Non Linear Systems Phase Portrait

Author:
Georgios Panagiotou

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1 Exercise 1: Phase-plane portrait

1.1

Consider the second-order differential equation of a pendulum:

$$\ddot{\theta} = -k\dot{\theta} - \sin(\theta) \tag{1}$$

To represent this system in state-space form, we introduce the variables:

$$x_1 = \theta, \quad x_2 = \dot{\theta} \tag{2}$$

This leads to the following first-order system:

$$\dot{x}_1 = x_2 \tag{3}$$
$$\dot{x}_2 = -kx_2 - \sin(x_1) \tag{4}$$

Thus, the state-space representation in the form $\dot{x}(t) = f(x(t))$ is given by:

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -kx_2(t) - \sin(x_1(t)) \end{bmatrix} \tag{5}$$

1.2

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function dx = pendulum(t, x, k)
    dx1 = x(2);
    dx2 = -k*x(2) - sin(x(1));
    dx = [dx1; dx2];
end
```

1.3

Equilibrium occurs when $\dot{x}_1 = \dot{x}_2 = 0$, leading to:

$$0 = x_2 \tag{6}$$
$$0 = -kx_2 - \sin(x_1) \tag{7}$$

Solving gives $x_1 = n\pi$, $x_2 = 0$, for $n \in \mathbb{Z}$. Hence, the fixed points are:

$$(x_1, x_2) = (n\pi, 0), \ n \in \mathbb{Z} \tag{8}$$

For $k = 0$:

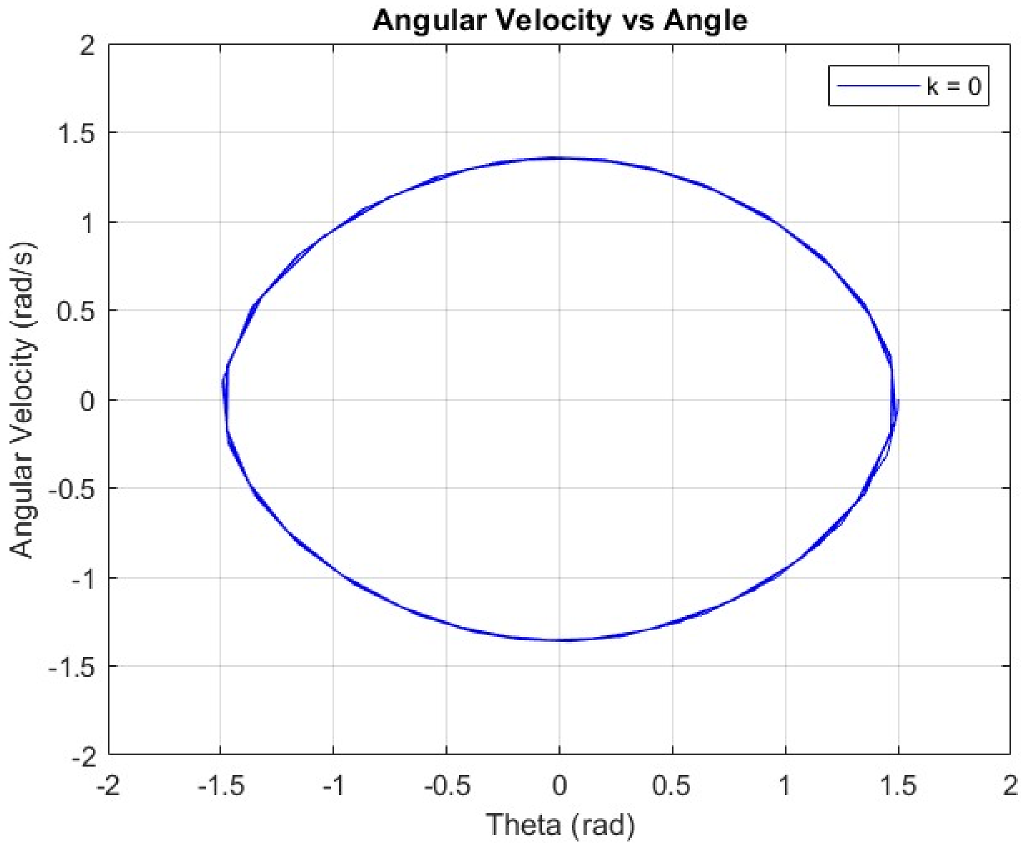

Figure 1: Solution of the second-order differential equation with initial conditions set to $[1.5, 0]$ radians, over a duration of 30 seconds.

The system oscillates harmonically between the positive and negative initial positions 1.5 and -1.5 radians. Thus, the ω-limit set is a periodic orbit, revisiting the initial condition every T, where T is the period of the oscillation.

$$\omega(x_0) = \left\{ x \in X : \{t_n\}_{n=1}^{\infty} \text{ with } t_n \rightarrow +\infty \text{ as } n \rightarrow +\infty \text{ and } \lim_{n\rightarrow+\infty} \varphi(t_n, x_0) = \nexists \right\}$$

$$x_0 = \varphi(nT, x_0) \text{ for the specific period } T > 0$$

$$\lim_{n\rightarrow+\infty} \varphi(nT, x_0) = x_0 \in \omega(x_0) \text{ for the specific period } T > 0$$

For $k = 1$:

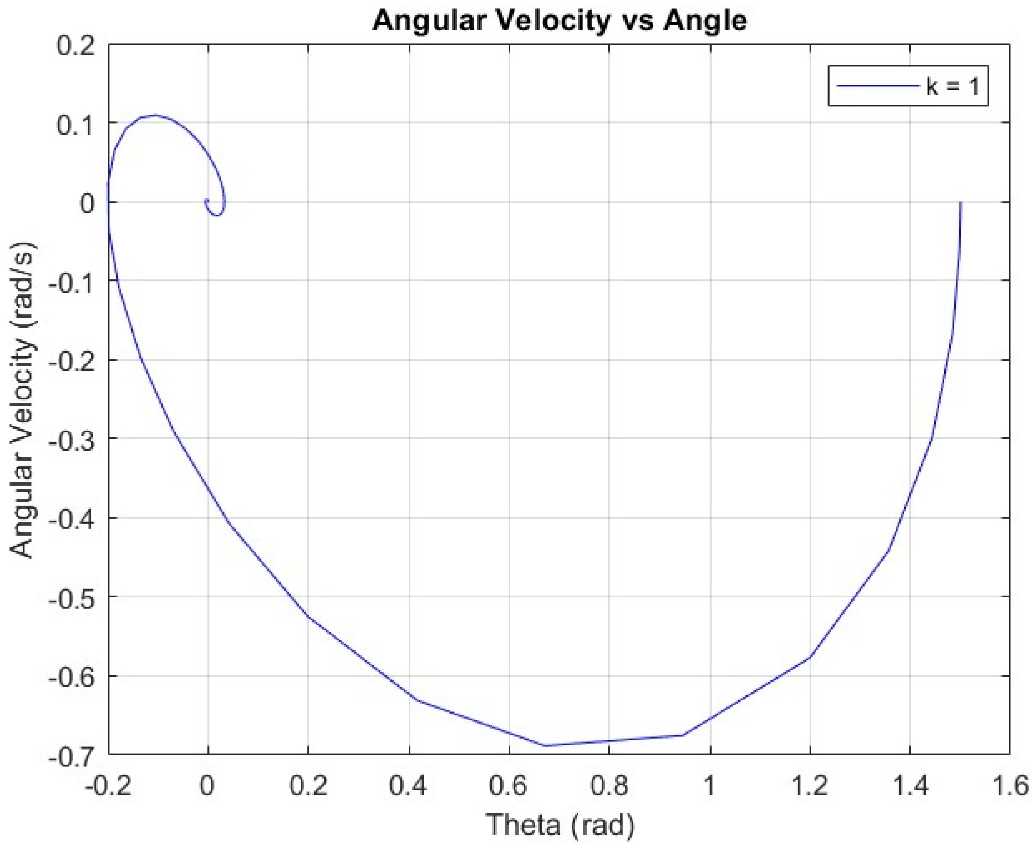

Figure 2: Solution of the second-order differential equation with initial conditions set to $[1.5, 0]$ radians, over a duration of 30 seconds.

The system quickly converges to the sole equilibrium point at the origin. The ω-limit set of the system is therefore the equilibrium point.

$$\omega(x_0) = \left\{ x \in X : \{t_n\}_{n=1}^{\infty} \text{ with } t_n \to +\infty \text{ as } n \to +\infty \text{ and } \lim_{n\to+\infty} \varphi(t_n, x_0) = x_0 \right\}$$

$$\omega(x_0) = x_e = (0, 0)$$

1.4

```
figure;
pendulum_phase_plane(0, -pi, pi, -4, 4);
figure;
pendulum_phase_plane(1, -pi, pi, -4, 4);

function pendulum_phase_plane(k, xmin, xmax, ymin, ymax)
    [X1, X2] = meshgrid(linspace(xmin, xmax, 1000), linspace(ymin,
        ymax, 1000));
    %The meshgrid function generates two 1000x1000 matrices: X1(
        values for the angle) and X2(values for the angular velocity).
    % state space equations for every coordinate from above
    DX1 = X2;
```

```
    DX2 = -k * X2 - sin(X1);
    %streamline plot
    streams = streamslice(X1, X2, DX1, DX2, 1);
    set(streams, 'Color', 'b');
    hold on;
    xlabel('Theta (radians)');
    ylabel('Angular Velocity (radians/s)');
    title(['Phase Plane for k = ', num2str(k)]);
    xlim([xmin xmax]);
    ylim([ymin ymax]);
    hold off;
end
```

For $k = 0$:
The phase portrait illustrates that the system exhibits harmonic oscillations, consistently alternating between positive and negative amplitudes without converging to an equilibrium point.

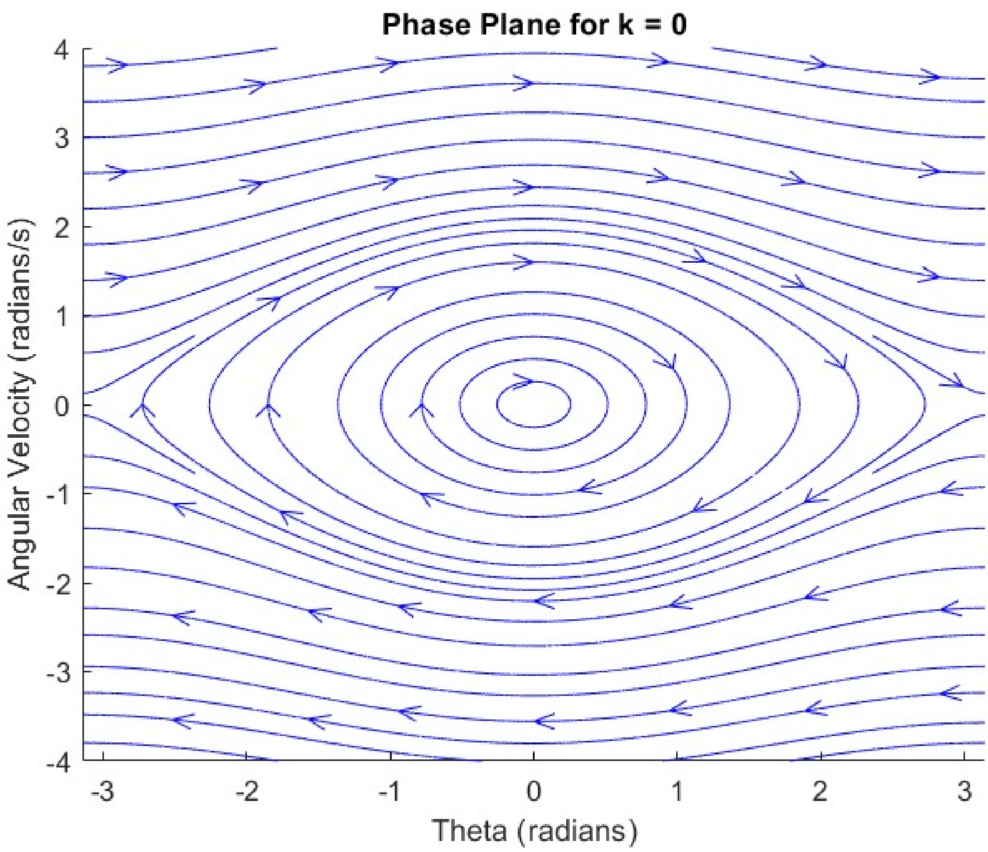

Figure 3: Phase plane of a pendulum system (with $k = 0$), illustrating pure harmonic oscillations without convergence to an equilibrium.

For $k = 1$:
The phase portrait indicates that the trajectories converge to the stable equilibrium point at (0,0). It is evident that the equilibrium point can be characterized as a node since all trajectories converge to it and remain there indefinitely. Furthermore, it

is apparent that, for any initial condition, the system converges to an equilibrium point.

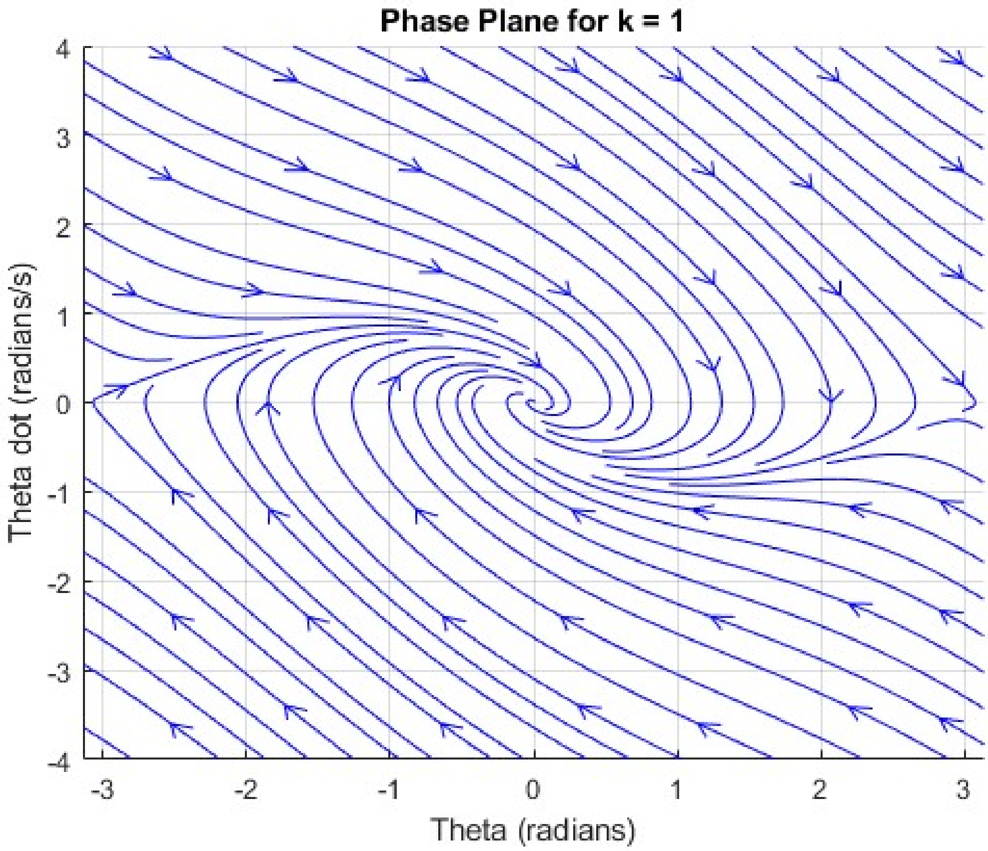

Figure 4: Phase plane of a pendulum system for $k = 1$, showing trajectories converging to the stable equilibrium at the origin.

1.5

Physical meaning of k:
It is already established that the second-order differential equation represents a mathematical model of a pendulum, with θ_0 representing the initial conditions of the system. According to the simulation in question 3, it is clear that when k=0, the system oscillates over time. However, for any nonzero k, the system quickly settles at zero degrees. The physical interpretation of k could be a damping coefficient that represents additional forces opposing the motion of the pendulum, causing it to settle quickly to the resting angle (equilibrium point).

2 Exercise 2: A counter-example to Poincare-Bendixson

2.1

We have been asked to run simulations for a given initial conditions, but without choosing initial conditions to be equal to an equilibrium point. For this reason, the first step is to compute the equilibrium points of the system.

Given system in the state-space representation form:

$$\dot{x}_1 = \sigma(x_2 - x_1) \tag{9}$$
$$\dot{x}_2 = 28x_1 - x_2 - x_1x_3 \tag{10}$$
$$\dot{x}_3 = x_1x_2 - \frac{8}{3}x_3 \tag{11}$$

To find the equilibrium points, we set all derivatives to zero:

$$0 = \sigma(x_2 - x_1) \qquad \Longrightarrow x_1 = x_2 \tag{12}$$
$$0 = 28x_1 - x_2 - x_1x_3 \qquad \Longrightarrow 27x_1 = x_1x_3 \tag{13}$$
$$0 = x_1x_2 - \frac{8}{3}x_3 \qquad \Longrightarrow x_1x_2 = \frac{8}{3}x_3 \tag{14}$$

From equations (12) and (13), we find two solutions for x_1 and x_3:

$$x_1 = 0, \qquad x_3 = 27$$

Substituting these into equation (14) gives us the following equilibrium points:

- For $x_1 = 0$, we find the equilibrium point at $(0,0,0)$.
- For $x_3 = 27$, solving $0 = x_1^2 - \frac{8}{3}\cdot 27$ leads to $x_1^2 = 72$, thus yielding two equilibrium points at $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$.

Now that we have the three equilibrium points, we choose the initial conditions to be [3 3 3].

2.1.1 For $\sigma = 1$:

The plot shows that the system starts at (3,3,3) point and then moves in a curling path towards an equilibrium point, particularly settling at $(6\sqrt{2}, 6\sqrt{2}, 27)$.

$$\omega(x_0) = \left\{ x \in X : \{t_n\}_{n=1}^{\infty} \text{ with } t_n \to +\infty \text{ as } n \to +\infty \text{ and } \lim_{n\to+\infty} \varphi(t_n, x_0) = x_e \right\}$$

$$\omega(x_0) = x_e = (6\sqrt{2}, 6\sqrt{2}, 27)$$

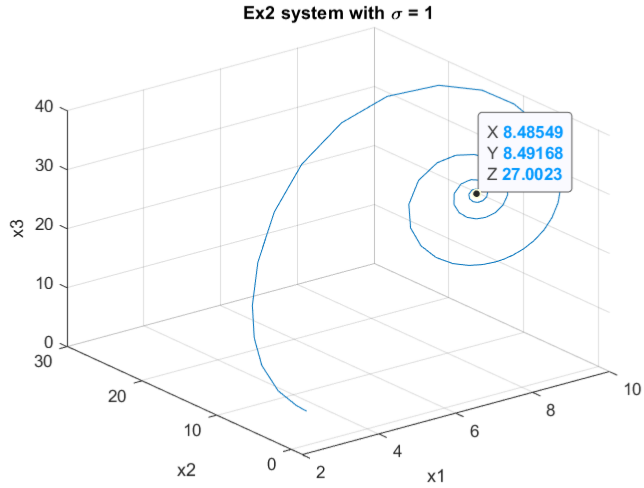

Figure 5: Plot of the solution with initial conditions [3 3 3] with $\sigma = 1$.

2.1.2 For $\sigma = 5$:

The plot shows that the system starts at the (3,3,3) point and then moves in a curling path towards an equilibrium point, particularly settling at $(-6\sqrt{2}, -6\sqrt{2}, 27)$.

$$\omega(x_0) = \left\{ x \in X : \{t_n\}_{n=1}^{\infty} \text{ with } t_n \to +\infty \text{ as } n \to +\infty \text{ and } \lim_{n\to+\infty} \varphi(t_n, x_0) = x_e \right\}$$

$$\omega(x_0) = x_e = (-6\sqrt{2}, -6\sqrt{2}, 27)$$

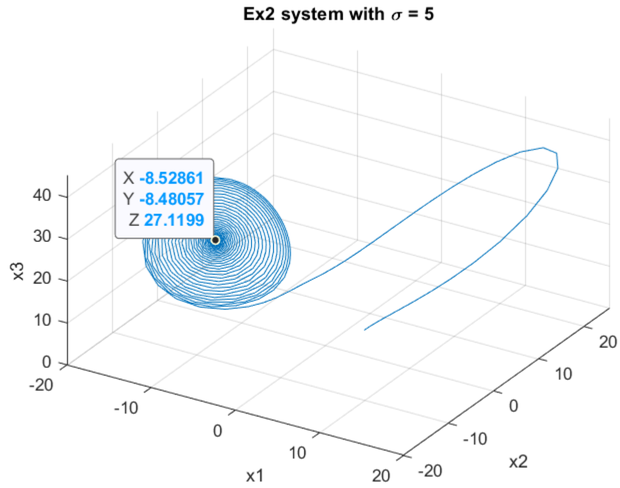

Figure 6: Plot of the solution with initial conditions [3 3 3] with $\sigma = 5$.

2.1.3 For $\sigma = 10$:

The figure presents a system with very complicated behavior. It is clear from the figure that the system's oscillations stay around the two equilibrium points $(6\sqrt{2}, 6\sqrt{2}, 27)$

and $(-6\sqrt{2}, -6\sqrt{2}, 27)$, without settling at either (They maintain a distance from them). The trajectory of the system does not converge to a single point nor is considered as periodic orbit but instead spread out in a specific pattern.

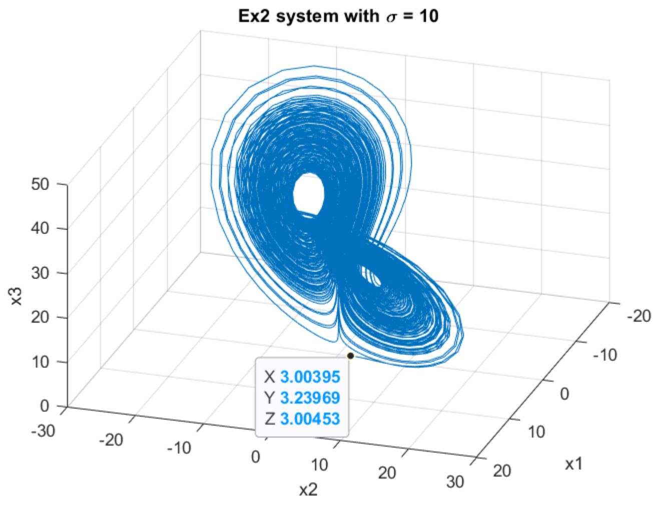

Figure 7: Plot of the solution with initial conditions [3 3 3] with $\sigma = 10$.

2.1.4 For $\sigma = 20$:

The plot shows that the system starts at (3,3,3) point and then moves in a curling path towards an equilibrium point, particularly settling at $(-6\sqrt{2}, -6\sqrt{2}, 27)$.

$$\omega(x_0) = \left\{ x \in X : \{t_n\}_{n=1}^{\infty} \text{ with } t_n \to +\infty \text{ as } n \to +\infty \text{ and } \lim_{n \to +\infty} \varphi(t_n, x_0) = x_e \right\}$$

$$\omega(x_0) = x_e = (-6\sqrt{2}, -6\sqrt{2}, 27)$$

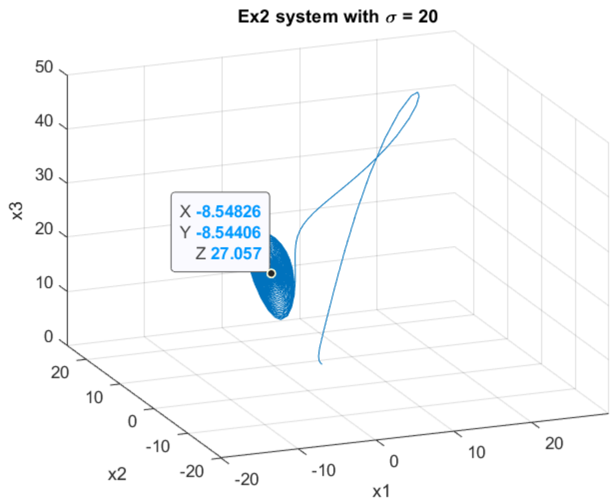

Figure 8: Plot of the solution with initial conditions [3 3 3] with $\sigma = 20$.

2.1.5 For $\sigma = -1$:

It is clear from the figure that the system's oscillations stay around the equilibrium point $(0, 0, 0)$, without settling to it. The trajectory of the system does not converge to a single point.

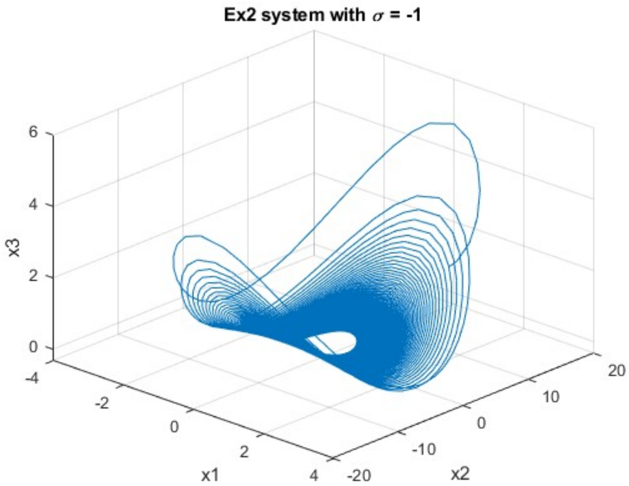

Figure 9: Plot of the solution with initial conditions [3 3 3] with $\sigma = -1$.

2.1.6 For $\sigma = -3$:

The figure presents a system with very complicated behavior. It is clear from the figure that the system's oscillations stay around the equilibrium point $(0, 0, 0)$, without settling to it. The trajectory of the system does not converge to a single point.

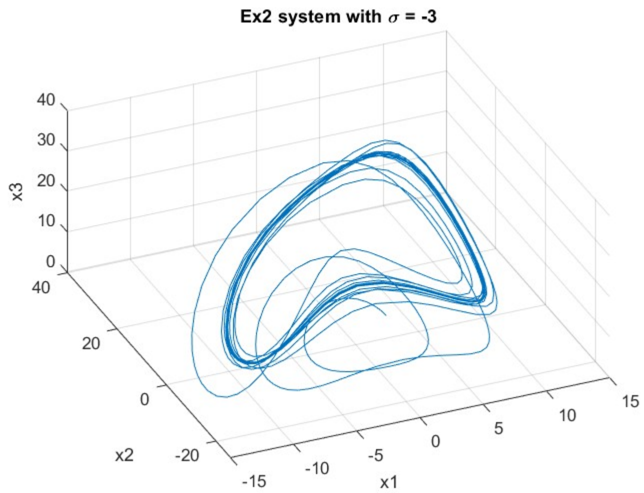

Figure 10: Plot of the solution with initial conditions [3 3 3] with $\sigma = -3$.

2.2 Find a value of σ for which the ω-limit set is neither a closed-orbit nor an equilibrium

From several simulations, it is found that for values of σ between 6 and 17 and for negative values of σ the system behaves as explained in the exercise above for $\sigma = 10, -1, -3$. It is not converges to an equilibrium point nor is it a closed-orbit trajectory as it does not cross second time from the initial condition along its path.