[Weighted Matroid Intersection]

- Given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set and a weight function $C : S \to \mathbb{Q}$, find a maximum weight common independent set.

Thm (Unweighted) Matroid Intersection Theorem

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} \left(r_{M_1}(U) + r_{M_2}(S \setminus U) \right)$$

⊗ Special Case : $C(e) = 1$ (Constant function)

Let $C_1, C_2 : S \to \mathbb{Q}$ s.t. $\forall i \in S$, $C_1(i) + C_2(i) = C(i)$

For any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we have

$$\Rightarrow \left(\begin{aligned} C(I) &= C_1(I) + C_2(I) \\ &\leq \max_{I_1 \in \mathcal{I}_1} C_1(I_1) + \max_{I_2 \in \mathcal{I}_2} C_2(I_2) \end{aligned} \right)$$

Lemma 2]
$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} C(I) \leq \min_{\substack{C_1, C_2 \\ \forall C_1(i) + C_2(i) = C(i)}} \left(\max_{I_1 \in \mathcal{I}_1} C_1(I_1) + \max_{I_2 \in \mathcal{I}_2} C_2(I_2) \right)$$

Theorem 3] Weighted Matroid Intersection Theorem

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} C(I) = \min_{\substack{C_1, C_2 \\ \forall C_1(i) + C_2(i) = C(i)}} \left(\max_{I_1 \in \mathcal{I}_1} C_1(I_1) + \max_{I_2 \in \mathcal{I}_2} C_2(I_2) \right)$$

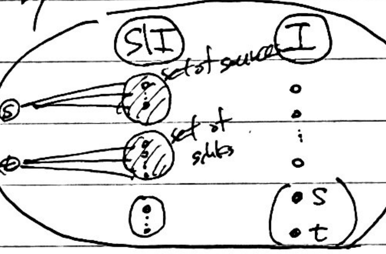
⊗ Remark) if C is integral, so can C_1 and C_2 be chosen to be satisfying the equality.
(if C integral $\Rightarrow \exists C_1, C_2$ integral))

• Def) Weighted Exchange Graph : $G_{M_1, M_2}(I)$

(Given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, a common independent set $I = \mathcal{I}_1 \cap \mathcal{I}_2$ and the two weight functions $c_1, c_2 : S \to \mathbb{Q}$)

$\Rightarrow$ the weighted exchange graph ($G_{M_1, M_2}(I)$) is the following directed bipartite graph

: the bipartition of the vertices & arcs are given by



① • $V_L = S \setminus I \Rightarrow x$

• $V_R = I \cup \{s, t\} \Rightarrow y$

② Where $x \in (S \setminus I)$, $y \in I$, $s, t \in I$

• $\langle \overrightarrow{y, x} \rangle \in A$ iff $I - y + x \in \mathcal{I}_1$ (operation maintains independence of I in M_1)

and its weight is $w(y, x) := c_1(y) - c_1(x)$

• $\langle \overrightarrow{x, y} \rangle \in A$ iff $I - y + x \in \mathcal{I}_2$ (" in M_2)

and its weight is $w(x, y) := c_2(y) - c_2(x)$

• $\langle s, x \rangle \in A$ iff $I + x \in \mathcal{I}_1$

and its weight is $w(s, x) := m_1 - c_1(x)$

where $m_1 := \max\{c_1(x) \mid x \in S \setminus I, I + x \in \mathcal{I}_1\}$

• $\langle x, t \rangle \in A$ iff $I + x \in \mathcal{I}_2$

and its weight is $w(x, t) := m_2 - c_2(x)$

where $m_2 := \max\{c_2(x) \mid x \in S \setminus I, I + x \in \mathcal{I}_2\}$

W $\Rightarrow$ Net Loss 의 개념.(해당 edge의 operation에 대응하는)

• Def) Let $\overline{G}_{M_1, M_2}(I)$ be the subgraph of $G_{M_1, M_2}(I)$ consisting of only the edges with zero costs. (※ Their sets of vertices are the same)

[Algorithm]

$I_0 \leftarrow \phi \quad (k=0)$

$C_1 \leftarrow c$

$C_2 \leftarrow 0$

repeat

Let R be the set of vertices reachable from s in $\overline{G}_{M_1, M_2}(I_k)$

if $t \in R$ ($\exists$ an s–t path)

then, Let $P = \{s, x_1, y_1, x_2, y_2 \cdots x_{m+1}, t\}$: an s–t path (shortest) with the fewest vertices in $\overline{G}$ (may not be a shortest path in $\overline{G}$)

$I_{k+1} \leftarrow I_k \triangle (P \setminus \{s, t\})$

$k \leftarrow k+1$

$\varepsilon \leftarrow 0$

else ($t \notin R$) ($\nexists$ any s–t path)

$\varepsilon \leftarrow \min \{ w(z_1, z_2) \mid \langle z_1, z_2 \rangle \in A,\ z_1 \in R,\ z_2 \notin R \}$

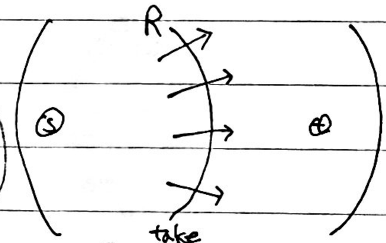
if $\varepsilon < \infty$ (ε is finite)

then each $z \in R \cap S$

$C_1(z) \leftarrow C_1(z) - \varepsilon$

$C_2(z) \leftarrow C_2(z) + \varepsilon$

$\Rightarrow$ update $\overline{G}$



$\varepsilon =$ take minimum

until $\varepsilon = \infty$

Return The maximum weight set among $I_0 \cdots I_k$.

<Optimality Proof>

Theorem 1] Let $(S, \mathcal{I})$ be a matroid with weight function $c : S \to \mathbb{R}$ and $k \in \mathbb{N}$.

For $X \in \mathcal{I}$, with $|X| = k$, we have $c(X) = \max\{c(Y) \mid Y \in \mathcal{I}, |Y| = k\}$

(iff) ① for $\forall j \in S \setminus X$ with $X + j \notin \mathcal{I}$ $\Rightarrow$ X is maximum independent (with k)

we have $c(i) \geq c(j)$ for $\forall i \in C(X, j)$ → unique circuit of $(X+j)$

② for $\forall j \in S \setminus X$ with $X + j \in \mathcal{I}$

we have $c(i) \geq c(j)$ for $\forall i \in X$

Lemma 2'] (With Cardinality Constraint)

For $k \in \mathbb{N}$, if $\exists$ some $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|I| = k$,

We have

$$\max_{\substack{I \in \mathcal{I}_1 \cap \mathcal{I}_2 \\ |I| = k}} c(I) \leq \min_{\substack{c_1, c_2 \\ \text{for } \forall i, \\ c_1(i) + c_2(i) = c(i)}} \left\{ \max_{\substack{I \in \mathcal{I}_1 \\ |I_1| = k}} c_1(I) + \max_{\substack{I_2 \in \mathcal{I}_2 \\ |I_2| = k}} c_2(I_2) \right\}$$

Lemma 4] For $x \in (S \setminus I)$ and $y \in I$,

we have $\langle y, x \rangle \in A$ (iff) ① $I + x \notin \mathcal{I}_1$ and $y \in C_1(I, x)$
or ② $I + x \in \mathcal{I}_1$

we have $\langle x, y \rangle \in A$ (iff) ① $I + x \notin \mathcal{I}_2$ and $y \in C_2(I, x)$
or ② $I + x \in \mathcal{I}_2$.

※ where C_1 → Unique circuit in M_1
C_2 → Unique circuit in M_2.

proof) ① Regarding M_1 (⇐) Trivial

(⇒) Suppose $(I - y + x) \in \mathcal{I}_1$ and $I + x \notin \mathcal{I}_1$

if) $y \notin C_1(I, x)$ then $C_1(I, x) \subseteq I + x - y$

$\therefore (I + x - y) \notin \mathcal{I}_1$ #.

② Regarding M_2. (⇐) Trivial

(⇒) Same way w.r.t $\mathcal{I}_2$

3] If $z_1 = s$ and $z_2 \in (S \setminus I_k)$ ($\overrightarrow{\langle s, x \rangle}$ case)

$\Rightarrow$ by definition, $w(z_1, z_2) = \max\{c_1(x) \mid x \in S \setminus I_k,\ I_k + x \in \mathcal{I}_1\} - c_1(z_2) \geq 0$.

since $z_2 \in \{x \mid x \in S \setminus I_k,\ I_k + x \in \mathcal{I}_1\}$

4] If $z_1 \in (S \setminus I_k)$ and $z_2 = t$ ($\overrightarrow{\langle x, t \rangle}$ case)

$\Rightarrow$ by definition, $w(z_1, z_2) = \max\{c_2(x) \mid x \in S \setminus I_k,\ I_k + x \in \mathcal{I}_2\} - c_2(z_1) \geq 0$

since $z_1 \in \{x \mid x \in S \setminus I_k,\ I_k + x \in \mathcal{I}_2\}$

$\Rightarrow$ By 1] ~ 4], Lemma 7 is implied by Lemma 6.

- Consider an iteration at the beginning of which all these lemmas hold.

[Clause 1] Suppose that $t \notin R$ ($\nexists$ s-t path in $\overline{G_1}$)

- From Lemma 7, $\varepsilon := \min\{w(z_1, z_2) \mid \langle z_1, z_2 \rangle \in A,\ z_1 \in R,\ z_2 \notin R_2\} > 0$

  ($w \geq 0$ in G_1 and $w \neq 0$ since z_2 is not in R_2)

- Suppose that $\varepsilon < \infty$ (O/W algo terminates)

  and let c_1' and c_2' denote the modified weight functions,

  whereas c_1 and c_2 denote the functions before modification.

- $\circledast$ [Thm 1] Let $(S, \mathcal{I})$ be a matroid with weight function $c: S \to \mathbb{R}$ and $k \in \mathbb{N}$.

  For $X \in \mathcal{I}$ with $|X| = k$, we have $c(X) = \max\{c(Y) \mid Y \in \mathcal{I}, |Y| = k\}$

  (iff) ① for $\forall j \in S \setminus X$ with $X + j \notin \mathcal{I}$. X is maximum independent set.

  we have $c(i) \geq c(j)$ for $\forall i \in C(X, j)$

  ② for $\forall j \in S \setminus X$ with $X + j \in \mathcal{I}$

  we have $c(i) \geq c(j)$ for $\forall i \in X$

- Let x be an arbitrary element in $S \setminus I_k$ such that $I_k + x \notin \mathcal{I}$ (case ① in thm 1)

  (Then, for $\forall y \in C_1(I_k, x)$, $\langle y, x \rangle \in G_{M_1, M_2}(I_k)$ (by def)

  Let x be an arbitrary element in $S \setminus I_k$ such that $I_k + x \in \mathcal{I}$ (case ② in thm 1)

  Then, for $\forall y \in I_k$, $\langle y, x \rangle \in G_{M_1, M_2}(I_k)$ (by def)

$\Rightarrow$ Then by [Thm 1] $\Rightarrow$ $c_1(y) \geq c_1(x)$

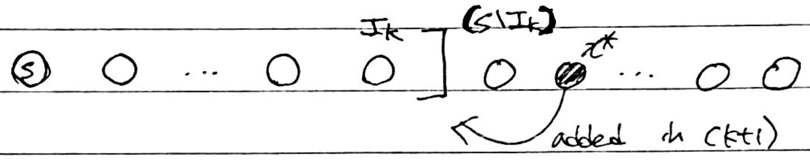
for $y \in R$ and $x \notin R$ $\Rightarrow$ $\varepsilon \leq c_1(y) - c_1(x)$

$\Rightarrow c_1'(y) = c_1(y) - \varepsilon \geq c_1(x) = c_1'(x)$ (x not reachable … no update)

$\Rightarrow c_1'(y) \geq c_1'(x)$ … by [Thm 1]. (by $\circledast$ on M_2, $c_2'(y) \geq c_1'(x)$) $\Rightarrow$ Lemma 6 Remains True.

[Clause 2] Suppose $t \in R$ ($\exists$ s-t path in $\bar{G}$)

- Fix an ordering of S such that when the greedy algo is run with it, the algo returns I_k (for M_1 and c_1)
- Under this ordering, let x^* be the 1st element in $\{x \mid \langle s, x\rangle \in \bar{G}\} \neq \phi$ (since $\exists$ s-t path in $\bar{G}$)
- Observe that, when the algorithm is run with $(k+1)$ instead, it returns $(I_k + x^*)$ instead.



This shows that $\max\{c_1(I) \mid I \in \mathcal{I}_1, |I| = k+1\}$

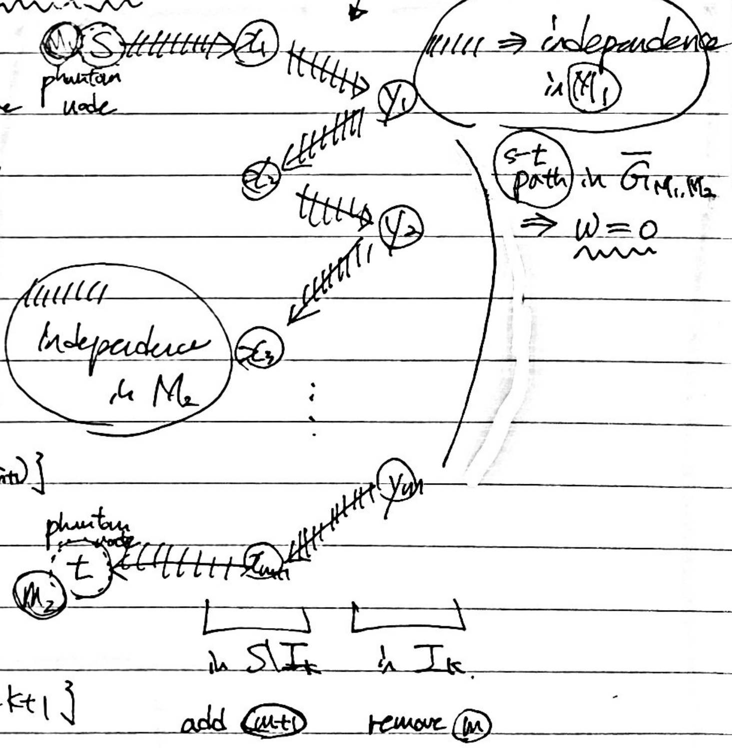
$= c_1(I_k + x^*)$

$= c_1(I_k) + m_1$ → $\max\{c_1(x) \mid x \in S \setminus I, I+x \in \mathcal{I}_1\}$

On the other hand, $c_1(I_{k+1}) = c_1(I_k) + c_1(x_1) - \sum_{i=1}^{m}\{c_1(y_i) - c_1(x_{i+1})\}$

$= c_1(I_k) + m_1$

- Showing that the 1st half of the conditions of Lemma 6 holds, assuming Lemma 5.



(likewise)

// $c_2(I_{k+1}) = c_2(I_k) - c_2(x_1) + \sum_{i=1}^{m}\{c_2(y_i) - c_2(x_{i+1})\}$

$= c_2(I_k) + m_2$

$= c_2(I_k + \tilde{x}^*)$

$= \max\{c_2(I) \mid I \in \mathcal{I}_2, |I| = k+1\}$

- 2nd half of the conditions of Lemma 6 holds.

⇒ Lemma 6 remains true.

In both [Clause 1] and [Clause 2] Lemma 6 holds (and Lemma 1 is implied by it)

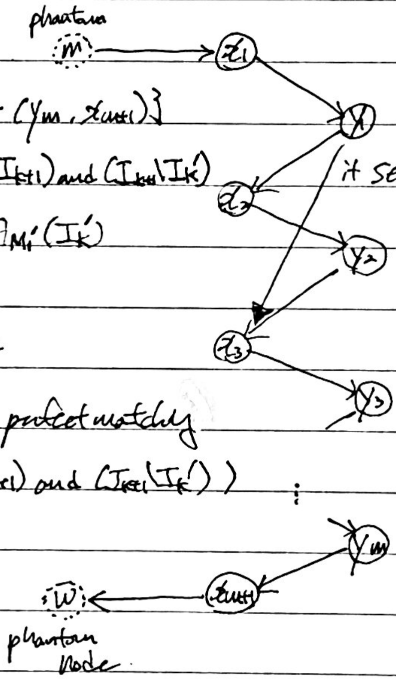
Part 1

• Now, we show that $I_{k+1} \in \mathcal{I}_1 \cap \mathcal{I}_2$ (Lemma 4)

① • Consider an auxiliary matroid

$M_1' := (S+m, \mathcal{I}_1')$ where $\mathcal{I}_1' := \{ I \subseteq (S+m) \mid I \setminus \{m\} \in \mathcal{I}_1 \}$

Then, $I_k' := (I_k + m) \in \mathcal{I}_1'$ where $I_k \in \mathcal{I}_1$.



• Note that $N_1 := \{(m, z_1)\ (y_1, z_2) \cdots (y_m, z_{m+1})\}$ is a perfect matching between $(I_k' \setminus I_{k+1})$ and $(I_{k+1} \setminus I_k')$ in $G_{M_1'}(I_k')$

it seems that can exist in G (but it is not)

• Suppose towards a contradiction that

$\exists\ \bar{N_1}\ (\neq N_1)$ (is another perfect matching between $(I_k' \setminus I_{k+1})$ and $(I_{k+1} \setminus I_k')$)

• We observe that, by defining $c_1(m) := m_1$

we have that $\sum_{a \in N} w(a) = \sum_{y \in \{m, y_1 \cdots y_m\}} c_1(y) - \sum_{z \in \{z_1 \cdots z_{m+1}\}} c_1(z)$ for $N = N_1$ and $\bar{N_1}$

Since $\sum_{a \in N_1} w(a) = 0$, we have that $w(a) = 0$ for $\forall\ a \in \bar{N_1}$ by Lemma 7 (in $\bar{G}$) $(w(a) \geq 0)$

$\Rightarrow$ Contradicts the choice of P as a shortest s-t path

$\Rightarrow$ N_1 is the unique perfect matching $\therefore I_{k+1} \in \mathcal{I}_1$.

part 2

② // (likewise) • $M_2' := (S+w, \mathcal{I}_2')$ where $\mathcal{I}_2' := \{ I \subseteq (S+w) \mid I \setminus \{w\} \in \mathcal{I}_2 \}$

Then $I_k' := (I_k + w) \in \mathcal{I}_2'$ where $I_k \in \mathcal{I}_2$.

• $N_2 := \{(z_1, y_1)\ (z_2, y_2) \cdots (z_m, y_m)\ (z_{m+1}, w)\}$ is a perfect matching btw $(I_k' \setminus I_{k+1})$ and $(I_{k+1} \setminus I_k')$ in $G_{M_2'}$

$\Rightarrow$ Then its unique. $\therefore I_{k+1} \in \mathcal{I}_2$

by ① and ② $I_{k+1} \in \mathcal{I}_1 \cap \mathcal{I}_2$, Lemma 4 holds.

<Lemma 8> The final value of k is equal to $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I|$

proof) Note that $\varepsilon = \infty$ implies that $G_{M_1,M_2}(I_k)$ has no s-t path.
The desired conclusion follows from the analysis of the maximum cardinality matroid intersection algorithm ∎

<Lemma 9> The given algorithm performs a polynomial # of basic operations and oracle queries.

proof) • If $t \in R$ ($\exists$ an s-t path) in an iteration,
k increases by 1 and k cannot exceed $|S|$

• If $t \notin R$ and $\varepsilon < \infty$ in an iteration "z_2 reachable"
Let $\langle z_1, z_2 \rangle$ be an arbitrary arc ~~in~~ $\bar{G}_{M_1,M_2}(I_k)$ at the beginning of the iteration.
⇒ We claim that if $z_1 \in R$, $\boxed{\langle z_1, z_2 \rangle \text{ does not leave } \bar{G}_{M_1,M_2}(I_k) \text{ during the iteration}}$

m_1 shrink

⊛ c_1, c_2 : Costs beginning of iter, c_1', c_2' : " end of iter.
m_1, m_2 : variables beginning of iter, m_1', m_2' : " end "

??, ⇒

• Case 1] $z_1 = s$ and $z_2 \in S \setminus I$

$\max\{c_1(x) \mid x \in S \setminus I, I + x \in \mathcal{I}_1\}$

• Since $\langle z_1, z_2 \rangle \in \bar{G}$ at the beginning ⇒ $m_1 - c_1(z_2) = 0$

• Since $z_2 \in R$ at the beginning ⇒ $c_1'(z_2) = m_1 - \varepsilon$

$\min\{w(z_1, z_2) \mid \langle z_1, z_2 \rangle \in A, z_1 \in R, z_2 \notin R\}$

since ⇒ $m_1' = m_1 - \varepsilon$
⇒ $m_1' - c_1'(z_2) = m_1 - c_1'(z_2) = \underline{0}$.
⇒ $\langle z_1, z_2 \rangle$ remains ∎

⊛ We claim that for $\forall x$ s.t. $(I_k + x) \in \mathcal{I}_1$
We have $c_1'(x) \leq m_1 - \varepsilon$

• Suppose not // This implies that $(c_1'(x) > m_1 - \varepsilon) \cdots (c_1'(x) \neq m_1 - \varepsilon)$
⇒ $x \notin R$
⇒ $c_1'(x)$ not updated
⇒ $c_1'(x) = \underline{c_1(x) > m_1 - \varepsilon}$

However, since $\langle s, x \rangle \in A$,
⇒ $x \in R$ ※.

⇒ $\varepsilon \leq m_1 - c_1'(x)$
and $c_1'(x) = c_1(x) - \varepsilon$
$w(s, x) = m_1' - c_1'(x)$
=

• Case 2] $z_1 \in I$ and $z_2 \in S \setminus I$
• Since $z_1, z_2 \in R$
⇒ $c_1(z_1) - c_1(z_2) = c_1'(z_1) - c_1'(z_2) = 0$

• Case 3] $z_1 \in S \setminus I$ and $z_2 \in I$
//

• Case 4] $z_1 \in S \setminus I$ and $z_2 = t$
since $t \notin R$, case 4 does not exist.
⇒ This shows that $|R|$ never shrinks.

• Now, if $\varepsilon = m_2 - c_2(x)$ for some $x \in R$,

$\langle x, t \rangle$ enters $\bar{G}$ in this iteration.

(since $w(x,t) = m_2' - c_2'(x)$, t is not reachable now,

$(\Rightarrow m_2' = m_2, \; c_2'(x) = c_2(x) + \varepsilon)$

$= m_2 - c_2(x) - \varepsilon$

$= \varepsilon - \varepsilon = 0$ ... enters)

$\Rightarrow$ hence $t \in R$ would hold in the next iteration.

• (Note that

(if) $m_2 = c_2(x)$ for some x, then $x \notin R$)

Q(W) we claim that $|R|$ strictly increases)

Case 1] $\varepsilon = m_1 - c_1(x)$ for some $x \notin R$ $\quad c_1'(x) = c_1(x)$

$\Rightarrow$ we have $m_1' - c_1'(x) \leq (m_1 - \varepsilon) - c_1(x) = 0$.

$\Rightarrow \quad \underbrace{0 \leq}_{\text{nonnegative Lemma 7}} m_1' - c_1'(x) \leq 0$

$\Rightarrow \quad m_1' - c_1'(x) = 0$

$\Rightarrow \quad \langle s, x \rangle$ enters A, x enters R.

$c_1'(x) = c_1(x)$ $\quad c_1'(y) = c_1(y) - \varepsilon$

Case 2] $\varepsilon = c_1(y) - c_1(x)$ for some $x \notin R$ and $y \in R$.

$\Rightarrow$ we have $c_1'(y) - c_1'(x) = c_1(y) - \varepsilon - c_1(x) = 0$.

$\Rightarrow$ hence $\langle y, x \rangle$ enters A, x enters R

$c_2'(x) = c_2(x) + \varepsilon$

Case 3] $\varepsilon = c_2(y) - c_2(x)$ for some $x \in R$, and $y \notin R$

$\Rightarrow$ we have $c_2'(y) - c_2'(x) = c_2(y) - c_2(x) - \varepsilon$ $\quad c_2'(y) = c_2(y)$

$= 0$.

$\Rightarrow$ hence $\langle x, y \rangle$ enters A, y enters R ∎

(Thm) The weighted matroid intersection algorithm is correct.

proof) Lemma 2', 5, 6, 7, 8, 9 ∎