Approximation Alg (Claire Mathieu)

- P (PTIME) Deterministic Turing Machine 으로 Polynomial Time 안에 풀수있는 결정문제 (Decision Problem)
  $\Rightarrow$ Answer $\in$ {Yes, No}
- NP (NPTIME) Nondeterministic Turing Machine 으로 Polynomial Time 안에 풀수 있는 결정문제 (Decision Problem)

$\Rightarrow P \subseteq NP$

NP-Hard : NP $\to$ NP-hard "Polynomial Time Reduction" 이 가능한 문제집합.

NP-Complete := (NP) $\cap$ (NP-Hard)

ex) : NP-hard (& Not NP-Complete)
$\Rightarrow$ Halting Problem

ex) : NP-hard & NP-Complete
$\Rightarrow$ Subset Sum Problem.

[Sum(finite integer set $\Rightarrow$ subset)
$\cdots \exists S \subseteq A$ s.t $sum(S) = 0$??.]

⊛ problem $H \in$ NP-hard $\Rightarrow$
① Any problem $L \in P (\subseteq NP)$
$\Rightarrow \exists$ polytime reduction $[L \to H]$
② if $\exists$ poly Alg for H (NP-hard)
$\Rightarrow \exists$ poly Alg for Hamiltonian Path Problem (NP-complete). 역은 성립X
③ For any problem $L \in NP$
$\Rightarrow \exists$ polytime reduction $[L \to H]$ ($\because$ by definition)

⊛ problem $H \in$ NP-Complete $\Rightarrow$
① An output of $H \in \{T, F\}$ $\because$ NP-Complete $\subseteq$ NP $\subseteq$ Decision Problem
⊛ NP-hard $\not\subseteq$ Decision Problem.
② if $\exists$ poly Alg for H
$\Rightarrow \exists$ poly Alg for Hamiltonian Path Problem
if $\exists$ poly Alg for Hamiltonian Path Problem
$\Rightarrow \exists$ poly Alg for H.
③ $H \subseteq NP$ ($\because NPC \subseteq NP$)
④ Any problem $L \in P \Rightarrow \exists$ polytime reduction $[L \to H]$
" $L \in NP \Rightarrow \exists$ polytime reduction $[L \to H]$

⊛ Linear Program을 Worst Case Poly Time에 풀수있는 Algorithm : ex) · Ellipsoid Method
· Projective Algorithm
⊛ $\Rightarrow$ Simplex Algorithm은 Worst Case에 Poly X.

⊛⊛ Unique Game Conjecture (UGC) (2002)

- Unique Game [NP-hard] // if True $\Rightarrow$ There is no polytime alg for "good" approximation.

Approx Algo 1 Exercise #1

① Maximum Matchings for Vertex Cover

[Algo.]

$VC \leftarrow \phi$, Given $G=(V,E)$

Compute a maximal matching of G, $M \subseteq E$ ("Maximal" ... Select Greedily)

For each edge $e=(x,y) \in M$.

  VC.add(x) and VC.add(y)

Return VC.

LP) Minimize $\sum_{u \in V} C_u x_u$ (s.t.) $0 \le x_u \le 1$, $x_u + x_v \ge 1$ for $\forall (u,v) \in E$ → polytime

⇒(Round) X ... $x_i' = 1$ if $x_i \ge 0.5$, $x_i' = 0$ o/w

linear time

$x_i \ge \frac{1}{2} x_i'$

$\boxed{x_i' \le 2x_i}$

$|X'| \le 2\,Opt$

<Correctness> : Claim) VC is a vertex cover (for $\forall e=(x,y) \in E$, $x \in VC$ or $y \in VC$)

proof) Suppose toward a contradiction that VC is not a vertex cover.

⇒ $\exists$ an edge e such that $e=(x,y)$ and $x \notin VC$ and $y \notin VC$.

Then M is not maximal, since $M \cup \{e\}$ is also a matching with cardinality $|M|+1$

※

<Approximation Guarantee> (Worst Case)

• Suppose that VC^* is the minimum vertex cover.

By König's Thm, $VC^* = M^*$ where M^* is maximum matching.

Our algorithm, given M^*, our algorithm returns VC where $|VC| = 2|M^*| = 2|VC^*|$

since any $M \subseteq E$ ⇒ $|M| \le |M^*|$ ∴ returns $|VC| = 2|M| \le 2|M^*| = 2|VC^*|$

∴ output $|VC| \le 2\,Opt$

<Running Time> given computing a maximal matching takes $O(n+m)$ ($n = |V|$, $m = |E|$)

⇒ Bounded by $O(n+m)$.

② Triangles of a Graph.

[Problem] Given a graph $G=(V,E)$, we say that a triple of vertices A,B,C forms a triangle if the edges $(AB),(BC),(CA) \in E$.

Minimum $S \subseteq V$ that $G \setminus S$ does not contain any $\triangle$.

• if $\exists$ a triangle $\triangle ABC \subseteq S$, $x_a + x_b + x_c = 3$

• LP Formulation>

$$\left[\begin{array}{ll} \text{Minimize } \sum_{u \in V} x_u & \text{(s.t.) } 0 \le x_u \le 1 \text{ for } \forall u \in V \\ & x_a + x_b + x_c \ge 1 \text{ for } \forall \triangle abc \in G \end{array}\right]$$

Minimum value of

⇒ $\max(x_a, x_b, x_c)$ for $\triangle abc \in G$ and X returned solution.

$= \frac{1}{3}$ ⇒ $\max(x_a, x_b, x_c) \ge \frac{1}{3}$ ☆ if $x_i \ge \frac{1}{3}$ ⇒ $x_i' = 1$

o/w ⇒ $x_i' = 0$

[Alg]

Given $G=(V,E)$.

$X \leftarrow$ Solve: $\min \{ \|X\|_1 \mid 0 \le x_i \le 1$ for $\forall i \in V$, $x_a + x_b + x_c \ge 1$ for $\forall \triangle abc \in G \}$

$X' \leftarrow \begin{cases} x_i' = 1 & \text{if } x_i \ge \frac{1}{3} \\ x_i = 0 & \text{o/w.} \end{cases}$ $\qquad \underbrace{x_i'}_{\text{Rounded IP}} \le \underbrace{3x_i}_{\text{LP}}$

if $x_i' = 1$ then $i \in S$

Return S

<Correctness> Claim $\nexists \triangle abc$ in $G \setminus S$.

prf) suppose toward a contradiction that $\exists$ a triangle $\triangle abc \in G \setminus S$.

i.e. $\triangle abc$ is not removed by S. $\Rightarrow x_a' = 0,\ x_b' = 0,\ x_c' = 0.$

$\Rightarrow x_a < \frac{1}{3},\ x_b < \frac{1}{3},\ x_c < \frac{1}{3}$

$\Rightarrow x_a + x_b + x_c < 1$ for $\triangle abc \in G$. #

(not a proper solution of LP).

<Approximation Guarantee>

- Suppose that S^*, $|S^*| = OPT$ ... Solution of our LP

Our solution $|S| = \sum_u x_u' \le 3 \sum_u x_u^* = 3(OPT)$

$\Rightarrow |S| \le 3\,opt$

③ 4-colorable graph.

- We propose to derive a 3/2-approx algorithm for a more restricted class of graph

given $G=(V,E)$, it is 4-colorable iff for every $e=(x,y) \in E$,

we can assign number $n(x) \ne n(y)$

and $n(v) \in \{1,2,3,4\}$ for $\forall v \in V$.

- from LP (Vertex Cover) $\Rightarrow$ $V_k^{1/2} = \{ v \mid x_v = 1/2$ and $C(v) = k \}$

$V_k^{1} = \{ v \mid x_v = 1$ and $C(v) = k \}$

$V_k^{0} = \{ v \mid x_v = 0$ and $c(v) = k \}$

Assume that $|V_0^{1/2}| \le |V_1^{1/2}| \le |V_2^{1/2}| \le |V_3^{1/2}|$

<Approximation Ratio>

Q1) Give a relation between the value val(x) and the cardinality of the sets $V^{1/2}$ and V^1.

Answer) $Val(x) = \sum_{v \in V} x_v = \sum_{v \in V^{1/2}} \left(\frac{1}{2}\right) + \sum_{v \in V^1} (1)$

$$= \frac{1}{2} \times |V^{1/2}| + |V^1|$$

$$\Rightarrow Val(x) = \frac{1}{2}|V^{1/2}| + |V^1|$$

Q2). Give a tight lower bound on the cardinality of $V_3^{1/2}$ based on the cardinality of $V^{1/2}$.

Answer) $|V^{1/2}| = |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| + |V_3^{1/2}|$ (since they are disjoint sets)

since $|V_3^{1/2}|$ is minimized when $|V_0^{1/2}| = |V_1^{1/2}| = |V_2^{1/2}| = |V_3^{1/2}| = \frac{1}{4}|V^{1/2}|$

$$\Rightarrow |V_3^{1/2}| \geq \frac{1}{4}|V^{1/2}|$$

Q3) Deduce from Q2 an upperbound on the cardinality of $|V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}|$ based on the cardinality of $V^{1/2}$.

Answer) $|V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| = |V^{\frac{1}{2}}| - |V_3^{\frac{1}{2}}| \leq |V^{1/2}| - \frac{1}{4}|V^{1/2}|$ (by Q2)

$$= \frac{3}{4}|V^{1/2}|$$

$$\Rightarrow |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| \leq \frac{3}{4}|V^{1/2}|$$

Q4) Combine Q1 and 3 to give an upperbound on the number of vertices in S based on val(x).

Answer) $|S| = |V^1| + |V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}| \leq |V^1| + \frac{3}{4}|V^{1/2}|$

$$\leq \frac{1}{2}|V^1| + (|V^1| + \frac{3}{4}|V^{1/2}|)$$

$$= \frac{3}{2} Val(x)$$

$$\Rightarrow |S| \leq \frac{3}{2} Val(x)$$

Q5) Combine Q4 and the property of val(x) to conclude on the approximation ratio of the rounding procedure. Recall that val(x) is the value of the optimal fractional solution to the LP.

Answer) by Q4, $|S| \leq \frac{3}{2} OPT$. (since $val(x) = OPT$)

<Correctness>

We now show that S is a correct vertex cover. Namely, we want to show that for each edge (u, v), u or v (or both of them) are in S. We will proceed by contradiction and assume that neither u or v are in S.

(Q6) Suppose, in addition, that $u \in V^0$, to which set does v belong? Recall that X is the solution to the LP.

Answer) Since S is a correct vertex cover, and $u \notin S$ (since $u \notin V^1 \cup V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2}$)

Then there exists a constraint in the LP : $x_u + x_v \geq 1$ and $x_u = 0$

Therefore $x_v = 1$ and $v \in V^1$

(Q7) Deduce from the previous question to which set of $V^1, V^0, V^{1/2}$ do u and v belong.

Answer) • If $u \in V^0$, then $v \in V^1$ (by Q6)

• If $u \in V_3^{1/2}$, then $v \in (V^1 \cup V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2})$

If $u \in (V_0^{1/2} \cup V_1^{1/2} \cup V_2^{1/2})$, then $v \in (V^1 \cup V^{1/2})$

$\Rightarrow$ If $u \in V^{1/2}$ then $v \in (V^1 \cup V^{1/2})$

• If $u \in V^1$, then $v \in V_0 \cup V_1 \cup V_{1/2}$

(Q8) If u and v belong to $V^{1/2}$, to which subset of V do they belong, if they donot belong to S.

Answer) Then $u \in V_3^{1/2}$ and $v \in V_3^{1/2}$ (Otherwise, $u \in S$ or $v \in S$)

(Q9) Recall that C is a 4-colouring. Explain the contradiction.

Answer) Since C is a 4-colouring of G, $c(u) \neq c(v)$ for $\forall e = (u, v) \in E$.

Therefore ($u \in V_3^{1/2}$ and $v \in V_3^{1/2}$) is a contradiction, because $c(u) = c(v) = 3$

(Q10) Give an example of a well-known class of graphs that is 4-colorable.

Answer) Bipartite Graphs are two-colourable, therefore 4-colorable.

• Knapsack Problem] (input rounding technique)

Given B : capacity, n items (size s_i & Value v_i) ⇒ Maximize value (NP-hard)

① ⇒ Slope $= \frac{s_i}{v_i}$ ... Greedy Approach. ⇝ Not really good

② • $s_i = s$ for $\forall i$ ⇒ greedy is good.

• $v_i = v$ for $\forall i$ ⇒ "

• $s_i = v_i$ for $\forall i$ ⇒ slope $= 1$ for $\forall i$ ... decreasing order

1] 2-approximation approach. (Greedy)

· $OPT \leq B$

· Output + (1 item I_{k+1}) > B ... output has at least 1 item.

... $I_0 > I_i$ (selected item I에 있는)

... $I_0 > I_{k+1}$

For the worst case ⇒ $I = \{I_0\}$, $s(I_0) < B$, $s(I_1) + s(I_0) > B$

, $s(I_0) > s(I_1)$

, $2s(I_0) > B$

, $s(I_0) > \frac{B}{2}$

$s(I) > \frac{B}{2}$

∴ Output $> \frac{B}{2} \geq \frac{OPT}{2}$

⇒ $\boxed{\text{Output} > \frac{1}{2} OPT}$ ⇒ 2-approximation

polynomial in (n)

• $s_i = v_i$ for $\forall i$ & $s_i \in \{1, 2, \cdots, B\}$ and B is a small integer

<Dynamic Programming approach>

Boolean

$A[i, v]$ = Whether v achievable with subset of first i items. : S

⇒ if $i \in S$ ... $(v - v_i)$ should be reached by first $(i-1)$ items

⇒ if $i \notin S$ ... v should be "

$$A[i,v] = A[i-1, v] \text{ or } \left(A[i-1, v-v_i] \text{ and } (v \geq v_i)\right)$$

Alg] For $v = 0, \cdots, B$: $A[1,v] \leftarrow$ false.

$A[1, v_1] \leftarrow$ True, $A[1, 0] \leftarrow$ True.

For $i = 2, \cdots n$

For $v = 0, \cdots, v_i - 1$: $A[i,v] \leftarrow A[i-1, v]$

For $v = v_i \cdots B$

$A[i,v] \leftarrow A[i-1, v]$ or $A[i-1, v-v_i]$

Output $\max\{v : A[n,v] \text{ is True}\}$ → Opt

$O(n \cdot B)$