# Assignment

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Stationary Stochastic Processes

September 26, 2024

# 1 Analysis

## 1.1 Analysis of the stationary process

Given a stationary process:

$$y(t) = ay(t-1) + e(t) + ce(t-1)$$

where $e(\cdot) \sim WN(0, \lambda^2)$, $a \in \mathbb{R}, |a| < 1$, and $c \in \mathbb{R}, |c| < 1$.

Computation of the expected value:

$$\bar{y} = E[y(t)] = aE[y(t-1)] + \underbrace{E[e(t)]}_{0} + c\underbrace{E[e(t-1)]}_{0}$$
$$\bar{y} = E[y(t)] = aE[y(t-1)] \Rightarrow \bar{y} = E[y(t)] = 0$$

The explanation behind the reason why the expected value is constant lies behind the definition of the stationary process. It is already given that the process y(t) is stationary and due to that the First-order moment which is equal to the expected value ($m(t) = E[y(t)]$) is constant.

For the same reason $E[y(t)] = E[y(t-n)]\ \forall n \in \mathbf{Z}$

The variance is given by:

$$var[y(t)] = E\{[y(t) - \bar{y}]^2\}$$

and because $\bar{y} = 0$ the equation becomes:

$$var[y(t)] = E[y(t)^2]$$

$$
\begin{aligned}
Var[y(t)] &= E\left\{[y(t)-\bar{y}]^2\right\} = E\left\{[ay(t-1)+e(t)+ce(t-1)]^2\right\} \\
&= E\left\{a^2y(t-1)^2\right\} + E\left\{e(t)^2\right\} + E\left\{c^2e(t-1)^2\right\} \\
&\quad + 2aE\left\{y(t-1)e(t)\right\} + 2acE\left\{y(t-1)e(t-1)\right\} + 2cE\left\{e(t)e(t-1)\right\} \\
&= E\left\{a^2y(t-1)^2\right\} + \underbrace{E\left\{e(t)^2\right\}}_{\lambda^2} + c^2\underbrace{E\left\{e(t-1)^2\right\}}_{\lambda^2} \\
&\quad + 2aE\left\{y(t-1)e(t)\right\} + 2ac\underbrace{E\left\{y(t-1)e(t-1)\right\}}_{\lambda^2} + 2cE\left\{e(t)e(t-1)\right\} \\
Var[y(t)] &= a^2Var[y(t-1)] + \lambda^2 + c^2\lambda^2 + 2ac\lambda^2 \\
Var[y(t)] &= \frac{\lambda^2(1+2ac+c^2)}{1-a^2}
\end{aligned}
$$

Proof of: $E[y(t-1)e(t-1)] = \lambda^2$

$$
\begin{aligned}
E\left\{y(t-1)e(t-1)\right\} &= \left\{(ay(t-2)+e(t-1)+ce(t-2))e(t-1)\right\} \\
&= aE[y(t-2)e(t-1)] + E[e(t-1)^2] + cE[e(t-2)e(t-1)] \\
&= \lambda^2
\end{aligned}
$$

The covariance function is given by the following equation:

$$\gamma_y(\tau) = E[y(t)-\bar{y}][y(t-\tau)-\bar{y}] \text{ for } \tau = 1,2$$

and due to the reason that $\bar{y} = 0$ the equation becomes:

$$\gamma_y(\tau) = E[y(t)][y(t-\tau)] \text{ for } \tau = 1,2$$

Hence:

$$
\begin{aligned}
\gamma_y(0) &= E\{[y(t)]^2 = Var[y(t)] = \frac{\lambda^2(1+2ac+c^2)}{1-a^2} \\
\gamma_y(1) &= E\{[y(t)][y(t-1)]\} \\
&= E\{[ay(t-1)+e(t)+ce(t-1)][y(t-1)]\} \\
&= a\underbrace{E[y(t-1)^2]}_{\gamma_y(0)=Var[y(t)]} + E[e(t)y(t-1)] + c\underbrace{E[e(t-1)y(t-1)]}_{\lambda^2} \\
&= a\gamma_y(0) + c\lambda^2 \\
\gamma_y(2) &= E\{[y(t)][y(t-2)]\} \\
&= E\{[ay(t-1)+e(t)+ce(t-1)][y(t-2)]\} \\
&= a\underbrace{E[y(t-1)y(t-2)]}_{\gamma_y(1)} + E[e(t)y(t-2)] + cE[e(t-1)y(t-2)]\} \\
&= a\gamma_y(1)
\end{aligned}
$$

## 1.2 PEM Identification of AR Models

Estimation using AR(1) model:

$$
\begin{aligned}
\mathcal{M}_1(\theta_1): &\quad y(t) = a_1 y(t-1) + \xi(t), \qquad\qquad\qquad\qquad \theta_1 a_1 \\
\hat{\mathcal{M}}_1(\theta_1): &\quad \hat{y}(t) = a_1 y(t-1)
\end{aligned}
$$

The error is given by:

$$\epsilon(t) = y(t) - \hat{y}(t)$$

The cost function that we need to minimize is given by the equation:

$$
\begin{aligned}
J_1(\theta_1) = E\left[\epsilon(t)^2\right] =& E\left[(y(t) - \hat{y}(t))^2\right] \\
=& E\left[y(t)^2\right] - 2a_1 E\left[y(t)y(t-1)\right] + a_1^2 E\left[y(t-1)^2\right] \\
=& \gamma_y(0) - 2a_1\gamma_y(1) + a_1^2\gamma_y(0) \\
J_1(\theta_1) =& (1 + a_1^2)\gamma_y(0) - 2a_1\gamma_y(1)
\end{aligned}
$$

To find the solution for which the cost function is minimized we have to set the derivative of the cost function to zero:

$$\frac{dJ_1(\theta_1)}{da_1} = 0 = 2a_1\gamma_y(0) - 2\gamma_y(1)$$

Hence the optimal value of $\theta_1$ is:

$$a_1 = \frac{\gamma_y(1)}{\gamma_y(0)}$$

The variance of the error for the optimal value of $\theta_1$ is given by the cost function:

$$\mathrm{var}[\epsilon^o_{\theta_1}(t)] = J_1(a_1^o) = (1 + (\frac{\gamma_y(1)}{\gamma_y(0)})^2)\gamma_y(0) - 2\frac{\gamma_y(1)}{\gamma_y(0)}\gamma_y(1)$$

Which is simplified to:

$$\mathrm{var}[\epsilon^o_{\theta_1}(t)] = J_1(a_1^o) = \frac{\gamma_y(0)^2 - \gamma_y(1)^2}{\gamma_y(0)}$$

Estimation using the AR(2) model:

$$
\begin{aligned}
\mathcal{M}_2(\theta_2): &\quad y(t) = a_1 y(t-1) + a_2 y(t-2) + \xi(t), \qquad\qquad \theta_2 [a_1, a_2]^T \\
\hat{\mathcal{M}}_\in(\theta_2): &\quad \hat{y}(t) = a_1 y(t-1) + a_2 y(t-2)
\end{aligned}
$$

The error is given by:

$$\epsilon(t) = y(t) - \hat{y}(t)$$

We introduce the cost function that we have to minimize:

$$\begin{aligned} J_2(\theta_2) =& E\left[\epsilon(t)^2\right] = E\left[(y(t) - \hat{y}(t))^2\right] \\ =& (1 + a_1^2 + a_2^2)\gamma_y(0) - 2a_1\gamma_y(1) - 2a_2\gamma_y(2) + 2a_1a_2\gamma_y(1) \end{aligned}$$

We set the gradient of the cost function to zero in order to get the minimization point. Thus, now we have 2 equations with 2 unknowns:

$$\begin{aligned} \frac{dJ_2(\theta_2)}{da_1} =& 0 \Rightarrow 2a_1\gamma_y(0) - 2\gamma_y(1) + 2a_2\gamma_y(1) = 0 \\ \frac{dJ_2(\theta_2)}{da_2} =& 0 \Rightarrow 2a_2\gamma_y(0) - 2\gamma_y(2) + 2a_1\gamma_y(1) = 0 \end{aligned}$$

we solve for $a_1$ and $a_2$ in terms of the correlation functions $\gamma_y(0)$, $\gamma_y(1)$, and $\gamma_y(2)$. The solutions is:

$$\begin{aligned} a_1 &= \frac{\gamma_y(0) \cdot \gamma_y(1) - \gamma_y(1) \cdot \gamma_y(2)}{\gamma_y(0)^2 - \gamma_y(1)^2}, \\ a_2 &= \frac{\gamma_y(0) \cdot \gamma_y(2) - \gamma_y(1)^2}{\gamma_y(0)^2 - \gamma_y(1)^2}. \end{aligned}$$

Substituting these values into the equation for the variance of the prediction error, we get:

$$\mathrm{var}[\epsilon^o_{\theta_2}(t)] = J_2(\theta_2) = (1 + a_1^2 + a_2^2)\gamma_y(0) - 2a_1\gamma_y(1) - 2a_2\gamma_y(2) + 2a_1a_2\gamma_y(1),$$

which simplifies to:

$$\mathrm{var}[\epsilon^o_{\theta_2}(t)] = \frac{\gamma_y(0)^3 - 2\gamma_y(0)\gamma_y(1)^2 - \gamma_y(0)\gamma_y(2)^2 + 2\gamma_y(1)^2\gamma_y(2)}{\gamma_y(0)^2 - \gamma_y(1)^2}.$$

## 1.3 Not zero-mean white noise

For this exercise the white noise: $e(\cdot)$ have been replaced by $\tilde{e}(\cdot) = 1 + e(\cdot)$ with $e(\cdot) \sim WN(0, \lambda^2)$

The expected value of $\tilde{e}(\cdot)$ can be computed by:

$$\begin{aligned} E[\tilde{e}(t)] &= E[1] + E[e(t)] \\ &= 1 + 0 = 1 \end{aligned}$$

And the variance by:

$$\begin{aligned} Var[\tilde{e}(t)] &= Var[1] + Var[e(t)] \\ &= \lambda^2 \end{aligned}$$

So the new $\tilde{e}(t)$ will be a non-zero mean white noise with expected value 1 and variance $\lambda^2$.

Computation of the new expected value of $y(t)$:

$$\bar{y} = E[y(t)] = aE[y(t-1)] + \underbrace{E[\tilde{e}(t)]}_{1} + c\underbrace{E[\tilde{e}(t-1)]}_{1}$$

$$\bar{y} = a\bar{y} + 1 + c \Rightarrow \bar{y} = \frac{1+c}{1-a}$$

We let: $\bar{y} = \frac{1+c}{1-a}$, $\tilde{y}(t) = y(t) - \bar{y}$, $\bar{e} = 1$ and $\tilde{e}_{\text{new}}(t) = \tilde{e}(t) - \bar{e}$

We substitute the above assumptions in the stationary process:

$$y(t) = ay(t-1) + \tilde{e}(t) + c\tilde{e}(t-1)$$

$$\begin{aligned} \tilde{y}(t) + \frac{1+c}{1-a} &= a(\tilde{y}(t-1) + \frac{1+c}{1-a}) + (\tilde{e}_{\text{new}}(t) + 1) + c(\tilde{e}_{\text{new}}(t-1) + 1) \\ \tilde{y}(t) + \frac{1+c}{1-a} - a(\frac{1+c}{1-a}) - 1 - c &= a\tilde{y}(t-1) + \tilde{e}_{\text{new}}(t) + c\tilde{e}_{\text{new}}(t-1) \\ \tilde{y}(t) + \frac{(1+c)(1-a)}{(1-a)} - 1 - c &= a\tilde{y}(t-1) + \tilde{e}_{\text{new}}(t) + c\tilde{e}_{\text{new}}(t-1) \end{aligned}$$

Thus:

$$\tilde{y}(t) = a\tilde{y}(t-1) + \tilde{e}_{\text{new}}(t) + c\tilde{e}_{\text{new}}(t-1)$$

Where, $E[\tilde{y}(t)] = 0$ and $\tilde{e}_{\text{new}}(\cdot) \sim WN(0, \lambda^2)$

The new zero-mean stationary process, $\tilde{y}(t)$, has the same correlation function as the original process $y(t) = ay(t-1) + \tilde{e}(t) + c\tilde{e}(t-1)$. Since the new zero-mean stationary process has identical properties to those discussed in questions A1 and A2, the analysis for the new process can proceed similarly. Due to that the only difference between the two stationary processes is the expected value.