Realisation of SO(3) complementary filter from inertial measurements of acceleration, angular velocuty and magnetic field as functions of time

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1 Introduction

Attitude estimation of an aerial controlled object consists on estimation of the so called Euler angles (roll, pitch and yaw) which are angles of rotation around 3 perpendicular axes linked to the object. Most generally, the Euler angles are deduced by the so called Inertial Measurement Units (IMU). An IMU is a device which can measure angular velocity provided by gyrometers as a basic data, and magnetic field or specific acceleration, provided by magnetometers and accelerometers, respectively, as the supplementary data. The main principle of work of an IMU is based on the inertial properties of the device and it is important that an IMU does not need any external signals or references for the measurements. For instance, the aforementioned term 'specific acceleration' does not designate the actual mechanical acceleration in an inertial frame, but designates the acceleration which is "felt" by a sensor. In fact, specific acceleration is a difference between the mechanical acceleration and the gravity [1] and is the value which is measured by an accelerometer.

As it has been said before, the main objects of the interest in attitude estimation are Euler angles. In the simplest case, they can be found by integration of the angular velocity around three perpendicular axes. The main problem laying behind this way of solution is the error accumulation. In fact, the error is being accumulated because of the integration in time. The gyrometers are thus vulnerable to the drift of zero position. In other words, the read-outs of a gyrometer are not equal to zero even if the system is returned to the initial position. The read-outs of gyrometers are thus reliable only on a short-term scale. In contrast, the measurements of accelerometers are reliable only on a long-term scale since an accelerometer detects any perturbation impacting the object such wind, engine vibrations and so one. However, an accelerometer is resistant to the drift of zero since there is no need in time integration. Therefore, it would be fruitful to use both a gyrometer and accelerometer for estimation of attitude of the aerial object and combine their advantages to make a good deal. To do so, a concept of a complementary filter has been suggested. One of the reviews on basic principles of a classical complementary filter is given in [2]. The idea of a classic complementary filter is the following: if ϕ^{k+1} is an angle of interest at the $k+1$-st time instant, Ω^k is the angular velocity at k-th time instant provided by gyrometers and ϕ^k_{acc} is the angle from accelerometer read-outs at k-th time instant, the angle of interest can be expressed as:

$$\phi^{k+1} = \alpha_{gyr}\Omega^k \Delta t_k + \alpha_{acc}\phi^k_{acc}, \tag{1}$$

where $\alpha_{gyr} + \alpha_{acc} = 1$; $\alpha_{gyr}, \alpha_{acc}$ designate the relative confidence weights for each of the devices and t_k is the discretization time.

The idea proposed in the paper of Hamel and Mahony [3] is the development of the classic complementary filter. It is the so called SO(3)-based complementary filter, where SO(3) is a designation for a group of orthogonal $\mathbb{R}^3 \to \mathbb{R}^3$ maps preserving the orientation. The main idea suggested by Mahony was to use a 3-axial accelerometer and a 3-axial magnetometer as supplementary 'tilt sensors' [4] provided that the average inertial acceleration of an aerial object is equal to zero. The explanation of the details of the algorithm, the way of the treatment of data from 3-axial gyrometer, magnetometer and accelerometer, demonstration of the results and possible tuning and optimization of the algorithm will be therefore given in the following sections of the report.

2 Description of the algorithm

To begin, let consider two reference frames. The first one (A) is the the inertial reference frame fixed to the Earth; the second one (B) is a moving reference frame fixed to the body. Let $R(t)$ be a transformation matrix from the A-frame to the B-frame at time instant t – in other words, the columns of the matrix $R(t)$ are the coordinates of the body-fixed axis vectors expressed in the coordinates of the A-frame. Let also the bases of A- and B-frames be orthonormal ones. In this case, $R(t) \in SO(3)$ is a composition of rotations around three perpendicular axes and contains all the information about trigonometric functions of the Euler angles. Therefore, the attitude estimation problem can be reformulated as a search of the matrix $R(t) \in SO(3)$ as a function of time.

The main equation describing the temporal evolution of the $R(t)$ matrix is:

$$\dot{R}(t) = R(t)\Omega_\times(t), \tag{2}$$

where $\Omega_\times(t)$ is a skew-symmetric matrix:

$$\Omega_\times = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}. \tag{3}$$

Here, $\Omega_{1,2,3}$ are the angular rotation velocities (of roll, pitch and yaw) around x, y, z perpendicular axes in the body frame. The main goal of the algorithm is to find the best smooth approximation $\hat{R}(t)$ of the $R(t)$ matrix as a function of time. Thus, the $\hat{R}(t)$ matrix is a transformation matrix between the A-frame and the new E-frame which is called 'estimator frame'. It should be noted that since both $\hat{R}(t)$ and $R(t)$ are orthogonal matrices, the inverse transformations from E to A and from B to A frames can be expressed as their adjoints $\hat{R}^T(t)$ and $R^T(t)$. Therefore, the matrix $\tilde{R}(t) = \hat{R}^T(t)R(t)$ is the transformation matrix between the E-frame and the body frame. Since the E-frame should be as closer as possible to the body frame, the $\tilde{R}(t)$ should uniformly converge to identity $||\tilde{R}(t) \longrightarrow I_3||$. Like this, a basic cost function E_c to be minimized can be defined:

$$E_c = ||\tilde{R}(t) \longrightarrow I_3||^2 = \frac{1}{2}\mathrm{tr}(\tilde{R}(t) - I_3). \tag{4}$$

The paper [3] describes two different ways of solution to the problem. The first one is error-based passive complementary filter. The main idea of the algorithm consists on the fact that a gyrometer measures the angular velocity not in a precise way but with an additional bias vector changing within the time. Therefore, the read-outs of the gyrometers should be corrected in order to eliminate the bias. In its turn, the bias vector can be derived from the read-outs of magnetometers and accelerometers. The error-based complementary filter uses an additional angular velocity as an innovation term which is derivable from the read-outs of magnetometers and accelerometers as well. Another opportunity is a passive filter which can be called as 'minimization-based' passive complementary filter. A slight difference between the two filters consists on the fact that the first one does not need an initial estimate of rotation matrix $R(t)$. Instead, it is necessary to estimate $R(t)$ at each time instant in order to implement the passive complementary filter. The estimation of $R(t)$ can be done by an optimization process which is not necessarily well defined though. Both the error-based complementary filter and the passive complementary filter have been implemented in the frame of the current report. Now, the details of both algorithms and their results will be demonstrated and discussed.

2.1 Error-based passive complementary filter

Let a_0 and m_0 be the normalized vectors of gravity and the external magnetic field, respectively. Both vectors are expressed in the inertial frame. Therefore, since $R(t)$ is the matrix of transformation from body frame to the inertial one,

$$a(t) = R(t)a_0 \tag{5}$$

and

$$m(t) = R(t)m_0 \tag{6}$$

are true vectors of the acceleration and the magnetic field in the body frame. On the other hand, if $\hat{a}(t)$ and $\hat{m}(t)$ are estimations of the acceleration and magnetic field felt by the sensors of the aerial object, they can be expressed via the $\hat{R}$ matrix in a similar way:

$$\hat{a}(t) = \hat{R}(t)a_0 \tag{7}$$

and

$$\hat{m}(t) = \hat{R}(t)m_0. \tag{8}$$

The main idea of the error-based complimentary filter can be reformulated in such a way that the estimates of the acceleration and magnetic field $\hat{a}(t)$ and $\hat{m}(t)$ have to be as closer as possible to true values $a(t)$ and $m(t)$. To reach the goal, let focus on the expression for the angular velocity and its evolution within time. Let Ω_y be the read-outs from the gyrometers. According to [2, 3] and the Equation 2, the modified dynamics of the rotation matrix $\hat{R}(t)$ can be expressed in the following way:

$$\dot{\hat{R}}(t) = \hat{R}(t)\left(\Omega_y(t) - \hat{b}(t) + \omega(t)\right)_\times, \tag{9}$$

where $\Omega(t) = \Omega_y(t) - \hat{b}(t) + \omega(t)$ is the true value of the angular velocity, $\hat{b}(t)$ is the estimate of the bias vector as a function of time and $\omega(t)$ is a correction angular velocity in the estimator frame. It is shown in [3] that if the temporal dynamics of the estimation of the correction angular velocity $\omega(t)$ and estimate of the bias vector $\hat{b}(t)$ obey the following expressions:

$$\omega(t) = -k_P\left(k_A a_y(t) \times \hat{R}^T(t)a_0 + k_M m_y(t) \times \hat{R}^T(t)m_0\right); \tag{10}$$

$$\frac{d\hat{b}(t)}{dt} = -\frac{k_I}{k_P}\omega(t). \tag{11}$$

where k_P and k_I are positive constants, k_A and k_M are weights expressing the confidence of accelerometers and magnetometers measurements, $k_A + k_M = 1$ (most generally, $k_A \gg k_M$ since magnetometers are more susceptible to external perturbations), the $\hat{R}(t)$ and $\hat{b}(t)$ converge asymptotically to their true values $R(t)$ and $b(t)$. The criterion of the convergence can be roughly formulated in such a way that the initial measurements and the initial bias vector should not be too far away from their true values (see Expression 7 in [3]).

It is now possible to formulate the steps of the implementation of the error-based complementary filter:

1. To normalize the measurement vectors $a_y(t)$ and $m_y(t)$ of the acceleration and magnetic field for each time instant.
2. To introduce $\hat{R}(t=0) = \hat{R}_0^T = I_3$ as the initial approximation of the rotation matrix.
3. To introduce $\hat{b}_0(t=0) = [0\ 0\ 0]^T$ as the initial approximation of the bias vector. A possible optimization of the approximation will be described in the section 4.
4. To introduce $a_0 = a_y(t=0)$ and $m_0 = m_y(t=0)$ as the true vectors of the acceleration and the magnetic field in the inertial frame since it coincides with the body frame at $t=0$. According to the Equation 10, $\omega_0 = [0\ 0\ 0]^T$.
5. To introduce $k_A = 0.9$ and $k_M = 0.1$ as initial approximation of weights for accelerometer and magnetometer read-outs. A possible optimization of the approximation will be described in the section 4. The values of $k_I = 1$ rad/s and $k_P = 0.3$ rad/s can be taken from [3].
6. To calculate the correction term ω_k according to the Equation 10:

$$\omega_k = -k_P \left(k_A a_{y,k} \times \hat{R}_{k-1}^T a_0 + k_M m_{y,k} \times \hat{R}_{k-1}^T m_0 \right) \tag{12}$$

7. To calculate estimation of the bias vector $\hat{b}_k$ according to the discretized Equation 11:

$$b_k = b_{k-1} - T \frac{k_I}{k_P} \omega_{k-1} \tag{13}$$

where T is the discretization period.

8. To calculate $\hat{R}_k$ according to the discretized Equation 9:

$$\hat{R}_k = \hat{R}_{k-1} \exp \left((\Omega_y(t) - \hat{b}(t) + \omega(t))T \right)_\times , \tag{14}$$

where the matrix exponential can be calculated according to the Rodrigue's formula [5].

9. To calculate the cost function at k-th time instant according to the following formula:

$$E_{cost,k} = k_A(1 - < \hat{R}_k^T a_0, a_{y,k} >) + k_M(1 - < \hat{R}_k^T m_0, m_{y,k} >) \tag{15}$$

10. To repeat steps 6-9 for all $1 \le k \le N$, where N is the number of measurements.

2.2 Passive complementary filter (minimization-based)

The passive complementary filter can be also reformulated in terms of minimization problem ([2, 3]). In this case, it is necessary to calculate the initial approximation of the rotation matrix R_y at each time step. The problem of R_y search can be expressed as follows:

$$R_y = argmin_{R \in SO(3)} \left(k_A ||a_0 - Ra_y||^2 + k_M ||m_0 - Ra_y||^2 \right) . \tag{16}$$

The solution to this problem known as Wahba's problem is given in [6] and consists on finding the Singular Value Decomposition of the matrix $B = (k_A a_0 a^T + k_m m_0 m^T)$. If $B = USV^T$, the optimal rotational matrix is:

$$R_y = U * diag([1\ 1\ detUdetV]) * V^T . \tag{17}$$

The steps of the implementation of the passive complementary filter are the following:

1. To introduce the initial approximation of the rotational matrix $\hat{R}_0 = I_3$
2. To calculate $R_{y,i}$ using the Equation 17.
3. To calculate $\tilde{R}_i = \hat{R}_i^T R_{y,i}$
4. To use the asymmetric projection map $k\pi(R) = \frac{k}{2}(R - R^T)$ on the $\tilde{R}_i$, where $k > 0$ is a parameter of the algorithm. The result will be taken as the correction angular velocity ω_i.
5. To calculate the corrected matrix of angular velocity:
$$A_i = \omega_i + (\Omega_{y,i})_\times \tag{18}$$
and to make time integration in order to renew the matrix $\hat{R}_{i+1} = \hat{R}\exp(A_i T)$ using the Rodrigue's formula.
6. To repeat steps 2-5 for all $1 \leq i \leq N$, where N is the number of measurements.

It should be noted that this approach has a disadvantage. It consists on the fact that the bias vector is not being explicitly calculated at each time instant whereas the implementation error-based passive complementary filter provides it.

2.3 Calculation of Euler angles

As soon as the $\hat{R}_k$ matrices and the corrected angular velocities are calculated, it is possible to calculate Euler angles. The first way to do it is to integrate the corrected angular velocity (see Equation (9) or (18)) on time. In this case, the Euler angles will be defined unambiguously since the Euler angles will be calculated themselves and not their trigonometric functions. The time integration can introduce additional error which is though compensated by the filter. Another way to define the Euler angles is to derive the the trigonometric functions of them from the $\hat{R}_k$ matrices. In this case, there will be no time integration but there will be the unambiguity since the trigonometric functions are periodic. The way to do it consists on angle-axis representation of a rotation. It is shown in [5] that the pitch angle β, roll angle α and yaw angle γ can be derived from the elements of the matrix $\hat{R}$. The subscripts in the following expressions designate the indices of the matrix elements:

$$\beta = atan2\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right);\ \alpha = atan2\left(\frac{r_{23}}{\sin\beta}, \frac{r_{13}}{\sin\beta}\right);\ \gamma = atan2\left(\frac{r_{32}}{\sin\beta}, -\frac{r_{31}}{\sin\beta}\right). \tag{19}$$

Here, $-\pi/2 \leq \alpha \leq \pi/2$; $-\pi \leq \beta \leq \pi$ and $-\pi \leq \gamma \leq \pi$. The results of the implementation of the algorithm will be given in the next section. The time integration of corrected angular velocity is chosen for calculation of Euler angles.

3 Implementation of the algorithm on the given data

Figures 1-3 represent the result of calculations of the roll, pitch and yaw angle as functions of time, respectively. The calculations with both types of complementary filter as well as without any filter are demonstrated. As it can be seen, the results for the yaw and roll angles do not have a lot of difference, whereas the pitch angle curves have a significant discrepancy. It should be noted that the error-based algorithm provides the least values of the pitch angle after 15 s. It can be caused by the fact that this kind of a filter is the most efficient one, and therefore is less susceptible to the drift of zero. Still, the fact that the yaw and roll angles curves have an obvious zero drift whether a filter is implemented or not, witnesses the necessity of the further optimization of initial conditions of the algorithm. The optimization process will be described in the next section.

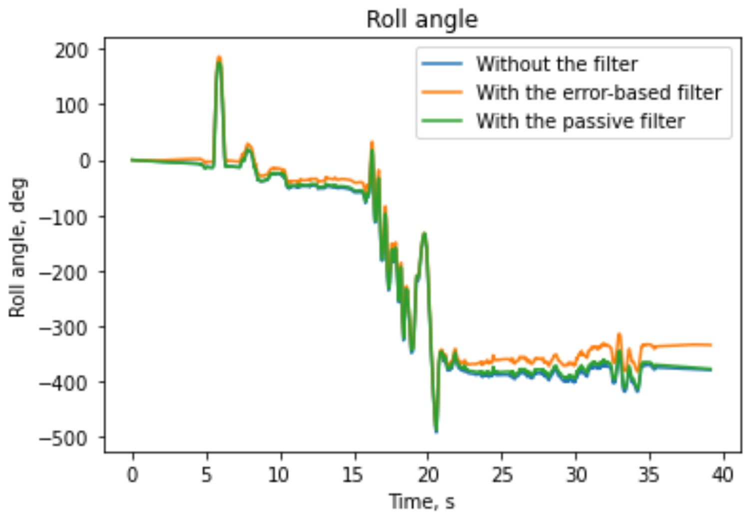

Figure 1: Roll angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based passive filter used, green curve: minimization-based passive filter used.

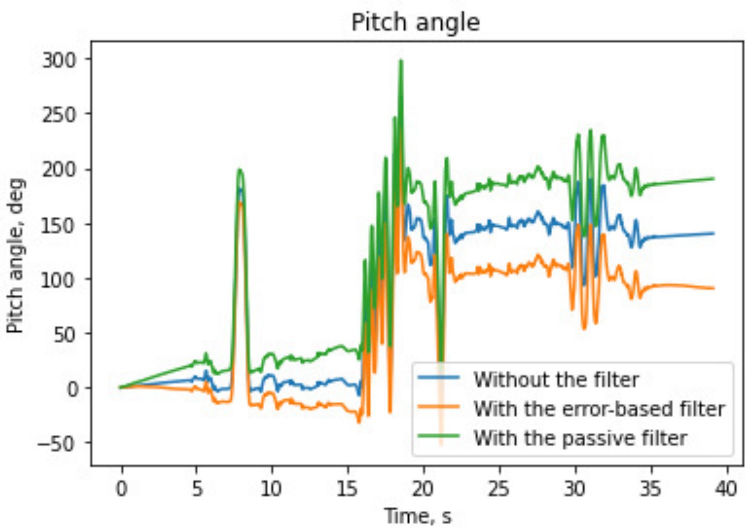

Figure 2: Pitch angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based passive filter used, green curve: minimization-based passive filter used.

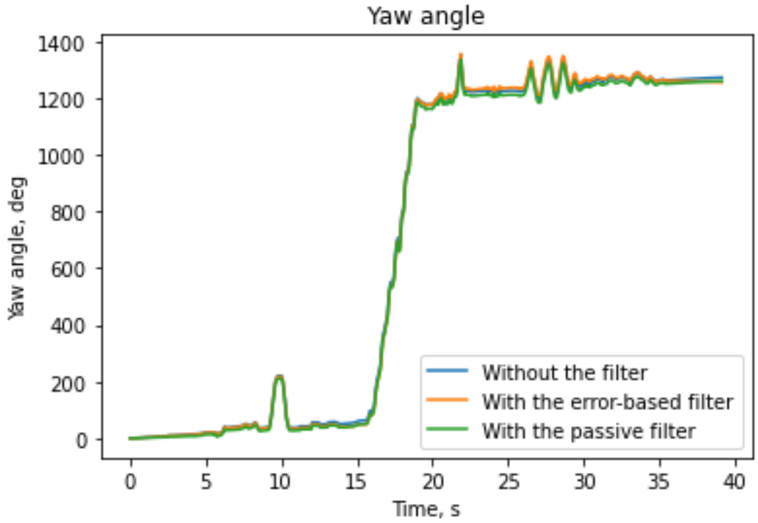

Figure 3: Yaw angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based passive filter used, green curve: minimization-based passive filter used.

4 Optimization of the algorithm

One of the ideas of the optimization process consists on the minimization of the sum of the cost function calculated at each time instant using the Equation 15. As it has been said in the description of the algorithm (see Section 2.1), it is possible to use a global minimum search algorithm in order to find a better approximation of the initial bias vector. This parameter is important since it influences the condition of exponential convergence of the $\hat{R}(t), \hat{b}(t)$ approximations to their exact temporal profiles $R(t), b(t)$ [3]. The Nelder-Mead method (or the simplex method) has been chosen since it is relatively fast and the cost function is not smooth. Therefore, gradient descent methods are not applicable in the current case. The initial approximation of the bias vector was chosen as $[0\ 0\ 0]^T$. After the use of the algorithm, the optimal bias vector was found to be equal to $\hat{b}_{opt} = [-0.058, 0.005, 0.066]^T$. The results for the new calculations are given in the Figures 4-6.

The enhancement of the results can be seen most clearly in the Figure 6 which demonstrates the temporal profile of the yaw angle. It can be seen that the drift of zero before 15 s has disappeared and the results became more reliable.

Other input parameters which influence the algorithm are k_I and k_P. The initial approximation of them has been taken as 1.0 rad/s and 0.3 rad/s, respectively, just like it has been proposed in [3]. Again, the Nelder-Mead algorithm has been used in order to find the optimal values of these parameters. The optimal parameters were found to be: $k_{I,opt} = 0.3$ rad/s and $k_{P,opt} = 3.0$ rad/s. The Figures 7-9 demonstrate the new results of calculations. It can be seen that the results became more reliable than previously in terms of zero drift. The most significant change is the disappearance of the negative bias of the pitch angle between 0 and 15 s. As for the yaw angle, a decrease of bias between 15 and 40 s is also noticeable.

5 Conclusion

Mathematical concept of two variations of a passive complementary filter has been briefly described. Then, the steps of both algorithms were given and then the algorithms were tested on the real raw

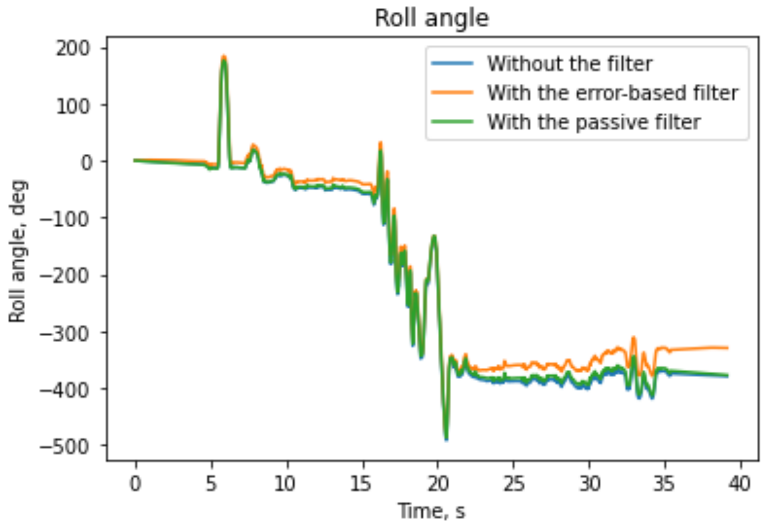

Figure 4: Roll angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based **optimized** passive filter used, green curve: minimization-based passive filter used.

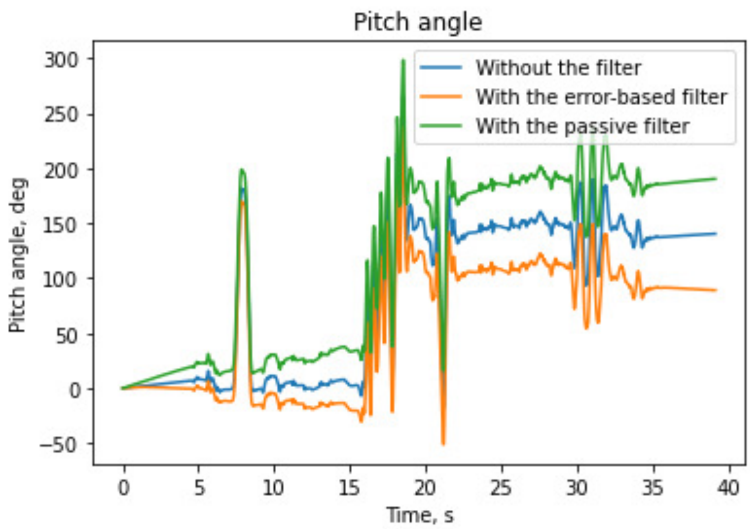

Figure 5: Pitch angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based **optimized** passive filter used, green curve: minimization-based passive filter used.

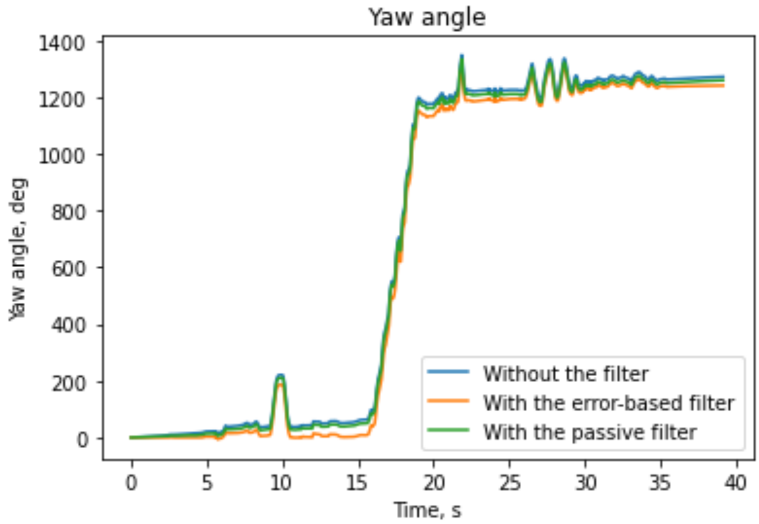

Figure 6: Yaw angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based **optimized** passive filter used, green curve: minimization-based passive filter used.

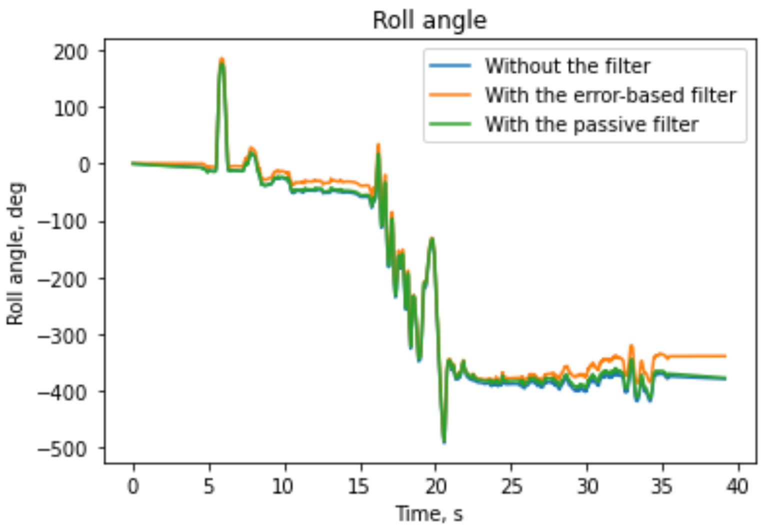

Figure 7: Roll angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based $\mathbf{k}_I$, $\mathbf{k}_P$**-optimized** passive filter used, green curve: minimization-based passive filter used.

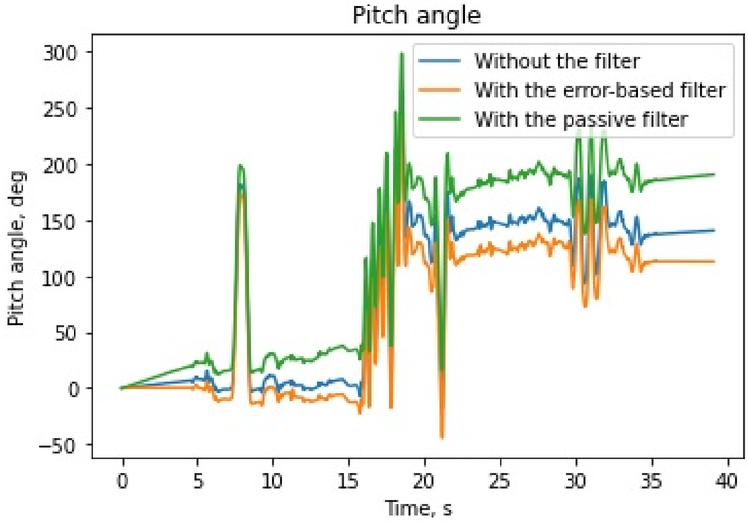

Figure 8: Pitch angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based $\mathbf{k}_I$, $\mathbf{k}_P$**-optimized** passive filter used, green curve: minimization-based passive filter used.

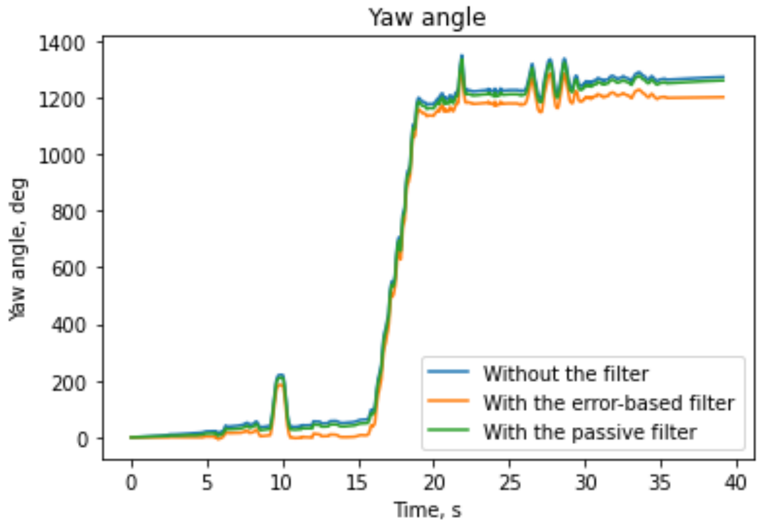

Figure 9: Yaw angle: integrated angular velocity as a function of time. Blue curve: without any filter (raw data), orange curve: error-based $\mathbf{k}_I$, $\mathbf{k}_P$**-optimized** passive filter used, green curve: minimization-based passive filter used.

data from Inertial Measurement Units (gyrometers, accelerometers and magnetometers). The Euler angles (roll, pitch and yaw) have been deduced by time integration of corrected angular velocity which, in its turn, was obtained due to the filter. It has been shown that the error-based variation of the passive complementary filter is the optimal algorithm among the two variations. A way of tuning of the input parameters of the algorithm based on the minimization of the cost function has been suggested and its relevance has been demonstrated.

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