<Approximation Ratio>

Q1) Give a relation between the value val(x) and the cardinality of the sets $V^{1/2}$ and V^1.

Answer) $Val(x) = \sum_{v \in V} x_v = \sum_{v \in V^{1/2}} (\frac{1}{2}) + \sum_{v \in V^1} (1)$

$= \frac{1}{2} \times |V^{1/2}| + |V^1|$

$\Rightarrow Val(x) = \frac{1}{2}|V^{1/2}| + |V^1|$

Q2). Give a tight lower bound on the cardinality of $V_3^{1/2}$ based on the cardinality of $V^{1/2}$.

Answer) $|V^{1/2}| = |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| + |V_3^{1/2}|$ (since they are disjoint sets)

since $|V_3^{1/2}|$ is minimized when $|V_0^{1/2}| = |V_1^{1/2}| = |V_2^{1/2}| = |V_3^{1/2}| = \frac{1}{4}|V^{1/2}|$

$\Rightarrow |V_3^{1/2}| \geq \frac{1}{4}|V^{1/2}|$

Q3) Deduce from Q2 an upperbound on the cardinality of $|V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}|$ based on the cardinality of $V^{1/2}$.

Answer) $|V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| = |V^{\frac{1}{2}}| - |V_3^{\frac{1}{2}}| \leq |V^{1/2}| - \frac{1}{4}|V^{1/2}|$ (by Q2)

$= \frac{3}{4}|V^{1/2}|$

$\Rightarrow |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| \leqq \frac{3}{4}|V^{1/2}|$

Q4) Combine Q1 and 3 to give an upperbound on the number of vertices in S based on val(x).

Answer) $|S| = |V^1| + |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| \leqq |V^1| + \frac{3}{4}|V^{1/2}|$

$\leq \frac{1}{2}|V^1| + (|V^1| + \frac{3}{4}|V^{1/2}|)$

$= \frac{3}{2} Val(x)$

$\Rightarrow |S| \leqq \frac{3}{2} Val(x)$

Q5) Combine Q4 and the property of val(x) to conclude on the approximation ratio of the rounding procedure. Recall that val(x) is the value of the optimal fractional solution to the LP.

Answer) by Q4, $|S| \leqq \frac{3}{2} OPT$. (since $val(x) = OPT$)

<Correctness>

We now show that S is a correct vertex cover. Namely, we want to show that for each edge (u, v), u or v (or both of them) are in S. We will proceed by contradiction and assume that neither u or v are in S.

Q6) Suppose, in addition, that $u \in V^0$, to which set does v belong? Recall that X is the solution to the LP.

Answer) Since S is a correct vertex cover, and $u \notin S$ (since $u \notin V^1 \cup V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2}$)

Then there exists a constraint in the LP: $x_u + x_v \geq 1$ and $x_u = 0$

Therefore $x_v = 1$ and $v \in V^1$

Q7) Deduce from the previous question to which set of $V^1, V^0, V^{1/2}$ do u and v belong

Answer)
- If $u \in V^0$, then $v \in V^1$ (by Q6)
- If $u \in V_3^{1/2}$, then $v \in (V^1 \cup V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2})$
  If $u \in (V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2})$, then $v \in (V^1 \cup V^{1/2})$
  $\Rightarrow$ If $u \in V^{1/2}$ then $v \in (V^1 \cup V^{1/2})$
- If $u \in V^1$, then $v \in V_0 \cup V_1 \cup V_{1/2}$

Q8) If u and v belong to $V^{1/2}$, to which subset of V do they belong, if they donot belong to S.

Answer) Then $u \in V_3^{1/2}$ and $v \in V_3^{1/2}$ (Otherwise, $u \in S$ or $v \in S$)

Q9.) Recall that C is a 4-colouring. Explain the contradiction.

Answer) Since C is a 4-colouring of G, $C(u) \neq C(v)$ for $\forall e = (u, v) \in E$.

Therefore ($u \in V_3^{1/2}$ and $v \in V_3^{1/2}$) is a contradiction, because $C(u) = C(v) = 3$

Q10.) Give an example of a well-known class of graphs that is 4-colorable.

Answer) Bipartite Graphs are two-colourable, therefore 4-colorable.