[Problem] Max weight independent set problem

given a matroid $(S, \mathcal{I})$ and a weight function $c : S \to \mathbb{Q}$

find an independent set I that maximizes $\sum_{x \in I} c(x)$

⇒ LP Relaxation)

[P] Maximize $\sum_{i \in S} c(i) x_i$

subject to $\sum_{i \in U} x_i \leq r(U)$ for $\forall U \subseteq S$: independent set constraint.

$\underline{\underline{x_i \geq 0}}$ $\forall i \in S$.

원래는 $\underline{\underline{x_i \in \{0, 1\}}}$

[D] Minimize $\sum_{U \subseteq S} r(U) y_U$

subject to $\sum_{U : i \in U} y_U \geq c(i)$ for $\forall i \in S$

$y_U \geq 0$ for $\forall U \subseteq S$ $y_\emptyset$ 는 존재하지만 고려 X.

WLOG, assume that $S = \{1, \dots, n\}$

with $c(1) \geq \dots \geq c(n)$: sorted. Let $n' := \max\{i \in S \mid c(i) \geq 0\}$

"greedy" // ⊛ Kruskal's

Alg $I \leftarrow \emptyset$

⊛ $y \leftarrow \vec{0}$ ↗ Dual infeasible. ~~set~~ $y_\emptyset$ to a sufficiently large number

for $i \leftarrow \{1, \dots, n'\}$ — dual alg.

⊛² Simultaneously decrease $y_{\{1, \dots, i-1\}}$

and increase $y_{\{1, \dots, i\}}$ at the same rate — Dual feasible로 감.

until the correspond dual constraint of i becomes tight. (*)

if $(I + i) \in \mathcal{I}$, then $I \leftarrow I + i$

→ ⊛ independence check.

output I.

// primal은 계속 feasible

Thm Alg returns a maximum weight independence set

proof] It is clear that the alg returns an independent set ⊛

Let $x^* \in \{0,1\}^{|S|}$ be a vector satisfying $x_i^* = 1$ iff $i \in I$

Let y^* denote the value of y upon termination.

claim 1> For $i = 0, \cdots, n'$ at the beginning of the $(i+1)^{st}$ iteration (= at the end of the i^{th} iteration)

① I is a ~~base~~ maximal indep set for $\{1, \cdots, i\}$

② $y_{\{1,\cdots,i\}} = C(i)$ if $i \geq 1$.

③ the dual constraint corresponding to $1 \sim i$ are tight.

④ for $\forall U$ s.t. $U \cap \{i+1, \cdots, n\} \neq \phi$, $y_U = 0$

⑤ y is nonnegative.

Moreover, for $i \neq 0$, step ⊛^{2} at the ith iteration is well-defined
ie. there exist some finite $\delta \geq 0$ by which $y_{\{1,\cdots,i-1\}}$ decrease and $y_{\{1,\cdots,i\}}$ increase.

[proof of claim 1] by induction on i

when $i=0$, ① ~ ⑤ trivially hold.

Suppose that $i \geq 0$, (Let I_0 denote the value of I at the beginning of ith and I_1 denote the value of I at the end of ith iter

- If $r(\{1 \cdots i-1\}) \neq r(\{1 \cdots i\})$: I_0 not a base. (but indep)
  $\Rightarrow$ then $\{I_0 + i\}$ is a base for $\{1 \cdots i\}$

  If $r(\{1 \cdots i-1\}) = r(\{1 \cdots i\})$
  $\Rightarrow$ then I_0 is a base for $\{1 \cdots i\}$
  $\Rightarrow$ I_1 is a base for $\{1 \cdots i\}$: ① proven.

- At the beginning of the ith iter, the LHS of the dual constraint corresponding to i is 0 (by IH).
  $\Rightarrow$ $y_{\{1 \cdots i\}}$ needs to increase from 0 to $C(i) \geq 0$.
  in order to be tight $\therefore y_{\{1\cdots i\}} = C(i)$: ② proven

- in addition, to show that step ⊛^{2} is well-defined.
  Also note that $y_{\{1,\cdots,i\}}$ does not go below 0 (from IH)
  since $c(i+1) \geq c(i)$ ⇒ decrease 된 차이에 0 밑으로 X.

- again at the beginning of the ith iter, the dual constraint corresponding to $1 \sim i-1$ are tight. by IH,

$k = i-1$ $\quad \sum_{U:k\in U} y_U \geq c(k)$ for $\forall U \subseteq S$

$\Rightarrow$ tight $\sum_{U:k\in U} y_U = c(k)$ when $k = i-1$

$\Rightarrow \left(\begin{array}{l} y_{\{1,\cdots,i-1\}} \Rightarrow \text{decrease by } \delta \\ y_{\{1,\cdots,i-1,i\}} \Rightarrow \text{increase by } \delta \end{array} \right] \leftarrow 0.$ ⋯ 계속 tight.

$k = i$ $\quad \sum_{U:k\in U} y_U = 0 \Rightarrow \sum_{U:k\in U} y_U = c(k)$ ⋯ tight

⋯ ⓐ proven

$\Rightarrow$ $y \geq 0$ ⋯ ⓑ proven

$\Rightarrow$ ⓐ is trivial.

∴ claim 1 is proven.

Claim 2> For $i = 0, \cdots, n'$ at the beginning of the $(i+1)^{th}$ iteration,

$$\underbrace{\sum_{i\in I} c(i)}_{\text{value of } x^*} = \underbrace{\sum_{U\subseteq S} r(U)\, y_U}_{\text{value of } y^*} \quad \text{holds.}$$

⊛ strong duality.

proof] by induction on i, the base case is trivial.
Let δ denote the amount of decrease & increase in step ⊛^{2} of ith iter

The LHS, $\sum_{i\in I} c(i)$ either remains the same if $(I+i) \notin \mathcal{I}$
or increases by $\delta = c(i)$ if $(I+i) \in \mathcal{I}$

The RHS, $\sum_{U\subseteq S} r(U) y_U$ either remains the same if $(I+i) \notin \mathcal{I}$
since> $r(\{1 \cdots i-1\}) = r(\{1 \cdots i\})$
or increase by $\delta = c(i)$ if $(I+i) \in \mathcal{I}$
since> $r(\{1 \cdots i\}) = r(\{1 \cdots i-1\}) + 1$

by Claim 1 and Claim 2>
x^* and y^* are respectively feasible & LHS = RHS.
by Strong duality thm $\Longrightarrow$ x^* is optimal.

⊛⊛ Claim 2> 대신

Note that all dual constraints corresponding to $1 \cdots n'$ are tight in y^*

and $x_i^* \neq 0$ implies $i \in \{1, \cdots, n'\}$;

Moreover, if $y_U^* > 0$ for some $U \subseteq S$,

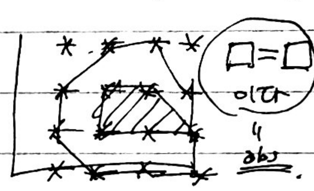
we have $U = \{1 \cdots i\}$ for some $i \leq n'$ and hence $|U \cap I| = r(U)$

Thus, x^* and y^* are feasible that satisfy the complementary slackness.

Corr The feasible region of (P) is an <u>integral polyhedron</u>.

Obs An <u>integral solution</u> x^* of (P) is (the incidence vector of) an <u>independent set</u>

(Integral Polyhedron & x^* ... is an independent set)



proof of obs] Note that $x^* \in \{0, 1\}^{|S|}$.

since for each $i \in S$, $r(\{i\}) \in \{0, 1\}$

Suppose to contradiction that $I^* := \{i \mid x_i^* = 1\}$ is <u>dependent</u>.

Then I^* has a circuit C^* s.t. $\sum_{i \in C^*} x_i^* = |C^*|$

$> r(C^*)$

$\sum_{i \in C^*} x_i^* > r(C^*)$ ... constraint violation = contradiction.

Def Given $U \subseteq S$,

- an independence oracle $\Rightarrow$ returns whether $U \subseteq \mathcal{I}$ or not.
- a rank oracle $\Rightarrow$ returns $r(U)$.

Lemma Given an indep oracle we can simulate a rank oracle using at most poly(|S|) (# basic oper ~~...~~)

& (" rank " , " indep oracle ~)

proof] given indep. $\Rightarrow$ greedily construct a base B for U

$\Rightarrow r(U) = |B|$.

given rank $\Rightarrow$ if $r(U) = |U|$ ... indep. O/W not.

[Problem] Max weight base (maximal independent set) problem.

given a matroid $(S, \mathcal{I})$ and a weight function $c : S \to \mathbb{Q}$

find an independent set B that maximizes $\sum_{x \in B} c(x)$.

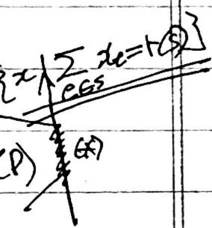
(maximal (a Base))

<polyhedral result>

Thm The convex combination of (the incidence vector of) bases of a matroid $(S, \mathcal{I})$ with rank function $r(\cdot)$ is

given by $(*)$
$$\begin{cases} \sum_{e \in U} x_e \leq r(U), & \forall U \subsetneq S \quad (** \ U \subseteq S \text{ but } U \neq S) \\ \sum_{e \in S} x_e = r(S) & \\ x_e \geq 0 & \forall e \in S. \end{cases}$$

max weight indep set problem



proof It is clear that $(*)$ is a relaxation.

Note that $\{x \mid \sum_{e \in S} x_e = r(S)\}$ is a supporting hyperplane of the feasible region of (P).

since a face of a face is a face $(*)$ is an integral polyhedron.

since (P) is an integral polyhedron.

Observe that any integral solution to $(*)$ is an integral solution to (P) and thus, (the incidence vector of) a base

<algorithmic result>

Alg WLOG, assume $c(1) \geq \cdots \geq c(n)$

$I \leftarrow \emptyset$

for $i \leftarrow \{1 \cdots n\}$

  if $(I + i) \in \mathcal{I}$ then $I \leftarrow (I + i)$

Obs given an independent oracle, (Alg) performs poly # of basic operations and oracle queries.

Obs (Alg) is Kruskal's Algorithm when applied to a graphic matroid

Thm **Alg** returns a max-weight base.

proof Let $c' : S \to \mathbb{Q}$ be a weight function defined by $c'(i) := c(i) + A$ for some const A

by choosing A sufficiently large, we can have $\forall i \in S$, $c'(i) \geq 0$.

Since $c'(1) \geq \cdots \geq c'(n) \geq 0$, we can interpret the transcript of Alg's execution as that of <Alg: max ind set problem> with c' in lieu of c.

$\Rightarrow$ Therefore
- I : max weight independent set under c'
- $|I| = r(S)$ by Claim 1 - ① of MWIS problem.

Note that, for any two bases B and B'

$c(B) \geq c(B')$ implies $c'(B) \geq c'(B')$

and vice versa.

[Problem] Max weight under given rank k independent set problem.

given a matroid $(S, \mathcal{I})$ and a weight function $c : S \to \mathbb{Q}$.

find an independent set I with $|I| = k$ that maximizes $\sum_{x \in I} c(x)$

(where $k \in \mathbb{N}$ and $k \leq r(S)$)

Lemma Let r be the rank function of $(S, \mathcal{I})$, and $\mathcal{I}' = \{I \mid I \in \mathcal{I}$ and $|I| \leq k\}$

Then $(S, \mathcal{I}')$ is a matroid whose rank function is given by

$\Rightarrow r'(U) := \min(r(U), k)$

proof)

<polyhedral result>

Thm The convex combination of (the incidence vector of) the independent sets of cardinality k is given by

$$(*)\quad \begin{cases} \sum_{e \in U} x_e \leq \underline{r(U)} & \text{for } \forall U \subsetneq S \\ \sum_{e \in S} x_e = k \end{cases}$$

$$x_e \geq 0. \quad \forall e \in S$$

proof [by thm of polyhedral result of MWB problem].

$\{x \mid \sum_{e \in S} x_e = k\}$ is a supporting hyperplane of $\{\sum_{e \in U} x_e \leq r'(U)\}$ $(\leq k)$

$\Rightarrow (**)$ is an integral polyhedron (face of face = face).

$\Rightarrow$ independent integral solution with $r(I) = k$.

where $(**)$ $\begin{cases} \sum_{e \in U} x_e \leq r'(U) & \text{for } \forall U \subsetneq S \\ \sum_{e \in S} x_e = k \ (= r'(S)) \end{cases}$ $\Rightarrow (=)\ \sum_{e \in U} x_e \leq r(U)$ for $\forall U \subsetneq S$

$x_e \geq 0$ for $\forall e \in S$

애매하게.

& $(**)$ is equivalent to $(*)$

<Algorithmic result>

Alg WLOG, assume $c(1) \geq \cdots \geq c(n)$

$I \leftarrow \emptyset$

for $i \leftarrow \{1, \cdots, n\}$

  if $(I + i) \in \mathcal{I}$ then $I \leftarrow (I + i)$

  if $|I| = k$ then return I

Thm Alg returns a maxweight independent set of cardinality k

proof Note that the transcript of Alg's execution can be interpreted as that of MWB Alg for $\underline{\underline{(S, \mathcal{I}')}}$