordered set). In the first case, we have that $\max\{a, b\} = a$ and also that $|a - b| = a - b$, which means $\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = a = \max\{a, b\}$. The second case is almost exactly the same, except for $|a - b|$ evaluating to $b - a$.

- $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$: This is done in the same manner as above, again by utilizing the fact that for any two a, b, it either holds that $a \leq b$ or that $a \geq b$.

Exercise 18
(a) Given $a > -1, a \neq 0$, use induction to show that $(1 + a)^n > 1 + na$ for any integer $n > 1$.
(b) Use (a) to show that, for any $x > 0$, the sequence $(1 + \frac{x}{n})^n$ increases.
(c) If $a > 0$, show that $(1 + a)^r > 1 + ra$ holds for any *rational* exponent $r > 1$.
[Hint: If $r = \frac{p}{q}$, then apply (a) with $n = q$ and (b) with $x = ap$.]
(d) Finally, show that (c) holds for any *real* exponent $r > 1$.

Solution.
(a) The base case of the induction is $n = 2$. We then have that:

$$(1 + a)^2 = 1 + 2a + a^2 > 1 + 2a$$

, since a is not zero, meaning that $a^2 > 0$. Suppose now that the inequality holds for $n = k > 1$. Then we have that:

$$(1 + a)^{k+1} = (1 + a)(1 + a)^k > (1 + a)(1 + ka) = 1 + ka + a + ka^2 = 1 + (k + 1)a + ka^2 > 1 + (k + 1)a$$

, where, crucially, we used the fact that $a > -1$, thus that $a + 1 > 0$, thus that we can multiply the inequality for $n = k$ with $(1 + a)$. In the last step we again use the fact that $a \neq 0$, thus that $ka^2 > 0$.

(b) We'll work through this the same way as the book's examples, i.e., compute the ratio between two successive terms of the sequence:

$$\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^n} = \left(1+\frac{x}{n}\right)\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^{n+1}} = \left(1+\frac{x}{n}\right)\left(\frac{\frac{n+1+x}{n+1}}{\frac{n+x}{n}}\right)^{n+1}$$

$$= \left(1+\frac{x}{n}\right)\left(\frac{n^2+n+nx}{n^2+x+n+nx}\right)^{n+1} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n^2+x+n+nx}\right)^{n+1} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{(x+n)(n+1)}\right)^{n+1}$$

Now, for any $x > 0$ and n positive integer we have that:

$$xn + n^2 + n > 0 \implies xn + n^2 + n + x > x \implies (x+n)(n+1) > x \implies -(x+n)(n+1) < -x$$

$$\implies -1 < -\frac{x}{(x+n)(n+1)}$$

Also, $x > 0$ thus this quantity is never zero. Therefore, using it as an a in the Bernoulli inequality (part (a)) we can obtain that:

$$\frac{(1+\frac{x}{n+1})^{n+1}}{(1+\frac{x}{n})^n} > \left(1+\frac{x}{n}\right)(1-(n+1)\frac{x}{(x+n)(n+1)}) = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{x+n}\right) = \frac{x+n}{n}\cdot\frac{n}{x+n} = 1$$

, which means that the sequence $\left(1+\frac{x}{n}\right)$ does indeed increase.
(c)Consider any rational $r > 1$. r can be written as $\frac{p}{q}$, where p, q are positive integers, and in fact $p > q$. Setting $x = ap$ (for this part, $a > 0$) and observing that $p > q$, from part (b) we have that:

$$\left(1+\frac{ap}{q}\right)^q < \left(1+\frac{ap}{p}\right)^p \implies \left(1+\frac{ap}{q}\right)^q < (1+a)^p$$

For $a > 0$, both quantities inside the parentheses are positive, and thus we can take q-th roots and obtain:

$$\left(1+\frac{ap}{q}\right) < (1+a)^{\frac{p}{q}}$$

, which, since $r = \frac{p}{q}$, is the equivalent of the Bernoulli inequality for a rational exponent $r > 1$ and $a > 0$.
(d) We know that we can approach any real number $r > 1$ with a sequence of rationals. Furthermore, the Bernoulli inequality will hold for all of those rationals that are larger than 1, so as we approach $r > 1$, the Bernoulli inequality will hold. Now, at the limit, this strict inequality holds as a non-strict inequality, thus we can conclude that:

$$(1+a)^r \geq 1 + ra$$

Now pick a rational q such that $1 < q < r$ (such a rational always exists). Observe that:

$$(1+a)^r = (1+a)^{\frac{rq}{q}} = ((1+a)^q)^{\frac{r}{q}} > (1+qa)^{\frac{r}{q}}$$

, where we applied the Bernoulli inequality for q. Now, however, $\frac{r}{q}$ is a real number greater than 1, and hence we can use our non-strict inequality above to arrive at the desired result:

$$(1+a)^r > (1+qa)^{\frac{r}{q}} \geq 1 + q\frac{r}{q}a = 1 + ra \implies (1+a)^r > 1 + ra$$

Exercise 21
Let $p \geq 2$ be a fixed integer, and let $0 < x < 1$. If x has a finite-length base p decimal expansion, that is, if $x = a_1/p + \ldots + a_n/p^n$ with $a_n \neq 0$, prove that x has precisely two base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique. Characterize the numbers $0 < x < 1$ that have repeating base p decimal expansions. How about eventually repeating?

Solution.
We consider first the decimal expansion $0.a_1a_2\ldots a_n00\ldots$, with an infinite number of trailing zeros after the n-th decimal, and call the i-th decimal here b_i. As per the book's definition, this corresponds to an infinite series defined as $\sum_{i=1}^{\infty} b_i/p^i$. It's clear that the limit of this sum is x, since x is precisely the sum of the first n terms, and all of the terms after n are zero. Thererefore, the expansion above indeed corresponds to x.
Now consider the decimal expansion corresponding to the series $\frac{a_1}{p} + \frac{a_2}{p^2} + \ldots + \frac{a_n-1}{p^n} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^k}$. This would yield a series of decimals the first n of which would equal $a_1, \ldots a_n - 1$ (note that because $a_n \neq 0, a_n - 1$ causes no problems), while all decimals after that would equal $p-1$ (call the sequence of digits c_i). If we consider the limit of this series, we can see that the last term is a series summing to $\frac{1}{p^n}$, and that the first n terms sum to $x - \frac{1}{p^n}$. Thus, the series as a whole tends to x, which means that it *is* a decimal expansion for x.
We now have to show that there is no other decimal expansion for x. We will do this by selecting a decimal expansion $0.b_1b_2\ldots b_nb_{n+1}\ldots$ that is not equal to $0.a_1a_2\ldots a_n00\ldots$ and show that it must necessarily equal $0.a_1a_2\ldots(a_n-1)(p-1)(p-1)\ldots$. Let us then examine such a decimal expansion. Since it differs from $0.a_1a_2\ldots a_n00\ldots$, there must exist a first j such that $a_j \neq b_j$. Because both decimal expansions correspond to x, for the two series it must hold that:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, where we have erased from both sides the first $j-1$ digits that are equal (note that the set of these may be empty). Because we are manipulating convergent series, we can write:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i} = \sum_{i=j+1}^{n} \frac{b_i - a_i}{p^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{p^i}$$

, where we noted that after the n-th digit, all a_i are zero. Now, note that if $b_j > a_j$, the LHS here equals at most $-\frac{1}{p^j}$. At the same time, the RHS equals at least 0, which happens when $b_i = 0, i \geq n+1$ and all $b_i - a_i = 0, j+1 \leq i \leq n$. Clearly then, they can never be equal. On the other hand, if $b_j < a_j$ the LHS equals at least $\frac{1}{p^j}$. The RHS equals at most $\sum_{i=j+1}^{\infty} \frac{p-1}{p^i} = \frac{1}{p^j}$, and, crucially, this happens if all $b_i = p-1, i \geq n+1, b_i - a_i = p-1, j+1 \leq i \leq n$. Because the value is achieved when all of the digits fulfill these conditions, the RHS will be strictly smaller than the LHS in all other cases. The consequence of this is that all b_i starting at $i = n+1$ are equal to $p-1$. Additionally, all $a_i, j+1 \leq i \leq n$ have to be zero. But because $a_n \neq 0$, it must hold that $j+1 > n$. That is, the first digit where the two expansions differ must be at least at position n. If this was strictly larger than n, the LHS would not achieve its minimum value (a_j would be 0 and b_j would be $p-1$), thus it could not equal the RHS, contradiction. Thus $j = n$. By these observations, the only digit for which we have not yet determined a value is at position n. Recall that $a_j - b_j$ must equal 1, which means that $b_j = a_j - 1$. In other words, $b_n = a_n - 1$, all $b_i = a_i, i < n$ and all $b_i = p-1, i > n$, leading us to conclude that there exists no third possible representation.
Now we proceed to examine an x with no finite-length base p decimal expansion. Let $a_i, i = 1, 2, \ldots$ be one of its expansions, which is necessarily infinite in length. We will show that this is unique by taking another expansion b_i and showing that it must equal a_i at all digits. Indeed, suppose that they differ, and that the first digit at which this happens is at position j. As we did above, we can write:

$$\sum_{i=1}^{j-1} \frac{a_i}{p^i} + \sum_{i=j}^{\infty} \frac{a_i}{p^i} = \sum_{i=1}^{j-1} \frac{b_i}{p^i} + \sum_{i=j}^{\infty} \frac{b_i}{p^i} \implies \frac{a_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{a_i}{p^i} = \frac{b_j}{p^j} + \sum_{i=j+1}^{\infty} \frac{b_i}{p^i}$$

, and, again because the series converge, we can rewrite this as:

$$\frac{a_j - b_j}{p^j} = \sum_{i=j+1}^{\infty} \frac{b_i - a_i}{p^i}$$

If $b_j > a_j$ we again observe that the LHS equals at most $-\frac{1}{p^j}$. The RHS equals at least $\sum_{i=j+1}^{\infty} \frac{-(p-1)}{p^i} = -\frac{1}{p^j}$, which happens if all $b_i - a_i = -(p-1), i > j$. This cannot otherwise be true because this minimum value is achieved when all $b_i - a_i$ are minimized. The consequence is each $b_i = 0, a_i = p-1, i > j$. However, this implies that x can then be written as $0.b_1 b_2 \dots b_j 00\dots$, which is a finite-length expansion since all trailing digits after j are zeros. This is a contradiction. An exactly symmetrical argument applies when $b_j < a_j$, leading us to conclude that $b_j = a_j$ for *all* j, thus a_i is a unique expansion for x.
Now we examine $0 < x < 1$ that have an eventually repeating base p decimal expansion. This means that there exists a finite-length "unique" first part of the number, consisting of m digits, as well as a minimum "period" of repetition n (this can also be zero, in which case the part below does not apply but the number is trivially rational), such that the number can be written as:

$$0.a_1 a_2 \dots a_m a_{m+1} a_{m+2} a_{m+3} \dots a_{m+n} a_{m+1} \dots a_{m+n} \dots$$

, where we have defined n as the minimum integer for which this holds. Then observe that we can write:

$$x = (a_1 a_2 \dots a_m) p^{-m} + (a_{m+1} a_{m+2} \dots a_{m+n}) p^{-m-n} + (a_{m+1} a_{m+2} \dots a_{m+n}) p^{-m-2n}$$
$$+ (a_{m+1} a_{m+2} \dots a_{m+n}) p^{-m-3n} + \dots$$
$$= \frac{a_1 a_2 \dots a_m}{p^m} + (a_{m+1} a_{m+2} \dots a_{m+n}) (\sum_{i=1}^{\infty} p^{-m-in}) =$$
$$\frac{a_1 a_2 \dots a_m}{p^m} + \frac{a_{m+1} a_{m+2} \dots a_{m+n}}{p^m (p^n - 1)}$$

, which is a sum of two quotients of integers, and therefore is a rational number. Note that for the case of repeating expansions everything above simplifies to $m = 0$. Now we need to examine whether *every* rational number can be written in this way, in other words, whether every rational number has an eventually repeating (which includes the trivially "eventually repeating" finite-length digit sequences) base p decimal expansion.
Recall that a rational number can be written as the quotient of two integers. Furthermore, because we are interested in numbers in $[0, 1]$, we can restrict ourselves to $x = \frac{a}{b}$ where a, b natural, non-zero numbers with $a < b$. We now recall that for finding a base p decimal expansion for x we can use the division algorithm iteratively. Namely, we first find the smallest power p_1^n such that $ap_1^n \geq b$, and then we use the division algorithm on ap^n, b. According to it, there will exist unique $q_1, r_1, 0 \leq r_1 < b \in \mathbb{N}$ such that $ap^{n_1} = bq_1 + r_1$. Notice that this implies that $\frac{a}{b} = \frac{q}{p^{n_1}} + \frac{r_1}{bp^{n_1}}$. The first $n_1 - 1$ digits of the decimal expansion of x will be zeros, since for all powers $p^k, k < n$ it has to be that $ap^k < b \implies x < \frac{1}{p^k}$. Now notice that $q_1 \leq \frac{ap^{n_1}}{b} < p$ because $ap^{n_1 - 1} < b$ due to the choice of n_1. Therefore q_1 can be thought of as the n_1-th digit in the decimal expansion of x. If $r_1 = 0$, the decimal expansion is finished, and finite in length. Otherwise, we can apply the same procedure to r_1, bp^{n_1} to obtain that $r_1 p^{n_2} = bp^{n_1} q_2 + r_2$, where n_2 has been chosen in the same manner as above, and signifies that the next $n_2 - 1$ digits of x will be zero, while the n_2-th will be q_2. Now we observe that if at any point r_k becomes zero, the expansion is finite in length. Otherwise, we record this sequence of remainders. Each of them is in the range $(0, b)$. Therefore, after at most $b - 1$ steps we will encounter a remainder that has been encountered before. The consequence is that after this point, the sequence of digits is fully known, because we are always performing a division with b. Therefore, from that point on the decimal expansion will indeed be infinitely repeating, thus completing the proof.

Exercise 23
If a_n is convergent, show that $\text{liminf}_{n\to\infty} a_n = \text{limsup}_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

Solution.
We begin by showing $\text{liminf}_{n\to\infty} a_n = \lim_{n\to\infty} a_n$. Let L be the limit of a_n. By definition, $\text{liminf}_{n\to\infty} a_n = \sup_{n\geq 1}\{\inf\{a_n, a_{n+1}, \dots\}\}$. Suppose first that this supremum did not exist, i.e. that it "equals infinity". Then for every $M > 0$ there must exist $N > 0$ such that $\inf\{a_N, a_{N+1}, \dots\} > M$. But this in turn would mean that a_n is not bounded and yet converges to L, a contradiction. Now suppose that the supremum

equals $L+\epsilon, \epsilon > 0$. Then $L+\epsilon', \epsilon' < \epsilon$ cannot constitute an upper bound for the infimums. Thus there exists $N > 0$ such that $\inf\{a_N, a_{N+1}, \ldots\} > L+\epsilon'$, which means that for $n \geq N, a_n > L+\epsilon'$. This is a clear contradiction of the definition of limit, since a_n cannot get more than ϵ'-close to L. Therefore, the liminf of a_n is at most L.
Now, by the definition of the limit, for any $\epsilon > 0$ there exists $N > 0$ such that for $n > N, |a_n - L| < \epsilon \implies a_n > L - \epsilon$. Notice that this means that $\inf\{a_N, a_{N+1}, \ldots\} \geq L - \epsilon$, which in turn means that the liminf of a_n, as an upper bound for these infimums, must equal at least $L - \epsilon$ for *any* $\epsilon > 0$. At the same time, it equals at most L. The only possibility then is that $\text{liminf}_{n\to\infty} a_n = L$, which is what we are asked to prove.
The argument for showing that $\text{limsup}_{n\to\infty} a_n = L$ is exactly symmetrical.

Exercise 24
Show that $\lim\,\sup_{n\to\infty}(-a_n) = -\lim\,\inf_{n\to\infty} a_n$.

Solution.
We begin by the definition of lim sup:

$$\lim\,\sup_{n\to\infty}(-a_n) = \inf_{n\geq 1}\{\sup\{-a_n, -a_{n+1}, \ldots\}\}$$

We recall from exercise 1 that if $-A = \{-a, a \in A\}$ for some set A, then $\inf A = -\sup(-A)$. This means that for any of the sets $\{-a_n, -a_{n+1}, \ldots\}$ it holds that $\sup\{-a_n, -a_{n+1}, \ldots\} = -\inf\{a_n, a_{n+1}, \ldots\}$. Thus:

$$\lim\,\sup_{n\to\infty}(-a_n) = \inf_{n\geq 1}\{-\inf\{a_n, a_{n+1}, \ldots\}\}$$

Now observe that the "outer" infimum here is taken over the set $\{-\inf\{a_1, a_2, \ldots\}, -\inf\{a_2, a_3, \ldots\}, \ldots\}$. If we now set $A = \{-\inf\{a_1, a_2, \ldots\}, -\inf\{a_2, a_3, \ldots\}, \ldots\}$, then $-A = \{\inf\{a_1, a_2, \ldots\}, \inf\{a_2, a_3, \ldots\}, \ldots\}$. Therefore we can again apply the result of exercise 1 to get that:

$$\lim\,\sup_{n\to\infty}(-a_n) = \inf A = -\sup(-A) = -\sup_{n\geq 1}\{\inf\{a_1, a_2, \ldots\}, \inf\{a_2, a_3, \ldots\}, \ldots\} = -\lim\,\inf_{n\to\infty} a_n$$

Exercise 25
If $\lim\,\sup_{n\to\infty} a_n = -\infty$, show that a_n diverges to $-\infty$. If $\lim\,\sup_{n\to\infty} a_n = +\infty$, show that a_n has a *subsequence* that diverges to $+\infty$. What happens if $\lim\,\inf_{n\to\infty} = \pm\infty$?

Solution.
We have that $\lim\,\sup_{n\to\infty} a_n = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$. If this "equals negative infinity", this means that the set over which the infimum is computed is not bounded below. In other words, for any $M < 0$, we can find $n \geq 1$ such that $\sup\{a_n, a_{n+1}, \ldots\} < M$. If the least upper bound of such a set is less than M, then clearly all elements of the set must also be less than M. This means that given an $M < 0$ we can always find an n such that $a_k < M$ for all $k > n$, by appropriately picking the set as above. This means precisely that a_n diverges to $-\infty$.
Now let us examine what happens if lim sup "equals positive infinity". This means that the set over which the infimum is computed is empty. To see why this is the case, suppose that it was not. Then either it has an lower bound or it does not (law of excluded middle). In the first case, the (corollary of the) Least Upper Bound axiom tells us that the infimum would have to equal some real number. In the second case, by definition we would write that the infimum equals $-\infty$. The only case left is thus for the set to be empty, and in this case we have defined (page 3-4 for the supremum, symmetric for the infimum) that $\inf\emptyset = +\infty$, since every real number is a lower bound for the empty set. For the set to be empty, it must be the case that *all* $\sup\{a_n, a_{n+1}, \ldots\}$, i.e. for every n, are $+\infty$. This means that all of these sets are unbounded, i.e. that for every n, it is the case that $\{a_n, a_{n+1}, \ldots\}$ contains, for every $M > 0$, an element a_{n_k} such that $a_{n_k} > M$.
Form then a subsequence b_j of a_n in the following way. Select b_1 to be the first element of $\{a_1, a_2, \ldots\}$ such that $b_1 > 1$, which is guaranteed to exist. Then select b_2 to be the first element of $\{a_2, a_3, \ldots\}$ such

that $b_2 > 2, b_2 \neq b_1$. This is also guaranteed to exist, because if it did not, then all elements of a_n after b_1 would be at most 2, which would mean that after that point a_n is bounded and then at least one of the suprema above would be a real number, in which the set over which lim sup is computed would not be empty, a contradiction. Continue this way to pick any b_j to never equal any of the previously selected elements, and then observe that this is a subsequence of a_n that indeed diverges to $+\infty$.
For lim inf the cases are symmetrical.

Exercise 26
Prove the characterization of lim sup given above. That is, given a bounded sequence (a_n), show that the number $M = \limsup_{n\to\infty} a_n$ satisfies (*) (page 12) and, conversely, that any number M satisfying (*) must equal $\limsup_{n\to\infty} a_n$. State and prove the corresponding result for $m = \liminf_{n\to\infty} a_n$.

Solution.
Suppose M is the limit supremum of a bounded sequence (a_n). This means that:

$$M = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$$

Now, pick any $\epsilon > 0$. Recall from exercise 3 that there must exist an element s of the set of suprema such that $s < M + \epsilon$. This means that there must exist N such that $s = \sup\{a_N, a_{N+1}, \ldots\} < M + \epsilon$. Now, by the definition of supremum, this means that for all $a_n, n \geq N$ it must hold that $a_n < M + \epsilon$. These are clearly infinitely many elements of the sequence, and thus the inequality stated here *may* not be true only for *finitely* many elements (up to and not including a_N).
Furthermore, suppose that $M - \epsilon < a_n$ is true for finitely many elements of the sequence, and call N' the largest integer for which $M - \epsilon < a_{N'}$ (well defined due to the set containing finitely many elements). Then for all $n > N'$ it must hold that $M - \epsilon \geq a_n$, which means that $M - \epsilon < M$ constitutes an upper bound for all subsequences $a_n, a_{n+1}, \ldots, n > N'$, and thus $\sup\{a_n, a_{n+1}, \ldots\} \leq M - \epsilon$. But by the definition of lim sup, M is the infimum of the set of suprema, yet is larger than any $\sup\{a_n, a_{n+1}, \ldots\}, n > N'$, a contradiction. Therefore $M - \epsilon < a_n$ has to hold for infinitely many n.
In the other direction, suppose M satisfies (*). We need to show that $M = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$. Suppose first that M is greater than the lim sup. By the definition of infimum, M cannot then constitute a lower bound for the set of suprema. Therefore there exists $N > 0$ such that $\sup\{a_N, a_{N+1}, \ldots\} < M$. Set then $\epsilon = M - \sup\{a_N, a_{N+1}, \ldots\}$, in which case by (*) there must exist infinitely many n such that $M - \epsilon < a_n \implies \sup\{a_N, a_{N+1}, \ldots\} < a_n$. But because n are infinitely many, some of them are greater than N, and thus this inequality contradicts the definition of supremum for the subsequence starting at N.
Now suppose M is smaller than the lim sup, in which case $\limsup_{n\to\infty} a_n > M$. Then $\limsup_{n\to\infty} a_n - M = \epsilon > 0$. This furthermore means that there exists $\epsilon' > 0, \epsilon' < \epsilon$. Now, there must exist infinite n such that $a_n < M + \epsilon' < M + \epsilon = \limsup_{n\to\infty} a_n$. Observe that this means the following. First, that for some $N > 0$, for all $n \geq N, M + \epsilon'$ constitutes an upper bound for all a_n, i.e. $M + \epsilon' \geq \sup\{a_N, a_{N+1}, \ldots\}$. Second, that $M + \epsilon' < \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}$, which means that more specifically $M + \epsilon' < \sup\{a_N, a_{N+1}, \ldots\}$. We have arrived at a contradiction. Therefore M cannot be smaller than the lim sup, leading us to conclude that these must in fact be equal.
For lim inf, the corresponding result is as follows. m equals $\liminf_{n\to\infty} a_n$ if and only if for every $\epsilon > 0, a_n < m + \epsilon$ for infinitely many n and $a_n > m - \epsilon$ for all but finitely many n. The proof would use symmetrical arguments.

Exercise 27
Prove that every sequence of real numbers (a_n) has a subsequence (a_{n_k}) that converges to

$$\lim\ \sup_{n\to\infty} a_n$$

[Hint: If $M = \lim\ \sup_{n\to\infty} a_n = \pm\infty$, we must interpret the conclusion loosely; this case is handled in exercise 25. If $M \neq \pm\infty$, use (*) to choose (a_{n_k}) satisfying $|a_{n_k} - M| < 1/k$, for example. There is also a subsequence that converges to $\lim\ \inf_{n\to\infty} a_n$. Why?]

Solution.
Firstly, as mentioned in the hint let us categorize the cases of $\pm\infty$. One, if lim sup equals $+\infty$ then (a_n) has a subsequence diverging to $+\infty$. Two, if lim sup equals $-\infty$, then the sequence itself diverges to $-\infty$. Now let's examine the case where lim sup equals some real number M. We consider the following sequence: $\epsilon_k = \frac{1}{k}$. By the characterization of the supremum, there exist infinitely many n such that $M - \epsilon_1 < a_n$, and it also holds that *for all but* finitely many $n, a_n < M + \epsilon_1$. Crucially, the second observation means that there exists a maximum N for which the second inequality *does not* hold. Therefore, the second inequality holds for all $n > N$ for some N. Additionally, the first inequality holds for infinitely many n, and thus infinitely many of those have to be greater than N. Therefore, there exist infinitely many n such that $|a_n - M| < \epsilon_1$. Pick the first of those and call it a_1.
Precisely because these elements are always infinite in number, for any $k > 1$ we will always be able to pick an a_k as above with the additional constraint that a_k has not been picked before. Repeating this procedure will yield a subsequence (a_{n_k}) that clearly converges to M because $e_k \to 0$ (and also decrease monotonically). This concludes the proof.
Because a symmetric characterization exists for lim inf, and because a symmetric version of exercise 25 also exists for it, there will also be a subsequence that converges to lim inf, regardless of whether it equals $\pm\infty$ or some real number.

Exercise 29
If (a_{n_k}) is a convergent subsequence of (a_n), show that $\lim\ \inf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \lim\ \sup_{n\to\infty} a_n$.

Solution.
By exercise 23, we know that $\lim\ \inf_{k\to\infty} a_{n_k} = \lim\ \sup_{k\to\infty} a_{n_k} = \lim_{k\to\infty} a_{n_k}$, since (a_{n_k}) converges. Recall the definition of lim sup for $(a_n), (a_{n_k})$:

$$\lim\ \sup_{n\to\infty} a_n = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, \ldots\}\}, \lim\ \sup_{k\to\infty} a_{n_k} = \inf_{k\geq 1}\{\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\}\}$$

We claim that the second lim sup equals at most the first one. To see why this is the case, observe that any $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\}$ is the supremum of a number of elements of (a_n) starting at the n_k-th one and possibly excluding some elements after that. This means that $\{a_{n_k}, a_{n_{k+1}}, \ldots\} \subset \{a_{n_k}, a_{n_k+1}, \ldots\}$, and thus by exercise 2, $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\} \leq \sup\{a_{n_k}, a_{n_k+1}, \ldots\}$. Now suppose that the second lim sup was larger than the first. This would mean that at least one of $\sup\{a_n, a_{n+1}, \ldots\}$ is smaller than $\lim\ \sup_{k\to\infty} a_{n_k}$ (such that it cannot be the largest *lower* bound for the set of suprema of subsequences of (a_n)). Suppose that this happens for $n = N$.
But then because (a_{n_k}) has infinite terms, it has to be the case that for some $k, n_k > N$. Then $\sup\{a_{n_k}, a_{n_{k+1}}, \ldots\} \leq \sup\{a_{n_k}, a_{n_k+1}, \ldots\} \leq \sup\{a_N, a_{N+1}, \ldots\} < \lim\ \sup_{k\to\infty} a_{n_k}$. The first equality here was proved above, while the second is again based on exercise 3 applied on two subsequences, one starting at N and one starting at n_k. But then $\lim\ \sup_{k\to\infty} a_{n_k}$ is not a lower bound for the suprema of the subsequences of (a_{n_k}), which is a contradiction. Therefore, it has to be the case that $\lim\ \sup_{n\to\infty} a_n \leq \lim\ \sup_{k\to\infty} a_{n_k}$. As stated in the beginning, by the convergence of (a_{n_k}) this also implies that $\lim_{k\to\infty} a_{n_k} \leq \lim\ \sup_{n\to\infty} a_n$. The proof for the inequality involving the infimum makes use of exercise 23 and a completely symmetric argument.

Exercise 31
If (a_n) is convergent and (b_n) is bounded, show that $\text{limsup}_{n\to\infty}(a_n+b_n) \le \lim_{n\to\infty} a_n + \text{limsup}_{n\to\infty} b_n$.

Solution.
First of all, because a_n is convergent, it is also bounded. Since b_n is bounded as well, their sum is bounded. This means that $a_n + b_n$ has a limit supremum that is indeed a real number. Then, by exercise 27, we know that the sequence $a_n + b_n$ has a subsequence $a_{n_k} + b_{n_k}$ that converges to $\text{limsup}_{n\to\infty}(a_n + b_n)$. Because a_n converges, the subsequence a_{n_k} that corresponds to the subsequence of the sum also converges to the limit of a_n. Consequently, the subsequence b_{n_k} that corresponds to the subsequence of the sum also converges as a difference of convergent sequences. Observe that this means that:

$$\text{limsup}_{n\to\infty}(a_n + b_n) = \lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k} = \lim_{n\to\infty} a_n + \lim_{k\to\infty} b_{n_k}$$

Now, from exercise 29, because b_{n_k} is a convergent subsequnce of b_n we conclude that $\lim_{k\to\infty} b_{n_k} \le \text{limsup}_{n\to\infty} b_n$. Combining these two observations we obtain that:

$$\text{limsup}_{n\to\infty}(a_n + b_n) \le \lim_{n\to\infty} a_n + \text{limsup}_{n\to\infty} b_n$$

Exercise 33
Show that (x_n) converges to $x \in \mathbb{R}$ if and only if every subsequence (x_{n_k}) of (x_n) has a *further* subsequence $(x_{n_{k_l}})$ that converges to x.

Solution.
$\Longrightarrow$: First suppose that (x_n) converges to $x \in \mathbb{R}$. Pick any subsequence (x_{n_k}), and pick any $\epsilon > 0$. There exists $N > 0$ such that for all $n > N$ it holds that $|x_n - x| < \epsilon$. Because (x_{n_k}) has infinite terms, there exist infinitely many $n_k > N$, and for all of these, by the definition of subsequences, the above inequality holds. Therefore, any subsequence of (x_n) converges to x. Any sequence is a trivial subsequence of itself, and thus we have completed the proof in this direction.
$\Longleftarrow$: Conversely, suppose that every subsequence (x_{n_k}) of a sequence (x_n) has a subsequence $(x_{n_{k_l}})$ that converges to $x \in \mathbb{R}$. Suppose by way of contradiction that (x_n) does not converge to $x \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that for all $N > 0$ there exists $n > N$ for which $|x_n - x| \ge \epsilon$. Construct the following subsequence (x_{n_k}) of (x_n): the i-th term is the first element of (x_n) with $n > i$ such that $|x_n - x| \ge \epsilon$. By the observation above, this is always well defined.
By construction, all elements of this (x_{n_k}) are at least ϵ-away from x. Therefore, no subsequence $(x_{n_{k_l}})$ of (x_{n_k}) can ever converge to x. which directly contradicts our hypothesis, leading us to conclude that (x_n) converges to $x \in \mathbb{R}$.

Exercise 34
Suppose that $a_n \ge 0$ and that $\sum_{n=1}^{\infty} a_n < \infty$.
(i) Show that $\liminf_{n\to\infty} na_n = 0$.
(ii) Give an example showing that $\limsup_{n\to\infty} na_n > 0$ is possible.

Solution.
(i) In order to show that $\liminf_{n\to\infty} na_n = 0$, we need to show two things. One, that for every $\epsilon > 0, na_n > 0 - \epsilon$ holds for all but finitely many n and two, that for every $\epsilon > 0, na_n < 0 + \epsilon$ holds for infinite n. The first inequality is obvious, since $a_n \ge 0$. Suppose then that the second inequality does not hold. This means that there exists $\epsilon > 0$ for which $na_n \ge \epsilon$ for only a finite number of n. Call these $n_1, n_2, \dots n_k$. Then, observe that for all $n > n_k$, it must be the case that $na_n \ge \epsilon \implies a_n \ge \frac{\epsilon}{n}$. Then:

$$\sum_{n>n_k} a_n \ge \sum_{n>n_k} \frac{\epsilon}{n} = \epsilon \sum_{n>n_k} \frac{1}{n}$$

We know, however, that this infinite sum diverges (since the infinite sum of all $\frac{1}{n}$ diverges), whereas our LHS here was assumed to converge. We have thus arrived at a contradiction, which means that the second inequality must indeed hold, and thus $\liminf_{n\to\infty} a_n = 0$.

(ii) Consider the sequence $a_n = \frac{1}{n}$ for $n = 2^k, k = 1, 2, \ldots$ and $a_n = 0$ for every other n. Observe then that the corresponding series is a geometric series, and thus converges. At the same time, observe that $na_n = 1$ for $n = 2^k, k = 1, 2, \ldots$ and $na_n = 0$ for every other n. Clearly then it is the case that $\limsup na_n = 1 > 0$.

Exercise 35
(The ratio test): Let $a_n \geq 0$.
(i) If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, show that $\sum_{n=1}^{\infty} a_n < \infty$.
(ii) If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, show that $\sum_{n=1}^{\infty} a_n$ diverges.
(iii) Find examples of both a convergent and a divergent series having $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

Solution.
(i) Observe, first of all, that $a_n \geq 0$. This means that the corresponding sequence of partial sums is non-decreasing. If we can show that it is also bounded, then we will have proved that it converges, which by definition means that $\sum_{n=1}^{\infty} a_n < \infty$.
By our hypothesis, we have that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. Call this quantity M. Then we can always find some $\epsilon > 0$ such that $M + \epsilon < 1$. For this ϵ, we apply the characterization of lim sup on the sequence $\frac{a_{n+1}}{a_n}$. This means that for all but finitely many n, it holds that $\frac{a_{n+1}}{a_n} < M + \epsilon$. Call N the largest n for which this does not hold, and then observe that we have that $a_{N+2} < a_{N+1}(M + \epsilon)$. This implies then that $a_{N+3} < a_{N+2}(M + \epsilon) < a_{N+1}(M + \epsilon)^2$. More generally, if $N' = N + 1$ for $k \geq 1$ we have that $a_{N'+k} < a_{N'}(M + \epsilon)^k$.
Then for the sequence of partial sums we have that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{N} a_i + \sum_{i=N'}^{n} a_i < \sum_{i=1}^{N} a_i + \sum_{k=1}^{n-N'} a_{N'}(M + \epsilon)^k$$

The first term here is clearly bounded as a finite sum. The second term can also be bounded by the *infinite* sum for k, precisely because it corresponds to a geometric series with ratio $M + \epsilon < 1$. But then this means that the sequence of partial sums is indeed bounded, and thus it must converge.
(ii) Call the lim inf m. Then $M > 1$ and we can always find $\epsilon > 0$ such that $m - \epsilon > 1$. Then, by the characterization of lim inf, for all but finitely many n it holds that $\frac{a_{n+1}}{a_n} > m - \epsilon$. Call N the first n after which (and including it) this holds. Then observe that, similarly to (i), for $k \geq 1$ we have that $a_{N+k} > (m - \epsilon)^k a_N$. But because $m - \epsilon > 1$ this implies that the terms of the sequence are not bounded, and hence the infinite series must necessarily diverge.
(iii) Consider the series corresponding to the sequence $a_n = 1$. Clearly, the ratio of two subsequent terms is 1, but the series obviously diverges.
Consider also the series corresponding to the sequence $a_n = \frac{1}{n^2}$, which we take as known that it converges to $\frac{\pi^2}{6}$. Observe that:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{n^2 + 2n + 1}$$

, which can easily be shown to converge to 1 as $n \to \infty$.

Exercise 37
If (E_n) is a sequence of subsets of a fixed set S, we define

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k) \text{ and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} E_k)$$

Show that

$$\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n \text{ and that } \liminf_{n\to\infty} (E_n^c) = (\limsup_{n\to\infty} E_n)^c$$

Solution.
For the first part, we have the following. Suppose $x \in \liminf_{n\to\infty} E_n$. We need to show that $x \in \limsup_{n\to\infty} E_n$. By this definition of lim inf, it must be the case that $x \in \bigcap_{k=n}^{\infty} E_k$ for at least one $n \geq 1$.

This in turn means that there exists $n \geq 1$ such that $x \in E_n, E_{n+1}, \dots$. Now observe that the following hold:

- For $l \leq n$, it is the case that $E_n \subset \bigcup_{k=l}^{\infty} E_k$. Therefore, $x \in \bigcup_{k=l}^{\infty} E_k, l \leq n$.
- For $i \geq 1$, it is the case that $E_{n+i} \subset \bigcup_{k=n+i}^{\infty} E_k$. Therefore, $x \in \bigcup_{k=n+i}^{\infty} E_k$. This can equivalently be written as $x \in \bigcup_{k=l}^{\infty} E_k, l > n$.

But then we have shown that $x \in \bigcup_{k=n}^{\infty} E_n$ for any $n \geq 1$, which means that, by the definition of lim sup used here, $x \in \limsup_{n\to\infty} E_n$ (since this is the intersection of all of these sets). Since x was selected as an arbitrary element of $\liminf_{n\to\infty} E_n$, we've shown that $\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n$.
For the second part, we will use De Morgan's laws, which we know hold for infinite unions and intersections as well (this is easy to prove using the same arguments as for finite unions and intersections). Namely, we have that:

$$\liminf_{n\to\infty}(E_n^c) = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} E_k^c) = \bigcup_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k)^c = (\bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k))^c = (\limsup_{n\to\infty} E_n)^c$$

1.2 Limits and Continuity

Exercise 40
Prove the following theorem (1.17):
Let f be a real-valued function defined in some punctured neighborhood of $a \in \mathbb{R}$. Then, the following are equivalent:
(i) There exists a number L such that $\lim_{x\to a} f(x) = L$ (by the $\epsilon - \delta$ definition).
(ii) There exists a number L such that $f(x_n) \to L$ whenever $x_n \to a$, where $x_n \neq a$ for all n.
(iii) $(f(x_n))$ converges (to something) whenever $x_n \to a$, where $x_n \neq a$ for all n.

Solution.
(i) $\implies$ (ii): Pick any sequence $x_n \to a$, with $x_n \neq a$ for all of its terms. For any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x_n - a| < \delta$ we have that $|f(x_n) - L| < \epsilon$. Since $x_n \to a$, for this $\delta > 0$ there exists $N > 0$ such that for $n > N$ we have that $0 < |x_n - a| < \delta$. But then it also holds that for $n > N$, $|f(x_n) - L| < \epsilon$, which means that for any $\epsilon > 0$ we are able to find $N > 0$ such that the definition of the limit for $(f(x_n))$ holds, with the limit value being L.
Therefore, whenever $x_n \to a, x_n \neq a$, it is also the case that $f(x_n) \to L$.
(ii) $\implies$ (iii): Pick any sequence $(f(x_n))$ such that the corresponding (x_n) converges to a and such that $x_n \neq a$. Then, applying (ii) one can obtain that $f(x_n) \to L$, which means indeed that $(f(x_n))$ converges to *something*.
(iii) $\implies$ (ii): Pick any two sequences $x_n \to a, y_n \to a$, such that $x_n \neq a, y_n \neq a$ for all n. We then have that $f(x_n) \to L_1, f(y_n) \to L_2$. As pointed out in the book, construct the sequence $x_1, y_1, x_2, y_2, \dots$ by "interlacing" $(x_n), (y_n)$. It's easy to see that this sequence converges to a as well. Therefore, $f(z_n) \to L_3$. But now observe that both $f(x_n)$ and $f(y_n)$ are subsequences of $f(z_n)$, which means more specifically that all three of them must converge to the same limit, i.e. $L_1 = L_2 = L_3$.
We have thus shown that for any two sequences $x_n \to a, y_n \to a$, the corresponding $(f(x_n)), (f(y_n))$ always converge to the same limit, which is an equivalent way of stating (ii).
To complete the full equivalence of the theorem, we will now show that (ii) $\implies$ (i). To do this, suppose that (i) does not hold. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ it holds that for some x it is the case that both $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. Construct a sequence of $\delta_i = \frac{1}{i}$, and from that construct a corresponding sequence of $i \to x_i$, where each x_i fulfills the above mentioned conditions. Then clearly the sequence (x_i) converges to a, but the corresponding $(f(x_i))$ does not converge to L. This directly contradicts (ii), and we have thus shown that (ii) does indeed imply (i).

Exercise 45
Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(x) = 0$ whenever x is rational. Show that $f(x) = 0$ for every $x \in [a, b]$.

Solution.
Pick any $x \in [a, b]$. If $x \in \mathbb{Q}$, we immediately know that $f(x) = 0$. If $x \notin \mathbb{Q}$, then we know from a previous result that we can always approach the real number x with a sequence of rational numbers (x_n). In other words, $x_n \to x, x_n \in \mathbb{Q}$ for all x_n. Because f is continuous, it must then hold that $f(x_n) \to f(x)$. Observe that the sequence of $f(x_n)$ is a sequence of zeros, since all $x_n \in \mathbb{Q}$. Clearly then the limit has to be zero as well, which directly means that $f(x) = 0$, thus showing that $f(x)$ is indeed the zero function on $[a, b]$.

Exercise 46
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some open interval $(-a, a)$.
(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≥ 0" replaced by "> 0"? Explain.

Solution.
(a) Suppose for the sake of contradiction that this does not hold. This is equivalent to saying that for any $a \in \mathbb{R}$, there exist some $x_a \in (-a, a)$ such that $f(x_a) \leq 0$. Consider constructing the following sequence of a_i: $a_i = \frac{1}{n}$. Then, in every one of these intervals there has to exist x_i such that $f(x_i) \leq 0$. Observe that $a_i \to 0$, which means also that $x_i \to 0$. Since f is continuous, this means that it has to be the case that $f(x_i) \to f(0) > 0$. However, this is by construction a sequence of non-positive numbers, and because limits preserve non-strict inequalities, it cannot be the case that the limit of $(f(x_i))$ is positive. We arrive at a contradiction, and therefore it must be the case that for some a, $f(x) > 0$ for all $x \in (-a, a)$.
(b) Again, for an $x \in \mathbb{R} \setminus \mathbb{Q}$, we can approach it with a sequence of rationals (x_n). Then it is the case that $f(x_n) \geq 0$ for all n, and because f is continuous it also holds that $f(x_n) \to f(x)$. Again, limits preserve non-strict inequalities, and thus $f(x) \geq 0$ holds for all real x.
Now we consider whether this is true if we alter "$\geq$" to "$>$". Consider the function $f(x) = |x - \sqrt{2}|$. Clearly, this function is continuous, and also positive whenever $x \neq \sqrt{2}$ and zero for $x = \sqrt{2}$. More specifically, it's a function that is strictly positive for rational x. However, for the real $x = \sqrt{2}$, $f(\sqrt{2}) = 0$, thus providing a counterexample to the claim "if $f(x) > 0$ for every rational x, show that $f(x) > 0$ for all real x".

Exercise 50
Let D denote the set of rationals in $[0, 1]$ and suppose that $f : D \to \mathbb{R}$ is increasing. Show that there is an increasing function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ whenever x is rational. [Hint: For $x \in [0, 1]$, define $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}]$.

Solution.
Define g as in the provided hint. We first show that if $x \in \mathbb{Q}, f(x) = g(x)$. We have that $g(x) = \sup\{f(t) : 0 \leq t \leq x, t \in \mathbb{Q}\}$. Recall that f is increasing, therefore, for all $0 \leq t \leq x, t \in \mathbb{Q}$ it has to hold that $f(t) \leq f(x)$. Therefore $f(x)$ is an upper bound for the set over which we are computing the supremum. For any $s < f(x)$, it is clear that s cannot be an upper bound for this set, because $f(x)$ belongs in it. We conclude that $f(x)$ is indeed the supremum, and thus that $g(x) = f(x)$.
Now to show that g is increasing, pick $x_1 < x_2, x_1, x_2 \in [0, 1]$ and call the sets over which $g(x_1), g(x_2)$ are computed S_1, S_2 respectively. If $y \in S_1$, we have that $y = f(t)$ for some $0 \leq t \leq x_1, t \in \mathbb{Q}$. Since $x_1 < x_2$, this means that $y \in S_2$ as well. Therefore $S_1 \subset S_2$, which means that $g(x_1) \leq g(x_2)$ by the properties of the supremum. Therefore g is indeed an increasing function.

Exercise 51
Let $f : [a, b] \to \mathbb{R}$ be increasing and define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x+)$ for $a \leq x < b$. and $g(b) = f(b)$. Prove that g is increasing and right-continuous.

Solution.
Before we begin we note that by preposition 1.19, $f(x+)$ is well-defined for all $x \in [a, b)$. First we will show that g is increasing. Pick $x_1, x_2 \in [a, b], x_1 < x_2$. We need to show that $g(x_1) \leq g(x_2)$, or, equivalently, that $f(x_1+) \leq f(x_2+)$. We will do this in two steps.
First, we will show that for every $x \in [a, b), f(x+) \geq f(x)$. Suppose this was not the case for some x, which would mean that $f(x+) < f(x) \implies 0 < f(x) - f(x+) = \epsilon$. Let $L = f(x+)$ to ease notation. Because L is well-defined, for this ϵ there exists $\delta > 0$ such that for all $y, x < y < x + \delta$ it holds that $|f(y) - L| < \epsilon$. More specifically, this implies that $f(y) < L + \epsilon = L + f(x) - L \implies f(y) < f(x)$. This contradicts the fact that f is increasing.
Secondly, we will show that for all $x \in [a, b), y \in [a, b], x < y$ it holds that $f(x+) \leq f(y)$. Suppose again that for some such x, y this does not hold, which means that $f(x+) > f(y)$, and again for ease of notation let $L = f(x+)$. Firstly, let $\delta_1 = y - x > 0$. Secondly, let $\epsilon = L - f(y) > 0$. Because L is well-defined, for this ϵ there exists $\delta_2 > 0$ such that for all $z, x < z < x + \delta_2$ it holds that:

$$|f(z) - L| < \epsilon \implies f(z) > L - \epsilon = L - L + f(y) \implies f(z) > f(y)$$

Set $\delta = \min\{\delta_1, \delta_2\}$, which means that $z < x + \delta \leq x + \delta_1 = y$ and also that, as shown above, $f(z) > f(y)$. Again, this directly contradicts the fact that f is increasing.
We can now complete the proof by observing that $g(x_1) = f(x_1+) \leq f(x_2)$ by using the second of the two proven lemmas, and then that $f(x_2) \leq f(x_2+) = g(x_2)$ by using the first of the two proven lemmas. Note that the last step is omitted if $x_2 = b$, in which case $f(x_2) = f(b) = g(b)$.
Now, because g is increasing in $[a, b]$, we know by proposition 1.19 that $g(x+)$ always exists for $x \in [a, b)$. We thus only need to show that $g(x+) = g(x)$. Because $f(x+)$ is well defined, for any given $\epsilon > 0$, there exists $\delta > 0$ such that for $x < y < x + \delta$ it holds that $|f(y) - f(x+)| < \epsilon$. Set $\delta' = \frac{\delta}{2} < \delta$. Then for all $y, x < y < x + \delta'$ it is the case that there exists $z \in (x, x + \delta)$ such that $y < z$. By the second of the two lemmas above, $g(y) = f(y+) \leq f(z) < f(x+) + \epsilon$. By the first of the two lemmas above, it is furthermore true that $g(y) = f(y+) \geq f(y) > f(x+) - \epsilon$.
Putting these together we obtain that $f(x+) - \epsilon < g(y) < f(x+) + \epsilon$, and since $f(x+) = g(x)$ this means that by picking $\delta' = \frac{\delta}{2}$ the definition of right continuity for g at x is satisfied.

Chapter 2

Countable and Uncountable Sets

2.1 Equivalence and Cardinality

Exercise 3
Given finitely many countable sets $A_1, \ldots, A_n$, show that $A_1 \cup \ldots \cup A_n$ and $A_1 \times \ldots \times A_n$ are countable sets.

Solution.
We will use induction on n:

- Base case, for $n = 2$: Consider two countable sets, A_1, A_2. If both of them are finite, then their union and Cartesian product are also finite, and thus trivially countable.

  If exactly one is finite, say $A_2 = \{a'_1, a'_2, \ldots, a'_n\}$, then observe that $A_1 \cup A_2$ contains at most all elements of A_1 and all elements of A_2, and possibly fewer if their intersection is not empty. In any case, suppose $|A_1 \cap A_2| = m$, and name f the bijection from A_1 to $\mathbb{N}$. Then let $S = A_2 \setminus A_1 = \{a'_{k_1}, \ldots a'_{k_{n-m}}\}$, and $f'(a'_{k_1}) = 1, \ldots f'(a'_{k_{n-m}}) = n - m$, in which case $A_1 \cup A_2 = A_1 \cup S$, and A_1, S have an empty intersection. Let $g : A_1 \cup S \to \mathbb{N}$ be such that:

  $$g(a) = \begin{cases} f(a) + (n-m) & , a \in A_1 \\ f'(a) & , a \in S \end{cases}$$

  Because f is a bijection, one can clearly see that g is surjective. Furthermore, g is one-to-one because f, f' are one-to-one and because the two “branches” of g have no overlapping values ($\min\{f(a) + (n-m)\} = n - m + 1 > n - m = \max\{f'(a)\}$). Therefore g is a bijection from $A_1 \cup A_2$ to $\mathbb{N}$, which means precisely that the union is countable.

  Now for the case where both A_1, A_2 are infinite, there exist bijections $f_1 : A_1 \to \mathbb{N}, f_2 : A_2 \to \mathbb{N}$. These impose orders $f_1(a_1) = 1, f_1(a_2) = 2, \ldots$ for $a'_i \in A_1$ and $f_2(a'_1) = 1, f_2(a'_2) = 2$ for $a_i \in A_2$. One can again set $S = A_2 \setminus A_1$, and f_2 can again be used to extract an order for the elements $s_1, s_2, \ldots$ of S. If S is finite, the problem reduces to the case above. If S is also infinite, we know that S is also countable, with $f'_2 : S \to \mathbb{N}$ the corresponding bijection. Additionally, $A_1 \cup A_2 = A_1 \cup S = \{a_1, s_1, a_2, s_2, \ldots\}$ ands $A_1 \cap S = \emptyset$. Use the orders imposed by f_1, f'_2 to sort the elements of A_1 in the order $a_1, a_2, \ldots$ and the elements of S in the order $s_1, s_2, \ldots$. Then define $g : A_1 \cup A_2 \to \mathbb{N}$ as:

  $$g(a) = \begin{cases} 2i, & a = a_i \in A_1 \\ 2i + 1, & a = s_i \in S \end{cases}$$

  For $n \in \mathbb{N}$, if n is even the equation $n = g(a)$ has a unique solution for $a = a_n$, due to the bijectivity of f_1. If n is odd, the equation $n = g(a)$ has a unique solution for $a = s_n$, due to the bijectivity of f'_2. In any case, $A_1 \cup A_2$ has been shown to be equivalent to $\mathbb{N}$.

  For the Cartesian product, the case where both A_1, A_2 are finite is again trivial. If A_1 is infinite and A_2 finite, then:

  $$A_1 = \{a_1, a_2, \ldots\}, A_2 = \{a'_1, a'_2, \ldots, a'_n\}$$

Let then $g : A_1 \times A_2 \to \mathbb{N}$:

$$g(a_i, a'_j) = n(i-1) + (j-1),\ j = 1, 2, \ldots, n$$

Observe that the fact that $A_1 \sim \mathbb{N}$ and j only takes a finite number of values makes g a surjection. Indeed, for $x = k \cdot n + l, l = 0, 1, \ldots, n-1$ (here we use the division algorithm for integers), one has but to set $i = 1 + k$ (possible due to $A_1 \sim \mathbb{N}$ and $j = l + 1$ (always possible due to the range of values l can achieve) to obtain $g(a_i, a'_j) = x$.

To show that g is injective, suppose $g(a_i, a'_j) = g(a_k, a'_l)$, and we then have that:

$$n(i-1) + (j-1) = n(k-1) + (l-1) \implies n(i-k) = l - j$$

This implies that the RHS is a multiple of n, which, because $1 \leq l, j \leq n$ is only possible if $l = j$. But then we also have that $i = k$, thus that $(a_i, a'_j) = (a_k, a'_l)$, i.e. that g is injective.

g is therefore a bijection, and thus $A_1 \times A_2 \sim \mathbb{N}$.

If both A_1, A_2 are infinite, then consider the function $g : A_1 \times A_2 \to \mathbb{N}$:

$$g(a_i, a'_j) = 2^i(2j-1)$$

, which, because of the fact that $A_1 \sim \mathbb{N}, A_2 \sim \mathbb{N}$ (thus i, j take all natural numbers as values), and by a proof completely analogous to $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ can be shown to be a bijection, thus proving that $A_1 \times A_2 \sim \mathbb{N}$.

- Inductive step: If this holds for $n = k \geq 2$, then observe that $A_1 \cup \ldots \cup A_{k+1} = (A_1 \cup \ldots \cup A_k) \cup A_{k+1}$, and we can therefore apply the inductive hypothesis to the sets $A_1, \ldots A_k$, and the base case to the union of those with A_k to obtain the statement for $n = k + 1$ as well. The same argument applies to the Cartesian product as well, thus concluding the proof.

Exercise 5
Prove that a set is infinite if and only if it is equivalent to a proper subset of itself.
[Hint: If A is infinite and $x \in A$, show that A is equivalent to $A \setminus \{x\}$.]

Solution.
$\implies$: Suppose A is infinite. Then A contains an element x, and the set $S = A \setminus \{x\}$ must also be infinite. By exercise 4, the set S contains a countably infinite subset S'. We then have that:

$$S = S' \cup (S \setminus S'), A = S' \cup (S \setminus S') \cup \{x\}$$

, where all of the sets used in the unions are disjoint. By exercise 3, it holds that $S' \cup \{x\} \sim S'$, and this yields a corresponding bijection $g : S' \cup \{x\} \to S'$. Consider then the following function $f : A \to S$:

$$f(z) = \begin{cases} z, & z \in S \setminus S' \\ g(z), & z \in S' \cup \{x\} \end{cases}$$

, which, by the above observations regarding the bijectivity of g and the disjointness of the sets $S', S \setminus S', \{x\}$, means that f is a bijection as well, proving that $A \sim S$.
$\impliedby$: Now suppose that a set A is equivalent to S, S being a proper subset of A. By contradiction, suppose A is not infinite and that it contains n elements. Then S must contain at most $n-1$ elements. By the pigeonhole principle, there cannot exist a bijection from A to S, which means that $A \sim S$ cannot be true, a contradiction. Therefore A must be infinite.

Exercise 8
Show that $(0, 1)$ is equivalent to $[0, 1]$ and to $\mathbb{R}$.

Solution.
First, observe that $[0,1] = \{0\} \cup \{1\} \cup (0,1)$, and that $[0,1], [0,1), (0,1)$ are all infinite sets. By exercise 5, we have that $[0,1) \sim (0,1)$ (by explicitly picking $x = 0$ and $S = (0,1)$. Similarly, we also have that $[0,1] \sim [0,1)$ (by explicitly picking $x = 1$ and $S = [0,1)$). The transitivity property of the equivalence relation "is equivalent to" (exercise 1) yields then that $[0,1] \sim (0,1)$.
For $(0,1) \sim \mathbb{R}$, consider the function $f : \mathbb{R} \to (0,1), f(x) = \frac{1}{1+e^{-x}}$, which is known to be both injective and surjective, thus sufficing to show that $(0,1) \sim \mathbb{R}$.

Exercise 9
Show that $(0,1)$ is equivalent to the unit square $(0,1) \times (0,1)$.
[Hint: "Interlace" decimals, but carefully!]

Solution.
Pick any $x \in (0,1) \times (0,1)$. Then $x = (a,b), a,b \in (0,1)$. If a or b can be written as a decimal ending in infinite 9's (e.g. $0.3999\ldots$), write them in the equivalent form that features finite decimals (in this example, 0.4). Now consider the function $f : (0,1) \times (0,1) \to (0,1)$:

$$f(0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots) = 0.a_1b_1a_2b_2a_3b_3\ldots$$

Observe first that for any $x \in (0,1)$, x can be written as $x = 0.x_1x_2x_3\ldots$, and again if this can be written as ending in infinite 9's, "round it up". Then $f(0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots) = x$. Furthermore, suppose that $f(0.y_1y_2y_3\ldots, 0.z_1z_2z_3\ldots) = x = 0.x_1x_2x_3\ldots$. By definition, it is also the case that

$$f(0.y_1y_2y_3\ldots, 0.z_1z_2z_3\ldots) = 0.y_1z_1y_2z_2y_3z_3\ldots$$

Because we always choose to write decimals ending in infinite 9's as "rounded up", there can be no ambiguities here, in the sense that the equality of these two numbers directly implies that $y_1 = x_1, z_1 = x_2, y_2 = x_3, z_2 = x_4\ldots$ (i.e. any two decimals that do not end in infinite 9's can only be equal if all corresponding digits are equal). These two observations yields that f is bijective, thus proving the desired equivalence.

Exercise 10
Prove that $(0,1)$ can be put into one-to-one correspondence with the set of all *functions* $f : \mathbb{N} \to \{0,1\}$.

Solution.
To begin, we observe that a function from $\mathbb{N}$ to $\{0,1\}$ can be thought of a set of the form:

$$\{(0,v_0), (1,v_1), (2,v_2), \ldots\}$$

, where each v_i is the value of the function on i, and as such can be either 0 or 1. We now observe that the set of all such functions can be written as $S \cup \overline{S}$, where $S = \{f \in \{0,1\}^{\mathbb{N}} | \exists k \in \mathbb{N} \cup \{-1\}, f(i) = 1, i > k\}$ (in other words, a function either becomes 1 "forever" after some point, or it does not). Note that from now on we will use the binary system in this exercise.
Observe that all functions that belong in S are uniquely characterized by a "finite-length prefix" of values. This means that we can list all of them: start with a prefix of length 0, list all possible f such that in the above definition $k = -1$, then list all f with a prefix of length 1, etc. Since each length N yields only a finite number of functions (each corresponds to a binary number with N digits such that the last digit is not 1), we conclude that S is in fact countable, and that there exists a bijection $g : S \to \mathbb{N}$. Define now the following mapping $F : \{0,1\}^{\mathbb{N}} \to (0,1) \cup \mathbb{N}$:

$$F(f) = \begin{cases} 0.v_0v_1v_2\ldots, & f \in \overline{S} \\ g(f), & f \in S \end{cases}$$

By the bijectivity of g, F achieves all natural numbers. Furthermore, for any $x \in (0,1)$, we can always write $x = 0.b_0b_1\ldots$ such that it does not end in infinite ones, and then we can define an $f : \mathbb{N} \to \{0,1\}$

such that $f(i) = b_i$, which means that $F(f) = x$. We have thus shown that F is surjective. Additionally, we observe that due to our construction, if $f_1 \in \overline{S}, f_2 \in S, F(f_1), F(f_2)$ can never be equal: one is always in $(0, 1)$ and the other is a non-zero natural number. For $f_1, f_2 \in \overline{S}, F(f_1) = F(f_2)$ either implies that all corresponding digits are equal, in which case clearly $f_1 = f_2$, or that one of them ends in infinite ones, which is impossible since $f_1, f_2 \in \overline{S}$. Finally, for $f_1, f_2 \in S$, the bijectivity of g implies that $f_1 = f_2$.
This shows that F is also injective, which means it is a bijection from $\{0, 1\}^{\mathbb{N}}$ to $(0, 1) \cup \mathbb{N}$. Now , $\mathbb{N}$ is clearly countable, and from exercise 6 we then have that $(0, 1) \cup \mathbb{N} \sim (0, 1)$, which by exercise 1 also gives us our desired result, $\{0, 1\}^{\mathbb{N}} \sim (0, 1)$.

Exercise 13
Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

Solution.
We know that there exist infinitely many primes, $p_1, p_2, \ldots$. Consider then forming the sets $S_i = \{p_i^k | k \in \mathbb{N} \setminus \{0\}\}$. These are clearly infinitely many, and each of them is infinite. Suppose that two of them, S_i, S_j had a non-empty intersection. Then there exists $x \in \mathbb{N}$ such that $x = p_i^k = p_j^l, k, l \geq 1$. But then this number has two different prime factorizations, which is a contradiction. Therefore all of these sets are pairwise disjoint.

Exercise 15
Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable. [Hint: Each one contains a rational!]

Solution.
Suppose that there exists an uncountable collection of pairwise disjoint and nonempty intervals of the form $(a, b), a, b \in \mathbb{R}$. As indicated in the hint, each such interval must contain a rational, regardless of whether a, b are rationals or not. Furthermore, because the intervals are all disjoint, no two such rationals can be equal. This means that there exist at least as many rationals as intervals, therefore an uncountable number of them, which we know is a contradiction.

2.2 The Cantor Set

Exercise 21
Show that any ternary decimal of the form $0.a_1a_2 \ldots a_n11$ (base 3), i.e., any finite-length decimal ending in two (or more) 1s is *not* an element of Δ.

Solution.
For this problem we will be using the following formalization of the Cantor set $\mathcal{C}$:

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \Big(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\Big)$$

We now consider a ternary decimal x of the form $x = 0.a_1 \ldots a_n11$, where n may be zero, in which case the "prefix" has zero length. Consider then what x equals:

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} = \frac{a_1 3^{n+1} + a_2 3^n + \ldots + a_n 3^2 + 3 + 1}{3^{n+2}}$$

$$= \frac{1}{3^{n+1}}(a_1 3^n + a_2 3^{n-1} + \ldots + a_n 3 + \frac{4}{3}) = \frac{1}{3^{n+1}}(3(a_1 3^{n-1} + a_2 3^{n-2} + \ldots + a_n) + \frac{4}{3})$$

Notice now that if $y = a_1 3^{n-1} + a_2 3^{n-2} + \ldots + a_n$, y is an integer that is at least 0 and at most $6\frac{3^n-1}{2} = 3(3^n - 1)$ (by using the geometric progression sum). But then this range in which y can move means that x is precisely equal to *one* of the left endpoints in the n-th union in the definition of $\mathcal{C}$ plus $\frac{4}{3}$, which, because $1 < \frac{4}{3} < 2$, means that x will always lie *inside* one of the intervals that comprise this union, which means of course that it lies outside of the Cantor set.

Exercise 22
Show that Δ contains no (nonempty) open intervals. In particular, show that if $x, y \in \Delta$, then there is some $z \in [0,1] \setminus \Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small".)

Solution.
Let $x, y \in \Delta$. Following the notation of the book, we name I_k the union of closed intervals that comprises the k-th "level" of the "tree" that creates the Cantor set. Then recall that $\Delta = \bigcap_{k=0}^{\infty} I_k$. Since $x, y \in \Delta$, it holds that $x, y \in I_k$ for all k. We now claim that there exists k such that x, y belong in two disjoint intervals that comprise the union I_k. This will of course imply that there exists a "removed middle third" interval between x, y at the k-th level, any of whose elements fulfill the condition $z \notin \Delta$. We do this by contradiction. Namely, suppose that x, y belong in the same closed interval at every level of the "tree", and call those intervals $I_1, I_2, \ldots$. By construction, these are nested intervals, and we know that I_n's length goes to zero as $n \to \infty$. But then the nested interval theorem dictates that the infinite intersection of $I_1, I_2, \ldots$ contains precisely one element, which means $x = y$, a contradiction.

Exercise 23
The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in Δ of the form $a/3^n$ for some integers n and $0 \leq a \leq 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as $0.a_1a_2\ldots a_{n+1}$ (base 3) where each a_k is 0 or 2, except a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2\ldots a_n1, 0.a_1a_2\ldots a_n2)$ where both entires are points of Δ written in base 3.

Solution.
We will apply induction on n, which corresponds to the "level" of the "tree" of the Cantor set. More specifically, we will prove the following.

- If $x \neq 0$ is a left endpoint of the Cantor set, that is, the the intersection forming the Cantor set contains it in the n-th "tree-level union " in some interval $[x, y]$, then it can be written in the form $0.a_1\ldots a_n2$ where $a_i \in \{0, 2\}, i = 1, \ldots, n$.
- If $x \neq 1$ is a right endpoint of the Cantor set, that is, the intersection forming the Cantor set contains it in the n-th "tree-level union" in some interval $[y, x]$, then it can be written in the form $0.a_1\ldots a_n1$ where $a_i \in \{0, 2\}, i = 1, \ldots, n$.

As stated, we will use induction on n starting at 0, which corresponds to the first level of the tree.

- **Base case**: if $n = 0$, then the corresponding union is $I_1 = [0, 0.1] \cup [0.2, 1]$ (written in ternary). Obviously, the only non-zero left endpoint is 0.2, which satisfies the claim stated above. Similarly, the only right endpoint not equal to 1 is 0.1, which also satisfies the claim.
- **Inductive hypothesis**: Suppose that for $n = k, k \geq 0$ it holds that $I_n = [0, x_1] \cup [x_2, x_3] \cup \ldots \cup [x_M, 1]$, where each left and right endpoint fulfill the conditions stated above (note that $M = 2^{n+1}$).
- **Inductive step**: Observe then that the intervals comprising I_{n+1} are formed by taking each interval $[x, y]$ comprising I_n and replacing it by $[x, x + 0.\underbrace{0\ldots0}_{n+1}1] \cup [y - \underbrace{0.0\ldots0}_{n+1}1, y]$. Therefore, new right endpoints are formed by taking either 0 or a previous left endpoint and adding $0.\underbrace{0\ldots0}_{n+1}1$. Clearly, adding such a number to 0 results in a right endpoint fulfilling the claim stated in the beginning. In the other case, we have an endpoint of the form $0.a_1\ldots a_n2 + 0.\underbrace{0\ldots0}_{n+1}1 = 0.a_1\ldots a_n21$, which based on the inductive hypothesis also fulfills the initial claim. Similarly, new left endpoints are formed by taking either 1 or previous right endpoints and subtracting $0.\underbrace{0\ldots0}_{n+1}1$. Subtracting this quantity from 1

results in $0.\underbrace{2\ldots2}_{n+1}2$, which fulfills the claim for a left endpoint. On the other hand, subtracting it from a previous right endpoint results in an endpoint of the form $0.a_1\ldots a_n1 - 0.\underbrace{0\ldots0}_{n+1}1 = 0.a_1\ldots a_n02$, which based on the inductive hypothesis also satisfies the claim for left endpoints.

This concludes the proof that endpoints of Δ have a ternary decimal expansion described in the exercise.

Exercise 26
Let $f : \Delta \to [0,1]$ be the Cantor function and let $x, y \in \Delta$ with $x < y$. Show that $f(x) \leq f(y)$. If $f(x) = f(y)$, show that x has two distinct ternary decimal representations. Finally, show that $f(x) = f(y)$ if and only if x, y are "consecutive" endpoints of the form $x = 0.a_1a_1\ldots a_n1$ and $y = a_1a_2\ldots a_n2$ (base 3).

Solution.
Let $x = 0.a_1a_2\ldots$ and $y = 0.b_1b_2\ldots$. Because $x, y \in \Delta$, we know that we can write x, y such that each of a_i, b_i is either 0 or 1. Now, because $x < y$ it has to be the case that for some $i \geq 1, a_i < b_i$ and $a_k = b_k$ for $k < i$. This can be seen by making a "digit-wise" comparison: the first decimal digit in which x, y differ determines which one is larger, since after that point the decimal digits get multiplied by smaller powers of 3 and the smaller number can never "catch up". The only possible exception would be if $x = 0.a_1\ldots a_i00\ldots$ and $y = 0.a_1\ldots(a_i-1)22\ldots$, but we know that this cannot be true since it would require $a_i = 1$ or $a_i - 1 = 1$, which is impossible due to the way we've written x, y. Now consider the decimal strings $x' = 0.(\frac{a_1}{2})(\frac{a_2}{2})\ldots, y' = 0.(\frac{b_1}{2})(\frac{b_2}{2})\ldots$. What the Cantor function does is interpret these as binary instead of ternary numbers, and set then $f(x) = x'_{\text{bin}}, f(y) = y'_{\text{bin}}$. But then $\frac{a_k}{2} = \frac{b_k}{2}$ for $k < i, \frac{a_i}{2} < \frac{b_i}{2}$. By this we have established that it is impossible that $f(x) > f(y)$, thus $f(x) \leq f(y)$. If it is the case that $f(x) = f(y)$, then for $k > i, \frac{a_k}{2} = 1, \frac{b_k}{2} = 0$ and $\frac{a_i}{2} = 0, \frac{b_i}{2} = 1$. Going back to the ternary numbers x, y, this implies that x is of the form:

$$x = 0.a_1\ldots a_{i-1}0222\ldots,$$

which can also be written as $x = 0.a_1\ldots a_{i-1}1$. Furthermore, it implies that y is of the form:

$$y = 0.a_1\ldots a_{i-1}2000\ldots,$$

which completes the proof that a) x has two distinct ternary decimal expansions, and that whenever $f(x) = f(y), x, y$ are endpoints of "discarded" middle third intervals. For the other direction of the last equivalence, if x, y are endpoints of "discarded" middle third intervals, write both of them such that they only contain 0s and 2s in ternary, apply f and observe that $f(x)$ ends in infinite 1s, $f(y)$ in 0s, and for some $i, f(x)_i0, f(y)_i = 1$ and $f(x)_k = f(y)_k, k < i$, which of course means that $f(x) = f(y)$.

Exercise 29
Prove that the extended Cantor function $f : [0,1] \to [0,1]$ (as defined above) is increasing. [Hint: consider cases.]

Solution.
For any two $x, y \in [0,1]$, with $x < y$, we have the following:

- If $x, y \in \Delta$, we have shown in exercise 23 that $f(x) \leq f(y)$.
- If $x \in \Delta, y \notin \Delta$, then $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$. This means that $f(x) \in \{f(z), z \leq y, z \in \Delta\}$, and thus it must be the case that $f(x) \leq f(y)$.
- If $x, y \notin \Delta$, then $f(x) = \sup\{f(z), z \leq x, z \in \Delta\}$ and $f(y) = \sup\{f(z), z \leq y, z \in \Delta\}$. But then since $x < y$, the set over which the second supremum is computed is a superset of the set over which the first supremum is computed, meaning that $f(x) \leq f(y)$.

2.3 Monotone Functions

Exercise 34
Let $D = \{x_1, x_2, \ldots\}$, and let $\epsilon_n > 0$ with $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Define $f(x) = \sum_{x_n \le x} \epsilon_n$ (as above). Check the following:
(i) f is discontinuous at the points of D
(ii) f is right-continuous everywhere
(iii) f is continuous at each point $x \in \mathbb{R} \setminus D$
How might this construction be modified to yield a *strictly* increasing function with these same properties?

Solution.
(i) Notice that we've already seen in the main text that for $x_k \in D, f(x_k-) = f(x_k) - \epsilon_k$, that is, the left limit of f at each point of D can never equal its value on the respective point, since $\epsilon_k > 0$. Therefore f is not continuous at the points of D.
(ii) Again, in the main text we've seen that f is right-continuous at the points of D. It suffices therefore to check that for $x \in \mathbb{R} \setminus D, f(x+) = f(x)$. Expanding this, we have:

$$f(x+) = \lim_{y \to x+} f(y)$$

Pick any $\epsilon > 0$. We need to show then that there exists $\delta > 0$ such that whenever $y - x < \delta$ it holds that $f(y) - f(x) < \epsilon$ (note that the fact that f is increasing and the fact that we are taking a right limit allows us to omit absolute values). We have that:

$$f(y) - f(x) = \sum_{\{n : x_n \le y\}} \epsilon_n - \sum_{\{n : x_n \le x\}} \epsilon_n = \sum_{\{n : x < x_n \le y\}} \epsilon_n$$

As has already been observed in the main text, since the series corresponding to ϵ_n converges, it is the case that $\sum_{n=N}^{\infty} \epsilon_n = 0$ as $N \to \infty$. This means that given our chosen $\epsilon > 0$, we can find an $N > 0$ such that $\sum_{n=N}^{\infty} \epsilon_n < \epsilon$. Therefore, after excluding a *finite* number of terms from the series, namely $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\}$ we can make the corresponding remaining infinite sum arbitrarily small. Because the set is finite, we can then pick $\delta > 0$ such that none of these terms are in the interval $(x, y]$: simply find the closest term to x and pick y to be closer to x than that. Then by necessity:

$$\sum_{\{n : x < x_n \le y\}} \epsilon_n \le \sum_{n=N}^{\infty} \epsilon_n < \epsilon,$$

which means precisely that $f(y) - f(x) < \epsilon$, thus completing the proof that f is right-continuous everywhere.
(iii) Due to part (ii) we only need to prove that f is left-continuous at $x \in \mathbb{R} \setminus D$. That is, we need to prove that for $x \in \mathbb{R} \setminus D$:

$$\lim_{y \to x-} f(y) = f(x)$$

Pick any $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $x - y < \delta$:

$$f(x) - f(y) < \epsilon \implies \sum_{\{n : x_n \le x\}} \epsilon_n - \sum_{\{n : x_n \le y\}} \epsilon_n < \epsilon \implies \sum_{\{n : y < x_n \le x\}} \epsilon_n < \epsilon$$

This means that we can now use the exact same observation as in (ii) to bring y sufficiently close to x such that the LHS sum is strictly smaller than ϵ, thus proving that f is indeed left-continuous at x. Note that the important difference from the case $x \in D$ examined in the book is that here $x_n \le x$ does not introduce a fixed term in the sum, which would of course impose a lower bound on it. In contrast, when $x \in D$ it is not guaranteed that we can always exclude the finite set of terms obtained in (ii) since one of them *may be x itself.*
Now for the question posed in the exercise, consider the function $g(x) = f(x) + x$. Because the identity function is continuous everywhere, g will maintain all properties of f shown in the exercise: it will be right-continuous everywhere as a sum of right-continuous functions, it will be continuous at $\mathbb{R} \setminus D$ and at the points of D it will be discontinuous (since otherwise $f(x) = g(x) - x$ would be continuous as a difference of continuous functions). In addition, for any two $x_1 < x_2$, we have that $f(x_1) \le f(x_2)$, and therefore clearly $g(x_1) < g(x_2)$, meaning that g is strictly increasing.

Exercise 35
Let $f : [a, b] \to \mathbb{R}$ be increasing, and let (x_n) be an enumeration of the discontinuities of f. For each n, let $a_n = f(x_n) - f(x_n-)$ and $b_n = f(x_n+) - f(x_n)$ be the left and right "jumps" in the graph of f, where $a_n = 0$ if $x_n = a$ and $b_n = 0$ if $x_n = b$. Show that $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$ and $\sum_{n=1}^{\infty} b_n \leq f(b) - f(a)$.

Solution.
We will first use exercise 33 (without the absolute values since we have an *increasing* function) to obtain an intermediate result about the first n terms of the sums that appear in this exercise. Namely, applying exercise 33 on the points (x_n) we have that:

$$\sum_{i=1}^{n} f(x_i+) - f(x_i-) \leq f(b) - f(a) \implies \sum_{i=1}^{n} f(x_i+) - f(x_i) + f(x_i) - f(x_i-) \leq f(b) - f(a)$$

$$\implies \sum_{i=1}^{n} f(x_i+) - f(x_i) + \sum_{i=1}^{n} f(x_i) - f(x_i-) \leq f(b) - f(a) \implies \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i \leq f(b) - f(a)$$

Since we are dealing with non-negative quantities from this we can conclude that

$$\sum_{i=1}^{n} b_i \leq f(b) - f(a), \sum_{i=1}^{n} a_i \leq f(b) - f(a)$$

Observe therefore that if the left or right discontinuities are finite in number, this result proves what is needed in the exercise (for a_i or b_i respectively). We now examine the case where the left discontinuities are infinite in number. Observe that each a_n is non-negative, and therefore the sequence of partial sums is non-decreasing, while we've already shown that it is bounded. We thus know that it converges, and since limits maintain non-strict inequalities, we obtain that $\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$. The exact same reasoning can also be applied to right discontinuities.

Exercise 36
In the notation of Exercise 35, define $h(x) = \sum_{\{n: x_n \leq x\}} a_n + \sum_{\{n: x_n < x\}} b_n$. Show that h is increasing and that $g = f - h$ is both continuous and increasing. Thus, each increasing function f can be written as the sum of a continuous increasing function g and a "pure jump" function h.

Solution.
We begin by showing that h is increasing. Consider $y_1, y_2 \in [a, b], y_1 < y_2$. Then:

$$\begin{aligned} h(y_2) - h(y_1) &= \sum_{\{n: x_n \leq y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \leq y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n \\ &= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n \geq 0, \end{aligned}$$

since each term of these sums is non-negative (recall that f is increasing). Therefore h is increasing. Now we examine the continuity of g, for which we'll have to examine the left and right limits at the points of discontinuity of f and at the points where f is continuous.
First let's consider $x_i \in \{x_1, x_2, \ldots\}$, i.e. a point of discontinuity of f:

- For $y \to x_i+$, we are interested in the following limit:

$$\lim_{y \to x_i+} g(y) = \lim_{y \to x_i+} (f(y) - h(y)) = \lim_{y \to x_i+} \Big(f(y) - \sum_{\{n: x_n \leq y\}} a_n - \sum_{\{n: x_n < y\}} b_n \Big)$$

  Now notice that in this case it is always true that $y > x_i$, leading us to decompose the sums as follows:

$$f(y) - \sum_{\{n: x_n \leq x_i\}} a_n - \sum_{\{n: x_i < x_n \leq y\}} a_n - \sum_{\{n: x_n < x_i\}} b_n - \sum_{\{n: x_i \leq x_n < y\}} b_n$$

$$= f(y) - h(x_i) - \Big(\sum_{\{n: x_i < x_n \le y\}} a_n + \sum_{\{n: x_i \le x_n < y\}} b_n \Big)$$

Now we observe that since we saw in exercise 35 that the series of the non-negative (a_n) converges, the sum of a_n inside the parenthesis can be made to go to zero as $y \to x_i+$: the argument here is the same as in exercise 34, where we show that we need but to exclude a finite number of terms of the series. The same applies to the sum of b_n, except that it tends to b_i instead (due to the "less than or equal to x_i" in its sum, where x_i does correspond to a term of the sequence). Consequently, the limit of this expression as $y \to x_i+$ must be:

$$f(x_i+) - h(x_i) - 0 - b_i = f(x_i+) - h(x_i) - f(x_i+) + f(x_i) = f(x_i) - h(x_i),$$

where the first term comes from the definition of the right limit of f at x_i, the second term is constant and the third and fourth were explained above. Therefore we conclude that $\lim_{y \to x_i+} g(y) = f(x_i) - h(x_i) = g(x_i)$.

- For $y \to x_i-$, the proof will be highly similar:

$$\lim_{y \to x_i-} g(y) = \lim_{y \to x_i-} \Big(f(y) - \sum_{\{n: x_n \le y\}} a_n - \sum_{\{n: x_n < y\}} b_n \Big)$$

  The decomposition now looks like:

$$f(y) - \sum_{\{n: x_n \le x_i\}} a_n + \sum_{\{n: y < x_n \le x_i\}} a_n - \sum_{\{n: x_n < x_i\}} b_n + \sum_{\{n: y \le x_n < x_i\}} b_n$$
$$= f(y) - h(x_i) + \Big(\sum_{\{n: y < x_n \le x_i\}} a_n + \sum_{\{n: y \le x_n < x_i\}} b_n \Big)$$

  A similar argument as above yields that the first sum here tends to a_i and the second to zero. Therefore the desired limit will equal:

$$f(x_i-) - h(x_i) + a_i = f(x_i-) - h(x_i) + f(x_i) - f(x_i-) = f(x_i) - h(x_i),$$

  which is again the desired quantity.

Now we consider $x \notin \{x_1, x_2, \ldots\}$:

- For $y \to x+$, we are interested in:

$$\lim_{y \to x+} g(y) = \lim_{y \to x} (f(y) - h(y))$$

  Carrying out the calculations will lead to an expression identical to the first bullet above, except that since $x \notin \{x_1, x_2, \ldots\}$ all sums that involve a_n, b_n will now go to zero (unlike before, they never contain a "fixed" term corresponding to x_i). Therefore we'll eventually arrive at $\lim_{y \to x+} g(y) = f(x+) - h(x) = f(x) - h(x)$, since f is now continuous at x.

- For $y \to x-$, everything is exactly symmetrical to the above, and therefore once more $\lim_{y \to x-} g(y) = f(x) - h(x)$.

This completes the proof that g is in fact continuous in all of $[a, b]$.
All that remains is to show that g is increasing as well. Take $y_1, y_2 \in [a, b]$ with $y_1 < y_2$. We want to show that $g(y_1) \le g(y_2)$. We have that:

$$g(y_2) - g(y_1) = f(y_2) - h(y_2) - f(y_1) + h(y_1) = f(y_2) - f(y_1) - (h(y_2) - h(y_1))$$

We examine $h(y_2) - h(y_1)$:

$$h(y_2) - h(y_1) = \sum_{\{n: x_n \le y_2\}} a_n + \sum_{\{n: x_n < y_2\}} b_n - \sum_{\{n: x_n \le y_1\}} a_n - \sum_{\{n: x_n < y_1\}} b_n$$

$$= \sum_{\{n: y_1 < x_n \leq y_2\}} a_n + \sum_{\{n: y_1 \leq x_n < y_2\}} b_n$$

Notice that every term that appears in this sum corresponds to a $x_n \in [y_1, y_2]$. Recall also from exercise 35 that we had $\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i \leq f(b) - f(a)$ and that taking the limit of each of the two sums was well-defined, which means that this inequality as well holds at the limit. The only detail that may potentially be problematic here is that we may also be summing over the terms $f(y_1+) - f(y_1), f(y_2) - f(y_2-)$ in case one or both endpoints y_1, y_2 are also discontinuities. However, it can be shown fairly easily that with a small modification to 33 the inequality given there can also be shown to hold for n points in the closed interval [a, b] instead of just the open (one just has to take a finite number of $x_1, \ldots, x_n$ and consider $a < a' < x_1, x_n < b' < b$). Then 35 can be applied to generalize this to countably infinite points which may be the case here.
All of this results in the fact that the sum above is at most $f(y_2) - f(y_1)$ (since these are the endpoints of the relevant interval), and therefore also $g(y_2) \geq g(y_1)$, which shows that g is increasing.

Chapter 3

Metrics and Norms

3.1 Metric Spaces

Exercise 2
If d is a metric on M, show that $|d(x,z) - d(y,z)| \leq d(x,y)$ for any $x, y, z \in M$.

Solution.
First, we apply the triangle inequality as follows:

$$d(x,z) \leq d(x,y) + d(y,z) \implies d(x,z) - d(y,z) \leq d(x,y)$$

Another application of it yields:

$$d(y,z) \leq d(y,x) + d(x,z) \implies -d(y,x) \leq d(x,z) - d(y,z) \implies -d(x,y) \leq d(x,z) - d(y,z),$$

where we used the symmetry property of metric d. Putting the two inequalities together results in:

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

Exercise 3
As it happens, some of our requirements for a metric are redundant. To see why this is so, let M be a set and suppose $d : M \times M \to \mathbb{R}$ satisfies $d(x,y) = 0$ if and only if $x = y$, and $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in M$. Prove that d is a metric; that is, show that $d(x,y) \geq 0$ and $d(x,y) = d(y,x)$ hold for all x, y.

Solution.
Pick any two $x, y \in M$. We know then that for any $z \in M$ it holds that $d(x,z) \leq d(x,y) + d(z,y)$. More specifically, this holds for $z = x$, in which case we obtain:

$$d(x,x) \leq d(x,y) + d(x,y) \implies 0 \leq 2d(x,y),$$

which means that for any two $x, y \in M, d(x,y) \geq 0$.
For the symmetry propety, once again pick any two $x, y \in M$. Then $d(x,y) \leq d(x,z) + d(y,z)$ for any z. Pick then $z = x$, in which case we obtain $d(x,y) \leq d(x,x) + d(y,x) \implies d(x,y) \leq d(y,x)$. By exchanging the roles of x, y, we have that $d(y,x) \leq d(y,z) + d(x,z)$ for any z. Pick $z = y$ to observe that $d(y,x) \leq d(y,y) + d(x,y) \implies d(y,x) \leq d(x,y)$. But then $d(x,y) \leq d(y,x) \leq d(x,y)$, thus the only possibility is that $d(x,y) = d(y,x)$. Therefore d is indeed a metric.

Exercise 6
If d is any metric on M, show that $\rho(x,y) = \sqrt{d(x,y)}, \sigma(x,y) = d(x,y)/(1 + d(x,y))$ and $\tau(x,y) = \min\{d(x,y), 1\}$ are also metrics on M. [Hint: $\sigma(x,y) = F(d(x,y))$, where F is as in exercise 5.]

Solution.
We will solve this as a simple application of 7. If $F_1 : [0,\infty) \to [0,\infty), F_1(x) = \sqrt{x}$, then $\rho(x,y) = F_1(d(x,y))$. Then $F_1(x)/x = 1/\sqrt{x}$, which is clearly decreasing for $x > 0$, and therefore exercise 7 guarantees that ρ is a metric.
Similarly, if $F_2 : [0,\infty) \to [0,\infty), F_2(x) = \frac{x}{1+x}$, then $F_2'(x) = \frac{x+1-x}{(1+x)^2} = \frac{1}{(x+1)^2}$, which is clearly decreasing as for $x \geq 0$, and therefore σ is also a metric.
Lastly, if $F_3 : [0,\infty) \to [0,\infty), F_3(x) = \min\{x, 1\}$, then $\tau(x,y) = F_3(d(x,y))$ and:

$$F_3(x)/x = \begin{cases} 1, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases},$$

which is of course a decreasing function for $x > 0$, and therefore τ is a metric.

Exercise 7
Here is a generalization of exercises 5 and 6. Let $f : [0,\infty) \to [0,\infty)$ be increasing and satisfy $f(0) = 0$, and $f(x) > 0$ for all $x > 0$. If f also satisfies $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric. Show that each of the following conditions is sufficient to ensure that $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$:
(a) f has a second derivative satisfying $f'' \leq 0$
(b) f has a decreasing first derivative
(c) $f(x)/x$ is decreasing for $x > 0$
[Hint: First show that (a) $\implies$ (b) $\implies$ (c).]

Solution.
As indicated in the hint, we first show that (a) $\implies$ (b) $\implies$ (c). If, after that, we can show that (c) implies the triangle inequality for f, we know also that any of (a), (b) imply it as well (since they imply (c)).
(a) $\implies$ (b): Since $f'' \leq 0$, f more specifically has a first derivative on every point of $[0,\infty)$. From calculus 1, it's known also that this implies that f' is decreasing (one would prove this via the Mean Value Theorem).
(b) $\implies$ (c): (b) implies that f is differentiable, thus $g(x) = f(x)/x, x > 0$ is also differentiable, with $g'(x) = \frac{f'(x)x - f(x)}{x^2}$. If we can show that the numerator here is non-positive for $x > 0$, we'll have shown that g is decreasing. Pick any $x > 0$, and apply the Mean Value Theorem on f in the interval $[0, x]$. This means that there exists $y \in (0, x)$ such that:

$$f'(y) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x},$$

where we used the fact that $f(0) = 0$. Therefore, we have that:

$$f'(x)x - f(x) = f'(x)x - xf'(y) = x(f'(x) - f'(y)) \leq 0,$$

since f' is decreasing and $y < x$. This completes the proof that (b) $\implies$ (c).
We now need to show that the triangle inequality for f follows from (c). Pick any two $x, y \geq 0$. If any of them, or both, are zero, the triangle inequality follows from the fact that $f(0) = 0$. If $x, y \neq 0$, then we know that $x \leq x + y, y \leq x + y$, and thus that $\frac{f(x)}{x} \geq \frac{f(x+y)}{x+y}, \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$ (by the fact that $f(x)/x$ is decreasing). We can rewrite these as:

$$(x+y)f(x) \geq xf(x+y), (x+y)f(y) \geq yf(x+y),$$

and now we can add them to obtain that:

$$(x+y)(f(x) + f(y)) \geq f(x+y)(x+y) \implies f(x) + f(y) \geq f(x+y),$$

which is of course what we wanted to show.

Exercise 10
The *Hilbert cube*, H^∞, is the collection of all real sequences $x = (x_n)$ with $|x_n| \leq 1, n = 1, 2, \ldots$.
(i) Show that $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞.
(ii) Given $x, y \in H^\infty$ and $k \in \mathbb{N}$, let $M_k = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. Show that $2^{-k}M_k \leq d(x, y) \leq M_k + 2^{-k+1}$.

Solution.
(i) We will make use of exercise 3, which allows us to conclude that d is a metric if the following three things hold:

- First, that $d(x, y)$ is a non-negative real numer for any x, y, which, due to the infinite sum, is non-trivial here. Consider any two sequences x, y. We know that $|x_n| \leq 1, |y_n| \leq 1$, for all n, therefore the maximum value that $|x_n - y_n|$ is 2. This means that $2^{-n}|x_n - y_n| \leq 2^{-n+1}$. The partial sums that correspond to d are therefore non-decreasing and upper bounded by a convergent geometric series, therefore the series that defines d also converges.

- Second, that $d(x, y) = 0$ iff $x = y$. If two sequences x, y are equal, then $x_n = y_n$ for all n, thus $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - x_n| = 0$. Conversely, if $d(x, y) = 0$, then we have a non-decreasing sequence of non-negative partial sums that converges to zero. The only way this can happen is if each term of the sequence is zero, i.e. if $|x_n - y_n| = 0$ for all n, that is, if $x_n = y_n$ for all n. But this means precisely that $x = y$.

- Third, that the triangle inequality holds. Pick any three sequences x, y, z, and $k \in \mathbb{N}^+$. Then:

$$\sum_{n=1}^{k} 2^{-n}|x_n - y_n| = \sum_{n=1}^{k} 2^{-n}|x_n - z_n + z_n - y_n| \leq \sum_{n=1}^{k} 2^{-n}|x_n - z_n| + \sum_{n=1}^{k} 2^{-n}|z_n - y_n|,$$

  by the triangle inequality. Since this holds for any k and all series converge (by the first item proven), we know that the inequality holds for the infinite series as well, i.e. that:

$$d(x, y) \leq d(x, z) + d(y, z)$$

Therefore d is a metric on the Hilbert cube.
(ii) Pick any two sequences x, y in the Hilbert cube and $k \in \mathbb{N}$. We know then by definition that $M_k \geq |x_n - y_n|$ for $i \leq k$. In addition, for $n > k$, the n-th term of the series that forms the metric d is $2^{-n}|x_n - y_n| \leq 2^{-n+1}$. Thus:

$$d(x, y) = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n| \leq M_k \sum_{n=1}^{k} 2^{-n} + \sum_{n=k+1}^{\infty} 2^{-n+1}$$

$$= M_k(1 - 2^{-k}) + 2(2 - 1 - (1 - 2^{-k})) = M_k - 2^{-k}M_k + 2^{-k+1} \leq M_k + 2^{-k+1},$$

where we used the fact that the geometric series with ratio $r = 1/2$ sums to 2, and our second sum here was thus 2 minus the sum of the first k terms minus 1 since the series starts at 1 instead of 0. In the last step we also use the fact that $M_k \geq 0$. For the other inequality, we have again that:

$$d(x, y) = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n>k}^{\infty} 2^{-n}|x_n - y_n|$$

Notice that in order for x, y to minimize this, it should be the case that $x_n - y_n = 0, n > k$, and that the maximum absolute difference M_k is multiplied with the smallest possible quantity up to k, that is, with 2^{-k}, while for all other $n < k, x_n - y_n = 0$. This would yield that $d(x, y) \geq M_k 2^{-k} + 0 + 0 = M_k 2^{-k}$.

Exercise 12
Check that $d(f,g) = \max_{a\le t\le b}|f(t) - g(t)|$ defines a metric on $C[a,b]$, the collection of all continuous, real valued functions defined on the closed interval $[a,b]$.

Solution.
Note that here continuity is important to ensure that the difference of any two functions is continuous, and as such achieves a maximum value in any closed interval. Using exercise 3, we need to check the following:

- One, that $d(f,g) = 0$ iff $f = g$. Suppose first that $d(f,g) = 0$, that is, $\max_{a\le b}|f(t) - g(t)| = 0$. If f,g were not equal, there would exist $x \in [a,b]$ such that $f(x) \neq g(x)$. Clearly then $\max_{a\le t\le b}|f(t) - g(t)| \ge |f(x) - g(x)| > 0$, which is a contradiction. Therefore $f = g$. In the other direction, if $f = g$, we equivalently have that $f - g$ is the zero function on $[a,b]$, and therefore $d(f,g) = \max_{a\le t\le b}|f(t) - g(t)| = 0$.

- Two, that for any three $f,g,h \in C[a,b], d(f,h) \le d(f,g) + d(g,h)$. We have first that $d(f,h) = \max_{a\le t\le b}|f(t) - h(t)|$. Suppose that this function achieves its maximum value on $t_M \in [a,b]$ (if there are more than one, pick any). Then $d(f,h) = |f(t_M) - h(t_M)| = |f(t_M) - g(t_M) + g(t_M) - h(t_M)| \le |f(t_M) - g(t_M)| + |g(t_M) - h(t_M)|$. By definition, it holds that $|f(t_M) - g(t_M)| \le \max_{a\le t\le b}|f(t) - g(t)| = d(f,g)$, and similarly $|g(t_M) - h(t_M)| \le d(g,h)$. We therefore obtain that $d(f,h) \le d(f,g) + d(g,h)$.

Therefore d is indeed a metric on $C[a,b]$.

Exercise 14
We say that a subset A of a metric space is **bounded** if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a,x_0) \le C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.

Solution.
Suppose we have the finite union

$$U = A_1 \cup A_2 \cup \ldots \cup A_n$$

of n bounded subsets of a metric space M. Let then $x_1, x_2, \ldots, x_n, C_1, \ldots, C_n$ be such that $d(a,x_i) \le C$ for $a \in A_i$. Form the set $\{d(x_1,x_1), d(x_1,x_2), \ldots, d(x_n,x_1)\}$, which, crucially, has a finite number of elements, and therefore also has a maximum element, M. By the same reasoning, there exists C such that $C = \max\{C_1, \ldots, C_n\}$. Now pick any $a \in U$, which must belong in at least one A_i. Therefore:

$$d(a,x_1) \le d(a,x_i) + d(x_i,x_1) \le C_i + M \le C + M$$

If we thus set $x_U = x_1$ and $C_U = C + M$, we have shown precisely that U is bounded.

Exercise 15
We define the **diameter** of a nonempty subset A of M by $\text{diam}(A) = \sup\{d(a,b) : a,b \in A\}$. Show that A is bounded if and only if $\text{diam}(A)$ is finite.

Solution.
$\implies$: Suppose first that A is bounded. Then there exists $x_0 \in M, C \in \mathbb{R}$ such that $d(a,x_0) \le C$ for all $a \in A$. Pick any two $a,b \in A$. We then have that $d(a,b) \le d(a,x_0) + d(x_0,b) \le 2C$. Therefore the set $\{d(a,b) : a,b \in A\}$ has an upper bound, namely, $2C$, and therefore also a finite least upper bound, which means precisely that $\text{diam}(A)$ is finite.
$\impliedby$: Now suppose $\text{diam}(A) = C \in \mathbb{R}$. Fix an $a \in A$ (which exists since A is nonempty). Pick any $b \in A$, in which case we have that:

$$d(a,b) \in \{d(x,y) : x,y \in A\} \implies d(a,b) \le \text{diam}(A) = C$$

If thus set $x_0 = a$ and C the respective constant, we see that the definition of A being bounded is fulfilled.

3.2 Normed Vector Spaces

Exercise 16
Let V be a vector space, and let d be a metric on V satisfying $d(x, y) = d(x - y, 0)$ and $d(ax, ay) = |a|d(x, y)$ for every $x, y \in V$ and every scalar a. Show that $||x|| = d(x, 0)$ defines a norm on V (that has d as its "usual" metric). Give an example of a metric on the vector space $\mathbb{R}$ that fails to be associated with a norm in this way.

Solution. Let us examine the properties that would make $||\cdot||$ one by one:

- Suppose $x \in V$. Then $||x|| = d(x, 0) \geq 0$, since d is a metric (and of course $||x||$ is well-defined as a finite real number for the same reason).
- Suppose $||x|| = 0$. Then $d(x, 0) = 0$, which by the properties of metrics we know is true iff $x = 0$. Conversely, if $x = 0$, by the same argument $d(x, 0) = 0 \implies ||x|| = 0$. Thus $||x|| = 0 \iff x = 0$.
- For any scalar a, we have that $||ax|| = d(ax, 0) = d(ax, a0) = |a|d(x, 0) = |a|||x||$ by the exercise hypothesis.
- Pick any two $x, y \in V$. Then:

  $$||x+y|| = d(x+y, 0) \leq d(x+y, y)+d(y, 0) = d(x+y-y, 0)+d(y, 0) = d(x, 0)+d(y, 0) = ||x||+||y||,$$

  where we used the triangle inequality property for metrics and the fact that $d(x, y) = d(x - y, 0)$. Therefore the triangle inequality holds for the proposed norm.

We have thus shown that $||\cdot||$ is indeed a norm, and its usual metric will of course be $d'(x, y) = ||x - y|| = d(x - y, 0) = d(x, y)$, i.e., d.
For the requested example, consider the metric $\sigma(x, y) = |x - y|/(1 + |x - y|)$ from exercise 6 of section 3.1. Notice that the "proposed" norm would then be $||x|| = \sigma(x, 0) = |x|/(1 + |x|)$. Then for a scalar a:

$$||ax|| = |ax|/(1 + |ax|) = |a| \cdot |x|/(1 + |a| \cdot |x|),$$

which we can see does not necessarily equal $|a| \cdot ||x||$, and therefore the "proposed" norm is not really a norm. The cause for this is the fact that $\sigma(ax, ay) \neq |a|\sigma(x, y)$ in general, for similar reasons.

Exercise 18
Show that $||x||_\infty \leq ||x||_2 \leq ||x||_1$ for any $x \in \mathbb{R}^n$. Also check that $||x||_1 \leq n||x||_\infty$ and $||x||_1 \leq \sqrt{n}||x||_2$.

Solution.
We have that $||x||_\infty = \max_{1\leq i\leq n}|x_i|$, while $||x||_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2}$. By definition, there must exist at least one $j \in \{1, 2, \ldots, n\}$ such that $|x_j| = \max_{1\leq i\leq n}|x_i|$. In addition, the square is an increasing function for $x \geq 0$, which would mean that $(\max_{1\leq i\leq n}|x_i|)^2 = \max_{1\leq i\leq n}|x_i|^2$. We therefore have that:

$$\sum_{i=1}^{n}||x_i||^2 = (\max_{1\leq i\leq n}|x_i|)^2 + X,$$

for $X \geq 0$ (the sum of the absolute values of the remaining coordinates). Thus $(||x||_2)^2 \geq (||x||_\infty)^2$, and by taking square roots we have the first desired result.
We also have that:

$$(||x||_1)^2 = \left(\sum_{i=1}^{n}|x_i|\right)\left(\sum_{i=1}^{n}|x_i|\right) \geq \sum_{i=1}^{n}|x_i|^2 = (||x_2||)^2,$$

which we obtain by observing that the product results in a sum of non-negative terms over all pairs of indices $i, j \in \{1, 2, \ldots, n\}$, which of course includes all terms for which $i = j$. Taking square roots yields the second desired result.
For the third result, observe that for each i, $|x_i| \leq \max_{1\leq i\leq n}|x_i| = ||x||_\infty$. This means of course that:

$$||x||_1 = \sum_{i=1}^{n}|x_i| \leq n||x||_\infty$$

Lastly, we consider the following "trick" for the last part. For a given $x = (x_1, x_2, \ldots, x_n)$, consider the vectors $y, z \in \mathbb{R}^{n^2}$:

$$y = \begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \\ \vdots \\ |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix}, z = \begin{pmatrix} |x_1| \\ |x_1| \\ \vdots \\ |x_1| \\ \vdots \\ |x_n| \\ |x_n| \\ \vdots \\ |x_n| \end{pmatrix}$$

We have that $||z||_2 = ||y||_2 = \sqrt{\sum_{i=1}^n n|x_i|^2} = \sqrt{n}||x||_2$, while $\langle y, z\rangle = \sum_{i=1}^n \sum_{j=1}^n |x_i| \cdot |x_j| = (\sum_{i=1}^n |x_i|)^2 = (||x||_1)^2$. Applying the Cauchy-Schwarz inequality on y, z, we then have that:

$$\langle y, z\rangle \leq ||y|| \cdot ||z|| \implies (||x||_1)^2 \leq \sqrt{n}||x||_2\sqrt{n}||x||_2,$$

which, once more by taking square roots, gives us the last desired result.

Exercise 19
Show that we have $\sum_{i=1}^n x_i y_i = ||x||_2||y||_2$ (equality in the Cauchy-Schwarz inequality) if and only if x, y are *proportional*, that is, if and only if $x = ay$ or $y = ax$ for some $a \geq 0$.

Solution.
$\Longleftarrow$: Suppose $x = ay$ for some $a \geq 0$ (everything is symmetric for $y = ax$). Then:

$$||x||_2||y||_2 = ||ay||_2||y||_2 = |a| \cdot ||y||_2^2,$$

where we used the "scalar multiplication"/positive homogeneity property of the norm. Furthermore:

$$\sum_{i=1}^n x_i y_i = \sum_{i=1}^n (ay_i)y_i = a\sum_{i=1}^n y_i^2 = a||y||_2^2,$$

by the definition of the 2-norm on sequences, which yields the desired equality.
$\Longrightarrow$: First, if $y = 0$ then this is trivially true. If $y \neq 0$, it's also the case that $||y||_2 \neq 0$ and we have the following. Consider the beginning of the proof of the Cauchy-Schwarz inequality:

$$0 \leq ||x + ty||_2^2 = ||x||_2^2 + 2t\langle x, y\rangle + t^2||y||_2^2,$$

for any $t \in \mathbb{R}$. As we saw, the discriminant here is $\Delta = (2\langle x, y\rangle)^2 - 4||x||_2^2||y||_2^2 = 0$, by our hypothesis that $\sum_{i=1}^n x_i y_i = ||x||_2||y||_2$ (since $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$). Then this means that the corresponding second-degree polynomial of t has a *unique* solution a:

$$a = \frac{-2\langle x, y\rangle}{2||y||_2^2} = -\frac{\langle x, y\rangle}{||y||_2^2}$$

By the above equation, it must also be the case that:

$$0 = ||x + ay||_2^2,$$

which, by the properties of the norm, means that by necessity $x = -ay$. Now we only need to show that $a \leq 0$. Observe that the denominator of a is clearly positive as the norm of y. The numerator equals, by definition, $\sum_{i=}^n x_i y_i = ||x||_2||y||_2$ (by the hypothesis), so it's clearly also non-negative. The minus sign in front of the fraction makes a non-positive, which completes the proof.

Exercise 21
Recall that we defined l_1 to be the collection of all absolutely summable sequences under the norm $||x||_1 = \sum_{n=1}^{\infty}|x_n|$, and we defined l_∞ to be the collection of all bounded sequences under the norm $||x||_\infty = \sup_{n\geq 1}|x_n|$. Fill in the details showing that each of these spaces is in fact a normed vector space.

Solution.
For each of the two proposed norms, we need to show that they are in fact norms, and also that the sequences which converge under them form a vector space. We begin with showing that l_1 defines a norm:

- It's obvious from the definition that $||x||_1 \geq 0$ whenever it exists, as a sum of non-negative terms.
- For any scalar a, we have that if for a sequence x the series $\sum_{n=1}^{\infty}|x_n|$ converges to $||x||_1$, then by well-known properties of limits, the series $\sum_{n=1}^{\infty}|ax_n|$ will converge to $|a| \cdot ||x||_1$.
- If $\sum_{i=0}^{\infty}|x_n|$ converges to zero, then since all terms are strictly non-negative, it must be the case that all of them are zero, thus that $||x||_1 = 0$ implies $x = 0$. The converse is obvious.
- We now need to prove the triangle inequality. Pick $x, y \in l_1$ and any $n > 0$. Then, by the triangle inequality for absolute values:

$$\sum_{i=1}^{n}|x_i + y_i| \leq \sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}|y_i|$$

  Notice that as $n \to \infty$, the two sums on the RHS converge to $||x||_1, ||y||_1$ respectively. This imposes a bound on the series corresponding to the LHS, whose partial sums are also non-decreasing. Hence, the LHS also converges, and in fact the inequality holds at infinity, yielding $||x+y||_1 \leq ||x||_1 + ||y||_1$.

Therefore, $||x||_1$ is a norm. The third point above showed that $0 \in l_1$, while the triangle inequality proof showed that l_1 is closed under addition. Lastly, the second point above showed that l_1 is closed under scalar multiplication. Therefore, we've already shown that l_1 is a vector space, and thus $(l_1, ||\cdot||_1)$ is a normed vector space.
Now we examine $||\cdot||_\infty$ in a similar fashion.

- Again, from the definition it is obvious that whenever x is a bounded sequence, $||x||_\infty \geq 0$.
- For any scalar a, and any bounded sequence x, the sequence $(|ax_i|)$ will have as supremum $|a| \cdot ||x||_\infty$. If this were not the case, one would get a contradiction for x by dividing with $|a|$ (and if $a = 0$, then the supremum of $(|ax_i|)$ is obviously zero).
- It's clear that $||x||_\infty$ is zero iff x is the zero sequence, since the supremum of a set of non-negative numbers is zero iff all of them are zero.
- Lastly, the triangle inequality in this case arises as follows. Pick any $x, y \in l_\infty$. Then, for any i we have that $|x_i + y_i| \leq |x_i| + |y_i| \leq ||x||_\infty + ||y||_\infty$ by the definition of the infinity norm. But then this sum constitutes an upper bound for $|x_i + y_i|$ for all i, which means that by definition the supremum $\sup_i|x_i + y_i|$ both exists and is such that $||x+y||_\infty \leq ||x||_\infty + ||y||_\infty$.

The above shows both that $||\cdot||_\infty$ is a norm, and that l_∞ is a vector space, thus that $(l_\infty, ||\cdot||_\infty)$ is a normed vector space.

3.3 More Inequalities

Exercise 24
The conclusion of Lemma 3.7 (Hölder's inequality) also holds in the case $p = 1$ and $q = \infty$. Why?

Solution.
As a reminder, we say that a sequence y is in l_∞ if y is bounded above, and in that case we define $||y||_\infty = \sup_n |y_n|$. Let us first formally state what Hölder's inequality would assert in this case:
Given $x \in l_1$ and $y \in l_\infty$, we have $\sum_{i=1}^{\infty} |x_i y_i| \leq ||x||_1 ||y||_\infty$.
Inded, pick any $n \geq 1$. We then have that:

$$\sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i| \cdot |y_i| \leq \sum_{i=1}^{n} |x_i| \cdot \sup_k |y_k| = \sup_k |y_k| \cdot \sum_{i=1}^{n} |x_i| \leq ||y||_\infty \cdot ||x||_1,$$

where we used the definition of the supremum and the fact that $||x||_1$ is well-defined. Since the partial sums are non-decreasing and bounded, the LHS converges and the inequality (i.e., Hölder's) holds for the infinite series as well.

Exercise 25
The same techniques can be used to show that $||f||_p = (\int_0^1 |f(t)|^p dt)^{1/p}$ defines a norm on $C[0, 1]$ for any $1 < p < \infty$. State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for $p = 1, q = \infty$?)

Solution.
We begin by noting that due to the absolute value, the functions being integrated are always non-negative. For any $t \in (0, 1)$, we can thus apply Young's inequality on $a = \frac{|f(t)|}{||f||_p}, b = \frac{|g(t)|}{||g||_q}$:

$$\frac{|f(t)|}{||f||_p} \cdot \frac{|g(t)|}{||g||_q} \leq \frac{1}{p} \cdot \frac{|f(t)|^p}{||f||_p^p} + \frac{1}{q} \cdot \frac{|g(t)|^q}{||g||_q^q}$$

Since this is true for all $t \in (0, 1)$, we can integrate both sides wrt. t to obtain:

$$\int_0^1 \frac{|f(t)|}{||f||_p} \cdot \frac{|g(t)|}{||g||_q} dt \leq \frac{1}{p} \cdot \int_0^1 \frac{|f(t)|^p}{||f||_p^p} dt + \frac{1}{q} \cdot \int_0^1 \frac{|g(t)|^q}{||g||_q^q} dt \implies$$

$$\frac{1}{||f||_p \cdot ||g||_q} \int_0^1 |f(t)g(t)| dt \leq \frac{1}{p} \cdot \frac{1}{||f||_p^p} \int_0^1 |f(t)|^p dt + \frac{1}{q} \cdot \frac{1}{||g||_q^q} \int_0^1 |g(t)|^q dt = \frac{1}{p} + \frac{1}{q} = 1$$

This yields $\int_0^1 |f(t)g(t)| dt \leq ||f||_p \cdot ||g||_q$, which would be the equivalent of Hölder's inequality.
Now, Minkowski's inequality would state firstly that for any $1 < p < \infty$, if the p-norm exists for $f, g \in C[0, 1]$, it also exists for $f + g$, and secondly that $||f + g||_p \leq ||f||_p + ||g||_p$.
For the analogue of Minkowski's inequality ("if $f, g \in C[0, 1]$ and $||f||_p, ||g||_p$ both exist, then $||f+g||_p$ exists and $||f+g||_p \leq ||f||_p + ||g||_p$"), we first prove an analogue of Lemma 3.5 that allows us to immediately obtain that $||f + g||_p$ exists. More specifically, for any $t \in (0, 1)$, from Lemma 3.5 applied on $a = |f(t)|, b = |g(t)|$, we have that:

$$(|f(t)| + |g(t)|)^p \leq 2^p (|f(t)|^p + |g(t)|^p)$$

By the triangle inequality of the absolute value, we also have that $|f(t) + g(t)|^p \leq (|f(t)| + |g(t)|)^p$ since $p > 1$. Therefore, since these hold for any $t \in (0, 1)$:

$$|f(t) + g(t)|^p \leq 2^p (|f(t)|^p + |g(t)|^p) \implies \int_0^1 |f(t) + g(t)|^p dt \leq 2^p (\int_0^1 |f(t)|^p dt + \int_0^1 |g(t)|^p dt)$$
$$\implies ||f + g||_p^p \leq 2^p (||f||_p^p + ||g||_p^p),$$

which imposes a bound on $||f + g||_p$, thus showing it exists.
Now, following the proof in the book, we observe the following regarding $||f||_p^{p-1}$, for $1/p + 1/q = 1$:

$$||f^{p-1}||_q = (\int_0^1 |f^{q(p-1)}(t)|dt)^{1/q} = (\int_0^1 |f(t)|^p dt)^{1/q} = (\int_0^1 |f(t)|^p dt)^{\frac{p-1}{p}} = ||f||_p^{p-1}$$

Now, for any $t \in (0,1)$:

$$|f(t)+g(t)|^p = |f(t)+g(t)| \cdot |f(t)+g(t)|^{p-1} \leq |f(t)| \cdot |f(t)+g(t)|^{p-1} + |g(t)| \cdot |f(t)+g(t)|^{p-1},$$

which means we can integrate to obtain that:

$$\int_0^1 |f(t)+g(t)|^p dt \leq \int_0^1 |f(t)| \cdot |f(t)+g(t)|^{p-1} dt + \int_0^1 |g(t)| \cdot |f(t)+g(t)|^{p-1} dt$$

Now, define q such that $1/p + 1/q = 1$, and by Hölder's inequality and the observation above:

$$\int_0^1 |f(t)+g(t)|^p dt \leq ||f||_p \cdot ||(f+g)^{p-1}||_q + ||g||_p \cdot ||(f+g)^{p-1}||_q \leq ||f||_p \cdot ||f+g||_p^{p-1} + ||g||_p \cdot ||f+g||_p^{p-1}$$

$$\implies ||f+g||^p \leq ||f+g||_p^{p-1}(||f||_p + ||g||_p),$$

from which Minkowski's inequality follows directly. For the case $p = 1, q = \infty$, we have that for any $t \in (0,1)$:

$$|f(t)g(t)| = |f(t)| \cdot |g(t)| \leq |f(t)| \cdot \max_{0 \leq t \leq 1} |g(t)| = |f(t)| \cdot ||g||_\infty$$

$$\implies \int_0^1 |f(t)g(t)| dt \leq ||g||_\infty \int_0^1 |f(t)| dt = ||g||_\infty \cdot ||f||_1,$$

showing that Hölder's inequality does indeed hold again.

Exercise 26
Given $a, b > 0$, show that $\lim_{p \to \infty} (a^p + b^p)^{1/p} = \max\{a, b\}$. [Hint: If $a < b$ and $r = a/b$ show that $(1/p)\log(1 + r^p) \to 0$ as $p \to \infty$.] What happens as $p \to 0$? as $p \to -1$? as $p \to -\infty$?

Solution.
First, if $a = b$, the statement is obvious: the quantity inside the limit simplifies to $(2a^p)^{1/p} = 2^{1/p}a$, and as $p \to \infty, 1/p \to 0$ thus $2^{1/p}a \to 1a = a = \max\{a, b\}$. Therefore we can assume from now on that $a < b$, and, as indicated in the hint, set $r = a/b < 1$. Since $r < 1$, we have that $\lim_p \to \infty r^p \to 0$ (an exponential with a base less than 1). Therefore, by standard limit rules, $\lim_{p\to\infty} \log(1 + r^p) = \log(1) = 0$. Furthermore $\lim_{p\to\infty} \frac{1}{p} = 0$, which means $\lim_{p\to\infty}(1/p)\log(1 + r^p) = 0$. Now we apply the following "trick":

$$(a^p + b^p)^{1/p} = e^{\log(a^p+b^p)^{1/p}} = e^{\frac{\log(a^p+b^p)}{p}} = e^{\frac{\log((rb)^p+b^p)}{p}} = e^{\frac{\log(b^p)+\log(1+r^p)}{p}} = e^{\log(b)+\frac{\log(1+r^p)}{p}}$$

Clearly, what we showed based on the hint allows us to take limits on both sides and easily obtain that:

$$\lim_{p\to\infty} (a^p + b^p)^{1/p} = e^{\log(b)} = b = \max\{a, b\}$$

Now, for the case of $p \to 0$, if $a = b, \lim_{p\to 0}(a^p + b^p)^{1/p} = \lim_{p\to 0}(2a^p)^{1/p} = \lim_{p\to 0}(2^{1/p}a)$, and as $p \to 0+$ this quantity will tend to positive infinity, whereas if $p \to 0-$ it will tend to zero, thus the limit does not exist. If $a < b$, based on the above observation what interests us is $\lim_{p\to 0} \frac{\log(1+r^p)}{p}$. In this case, $r^p \to 1$ as $p \to 0$, and thus the numerator tends to $\log(2)$, thus the entire fraction tends to positive infinity as $p \to 0+$ and negative infinity as $p \to 0-$, which means the limit does not exist.
The expression inside the limit is continuous as a function of p at -1, since $a, b > 0$. Therefore $\lim_{p\to -1}(a^p + b^p)^{1/p} = (\frac{1}{a} + \frac{1}{b})^{-1} = \frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{ab}{a+b}$.
Lastly, for $p \to -\infty$, for $a = b$ we are interested in $\lim_{p\to -\infty}(2^{1/p}a)$, which is easily seen to equal a. For $a < b$, we are interested in $\lim_{p\to -\infty} \frac{\log(1+r^p)}{p}$. Because $0 < r < 1$, the quantity inside the logarithm will

tend to positive infinity, whereas the denominator tends to negative infinity. Applying L'Hospital's rule and calling this limit L we have that:

$$L = \lim_{p \to -\infty} \frac{\log(r) r^p}{1 + r^p} = \lim_{p \to -\infty} \frac{\log(r)}{1 + \frac{1}{r^p}}$$

Here the numerator is constant, whereas since $0 < r < 1, r^p \to \infty$, and thus the denominator will tend to 1. This means that $L = \log(r) = \log(a/b) = \log(a) - \log(b)$. Going back to our original limit we would have that $\lim_{p \to -\infty}(a^p + b^p)^{1/p} = e^{\log(b)+\log(a)-\log(b)} = a = \min\{a, b\}$, which we can see was also true for $a = b$.

Exercise - Unlisted; Arose from a discussion of exercise 26
a) Prove that if $1 \leq p \leq q \leq \infty$, then $l^p \subset l^q$.
b) If $x \in l_p \subset l_q$ for $1 \leq p \leq q \leq \infty$, then $||x||_p \geq ||x||_q$.
c) If $x \in l^{p_0}$ for some p_0, prove that $\lim_{p \to \infty} ||x||_p = ||x||_\infty$.

Solution.
a) Let x be a sequence in l^p. If $p = q$, then the statement is obvious, so we continue the proof assuming that $p \neq q$. In the case where $q = \infty$, we know that if, for some $1 \leq p < \infty$, $||x||_p$ exists, then the sequence of partial sums corresponding to $(|x_i|^p)$ must be bounded and non-decreasing. But then the same must hold true for the sequence $(|x_i|)$, which leads us to conclude that x is in fact bounded, and thus has a supremum. This means then that $x \in l_q$, since $q = \infty$ and the infinity norm is defined as the supremum of absolute values.
Now, in the remaining case we have that $1 \leq p < q$ and both are real numbers. Note that, by using e.g. the Archimidean property of $\mathbb{R}$, we can write $q = np + \epsilon$, where n is a positive integer and $\epsilon > 0$. Consider now examining the first m terms of x in the following way:

$$\sum_{i=1}^{m} |x_i|^q = \sum_{i=1}^{m} |x_i|^{np+\epsilon} = \sum_{i=1}^{m} |x_i|^{np} \cdot |x_i|^\epsilon$$

Recall that we already showed that if $x \in l_p$, then $|x_i|$ are bounded above, and thus we can say that for every $i, |x_i|^\epsilon \leq S^\epsilon, S = \sup_j |x_j|$. Furthermore, since n is a positive integer, we have that:

$$\sum_{i=1}^{m} |x_i|^{np} \leq (\sum_{i=1}^{m} |x_i|^p)^n,$$

since if one expands the power of the RHS, we get at least all terms of the LHS plus possibly more, all of which are non-negative. Combining these two facts, we have that:

$$\sum_{i=1}^{m} |x_i|^q \leq (\sum_{i=1}^{m} |x_i|^p)^n S^\epsilon \leq ||x||_p^{pn} S^\epsilon,$$

which we can safely conclude since the p-norm exists. But then the partial sums are bounded above for each m and are non-decrasing, meaning that the LHS converges as $m \to \infty$, and this equals precisely $||x||_q^q$. Thus x is indeed also in l^q.
b) Consider a sequence $x = (x_1, x_2, \dots) \in l_p \subset l_q$. The statement is obvious if $p = q$, and also if x is the zero sequence. Therefore from now on we assume $p < q, x \neq 0$, which means also $||x||_p > 0, ||x||_q > 0$. In the case where $q = \infty$, we have that $||x||_q = \sup_i |x_i|$. For any $m > 0$, we have that:

$$\sum_{i=1}^{m} |x_i|^p \geq \max_{1 \leq i \leq m} |x|^p$$

If we take limits on both sides, and by thinking about the ϵ-based definition of the supremum, we can see that this yields $||x||_p^p \geq ||x||_\infty^p \implies ||x||_p \geq ||x||_\infty$.
In the case where $p < q < \infty$, consider the following. Let y be the sequence formed by $y_i = \frac{x_i}{||x||_p}$. Notice first that $||y||_p = 1$. Notice also that for any $i, |y_i|^p \leq 1$, since $||y||_p = 1 \implies \sum_{i=1}^{\infty} |y_i|^p = 1^p$, so any

individual term of the series must be at most 1. Since $q > p \geq 1$, we have that it must be the case that $q = rp$ for some $r > 1$. Then, by using properties of powers, for any i:

$$|y_i|^p \leq 1 \implies |y_i|^{rp} \leq |y_i|^p \implies |y_i|^q \leq |y_i|^p$$

This in turn implies that for any $m > 0$:

$$\sum_{i=1}^{m} |y_i|^q \leq \sum_{i=1}^{m} |y_i|^p \leq ||y||_p^p = 1$$

Since this holds for any m, it also holds at infinity, meaning that $||y||_q^q \leq 1 = ||y||_p^q$, and thus by taking q-roots we obtain that $||y||_q \leq ||y||_p$. But then by using the definition of y:

$$||y||_q \leq ||y||_p \implies \left|\left|\frac{x}{||x||_p}\right|\right|_q \leq \left|\left|\frac{x}{||x||_p}\right|\right|_p \implies ||x||_q \leq ||x||_p,$$

where we used the “multiplication by scalar” property of norms. This completes the proof.
c) First of all, we observe from part (a) that since $||x||_{p_0}$ exists for some p_0, it will also be the case that $x \in l^p$ for all $p \geq p_0$, as well as that $x \in l^\infty$. We now have the following:

$$\lim_{p\to\infty} \frac{||x||_p}{||x||_\infty} = \lim_{p\to\infty} e^{\log\left(\frac{||x||_p}{||x||_\infty}\right)}$$

We focus on the exponent:

$$\lim_{p\to\infty} \log\left(\frac{||x||_p}{||x||_\infty}\right) = \lim_{p\to\infty} \log\left(\frac{(\sum_{i=1}^{\infty}|x_i|^p)^{1/p}}{(||x||_\infty^p)^{1/p}}\right) = \lim_{p\to\infty} \frac{1}{p} \cdot \log\left(\sum_{i=1}^{\infty} \frac{|x_i|^p}{||x||_\infty^p}\right)$$

Now we make the observation that since the series corresponding to $||x||_p^p$ converges, the individual terms tend to zero. This means that must exist at least one j such that $|x_j|$ equals the supremum of x. This is because the only other possibility would be for $|x_i|$ to tend to their supremum without achieving it, but then the series would not converge. Therefore, the argument of the logarithm is at least 1 (since $\frac{|x_j|^p}{||x||_\infty^p} = 1$). Additionally, the existence of $||x||_p$ implies that the argument does not go to infinity. Thus, the numerator of the last limit above tends to $L > 1$ and the denominator to positive infinity, which means that the limit tends to zero. But then:

$$\lim_{p\to\infty} \frac{||x||_p}{||x||_\infty} = e^0 = 1,$$

which concludes the proof since $||x||_\infty$ is constant with respect to the limit variable.

Exercise - Unlisted; Arose from a discussion of exercise 26
Recall that for a continuous function $f \in C([0,1])$ we define

$$||f||_p = (\int_0^1 |f(x)|^p dx)^{1/p}, ||f||_\infty = \max_{x\in[0,1]} |f(x)|$$

Show that:
a) If $f \in C[0,1]$, then for $1 \leq p \leq q \leq \infty$, $||f||_1 \leq ||f||_p \leq ||f||_q \leq ||f||_\infty$.
b) If $f \in C[0,1]$, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Solution.
a) We begin by showing that for any real number $p \geq 1, ||f||_p \leq ||f||_\infty$. We do this as follows. First, we have that:

$$||f||_p^p = \int_0^1 |f(x)|^p dx$$

Since $p \geq 1$, the p-th power is a non-decreasing function, and we know that by definition, $|f^p(x)| \leq \max_{0 \leq x \leq 1} |f^p(x)|$ for any $x \in [0,1]$. By integrating both sides we have that:

$$\int_0^1 |f^p(x)| dx \leq \max_{0 \leq x \leq 1} |f^p(x)| \implies ||f||_p^p \leq ||f||_\infty^p$$

By taking p-roots, we have the desired inequality. Now consider any two real numbers $p, q \geq 1$ such that $p < q$. This means that there exists $r > 1$ such that $q = rp$. Let $g(x) = f^P(x)$ and set r' to be the number satisfying $1/r + 1/r' = 1$. We now apply Hölder's inequality for the functions g and $h(x) = 1$:

$$\int_0^1 |g(x) \cdot 1| dx \leq ||g||_r \cdot ||1||_{r'} \implies \int_0^1 |f^p(x)| dx \leq (\int_0^1 |f^{rp}(x)|)^{(1/r)} \implies (\int_0^1 |f^p(x)| dx)^r \leq \int_0^1 |f^{rp}(x)| dx,$$

where for the last step we used the fact that $r > 1$. Continuing:

$$(\int_0^1 |f^p(x)| dx)^r \leq \int_0^1 |f^{rp}(x)| dx \implies (\int_0^1 |f^p(x)| dx)^{q/p} \leq \int_0^1 |f^q(x)| dx$$

$$\implies (\int_0^1 |f^p(x)| dx)^{1/p} \leq (\int_0^1 |f^q(x)| dx)^{1/q} \implies ||f||_p \leq ||f||_q$$

This completes the proof of the remaining inequalities, thus establishing that for $1 \leq p \leq q \leq \infty, ||f||_1 \leq ||f||_p \leq ||f||_q \leq ||f||_\infty$.

b) From part (a) we already know that as $p \to \infty, ||f||_p$ always exists, and in fact is bounded above by $||f||_\infty$ and is non-decreasing (since for $p \leq q, ||f||_p \leq ||f||_q$). This means that $\lim_{p \to \infty} ||f||_p$ exists, and will equal the supremum of the set $S = \{||f||_p, p \geq 1\}$. We thus only need to show that this supremum is in fact $||f||_\infty$. Since we already know that this constitutes an upper bound for S, assume that the supremum of S is $M < ||f||_\infty$. Then, by the definition of the infinity norm, there exists $x_0 \in [0,1]$ such that $|f(x_0)| = ||f||_\infty > M$. For the sake of simplicity, we shall assume that $x_0 \in (0,1)$: as will become clear, in the "edge cases" the only thing that changes is that some quantities lack a factor of 2. Set now $\epsilon = |f(x_0)| - M > 0$. Since f is continuous, there must exist $\delta > 0$ such that:

$$|x - x_0| < \delta \implies |f(x_0)| - |f(x)| < \epsilon,$$

where the absolute value in the second inequality can be omitted since $|f(x_0)|$ is the maximum value of $|f|$. Now, this can be rewritten as $|f(x)| > |f(x_0)| - \epsilon = |f(x_0)| - |f(x_0)| + M = M$, which means that in the interval $(x_0 - \delta, x_0 + \delta), |f(x)| > M$. Therefore, for all x in this interval we have that $|f(x)| > M \implies |f^p(x)| > M^p$. By integrating:

$$\int_{x_0-\delta}^{x_0+\delta} |f^p(x)| dx > 2M^p\delta \implies \left(\int_{x_0-\delta}^{x_0+\delta} |f^p(x)| dx\right)^{1/p} > (2\delta)^{1/p} M$$

We now have that $||f||_p$ is greater than or equal to the LHS here, since $(x_0 - \delta, x_0 + \delta) \subset (0,1)$. Furthermore, as $p \to \infty$, the RHS tends to M, since δ is constant. By taking limits, we can thus obtain that:

$$\lim_{p \to \infty} ||f||_p \geq M$$

Now we observe that we can repeat the entirety of the argument above for some N with $M < N < ||f||_\infty$. But then this means also that $\lim_{p \to \infty} ||f||_p \geq N$, and then the ϵ-based limit definition would allow us to find p such that $||f||_p > M$, which contradicts the defining property of M as the supremum of all $||f||_p$. Therefore, we have arrived at a contradiction, and thus it must be the case that $\lim_{p \to \infty} ||f||_p = ||f||_\infty$.

3.4 Limits in Metric Spaces

Exercise 30
If $A \subset B$, show that $\text{diam}(A) \leq \text{diam}(B)$.

Solution.
Consider the definition of $\text{diam}(A)$: $\text{diam}(A) = \{\sup\{d(a, b) : a, b \in A\}$. Because $A \subset B$, we have that any two $a, b \in A$ also belong in B. Therefore, the set S_1 over which the diameter is computed for B is a superset of the set S_2 over which the diameter is computed for A. But then exercise 2 of Chapter 1 guarantees that $\sup S_1 \leq \sup S_2$, which means precisely that $\text{diam}(A) \leq \text{diam}(B)$.

Exercise 32
In a normed vector space $(V, ||\cdot||)$ show that $B_r(x) = x + B_r(0) = \{x + y : ||y|| < r\}$ and that $B_r(0) = rB_1(0) = \{rx : ||x|| < 1\}$.

Solution.
Let us call $S = \{x + y : ||y|| < r\}$, in which case we asked to show that $B_r(x) = S$. First, let $z \in B_r(x)$. By definition, this means that $d(x, z) < r$. Recall that in a normed vector space we have that $||z - x|| = d(x, y)$ (unless a different metric is explicitly specified). Observe then that we can write $z = x + (z - x)$, where $||z - x|| = d(z, x) = d(x, z) < r$. By setting $y = z - x$, we obtain that $z \in S$. Therefore, $B_r(x) \subset S$.
In the other direction, suppose $z \in S$, which means that there exists $y, ||y|| < r$ such that $z = x + y$. Then we observe that $d(x, z) = ||z - x|| = x + y - x|| = ||y|| < r$. By definition, this means that $z \in B_r(x)$, and thus that $S \subset B_r(x)$, meaning that in fact $S = B_r(x)$.
For the second part of the exercise, set $S = \{rx : ||x|| < 1\}$. First, suppose $x \in B_r(0)$, which means $||x|| < r$. By using the scalar multiplication properties of vector spaces, we can then write $x = r \cdot \frac{x}{r}$. Set $y = \frac{x}{r}$, in which case $||y|| = ||\frac{x}{r}|| = \frac{1}{|r|}||x|| < 1$. This means that x can be written in the form $ry, ||y|| < 1$, thus that $x \in S$, thus that $B_r(0) \subset S$. Conversely, assume $x \in S$, which means $x = ry, ||y|| < 1$. But then $||x|| = ||ry|| = |r| \cdot ||y|| < r$, i.e. that $x \in B_r(0)$, and thus that $S \subset B_r(0)$, completing the proof that $B_r(0) = S$.

Exercise 33
Limits are unique. [Hint: $d(x, y) \leq d(x, x_n) + d(x_n, y)$.]

Solution.
Suppose that a sequence (x_n) in a metric space M converges to two $a, b \in M$ such that $a \neq b$. By the definition of metrics, we know then that it must be the case that $d(a, b) > 0$. Set $\epsilon = d(a, b) > 0$, in which case by the definition of the limit in a metric space there must exist N_1, N_2 such that $d(x_n, a) < \epsilon/4, d(x_n, b) < \epsilon/4$ for $n \geq N_1, n \geq N_2$ respectively. If we then pick $n > \max\{N_1, N_2\}$ we have that both of these inequalities hold for x_n. Recall the triangle inequality for metrics:

$$d(a, b) \leq d(a, x_n) + d(x_n, b) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon = d(a, b)$$

This is a clear contradiction, which means that if a sequence in a metric space has a limit, the limit has to be unique.

Exercise 34
If $x_n \to x$ in (M, d), show that $d(x_n, y) \to d(x, y)$ for any $y \in M$. More generally, if $x_n \to x, y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$.

Solution.
Pick $\epsilon > 0$. Because $x_n r \to x$, we know that there exists $N > 0$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. By the triangle inequality, for $n \geq N$ we have that:

$$d(x_n, y) \leq d(x, y) + d(x_n, x) < \epsilon + d(x, y) \implies d(x_n, y) - d(x, y) < \epsilon$$

By exercise 2 of Chapter 3, we also have that:

$$|d(x_n, x) - d(y, x)| \leq d(x_n, y) \implies -d(x_n, y) \leq d(x_n, x) - d(x, y) < \epsilon - d(x, y)$$

$$\implies d(x_n, y) - d(x, y) > -\epsilon$$

Combining these two inequalities we obtain that $|d(x_n, y) - d(x, y)| < \epsilon$ for all $n > N$, which is precisely the definition of $d(x_n, y) \to d(x, y)$.

For the second, more general statement, pick again $\epsilon > 0$. Then there exist N_1, N_2 such that $d(x_n, x) < \frac{\epsilon}{2}, d(y_n, y) < \frac{\epsilon}{2}$ whenever $n > N_1, n > N_2$ respectively. Set then $N = \max\{N_1, N_2\}$. By the triangle inequality for $n > N$ we have that:

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$\implies d(x_n, y_n) - d(x, y) < \epsilon$$

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + d(x_n, y_n)$$
$$\implies d(x, y) - d(x_n, y_n) < \epsilon \implies -\epsilon < d(x_n, y_n) - d(x, y)$$

Putting together these two inequalities results in $|d(x_n, y_n) - d(x, y)| < \epsilon$ for $n > N$, which is precisely the definition of $d(x_n, y_n) \to d(x, y)$.

Exercise 35

If $x_n \to x$, then $x_{n_k} \to x$ for any subsequence (x_{n_k}) of (x_n).

Solution.

Since $x_n \to x$, for any given $\epsilon > 0$ we can always find $N > 0$ such that for all $n \geq N, d(x_n, x) < \epsilon$. Because (x_{n_k}) contains infinite terms of (x_n) selected in an order which maintains the order of indices, it must be the case that for $n_k \geq N$, $d(x_{n_k}, x) < \epsilon$. We conclude that $x_{n_k} \to x$.

Exercise 36

A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set $\{x_n : n \geq 1\}$ is bounded).

Solution.

We begin by showing that a convergent sequence is Cauchy. Pick any $\epsilon > 0$. Because the sequence converges, say to x, there exists $N > 0$ such that whenever $n \geq N$ it is the case that $d(x_n, x) < \epsilon/2$. For any two such $n_1, n_2 \geq N$ we then have:

$$d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x) + d(x, x_{n_2}) < \epsilon,$$

which is precisely the definition of x being Cauchy.

Now assume (x_n) is Cauchy. Pick e.g. $\epsilon = 1$, and find $N > 0$ such that for $n_1, n_2 \geq N$ it holds that $d(x_{n_1}, x_{n_2}) < \epsilon$. Consider the set $\{d(x_n, x_1), 1 \leq n \leq N\}$. This has a finite number of non-negative elements, and as such a well-defined non-negative maximum M. For any $n > N$ we then have that:

$$d(x_n, x_1) \leq d(x_n, x_N) + d(x_N, x_1) < M + \epsilon,$$

and therefore all elements of the sequence satisfy $d(x_n, x_1) < M + \epsilon$, i.e. they are contained in the open ball $B_{M+\epsilon}(x_1)$, which is to say that the sequence is bounded.

Exercise 37

A Cauchy sequence with a convergent subsequence converges.

Solution.

Let (x_n) be a Cauchy sequence with a convergent subsequence $(x_{n_k}) \to x$. Pick any $\epsilon > 0$. Because $(x_{n_k}) \to x$, there exists $N_1 > 0$ such that for $n_k \geq N_1$ it holds that $d(x_{n_k}, x) < \epsilon/2$. Furthermore, because (x_n) is Cauchy, there exists $N_2 > 0$ such that for $i, j \geq N_2$ it holds that $d(x_i, x_j) < \epsilon/2$. Set $N = \max\{N_1, N_2\}$ and pick any $n > N$. Then, by the triangle inequality:

$$d(x_n, x) \leq d(x_n, x_{N_1}) + d(x_{N_1}, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

But then this means that (x_n) converges to x, which completes the proof.

Exercise 39
If every subsequence of (x_n) has a *further* subsequence that converges to x, then (x_n) converges to x.

Solution.
By way of contradiction, assume (x_n) does not converge to x. Then there exists $\epsilon > 0$ such that for all $N > 0$ there exists $n \geq N$ such that $d(x_n, x) \geq \epsilon$. Consider constructing the following subsequence of (x_n): for any $i > 0$, the i-th element of the subsequence equals the first x_n such that $d(x_n, x) \geq \epsilon, n \geq i$ and x_n has not been selected before. By the hypothesis above, this is always well-defined.
As a subsequence of (x_n), by the hypothesis of the exercise this subsequence will have a further subsequence converging to x. Notice, however, that by construction all elements of this "further subsequence" are such that $d(x_{n_{k_l}}, x) \geq \epsilon$, which contradicts convergence to x. Therefore, (x_n) itself must converge to x.

Exercise 41
Given $x, y \in l_2$, recall that $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$. Show that if $x^{(k)} \to x$ and $y^{(k)} \to y$ in l_2, then $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$.

Solution.
We begin first by verifying that if $a, b, c \in l_2$, then $\langle a + b, c\rangle = \langle a, b\rangle + \langle a, c\rangle$. We have that:

$$\langle a + b, c\rangle = \sum_{i=1}^{\infty}(a_i + b_i)c_i = \sum_{i=1}^{\infty}(a_i c_i + b_i c_i)$$

Notice that because a, b, c are all in l_2, we can safely split this series into a sum of the two series corresponding $\langle a, c\rangle, \langle b, c\rangle$. Now we move on to the main part of the proof by picking any $\epsilon > 0$. To begin with, for any $k, x^{(k)}, y^{(k)} \in l_2$ so any inner products involving them are well-defined:

$$|\langle x, y\rangle - \langle x^{(k)}, y^{(k)}\rangle| = |\langle x, y\rangle - \langle x, y^{(k)}\rangle + \langle x, y^{(k)}\rangle - \langle x^{(k)}, y^{(k)}\rangle| = |\langle x, y - y^{(k)}\rangle + \langle x - x^{(k)}, y^{(k)}\rangle|$$
$$\leq |\langle x, y - y^{(k)}\rangle| + |\langle x - x^{(k)}, y^{(k)}\rangle| \leq ||x||_2 \cdot ||y - y^{(k)}||_2 + ||x - x^{(k)}||_2 \cdot ||y^{(k)}||_2$$

Notice now that because $y^{(k)} \to y, x^{(k)} \to x$, we can, for this ϵ, find $K > 0$ such that for all $k \geq K, ||x - x^{(k)}||_2 < \epsilon, ||y - y^{(k)}||_2 < \epsilon$. By the extension of the triangle inequality, it will then also hold that $\big|||y^{(k)}||_2 - ||y||_2\big| < \epsilon$. Note that here we are using the association of the l_2 norm to the corresponding metric in l_2. Coupled with these observations, the above inequality implies:

$$|\langle x, y\rangle - \langle x^{(k)}, y^{(k)}\rangle| < ||x||_2 \cdot \epsilon + \epsilon \cdot (||y||_2 + \epsilon)$$

Now note that the RHS is a function of ϵ that tends to zero as ϵ tends to zero, which means (by the "elegance is not required" theorem from Hubbard & Hubbard) that indeed $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$.

Exercise 42
Two metrics d, ρ on a set M are said to be **equivalent** if they generate the same convergent sequences; that is, $d(x_n, x) \to 0$ iff $\rho(x_n, x) \to 0$. If d is any metric on M, show that the metrics ρ, σ, τ defined in Exercise 6 are all equivalent to d.

Solution.
We begin with $\rho(x, y) = \sqrt{d(x, y)}$. Suppose that some sequence x_n converges to x under d. Then, pick any $\epsilon > 0$ and find $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon^2$. We have then that $\rho(x_n, x) = \sqrt{d(x_n, x)} < \sqrt{\epsilon^2} = \epsilon$, which of course shows that $\rho(x_n, x) \to 0$, i.e. that x_n converges to x under ρ. Conversely, if $\rho(x_n, x) \to 0$ for some $(x_n), x$, then for any given $\epsilon > 0$, pick $N > 0$ such that for $n \geq N, \rho(x_n, x) < \sqrt{\epsilon} \implies \sqrt{d(x_n, x)} < \sqrt{\epsilon} \implies d(x_n, x) < \epsilon$, which means that $d(x_n, x) \to 0$.
Now, for $\sigma(x, y) = d(x, y)/(1 + d(x, y))$, suppose again $d(x_n, x) \to 0$ and pick first $\epsilon < 1$. Then there exists $N > 0$ such that for $n \geq N, d(x_n, x) < \epsilon/(1 - \epsilon) \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x) < \epsilon(1 + d(x_n, x)) \implies \sigma(x_n, x) < \epsilon$. Notice that for $\epsilon \geq 1$, it suffices to pick, say, the $N > 0$ for which the above holds when e.g. $\epsilon' = 1/2$. This shows that $\sigma(x_n, x) \to 0$. Conversely, for $\sigma(x_n, x) \to 0$, for any

given $\epsilon > 0$ there exists $N > 0$ such that for $n \geq N, \sigma(x_n, x) < \epsilon \implies d(x_n, x)/(1 + d(x_n, x)) < \epsilon \implies d(x_n, x) - \epsilon d(x_n, x) < \epsilon \implies d(x_n, x)(1-\epsilon) < \epsilon$. Notice that for $\epsilon < 1$, we obtain $d(x_n, x) < \epsilon/(1-\epsilon)$, which means that $d(x_n, x)$ is bound by a function of $\epsilon > 0$ which tends to zero as $\epsilon \to 0$. It furthermore holds that this function bounds $d(x_n, x)$ when $\epsilon < 1$. Also, for $\epsilon \geq 1$, it holds trivially that $d(x_n, x)/(1+d(x_n, x)) < \epsilon$. By defining $u(\epsilon) = \epsilon/(1 - \epsilon), \epsilon < 1$ and $u(\epsilon) = 1$ for $\epsilon \geq 1$, we have a function that fulfills the conditions of the "elegance is not required" theorem for limits, thus showing that $d(x_n, x) \to 0$.
Lastly, for $\tau(x, y) = \min\{d(x, y), 1\}$, if $d(x_n, x) \to 0$, then for $\epsilon > 1$ it trivially holds that $\tau(x_n, x) < \epsilon$ for $n \geq 1$, while for $\epsilon \leq 1$ we have that for sufficiently large $N > 0, d(x_n, x) < \epsilon$ and thus $\tau(x_n, x) = d(x_n, x) < \epsilon$, thus showing that $\tau(x_n, x) \to 0$. Conversely, if $\tau(x_n, x) \to 0$ then for any given $\epsilon > 0$, there exists $N > 0$ such that for $n \geq N, \tau(x_n, x) < \epsilon \implies \min\{d(x_n, x), 1\} < \epsilon$, thus for $\epsilon < 1$ it has to be that $d(x_n, x) < \epsilon$, which is enough to show that $d(x_n, x) \to 0$ (since for larger ϵ, N can trivially be found by using $\epsilon' < 1$).