Lab Assignments Computational Finance

Lecturer: Dr. Sumit Sourabh
Office: C3.146b
Address: Section Computational Science, Science Park 403, 1098 XH, Amsterdam
Email: s.sourabh@uva.nl

Teaching assistants: Dr. Sumit Sourabh
Office: C3.146b
Address: Section Computational Science, Science Park 403, 1098 XH, Amsterdam
Email: s.sourabh@uva.nl

Submission guidelines
These assignments can be done in groups of two students. Reports with a *clear description of the assignment, the methods, the results and discussion* should be submitted before the deadlines. You are free to choose the programming language/environment in which you would like to write your computer programs. If you have questions about the assignments do not hesitate to contact the teaching assistant or the lecturer.

Grading scheme

- Each of the three assignments carries equal weight of 20% and the exam is worth 40%;
- The score of the exam should be 5 points (on the scale of 1 to 10) and higher for passing the course;
- The fourth assignment is a bonus assignment and has to be submitted before the exam. With the bonus assignment the final grade can be increased by at most 1 point (on the scale of 1 to 10), but only if this assignment has been graded sufficient ($\geq 50\%$).

Assignment 1	Assignment 2	Assignment 3	Exam	Assignment 4 (Bonus)
20%	20%	20%	40%	10%

Assignment 4: The COS Method

Part I

Approximating the Density

For the COS method the characteristic function is needed.

A. Derive the characteristic function $\phi(u)$ of the standard normal distribution $N(0,1)$ by solving:

$$\phi(u) = \int_{\mathrm{R}} e^{iux} f(x) dx, \tag{1}$$

where $f(x)$ is the probability density function of $N(0,1)$ and $i := \sqrt{-1}$ is defined as the imaginary unit.

hint: first derive the expression on both sides w.r.t. u and use the identity $e^{i\theta} = \cos\theta + i\sin\theta$!

A density function $f(x)$ can be approximated by its Fourier-Cosine expansion:

$$f(x) \approx \frac{1}{2}F_0 + \sum_{n=1}^{N} F_n \cos\left(n\pi\frac{x-a}{b-a}\right) \tag{2}$$

where the coefficients F_n are defined as:

$$F_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(n\pi\frac{x-a}{b-a}\right) dx. \tag{3}$$

N.B. looks like characteristic function (Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$)

B. Use the expression of the characteristic function to get an approximation of the Fourier coefficients without an integral.

hint: approximate the characteristic function by the finite integral over a and b and take the real part.

C. Write a program that can approximate the standard normal density ($N(0,1)$) over the interval $[-5,5]$. Use $N = 2^k$ (where $k = 1, 2, \ldots, 6$) Fourier coefficients. Show your results in plots.

D. Show and explain why the convergence of the error is said to be exponential.

Part II
Calculating an option price

An asset price S_t (e.g. Stock or FX), can be modeled using Geometric Brownian motion:

$$dSt = rS_t dt + \sigma S_t dW_t^Q \tag{4}$$

where W_t^Q a Wiener process, r the interest rate and σ the volatility. For a European option with strike K, apply the following transformation:

$$x = \log(\frac{S_0}{K}) \text{ and } y = \log(\frac{S_t}{K}). \tag{5}$$

Then the value $V(x,t)$ of a European option must satisfy:

$$V(x,t) = e^{-rt} \int_{\mathbb{R}} g(y) f(y|x) dy, \tag{6}$$

where $g(y)$ is the payoff function and $f(y|x)$ the conditional density function.

A. Show that we can approximate $V(x,t)$ by a finite sum of Fourier cosine coefficients F_n and G_n of the conditional density function and payoff function respectively as:

$$e^{-rt}\frac{b-a}{2}\sum_{k=0}^{N} F_k(x) G_k$$

B. Assume the Fourier coefficients for European call option to be given by :

$$G_k = \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right) dy = \frac{2}{b-a}K(\chi_k(0,b) - \psi_k(0,b)),$$

where $\chi_k(a,b)$ and $\psi_k(a,b)$ are given as:

$$\begin{aligned}\chi_k(c,d) :=& \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - \cos\left(k\pi\frac{c-a}{b-a}\right)e^c\right.\\ &\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right],\\ \psi_k(c,d) :=& \begin{cases}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi} & \text{if } k \neq 0;\\ d-c & \text{if } k = 0.\end{cases}\end{aligned}$$

Furthermore, the Characteristic function of $y-x$ equals :

$$\phi_{\text{GBM}}(u) = e^{iu(r-\frac{1}{2}\sigma^2)t-(\frac{1}{2}\sigma^2 tu^2)};$$

(N.B. the term $\frac{S_0}{K}$ should be in $F_k(x)$.) Now price a one year ($T=1$) call option with parameters from part IIB. Use up to $N=64$ Fourier cosine coefficients and define a and b as:

$$\begin{aligned}a &= \log\frac{S_0}{K} + rT - 12\sqrt{\sigma^2 T}\\ b &= \log\frac{S_0}{K} + rT + 12\sqrt{\sigma^2 T};\end{aligned}$$

Compare your results with the earlier derived exact Black Scholes pricing formula and comment on the error and speed of convergence (by increasing the Fourier cosine coefficients to $96, 128, 160, 192$