Problem 1) We say a graph $G=(V,E)$ is c-edge-colorable if we can assign an integer in $\{1, 2, \cdots, c\}$ to every edge so that no two edges sharing a common end point are assigned the same number. We say a graph $G=(V,E)$ is d-regular if the degree of every vertex is d.

Prove that every k-regular bipartite graph is k-edge-colorable.

<Proof>

[Claim 1) d-regular bipartite graph has $|V_L| = |V_R| \geq d$.]

① $|V_L| = |V_R|$

- Since $G=(V_L \cup V_R, E)$ is a d-regular, for each vertex in V_L, there exists d adjacent edges. Therefore there exists $|V_L| \times d$ edges between V_L and V_R.
  On the other hand, since G is d-regular, each vertex in V_R also has d adjacent edges. Therefore there exists $|V_R| \times d$ edges adjacent to V_R.
  Since our graph G is bipartite, $\underbrace{|V_L| \times d}_{\text{\# of edges adjacent to } V_L} = \underbrace{|V_R| \times d}_{\text{\# of edges adjacent to } V_R}$ and $|V_L| = |V_R|$ ■

② $|V_L| = |V_R| \geq d$.

- Suppose toward a contradiction, $|V_L| = |V_R| < d$, where G is d-regular bipartite. Since G is bipartite, a vertex $v \in V_L$ can have edge (v, w) only when $w \in V_R$ and because G is not a multigraph, maximum number of edges that is possible to be in the form of (v, w) for a certain v is $|V_R| < d$.
  This contradicts our assumption of d-regular ※

[Claim 2) 1-regular bipartite graph $G=(V_L \cup V_R, E)$ has a perfect matching E]

- Suppose that $|V_L| = |V_R| = k \geq 1$ and thus $|E| = k \times 1$.
- Suppose toward a contradiction, that E is not a matching.
  Then, at least one vertex $v \in V_L \cup V_R$ has two adjacent edge e_1 and e_2.
  If so, degree of v is 2, which contradicts our assumption. ※
- Because E is a matching and degree of v in $(V_L \cup V_R)$ is 1 (means matched), E is a perfect bipartite matching.

- Every 1-regular bipartite graph $G=(V_L \cup V_R, E)$ $(|V_L| = |V_R| \geq 1)$ has a perfect matching E

Claim 3) Given any n-regular bipartite graph $G=(V_L \cup V_R, E)$ and $2 \leq n \leq |V_L| = |V_R|$, for any perfect matching $M \subseteq E$, $G' = (V_L \cup V_R, E \setminus M)$ is $(n-1)$-regular bipartite.

- First, note that, given n-regular bipartite graph G, there exists a perfect matching $M \subseteq E$. by Hall's Theorem⊛

⊛ ⇒ Given a bipartite graph $G=(V, E)$ with bipartition V_L V_R, G has a matching of size $|V_L|$ if and only if for every $S \subseteq V_L$, we have $|N(S)| \geq |S|$, where $N(S) = \{ b \in B \mid \exists a \in S \text{ with } (a,b) \in E \}$

and Since G is n-regular, for every $S \subseteq V_L$, $|N(S)| \geq |S|$

- Suppose not, $|N(S)| < |S|$ and there exists $n \cdot |S|$ edges between S and $N(S)$ then $\sum_{v \in N(S)} deg(v) = n \cdot |S|$ and $|N(S)| < |S|$ therefore $\exists$ at least one vertex that has $deg(v) > n$. Contradiction.
- Therefore n-regular bipartite graph $G = (V_L \cup V_R, E)$ has a matching of size $|V_L|$ and since $|V_L| = |V_R|$, it is a perfect bipartite matching.

∴ n-regular bipartite graph $G = (V_L \cup V_R, E)$ includes at least one perfect matching. $M \subseteq E$.

- Since in the perfect matching M, every vertex is matched exactly once, in the graph $G' = (V_L \cup V_R, E \setminus M)$, vertex $v \in (V_L \cup V_R)$ has degree $(n-1)$

∴ $G' = (V_L \cup V_R, E \setminus M)$ is a $(n-1)$-regular bipartite graph.

In our problem, for a specific color x, a set of edges $M_x := \{ e \in E \mid \text{edge } e \text{ colored } x \}$ is a matching in the edge-colorable graph G, since no two edges with color x shares a common endpoint.

And because every edge in G should be colored,

Claim 4) For a graph $G = (V, E)$, if there exists a partition P of E such that $P := \{ M_1, M_2, \cdots, M_k \}$ and M_i is a matching for $i = \{1, 2, \cdots, k\}$ then, G is a k-edge-colorable graph.

- If we assign color i to each corresponding matching M_i in P, then

① since $M_i \cap M_j = \phi$ for any $i, j \in \{1, .. k\}$, $i \neq j$, no edge is colored by multiple colors

② every edge is colored, since $\bigcup_{i=1}^{k} M_i = E$

③ No two edges with same color i shares a common endpoint, since M_i is a matching

Therefore k-edge-colorable.

Claim 5) Given any n-regular bipartite graph $G=(V_L \cup V_R, E)$ and $n \leqq |V_L|=|V_R|$
there exists at least one partition $P := \{M_1, M_2, \cdots, M_n\}$ of E
such that every M_i (where $i \in \{1, \cdots, k\}$) is a perfect matching.

- ① if $n=1$) then 1-regular bipartite graph G has a perfect matching E.
  therefore there exists $P=\{E\}$ of E by Claim 2.
  Claim 5 holds.
- ② if $n \geqq 2$) By Claim 3,

$$E = (E \setminus M_n) \cup M_n$$

(where M_n is a perfect matching and $G_n = (V_L \cup V_R, E \setminus M_n)$ is $(n-1)$-regular bipartite.

$$= (E \setminus (M_n \cup M_{n-1})) \cup M_{n-1} \cup M_n$$

(where M_{n-1} is a perfect matching and $G_{n-1} = (V_L \cup V_R, E \setminus (M_n \cup M_{n-1}))$ is $(n-2)$-regular bipartite.

$$\vdots$$

$$= \left(E \setminus \left(\bigcup_{i=2}^{n} M_i\right)\right) \cup \bigcup_{i=2}^{n} M_i$$ and $G_1 = \left(V_L \cup V_R, E \setminus \left(\bigcup_{i=2}^{n} M_i\right)\right)$ is 1-regular bipartite.

$\Rightarrow$ Then $(E \setminus \bigcup_{i=2}^{n} M_i)$ is a perfect matching by Claim 2
and $M_2, M_3, \cdots, M_n$ are perfect matchings and pairwise disjoint by Claim 3.

$\Rightarrow$ Therefore $P = \{E \setminus \bigcup_{i=2}^{n} M_i, M_2, M_3, \cdots, M_n\}$ is a partition of E
whose every element is a perfect matching.

by ① and ②, any n-regular bipartite graph has at least one partition P of E
such that $P = \{M_1, M_2 \cdots M_n\}$ where M_i is a matching for $i \in \{1, 2, \cdots, n\}$

■

<Main Proof>

By Claim 5, every n-regular bipartite graph has at least one edge partition
whose elements are only matchings and the number of elements is n.
And by Claim 4, since n-regular bipartite graph has such a partition,
it is n-edge-colorable.

$\therefore$ Every k-regular bipartite graph is k-edge-colorable ■

⊗ (Proof of Hall's Theorem.)

⇒ part] if an edge set M is a matching, then any subset $\bar{M} \subseteq M$ is also a matching.
if G has a matching M of size $|V_L|$, then for a vertex set $S \subseteq V_L$,
there exists a corresponding matching $\bar{M} \subseteq M$. Then we have $|N(S)| \geq |S|$.
O/W, at least one pair of edges shares a common endpoint $v \in N(S)$ #.

⇐ part] For an arbitrary vertex cover C, let $U := V_L \setminus C$. then $N(U) \subseteq C$ (**)

(** Suppose not. $\Rightarrow \exists (x,y) \in E$ s.t $x \in U$ and $y \notin C$
then (x,y) is not covered by C #)

· We thus have
$$
\begin{aligned}
|C| &= |C \cap V_L| + |C \cap V_R| \\
&= |V_L| - |V_L - C| + |C \cap V_R| \\
&= |V_L| - |U| + |C \cap V_R| \\
&\geq |V_L| - |U| + |N(U)| \quad (\text{since } N(C) \subseteq (C \cap V_R)) \\
&\geq |V_L| - |U| + |U| \quad (\text{since we assume that } |N(U)| \geq |U|) \\
&= |V_L|
\end{aligned}
$$

$\Rightarrow |C| \geq |V_L|$

$\Rightarrow |C^*| \geq |V_L|$ where $C^* :=$ minimum cardinality vertex cover

By König's Theorem (Proven in the previous class)

$\Rightarrow |M^*| = |C^*|$ where $M^* :=$ maximum cardinality vertex cover.

$\therefore |M^*| \geq |V_L|$

or G has a matching of size $|V_L|$, $\bar{M} \subseteq M^*$ ▫

Problem 2) Suppose we are given a directed graph $G=(V,A)$ and we want to choose a subset of arcs $A' \subseteq A$ so that, in the resulting subgraph $G'=(V,A')$, the degree of every vertex equals to its outdegree. There is a trivial solution to this question: we choose $A'=\emptyset$ and the indegree and outdegree of every vertex will be zero.

Design and analyze an algorithm that, given G and $k \in \mathbb{N}$ as input, determines whether there exists $A' \subseteq A$ with $|A'| \geq k$ that yields a subgraph $G'=(V,A')$ where the indegree of every vertex is equal to its outdegree.

<Answer>

- Define an incidence matrix $M_{|V|\times|A|}$ of the given directed graph $G=(V,A)$ as follows : each row corresponds to a vertex $v \in V$, each column corresponds to an arc $a \in A$

$$\Rightarrow M_{v,a} = \begin{cases} 1 & \text{if } a = \langle v,w \rangle \text{ (or leaves } v) \\ -1 & \text{if } a = \langle w,v \rangle \text{ (or enters } v) \\ 0 & \text{o/w} \end{cases}$$

- Theorem 1) The incidence matrix M of a directed graph G is totally unimodular

proof) Let $\bar{M}$ be a $n \times n$ square submatrix of M

- if $n=1$, $\bar{M}$ is unimodular, since $M_{v,a} \in \{-1,0,1\}$ by definition.
- Otherwise
  1. if $\bar{M}$ has a zero-column, then by determinant expansion with the column, $\Rightarrow |\bar{M}| = 0$
  2. if $\bar{M}$ has a column with exactly one 1 or -1, then for the submatrix $\bar{M}_{ij}$ of $\bar{M}$ (obtained by deleting ith row and jth column) $\Rightarrow |\bar{M}| = |\bar{M}_{ij}|$ or $-|\bar{M}_{ij}|$ ($\Rightarrow$ recursively compute).
  3. if $\bar{M}$'s every column has one 1 and one -1, $\Rightarrow$ by row operations, 0-row can be obtained by adding every rows. $\Rightarrow |\bar{M}| = 0$

$\Rightarrow$ There is no other possibility, because every column corresponding to an arc has exactly one 1 and one -1.

$\therefore$ every $\bar{M}$ of M has determinant value $|\bar{M}| \in \{-1,0,1\}$.

Therefore M of a directed graph G is totally unimodular.

• Theorem 2.) If a matrix A is totally unimodular, then the polyhedron $P = \{x \mid Ax \leq b\}$ is an integer polyhedron for every integral vector b. (or $P = P_I$)

proof from our course)

- Consider an arbitrary minimal face of P, F.
- We have $F = \{x \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $\{Ax = b\}$
  (*)(Use this fact without proof)
- WLOG, we can assume that A' has full row rank.
  Then, by permuting rows and columns, we obtain $A' = [U, V]$, where U is a square nonsingular matrix.
- Since U is a submatrix of A (which is TU) and $\det(U) = +1$ or -1 $(\neq 0)$
  $x := \begin{bmatrix} U^{-1}b \\ \vec{0} \end{bmatrix}$ is an integral vector.

  $$A'x = [U\, V] \begin{bmatrix} U^{-1}b' \\ \vec{0} \end{bmatrix} = b'$$

  ~~therefore~~ $x \in F$ or every minimal face F of P has an integral vector.
- Then $P = P_I$ since P is convex hull of all integer vectors in P itself.

• Again, our problem can be formulated as follows:

[1]

$$\text{Maximize} \quad z = \sum_{a \in A} x_a$$

$$\text{Subject to} \quad \sum_{a \in \delta^+(\{v\})} x_a - \sum_{a \in \delta^-(\{v\})} x_a = 0 \quad : \text{conservation constraint for } \forall v \in V$$

$$x_a \in \{0, 1\} \text{ for } \forall a \in A$$

$\Rightarrow$ Decision variable $x_a = 1$ if $a \in A'$ and $x_a = 0$ O/W.

We maximize z and if $z \geq k$ then return True and otherwise, return False.

- If we relax integral constraints and transform the IP to LP, then

[2] Maximize $z = \sum_{a\in A} x_a$

Subject to $\sum_{a\in\delta^+(\{v\})} x_a - \sum_{a\in\delta^-(\{v\})} x_a \leq 0$ for $\forall v \in V$

$-\sum_{a\in\delta^+(\{v\})} x_a + \sum_{a\in\delta^-(\{v\})} x_a \leq 0$ for $\forall v \in V$

$-x_a \leq 0$ for $\forall a \in A$

$x_a \leq 1$ for $\forall a \in A$

- [2] is equal to : $\mathrm{Max}\{\vec{1}X \mid MX \leq \vec{0}, -MX \leq \vec{0}, -IX \leq \vec{0}, IX \leq \vec{1}\}$

$$= \mathrm{Max}\left\{\vec{1}X \,\middle|\, \begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} X \leq \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{1} \\ \vec{0} \end{bmatrix}\right\}$$

Therefore, if $\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix}$ is totally unimodular, then (by theorem 2) [2] has a integral polyhedron. In other words, by solving [2] we can obtain integral solution of [1]. And we can safely relax integral constraints.

- Since M, $-M$, I, $-I$ are trivially totally unimodular.

And in addition to theorem 1, if submatrix M has multiple 1s or −1s (means two), then one of them must be in I or $-I$, and we can safely expand with the row.

Other parts of proof is equal to the proof.

$\Longrightarrow \therefore \begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix}$ is totally unimodular.

Therefore polyhedron $\left\{\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} X \leq \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{1} \\ \vec{0} \end{bmatrix}\right\}$ is integral.

- ILP is NP-hard but now we have an LP which is solvable in polynomial time.

<Algorithm>

- Given $G=(V,A)$ and $k \in \mathbb{N}$ $\Rightarrow$ Let $M :=$ Incidence Matrix of G

  (if $\max\{\vec{1}x \mid \begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} x \le \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{1} \\ \vec{0} \end{bmatrix}\} \ge k$ then return "True"

  Otherwise return "False")

- $\max\{\vec{1}x \mid \begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} x \le \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{1} \\ \vec{0} \end{bmatrix}\}$ can be solved ① by simplex method which is efficient in practice.

  ② by ellipsoid method which guarantees polynomial time.

  : $O(|A|^4 \log(|A|))$

Problem 3) You have purchased a nice little café near the campus of Yonsei Univ. In order to estimate the demand, you hired a consultant, and now you know that you need to have a_i employees working on day i, where $i = 1, \cdots, n$. You have also interviewed k-candidates to fill these positions — if you hire candidate j, you will need to pay $p_j > 0$ for $j = 1, \cdots, k$. Note that this is a "flat" wage : no matter how many days he/she works, he/she will be paid p_j in total.

You would want to hire $\max_{1 \leq i \leq n} a_i$ employees whose wages are the cheapest; however, each candidate can work only on a certain consecutive period of days. For each $j = 1, \cdots, k$, let s_j and t_j denote the first and the last day candidate j can work on, respectively : i.e., candidate j can work on day i if and only if $s_j \leq i \leq t_j$.

Design and analyze an algorithm that, given n, k, a, p, s, t, calculates the cheapest way to hire some candidates so that all the requirements are satisfied. The running time must be bounded by a polynomial in the input size.

<Answer>

- Our problem can be formulated as a following Integer program :

[1] $\text{Min} \{ pX \mid MX \geq a \text{ and } X_i \in \{0, 1\} \}$

where → M is a $n \times k$ matrix

$$\begin{cases} M_{ij} = 1 & \text{if } s_j \leq i \leq t_j \\ M_{ij} = 0 & \text{o/w} \end{cases}$$

→ $$\begin{cases} X_i = 1 & \text{if decision is to hire candidate } i \\ X_i = 0 & \text{if decision is not to hire candidate } i \end{cases}$$

- Program [1]'s integer constraint can be relaxed as follows :

[2] $\text{Min} \{ pX \mid MX \geq a \text{ and } IX \leq \vec{1} \text{ and } -IX \leq \vec{0} \}$

$$= \text{Min} \left\{ pX \;\middle|\; \begin{bmatrix} -M \\ I \\ -I \end{bmatrix} X \leq \begin{bmatrix} a \\ \vec{1} \\ \vec{0} \end{bmatrix} \right\}$$

Since $\begin{bmatrix} a \\ \vec{1} \\ \vec{0} \end{bmatrix}$ is integral, by "theorem 2 of Problem 2", If M is totally unimodular, then polyhedron of program [2] is integral.

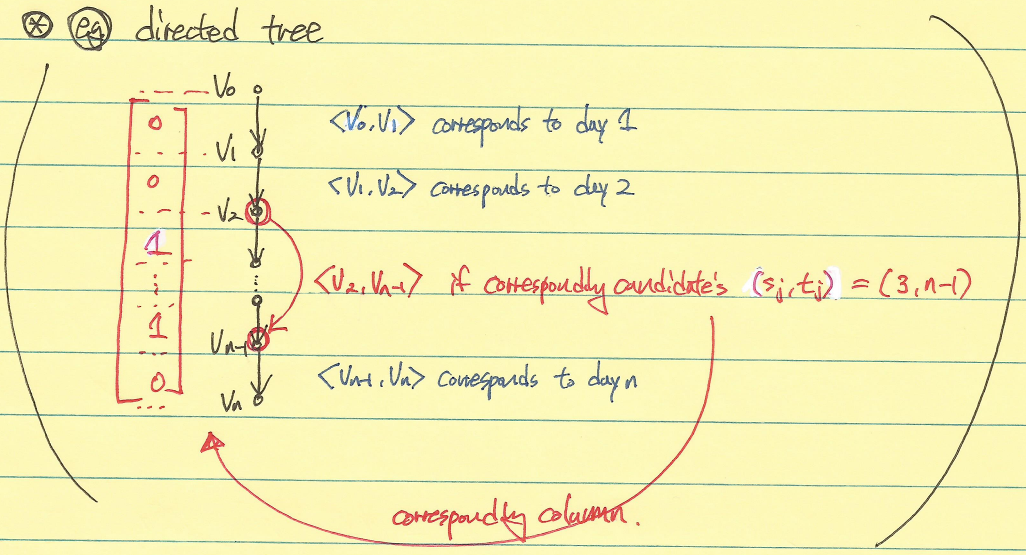
Therefore by solving [2], we can obtain integral solution of [1] if M is TU.

(※ If M is TU, then $\begin{bmatrix} -M \\ I \\ -I \end{bmatrix}$ is also TU. proven in problem 2)

• M is a network matrix defined on $G=(V,A)$ and $T=(V,A')$
if we define G and T as follows :

given $n, k, \vec{s}, \vec{t}$, $V = \{v_0, v_1, \cdots, v_n\}$
and $A' = \{\langle v_0, v_1\rangle, \langle v_1, v_2\rangle, \cdots, \langle v_{n-1}, v_n\rangle\}$
and $A = \{\langle v_a, v_b\rangle \mid a = s_j - 1 \text{ and } b = t_j \text{ for } \forall j \in \{1, 2, \cdots, k\}\}$

⊗ eg directed tree



Then, a network matrix defined on $G(V,A)$ and $T(V,A')$ is equal to M
since for candidate j, arc is defined by $\langle v_{s_j - 1}, v_{t_j}\rangle$ and then $M_{ij} = 1$ if and only if
$s_j \leq i \leq t_j$ for $\forall j$. and $M_{ij} = 0$ O/W.

⊗ (Proven from one of precious classes)
• A network matrix is totally unimodular
⇒ Therefore M is totally unimodular and by solving [2] (which is LP), we can obtain a solution of [1] (which is ILP)

• A linear program can be solved by ellipsoid method in polynomial time.

<Algorithm>

• Given $n, k, \alpha, p, \vec{s}, \vec{t}$.

solve $\min\{ pX \mid \begin{bmatrix} -M \\ I \\ -I \end{bmatrix} X \leq \begin{bmatrix} \alpha \\ \vec{1} \\ \vec{0} \end{bmatrix} \}$ where $M_{ij} = 1$ if $s_j \leq i \leq t_j$, $M_{ij} = 0$ O/W

Return X^* (optimal solution) and pX^* (minimized cost)