Reductive Groups and GIT

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Throughout, let k be a held of char O, (in char p 70, reductive groups aren't too interesting).

Quohents

Often it is useful to consider the following Situation: Let Gr be a group object acting on an object X.

When can we form the quotient object ×/G? What should the quotient be?

£x: In Set, we just take X/G to be the Set of orbits. Then G aets trivially on X/G and the fibre of each point in X/G is an orbit in X.

Ex: In topology, we can endow the set of orbits with the quotient topology, so that $X \to X/G$ is Continous.

Ex: In Smooth manfolds, G is now a Lie Group and X a smooth manifold. Then without restricting the action of G, the sect of orbits will not be a smooth manifold (usually because non-closed orbits give a non-Hausdorff quotient).

We restrict to Gracking freely and properly and everything works out.

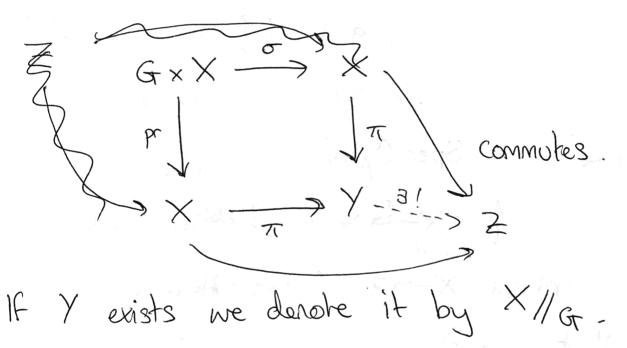
Now we look int the category of affine Schemes of hite type over k (or varieties).

Let & be an other algebraie group (that is, athre schene with group structure given by algebraie morphisms).

To understand this, let's abstract what a quotient is.

A categorical quotient of X by GT is a morphism

T: X -> Y St.



Note that $Y(X//G_T)$ is not recessarily an orbit space, (i.e. fibres of π may not be closed orbits).

In this sense, a carregorical quotent is different to a geometrie quotient, X/G, which is an orbit space.

When does X/1G exist?

Here is a recipe when X = Spee A is offine. Graeks on X = Spee A which induces an action on the finitely generated k-algebra $A = O_X(X) := O(X)$. Explicitly, $g \cdot f(p) = f(\bar{g}'p)$.

Let $O(x)^G = 2f \in O(x) \mid g \cdot f = f$ be the Subalgebra of G-invariant elements.

Then define $Y = \text{Spec } O(x)^G$. Then we get a G-invariant map $X \xrightarrow{\pi} Y$ s.t. any other G-invariant morphism $X \to Z$ factors through π . It seems that Y = X//G. However, Y is not necessarily of finite type $(O(x)^G$ is not necessarily finitely generated k-algebra).

Hulbert's 14th Problem: A fig k-algebra When is A finitely generated?

This is classical invarant theory. Hulbert proved for G=GLn over C.

- · Nagata Showed False in general (by taking Copies of Ga)
- · Nagata proved true for a large class of groups, "Reductive groups".
- "Reductive -> Reynold's Operator -> Reduce to A polynomial algebra + Hilbert basis."

for k=k char O, and & a smooth affine algebraie group, the notions of Reductive and linearly reductive coincide. We give the latter definition as it is micer.

Det: An affine algebraie group GT is hiearly reductive if every finite diversional linear representation p: GT -> GL(V) is completely reducible, i.e. decomposes as a direct sum of meducible par reps.

Ex: All histo graps, Gla, Sla, PSL, and Gm are linearly reductive.

Ex: The additive group $G_a = \operatorname{Spec} k[t]$ is not linearly reductive.

Thus we see that if Gris redbehire, then X//GT is exists and we call it the GIT quotient.

Reductive groups were not known to Hilbert one the representation theory had not yet been fully developed.

Connerts on Reductive groups

- · Very inhereshing in their own right!
- * Chevalley showed that the classification of reductive groups does not depend on k ($k=\overline{k}$).
- · Reconnered Milne's book Algebraia Groups.

Geometric Invarant Theory

We constructed the GIT quotent X//G when G is reductive. But X//G is not always a geometric quotent X/G, i.e. orbits are closed.

Ex: Consider $G_m = \operatorname{Speck}[t_i f']$ achieve on $X = A^2 = \operatorname{Speck}[x_i y]$ by $t \cdot (x_i y) = (t x_i f' y)$. The orbits are

- · Conics {(x,y) | xy = \alpha } for \alpha \end{all} 1-\{0}
- · punctured x-axis
- · punctured y-axis
- · the origin.

The origin and comes are closed orbits, but the punchured axis are not, and since they contain the origin in their closure, then these orbits will be identified in $X//_G = 80$ Speek[xy]. So $X//_G \neq X/_G$ (here the quotient is $\pi_i(x_iy) \rightarrow xy$).

However, Muntord's innovation was the notion of stability:

Def: A point XEX is said to be stable for Gr

- i) The orbit G.xc = X is closed
- 2) dim Gx = 0 (the stabiliser subgroup).

let X3 be the stable points.

Theorem: Let G be reductive acting on an other schene X and let $\pi: X \to X/G = Y$ be the GIT quotient. Then $X^s \subset X$ is open, G-invarant, and $\pi|_{X^s}$ is a geometric quotient.

"Muntord also "compactitical" X's by means of his notion of Semi-stability.

Muntord's GIT and stable lows are key for Constructing Model's spaces in algebraic geometry.

Ex: let $G = G_m$ and $X = Speek[x,y] = A^2$ again. Then origin and conics are closed orbits, but dim $G - \frac{20}{3} = \dim \frac{20}{3} = 0 < \dim G_m = 1$. Heree $\dim G_{(0,0)} > 0$, so (0,0) is not stable. Heree $X = \frac{2}{3}(x,y) \in A^2 | xy \neq 0 = X_{xy}$.