Def A $\boxed{\text{hyperplane}}$ in $\mathbb{R}^n$ is the set of all pts $x \in \mathbb{R}^n$ that satisfies $a^T x = b$
for some $a \in \mathbb{R}^n$, $a \neq \vec{0}$ and $b \in \mathbb{R}$

Def A $\boxed{\text{half space}}$ in $\mathbb{R}^n$ is the set of all pts $x \in \mathbb{R}^n$ that satisfies $a^T x \leq b$ "

Def $\boxed{\text{Polyhedron}}$:= intersection of finite # of half spaces
$\boxed{\text{Polytope}}$:= bounded polyhedron.

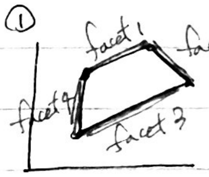
Def An inequality $a^T x \leq b$ $(a \neq \vec{0})$ is called $\boxed{\text{valid inequality}}$ for a polyhedron $P = \{x \mid Ax \leq b\}$
if for $\forall x \in P$ satisfy $a^T x \leq b$

Def $\{x \mid a^T x = b\}$ $(a \neq \vec{0})$ is called a $\boxed{\text{supporting hyperplane (접평면) of } P}$
if $a^T x \leq b$ is $\boxed{\text{valid}}$ for P ①
and the hyperplane has a nonempty intersection with P ②

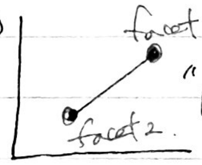
Def A $\boxed{\text{face}}$ of a polyhedron P := intersection of (P and a supporting hyperplane of P)
다면체 다면체의 접평면.

Def A $\boxed{\text{facet}}$ of a polyhedron P := inclusion-wise maximal face.
which is distinct from P ☆

ex ①



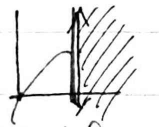
②



"P가 직선이라서" ⇒ facet ≠ P 이므로.

A face of P is called a facet iff dim(face F) = dim(polyhedron P) − 1

Def A face of P is said to be $\boxed{\text{minimal}}$ if it does not contain any other face.
⊛ 일반적으로는 pt가 minimal,



⊛ Given a rational polyhedron P, Let P_I denotes the convex hull of all integer vectors in P

→ Obs For a rational polyhedron P, P_I is also a polyhedron.

Def We say a rational polyhedron P is an integer polyhedron if $P = P_I$.

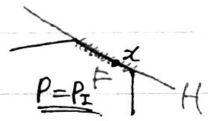
(Thm) TFAE (1) $P = P_I$

(2) Every face of P has an integer vector.

(3) Every minimal face of P has an integer vector.

(4) if $\max\{c^T x \mid x \in P\}$ is finite, it is attained by an integer vector x_I.

proof) (1) → (2) Let F be a face. Then, by def $F = P \cap H$ for a supporting hyperplane H.
Let x be an arbitrary pt in F. since $P = P_I$, $F = P_I \cap H$ and x is a convex combination of integral pts in P.

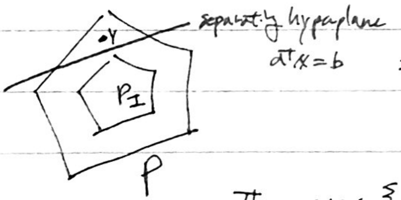


(2) → (3) Trivial

(3) → (4) Let $f = \max\{c^T x \mid x \in P\}$

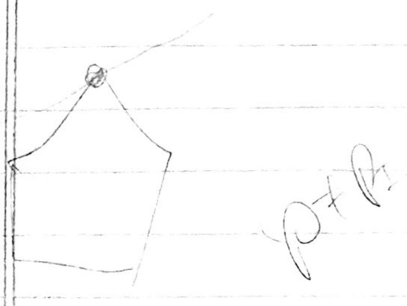
Then, $F = \{x \in P \mid c^T x = f\}$ is a face of P, which has an integer vector.

(4) → (1) Suppose towards contradiction that $\exists\ y \in P \setminus P_I$ (i.e. y not integral).
Then, there is a separating hyperplane between y and P_I.



$\Rightarrow \exists\ a$ and b s.t. $a^T x \leq b$ is valid for $\forall x \in P_I$
and $a^T y \geq b$

Then, $\max\{a^T x \mid x \in P_I\} \leq b$
whereas $\max\{a^T x \mid x \in P\} \geq b$ contradicting (4).
(not attained by an int vector).



For $A \in \mathbb{R}^{n\times n}$

Def $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$

⊗ 1st row fixed calculation

$\Rightarrow \det(A) = \sum_{i=1}^{n} (-1)^{i-1} a_{1i} \det A_{1i}$

Where (A_{ij} denote the submatrix A obtained by deleting ith row and jth column of A)

e.g. $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \end{bmatrix}\right) = a_{11}\cdot\det(\ ) - a_{12}\cdot\det(\ ) + a_{13}\cdot\det(\ )$

Thm If $A \in \mathbb{R}^{n\times n}$ is an invertable matrix, then we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) & \cdots \\ -\det(A_{12}) & \det(A_{22}) & \cdots & \\ \det(A_{13}) & & \ddots & \\ \vdots & & & \end{bmatrix}$$

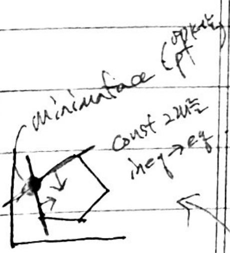
Def A matrix A totally unimodular (TU) (⊗ A가 square 일 필요 X)

if the determinant of every square submatrix of A is in $\{-1, 0, 1\}$

→ ⊗ Every TU matrix has only $\{0, -1, +1\}$ entries. (The opposite is not true).

Lemma) If A is TU, Then, for every nonsingular square submatrix A' of A

$\Rightarrow (A')^{-1}$ is integral.



Thm) If A is TU, then the polyhedron $P = \{x \mid Ax \le b\}$ is an integer polyhedron for ∀ integral vector b

⊗⊗ LP에 대해서 Matrix가 TU ⇒ Then $P = P_I$

proof) Consider an arbitrary minimal face of P. : F.

⊗ We have $F = \{x \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $(Ax = b)$

↳ (Use this fact without proof)

(WLOG), we can assume that A' has full row rank. ☆

Then by permuting rows & columns, we obtain $A' = [U, V]$

where U is a square nonsingular matrix.

e.g. $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

↳ nonsingular. U

Since U is a submatrix of A, and $\det(U) = +1$ or -1

$\Rightarrow x := \begin{pmatrix} U^{-1}b' \\ \vec{0} \end{pmatrix}$ is an integral vector $A'x = [U, V]\begin{bmatrix} U^{-1}b' \\ 0 \end{bmatrix} = b'$

$\Rightarrow x \in F$.

Since Minimal Face (F) of P is integral $\Rightarrow$ (1) $P = P_I$

Total Unimodularity Preserved under following operations.

① Transpose.

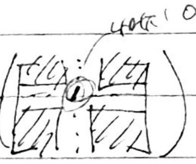
- If A is TU, so is A^T (proof) every submatrix of A^T : A'' is a transpose of a submatrix of A $\Rightarrow A'' = \text{trans}(A')$

② Augmenting with the identity I since $\det(A) = \det(A^T) \Rightarrow$ Total Unimodularity Preserved

(e.g.) $\begin{bmatrix} A \\ I \end{bmatrix}$ is also TU

proof) for a square submatrix of $A' = \begin{bmatrix} A \\ I \end{bmatrix}$, namely B

$\Rightarrow$ (i) B is consist of parts of A ---- ok.

(ii) B contains an element vector $\Rightarrow$ det  $= \pm\det(\text{▨})$.

$\therefore$ TU preserved.

③ Multiplying a row or column by -1.

④ Interchanging 2 rows or columns.

⑤ Duplication of rows or columns.

(Corr) Let A be a TUM, and let b and c be integral vectors,
Then, both problems in the LP-duality equation.

$$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0 \text{ and } yA = c\}$$

(P) (D)

have both integral optimum solutions.

proof) $A' = \begin{bmatrix} I \\ A^T \\ -A^T \end{bmatrix}$ is also totally unimodular.

$\Rightarrow P = \{ \mid A'x \leq b' \}$ .. integral.

Cor (Hoffman & Kruskal's Theorem)

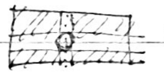
- Let A be an integer matrix. then A is TU **iff** for each integral b, $\{x \mid x \geq 0, Ax \leq b\}$ is integral.

proof) ← integral matrix.

- $\underset{m\times n}{A}$ is TU **iff** matrix $\underset{m\times(m+n)}{[I\,A]}$ is unimodular.

proof) • if A is TU.

square submatrix with rank $= m$ is $\det(\underset{m\times m}{A'}) \in \{-1, 0, 1\}$.

A'에 대해 하나의 column을 elementary vector로 교체하면.  $\Rightarrow \det(\ ) \in \{-1, 0, 1\}$.

• if $[I, A]$ is unimodular. ... trivial.

def $\underset{m\times n}{A}$ 에 대해 $\mathrm{rank}(A) = m$ & every square submatrix of A $\underset{m\times m}{A'}$ 에 대해 $\det(A') \in \{0, 1, -1\}$

$\Rightarrow$ A is unimodular.

& For any integral vector b, the vertices of the polyhedron $\{x \mid x \geq 0; Ax \leq b\}$ are integral if & only if " $\{x \mid x \geq 0; [I\ A]z = b\}$ are integral.

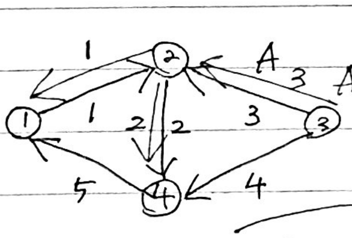
$\Rightarrow$ ~~D.~~ <not completed ※> proof.

Def A network matrix is defined by a directed graph $G = (V, A)$ and a directed tree $T = (V, A')$ on the same set of vertices V.

$\Rightarrow$ The matrix is an $|A'| \times |A|$ matrix whose (a', a)th entry is given by

$$M_{a',a} = \begin{cases} 1 & \text{unique path from } u \text{ to } v \text{ through } a' \text{ forward} \\ -1 & \text{" } a' \text{ backward} \\ 0 & \text{" not through } a' \end{cases}$$

eg



graph edge

$$\Rightarrow \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1\to 2 & 2\to 4 & 3\to 2 & 3\to 4 & 4\to 1 \end{array} \\ \begin{array}{c} \text{tree edge} \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(Thm) Every network matrix is totally unimodular. (Tutte 정리)

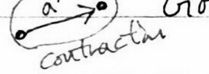
(Lemma) Let S be an arbitrary square submatrix of a network matrix M.
Then either $\det(S) = 0$ or S is a network matrix (or both).

proof of lemma)

① deleting a column corresponding to $a \in A$ from M.
$\Rightarrow$ $G \setminus a := \{V, A \setminus \{a\}\}$ and T ... results a network matrix

② " a row " $a' \in A'$ form M.

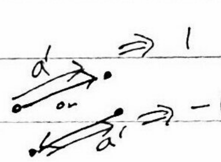
②-1> Suppose G does not have an arc between the same endpoint as that of a' (i.e. $\mathrm{und}(a') \notin \mathrm{und}(G)$).

(∵ 

G에 없는 edge 이므로 care X, contraction 해버리면 고민 X.
$\Rightarrow$ results a network matrix.

②-2> Suppose $\exists$ a column corresponding to an arc with the same end points as a'.

(∵ Observe that this column in M contains 0 only.
Except for the row corresponding to a'
$\Rightarrow$ row deletion ... the column is $\vec{0}$.
$\Rightarrow$ $\det(S) = 0$.



From the lemma, it suffices to show any square network matrix $C \Rightarrow \det(C) \in \{0, -1, 1\}$
($\det(S) = 0$ 이고 square submatrix 또는 network matrix 이므로).

Let k be the size of the matrix $M \in \mathbb{R}^{k \times k}$

(i) $k = 1$ ... trivial

(ii) $k \geq 2$. choose an arbitrary leaf node u of T
and let a' be the arc incident with u
(WLOG, Assume a' is from u) ※ reorientation $\Rightarrow$ det 의 부호만 바뀜.

Now, consider the set of arcs in G that are incident with u.
(Again, WLOG, we assume that all these arcs are incident from u)

$a_1 = \langle u, v_1 \rangle$
$a_2 = \langle u, v_2 \rangle$
$\vdots$
$a_h = \langle u, v_h \rangle$

if $h = 0 \Rightarrow \det(M) = 0$

if $h \geq 2$. Consider substracting the column of M corresponding to $a_1 \in A$ from the column corresponding to a_2

The resulting matrix M' satisfies $\det(M) = \det(M')$

since, $M' = M \cdot \underline{A}$ → 대각 1행열에서 어딘가에 0대신 -1이 있음.

Moreover, we have that M' is a network matrix defined by

$G' = (V, A \setminus \{a_2\} \cup \{\langle v_1, v_2 \rangle\})$ and T

[illegible] ??

Let w be the last common vertex of the unique u-v_1 path and the u-v_2 path. We have $m'_{a^*, a_1} = m_{a^*, a_2}$ if a^* is on the w-v_2 path.

$$
M'_{a^*, \langle v_1, v_2 \rangle} = \begin{cases}
m'_{a^*, \langle v_1, v_2 \rangle} = m_{a^*, a_2} = m_{a^*, a_2} - M_{a^*, a_1} & \text{if } a^* \text{ is on the } w\text{-}v_2 \text{ path} \\
m'_{a^*, \langle v_1, v_2 \rangle} = -m_{a^*, a_1} = m_{a^*, a_2} - m_{a^*, a_1} & \text{if } a^* \text{ is on the } w\text{-}v_1 \text{ path.} \\
m'_{a^* \langle v_1, v_2 \rangle} = 0 & \text{o.w}
\end{cases}
$$

By repeating this procedure, for $a_1 \cdots a_n$, we obtain a matrix M'' s.t $\det(M'') = \det(M)$ and M'' is defined by T and $\tilde{G}$ where $\tilde{G}$ has only one arc incident with u.

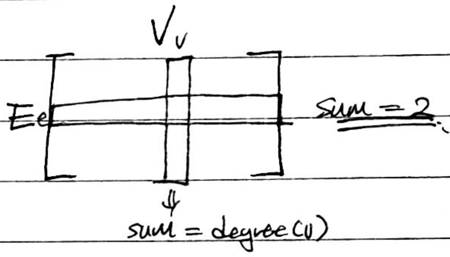
Thus, the row of M'' corresponding to a' contains only zeroes except for a single entry of ± 1.

This shows that $\det(M'') = \pm \det(M''')$ for a submatrix M''' of M'' which is a network matrix itself (or $\det M''' = 0$) from I.H $\square$.

[Obs] If A is TU, so is
① $A' = \begin{bmatrix} A \\ I \end{bmatrix}$
② $A' = \begin{bmatrix} A \\ A \end{bmatrix}$
③ $A' = \begin{bmatrix} A \\ -A \end{bmatrix}$
④ A^T

Lemma) For a bipartite graph $G = (V_L \cup V_R, E)$
$\Rightarrow$ Consider its incidence matrix.

$A = \{a_{ev}\}_{e \in E,\ v \in (V_L \cup V_R)}$



$\begin{cases} a_{ev} = 1 & \text{if } e \text{ is incident to } v \\ a_{ev} = 0 & \text{O/W} \end{cases}$

Then A is totally Unimodular.

Remark) ⊛ Our LP of bipartite matching

$v\,[\ {}^{E}\ ]$ = incidence matrix$)^T$

$\Rightarrow$ maximize $\sum_{e \in E} x_e$

subject to $\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V_L \cup V_R \quad \forall e \in E$

$x_e \geq 0 \quad \forall e \in E.$

Where $G = (V_L \cup V_R, E)$ is bipartite.

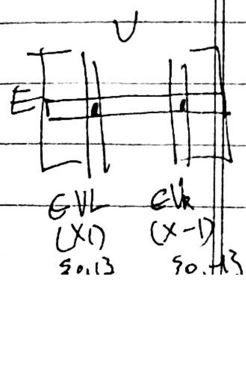
따라서 A가 TU ⇒ 이 LP는 $P = P_I$

proof) Consider an arbitrary square submatrix
$S \in \{0,1\}^{m \times m}$ of A

Observe that every row have at most 2 ones.

Case 1) If S has a row with zeros. $\Rightarrow \det(S) = 0$.
Case 2) 〃 with only one 1 $\Rightarrow \det(S) = \det(S')$
(S' is a smaller submatrix)

Case 3) Every row has exactly 2 ones.
For each row, one of the two 1s lies on a column corresponding to a vector in V_L
and the other 1 in V_R. thus by multiplying
[① to every column corresponding to a vertex in V_L
〃 ⊖1 〃 in V_R)



We obtain a matrix whose column sum to a zero vector
$\Rightarrow \det(S) = 0$

Thm For a given bipartite graph $G = (V_L \cup V_R, E)$, the maximum cardinality of its matching equals to the optimal value of the following LP.

$$[P] \quad \text{Maximize} \sum_{e \in E} x_e$$
$$\text{s.t.} \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in (V_L \cup V_R)$$
$$x_e \geq 0 \quad \text{for } \forall e \in E$$

proof) Observe that the coefficient matrix of $[P]$ is $\begin{bmatrix} A^T \\ -I \end{bmatrix}$ where A is the incidence matrix of G.

(From the observation ① and ④) of total unimodality theorem &.

$\Rightarrow \begin{bmatrix} A^T \\ -I \end{bmatrix}$ is totally unimodular.

$\Rightarrow$ Polyhedron of $[P] = P_I$. ... has integral optimal solution

Moreover, note that any integral feasible solution x^* to $[P]$ can be interpreted as a valid bipartite matching of cardinality $\sum_{e \in E} x_e^*$

Thm The feasible region of $[P]$ is an integral polyhedron

proof) $\begin{bmatrix} A^T \\ -I \end{bmatrix}$ is totally unimodular $\Rightarrow P = P_I$.
since A is T.U.

Remark A maximum matching itself (& it cardinality) can be retrieved from the LP $[P]$ in poly time.

Lemma If A is totally unimodular, then for any integral vector b, $\{x \mid Ax \leq b, \underbrace{x \geq 0}_{= -x \leq 0}\}$ is integral.

proof) $\begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ 0 \end{bmatrix} \Rightarrow$ since $\begin{bmatrix} A \\ I \end{bmatrix}$ is totally unimodular, its polyhedron is integral.

Lemma If A is totally unimodular, then for any integral vector a, b, c and d, $\{x \mid a \leq x \leq b, c \leq Ax \leq d\}$ is integral

proof)
$$\begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$
totally unimodular.

Thm For a totally unimodular matrix A and integral vectors b and c,

$\max\{c^T x \mid Ax \le b, x \ge 0\}$ has an optimal solution.

then, both $\left[\begin{array}{ll} \max\{c^T x \mid Ax \le b, x \ge 0\} & \text{: primal} \\ \min\{b^T y \mid A^T y \ge c, y \ge 0\} & \text{: dual} \end{array}\right.$

① have integral optimal solution

② and their values are the same : Strong Duality

Proof) From the strong duality of LP & the fact that A, A^T are both TUM.

Thm König's Theorem

given a bipartite graph $G = (V_L \cup V_R, E)$

(the maximum cardinality of matching
= the minimum cardinality of vertex cover)

proof 1) Let us write the dual of [P] ($\text{Max} \sum_{e \in E} x_e$ s.t $\sum_{e \in \delta(v)} x_e \le 1$, $x_e \ge 0$)

⊛ ⇒ [P] is Integral. and [D] is also integral

⇒ both have integral opt solution.

on the other hand ⇒ [D] ($\text{Min} \sum_{v \in (V_L \cup V_R)} y_v$ s.t $y_v \ge 0$ and $y_u + y_v \ge 1$ $\forall (u,v) \in E$)

integral &

edge 에 양끝 중 하나는 cover
⇒ edge cover.

⇒ vertex cover.

∴ proven.

proof 2] Algorithmic Proof

Given $G = (V_L \cup V_R, E)$

[P] Maximize $\sum_{e \in E} x_e$

s.t $\sum_{e \in \delta(v)} x_e \le 1$ for $\forall v \in V_L \cup V_R$ & $\forall e \in E$

$x_e \ge 0$ for $\forall e \in E$

[D] Minimize $\sum_{v \in (V_L \cup V_R)} y_v$

st $y_v \ge 0$ for $\forall v \in V$

$y_u + y_v \ge 1$ for $\forall (u,v) \in E$

We design an algorithm that finds an integral optimal solution of [P]

The algorithm, as a side product, finds an optimal solution of [D] as well.

⇒ The construction of the primal solution will be guided by the dual solution

⇒ Alg maintains a dual feasible solution.

... Throughout the entire execution of the alg

참 ⊛ Alg $M \leftarrow \phi$

while ($\exists$ Augmenting Path w.r.t M) {

let p be an arbitrary augmenting path.

$M \longleftarrow (M \setminus P) \cup (P \setminus M)$

}

return M (set of edges)

- Initially, we choose an arbitrary dual feasible solution

  ex) $y_u = 0$ for $\forall u \in V_L$

  $y_v = 1$ for $\forall v \in V_R$. (if $\exists (u,v)$) $\Rightarrow$ dual feasible

  o/w ($= 0$)

- Now Let E^* be the set of edges whose corresponding dual constraints are tight.

  i.e. ($y_u + y_v = 1$)

  $\Rightarrow E^* := \{ (u,v) \mid y_u + y_v = 1 \}$

- If E^* has a perfect matching (primal feasible)

  $\Rightarrow$ corresponding solution X is feasible

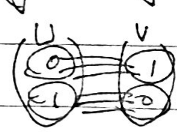
  $\Rightarrow$ by complementary slackness ... done.

  O/W We will modify the dual solution to admit new edges to E^*

  so that the maximum cardinality of a matching can be increased.

- For a matching with with max cardinality M on $(V_L \cup V_R, E^*)$ — ($y_u = 1$, $y_v = 0$ or $y_u = 0$, $y_v = 1$) 인정된 edge

  and let be the set of vertices that can be reached in the digraph.

  (Constructed by the alg to find an augmenting path from an exposed vertex in V_L)

- Let $C^* := (V_L \setminus X) \cup (V_R \cap X)$

  E^* does not have an edge between $(V_L \cap X)$ and $(V_R \setminus X)$

  $\therefore |C^*| = |M^*|$