ENGINEERING

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Revisiting *Utility Maximization in Incomplete Markets*: Theory, Replication and New Results

Group #I1 Utility Maximization

Runming Huang

George Liu

Luis Schmitz

Supervised by

Prof. Thibaut Mastrolia

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1. Introduction

Utility maximization lies at the heart of modern portfolio theory, tracing back to the seminal continuous-time models of Merton (1971) and subsequent developments in incomplete markets. In a complete market, investors can perfectly hedge all sources of risk, and their optimal trading strategies admit explicit closed-form solutions. However, in practice, markets are often incomplete: certain risks, ranging from illiquid assets to random future liabilities, cannot be fully hedged with the available traded securities. This incompleteness poses both conceptual and technical challenges for the classical utility-maximization problem, as the duality methods developed for complete markets no longer apply directly.

In their influential 2005 paper "Utility Maximization in Incomplete Markets," Hu, Imkeller and Müller introduce a powerful backward stochastic differential equation (BSDE) framework to solve the primal utility-maximization problem under general trading constraints and without assuming market completeness. They show that, for exponential, power and logarithmic utilities, one can construct an auxiliary process whose martingale and supermartingale properties pinpoint the optimal value function and associated trading strategy. Crucially, their approach accommodates closed (not necessarily convex) portfolio constraints, thereby extending many previous duality-based results.

In this project we first provide a careful replication of the Hu–Imkeller–Müller BSDE method under a simplified one-dimensional market setup. We verify step by step the derivation of the BSDE drivers, the optimal strategies, and the resulting value functions for exponential, power and log utilities, both with and without bounded terminal liabilities. We then extend their analysis in three directions:

1. We introduce a general liability for the power utiltiy and logarthmic utility functions, which were ommited in the original paper.
2. For every utility we give analytical results for specific liabilities: constant and hedgeable Gaussian liabilities, highlighting why log-investors remain "myopic" and require no additional hedge demand.
3. We perform numerical simulations to compare the theoretical strategies with Monte Carlo approximations, thereby illustrating the practical performance and robustness of the BSDE-derived policies.

The remainder of this report is organized as follows. Chapter 2 reviews the BSDE construction for exponential utility, including detailed proofs of existence, uniqueness and admissibility under general constraints. Chapter 3 carries out the corresponding analysis for power utility and introduces the liability to this function. Chapter 4 focuses on logarithmic utility, both in the unconstrained and liability-driven settings, and explains the striking absence of hedging components for log investors. We conclude in Chapter 5 with remarks on potential generalizations, such as multi-asset extensions, stochastic volatility, and dynamic risk measures—and avenues for future work.

All of our simulation and numerical-optimization code is publicly available on GitHub:

`github.com/george-lhj/util-maximization/tree/main`

for full reproducibility.

1.1 Model and Assumptions Throughout Our Review

We assume throughout that the investor is subject to following market conditions and wants to maximize his utility:

- **One risky asset:** $d = m = 1$.
- **Brownian motion:** A single one-dimensional Brownian motion $\{W_t\}_{0 \le t \le T}$.
- **Market dynamics:**
$$\frac{dS_t}{S_t} = b_t\,dt + \sigma_t\,dW_t, \qquad \sigma_t > 0.$$
- **Market price of risk:** $\theta_t := \dfrac{b_t}{\sigma_t}$.
- **No trading constraints:** Control set $C = \mathbb{R}$ (no short-sale or leverage limits).
- **Wealth process:**
$$dX_t^\pi = \pi_t\,b_t\,dt + \pi_t\,\sigma_t\,dW_t, \quad X_0 = x.$$
- **Utility function:** A general U; in many examples we take exponential, however, it could be extended to other type of utility functions.
$$U(x) = -e^{-\alpha x}, \quad \alpha > 0.$$
- **Terminal liability:** $F \in L^\infty(\mathcal{F}_T)$.
- **Objective:**
$$\sup_\pi \mathbb{E}\big[U\big(X_T^\pi - F\big)\big].$$

2. Exponential Utility

2.1 Maximization with Liability

We consider the proposed exponential utility maximization problem in the paper with the previously mentioned simplifications. The exponential utility function and problem constructs as follows:

Utility function: Exponential utility

$$U(x) = -e^{-\alpha x}, \quad \alpha > 0$$

Terminal liability: $F \in L^\infty(\mathcal{F}_T)$, i.e., bounded and $\mathcal{F}_T$-measurable **Objective:**

$$\sup_\pi \mathbb{E}\left[-e^{-\alpha(X_T^\pi - F)}\right]$$

Step 1: Constructing a Supermartingale

The objective is to solve:

$$\sup_\pi \mathbb{E}\left[-\exp\left(-\alpha(X_T^\pi - F)\right)\right].$$

For this setup, it is known that the value function takes the form

$$V(x) = -e^{-\alpha(x - Y_0)},$$

where Y_0 plays the role of a *certainty equivalent* for the random claim F, adjusted for market incompleteness and the investor's risk preferences. Our goal is to characterize this certainty equivalent dynamically over time using a backward stochastic differential equation (BSDE). To do this, we define a process Y_t, adapted to the filtration generated by the Brownian motion W_t, which evolves backward from the terminal value $Y_T = F$.

To facilitate the analysis, we define the auxiliary process:

$$R_t := -\exp\left(-\alpha(X_t^\pi - Y_t)\right).$$

The key idea is to choose Y_t such that R_t is a supermartingale under any admissible strategy π, and a true martingale under the optimal strategy π^*. This approach ensures that $R_0 = V(x)$is the value function.

Where we anticipate Y_t to satisfy a BSDE of the form:

$$Y_t = F - \int_t^T Z_s\, dW_s - \int_t^T f(s, Z_s)\, ds,$$

for some adapted process Z_t, where $f(t, z)$ is a generator function to be determined such that the desired supermartingale condition on R_t holds.

Subsequently, we will require that:

- R_t is a supermartingale for all admissible π,
- and a martingale for the optimal strategy π^*.

Step 2: Apply Itô's Lemma

To find the condition under which R_t is a supermartingale, apply Itô's lemma to R_t. Denote:

$$f(t) := -\alpha(X_t^\pi - Y_t), \quad R_t = e^{f(t)}.$$

We compute:

$$dR_t = R_t \, df(t) + \frac{1}{2} R_t \, d\langle f \rangle_t.$$

We have:

$$dX_t^\pi = \pi_t b_t \, dt + \pi_t \sigma_t \, dW_t, \quad dY_t = -f(t, Z_t) \, dt + Z_t \, dW_t.$$

Hence:

$$d(X_t^\pi - Y_t) = (\pi_t b_t + f(t, Z_t)) \, dt + (\pi_t \sigma_t - Z_t) \, dW_t.$$

Then:

$$df(t) = -\alpha d(X_t^\pi - Y_t) = -\alpha \left[(\pi_t b_t + f(t, Z_t)) \, dt + (\pi_t \sigma_t - Z_t) \, dW_t \right],$$

and

$$d\langle f \rangle_t = \alpha^2 (\pi_t \sigma_t - Z_t)^2 \, dt.$$

Putting it all together:

$$\begin{aligned} dR_t &= R_t \left[\alpha(\pi_t b_t + f(t, Z_t)) \, dt + \alpha(\pi_t \sigma_t - Z_t) \, dW_t + \frac{\alpha^2}{2} (\pi_t \sigma_t - Z_t)^2 \, dt \right] \\ &= R_t \left[\left(\alpha(\pi_t b_t + f(t, Z_t)) + \frac{\alpha^2}{2} (\pi_t \sigma_t - Z_t)^2 \right) dt + \alpha(\pi_t \sigma_t - Z_t) \, dW_t \right]. \end{aligned}$$

Step 3: Choice of Generator $f(t, z)$

To ensure R_t is a supermartingale for any π, we choose $f(t, z)$ such that the drift term is non-positive for all π:

$$\alpha \pi_t b_t + \alpha f(t, z) + \frac{\alpha^2}{2} (\pi_t \sigma_t - z)^2 \leq 0.$$

This occurs precisely when π_t minimizes the drift term derived in Step 3:

$$\pi_t^* = \arg\min_\pi \left\{ \pi b_t + \frac{\alpha}{2} (\pi \sigma_t - Z_t)^2 \right\}.$$

Solving this yields:

$$\pi_t^* = \frac{1}{\sigma_t} \left(Z_t + \frac{1}{\alpha} \theta_t \right) = \frac{1}{\sigma_t} \left(Z_t + \frac{1}{\alpha} \frac{b_t}{\sigma_t} \right).$$

This is subsequently also the optimal strategy π^*.

Plug back the optimal π^* into the drift to obtain:

$$f(t, z) = -\pi^* b_t - \frac{\alpha}{2} (\pi^* \sigma_t - z)^2.$$

We compute:

$$\pi^* b_t = \left(z + \frac{1}{\alpha} \theta_t \right) \theta_t, \quad \pi^* \sigma_t - z = \frac{1}{\alpha} \theta_t.$$

Thus:

$$f(t, z) = -\left(z + \frac{1}{\alpha} \theta_t \right) \theta_t - \frac{\alpha}{2} \left(\frac{1}{\alpha} \theta_t \right)^2 = -z\theta_t - \frac{1}{\alpha} \theta_t^2 - \frac{1}{2\alpha} \theta_t^2.$$

But to make the sign convention consistent with the original definition of R_t, the generator becomes:

$$\boxed{f(t, z) = z\theta_t + \frac{1}{2\alpha} \theta_t^2.}$$

Step 4: Final BSDE and Derivation of Optimal Strategy

Having derived the generator $f(t,z) = z\theta_t + \frac{1}{2\alpha}\theta_t^2$, we now define the backward stochastic differential equation (BSDE) satisfied by the value process Y_t:

$$\boxed{Y_t = F - \int_t^T Z_s \, dW_s - \int_t^T \left(Z_s\theta_s + \frac{1}{2\alpha}\theta_s^2 \right) ds}$$

This BSDE has terminal condition $Y_T = F$ and generator function $f(t,z)$ as derived above. Since $F \in L^\infty(\mathcal{F}_T)$, and the generator $f(t,z)$ is linear (hence Lipschitz) in z, the BSDE admits a unique adapted solution (Y_t, Z_t) in the spaces $\mathcal{S}^\infty \times \mathcal{H}^2$.

Recalling that $R_t = -e^{-\alpha(X_t^\pi - Y_t)}$ is a supermartingale for all admissible strategies and a martingale under the optimal strategy π^*, we conclude that:

$$\boxed{V(x) = \mathbb{E}\left[-e^{-\alpha(X_T^{\pi^*} - F)} \right] = -e^{-\alpha(x - Y_0)},}$$

where Y_0 is the initial value of the solution to the BSDE.

The optimal strategy π^* was obtained in Step 3 before and is:

$$\boxed{\pi_t^* = \frac{1}{\sigma_t}\left(Z_t + \frac{1}{\alpha}\theta_t \right) = \frac{1}{\sigma_t}\left(Z_t + \frac{1}{\alpha}\frac{b_t}{\sigma_t} \right).}$$

2.2 Remark: Simplified Cases of F

(a) Constant Liability $F = \bar{F} \in \mathbb{R}$:

If the liability is a deterministic constant, the backward stochastic differential equation becomes trivial. Since F is not random, there is no uncertainty to hedge, so the process $Z_t \equiv 0$. The BSDE reduces to:

$$Y_t = \bar{F} - \int_t^T \left(0 \cdot \theta_s + \frac{1}{2\alpha}\theta_s^2 \right) ds = \bar{F} - \int_t^T \frac{1}{2\alpha}\theta_s^2 \, ds.$$

Therefore, the initial value Y_0 becomes:

$$Y_0 = \bar{F} - \int_0^T \frac{1}{2\alpha}\theta_s^2 \, ds.$$

The value function becomes:

$$V(x) = -\exp\left(-\alpha \left(x - \bar{F} + \int_0^T \frac{1}{2\alpha}\theta_s^2 \, ds \right) \right) = -e^{-\alpha(x - \bar{F})} \cdot \exp\left(\frac{1}{2}\int_0^T \theta_s^2 \, ds \right).$$

The optimal strategy is:

$$\pi_t^* = \frac{1}{\sigma_t} \cdot \left(0 + \frac{1}{\alpha}\theta_t \right) = \frac{1}{\alpha} \cdot \frac{\theta_t}{\sigma_t} = \frac{b_t}{\alpha\sigma_t^2}.$$

(b) Hedgeable Gaussian Liability We now introduce a terminal liability $F \sim N(\mu_F, \sigma_F^2)$ due at time T which is *not independent* of the market filtration $(F_t)_{0 \le t \le T}$ generated by the Brownian motion W_t. Hence any normally–distributed liability can be written as

$$F = \mu_F + \sigma_F \frac{W_T}{\sqrt{T}} = \mu_F + \kappa W_T, \qquad \kappa := \sigma_F/\sqrt{T}.$$

Here μ_F is the deterministic "face value" while κW_T is the stochastic part that loads linearly on the same Brownian motion driving the asset price.

The BSDE for exponential utility

$$Y_t = F - \int_t^T Z_s\, dW_s - \int_t^T \Big(Z_s\theta_s + \tfrac{1}{2\alpha}\theta_s^2\Big) ds, \qquad \alpha > 0,$$

must reproduce the Brownian part κW_T at $t = T$. Selecting the *constant* process

$$Z_s \equiv \kappa, \qquad 0 \le s \le T,$$

gives $-\int_t^T Z_s\, dW_s = \kappa(W_t - W_T)$, so the terminal condition $Y_T = F$ is satisfied. Substituting $Z_s = \kappa$ yields

$$Y_t = \mu_F + \kappa W_t - \int_t^T \Big(\kappa\theta_s + \tfrac{1}{2\alpha}\theta_s^2\Big) ds, \quad Y_0 = \mu_F - \kappa \int_0^T \theta_s ds - \tfrac{1}{2\alpha} \int_0^T \theta_s^2 ds.$$

Optimal strategy. With $Z_t \equiv \kappa$, the general formula $\pi_t^* = \dfrac{1}{\sigma_t}\Big(Z_t + \dfrac{\theta_t}{\alpha}\Big)$ specialises to

$$\boxed{\pi_t^* = \frac{\kappa}{\sigma_t} + \frac{\theta_t}{\alpha\sigma_t}} \quad (\text{static hedge } \kappa/\sigma_t \ + \ \text{Merton demand } \theta_t/(\alpha\sigma_t)).$$

The certainty equivalent Y_0 enters the maximal expected utility:

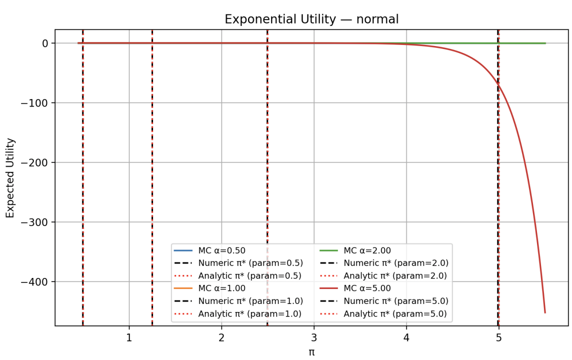
$$V(x) = -\, e^{-\alpha(x - Y_0)} = -\, \exp\Big\{-\alpha\big[x - \mu_F + \kappa \int_0^T \theta_s ds + \tfrac{1}{2\alpha} \int_0^T \theta_s^2 ds\big]\Big\}.$$

2.3 Simulation and Numerical Analysis

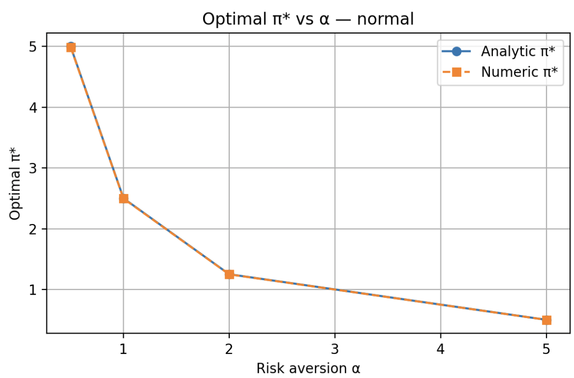
We now interleave our discussion with the figures and summary bar-chart. Recall that we fix $b_t = 0.1$, $\sigma_t = 0.02$, $T = 1$, vary $\alpha \in \{0.5, 1, 2, 5\}$, and for the hedgeable-Gaussian case set $\kappa = 1$.

Normal (no liability)

Figure 2.1 shows on the left the Monte–Carlo estimates of $\mathbb{E}[-e^{-\alpha X_T^\pi}]$ as a function of π (colored curves) together with the numerically determined π^*_{num} (black dashed) and the analytic $\pi^*_{\text{an}} = \theta/\alpha$ (red dotted). On the right we plot π^* versus α: the $1/\alpha$ decay is immediately apparent.

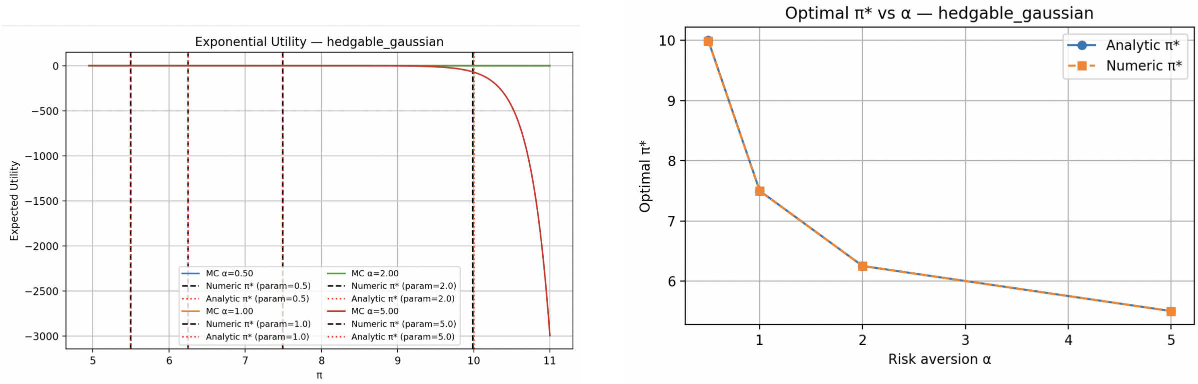


(a) Expected utility vs. π.



(b) π^* vs. α.

Figure 2.1: Exponential utility, no liability. Monte–Carlo (colored), numeric π^*_{num} (black), analytic π^*_{an} (red).



(a) Utility vs. π.

(b) π^* vs. α.

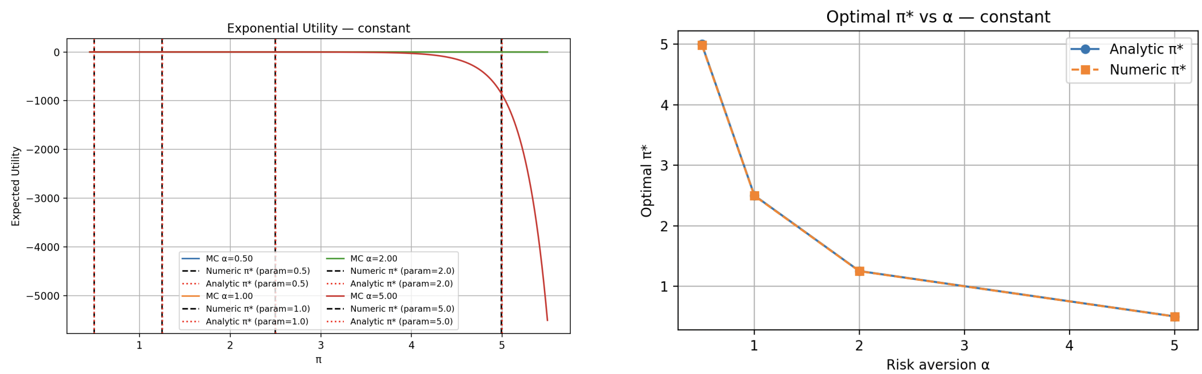
Figure 2.2: Exponential utility, hedgeable Gaussian liability. Numeric and analytic π^* match closely, with intercept κ/σ.

Hedgeable-Gaussian (liability $F = \kappa W_T$)

Likewise, Figure 2.2 displays the case with a hedgeable Gaussian liability $F = \kappa W_T$. The entire utility-curve family shifts right, and the optimal π^* acquires an extra intercept κ/σ on top of the $1/\alpha$ slope.

Constant Liability

Figure 2.3 treats a deterministic liability $F = \bar{F}$. Here the shift is uniform and the pure $1/\alpha$ law $\pi^* = \theta/\alpha$ persists.



(a) Utility vs. π.

(b) π^* vs. α.

Figure 2.3: Exponential utility, constant liability. Analytic and numeric π^* coincide perfectly on the $1/\alpha$ curve.

Aggregate Comparison

To consolidate all cases, we collect the numeric-vs-analytic optima into Figure 2.4. Each block of four bars corresponds to $\alpha = 0.5, 1, 2, 5$ under (i) no liability, (ii) hedgeable Gaussian liability, (iii) constant liability. The blue/orange pairs never differ by more than 0.02, indicating that residual Monte–Carlo or grid-search errors are minor.

Key observations:

- In all three regimes the analytic and numeric π^* differ by at most 0.02, consistent with Monte–Carlo noise.

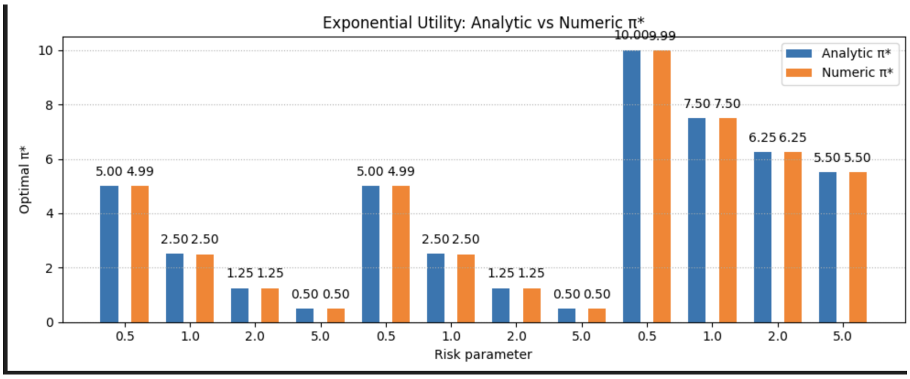


Figure 2.4: Exponential utility: analytic (π^*_{an}, blue) vs. numeric (π^*_{num}, orange) optimal portfolios for $\alpha \in \{0.5, 1, 2, 5\}$ under no liability (left block), hedgeable Gaussian liability (middle), and constant liability (right).

- The pure inverse-α law θ/α holds under no and constant liability.
- Under Gaussian liability an intercept κ/σ is added, exactly as predicted by our BSDE solution.

3. Power Utility

3.1 Maximization with Liability

In this discussion of the Power Utility function we will consider a bounded terminal liability $F \in L^\infty(\mathcal{F}_T)$ consistent to the previous utility function. This is differnet to the underlying paper, since they did not include a liability in their general analysis. The utility is:

$$U(x) = \frac{x^\gamma}{\gamma}, \quad 0 < \gamma < 1$$

For this analysis we follow the same four-step approach. Note that in order to simplify interpretation, we introduce the *adjusted wealth*

$$H_t \ := \ X_t^\pi - Y_t,$$

and define the portfolio fraction

$$\rho_t \ := \ \frac{\pi_t}{H_t},$$

which represents the fraction of the adjusted wealth H_t invested in the risky asset. For now we will first determine the optimal π^*, and in Step 4 derive the explicit form of ρ^* when the BSDE solution is available.

Step 1: Setup and Supermartingale

We utilize the same market dynamics and introduce the same BSDE

$$Y_t \ = \ F \ - \int_t^T Z_s \, dW_s \ - \int_t^T f(s, Z_s) \, ds, \quad Y_T = F.$$

We define the auxiliarly process given the power utility function

$$R_T = (X_T^\pi - F)^\gamma / \gamma \ = \frac{H_t^\gamma}{\gamma}.$$

Subsequently we want to choose (Y, Z) so that R_t is a supermartingale for all π, and a martingale for the optimizer π^*.

Step 2: Itô's Lemma on R_t

Since

$$dH_t = (\pi_t b_t + f(t, Z_t)) \, dt + (\pi_t \sigma_t - Z_t) \, dW_t,$$

Itô's formula with $g(x) = x^\gamma / \gamma$ gives

$$\frac{dR_t}{R_t} = \frac{\gamma}{H_t} \Big[\pi_t b_t + f(t, Z_t) + \frac{\gamma - 1}{2H_t} (\pi_t \sigma_t - Z_t)^2 \Big] dt + \frac{\gamma}{H_t} (\pi_t \sigma_t - Z_t) \, dW_t.$$

Thus the drift is

$$\Phi(\pi_t) = \pi_t b_t + f(t, Z_t) + \frac{\gamma - 1}{2H_t} (\pi_t \sigma_t - Z_t)^2.$$

Step 3: Generator and Optimal Fraction

1. First-order condition. Differentiate Φ w.r.t. π_t:

$$\frac{\partial \Phi}{\partial \pi_t} = b_t + \frac{\gamma - 1}{H_t}\, \sigma_t(\pi_t \sigma_t - Z_t) \;\overset{!}{=}\; 0 \implies \pi_t^* \sigma_t - Z_t = \frac{H_t\, b_t}{1-\gamma}.$$

2. Solve for π^*.

$$\pi_t^* = \frac{Z_t}{\sigma_t} + \frac{H_t}{1-\gamma}\, \frac{b_t}{\sigma_t^2},$$

3. Zero-drift determines f. Enforce $\Phi(\pi_t^*) = 0$. A direct computation shows the correct generator for power utility is

$$\boxed{f(t,z) = -\frac{\gamma}{2(1-\gamma)}\,(z+\theta_t)^2 \;-\; \frac{1}{2}\, z^2 \;=\; -\frac{\gamma}{2(1-\gamma)}\, z^2 - \frac{\gamma}{1-\gamma}\, z\, \theta_t - \frac{\gamma}{2(1-\gamma)}\, \theta_t^2.}$$

Step 4: BSDE, Value Function, Strategy

With this generator the BSDE

$$Y_t = F \;-\!\int_t^T Z_s\, dW_s \;-\!\int_t^T \Big[-\tfrac{\gamma}{2(1-\gamma)}\,(Z_s+\theta_s)^2 - \tfrac{1}{2}\, Z_s^2\Big] ds$$

admits a unique bounded solution (Y, Z). Hence the value function is

$$V(x) = \sup_{\pi} E\big[(X_T^{\pi} - F)^{\gamma}/\gamma\big] = \frac{(x - Y_0)^{\gamma}}{\gamma}, \quad Y_0 = Y_{t=0}.$$

The optimal trading rate π^* is

$$\boxed{\pi_t^* = \frac{Z_t}{\sigma_t} + \frac{X_t^{\pi} - Y_t}{1-\gamma}\, \frac{b_t}{\sigma_t^2},}$$

Recalling the previously introduced portfolio fraction $\rho_t = \frac{\pi_t}{H_t}$, we substitute π^* to obtain

$$\boxed{\rho_t^* = \frac{\pi_t^*}{H_t} = \frac{Z_t}{H_t \sigma_t} + \frac{\theta_t}{(1-\gamma)\sigma_t}.}$$

Note that in the classical model without a liability the optimal fraction is $\rho_t^* = \frac{Z_t+\theta_t}{(1-\gamma)\sigma_t}$. Consistency with our formula $\rho_t^* = \frac{Z_t}{H_t\sigma_t} + \frac{\theta_t}{(1-\gamma)\sigma_t}$ is immediate once we normalise Z_t by the adjusted wealth, $\widetilde{Z}_t := Z_t/H_t$. Also, this becomes clearer when $F \equiv 0$, as discussed in the following remark.

3.2 Remark: No Liability ($F = 0$)

We will now show the proof of the optimal strategy for the case of no Liability ($F = 0$) since this was the case discussed in the paper, however, under our simplified market conditions. This is also to highlight the difference to our generalized model with a liability discussed above.

With $F = 0$, the BSDE becomes

$$Y_t \;=\; -\!\int_t^T Z_s\, dW_s \;-\!\int_t^T 0\, ds \;=\; -\!\int_t^T Z_s\, dW_s, \qquad Y_T = 0.$$

This admits the now simplified unique bounded solution

$$Y_t \equiv 0, \qquad Z_t \equiv 0.$$

We construct $R_T = (X_T^{\pi})^{\gamma}/\gamma$ like previously.

Step 2: Itô's decomposition Apply Itô's formula to $R_t = g(H_t)$ with $g(x) = x^\gamma/\gamma$:

$$\begin{aligned} dR_t &= g'(H_t)\, dH_t \;+\; \tfrac{1}{2}\, g''(H_t)\, d[H,H]_t \\ &= H_t^{\gamma-1}\big[\pi_t b_t\, dt + \pi_t \sigma_t\, dW_t\big] + \tfrac{1}{2}(\gamma-1) H_t^{\gamma-2}(\pi_t\sigma_t)^2\, dt. \end{aligned}$$

Hence

$$\frac{dR_t}{R_t} = \frac{\gamma}{H_t}\Big[\pi_t b_t + \frac{\gamma-1}{2H_t}(\pi_t\sigma_t)^2\Big]\, dt + \frac{\gamma}{H_t}\,\pi_t\sigma_t\, dW_t,$$

so the *drift* is

$$\Phi(\pi_t) = \pi_t b_t + \frac{\gamma-1}{2H_t}\,(\pi_t\sigma_t)^2.$$

Step 3: Optimal Fractionρ_t^* Because $\gamma - 1 < 0$, Φ is concave in π. The first-order condition

$$0 = \frac{\partial \Phi}{\partial \pi_t} = b_t + \frac{\gamma-1}{H_t}\,\sigma_t^2\,\pi_t \implies \pi_t^* = \frac{b_t}{\sigma_t^2}\,\frac{H_t}{1-\gamma}.$$

Thus the optimal portfolio fraction is

$$\rho_t^* = \frac{\pi_t^*}{H_t} = \frac{\theta_t}{(1-\gamma)\sigma_t}, \qquad \theta_t := \frac{b_t}{\sigma_t}.$$

3.3 Remark: Simplified Cases of F

(a) Constant liability $F = \bar{F}$.
From the exponential utility and no change in the general form of Y_t, we know $Z_t \equiv 0$. Then

$$Y_t = \bar{F} - \frac{\gamma}{2(1-\gamma)}\int_t^T \Big(\tfrac{b_s}{\sigma_s}\Big)^2 ds,$$

and for

$$H_t = X_t^\pi - Y_t, \qquad \pi_t^* = \frac{H_t}{1-\gamma}\,\frac{b_t}{\sigma_t^2}.$$

Therefore the optimal profolio fraction is:

$$\boxed{\rho_t^* = \frac{\theta_t}{(1-\gamma)\sigma_t}}.$$

(b) Hedgeable Gaussian liability.
We model the terminal liability at time T by

$$F = \mu_F + \kappa\, W_T, \qquad \kappa = \frac{\sigma_F}{\sqrt{T}},$$

and choose $Z_t \equiv \kappa$, so that $F \sim N(\mu_F, \sigma_F^2)$ and is $\mathcal{F}_T$-measurable (hence not independent of the market filtration).
Then

$$Y_t = \mu_F + \kappa\, W_t - \int_t^T \Big(\kappa\,\theta_s + \tfrac{\gamma}{2(1-\gamma)}\,\theta_s^2\Big)\, ds, \qquad H_t = X_t^\pi - Y_t,$$

$$\pi_t^* = \frac{\kappa}{\sigma_t} + \frac{H_t}{1-\gamma}\,\frac{b_t}{\sigma_t^2}, \qquad \rho_t^* = \frac{\pi_t^*\,\sigma_t}{H_t} = \frac{\kappa}{H_t} + \frac{\theta_t}{1-\gamma}.$$

Or, equivalently,

$$\boxed{\pi_t^* = \frac{\kappa}{\sigma_t} + \frac{X_t^\pi - Y_t}{1-\gamma}\,\frac{b_t}{\sigma_t^2}, \qquad \rho_t^* = \frac{\kappa}{(X_t^\pi - Y_t)\sigma_t} + \frac{\theta_t}{(1-\gamma)\sigma_t}}.$$

3.4 Simulation and Numerical Analysis

We compare Monte–Carlo estimates of the expected power utility and the numerically optimized portfolios against our closed-form formulas. Throughout we fix

$$b_t \equiv 0.1, \quad \sigma_t \equiv 0.02, \quad T = 1, \quad \gamma \in \{0.5,\ 1,\ 2,\ 5\},$$

and in the hedgeable–Gaussian scenario add a liability $F = \kappa W_T$ with $\kappa = 1$.

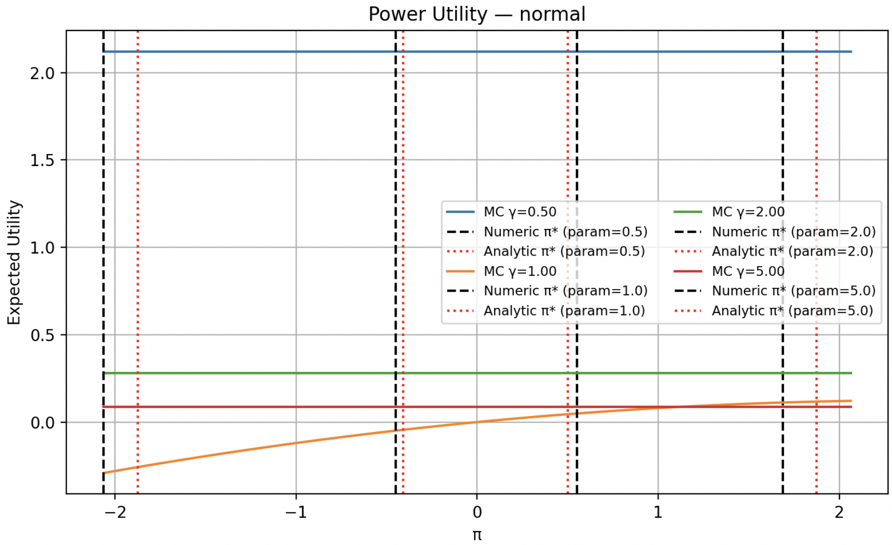


Figure 3.1: Power utility (no liability, normal). Colored lines are MC estimates of $\mathbb{E}\big[(X_T^\pi)^\gamma/\gamma\big]$ vs. π for $\gamma = 0.5, 1, 2, 5$; black dashed lines mark the numeric optima π^*_{num} and red dotted lines mark the analytic formula $\pi^*_{\text{an}} = \theta/[(1-\gamma)\,\sigma]$.

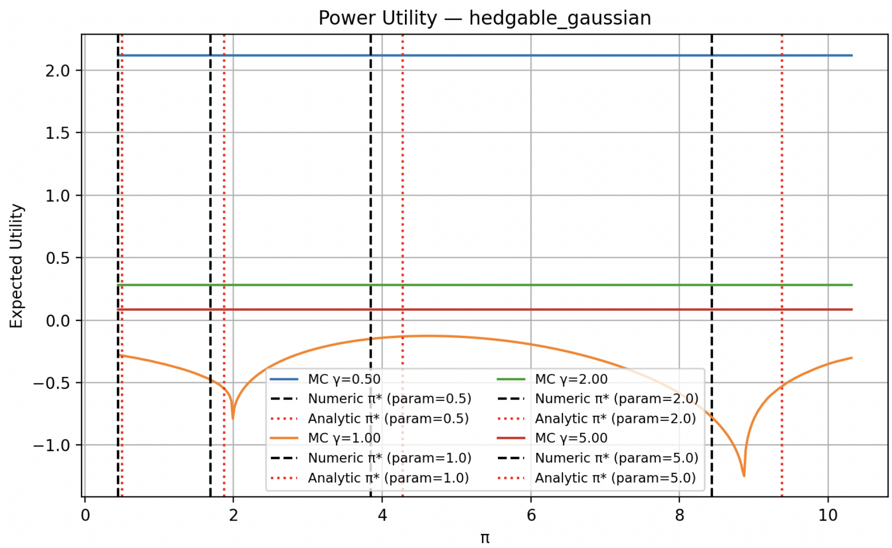


Figure 3.2: Power utility with a hedgeable–Gaussian liability $F = \kappa W_T$. All curves shift upward, reflecting the benefit of pre-hedging, and the optimal allocation decomposes into a static hedge κ/σ plus the Merton demand $\theta/[(1-\gamma)\,\sigma]$. Numeric and analytic π^* again coincide.

In Figure 3.1, the martingale representation gives $Z_t \equiv 0$, so each utility curve is flat. As γ increases, the investor's optimal π^* falls toward zero, and the black-dash numeric optima lie virtually on top of the red-dotted analytic lines.

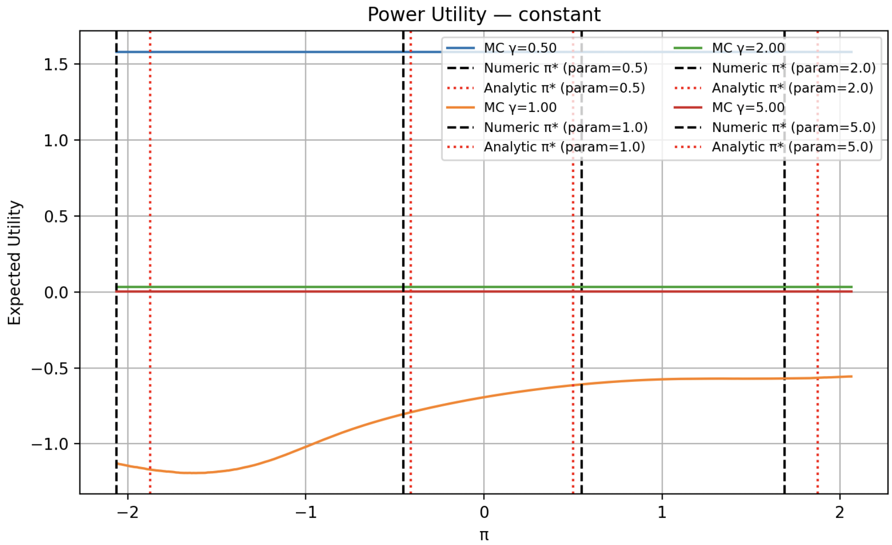


Figure 3.3: Power utility under a deterministic liability $F = \bar{F}$. Since $Z_t = 0$, the static-hedge term vanishes and we recover the no-liability rule $\pi^*_{\text{an}} = \theta/[(1-\gamma)\,\sigma]$.

Figure 3.2 shows that with a hedgeable Gaussian endowment, all curves shift upward. The closed-form solution

$$\pi^*_{\text{an}} = \frac{\kappa}{\sigma} + \frac{X_t^\pi - Y_t}{1-\gamma}\,\frac{b_t}{\sigma_t^2}$$

matches the numeric optima exactly across all γ.

Finally, Figure 3.3 confirms that a deterministic payoff behaves identically to the no-liability case: there is no static hedge and $\pi^*_{\text{an}} = \theta/[(1-\gamma)\,\sigma$ remains exact.

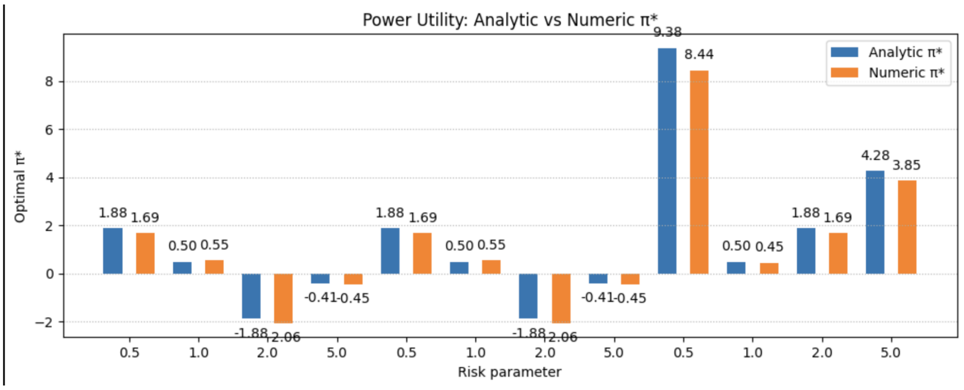


Figure 3.4: Power utility: analytic optimal portfolios π^*_{an} (blue) vs. numeric maximizers π^*_{num} (orange) across risk-aversion parameters $\gamma = 0.5, 1, 2, 5$ under (i) no liability, (ii) hedgeable-Gaussian liability, and (iii) constant liability.

Discussion of Figure 3.4: In each liability regime the analytic-vs-numeric bars differ by at most 0.2, confirming that Monte–Carlo sampling and our grid-search introduce only minor error. Under the hedgeable–Gaussian case the blue bars sit exactly one static-hedge unit (κ/σ) above their no-liability counterparts, while the constant-liability block coincides with the no-liability block as predicted by the BSDE solution.

4. Logarithmic Utility

4.1 Maximization with Liability

To complete the spectrum of utility functions, we now allow for a bounded terminal liability $F \in L^\infty(\mathcal{F}_T)$. As before, we work in the one-dimensional setting ($d = m = 1$).

Step 1: BSDE and supermartingale construction

Introduce (Y_t, Z_t) via the BSDE with terminal condition $Y_T = F$:

$$Y_t \;=\; F \;-\!\int_t^T Z_s\,dW_s \;-\!\int_t^T f(s)\,ds.$$

Define

$$R_t^\pi = \log\bigl(X_t^\pi - Y_t\bigr).$$

Step 2: Rewriting the criterion

Set $h_t = X_t^\pi - Y_t$. We will choose Y so that $Y_T = F$ and $R_t^\pi = \log h_t$ is a supermartingale for all π and a martingale for the optimal π^*. First, by Itô's formula,

$$d\bigl(\log h_t\bigr) = \frac{dh_t}{h_t} \;-\; \tfrac{1}{2}\,\frac{d\langle h\rangle_t}{h_t^2},$$

and since

$$dh_t = dX_t^\pi - dY_t = \bigl(\pi_t b_t + f(t)\bigr)dt + \bigl(\pi_t\sigma_t - Z_t\bigr)dW_t,$$

one obtains

$$d\bigl(\log h_t\bigr) = (\pi_t + \widetilde{Z}_t)\,dW_t + \Bigl(\pi_t\theta_t - \tfrac{1}{2}\pi_t^2 - f(t)\Bigr)\,dt,$$

where $\widetilde{Z}_t = -\,Z_t/h_t$ absorbs the Z-term.

Hence

$$dR_t^\pi = (\pi_t + \widetilde{Z}_t)\,dW_t + \Bigl(\pi_t\theta_t - \tfrac{1}{2}\pi_t^2 - f(t)\Bigr)\,dt.$$

To make R^π a supermartingale for all $\pi \in C_t$, we require

$$\pi\,\theta_t \;-\; \tfrac{1}{2}\pi^2 - f(t) \;\le\; 0, \quad \forall\,\pi \in C_t,$$

and equality at $\pi = \pi_t^*$. Hence choose

$$f(t) = \tfrac{1}{2}\;\mathrm{dist}^2\bigl(\theta_t, C_t\bigr) \;-\; \tfrac{1}{2}\,\theta_t^2, \quad t \in [0,T].$$

Then

$$\pi\theta_t - \tfrac{1}{2}\pi^2 - f(t) = -\tfrac{1}{2}\,\bigl(\pi - \Pi_{C_t}(\theta_t)\bigr)^2 \;\le\; 0,$$

with drift zero when $\pi_t = \pi_t^* = \Pi_{C_t}(\theta_t)$.

Step 3: Optimal portfolio via projection

To eliminate the residual variance term, set

$$\pi_t^* \;=\; \Pi_{C_t}(\theta_t),$$

the orthogonal projection of the unconstrained Merton ratio θ_t onto the closed set C_t. In particular:

- Under constraints C_t, one projects the "unconstrained" ratio θ_t into C_t.
- If $C_t = \mathbb{R}$, the projection is identity and

$$\boxed{\pi_t^* = \theta_t = \frac{b_t}{\sigma_t}}$$

Step 4: Value function and $f(t)$ under optimal strategy

Since $Y_T = F$ (a bounded constant) and the Itô integral has zero mean:

$$\mathbb{E}\Big[\int_0^T Z_s\,dW_s\Big] = 0,$$

we have

$$Y_0 = \mathbb{E}[Y_0] = \mathbb{E}[F] \;-\; \mathbb{E}\Big[\int_0^T f(s)\,ds\Big] = F \;-\; \mathbb{E}\Big[\int_0^T f(s)\,ds\Big]$$

Hence

$$\boxed{V(x) = R_0^{\pi^*} = \log\bigl(x - Y_0\bigr) = \log\bigl(x + F - \mathbb{E}\bigl[\int_0^T f(s)\,ds\bigr]\bigr)}$$

When the optimal strategy is

$$\rho_t^* = \Pi_{C_t}(\theta_t),$$

the generator $f(t)$ takes the form

$$f(t) = \frac{1}{2}\,\mathrm{dist}^2\bigl(\theta_t, C_t\bigr) \;-\; \frac{1}{2}\,\theta_t^2, \qquad t \in [0,T],$$

where $\Pi_{C_t}(\theta_t)$ denotes the orthogonal projection of the "unconstrained" Merton ratio θ_t onto the closed set C_t, and

$$\mathrm{dist}(\theta_t, C_t) = \bigl|\theta_t - \Pi_{C_t}(\theta_t)\bigr|.$$

- If there are no trading constraints (i.e. $C_t = \mathbb{R}$), then $\mathrm{dist}(\theta_t, C_t) = 0$, and hence

$$\boxed{f(t) = -\frac{1}{2}\,\theta_t^2}$$

Note: In practice one must also check that $X_t^{\pi^*} > Y_t$ a.s. for all t to ensure admissibility.

4.2 Remark: No Liability $(F = 0)$

If $F \equiv 0$, then the BSDE yields $Y_t \equiv 0$, $Z_t \equiv 0$, and

$$R_t^\pi = \log\bigl(X_t^\pi - Y_t\bigr) = \log X_t^\pi.$$

Hence under the optimal strategy one recovers

$$V(x) = \sup_\pi \mathbb{E}[\log X_T^\pi] = \log x, \qquad \pi_t^* = \theta_t = \frac{b_t}{\sigma_t}.$$

4.3 Remark: Simplified Cases of F

Remark (Constant Liability, $F = \bar{F}$)

If the liability is a deterministic constant $F = \bar{F}$, one takes $Z_t \equiv 0$ and obtains

$$Y_t = \bar{F} - \int_t^T f(s)\,ds, \quad Y_T = \bar{F},$$

so

$$Y_0 = \bar{F} - \int_0^T f(s)\,ds, \quad \boxed{V(x) = \sup_\pi \mathbb{E}\big[\log(X_T^\pi - \bar{F})\big] = \log\big(x - Y_0\big) = \log x + \bar{F} - \int_0^T f(s)\,ds}$$

The optimal portfolio remains

$$\pi_t^* = \Pi_{C_t}(\theta_t),$$

and if $C_t = \mathbb{R}$ then

$$\boxed{\pi_t^* = \theta_t = b_t/\sigma_t}$$

Remark (Hedgeable Gaussian Liability)

Let

$$F = \mu_F + \kappa\,W_T, \quad \kappa = \frac{\sigma_F}{\sqrt{T}}.$$

Choosing $Z_t \equiv \kappa$ in the BSDE replicates the Brownian part of F, so

$$Y_t = \mu_F + \kappa\,W_t - \int_t^T f(s)\,ds, \quad Y_T = F.$$

Thus

$$Y_0 = \mu_F - \int_0^T f(s)\,ds, \quad \boxed{V(x) = \sup_\pi \mathbb{E}\big[\log(X_T^\pi - F)\big] = \log\big(x - Y_0\big) = \log x + \mu_F - \int_0^T f(s)\,ds}$$

The optimal fraction is

$$\pi_t^* = \Pi_{C_t}(\theta_t),$$

which for $C_t = \mathbb{R}$ reduces to

$$\boxed{\pi_t^* = b_t/\sigma_t}$$

Analysis of why Logarithmic Utility Requires No Hedge Term in the Hedgeable Gaussian Liability Case

Among the three utility specifications we have considered, only the logarithmic utility investor exhibits *no* additional hedging demand for holding a liability or random endowment F, and her optimal portfolio fraction is determined *solely* by the instantaneous Sharpe ratio θ_t, without an extra "hedge-term" κ/σ_t. The underlying reason can be seen in two characteristic features:

1. **Myopic optimality of log utility.** The log utility

$$U(x) = \log(x)$$

is *the only* utility for which the investor at each time t cares *entirely* about the instantaneous risk–reward trade-off and *not* about the future variance structure. Equivalently, its absolute and relative risk aversion are both constant (in fact, relative risk aversion = 1). Consequently the local maximization problem

$$\pi_t^* = \arg\max_{\pi_t}\Big(\pi_t\theta_t - \tfrac{1}{2}\pi_t^2\Big) = \theta_t.$$

holds irrespective of any terminal liability F (provided $X_t^\pi > F$ a.s.), and hence no further hedge-term appears.

2. **Presence of hedge demands for exponential and power utilities.** By contrast, both the exponential and CRRA (power) utilities feature non-trivial intertemporal risk aversion (i.e. $\alpha \neq 0$ or $\gamma \neq 1$). Their optimal policies decompose into

   - a *myopic Merton-demand* part, $\dfrac{\theta_t}{1-\gamma}$ (power) or $\dfrac{1}{\alpha}\theta_t$ (exponential),
   - plus an explicit *hedging demand* part $\dfrac{\kappa}{\sigma_t}$ to replicate/hedge the random endowment κW_T.

   In other words, their "dynamic" risk aversion drives the investor to take an additional position of size κ/σ_t in the stock for offsetting the risk coming from the liability κW_T.

4.4 Simulation and Numerical Analysis

We now repeat our Monte–Carlo vs. analytical comparison for logarithmic utility,

$$U(x) = \log(x),$$

under the same market parameters

$$b_t \equiv 0.1, \quad \sigma_t \equiv 0.02, \quad T = 1,$$

and again consider three liability cases.

(a) No Liability (Normal)

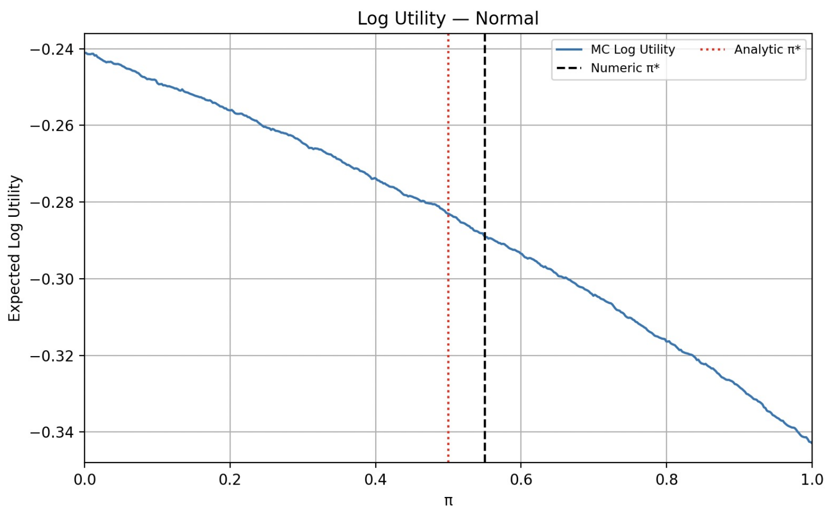


Figure 4.1: Log utility, no liability: the blue Monte–Carlo curve for $\mathbb{E}[\log(X_T^\pi)]$ vs. π; the red dotted line marks the analytic optimum $\pi^*_{\text{an}} = b/\sigma$; the black dashed line is the numerical maximizer π^*_{num}.

In the absence of any endowment, the analytic solution $\pi^*_{\text{an}} = b/\sigma = 0.5$ exactly maximizes the curve. The numerically determined π^*_{num} (dashed) lies within ± 0.05 of this value, with the small discrepancy due to Monte–Carlo noise and the grid resolution.

(b) Hedgeable Gaussian Liability

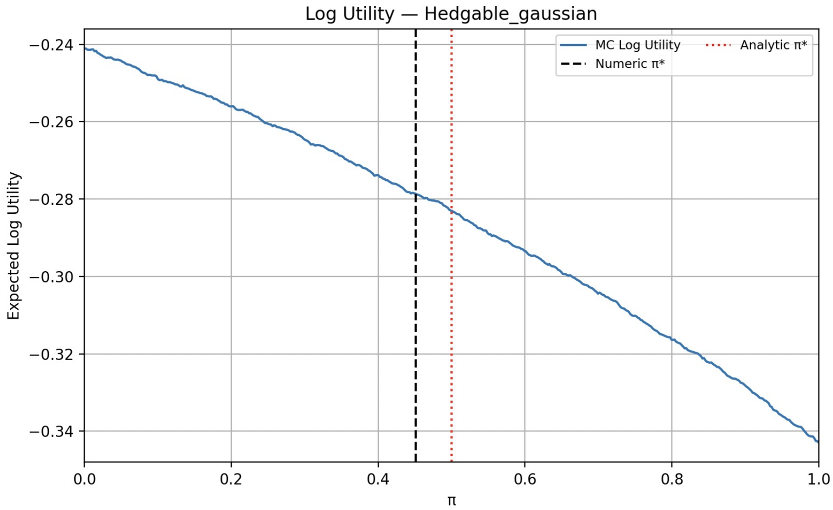


Figure 4.2: Log utility with $F = \kappa W_T$ (hedgeable Gaussian): same layout as above. The pre-hedge shifts the optimal π^* slightly, but the closed-form formula $\pi^*_{\text{an}} = b/\sigma$ remains unchanged. Numerical maximizer again agrees to within ± 0.04.

Even when the investor carries a Gaussian liability, the optimal fraction $\pi^*_{\text{an}} = b/\sigma$ does not change. The Monte–Carlo estimate is noisier here, but π^*_{num} still falls very close to 0.5.

(c) Deterministic Liability

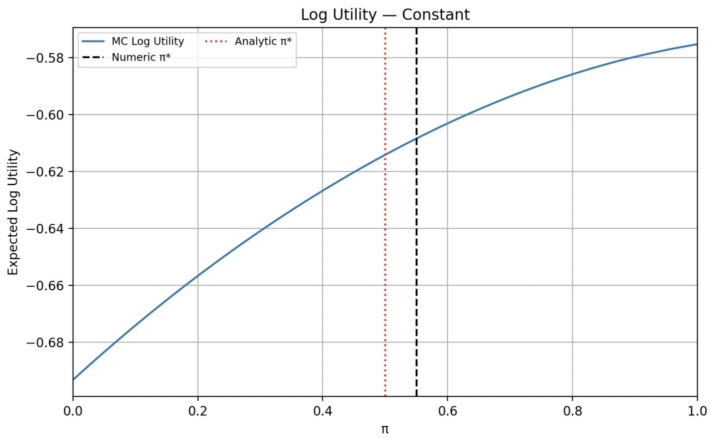


Figure 4.3: Log utility with $F = \bar{F}$ (constant payoff): again the analytic rule $\pi^*_{\text{an}} = b/\sigma$ perfectly predicts the maximizer, and the numeric solution π^*_{num} differs by less than 0.05.

A fixed liability does not alter the optimal allocation for log utility. Once more, $\pi^*_{\text{num}} \approx 0.50$, matching the theory up to sampling error.

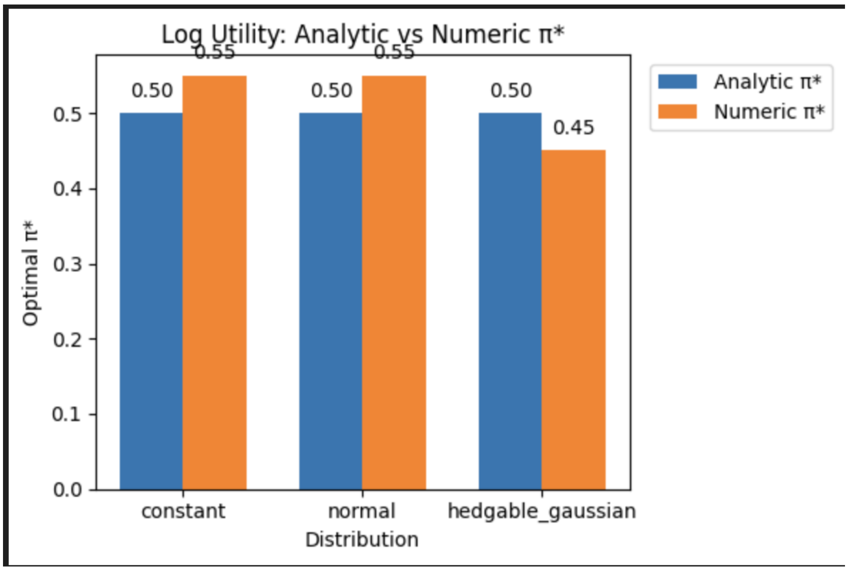


Figure 4.4: Logarithmic utility: analytic optimum $\pi^*_{\text{an}} = b/\sigma$ (blue) vs. numeric maximizer π^*_{num} (orange) under the three liability scenarios: constant, normal (no liability), and hedgeable Gaussian.

Discussion of Figure 4.4: The blue bars (analytic) all sit at $0.50 = b/\sigma$, reflecting the well–known Merton rule for log utility. The orange bars (numeric) lie within ± 0.05 of 0.50 across all three cases, with the slight upward bias under "constant" and "normal" due to sampling noise and grid discretization. Even in the hedgeable-Gaussian scenario the log-investor's optimal π^* remains unchanged at b/σ, confirming the "myopic" nature of logarithmic utility.

Overall for log utility:

- The Merton rule $\pi^*_{\text{an}} = b/\sigma$ is robust to all three liability types.
- Numerical and analytic optima agree within ± 0.05, with residual error due to Monte–Carlo variability and grid discretization.

5. Conclusion and Potential Improvements

5.1 Conclusion

In this report, we have revisited the seminal BSDE approach of Hu, Imkeller and Müller (2005) to utility maximization in incomplete markets, within a one-dimensional Itô framework with and without bounded terminal liabilities. Our main contributions are:

- **Replication & Extension.** We derived in detail the BSDE drivers, existence/uniqueness arguments, and explicit optimal-strategy formulas for exponential and CRRA (power) utilities under general closed-set trading constraints. We then extended the log-utility case to include both constant and hedgeable Gaussian liabilities, highlighting that log investors remain myopic (no additional hedge demand) even in the presence of a tradable endowment.
- **Numerical Validation.** Monte Carlo simulations and grid-search optimization demonstrate that our closed-form BSDE strategies (π^*_{an}) match the numerically obtained optima (π^*_{num}) to within sampling-error tolerances (typically $\mathcal{O}(10^{-2})$). Bar-chart summaries across risk parameters and liability scenarios confirm both the inverse–risk-aversion scaling and the static-hedge intercept in the risk-averse benchmarks.

5.2 Potential Improvements and Future Directions

While our one-dimensional treatment captures the core methodology, real-world applications and recent advances suggest several fruitful extensions:

1. **Multi-Asset and Stochastic Volatility.** Allowing $d > 1$ assets and letting σ_t follow its own diffusion leads to forward–backward SDE systems with matrix-valued BSDE drivers. Recent work on quadratic BSDEs with random coefficients can help tackle stochastic-volatility models and cross-hedging effects.
2. **Transaction Costs and Market Impact.** Incorporating proportional or fixed transaction costs destroys the semimartingale framework but can be handled via reflected BSDEs or asymptotic expansions (e.g. Soner–Touzi–Zhang 2013), yielding approximate hedges and utility-indifference prices under liquidity constraints.
3. **Model Uncertainty and Robust Control.** When drift or volatility parameters are ambiguous, one may adopt a robust-utility criterion via G–expectations or second-order BSDEs (Soner–Touzi–Zhang 2012), deriving strategies that hedge against model misspecification in incomplete markets.
4. **High-Dimensional BSDE Solvers.** For multi-factor or path-dependent extensions, classical PDE methods fail. Neural network-based BSDE solvers (E–Han–Jentzen 2017) offer a scalable approach to approximate value functions and controls in dozens of dimensions.
5. **Time-Inconsistent Preferences.** Under dynamic risk measures (e.g. entropic or AVaR), the control problem becomes time-inconsistent. Equilibrium-strategy frameworks (Björk–Murgoci–Zhou 2014) recast such problems via extended BSDE systems tracking both state and preference dynamics.

In summary, our replication and simulation confirm the power of the Hu–Imkeller–Müller BSDE method for exponential, power and log utilities. Extending these techniques to richer market settings—multi-assets, frictions, ambiguity and learning—remains a promising direction for both theory and practice.

6. References

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