① Bipartite Matching (Cardinality)

① Algo Proof ⇒ Integral Polyhedron

②ℓ. 2nd proof of König.

Min-cost bipartite matching.

(A). Totally Unimodular Matrix.

(thm) TUM ... polyhedron ⇒ ($P = P_I$).

① (Incidence matrix of bipartite graph ⇒ TU. or Network Mat)

② (Incidence matrix of digraph G ⇒ TU or Network Mat.)

Conversion 가능.

② Max flow Problem (with Integral ~~cost~~ bounds) ⇒ Integral polyhedron

Min Cost Max flow Problem (with Integral bounds) ⇒ $P = P_I$

Circulation polyhedron. (with Integral bounds) ⇒ $P = P_I$

(Net Mat ⇒ TU)

B229

"Bipartite Graph"

Def) A graph $G=(V,E)$ is called bipartite if $\exists$ a bipartition V_L and V_R of V s.t. every edge in E has one end pt in V_L and the other in V_R.

"Matching"

Def) Given a graph $G=(V,E)$,
We say, $M \subseteq E$ is a matching if every vertex in V is incident to $\boxed{\text{at most one}}^{*}$ $(=\{0,1\})$ edge in M

⊛ (If $G=(V,E)$ is bipartite, then M is a bipartite matching.)

(⊛⊛ Def) Given a graph $G=(V,E)$,
we say, $C \subseteq V$ is a vertex cover if every edge in E is incident to $\boxed{\text{at least one}}^{*}$ vertex in C)

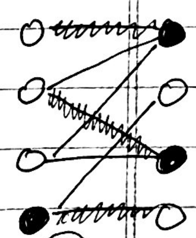
"matched" // "exposed"

Def) Given a graph $G=(V,E)$ and its matching M,
we say a vertex $v \in V$ is $\begin{cases} \text{matched} : \text{if } v \text{ is incident to an edge in } M \\ \text{exposed} : \text{O/W} \end{cases}$

[< Bipartite Matching of the Maximum Cardinality > → 여기선 # of edges $= |M|$

: Given a bipartite graph $G=(V_L \cup V_R, E)$, find a matching of the max cardinality]

e.g.)



G_1

Sol $= 3$

⊛ {Sol ≤ 4} (V_R)

proof) Let M^* be a max matching. Consider V_R, every edge in E is incident to a vertex in V_R. In particular, every edge in M^* is incident to a vertex in V_R.

- Now, "assign" every edge e in M^* to a vertex in V_R that is incident to e.
- We know that every edge can be assigned.
- On the other hand, no two edges in M^* can be incident to one vertex.

⇒ Therefore, Sol $= |M^*| \leq |V_R|$

⊛ {Sol ≤ 3} (V_{Blue}) ⇒ vertex cover ($|V_{Blue}| = 3$)

Lemma] Given a graph $G=(V,E)$, Let M be an arbitrary matching and S be an arbitrary vertex cover.
Then, we have $|M| \leq |S|$

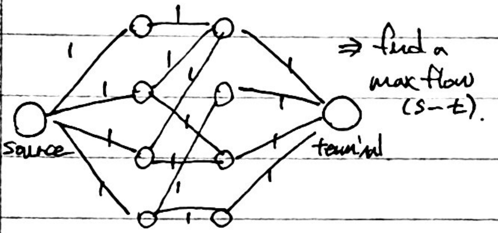
proof) Every edge in M is incident to a vertex in S.
Moreover, no two edges in M are incident to the same vertex.
This shows that every edge in M can be assigned to a vertex in S so that

Thm) König's theorem

[Given a bipartite graph $G=(V,E)$, and max matching M^* and min vertex cover C^*.
$\Rightarrow$ $|M^*| = |C^*|$]

proof) [Algorithmic Proof]

① Max flow algorithm

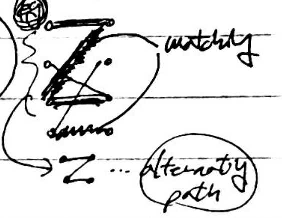


$\Rightarrow$ find a max flow $(s-t)$.

② Hungarian Method.

Def] Alternating Path

Given $G=(V,E)$ and a matching M.
[An alternating path with regard to M] is a path that alternates between edges in M and edges in $E \setminus M$



Def] Augmenting Path.

[An augmenting path with regard to M] is an alternating path whose first & last vertices are exposed.
$\Rightarrow$ 처음/끝 exposed + Alternating
(with regard to M)

[Algorithm]

$\{$ $M \leftarrow \emptyset$

While ($\exists$ Augmenting path w.r.t M) {
Let P be an arbitrary augmenting path
$M \leftarrow (M \setminus P) \cup (P \setminus M) = M \triangle P$
}
return M]

<Correctness of the Algorithm>

① Lemma) The algorithm returns a matching

Proof) Induction on # of iterations.

Note that $M' := (M \setminus P) \cup (P \setminus M)$ is a matching iff ① M is a matching and ② P is an augmenting path w.r.t M.

$\Rightarrow$ Every vertex is incident to the same number of edges in M and M' except for the first & last vertices of P

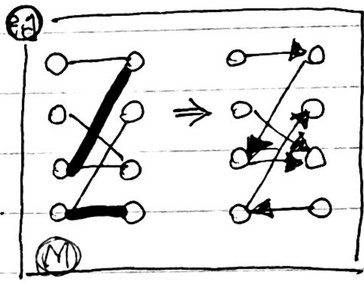
$\Rightarrow$ These two vertices are respectively incident to one edge in M',
moreover, $|M'| = |M| + 1$, establishing the algorithm's termination.

② Lemma) Given a matching M, we can find an augmenting path w.r.t M in polytime

proof) Given a directed graph obtained from G. by orienting its edges.

(For each edge $(x,y) \in E$, where $x \in V_L$ and $y \in V_R$

· we orient the edge $\begin{cases} \text{from } x \text{ to } y & \text{if } (x,y) \notin M \\ \text{from } y \text{ to } x & \text{if } (x,y) \in M. \end{cases}$



Now, every alternating w.r.t M is a directed path in the constructed digraph, and vice versa.

We can therefore, find an augmenting path by testing reachability between exposed vertices.

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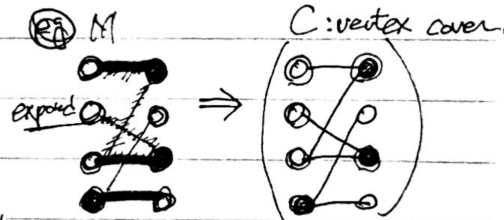
ex Let X be the set of vertices that can be reached from an $\boxed{\text{exposed vertex in } V_L}$ at the end of the algorithm.

$\Rightarrow$ Let $C^* = \underbrace{(V_L \setminus X)}_{\substack{V_L \text{ 중에서} \\ X\text{에서 Not} \\ \text{reachable}}} \cup \underbrace{(V_R \cap X)}_{\substack{V_R \text{ 중에서} \\ X\text{에서} \\ \text{reachable.}}}$



$\Rightarrow$ and M^* be the returned Matching of the algorithm.

③ Lemma) C^* is a vertex cover and we have $|C^*| = |M^*|$

proof) Suppose towards a contradiction that C^* is not a vertex cover.

Then, $\exists$ an edge $(x,y) \in E$ such that $\underbrace{x \in (V_L \cap X)}_{\text{"}V_L \text{ 중에서 exposed에서 reachable"}}$ and $y \in (V_R \setminus X)$

Since $x \in (V_L \cap X)$: V_L 중에서 reachable

$\Rightarrow \underbrace{(x \text{ itself is exposed})}_{\text{Case ①}}$ or $\underbrace{(\exists \text{ some } z \in (V_R \cap X) \text{ such that } (x,z) \in M)}_{\text{Case ②}}$ → therefore, $z \neq y$ any z.

(역. ← direction)

Case ① : since x is exposed $\Rightarrow$ $(x,y) \notin M$

Case ② : since M is a matching & $(x,z) \in M$ & $z \notin (V_R \setminus X)$ & $y \in (V_R \setminus X)$

$\Rightarrow$ x cannot be selected more than once.

$\Rightarrow$ $(x,y) \notin M$

This, however, implies that the digraph contains $\overrightarrow{\langle x,y \rangle}$ $\Rightarrow$ "y is reachable from an exposed vertex"

Contradicting $y \in (V_R \setminus X)$. (or $y \notin X$)

⊗⊗ to show the 2nd part of the proof $(|C^*| = |M^*|)$

(∵ we already showed that $|M| \leq |C|$ for arbitrary M & C)

we show that $|C^*| \leq |M^*|$

① by def of X, no vertex v is exposed if $v \in (V_L - X)$

(if exposed then $v \in X$)

② No vertex $v \in (V_R \cap X)$ is exposed

V_R 부터 V_L exposed vertex 로부터 reachable.

if exposed v exists then, ∃ augmenting path. ⇒ algorithm would not have terminated.

③ No edge $e \in M$, $e = (x, y)$ and $x \in (V_L \setminus X)$, $y \in (V_R \cap X)$

since $C^* = (\underbrace{V_L \setminus X}_{\text{by ①: not exposed}}) \cup (\underbrace{V_R \cap X}_{\text{by ②: not exposed}})$

⇒ C^* : every vertex **matched**

by ③ ∄ $(V_L \setminus X) \leftrightarrow (V_R \cap X)$ matching edge.

- $(V_L \setminus X)$ 모든 vertex 가 matching 에 사용됨
- $(V_R \cap X)$ 〃

⇒ $|C^*| \leq |M^*|$

∴ $|M^*| = |C^*|$

Thm) An LP can be solved in poly-time ⊗

- simplex method
- Ellipsoid Method (⇒ guarantees.)

[Min-cost bipartite matching]

(Def) We say a matching M is perfect if every vertex is matched ($|V_L| = |V_R|$)

[Problem] Given a complete bipartite graph $G = (V_L \cup V_R, E)$ and its edge costs $c : (V_L \times V_R) \to \mathbb{R}$, find a perfect matching M of G that minimizes

(Min) $\displaystyle\sum_{(u,v)\in M} c(u,v)$

⇒ (LP Relaxation)

[P] Minimize $\displaystyle\sum_{(u,v)\in(V_L \cup V_R)} c(u,v)\, x_{uv}$

subject to

$$\begin{cases} \displaystyle\sum_{v\in V_R} x_{uv} = 1 & \forall u \in V_L \quad \text{VL의 어떤정점은 VR중 딱하나랑!} \\ \displaystyle\sum_{u\in V_L} x_{uv} = 1 & \forall v \in V_R \quad \text{VR의 어떤정점은 VL중 딱하나랑!} \\ x_{uv} \geq 0 & \forall (u,v) \in V_L \times V_R \end{cases}$$

[D] Maximize $\displaystyle\sum_{u\in(V_L \cup V_R)} y_u$

Subject to

$$\begin{cases} y_u + y_v \leq c(u,v) & \text{for } \forall (u,v) \in (V_L \times V_R) \\ y \text{ free.} \end{cases}$$

※ Remark] It is easy to verify that the cost of a feasible solution to [D] is a lower bound on min cost of a bipartite matching.

(Thm) The feasibility region of [P] is integral.

Proof] We design an algorithm that finds an integral opt sol to [P]

The (alg) maintains a dual feasible solution throughout the entire execution of (Alg)

Initially, $y_u \leftarrow 0 \quad \forall u \in V_L$

$y_v \leftarrow \min_{u\in V_L} c(u,v) \quad \forall v \in V_R$.

$E^* := \{(u,v) \mid y_u + y_v = c(u,v)\}$: tight constraints.

[If E^* has a perfect matching (primal feasible) → Done

O/W Fix a maximum cardinality matching M and let X be the set of vertices reachable in the digraph from an exposed vertex in V_L

$C^* := (V_L \setminus X) \cup (V_R \cap X)$

E^* does not have an edge between $(V_L \cap X)$ and $(V_R \setminus X)$

This implies that $\delta := \min_{\substack{u\in V_L \cap X \\ v\in(V_R \setminus X)}} c(u,v) - y_u - y_v > 0$: not tight.

Now let • $y_u \leftarrow \begin{cases} y_u & \text{if } u \in V_L \setminus X \\ \underline{y_u + \delta \ \text{ if } u \in V_L \cap X} \end{cases}$ • $y_v \leftarrow \begin{cases} y_v & \text{if } v \in V_R \setminus X \\ y_v - \delta & \text{if } v \in V_R \cap X \end{cases}$

중요

Note that the new dual solution is also feasible.

$\Rightarrow \cdot\ \tilde{y}_u + \tilde{y}_v = y_u + y_v \qquad \leq c(u,v)$

$\cdot\ \tilde{y}_u + \tilde{y}_v = (y_u + \delta) + (y_v - \delta) \leq c(u,v)$

$\cdot\ \tilde{y}_u + \tilde{y}_v = (y_u) + (y_v - \delta) \leq c(u,v)$

$\star \cdot\ \tilde{y}_u + \tilde{y}_v = (\underbrace{y_u + \delta}_{u \in V_L \cap X}) + \underbrace{y_v}_{v \in V_R \setminus X}$ 이때, $\delta = \min_{\substack{u \in V_L \cap X \\ v \in V_R \setminus X}} c(u,v) - y_u - y_v > 0.$

$\leq c(u,v) \qquad \Rightarrow y_u + y_v + \delta = \min c(u,v)$

∴ 새로운 해도 feasible.

⊛ Note that the dual solution is also feasible & its value is strictly larger than the old one ☆.

difference : $\delta(|V_L \cap X| - |V_R \cap X|) = \delta(\underbrace{|V_L \cap X| + |V_L \setminus X|}_{=|V_L|} - (|V_L \setminus X| + |V_R \cap X|))$

$= \delta(|V_L| - |C^*|)$

$= \delta(\frac{n}{2} - |C^*|)$ since $\frac{2}{n} > |C^*|$ and $\delta > 0$

$> 0.$

⊛ Note that every edge in M remains in E^*. In order for an edge (u,v) to leave E^*

$\left[\begin{array}{l} u \in (V_L \setminus X) : \text{not reachable} \\ v \in (V_R \cap X) : \text{reachable.} \end{array}\right.$ must hold. (before the modification).

However, since $\langle v,u \rangle$ is in the digraph, this condition cannot be satisfied.
Thus, M remains a valid matching in the new E^* as well.

If E^* now has a matching of larger cardinality, we fix it as the new matching and repeat.

O/W Let $\left(\begin{array}{l} X_0 : X \text{ before Mod} \\ X_1 : X \text{ after mod} \end{array}\right)$ claim that $|V_R \cap X_0| < |V_R \cap X_1|$

This show that the entire algorithm terminates and successfully finds a perfect matching along with its optimality certificate in the form of a dual solution

⊛ Observe that the only edge that can leave E^* are in the form $(u,v) \left[\begin{array}{l} u \in V_L \setminus X_0 \\ v \in V_R \cap X_0 \end{array}\right]$

This only deletes $\langle u,v \rangle$ from the digraph, which does not affect reachability ☆ $(\because (u,v) \notin M)$

Moreover, at least one edge (u,v) with $\begin{array}{l} u \in (V_L \cap X_0) \\ v \in (V_R \setminus X_0) \end{array}$

newly enters E^*, introducing $\langle u,v \rangle$ making $v \in (V_R \cap X_1)$

$\therefore |V_R \cap X_0| < |V_R \cap X_1|$

$\Rightarrow$ This Alg finds integral opt solution to [P] ∴ P has P_I.