Problem 1. $(S, \mathcal{I})$, $c : S \to \mathbb{Q}$. $r(\cdot)$, C : maximum weight of base.

$$\left\{ \begin{array}{ll} \sum_{e \in S} c(e) x_e = C & \\ \sum_{e \in U} x_e \leq r(U) & \forall U \subsetneq S \\ \sum_{e \in S} x_e = r(S) & \\ x_e \geq 0 & \end{array} \right\} \overset{\text{let}}{=} P'$$

<Solution> Observe that the polyhedron $\{X \mid \sum_{e \in U} x_e \leq r(U), \forall U \subsetneq S, \sum_{e \in S} x_e = r(S), x_i \geq 0\} = P$

$\cdots$ $P = P_I$. // network graph.

Since $\{\sum_{e \in S} c(e) x_e = C\}$ defines a supporting hyperplane of P,

We have that the polyhedron defined, P' also is an integer polyhedron

since a face of a face is a face.

($\circledast$ since C exists, P' is nonempty).

Problem 2. $(G = (V_L \cup V_R, E))$, Prove Hall's Theorem.

Given G, G has a matching of size $|V_L|$ if and only if for every $S \subseteq V_L$,

We have $|N(S)| \geq |S|$, where $N(S) = \{b \in V_R \mid \exists a \in S \text{ with } (a,b) \in E\}$

<Proof> ($\Rightarrow$) Trivial.

모든 edge의 적어도 한 end를 포함

($\Leftarrow$) Let $U := V_L \setminus C$ for an arbitrary vertex cover C $\Rightarrow$ $N(U) \subseteq C$

$\circledast$ Suppose not. $\Rightarrow$ $\exists (x,y) \in E$ s.t. $x \in U$ and $y \notin C$. therefore $x \notin C$.

$\Rightarrow$ (x,y) is not covered by C $\cdots$ Contradicts our choice of C.

- We thus have

$$\begin{aligned} |C| &= |C \cap V_L| + |C \cap V_R| \\ &= |V_L| - |V_L - C| + |C \cap V_R| \\ &= |V_L| - |U| + |C \cap V_R| \\ &\geq |V_L| - |U| + |N(U)| \quad \text{since } (N(U)) \subseteq (C \cap V_R) \\ &\geq |V_L| - |U| + |U| \quad \text{since we supposed } |N(U)| \geq |U| \\ &= |V_L| \end{aligned}$$

- Maximum Cardinality of Matching in $G := |M^*| = |C^*|$ by Kőnig's Theorem

$\Rightarrow |M^*| \geq |V_L|$ $\cdots$ Therefore G has a matching of size $|V_L|$

Problem 3. (b) $S := V_L$, $\mathcal{I} := \{U \subseteq V_L \mid \exists$ matching where every vertex in U is matched$\}$

Prove that $(S, \mathcal{I})$ is a matroid.

proof). • If $I \in \mathcal{I}$, then for any $J \subseteq I$, the matching that certificates I also satisfies that $J \in \mathcal{I}$. ... ① satisfied

• Suppose that $I, J \in \mathcal{I}$ and $|I| < |J|$

Let $M_I, M_J \in E$ be the matching in which every vertex in I (and J) is matched. (respectively)

(WLOG) assume that $|M_I| = |I|$ and $|M_J| = |J|$

• Now Consider a directed bipartite graph on $(V_L \cup V_R)$,

Where we have an arc $\overrightarrow{\langle x, y \rangle}$ iff $(x, y) \in M_J$

and $\overrightarrow{\langle y, x \rangle}$ iff $(x, y) \in M_I$

$\Rightarrow$ Since every vertex in this graph $\rightarrow$ (indeg ≤ 1 , outdeg ≤ 1)

The graph is a union of disjoint directed paths and directed cycles.

• Observe that the number of vertices in V_L whose indg is 0. is $|J \setminus I|$ and the number of vertices in V_L whose outdg is 0 is $|I \setminus J|$

• Since $|J \setminus I| > |I \setminus J|$, among the $|J \setminus I|$ directed paths that starts from a vertex in $|J \setminus I|$, at least one of them ends at a vertex in V_R

• Let P this path, and $M_I \triangle P$ shows that $\exists x \in J \setminus I$, $I + x \in \mathcal{I}$.

Problem 5 (a). given $(S, \mathcal{I})$, The intersection of two flats is a flat.

proof). Let $F_1 = \text{span}(F_1)$, $F_2 = \text{span}(F_2)$ and $x \in \text{span}(F_1 \cap F_2)$

We then have $r((F_1 \cap F_2) + x) = r(F_1 \cap F_2)$

• and also. $r(F_1 + x) = r(F_1)$

$r(F_2 + x) = r(F_2)$ ($\because$ submodularity)

• This shows that $x \in \text{span}(F_1)$ and $x \in \text{span}(F_2)$ $\therefore$ $\underline{\underline{x \in F_1 \cap F_2}}$

(b) Let $F = \text{span}(F)$, $t \in S \setminus F$ and F' be a flat with the minimum cardinality that contains $(F + t)$ $\Rightarrow$ then $\nexists$ a flat F'' s.t $F \subsetneq F'' \subsetneq F'$

proof) First, we claim that $F' = \text{span}(F + t)$

$\rightarrow$ proof) $(F + t) \subseteq F'$ and suppose that $y \in \text{span}(F + t)$

$\Rightarrow$ Then $r(F + t + y) = r(F + t)$ ... implying $r(F + y) = r(F)$

This shows that $\text{span}(F + t) \subseteq F$ and hence $F' = \text{span}(F + t)$

- Note that span$(F+t)$ is a flat and the unique set that contains span$(F+t)$ has cardinality $\leq |\text{span}(F+t)|$.
- We have $r(F') = r(\text{span}(F+t))$
$$= r(F) + 1 \quad \text{since } t \notin F = \text{span}(F).$$

This implies that $r(F'') \in \{r(F), r(F)+1\}$

Case 1) $r(F'') = r(F)$

Suppose toward contradiction that $\exists\ x \in F'' \setminus F$

and then we have $r(F'') \geq r(F+x) > r(F)$. ※

Case 2) $r(F'') = r(F')$

Suppose toward contradiction that $\exists$ some $x \in F' \setminus F''$

and then we have $r(F') \geq r(F''+x) > r(F'')$ ※ ☒