Weekly Homework 10

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Problem 1) $\Sigma = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$
We must have Σ positive definite. By the Cauchy-Schwartz inequality:
$\mathrm{cov}(X_1, X_2)^2 \leq \mathrm{cov}(X_1, X_1)\mathrm{cov}(X_2, X_2)$
$\mathrm{cov}(X_1, X_2)^2 \leq \mathrm{var}(X_1)\mathrm{var}(X_2)$
$r^2 \leq 1$ ... $r \in [-1, 1]$
For both $r = -1, r = 1$, the determinant of the matrix is 0, therefore the matrix is degenerate. Thus: $r \in (-1, 1)$. We can use the following expansion for the joint probability density function:

$$\Pr(X = x) = p(X) = \frac{1}{2\pi} \det(\Sigma)^{-1} \exp(-q(X - \mu))$$

, where q is defined by the following:

$$q(Z) = \frac{\frac{z_1^2}{\Sigma_{1,1}^2} - \frac{2\sqrt{1-\det(\Sigma)}z_1 z_2}{\mathrm{tr}(\Sigma)} + \frac{z_2^2}{\Sigma_{2,2}^2}}{2\det(\Sigma)}$$

Substituting in the values form this problem:

$$p(X) = \frac{1}{2\pi(1 - r^2)} \exp(-\frac{(x_1 - \mu_1)^2 - 2r(x_1 - \mu_1)(x_2 - \mu_2) + (x_2 - \mu_2)^2}{2(1 - r^2)})$$

For the eigen values we must set the determinant of the matrix equal to 0:
$\det(\Sigma) = 0 = (1 - \lambda)^2 - r^2$, thus $\lambda = 1 \pm r$. Consider the positive case λ_1, and the negative case λ_2. For each λ we can find the corresponding eigen vectors accordingly:
$(\Sigma - \lambda I)v = 0$
$\begin{bmatrix} -r & r \\ r & -r \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, thus $v_1 = c(1, 1) : c \in \mathbb{R}$, after normalization: $v_1 = (\frac{\sqrt(2)}{2}, \frac{\sqrt(2)}{2})$
$\begin{bmatrix} r & r \\ r & r \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, thus $v_2 = c(1, -1) : c \in \mathbb{R}$, after normalization: $v_2 = (\frac{\sqrt(2)}{2}, \frac{-\sqrt(2)}{2})$