2017 / 9 / 11 AA #4

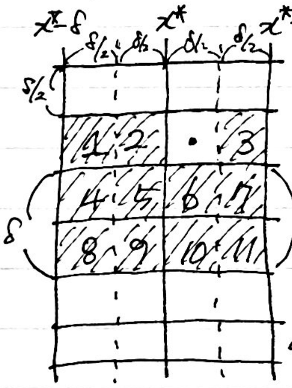
Lemma 1) If there exists $p_i \in L_x$ and $p_j \in R_x$ such that $d(p_i, p_j) < \delta$ then, both $x_i, x_j \in (x^* - \delta, x^* + \delta)$

pf) We know that $x_i \leq x^* \leq x_j$. Moreover, we have that

$$\left.\begin{array}{l} x^* - x_i \leq x_j - x_i \leq d(p_i, p_j) < \delta \Rightarrow x^* - \delta < x_i \\ \qquad\qquad \text{and trivially} \Rightarrow x_i \leq x^* \\ x_j - x^* \leq x_j - x_i \leq d(p_i, p_j) < \delta \Rightarrow x_j < x^* + \delta \\ \qquad\qquad \text{and trivially} \Rightarrow x^* \leq x_j \end{array}\right\} \Rightarrow x_i, x_j \in (x^* - \delta, x^* + \delta)$$

Lemma 2) If $p, q \in S_y$ satisfies $d(p, q) < \delta$, then, p and q are within 11 positions of each other.

pf) We partitioned the "band" into "boxes" which are the squares with horizontal and vertical sides of length $\delta/2$. One row of boxes are the four squares that have the same y-coords.



<Figure>

Each box can contain at most one point from S_y, because each box is completely contained in one of the two halves and any 2 points in the same box are at most $\delta/\sqrt{2}$ apart.

← diagonal length of each box.

Without loss of generality, we assume that p appears before q in S_y. Let B_p and B_q respectively be a box containing p and q.

Suppose towards a contradiction that p and q are at least 12 positions away in S_y. (Figure).

Then there are at least 2 rows between the row of B_p and the row of B_q. (Suppose not. To contain p, q and all the points in between in S_y. There at least 13 points and they must be contained in 12 or fewer boxes. From the pigeonhole principle, it is a contradiction.)

(Correctness of Alg 2)

Thm) The given algorithm finds a correct solution (a closest pair of points in P)

pf) We prove that the correctness of CPair() function. By induction on $|P_x|$, and it follows that the algorithm is correct.

It is clear that the function is correct if $|P_x| \leq 3$

Assume that $|P_x| \geq 4$. From the induction hypothesis, (p_{l_1}, p_{l_2}) (and (p_{r_1}, p_{r_2})) is a closest pair of points in L_x (R_x respectively).

Let (p_{o_1}, p_{o_2}) (: correct solution) be a closest pair of points in P_x.

Note that $\delta \geq d(p_{o_1}, p_{o_2})$ by definition.

Case 1) $\delta > d(p_{o_1}, p_{o_2})$

This implies that exactly one of the two points is in L_x and the other is in R_x.

Since O/W δ would have been greater than $d(p_{o_1}, p_{o_2})$

From Lemmas 1 and 2, p_{o_1} and p_{o_2} are in S_y, within 11 positions of each other.

The algorithm therefore finds (p_{m_1}, p_{m_2}) such that $d(p_{m_1}, p_{m_2}) = d(p_{o_1}, p_{o_2})$

and the best among the three is (p_{m_1}, p_{m_2})

Case 2) $\delta = d(p_{o_1}, p_{o_2})$

In this case, (p_{m_1}, p_{m_2}) may or may not be defined.

If they are defined, we know that (p_{m_1}, p_{m_2}) are chosen as two points in P_x such that $m_1 \neq m_2$.

Thus, the best among the 3 answers is a closest pair of points (which are δ apart) □.