<Matroid Intersection>

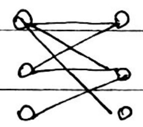
Given two matroids $M_1 = (S, \mathcal{I}_1)$ on the same groundset S.
$M_2 = (S, \mathcal{I}_2)$

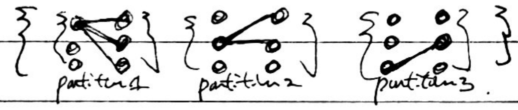
find a maximum weight (or cardinality) common independent set

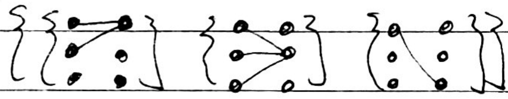
⊛ Matroid Intersection & Bipartite Matching

$S := E$ of G. $M_1 = (S, \mathcal{I}_1)$ where $\mathcal{I}_1 := \{I \mid |I \cap B_i| \leq 1 \text{ for } \forall i\}$.
$M_2 = (S, \mathcal{I}_2)$ where $\mathcal{I}_2 := \{I \mid |I \cap C_i| \leq 1 \text{ for } \forall i\}$

$\Rightarrow B_i :=$ 각 $v \in V_L$ 에 인접한 edge set
$\Rightarrow C_i :=$ 각 $u \in V_R$ 〃

ex) 

• Partition for M_1 = 

• partition for M_2 = 

$\Rightarrow$ Bipartite Matching ⋯ Matroid Intersection의 Special Case.

Lemma) If $X \subseteq Y \subseteq S$ then we have $span(X) \subseteq span(Y)$

proof) Consider an arbitrary element $z \in span(X)$

$\Rightarrow$ we have $r(X+z) = r(X)$ implying $r(Y+z) = r(Y)$

From the submodularity of $r(\cdot)$

Lemma) For $X \subseteq S$, $span(span(X)) = span(X)$

proof) • trivially $span(X) \subseteq span(span(X))$.

• since $\nexists\, y \in span(X) \setminus X$ such that $X + y \in \mathcal{I}$

we have $r(span(X)) = r(X)$

For $\forall z \in span(span(X))$, we have $r(span(X)+z) = r(span(X)) = r(X)$

$r(X) \leq r(X+z)$: monotonicity

$r(X+z) \leq r(span(X)+z)$

$\therefore \Rightarrow r(X) = r(X+z)$ ⋯ $z \in span(X)$

⊛ Let r_1 and r_2 be the rank function of M_1 and M_2 respectively.

For any $I \in (\mathcal{I}_1 \cap \mathcal{I}_2)$ and any $U \subseteq S$,

we have $|I| = |I \cap U| + |I \setminus U| \leq r_1(U) + r_2(S \setminus U)$

since $I \in \mathcal{I}_1 \Rightarrow (I \cap U) \in \mathcal{I}_1 \Rightarrow |I \cap U| = r_1(I \cap U) \leq r_1(U)$

$I \in \mathcal{I}_2 \Rightarrow T = S \setminus U$, $\because I \setminus U = I \cap T \in \mathcal{I}_2 \Rightarrow |I \cap T| = r_2(I \cap T) \leq r_2(T)$

$\Rightarrow |I \setminus U| \leq r_2(S \setminus U)$

$\therefore |I| = |I \cap U| + |I \setminus U| \leq r_1(U) + r_2(S \setminus U)$

Lemma 1 $\max_{I \in (\mathcal{I}_1 \cap \mathcal{I}_2)} |I| \leq \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U))$

$\Rightarrow$ **Thm** Matroid Intersection Theorem

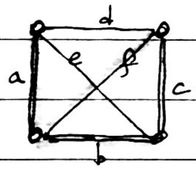
$$\max_{I \in (\mathcal{I}_1 \cap \mathcal{I}_2)} |I| = \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U))$$

Def) Given a matroid $M = (S, \mathcal{I})$ and an independent set $I \in \mathcal{I}$.

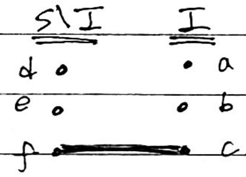
The exchange graph $G_M(I) = (V, E)$ is the following bipartite graph.

- $\Rightarrow$ bipartition given by $\begin{cases} V_L := S \setminus I \\ V_R := I \end{cases}$
- $\Rightarrow$ $(x, y) \in E$ iff $(I - y + x) \in \mathcal{I}$

ex Consider the graphic matroid defined by K_4



Let $I = \{a, b, c\}$ $\Rightarrow$



$\cdot (I - c + f) \in \mathcal{I}$.

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Lemma 2 [about Exchange Graph]

Let I and J be two independent sets with $|I| = |J|$, $(I \in \mathcal{I}, J \in \mathcal{I})$

Then, $G_M(I)$ has a (at least 1) (perfect matching between $J \setminus I$ and $I \setminus J$).

※ $\longrightarrow$ 아래를 가리킨.

proof) Induction on $|I \setminus J|$

if $|I \setminus J| = 0$: trivial. $\emptyset$ and $\emptyset$

- Suppose that $|I \setminus J| \geq 1$, Let x be an arbitrary element in $(J \setminus I)$.

Since $I \in \mathcal{I}$, $(J - x) \in \mathcal{I}$ and $|I| > |J - x|$

by def of matroid, $\exists\, y \in I \setminus (J - x)$ such that $(J - x + y) \in \mathcal{I}$.

Note that $y \in I \setminus J$ since $x \notin I$.

$\Rightarrow$ Let $J' := J - x + y$ and then we have $|I \setminus J'| < |I \setminus J| = k+1$

since $x \notin (J' \setminus I)$ and $y \notin (I \setminus J')$

$\Rightarrow$ by Induction Hypothesis, $|I \setminus J'| = k$ and $\exists$ perfect matching btw $(J' \setminus I)$ and $(I \setminus J')$ M

$\Rightarrow$ $(M + (x, y))$ yields to a perfect matching btw $(J \setminus I)$ and $(I \setminus J)$ in $G_M(I)$.

<Proposition> Consider the exchange graph $G_M(I)$ for some independent set I

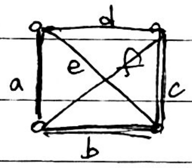
If $J \subseteq S$ is a set such that

$|I| = |J|$ and $G_M(I)$ has a perfect matching btw $(I \setminus J)$ and $(J \setminus I)$

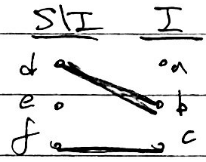
Then J is independent

• ⇒ False.

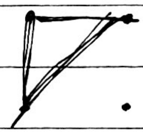
<Counter Example>



$I = \{a, b, c\}$



⇒ (J) $\{a, d, f\}$



⇒ $J \notin \mathcal{I}$.
not independent.

둘다. $\{a, d, c\} \in \mathcal{I}$.
$\{a, b, f\} \in \mathcal{I}$

Lemma 3) Consider the exchange graph $G_M(I)$ for some independent set I.
If $J \subseteq S$ is a set such that
$|I| = |J|$ and $G_M(I)$ has a $\boxed{\text{unique perfect matching}}$ btw $(I \setminus J)$ and $(J \setminus I)$
Then J is independent

Proof) Consider the subgraph G' of $G_M(I)$ induced by $I \Delta J = (J \setminus I) \cup (I \setminus J)$.
Let N be its unique perfect matching.

Orient the edges of G' as follows: every edge in N is oriented from $I \setminus J$ to $J \setminus I$. Note that G' is a DAG (unique perfect matching → acyclic) since O/W, N cannot be the unique perfect matching.

Now we claim that we can name the vertices in $I \Delta J$ so that $N = \{(x_1, y_1), \ldots, \ldots, (x_t, y_t)\}$ and every edge of $G' \setminus N$ is in the form (x_j, y_i) where $j > i$.

This ordering of vertices can be obtained from a topological order.
: Note that each $y \in I \setminus J$ has exactly one incoming arc and hence we can obtain a topological order when x_i and y_i are consecutive for $\forall (x_i, y_i) \in N$

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Suppose to a contradiction that $J \notin \mathcal{I}$.

Choose an arbitrary circuit : C in J

Let $x_i \in C$ be the element in $(J \setminus I) \cap C$ with the highest index.

(Note that $(J \setminus I) \cap C \neq \emptyset$ since o/w $C \subseteq I \in \mathcal{I}$)

For $\forall$ element in the circuit $z \in (C - x_i)$

- if $z \in J \setminus I$ then $z = x_j$ for some $j < i$
  and hence (x_j, y_i) cannot be in the exchange graph
  $\because C \notin G'$) and $x_j \in span(I - y_i)$
- if $z \in J$ then since $z \in J - y_i$
  we have $z \in span(I - y_i)$

hence we can obtain $(C - x_i) \subseteq span(I - y_i)$

On the other hand, since $(C - x_i) + x_i = C \notin \mathcal{I}$

we have $x_i \in span(C - x_i) \subseteq span(span(I - y_i))$
$= span(I - y_i)$

$\Rightarrow I - y_i + x_i \notin \mathcal{I}$ contradicting $(x_i, y_i) \in N$

$\therefore J$ is independent

Exchang Graph

Def) Given two matroids $\begin{cases} M_1 = (S, \mathcal{I}_1) \\ M_2 = (S, \mathcal{I}_2) \end{cases}$ and a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

the exchange graph $G_{M_1, M_2}(I)$ is the following directed bipartite graph.

$\Rightarrow \begin{cases} V_L := S \setminus I \\ V_R := I \end{cases}$ bipartition and. for $x \in V_L$ and $y \in V_R$

$V = V_L \cup V_R$

$\rightarrow \begin{cases} \langle \overrightarrow{y, x} \rangle \in A \text{ iff } I - y + x \in \mathcal{I}_1 \\ \langle \overrightarrow{x, y} \rangle \in A \text{ iff } I - y + x \in \mathcal{I}_2 \end{cases}$

$G_{M_1, M_2}(I) = (V, A)$

And we call $\begin{cases} X_1 := \{ x \in V_L (= S \setminus I) \mid I + x \in \mathcal{I}_1 \} & \underline{\text{Set of Sources}}. \\ X_2 := \{ x \in V_L (= S \setminus I) \mid I + x \in \mathcal{I}_2 \} & \underline{\text{Set of Sinks}}. \end{cases}$

x can be added without loosing independancy in $\mathcal{I}_1$

x can be added without loosing independancy in $\mathcal{I}_2$

[Algorithm] * Assume $\phi \in \mathcal{I}_1$ and $\phi \in \mathcal{I}_2$ (trivially).

(V_L은 계속 Algorithm이 작동하는 중!)

$I \leftarrow \phi$

While $G_{M_1, M_2}(I)$ has a path from X_1 to X_2

Let P be "the set of vertices" on the shortest path from X_1 to X_2

$I \leftarrow I \Delta P$

Output I.

<Correctness of the Algorithm>

Lemma] In each iteration of the algorithm, $I \Delta P \in (\mathcal{I}_1 \cap \mathcal{I}_2)$

Proof) Use induction on # iter. $\phi \in \mathcal{I}_1 \cap \mathcal{I}_2$ … base case

- Suppose that $I \in \mathcal{I}_1 \cap \mathcal{I}_2$

Let $\begin{cases} P = x_1 y_1 \cdots x_{k-1} y_{k-1} x_k \\ J = I \Delta P \end{cases}$

- Consider an auxiliary matroid $M_1' := (S+t, \underbrace{\{ I \subseteq (S+t) \mid I \setminus \{t\} \in \mathcal{I}_1 \}}_{\mathcal{I}_1'})$

$M_2' := (S+t, \underbrace{\{ I \subseteq (S+t) \mid I \setminus \{t\} \in \mathcal{I}_2 \}}_{\mathcal{I}_2'})$

for some $t \notin S$

This gives $(I+t) \in (\mathcal{I}_1' \cap \mathcal{I}_2')$

(XXX) M_1' is a matroid

$\Rightarrow$

- Now if we consider the set of arcs from $\underset{V_R}{I}$ to $\underset{V_L}{S\setminus I}$ on P.

∴ All these arcs (ignoring the orientation) ⇒ belongs to $G_{M_1'}(I+t)$

1>

[J is independent in $\mathcal{I}_1$]

- Let P_1 be the set of these edges

Moreover we have $(x_1, t) \in G_{M_1'}(I+t)$

and $\underline{N_1' = P_1 + (x_1, t)}$ is indeed a perfect matching between $(J\setminus(I+t))$ and $((I+t)\setminus J)$ in $G_{M_1'}(I+t)$

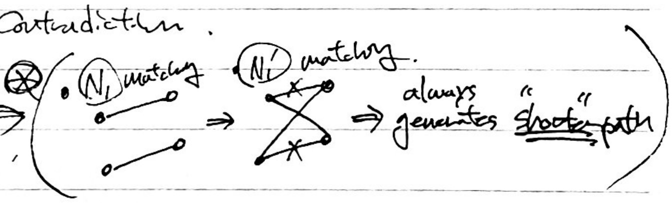
- Suppose towards contradiction that $N_1' \neq N_1$ (is not a unique perfect matching btw $(J\setminus(I+t))$ and $((I+t)\setminus J)$ in $G_{M_1'}(I+t)$)

⇒ ① If $(x_i, t) \in N_1'$ for some $i \geq 2$

⇒ ∃ a strictly shorter path $(\underset{\in X_1}{x_i} \to \underset{\in X_2}{x_k})$

⇒ P is not the shortest ∴ Contradiction.

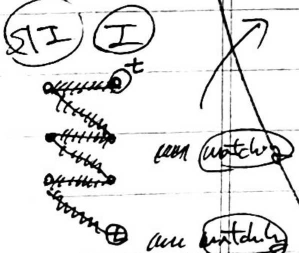
② If $(x_1, t) \in N_1'$



⇒ ∃ some $(x_i, y_j) \in N_1'$

such that $i > (j+1)$

∴ ∃ a shorter path ⇒ P is not the shortest ∴ Contradiction.



- Since in $G_{M_1'}(I+t)$, ∃ a unique perfect matching btw $(J\setminus(I+t))$ and $((I+t)\setminus J)$

and $(I+t)$ is independent (in $\mathcal{I}_1$) ⇒ J is independent (in $\mathcal{I}_1$) (by thm③)

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[J is independent in $\mathcal{I}_2$]

- Let P_2 be the set of the edges. (From $\underset{V_L}{\underline{S\setminus I}}$ to $\underset{V_R}{\underline{I}}$ on P)

and also we have $(x_k, t) \in G_{M_2'}(I+t)$

and $\underline{N_2' = P_2 + (x_k, t)}$ in $G_{M_2'}(I+t)$

is indeed a perfect matching between $(J\setminus(I+t))$ and $((I+t)\setminus J)$

- Suppose towards contradiction that $N_2' \neq N_2$ (N_2' is not a unique perfect matching)

⇒ ① If $(x_i, t) \in N_2'$ for some $i \leq k$ ⇒ Contradiction

② If $(x_k, t) \in N_2'$ ⇒ ∃ shorter path ⇒ Contradiction

- Since in $G_{M_2'}(I+t)$, ∃ a unique perfect matching btw $(J\setminus(I+t))$ and $((I+t)\setminus J)$

and $(I+t)$ is independent in $\mathcal{I}_2$ ⇒ J is independent in $\mathcal{I}_2$. (by thm③)

∴ $J \in (\mathcal{I}_1 \cap \mathcal{I}_2)$, and the result of our Algorithm is feasible.

<Optimality of the Algorithm>

Lemma] For $I \in (\mathcal{I}_1 \cap \mathcal{I}_2)$

Suppose that $G_{M_1,M_2}(I)$ does not have an $(X_1 - X_2)$ path. (so. Alg terminated)

Let $U := \{z \in S \mid G_{M_1,M_2}(I)$ has a path from z to $X_2\}$

and we have $|I| = r_1(U) + r_2(S \setminus U)$

proof). • We first claim that ① $r_1(U) = |I \cap U|$

• Suppose not. ($\rightarrow$. $r_1(U) > |I \cap U|$)

then $\exists x \in U \setminus I$ such that $(I \cap U) + x \in \mathcal{I}_1$ (by axiom of matroid)

($\because \exists \bar{U} \subseteq U$ s.t $\bar{U} \in \mathcal{I}_1$)

• Since $I \in \mathcal{I}_1$, we can repeatedly add elements of I to $(I \cap U) + x$ to obtain a set of the form $(I - y + x) \in \mathcal{I}_1$ for some $y \in I \setminus U$

• This implies the existence of $\langle y, x \rangle$ in $G_{M_1,M_2}(I)$ ⋯ $y \in$

$\Rightarrow$ $\exists$ a path ⋯ Contradiction ⋯ $\therefore r_1(U) = |I \cap U|$

• We claim that ② $r_2(S \setminus U) = |I \setminus U|$

• Suppose not ($\Rightarrow$ $r_2(S \setminus U) > |I \setminus U|$)

then $\exists x \in (S \setminus U) \setminus I$ such that $(I \setminus U) + x \in \mathcal{I}_2$.

($\because \exists T \subseteq (S \setminus U)$ s.t $T \in \mathcal{I}_2$) (by axiom of matroid)

• Since $I \in \mathcal{I}_2$, we can repeatedly add elements of I to $(I \setminus U) + x$ ($\in \mathcal{I}_2$) to obtain a set of the form $(I - y + x) \in \mathcal{I}_2$. for some $y \in (I \cap U)$

• This implies the existence of $\langle x, y \rangle$ in $G_{M_1,M_2}(I)$ and $x \in U$

Contradiction

$\Rightarrow$ We thus have $|I| = |I \cap U| + |I \setminus U| = r_1(U) + r_2(S \setminus U)$

Thm] "Matroid Intersection Theorem"

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} \left(r_1(U) + r_2(S \setminus U) \right)$$

proof) Note that in each iteration of the algorithm, $|I|$ increased by 1.

The algorithm always terminates, and the thm follows from the two lemmas.

Remark) $\Rightarrow$ Polynomial in $|S|$ // More than 3 Matroids ⋯ NP-hard.