2017/8/31 Algorithm Analysis (Week 1)

Example) 0, 100, -50, -60, 30, -10, 40, 5, 45, -2, 1

Problem) Given a sequence of integers $\{a_1, a_2, \cdots, a_n\}$,
find a contagious subsequence whose sum is the maximum.
(where sum values of empty subsequence is defined as 0)

Alg 1 (Exhaustive Method) ← very naive!

```
sol ← 0
for i ← 1, ···, n do
    for j ← i, ···, n do
        calculate sum(i~jth)
return sol;
```

$O(n^3)$ → Thm) Alg 1 runs in $O(n^3)$ time.

pf) # of basic operations is proportional to.

$$\left(\sum_{i=1}^{n}\sum_{j=i}^{n}(j-i+1)\right) = \sum_{i'=1}^{n}\sum_{j'=1}^{i'} j' = \frac{n(n+1)(n+2)}{6}$$

$\Rightarrow O(n^3)$

※ Proof of Algorithm
① running time analysis
② proof of correctness
partial proof 1 : sol is correct if alg terminates
partial proof 2 : alg always terminates.

Alg 2

```
sol ← 0
for i ← 1, ···, n do
    sum_{i,i-1} ← 0
    for j ← i, ··· n do
        sum_{i,j} ← sum_{i,j-1} + a_j
        if sum_{i,j} > sol then
            sol ← sum_{i,j}
return sol
```

$O(n^2)$ → Thm') Alg 2 runs in $O(n^2)$ time.

pf) # of basic operations is proportional to

$$\left(\sum_{i=1}^{n}\sum_{j=i}^{n} 1\right) = \sum_{i=1}^{n}(n-i+1) \Rightarrow O(n^2)$$

Thm²) Alg 2 is correct.

pf) The algorithm returns $\max\left[\{sum_{i,j} \mid 1 \le i \le j \le n\} \cup \{0\}\right]$

hence it suffices to prove that $sum_{i,j}$ is equal to $\left(\sum_{k=i}^{j} a_k\right)$: sum of subsequence ①

Induction

Basic Step) When $j-i=0$ then $sum_{i,j} = sum_{i,i} = \underbrace{sum_{i,i-1}}_{=0} + a_i = a_i$ therefore ① holds.

Induction Step) When $j-i = d_0+1$

then $sum_{i,j} = sum_{i,j-1} + a_j = \sum_{k=i}^{j-1} a_k + a_j$ (by induction hypothesis)

$= \sum_{k=i}^{j} a_k$. therefore ① holds. ∎

$\Rightarrow$ Since Alg 2 trivially terminates, Alg 2 is correct by thm² and runs in $O(n^2)$ by thm'

Alg 3 (Idea from merge sort)

function MaxSubseq(s, t) ⇒ return MaxSubseq(1, n)

if s = t then

returns max(as, 0)

O/W

$m \leftarrow \left[\frac{s+t}{2}\right]$

$sol_l \leftarrow$ MaxSubseq(s, m)

$sol_r \leftarrow$ MaxSubseq(m+1, t)

$subsol_l \leftarrow 0$, $subsol_r \leftarrow 0$

for $i \leftarrow m, m+1, \cdots, s$ do

if $\sum_{k=i}^{m} a_k > subsol_l$ then

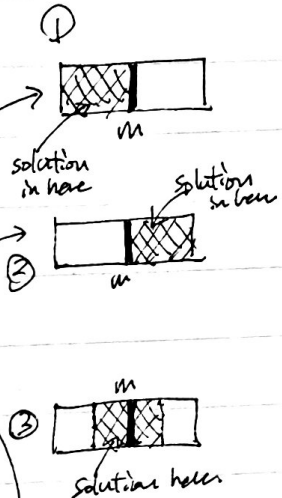
$subsol_l \leftarrow \sum_{k=i}^{m} a_k$

for $i \leftarrow m+1, \cdots, t$ do

if $\sum_{k=i}^{m+1} a_k > subsol_r$ then

$subsol_r \leftarrow \sum_{k=i}^{m+1} a_k$

return $\max\begin{cases} sol_l \\ sol_r \\ subsol_l + subsol_r \end{cases}$



Thm) Alg 3 runs in $O(n \log n)$ time. (Just like Mergesort)

pf) When $s = t$ Let $T(1) = 1$ for max(a, b) operation.

$T(n) = \underbrace{2T(n/2)}_{\text{Call two recurrences with half range.}} + n$ ← subsol operations.

Let $n = 2^k$ and $k = \log_2 n$.

$T(2^k) = 2T(2^{k-1}) + 2^k$

$= 2^k T(1) + k2^k$

$\Rightarrow T(n) = nT(1) + (\log_2 n)n$

$= n + n\log_2 n = O(n \log n)$ □

Alg 4 (linear time)

$subsol_0^{\ell} \leftarrow 0$, $subsol_0^{r} \leftarrow \max\left\{\sum_{i=0}^{n} a_i, 0\right\}$

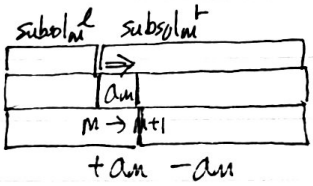
for $m \leftarrow 1, 2, \cdots, n-1$ do.

$subsol_m^{\ell} \leftarrow \max\left[\left\{\sum_{k=i}^{m} a_k \mid 1 \le i \le m\right\} \cup \{0\}\right]$

$subsol_m^{r} \leftarrow \max\left[\left\{\sum_{k=m+1}^{i} a_k \mid m+1 \le i \le n\right\} \cup \{0\}\right]$

$subsol_m^{\ell} \leftarrow \max\{subsol_{m-1}^{\ell} + a_m, 0\}$

$subsol_m^{r} \leftarrow \max\{subsol_{m-1}^{r} - a_m, 0\}$



"single run에 걸친 $subsol_m^{\ell}$, $subsol_m^{r}$을 연산."

$sol \leftarrow 0$

for $m \leftarrow 1, 2, 3 \cdots, n-1$ do.

if $(subsol_m^{\ell} + subsol_m^{r}) > sol$ then

$sol \leftarrow (subsol_m^{\ell} + subsol_m^{r})$

return sol.

(*) Obs) for $m > 1$, we have

$subsol_m^{\ell} = \max\{subsol_{m-1}^{\ell} + a_m, 0\}$

$subsol_m^{r} = \max\{subsol_{m-1}^{r} - a_m, 0\}$

pf)

⇒ If $subsol_m^{\ell} \neq 0$, the contiguous subsequence yields $subsol_m^{\ell}$ has to be obtained from max (subsequences ending at a_{m-1}) + a_m $= subsol_{m-1}^{\ell}$

⇒ If $subsol_m^{r} \neq 0$, the contiguous subsequence yields $subsol_m^{r}$ has to be obtained from max (subsequences start at a_{m-1}) − a_m. $= subsol_{m-1}^{r}$ ∎