

An introduction to survival analysis

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What is time-to-event (TTE) data?

We can measure **time** in:

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TTE data consists of $(time, \overset{\text{yes/no}}{\text{event}})$ tuples.

Time-to-event (TTE) data

TTE analysis is also known as:

- survival analysis
- failure time analysis
- reliability theory (engineering)
- duration modelling (economics)
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Use cases for TTE analysis:

- clinical research
- customer analytics (churn)
- hardware (equipment failure)

Example: Covid-19 treatment trial

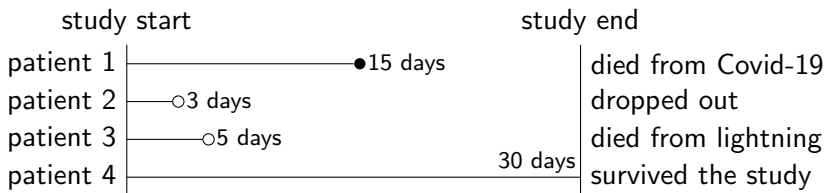
A randomised controlled trial ($n = 4$) was conducted to assess the efficacy of drug ABC in treating Covid-19. This is what happened to the patients:

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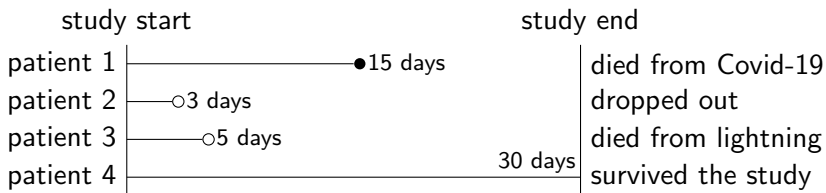
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patient	received ABC?	outcome
1	yes	died from Covid-19 on day 15
2	no	dropped out of the study after day 3
3	yes	died by a lightning stroke on day 5
4	no	survived the study (30 days)

Example: Covid-19 treatment trial

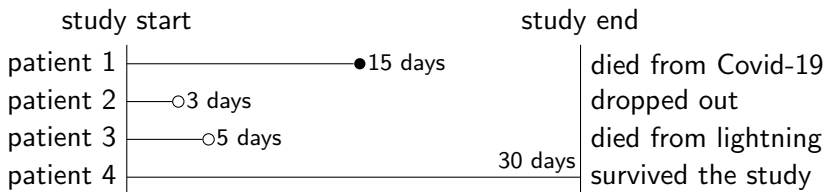


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Time-to-event data

patient	time	event
1	15	yes
2	[0, 3]	no
3	[0, 5]	no
4	[0, 30]	no

Censoring

Censoring occurs when we have some information about an individual's survival time, but don't know the exact time. Possible reasons include

- not experiencing the event before the study concludes;
- getting lost to follow-up during the study period;
- withdrawing from the study.

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We just saw examples of *right-censored* data.

Survival function

Let T be a continuous random variable representing survival time. The **survival function** $S(t)$ is the probability that an individual will survive past time t .

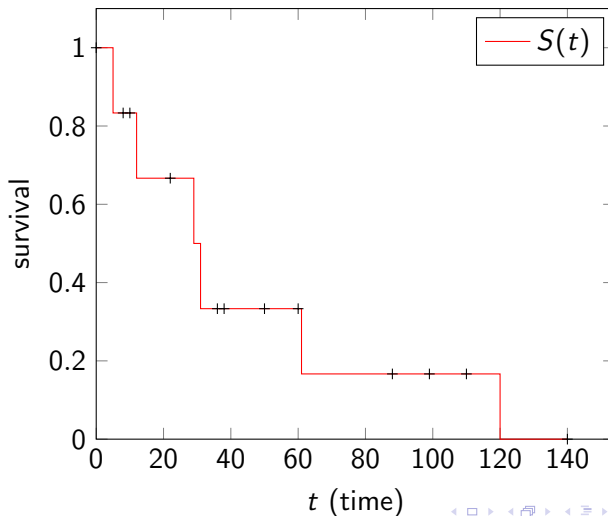
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Survival function

$$S(t) = \Pr(T > t)$$

Survival curve



Modelling the survival function

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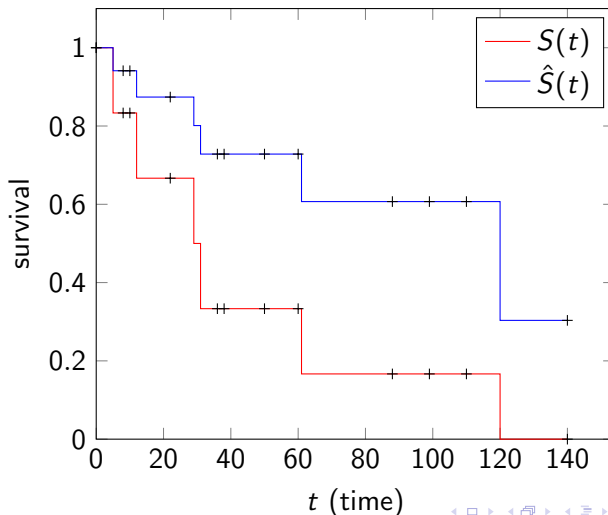
Kaplan-Meier estimator

$$\hat{S}(t) = \prod_{i: t_i \leq t} \left(1 - \frac{d_i}{n_i}\right)$$

where

- t_i is an event time
- d_i is the number of deaths at time t_i
- n_i is the number of individuals *known to have survived* until t_i

Survival curve and Kaplan-Meier estimator



Hazard function

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What does survival depend on?

Recall the survival function $S(t) = \Pr(T > t)$ as the probability that an individual will survive past time t .

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Recall the survival function $S(t) = \Pr(T > t)$ as the probability that an individual will survive past time t . Let's assume that $S(t)$ depends on

- 1 the **baseline hazard function** (how risk of event occurrence changes over time at baseline covariates); and
- 2 the **effect parameters** (how hazard varies due to the covariates), also known as the *partial hazard*.

Cox's proportional hazards model

Cox's proportional hazards model uses both factors to provide a semi-parametric estimate of the hazard function $\lambda(t)$ conditioned on the covariates \mathbf{x} .

Cox's proportional hazards model

$$\lambda(t|\mathbf{x}) = \underbrace{\lambda_0(t)}_{\text{baseline hazard}} \underbrace{\exp \left(\sum_{i=1}^n \beta_i \mathbf{x}_i \right)}_{\text{partial hazard}}$$

Proportional hazards assumption

The model assumes fixed **proportional hazards**, i.e. the hazard for an individual i in proportion to the hazard of any other individual j is fixed over time.

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Therefore,

- the baseline hazard $\lambda_0(t)$ is independent of the covariates, and
- the partial hazard is time-independent.

Partial likelihood

For each individual i , let

- T_i be a possibly censored survival time random variable, and
- \mathbf{X}_i denote the covariates.

Further, let the **risk set** $\mathcal{R}(t) = \{i : T_i \geq t\}$ be the set of individuals that are “at risk” at time t .

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$$L(\beta) = \prod_{j=1}^N \Pr(\text{individual } j \text{ dies} \mid \text{one death from } \mathcal{R}(T_j))$$

Partial likelihood formula

$$\begin{aligned} L(\beta) &= \prod_{j=1}^N \Pr(\text{individual } j \text{ dies} \mid \text{one death from } \mathcal{R}(T_j)) \\ &= \dots \\ &= \prod_{j=1}^N \frac{\lambda(T_j | \mathbf{x}_j)}{\sum_{k \in \mathcal{R}(T_j)} \lambda(T_j | \mathbf{x}_k)} \\ &= \prod_{j=1}^N \frac{\lambda_0(T_j) \exp(\beta \mathbf{x}_j)}{\sum_{k \in \mathcal{R}(T_j)} \lambda_0(T_j) \exp(\beta \mathbf{x}_k)} \\ &= \prod_{j=1}^N \frac{\exp(\beta \mathbf{x}_j)}{\sum_{k \in \mathcal{R}(T_j)} \exp(\beta \mathbf{x}_k)} \end{aligned}$$

Parameter estimation

We can estimate the parameters β by minimizing the negative partial log-likelihood, i.e. $-\log L(\beta)$, by taking the partial derivatives with respect to the parameters β and solving for the minimum using e.g. the Newton-Raphson algorithm.

Hazard ratios

The fraction used to express the proportional hazards assumption is actually the **hazard ratio**, measuring the risk of individual i relative to individual j :

$$HR = \frac{\lambda(t|\mathbf{X}_i)}{\lambda(t|\mathbf{X}_j)} = \exp(\beta(\mathbf{X}_i - \mathbf{X}_j)).$$

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We may be interested in the relative risk associated with a particular covariate c , specifically the risk of said covariate having value c_i compared to c_j . Consider two dummy individuals i and j differing only in the c^{th} covariate, i.e. $\mathbf{X}_{i,k} = \mathbf{X}_{j,k}$ for $k \neq c$. Then the relative risk associated with c_i compared to c_j is

$$HR = \exp(\beta_c(c_i - c_j)).$$

Interpretation of hazard ratios

- $HR = 1$: no effect
- $HR > 1$: increase in hazard
- $HR < 1$: reduction in hazard