Fourier Transform

Xuping Feng

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1. Orthogonality of trigonometric functions

The trigonometric function system is given by

$$\begin{gathered}\{0,1,sinx,cosx,sin2x,cos2x,\ldots,sinnx,cosnx,\ldots,\ldots\}\\ \Downarrow \\ \{sin0x,cos0x,sinx,cosx,sin2x,cos2x,\cdots,sinnx,cosnx,\cdots\}\end{gathered} \tag{1}$$

where, $n=0,1,2,\cdots,+\infty$**.**

Here, for given $n,m=0,1,2,\cdots,+\infty$**, we calculate the following integrations,**

$$\int_{-\pi}^{\pi} sin(nx)cos(mx)dx=0 \tag{2}$$

$sin(nx)cos(mx)$ **is an odd function, its integration between the symmetric interval** $[-\pi,\pi]$ **is zero.**

$$\begin{aligned}\int_{-\pi}^{\pi} sin(nx)sin(mx)dx &= \begin{cases}\int_{-\pi}^{\pi} 0dx, n=m=0\\ \int_{-\pi}^{\pi}\frac{[1-cos(2nx)]}{2}dx, n=m\neq 0\\ \int_{-\pi}^{\pi}\frac{1}{2}[cos(n-m)x-cos(n+m)x]dx, n\neq m\end{cases}\\ &= \begin{cases}0, n=m=0\\ \frac{x}{2}|_{-\pi}^{\pi}-\frac{sin(2nx)}{4n}|_{-\pi}^{\pi}, n=m\neq 0\\ \frac{1}{4(n-m)}sin(n-m)x|_{-\pi}^{\pi}-\frac{1}{4(n+m)}sin(n+m)x|_{-\pi}^{\pi}, n\neq m\end{cases}\\ &= \begin{cases}0, n=m=0\\ \pi, n=m\neq 0\\ 0, n\neq m\end{cases}\end{aligned} \tag{3}$$

$$\int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \begin{cases} \int_{-\pi}^{\pi} 1dx, n = m = 0 \\ \int_{-\pi}^{\pi} \frac{1 + cos(2nx)}{2}dx, n = m \neq 0 \\ \int_{-\pi}^{\pi} \frac{1}{2}[cos(n-m)x + cos(n+m)x]dx, n \neq m \end{cases}$$
$$= \begin{cases} 2\pi, n = m = 0 \\ \pi, n = m \neq 0 \\ \frac{1}{4(n-m)}sin(n-m)x|_{-\pi}^{\pi} + \frac{1}{4(n+m)}sin(n+m)x|_{-\pi}^{\pi}, n \neq m \end{cases} \tag{4}$$
$$= \begin{cases} 2\pi, n = m = 0 \\ \pi, n = m \neq 0 \\ 0, n \neq m \end{cases}$$

We proved the orthogonality of trigonometric functions.

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2. Fourier Series

We assume that a function $f(x)$ has a period 2π, namely $f(x) = f(x + 2\pi)$. If $f(x)$ satisfies $Dirichlet$ condition,

$$\boxed{\begin{array}{l} 1. f(x) \text{ has finite discontinuity points of the first kind in a period;} \\ 2. f(x) \text{ has finite extreme points in a period.} \end{array}}$$

$f(x)$ can be expanded as $Fourier$ series.

$$f(x) = \sum_{n=0}^{+\infty}[a_n cos(nx) + b_n sin(nx)] \tag{5}$$

Here, we usually write $eq.(5)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_n cos(nx) + b_n sin(nx)] \tag{6}$$

where, $\frac{a_0}{2} = [a_0 cos(0x) + b_0 sin(0x)]$.

$$\boxed{Finding\, a_0}$$

We integrate $f(x)$ on $[-\pi, \pi]$,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{+\infty}[a_n \int_{-\pi}^{\pi} cos(0n)cos(nx)dx + b_n \int_{-\pi}^{\pi} cos(ox)sin(nx)dx] \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{+\infty}[a_n \cdot 0 + b_n \cdot 0] \\ &= a_0\pi \end{aligned} \tag{7}$$

So

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{8}$$

$$\boxed{Finding\, a_n}$$

We integrate $f(x)cos(mx)$ **on** $[-\pi, \pi]$**,**

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)cos(mx)dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} cos(mx)dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n cos(nx)cos(mx)dx \\
&\quad + \int_{-\pi}^{\pi} \sum_{n=1}^{+\infty} b_n sin(nx)cos(mx)dx \\
&= \frac{a_0}{2} \int_{-\pi}^{\pi} cos(0x)cos(mx)dx + \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} a_n cos(nx)cos(mx)dx \\
&\quad + \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} b_n sin(nx)cos(mx)dx \\
&= 0 + [\int_{-\pi}^{\pi} a_1 cos(1x)cos(mx)dx + \int_{-\pi}^{\pi} a_2 cos(2x)cos(mx)dx \\
&\quad + \ldots + \int_{-\pi}^{\pi} a_n cos(1x)cos(mx)dx] + \sum_{n=1}^{+\infty} 0 \\
&= \int_{-\pi}^{\pi} a_n cos(nx)cos(nx)dx \; (n = m) \\
&= a_n \pi
\end{aligned} \tag{9}$$

We have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)cos(nx)dx \tag{10}$$

$$\boxed{Finding\, b_n}$$

Similarly, we integrate $f(x)sin(mx)$ **on** $[-\pi, \pi]$

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)sin(mx)dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} sin(mx)dx + \int_{-\pi}^{\pi} \sum_{n=1}^{+\infty} a_n cos(nx)sin(mx)dx \\
&\quad + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n sin(nx)sin(mx)dx \\
&= \frac{a_0}{2} \int_{-\pi}^{\pi} cos(0x)sin(mx)dx + \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} a_n cos(nx)sin(mx)dx \\
&\quad + \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} b_n sin(nx)sin(mx)dx \\
&= 0 + \sum_{n=1}^{+\infty} 0 + [b_1 \int_{-\pi}^{\pi} sin(1x)sin(mx)dx \\
&\quad + b_2 \int_{-\pi}^{\pi} sin(2x)sin(mx)dx + \cdots + b_n \int_{-\pi}^{\pi} sin(nx)sin(mx)dx] \\
&= \int_{-\pi}^{\pi} sin(nx)sin(nx)dx \; (n = m) \\
&= b_n \pi
\end{aligned} \tag{11}$$

So we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sin(nx) dx \tag{12}$$

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3. Series expansion of function with a period of $2L$

We have function $f(t)$, and $f(t) = f(t+2L)$. $2L$ is the minimum of positive period of $f(t)$.

Here, we let

$$x = \frac{\pi}{L}t \tag{13}$$

And

$$\begin{aligned} f(t) &= f(\frac{L}{\pi}x) \\ &= g(x) \end{aligned} \tag{14}$$

The Fourier series of $g(x)$ is given by

$$\begin{cases} g(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{+\infty}[a_n cos(nx) + b_n sin(nx)] \\ a_0 = \displaystyle\int_{-\pi}^{\pi} f(x)dx \\ a_n = \displaystyle\int_{-\pi}^{\pi} cos(nx)dx \\ b_n = \displaystyle\int_{-\pi}^{\pi} sin(nx)dx \end{cases} \tag{15}$$

We change x to t using $eq.(13)$ and we have

$$\begin{cases} cos(nx) = cos(n\dfrac{\pi}{L}t) \\ sin(nx) = sin(n\dfrac{\pi}{L}t) \\ \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} dx = \dfrac{1}{\pi}\int_{-L}^{L} d\dfrac{\pi}{L}t = \dfrac{1}{L}\int_{-L}^{L} dt \\ \pi \to L \\ -\pi \to -L \end{cases} \tag{16}$$

Using $eq.(16)$, the Fourier series of $f(t)$ can be described as

$$\begin{cases} f(t) = \dfrac{a0}{2} + \sum\limits_{n=1}^{+\infty}[a_n cos(n\dfrac{\pi}{L}t) + b_n sin(n\dfrac{\pi}{L}t) \\ a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)dt \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)cos(n\dfrac{\pi}{L}t)dt \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)sin(n\dfrac{\pi}{L}t)dt \end{cases} \tag{17}$$

Alternatively, we let $2L = T$, and we have

$$\begin{cases} f(t) = \frac{a0}{2} + \sum_{n=1}^{+\infty}[a_n cos(n\frac{2\pi}{T}t) + b_n sin(n\frac{2\pi}{T}t) \\ a_0 = \frac{2}{T}\int_0^T f(t)dt \\ a_n = \frac{2}{T}\int_0^T f(t)cos(n\frac{2\pi}{T}t)dt \\ b_n = \frac{2}{T}\int_0^T f(t)sin(n\frac{2\pi}{T}t)dt \end{cases} \tag{18}$$

Let $\omega = \frac{2\pi}{T}$, and we change $eq.\,(18)$ to

$$\begin{cases} f(t) = \frac{a0}{2} + \sum_{n=1}^{+\infty}[a_n cos(n\omega t) + b_n sin(n\omega t) \\ a_0 = \frac{2}{T}\int_0^T f(t)dt \\ a_n = \frac{2}{T}\int_0^T f(t)cos(n\omega t)dt \\ b_n = \frac{2}{T}\int_0^T f(t)sin(n\omega t)dt \end{cases} \tag{19}$$

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4. Complex number form of Fourier series

The $Euler's$ formula is given by

$$\begin{cases} e^{i\theta} = cos\theta + isin\theta \\ cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ sin\theta = -i\frac{e^{i\theta} - e^{-i\theta}}{2} \end{cases} \tag{20}$$

Here, we constitute $Euler's$ formula into $eq.\,(19)$ and we get

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_n\frac{e^{in\omega t} + e^{-in\omega t}}{2} - ib_n\frac{e^{in\omega t} - e^{-in\omega t}}{2}] \\ &= \frac{a_0}{2} + \sum_{n=1}^{+\infty}\frac{a_n - ib_n}{2}e^{in\omega t} + \sum_{n=1}^{+\infty}\frac{a_n + ib_n}{2}e^{-in\omega t} \\ &= \frac{a_0}{2} + \sum_{n=1}^{+\infty}\frac{a_n - ib_n}{2}e^{in\omega t} + \sum_{n=-\infty}^{-1}\frac{a_{-n} + ib_{-n}}{2}e^{in\omega t} \\ &= \sum_{n=0}^{0}\frac{a_0}{2}e^{in\omega t} + \sum_{n=1}^{+\infty}\frac{a_n - ib_n}{2}e^{in\omega t} + \sum_{n=-\infty}^{-1}\frac{a_{-n} + ib_{-n}}{2}e^{in\omega t} \\ &= \sum_{-\infty}^{+\infty}c_n e^{in\omega t} \end{aligned} \tag{21}$$

where

$$c_n = \begin{cases} \frac{a_0}{2}, n = 0 \\ \frac{a_n - ib_n}{2}, n = 1, 2, 3, \cdots, +\infty \\ \frac{a_{-n} + ib_{-n}}{2}, n = -1, -2, -3, \ldots, -\infty \end{cases} \tag{22}$$

Use $eq.$ (19) and the coefficient c_n is given by

$$c_n = \begin{cases} \frac{a_0}{2} = \frac{1}{2}\frac{2}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt \quad (n=0) \\ \frac{a_n - ib_n}{2} = \frac{1}{2}\frac{2}{T}\int_0^T f(t)cos(n\omega t)dt - \frac{i}{2}\frac{2}{T}\int_0^T f(t)sin(n\omega t)dt \\ \qquad = \frac{1}{T}\int_0^T f(t)[cos(n\omega t) - isin(n\omega t)]dt \\ \qquad = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt \quad (n = 1, 2, 3, \cdots, +\infty) \\ \frac{a_{-n} + ib_{-n}}{2} = \frac{1}{2}\frac{2}{T}\int_0^T f(t)cos(-n\omega t)dt + \frac{i}{2}\frac{2}{T}\int_0^T f(t)sin(-n\omega t)dt \\ \qquad = \frac{1}{T}\int_0^T f(t)[cos(n\omega t) - isin(n\omega t)]dt \\ \qquad = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt \quad (n = -1, -2, -3, \cdots, -\infty) \end{cases} \tag{23}$$

So

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt \tag{24}$$

Let $\omega_0 = \frac{2\pi}{T}$, here we call ω_0 basic frequency, and we have

$$\begin{cases} f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \\ \qquad = [\cdots + c_{-n}e^{i(-n)\omega_0 t} \\ \qquad\quad + \cdots + c_{-1}e^{i(-1)\omega_0 t} + c_0 e^{i0\omega_0 t} + c_1 e^{i(1)\omega_0 t} \\ \qquad\quad + \cdots + c_n e^{i(n)\omega_0 t} + \cdots] \\ c_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t)e^{-in\omega_0 t}dt \end{cases} \tag{25}$$

If the function $f(t)$ is non-period, namely, $T \to +\infty$ and $f(t) = \lim\limits_{T\to+\infty} f_T(t)$, then we have

$$\begin{aligned} \Delta\omega &= (n+1)\omega_0 - n\omega_0 \\ &= \omega_0 \\ &= \frac{2\pi}{T} \end{aligned} \tag{26}$$

and let

$$\begin{aligned} \omega &= n\omega_0 \\ &= n\Delta\omega \end{aligned} \tag{27}$$

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-\infty}^{\infty} f(t)e^{-in\Delta\omega t}dt \\ &= \frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-in\Delta\omega t}dt \\ &= \frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \end{aligned} \tag{28}$$

Constitute $eq.$ (28) to $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Delta\omega t}$, and $f(t)$ will be

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt e^{-i\omega t} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt] e^{i\omega t} \Delta\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt} e^{i\omega t} d\omega
\end{aligned} \tag{29}
$$

$NOTE$
The definition of integration is described by
$\int h(x)dx = \lim_{\Delta x \to 0} \sum h(x)\Delta x$
$So \sum_{-\infty}^{\infty} \boxed{\cdots} e^{i\omega t} \Delta\omega = \int_{-\infty}^{\infty} \boxed{\cdots} e^{i\omega d\omega}$ described in eq.(29).

Here, we define $F(\omega) = \mathscr{F}[f(t)]$ as the $Fourier\ Transform$ of function $f(t)$, and $F(\omega)$ is closed by black box in $eq(29)$.

Finally, we derive $\mathscr{F}[f(t)]$ and $\mathscr{F}^{-1}[F(\omega)]$

$$
\begin{cases}
F(\omega) = \mathscr{F}[f(t)] \\
\qquad = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
f(t) = \mathscr{F}^{-1}[F(\omega)] \\
\qquad = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega
\end{cases} \tag{30}
$$

where, $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$ is the inversed $Fourier\ Transform$ of $F(\omega)$.

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5. Bibliography

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