[Dual Matroid] Co-graphic Matroid $\Rightarrow$ $I$ ... acyclic.

$\Rightarrow$ Base of M (maximal) : spanning tree

Def) Given a matroid, $M = (S, \mathcal{I})$ its dual is defined as $M^* = (S, \mathcal{I}^*)$

Where $\mathcal{I}^* := \{ I \subseteq S \mid S \setminus I \text{ contains a base of } M \}$

$S/I \Rightarrow$ Contains spanning tree

$\Rightarrow S/I$ — $\boxed{\text{Connected}}$ ☆

Thm) $M^*$ is a matroid.

proof) It is trivial to see that

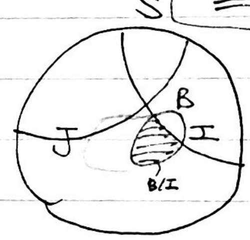
① If $I \in \mathcal{I}^*$ and $J \subseteq I$ then $J \in \mathcal{I}^*$ $((S \setminus I) \subseteq (S \setminus J)$ ... $(S \setminus J)$ always contains B)

② Suppose that $I, J \in \mathcal{I}^*$ and $|I| < |J|$ // 증명해야할것 : $\exists z \in J \setminus I$ s.t. $I + z \in \mathcal{I}^*$

$\Rightarrow$ $S \setminus (I+z)$ contains a base of M

- by definition, $\exists$ some base $B \subseteq S \setminus J$
- Since $S \setminus I$ also contains some base,

  $\exists$ some base $B'$ s.t. $B \setminus I \subseteq B' \subseteq S \setminus I$




- We have $J \setminus I \not\subseteq B'$

Since O/W $|B| = |B \cap I| + |B \setminus I|$

$\leq |I \setminus J| + |B \setminus I|$ (since $(B \cap I) \subseteq (I \setminus J)$)

$< |J \setminus I| + |B \setminus I|$ (since $|J| > |I|$)

$??\leq |B'|$

- Then $\exists z \in J \setminus I$, $z \notin B'$ therefore $(S \setminus (I+z))$ contains a base

$\Rightarrow I + z \in \mathcal{I}^*$

Thm) For $\forall U \subseteq S$, we have $r_{M^*}(U) = |U| - r_M(S) + r_M(S \setminus U)$

proof) $r_{M^*}(U) = \max \{ |U \setminus B| \mid B \text{ is a base of } M \}$

$= |U| - \min \{ |B \cap U| \mid B \text{ is a base of } M \}$

$= |U| - \min \{ |B| - |B \setminus U| \mid B \text{ is a base of } M \}$

$= |U| - r_M(S)^{\gamma |B|} + \max \{ |B \setminus U| \mid B \text{ is a base of } M \}$

$= |U| - r_M(S) + r_M(S \setminus U)$

⊛ Matroid Union.

Let $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ be matroids. (No restriction on $S_1$ and $S_2$)

The union of $M_1$ and $M_2$, denoted by $M_1 \vee M_2$, is $(S_1 \cup S_2, \mathcal{I})$

where $\mathcal{I} := \{ I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1 \text{ and } I_2 \in \mathcal{I}_2 \}$

Thm) • $M_1 \vee M_2$ is a matroid.

• whose rank function is given by $r_{M_1 \vee M_2}(U) = \min_{T \subseteq U} \{ |U \setminus T| + r_{M_1}(T) + r_{M_2}(T) \}$

$$= \min_{T \subseteq U} \{ |U \setminus T| + r_{M_1}(T \cap S_1) + r_{M_2}(T \cap S_2) \}$$

part 1 proof) We first show that $M_1 \vee M_2$ is a matroid.

① Suppose we have $I \in \mathcal{I}$ and $J \subseteq I$. By definition, $I = I_1 \cup I_2$ for some $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. And we have $J = J_1 \cup J_2$ where $J_1 \in \mathcal{I}_1$ and $J_2 \in \mathcal{I}_2$, implying $J \in \mathcal{I}$.

② Suppose that we have $I, J \in \mathcal{I}$ and $|I| < |J|$.

By definition, $\exists\, I_1, I_2, J_1, J_2$ such that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ and $I_1, J_1 \in \mathcal{I}_1$, $I_2, J_2 \in \mathcal{I}_2$. (WLOG) We can assume that $I_1 \cap I_2 = \emptyset$ and $J_1 \cap J_2 = \emptyset$.

• Among all such choice of $I_1, I_2, J_1$ and $J_2$, choose the one that

– Maximizes $|I_1 \cap J_1| + |I_2 \cap J_2|$

• Since $|J| > |I|$, for some $i \in \{1, 2\}$ we have $|J_i| > |I_i|$. thus there exists $t \in J_i \setminus I_i$ such that $I_i + t \in \mathcal{I}_i$, we have $t \in I_{3-i}$

• Since otherwise we could have chosen $I_i' := I_i + t$, $J_i' := J_i$

$I_{3-i}' := I_{3-i} - t$, $J_i' := J_i$

which yields $|I_1' \cap J_1'| + |I_2' \cap J_2'| > |I_1 \cap J_1| + |I_2 \cap J_2|$

• This shows that $t \notin I$ and $I + t \in \mathcal{I}$.

$\therefore$ $M_1 \vee M_2$ is a matroid.

part 2 proof) It is easy to see that $r_{M_1 \vee M_2}(U) \leq \min_{T \subseteq U} \{ |U \setminus T| + r_{M_1}(T \cap S_1) + r_{M_2}(T \cap S_2) \}$

• Suppose that $I \subseteq U$ be a set satisfying $|I| = r_{M_1 \vee M_2}(U)$ and $I = I_1 \cup I_2$

where $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$

For any $T \subseteq U$, we have $|I| = |I \setminus T| + |I \cap T| \leq |U \setminus T| + |I_1 \cap T| + |I_2 \cap T|$

$\leq |U \setminus T| + r_{M_1}(T \cap S_1) + r_{M_2}(T \cap S_2)$

- (WLOG), we can assume that $S_1 = S_2$.

(Since O/W we can consider $M_1' = (S', \mathcal{I}_1)$ and $M_2' = (S', \mathcal{I}_2)$ for $S' := S_1 \cup S_2$ and we have $M_1' \vee M_2' = M_1 \vee M_2$.)

Let $S := S_1 = S_2$.

Moreover, it suffices to prove $r_{M_1 \vee M_2}(U) \geq \min_{T \subseteq U} \{ |U \setminus T| + r_{M_1}(T \cap S_1) + r_{M_2}(T \cap S_2) \}$ only for $U = S$.

$$\Rightarrow r_{M_1 \vee M_2}(S) \geq \min_{T \subseteq S} \{ |S \setminus T| + r_{M_1}(T) + r_{M_2}(T) \}$$

(Since O/W we can consider the restriction of $M_1$ and $M_2$ to $U$.

$\Rightarrow$ Proof)

Consider the dual $M_2^* = (S, \mathcal{I}_2^*)$ of $M_2$.

$\forall$ $X \in \mathcal{I}_1 \cap \mathcal{I}_2^*$, we have a base $B_2$ of $M_2$ contained in $S \setminus X$, and thus, for $I := X \cup B_2$, we have $r_{M_1 \vee M_2}(S) \geq |I| = |X| + |B_2|$

$$\Rightarrow \text{yielding } r_{M_1 \vee M_2}(S) \geq \max_{X \in \mathcal{I}_1 \cap \mathcal{I}_2^*} |X| + r_{M_2}(X) ?$$

$$= \min_{T \subseteq S} (r_{M_1}(T) + r_{M_2^*}(S \setminus T) + \underbrace{r_{M_2}(X)}_{\text{constant}})$$

(⊛ Matroid Intersection theorem
$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U))$)

$$= \min_{T \subseteq S} (r_{M_1}(T) + |S \setminus T| - \cancel{r_{M_2}(S)} + r_{M_2}(T) + \cancel{r_{M_2}(S)})$$

(⊛ $r_{M^*}(U) = |U| - r_M(S) + r_M(S \setminus U)$)
Thm Dual Matroid.

$$= \min_{T \subseteq S} ( |S \setminus T| + r_{M_1}(T) + r_{M_2}(T) )$$ ◻

$$\Rightarrow \therefore r_{M_1 \vee M_2}(U) = \min_{T \subseteq U} ( |U \setminus T| + r_{M_1}(T \cap S_1) + r_{M_2}(T \cap S_2))$$

$O(|E|)$? polytime?

- Given a matroid $(S, \mathcal{I})$, $X \in \mathcal{I}$ and $y \in S \setminus X$ such that $X + y \notin \mathcal{I}$,
  Let $C(X, y)$ denote the unique circuit of $(X+y)$ → ⊗ 계속 쓰는 notation
  For $C : S \to \mathbb{R}$ and $X \subseteq S$, let $c(X) := \sum_{x \in X} c(x)$

**Thm)** Let $(S, \mathcal{I})$ be a matroid with weight function $C : S \to \mathbb{R}$ and $k \in \mathbb{N}$
for $X \in \mathcal{I}$ and $|X| = k$, we have $\underline{c(X) = \max\{c(Y) \mid Y \in \mathcal{I}, |Y| = k\}}$ "Maximum independent set with card k"

iff ⓘ for $\forall j \in S \setminus X$ with $X + j \notin \mathcal{I}$ (dependent),
we have $c(i) \geq c(j)$ for $\forall i \in \underline{C(X, j)}$
($\Rightarrow$ circuit 통해 $(+j - i)$ operation으로 $X'$를 만들면 weight $\Downarrow$)

ⓘⓘ for $\forall j \in S \setminus X$ with $X + j \in \mathcal{I}$ (independent with card $k+1$)
we have $c(i) \geq c(j)$ for $\forall i \in X$
($\Rightarrow$ 아무거나 빼면 card-$k$ … $(+j - i)$ operation으로 $X'$를 만들면 weight $\Downarrow$)

proof) ($\Rightarrow$) Trivially holds.
($\Leftarrow$) Let $e_1, e_2, \cdots, e_{|S|}$ denote the elements in $S$
(WLOG) we assume that
$$c(e_1) \geq c(e_2) \geq \cdots \geq c(e_{|S|})$$
and for $\forall i < j$ with $c(e_i) = c(e_j)$
$\Rightarrow$ ($j \in X$ implies $i \in X$)

Now, consider a new matroid, $\mathcal{I}_k := \{I \mid I \in \mathcal{I}, |I| \leq k\}$
Run the greedy algorithm to find a max weight base, considering $e_1 \cdots e_{|S|}$ in that order
We claim that the algorithm returns $X$.
In order to prove this claim, use induction on # of iterations
At the beginning of the $i$th iteration,
we have that the elements chosen so far is exactly $X_i = (e_1, \ldots, e_{i-1})$
Case ① If $e_i \in X$, the algorithm would choose $e_i$ as well.
Case ② Let $e_i \notin X$. Suppose to the contradiction that the algorithm
chooses to include $e_i$ in the solution.
This implies that $X \cap \{e_1, \cdots, e_{i-1}\} < k$

ⓐ Suppose $X + e_i \in \mathcal{I}$ Let $m := \max\{j \mid e_j \in X\}$
$\Rightarrow$ we have $c(e_i) > c(e_m)$ contradicting ⓘⓘ

ⓑ Suppose $X + e_i \notin \mathcal{I}$
$\Rightarrow$ we have that $C(X + e_i) \subseteq \{e_1, \cdots, e_i\}$, since OW algorithm
would not have chosen $e_i$. Let $e_m \in \{e_{i+1} \cdots e_{|S|}\}$ arbitrarily $\Rightarrow$ Cont ⓘⓘ