An introduction to survival analysis

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TTE data consists of $(\mathit{time}, \overbrace{\mathit{event}}^{\text{yes/no}})$ tuples.

must be a binary variable

Time-to-event (TTE) data

TTE analysis is also known as:

- survival analysis
- failure time analysis
- reliability theory (engineering)
- duration modelling (economics)
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Use cases for TTE analysis:

- clinical research
- customer analytics (churn)
- hardware (equipment failure)

Example: Covid-19 treatment trial

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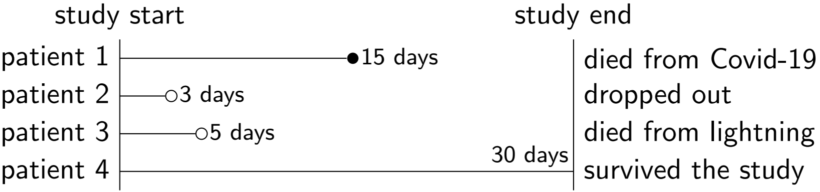
patient	received ABC?	outcome
1	yes	died from Covid-19 on day 15
2	no	dropped out of the study after day 3
3	yes	died by a lightning stroke on day 5
4	no	survived the study (30 days)

Example: Covid-19 treatment trial



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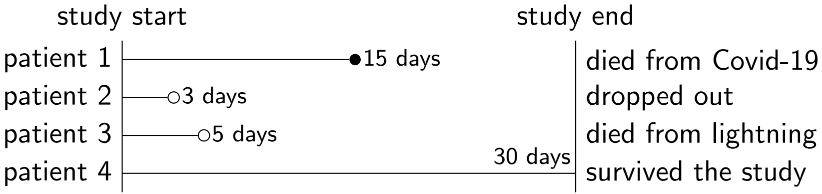

The **time** is the number of days since testing positive for Covid-19.
The **event** is whether the patient died due to Covid-19.

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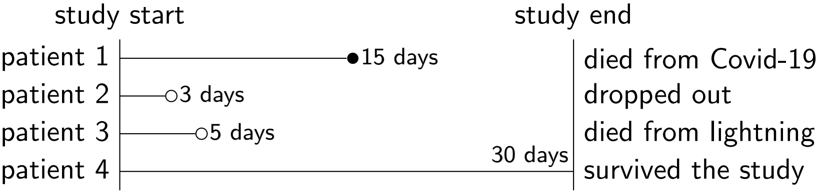

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Time-to-event data

patient	time	event
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2	?	?
3	?	?
4	?	no

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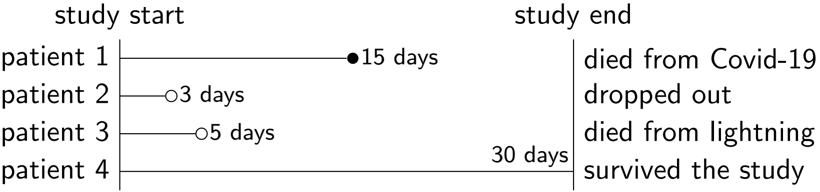

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Time-to-event data

patient	time	event
1	15	yes
2	$[0, 3]$	no
3	$[0, 5)$	no
4	$[0, 30]$	no

Censoring

Censoring occurs when we have some information about an individual's survival time, but don't know the exact time. Possible reasons include

- not experiencing the event before the study concludes;
- getting lost to follow-up during the study period;
- withdrawing from the study.

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We just saw examples of *right-censored* data.

Survival function

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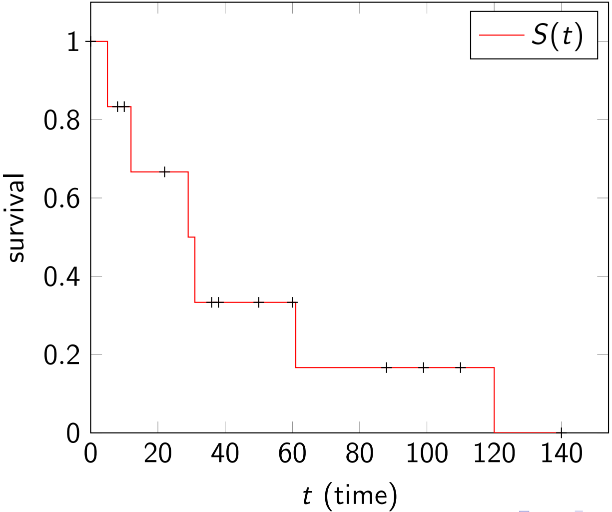
Survival function

$$S(t) = \Pr(T > t)$$

Survival curve



Modelling the survival function

The **Kaplan-Meier estimator** provides a non-parametric estimate of the survival function $S(t)$ using the survival curve.

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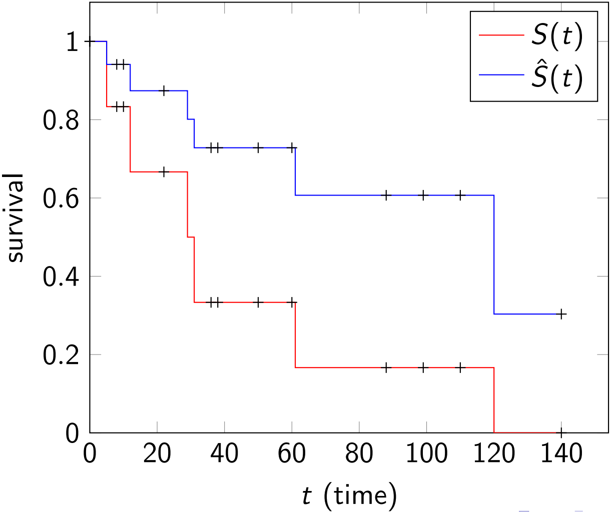
Kaplan-Meier estimator

$$\hat{S}(t) = \prod_{i:t_i \leq t} \left(1 - \frac{d_i}{n_i}\right)$$

where

- t_i is an event time
- d_i is the number of deaths at time t_i
- n_i is the number of individuals *known to have survived* until t_i

Survival curve and Kaplan-Meier estimator



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Hazard function

$$
\begin{aligned}
\lambda(t) &= \quad \frac{\Pr\left(\qquad\qquad\qquad \middle| \qquad \right)}{} \\
&= \lim_{dt \to 0} \frac{\Pr\left(t \leq T \leq t + dt\right)}{dt \cdot S(t)}
\end{aligned}
$$

What does survival depend on?

Recall the survival function $S(t) = \Pr(T > t)$ as the probability that an individual will survive past time t.

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Recall the survival function $S(t) = \Pr(T > t)$ as the probability that an individual will survive past time t. Let's assume that $S(t)$ depends on

1. the **baseline hazard function** (how risk of event occurence changes over time at baseline covariates); and
2. the **effect parameters** (how hazard varies due to the covariates), also known as the *partial hazard*.

Cox's proportional hazards model

Cox's proportional hazards model uses both factors to provide a semi-parametric estimate of the hazard function $\lambda(t)$ conditioned on the covariates $\mathbf{x}$.

Cox's proportional hazards model

$$\lambda(t|\mathbf{x}) = \overbrace{\lambda_0(t)}^{\substack{\text{baseline}\\ \text{hazard}}} \overbrace{\exp\left(\sum_{i=1}^{n} \beta_i \mathbf{x}_i\right)}^{\text{partial hazard}}$$

Proportional hazards assumption

The model assumes fixed **proportional hazards**, i.e. the hazard for an individual i in proportion to the hazard of any other individual j is fixed over time.

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Therefore,

- the baseline hazard $\lambda_0(t)$ is independent of the covariates, and
- the partial hazard is time-independent.

Partial likelihood

For each individual i, let

- T_i be a possibly censored survival time random variable, and
- $\mathbf{X}_i$ denote the covariates.

Further, let the **risk set** $\mathcal{R}(t) = \{i : T_i \geq t\}$ be the set of individuals that are "at risk" at time t.

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Cox proposed a **partial likelihood** for $\boldsymbol{\beta}$ without involving $\lambda_0(t)$. Maximising this function allows us to estimate the parameters $\boldsymbol{\beta}$.

$$L(\boldsymbol{\beta}) = \prod_{j=1}^{N} \Pr\left(\text{individual } j \text{ dies} \mid \text{one death from } \mathcal{R}(T_j)\right)$$

Partial likelihood formula

$$
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L(\beta) &= \prod_{j=1}^{N} \Pr\left(\text{individual } j \text{ dies} \mid \text{one death from } \mathcal{R}(T_j)\right) \\
&= \ldots \\
&= \prod_{j=1}^{N} \frac{\lambda(T_j|\mathbf{X}_j)}{\sum_{k\in\mathcal{R}(T_j)} \lambda(T_j|\mathbf{X}_k)} \\
&= \prod_{j=1}^{N} \frac{\lambda_0(T_j)\exp\left(\beta\mathbf{X}_j\right)}{\sum_{k\in\mathcal{R}(T_j)} \lambda_0(T_j)\exp\left(\beta\mathbf{X}_k\right)} \\
&= \prod_{j=1}^{N} \frac{\exp\left(\beta\mathbf{X}_j\right)}{\sum_{k\in\mathcal{R}(T_j)} \exp\left(\beta\mathbf{X}_k\right)}
\end{aligned}
$$

Parameter estimation

We can estimate the parameters $\boldsymbol{\beta}$ by minimizing the negative partial log-likelihood, i.e. $-\log L(\boldsymbol{\beta})$, by taking the partial derivatives with respect to the parameters $\boldsymbol{\beta}$ and solving for the minimum using e.g. the Newton-Raphson algorithm.

Hazard ratios

The fraction used to express the proportional hazards assumption is actually the **hazard ratio**, measuring the risk of individual i relative to individual j:

$$HR = \frac{\lambda(t|\mathbf{X}_i)}{\lambda(t|\mathbf{X}_j)} = \exp\left(\beta(\mathbf{X}_i - \mathbf{X}_j)\right).$$

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We may be interested in the relative risk associated with a particular covariate c, specifically the risk of said covariate having value c_i compared to c_j. Consider two dummy individuals i and j differing only in the c^{th} covariate, i.e. $\mathbf{X}_{i,k} = \mathbf{X}_{j,k}$ for $k \neq c$. Then the relative risk associated with c_i compared to c_j is

$$HR = \exp\left(\beta_c(c_i - c_j)\right).$$

Interpretation of hazard ratios

- $HR = 1$: no effect
- $HR > 1$: increase in hazard
- $HR < 1$: reduction in hazard