"Primal-Dual Algorithm 의 의의"

Thm) Let $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$ be matroids.

r_1, r_2 be corresponding rank functions.

The convex combination of the incidence vectors of the common independent sets of cardinality k is given by the following:

$$
\circledast \begin{cases} \sum_{e \in U} x_e \leq r_1(U) \quad \forall U \subseteq S \quad (1) & y_1^U \\ \sum_{e \in U} x_e \leq r_2(U) \quad \forall U \subseteq S \quad (2) & y_2^U \\ \sum_{e \in S} x_e = k \quad (3) & z \\ x_e \geq 0 \quad \forall e \in S \quad (4) & \end{cases}
$$

Dual Vars: y_1^U, y_2^U, z

(*) Assume that k is no greater than the maximum cardinality of a common independent set.

Proof) • It suffices to prove that $\circledast$ defines an integer polyhedron.

• For an arbitrary cost function $c: S \to \mathbb{R}$, Suppose that we run the algorithm with c.

• Let I_k be the solution returned by the algorithm. Then the incidence vector of I_k is a feasible solution to the following LP:

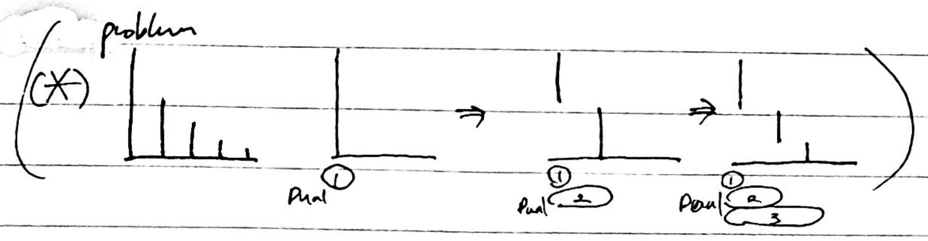
$\Rightarrow$ maximize $\sum_{e \in S} c(e) x_e$ subject to $\circledast$ [P]

and the value of the solution is $\sum_{e \in I_k} c(e)$

• Moreover, from the proof of "Correctness of the alg." (Lemma 6)

$\exists c_1, c_2$ such that $\begin{cases} c_1(I_k) = \max\{c_1(I) \mid I \in \mathcal{I}_1, |I| = k\} \\ c_2(I_k) = \max\{c_2(I) \mid I \in \mathcal{I}_2, |I| = k\} \end{cases}$

and $c_1(e) + c_2(e) = c(e)$ for $\forall e \in S$.



- Thus, the incidence vector of I_k is an optimal solution to ① and ②

① Maximize $\sum_{e \in S} c_1(e) x_e$

subject to $\sum_{e \in u} x_e \le r_1(u)$ $\quad (y_u^1)$ $\quad \forall u \subseteq S$

$\sum_{e \in S} x_e = k$ $\quad (z^1)$

$x_e \ge 0 \quad \forall e \in S$

② Maximize $\sum_{e \in S} c_2(e) x_e$

subject to $\sum_{e \in u} x_e \le r_2(u)$ $\quad (y_u^2)$ $\quad \forall u \subseteq S$

$\sum_{e \in S} x_e = k$ $\quad (z^2)$

$x_e \ge 0 \quad \forall e \in S.$

$\Downarrow$

①$^{\text{Dual}}$ Minimize $\sum_{u \subseteq S} r_1(u) y_u^1 + k z^1$

subject to $\sum_{\substack{u \subseteq S \\ e \in u}} y_u^1 + z^1 \ge c_1(e)$ for $\forall e \in S$

$y_u^1 \ge 0 \quad \forall u \subseteq S$

z^1 free.

②$^{\text{Dual}}$ Minimize $\sum_{u \subseteq S} r_2(u) y_u^2 + k z^2$

Subject to $\sum_{\substack{u \subseteq S \\ e \in u}} y_u^2 + z^2 \ge c_2(e)$ for $\forall e \in S$

$y_u^2 \ge 0 \quad \forall u \subseteq S$

z^2 free.

This implies that we have $\tilde{y}_u^1$ and $\tilde{z}^1$ satisfying ①$^{\text{Dual}}$ Constraints and $\sum_{u \subseteq S} r_1(u) \tilde{y}_u^1 + k \tilde{z}_1 = c_1(I_k)$

and have $\tilde{y}_u^2$ and $\tilde{z}_2$ satisfying ②$^{\text{Dual}}$ Constraints and $\sum_{u \subseteq S} r_2(u) \tilde{y}_u^2 + k \tilde{z}_2 = c_2(I_k)$

$\Rightarrow$ [D] Minimize $\sum_{u \subseteq S} r_1(u) y_u^1 + \sum_{u \subseteq S} r_2(u) y_u^2 + k z.$

subject to $\underbrace{z + \sum_{\substack{u \subseteq S \\ e \in u}} y_u^1 + \sum_{\substack{u \subseteq S \\ e \in u}} y_u^2 \ge c(e)}$ $\quad \forall e \in S.$

$y_u^1, y_u^2 \ge 0$. z free.

$\left(z_1 + \sum_{\substack{u \subseteq S \\ e \in u}} y_u^1\right) + \left(z - z_1 + \sum_{\substack{u \subseteq S \\ e \in u}} y_u^2\right) \ge c_1(e) + c_2(e)$

Observe that $(\tilde{y}_u^1, \tilde{y}_u^2, \tilde{z}_1 + \tilde{z}^2)$ is a feasible solution to [D]

and $\sum_{u \subseteq S} \left(r_1(u) \tilde{y}_u^1 + r_2(u) \tilde{y}_u^2 \right) + k(\tilde{z}_1 + \tilde{z}_2) = c_1(I_k) + c_2(I_k) = \underline{\underline{c(I_k)}}$

This shows that ⊛ defines an integer polyhedron. $\hookrightarrow$ so optimal.

(Def) For a matroid $M = (S, \mathcal{I})$ with rank function $r : 2^S \rightarrow \mathbb{N}$, let $P(M)$ denote its matroid polytope characterized as follows:

$$P(M) = \{ x \in \mathbb{R}^{|S|} \mid x \geq 0, \ \forall U \subseteq S, \ \sum_{e \in U} x_e \leq r(U) \}$$

(Def) Let $P(M_1 \wedge M_2)$ denotes a convex hull of the incidence vectors of the

(Thm) $P(M_1 \wedge M_2) = P(M_1) \cap P(M_2)$ (Regardless of k)

The convex hull of the incident vectors of the independence sets

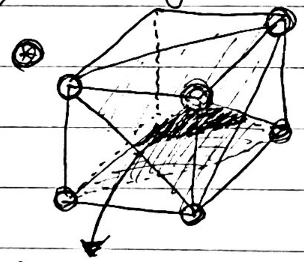
Lemma 1] Independent sets of cardinality at most k is given by

$$\{ x \in \mathbb{R}^{|S|} \mid x \geq 0, \sum_{e \in S} x_e \leq k, \sum_{e \in U} x_e \leq r(U) \text{ for } \forall U \subseteq S \}$$

Lemma 2] Independent sets of cardinality at least k is given by

$$\{ x \in \mathbb{R}^{|S|} \mid x \geq 0, \sum_{e \in S} x_e \geq k, \sum_{e \in U} x_e \leq r(U) \text{ for } \forall U \subseteq S \}$$

[{0} integral]
[{⊙} integral]



- Convex hull의 교점들
⇒ 이 경우, 교점들은 not integral
(ext pt 이 convex hull x)
- 그러나 Matroid는 X

Lemma 3] $\exists\, C_1$ and C_2 such that (where $I_{\bar{k}}$ be the solution returned)

$\bar{k}$ ··· Maximum weight 를 만드는 cardinality

$$C_1(I_{\bar{k}}) = \max \{ C_1(I) \mid I \in \mathcal{I}_1, \ |I| \leq \bar{k} \}$$
$$C_2(I_{\bar{k}}) = \max \{ C_2(I) \mid I \in \mathcal{I}_2, \ |I| \leq \bar{k} \}$$
and $C_1(e) + C_2(e) = C(e)$ for $\forall e \in S$

Lemma 4] $\exists\, C_1$ and C_2 such that

$$C_1(I_{\bar{k}}) = \max \{ C_1(I) \mid I \in \mathcal{I}_1, \ |I| \geq \bar{k} \}$$
$$C_2(I_{\bar{k}}) = \max \{ C_2(I) \mid I \in \mathcal{I}_2, \ |I| \geq \bar{k} \}$$
and $C_1(e) + C_2(e) = C(e)$ for $\forall e \in S$

Pf) It suffices to prove that $P(M_1) \wedge P(M_2)$ is an integer polyhedron.

Choose an arbitrary $C : S \to \mathbb{R}$. suppose run the algorithm with C.

(which returns best among $\{I_0 \sim I_{max}\}$)

k^* denote the final value of k and $I_{\bar{k}}$ be the solution returned by the algorithm.

proof of Lemma 3) Case 1) $\bar{k} = 0$ trivial.

Case 2) $\bar{k} \geq 1$.

Let $\bar{C}_1$ and $\bar{C}_2$ be the weight functions during the iteration $I_{\bar{k}}$ is computed from $I_{\bar{k}-1}$

Let $\bar{m}_1$ and $\bar{m}_2$ denote the values of m_1 and m_2 at the beginning of this iteration.

From the analysis of the algorithm,

$$C(I_{\bar{k}-1}) + \bar{m}_1 + \bar{m}_2 = C(I_{\bar{k}}) \geq C(I_{\bar{k}-1}) \quad (\text{since } I_{\bar{k}} \text{ is the best solution})$$

yielding $\bar{m}_1 + \bar{m}_2 \geq 0$.

Thus, for appropriate choice of constant δ

$$\Rightarrow \begin{cases} \tilde{C}_1(e) := C_1(e) + \delta \text{ and} \\ \tilde{C}_2(e) := C_2(e) - \delta \text{ for } \forall e \end{cases} \text{ ensures } \bar{m}_1 \geq 0 \text{ and } \bar{m}_2 \geq 0.$$

(defined w.r.t $\tilde{C}_1$ and $\tilde{C}_2$)

Moreover, this does not change the fact that $I_{\bar{k}-1}$ and $I_{\bar{k}}$ resp are max-weight independent sets of M_1 w.r.t. C_1

(and likewise in M_2 w.r.t C_2).

Now, observe that the desired conclusion follows from an execution of a greedy algorithm w.r.t. $\tilde{C}_1$ and $\tilde{C}_2$.

proof of Lemma 4) Case 1) $\bar{k} < k^*$

Consider the iteration where $I_{\bar{k}+1}$ is computed from $I_{\bar{k}}$

$\Rightarrow \bar{m}_1 + \bar{m}_2 \leq 0$.

Case 2) $\bar{k} = k^*$

(proof) suffices to prove that $P(M_1) \cap P(M_2)$ is integral polyhedron

Proof of Thm) $[\, P(M_1 \cap M_2) = P(M_1) \cap P(M_2) \,]$

best solution returned by the algorithm

- Let $c : S \to \mathbb{R}$ be an arbitrary function and x^* be the incidence vector of $I_{\bar{k}}$
- The convex hull of the incidence vectors of the independence set of cardinality at most $\bar{k}$ is given by $\{ x \in \mathbb{R}^{|S|} \mid x \geq 0, \sum_{e \in S} x_e \leq \bar{k}, \sum_{e \in U} x_e \leq r(U) \text{ for } \forall U \subseteq S \}$ (Lemma 1)

[P]

$\Rightarrow$ (LP)

$$\begin{aligned} &\text{Maximize } \sum_{e \in S} c(e) x_e \\ &\text{subject to } \begin{cases} \sum_{e \in U} x_e \leq r(U) & \forall U \subseteq S \\ \sum_{e \in S} x_e \leq \bar{k} \\ x_e \geq 0 & \forall e \in S \end{cases} \end{aligned}$$

[D]

$\Rightarrow$

$$\begin{aligned} &\text{Minimize } \sum_{U \subseteq S} r_i(U) y'_U + \bar{k} z' \\ &\text{subject to } \begin{cases} \sum_{U \subseteq S, e \in U} y'_U + z' \geq c_i(e) \\ y'_U \geq 0 & \forall U \subseteq S \\ z' \geq 0 \end{cases} \end{aligned}$$

- From Lemma 3, we can choose C_1^A, C_2^A such that
  $I_{\bar{k}}$ is the maximum weight independent set of cardinality at most $\bar{k}$ of M_1 w.r.t the cost function C_1^A
  (and of M_2 w.r.t the cost function C_2^A respectively)
  and $C_1^A(e) + C_2^A(e) = C(e)$ for $\forall e \in S$.
- From Lemma 1, there exists y^{1A}, z^{1A}, y^{2A} and z^{2A} such that

C_1^A에 대응 Optimal의 Dual solution

C_2^A에 대한 Optimal의 Dual solution

[1st Matroid]

(Minimized) $\sum_{U \subseteq S} r_1(U) y_U^{1A} + \bar{k} z^{1A} = \sum_{e \in S} C_1(e) x_e^*$

(1)

$$\text{s.t} \begin{cases} \sum_{U \subseteq S, e \in U} y_U^{1A} + z^{1A} \geq C_1^A(e) & \forall e \in S \\ y_U^{1A} \geq 0 & \forall U \subseteq S \\ z^{1A} \geq 0 \end{cases}$$

[2nd Matroid]

(Minimized) $\sum_{U \subseteq S} r_2(U) y_U^{2A} + \bar{k} z^{2A} = \sum_{e \in S} C_2(e) x_e^*$

(2)

$$\text{s.t} \begin{cases} \sum_{U \subseteq S, e \in U} y_U^{2A} + z^{2A} \geq C_2^A(e) & \forall e \in S \\ y_U^{2A} \geq 0 & \forall U \subseteq S \\ z^{2A} \geq 0 \end{cases}$$

$B \to$ at least F

(likewise)

- From Lemma 4, we can choose C_1^B, C_2^B such that
  I_F is the maximum weight independent set of cardinality at least F of M_1 w.r.t the cost function C_1^B
  (and M_2 w.r.t the cost function C_2^B)
  and $C_1^B(e) + C_2^B(e) = C(e)$ for $\forall e \in S$.
- From Lemma 2, there exists $\underbrace{y^{1B}, z^{1B}}_{C_1^B \text{el Dual Opt}}$, $\underbrace{y^{2B} \text{ and } z^{2B}}_{C_2^B \text{el Dual Opt.}}$
  such that

[1st Matroid]

(Minimized) $\sum_{u \subseteq S} r_1(u) y_u^{1B} + F z^{1B} = \sum_{e \in S} C_1(e) x_e^*$

(-λ)

st $\begin{cases} \sum_{u \subseteq S, e \in u} y_u^{1B} + z^{1B} \geq C_1^B(e) & \text{for } \forall e \in S \\ y_u^{1B} \geq 0 & \text{for } \forall u \subseteq S \\ z^{1B} \leq 0 & \end{cases}$

[2nd Matroid]

(Minimized) $\sum_{u \subseteq S} r_2(u) y_u^{2B} + F z^{2B} = \sum_{e \in S} C_2(e) x_e^*$

(-λ).

st $\begin{cases} \sum_{u \subseteq S, e \in u} y_u^{2B} + z^{2B} \geq C_2^B(e) & \forall e \in S \\ y_u^{2B} \geq 0 & \forall u \subseteq S \\ z^{2B} \leq 0 & \end{cases}$

- Now we write the dual of :

(P) $\left[\begin{array}{ll} \text{Maximize} & \sum_{e \in S} C(e) x_e \\ \text{s.t} & x \in P(M_1) \cap P(M_2) \end{array}\right]$

- Maximize $\sum_{e \in S} C(e) x_e$

  s.t $\begin{cases} \sum_{e \in u} x_e \leq r_1(u) & (y_u^1) \quad \text{for } \forall u \subseteq S \\ \sum_{e \in u} x_e \leq r_2(u) & (y_u^2) \\ x_e \geq 0 & \forall e \in S \end{cases}$

(D) Minimize $\sum_{u \subseteq S} (r_1(u) y_u^1 + r_2(u) y_u^2)$

s.t $\begin{cases} \sum_{u \subseteq S, e \in u} (y_u^1 + y_u^2) \geq C(e) & \text{for } \forall e \in S \\ y_u^1, y_u^2 \geq 0 & \text{for } \forall u \subseteq S \end{cases}$

1

- (D) Minimize $\sum_{u \in S} r_1(u) y_u^1 + \sum_{u \in S} r_2(u) y_u^2$

$$\text{s.t.} \begin{cases} \sum_{u \in S, e \in u} y_u^1 + \sum_{u \in S, e \in u} y_u^2 \geq c_1(e) + c_2(e) & \text{for } \forall e \in S \\ y_u^1, y_u^2 \geq 0 & \text{for } \forall u \subseteq S. \end{cases}$$

개별 등호성을 합쳐서도 feasible

Since $z^{1A}, z^{2A} \geq 0$ and $z^{1B}, z^{2B} \leq 0$

by choosing an appropriate $\lambda \in [0, 1]$

and letting

$$\begin{cases} y^1 = \lambda y^{1A} + (1-\lambda) y^{1B} \\ y^2 = \lambda y^{2A} + (1-\lambda) y^{2B} \end{cases} \qquad \begin{cases} z^1 = \lambda z^{1A} + (1-\lambda) z^{1B} \\ z^2 = \lambda z^{2A} + (1-\lambda) z^{2B} \end{cases}$$

We have $\lambda (\underbrace{z^{1A} + z^{2A}}_{\text{non negative}}) + (1-\lambda)(\underbrace{z^{1B} + z^{2B}}_{\text{non positive}}) = 0$

- $\sum_{u \in S} r_1(u) y_u^1 + \sum_{u \in S} r_2(u) y_u^2 + k z^1 + k z^2 = 0$

$$= \sum_{u \in S} r_1(u) \left(\lambda y_u^{1A} + (1-\lambda) y_u^{1B} \right) + \sum_{u \in S} r_2(u) \left(\lambda y_u^{2A} + (1-\lambda) y_u^{2B} \right)$$
$$+ k \lambda z^{1A} + k(1-\lambda) z^{1B} \qquad + k \lambda z^{2A} + k(1-\lambda) z^{2B}$$

$$= \lambda \left(\sum_{u \in S} r_1(u) y_u^{1A} + k z^{1A} \right) + (1-\lambda) \left(\sum_{u \in S} r_1(u) y_u^{1B} + k z^{1B} \right)$$
$$+ \lambda \left(\sum_{u \in S} r_2(u) y_u^{2A} + k z^{2A} \right) + (1-\lambda) \left(\sum_{u \in S} r_2(u) y_u^{2B} + k z^{2B} \right)$$

$$= \lambda \sum_{e \in S} c_1(e) x_e^* + (1-\lambda) \sum_{e \in S} c_1(e) x_e^*$$
$$+ \lambda \sum_{e \in S} c_2(e) x_e^* + (1-\lambda) \sum_{e \in S} c_2(e) x_e^*$$

$$= \sum_{e \in S} c_1(e) x_e^* + \sum_{e \in S} c_2(e) x_e^* = \sum_{e \in S} c(e) x_e^*$$

(*) maximized value of $\sum_{e \in S} c(e) x_e$ s.t. $x \in P(M_1 \cap M_2)$

This shows that x^* and (y^1, y^2) respectively are optimal solutions to [P] and [D] whose values are equal to $C(I_E)$ □.