Supplementary material for "WRAPD: Weighted Rotation-aware ADMM for Parameterization and Deformation"

GEORGE E. BROWN, University of Minnesota
RAHUL NARAIN, Indian Institute of Technology Delhi

ACM Reference Format:
George E. Brown and Rahul Narain. 2021. Supplementary material for "WRAPD: Weighted Rotation-aware ADMM for Parameterization and Deformation". *ACM Trans. Graph.* 40, 4, Article 82 (August 2021), 2 pages. https://doi.org/10.1145/3450626.3459942

1 THRESHOLD FOR REWEIGHTING

As discussed in Section 5.1 of the main paper, we update ADMM's global matrix only when the candidate weights differ significantly from the current weights. In particular, we flag the entire mesh for reweighting if the ratio of any element's squared candidate weight to its current squared weight exceeds γ (or falls below γ^{-1}). Small values of γ will trigger refactorization more often, improving convergence per iteration but increasing the computational cost, while larger values will do the opposite. Here, we evaluate the effect of the choice of γ on the overall performance of WRAPD.

In Figure 1, we plot the computation time needed for WRAPD to reach the early, middle, and late phases of convergence. As can be seen, the computational cost is fairly insensitive to γ over a broad range of values, only rising steeply when γ is very small (excessive refactorization) or very large (poor convergence).

2 MAXIMUM WEIGHT CLAMPING STRATEGY

When updating weights, we also clamp the squared candidate weight to lie in a bounded range $[\beta_{\min} w_0^2, \beta_{\max} w_0^2]$. For parameterization, we fix the lower bound $\beta_{\min} = 1$ and gradually increase the upper bound $\beta_{\max} = \min(10 \times 1.5^k, 10^9)$ as the iterations progress. Here we evaluate the benefits of this "easing" process.

Parameterization problems are initialized by mapping to a disk, giving many elements extremely high distortion in the initial state. Without weight clamping, the weights for such elements grow excessively large in the very earliest iterations. This leads to various numerical difficulties: the global system matrix becomes ill-conditioned in the presence of weights spanning several orders of magnitude, and convergence is slowed since elements with overestimated weights are unable to make progress.

An example is shown in Figure 2, where WRAPD without easing takes much longer to find a flip-free state. Among the nine meshes shown in Figure 11 of the main paper, this was the only one that exhibited this behavior. However, such difficulties occur more frequently in the larger parameterization dataset from Liu et al. [2018]. In Table 1, we report the time to early and late convergence, along with the percentage of meshes that failed to converge. Without easing, the failure rate for WRAPD+CM reaches 17.9% on the most challenging D3 dataset, indicating that such meshes failed to reach an injective state. With easing, the failure rate of WRAPD+CM on D3 drops to only 2.3%.

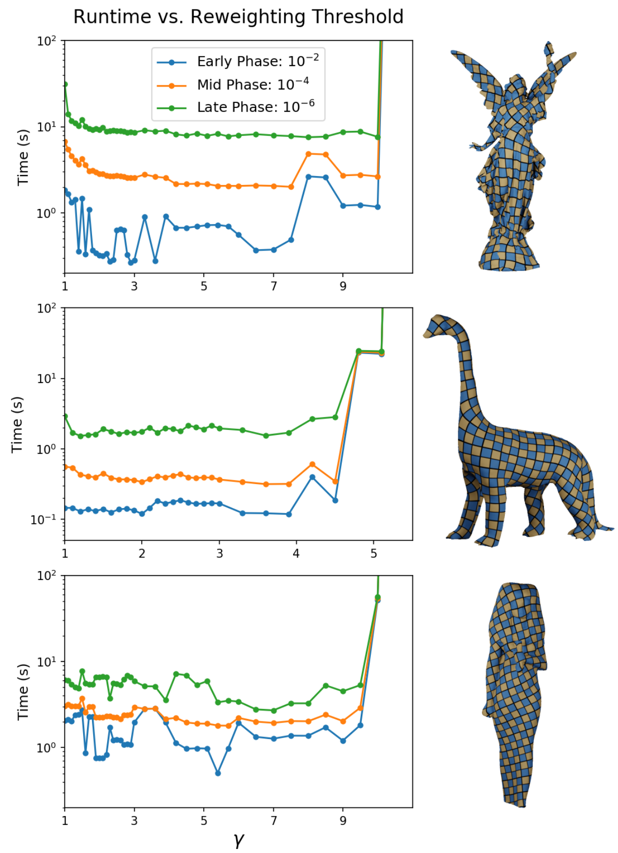

Fig. 1. Comparing reweighting thresholds using a symmetric Dirichlet parameterization example. *Left*: Runtime cost vs. reweighting threshold parameter γ. Plotted are the times required to reach the early, middle, and late phases of convergence. We define these thresholds at energy errors of 10^{-2}, 10^{-4}, and 10^{-6}, respectively. *Right*: Original mesh textured with the minimum energy solution.

Authors' addresses: George E. Brown, University of Minnesota, brow2327@umn.edu; Rahul Narain, Indian Institute of Technology Delhi, narain@cse.iitd.ac.in.

0730-0301/2021/8-ART82 $15.00
https://doi.org/10.1145/3450626.3459942

Set	Meshes	WRAPD (Easing)			WRAPD (Fixed)			WRAPD+CM (Easing)			WRAPD+CM (Fixed)		
		F.R.	τ (μs)	t_c (μs)	F.R.	τ (μs)	t_c (μs)	F.R.	τ (μs)	t_c (μs)	F.R.	τ (μs)	t_c (μs)
D1	5140	0.6%	5.5	21.1	1.0%	5.2	19.6	**0.4%**	3.2	12.2	0.7%	**3.2**	**11.9**
D2	6189	1.6%	9.6	44.1	5.4%	9.7	47.1	**1.3%**	4.7	18.0	6.3%	**4.5**	**17.9**
D3	4250	10.7%	20.5	95.1	18.2%	22.7	105.3	**2.3%**	8.4	**30.9**	17.9%	**7.7**	32.1

Table 1. Comparing maximum weight clamping strategies using large parameterization datasets and minimizing the symmetric Dirichlet energy. WRAPD (Easing) and WRAPD+CM (Easing) follow our proposed clamp easing strategy, while the (Fixed) counterparts always use the same maximum clamp value. The term F.R. stands for the failure rate, the percentage of meshes that failed to reach late convergence within 5000 iterations. τ (t_c) denote the resolution-normalized median time required to reach the early (late) phases of convergence, respectively. The lowest failure rates and fastest median times are bolded.

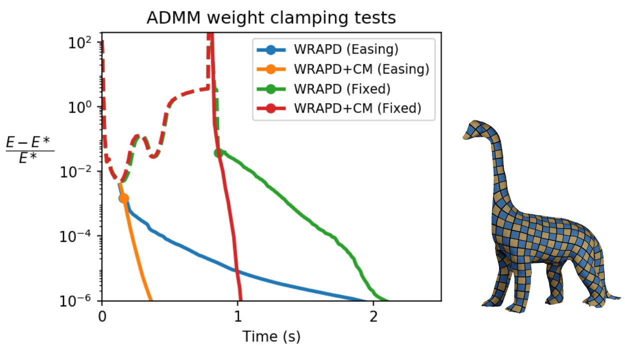

Fig. 2. Comparing weight clamping strategies using a symmetric Dirichlet parameterization example. WRAPD and WRAPD+CM were each run twice, first using our variable, geometrically increasing clamping heuristic, and again using a fixed clamp value. *Left*: Energy error vs. time. A dashed line indicates when an intermediate solution has triangle flips. *Right*: Original mesh textured with the minimum energy solution.

3 DERIVATION OF GRADIENT IN WRAPD GLOBAL STEP

First, we find the derivative of $\mathbf{S} = \mathrm{sym}(\mathbf{F})$, the stretching factor in the polar decomposition $\mathbf{F} = \mathbf{RS}$. We seek the differential $\delta\mathbf{S}$ such that $\mathrm{sym}(\mathbf{F} + \delta\mathbf{F}) = \mathbf{S} + \delta\mathbf{S}$ to first order. Derivatives of the polar decomposition have also appeared in previous work [Twigg and Kačić-Alesić 2010; Barbič 2012]; however, these focus only on differentiating the rotation factor $\mathbf{R}$ in polar decomposition.

We have $\mathbf{S}^2 = \mathbf{F}^T\mathbf{F}$, so

$$\mathbf{S}(\delta\mathbf{S}) + (\delta\mathbf{S})\mathbf{S} = \mathbf{F}^T(\delta\mathbf{F}) + (\delta\mathbf{F})^T\mathbf{F}. \tag{1}$$

Using the facts that $\mathbf{F} = \mathbf{U}\Sigma\mathbf{V}^T$ and $\mathbf{S} = \mathbf{V}\Sigma\mathbf{V}^T$, one can obtain

$$\Sigma(\delta\mathbf{T}) + (\delta\mathbf{T})\Sigma = \Sigma(\delta\mathbf{G}) + (\delta\mathbf{G})^T\Sigma. \tag{2}$$

where $\delta\mathbf{G} = \mathbf{U}^T(\delta\mathbf{F})\mathbf{V}$ and $\delta\mathbf{T} = \mathbf{V}^T(\delta\mathbf{S})\mathbf{V}$. Comparing entrywise,

$$(\sigma_i + \sigma_j)\delta t_{ij} = \sigma_i\,\delta g_{ij} + \sigma_j\,\delta g_{ji} \tag{3}$$

$$\implies \delta t_{ij} = \frac{\sigma_i\,\delta g_{ij} + \sigma_j\,\delta g_{ji}}{\sigma_i + \sigma_j}. \tag{4}$$

Thus, we have $\delta\mathbf{S} = \mathbf{V}(\delta\mathbf{T})\mathbf{V}^T$, where $\delta\mathbf{T}$ is given by (4).

With this knowledge, we can now differentiate the objective to be minimized in the global step. Recall that the objective is

$$L(\mathbf{X}) = \sum_i \frac{w_i^2}{2}\|\,\mathrm{sym}(\mathbf{D}_i\mathbf{X}) - \mathbf{P}_i\|_F^2 + \text{terms independent of } \mathbf{X}. \tag{5}$$

Considering a single element and differentiating with respect to $\mathbf{X}$ (dropping the subscript i for clarity), we have

$$\begin{aligned}\delta L &= w^2(\mathrm{sym}(\mathbf{D}_i\mathbf{X}) - \mathbf{P}) : \delta(\mathrm{sym}(\mathbf{D}_i\mathbf{X}))\\ &= w^2(\mathrm{sym}(\mathbf{F}) - \mathbf{P}) : \mathbf{V}(\delta\mathbf{T})\mathbf{V}^T\\ &= w^2(\Sigma - \mathbf{V}^T\mathbf{P}\mathbf{V}) : \delta\mathbf{T}\end{aligned} \tag{6}$$

with $\mathbf{F} = \mathbf{D}_i\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T$, and $\delta\mathbf{T}$ defined as above.

It can be shown further that since $\delta\mathbf{T}$ and $\delta\mathbf{G}$ are related by (4), we have $\Sigma : \delta\mathbf{T} = \Sigma : \delta\mathbf{G}$, and $\mathbf{V}^T\mathbf{P}\mathbf{V} : \delta\mathbf{T} = \mathbf{H} : \delta\mathbf{G}$ where $\mathbf{H}$ is the matrix with entries

$$h_{ij} = \frac{\sigma_i(\mathbf{v}_i^T\mathbf{P}\mathbf{v}_j + \mathbf{v}_j^T\mathbf{P}\mathbf{v}_i)}{\sigma_i + \sigma_j}. \tag{7}$$

Thus, we obtain

$$\begin{aligned}\delta L &= w^2(\Sigma - \mathbf{H}) : \delta\mathbf{G}\\ &= w^2\mathbf{D}^T(\mathbf{F} - \mathbf{U}\mathbf{H}\mathbf{V}^T) : \delta\mathbf{X}\end{aligned} \tag{8}$$

since $\delta\mathbf{G} = \mathbf{U}^T(\delta\mathbf{F})\mathbf{V} = \mathbf{U}^T\mathbf{D}(\delta\mathbf{X})\mathbf{V}$.

REFERENCES

Jernej Barbič. 2012. *Exact Corotational Linear FEM Stiffness Matrix.* Technical Report. University of Southern California.

Ligang Liu, Chunyang Ye, Ruiqi Ni, and Xiao-Ming Fu. 2018. Progressive Parameterizations. *ACM Transactions on Graphics(SIGGRAPH)* 37, 4 (2018).

Christopher D. Twigg and Zoran Kačić-Alesić. 2010. Point Cloud Glue: Constraining Simulations Using the Procrustes Transform. In *Proceedings of the 2010 ACM SIGGRAPH/Eurographics Symposium on Computer Animation (SCA '10).* Eurographics Association, Goslar, DEU, 45–54.