Bailey Pairs With Free Parameters, Mock Theta Functions and Tubular Partitions

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Abstract

This study began in an effort to find a simpler derivation of the Bailey pairs associated with the seventh order mock theta functions. It is shown that the introduction of a new parameter independent of both a and q leads to a much simpler treatment. It is noted that a previous treatment of the central fifth order mock theta function inadvertently uses this approach. The paper concludes by applying this method to find new surprising identities and new arithmetic objects, tubular partitions.

1 Introduction

Ramanujan's seventh order mock theta function proved to be one of his more enigmatic discoveries. In the 1930's A Selberg [13] described their behavior near $q = 1$. However, the real understanding of these functions started with D. Hickerson's major paper [11]. Hickerson based his studies on results in [4]. For example, let us define

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n}, \tag{1.1}$$

where

$$(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{1.2}$$

*Partially supported by National Security Agency Grant: H98230-12-1-0205

Then in [4, p. 132], we find

$$\mathcal{F}_0(q) = \frac{1}{(q)_\infty}\left\{\sum_{\substack{n=0\\|j|\le n}}^{\infty} q^{7n^2+n-j^2}\left(1-q^{12n+6}\right) - 2q\sum_{n=0}^{\infty}\sum_{j=0}^{n} q^{7n^2+8n-j^2}(1-q^{12n+13})\right\} \tag{1.3}$$

The proof of this result relies on the following instance of the weak form of Bailey's Lemma [5, p. 27, eq. (3.33)]

$$\sum_{n=0}^{\infty} q^{n^2}a^n\beta_n = \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty} q^{n^2}a^n\alpha(a,n) \tag{1.4}$$

where

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha(a;j)}{(q)_{n-j}(aq)_{n+j}}. \tag{1.5}$$

In the case of (1.3), the Bailey pair $(\alpha(1,n),\beta_n)$ with $a=1$ is given by

$$\beta_n = \frac{1}{\left(q^{n+1}\right)_n}, \tag{1.6}$$

and

$$\alpha(1,2n) = q^{3n^2+n}\sum_{|j|\le n} q^{-j^2} - q^{3n^2-n}\sum_{|j|<n} q^{-j^2}, \tag{1.7}$$

$$\alpha(1,2n+1) = -2q^{3n^2+4n+1}\sum_{j=0}^{n} q^{-j^2-j} + 2q^{3n^2+2n}\sum_{j=0}^{n-1} q^{-j^2-j}. \tag{1.8}$$

We note that the proof that these $(\alpha(1,n),\beta_n)$ satisfy (1.5) required a couple of pages of ad hoc recurrence arguments.

At the end of [4], it was suggested that insight might be gained by studying

$$\beta_n = \frac{(bq)_n}{(q)_{2n}}. \tag{1.9}$$

It was pointed out that $b=1$ yields (1.1). In addition, other famous identities arise from $b = 0, q^{-\frac{1}{2}}$, and -1. The first object of this paper is to explore and explain this phenomenon.

In section 2, we consider general aspects of Bailey pairs $(\alpha(a,n),\beta_n)$ given by (1.5), where β_n has a parameter b and β_n is independent of a. Section 3 illustrates how the treatment of the central fifth order mock theta function in [6, §12] fits into this approach. In section 4, we develop new identities using this approach. For example, we shall prove that

$$1+3\sum_{n=1}^{\infty}\frac{(-q)_{n-1}^2 q^{n^2}}{(q)_{2n}}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}\lfloor\frac{3n+2}{2}\rfloor q^{n(3n-1)/2}(1-q^{4n+2}), \tag{1.10}$$

and

$$1+2\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2}}{(q)_{n-1}(q)_{2n}}=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}q^{n(3n-1)/2}\left((-1)^{\lfloor\frac{n}{3}\rfloor}-(-1)^{\lfloor\frac{n-1}{3}\rfloor}q^n\right)\right). \tag{1.11}$$

The FIRST point to emphasize here is the essential difference between the current recurrence treatment of Bailey pairs and the derivation using specializations of q-hypergeometric series identities. The former method really started with L.J. Rogers in [16] and [17], while the latter method was begun by W.N. Bailey [8], [9] and used extensively by L.J. Slater [14], [15].

In Rogers' pioneering work, he did not employ a second parameter like the b in (1.9). When I raised the original question [4] about the β_n in (1.9), I was hoping one could find a useful q-hypergeometric representation of the related $\alpha(1,n)$. In section 4, the mystery of this $\alpha(1,n)$ will be solved via recurrences, and this will provide a very simple proof of (1.7) and (1.8) (the case $b=1$).

The results in section 5 (including (1.10) and (1.11)) help to illustrate the observation that there are natural identities that seem much more amenable to recurrence treatment than to q-hypergeometric treatment. Hopefully this paper will show that the ideas of L.J. Rogers merit continued study.

The SECOND point to emphasize is that the realm of q-hypergeometric series can be greatly expanded in surprising ways if we do not limit ourselves initially to the classic q-series definition as a series [10, p. xii]

$$\sum_{n=0}^{\infty}c_n,$$

where C_{n+1}/C_n is a rational function of q^n. There have been extensions of this nature in the past (e.g. [1]); however they have been of limited interest.

To provide an idea of where we are going, we shall define *tubular partitions.*

Definition. *The positive integer sequences* $\{n,n\}$, $\{n,n-1\}$, $\{n,n-1,n-1,n-2\}$, $\{n,n-1,n-1,n-2,n-2,n-3\},\dots,\{n,n-1,n-1,\dots,n-j+1,n-j+1,n-j\},\dots$ *are called tubes.*

Definition. *A tubular partition of* n *is a partition of* n *where parts form a disjoint union of tubes;* $T(n)$ *is the number of tubular partitions of* n*, and* $t_m(q)$ *is the generating function for tubular partitions whose parts are* $< m$.

For example, there are two tubular partitions of 6: $(3+3),(2+2)+(1+1)$. There are six tubular partitions of 12: $(6+6),(5+5)+(1+1)$, $(4+4)+(2+2),(5+4)+(2+1),(4+3+3+2)$, $(3+3)+(2+2)+(1+1)$.

In section 6, we shall prove that

$$\sum_{n\geq 0}\frac{t_n(q)q^{n^2}}{(q)_{2n}}=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{35n^2-3n}\left(1-q^{20n+2}\right). \tag{1.12}$$

We shall also consider augmented tubular partitions (to be defined in section 6), and $\tau_m(q)$ will be the generating function for augmented tubular partitions whose parts are all $< m$. We shall prove that

$$\sum_{n\geq 0}\frac{\tau_n(q)q^{n^2}}{(q)_{2n}}=\frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}q^{n(3n-1)/2}(1-q^n)\sum_{|4j|<n}(-1)^j q^{-2j^2}+\sum_{n\geq 1}(-1)^n q^{22n^2-2n}\right) \tag{1.13}$$

Here we see the Hecke-type series (so familiar in the study of mock theta functions) involving indefinite quadratic forms.

In Section 7, we note a simple bijection between tubular partitions and the partitions arising in the second Rogers-Ramanujan identity [12]. This will allow us to show that (1.12) is equivalent to:

$$1+\sum_{n,j\geq 0}\frac{q^{n^2+2jn+2n+2j^2+3j+1}}{(q)_j(q)_{2n}(q^{2n+j+1};q)_{j+2}}=\frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}q^{35n^2-3n}(1-q^{20n+2}) \tag{1.14}$$

In Section 8, we shall outline specific directions for future work, and Section 9 concludes with general questions.

2 Bailey Pairs With Independent β_n

Throughout the remainder of this paper, we shall stipulate that the β_n appearing in instances of (1.5) is independent of a. This seems to be highly unlikely in general given the right-hand side of (1.5). However, as was shown in [2], [3] there is as equivalent formulation of the defining relation between $\alpha(a,n)$ and β_n :

$$\alpha(a,n) = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q)_{n-j}}. \tag{2.1}$$

This relation is used when the β_j sequence is given first and the $\alpha(a,n)$ is to be determined by (2.1).

Usually we are most interested in Bailey pairs where $a = 1$ or q. So we define

$$\alpha_1(n) = \alpha(1,n), \tag{2.2}$$

$$\alpha_0(n) = \frac{(1-q)}{(1-q^{2n+1})}\alpha(q,n) \tag{2.3}$$

More generally, we define

$$A(a,n) := \frac{(1-aq)}{(1-aq^{2n+1})}\alpha(aq,n), \tag{2.4}$$

so that

$$\alpha_0(n) = A(1,n). \tag{2.5}$$

We also define $\bar{\alpha}(a,n)$ to be

$$\bar{\alpha}(a,n) = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}+j} \beta_j}{(q)_{n-j}}; \tag{2.6}$$

i.e. $(\bar{\alpha}(a,n), q^j\beta_j)$ form a Bailey pair.

Next

$$\alpha^*(a,n) = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_{j+1}}{(q)_{n-j}}, \tag{2.7}$$

i.e. $(\alpha^*(a,n), \beta_{n+1})$ form a Bailey pair.

and

$$A^*(a,n) = \frac{(1-aq)}{1-aq^{2n+1}}\alpha^*(a,q,n). \tag{2.8}$$

Theorem 1. *For $n \geq 1$,*

$$\alpha(a,n) - \frac{(1-aq)}{(1-aq^{2n+1})}\alpha(aq,n) = \frac{-aq^{2n-1}(1-aq)}{(1-aq^{2n-1})}\alpha(aq,n-1). \tag{2.9}$$

Remark. The proof crucially relies on the fact that β_n is independent of n.

Proof.

$$\begin{aligned}
&\alpha(a,n) - \frac{(1-aq)}{(1-aq^{2n+2})}\alpha(aq,n) \\
&\quad = \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q)_{n-j}} \left((1-aq^{2n}) - (1-aq^{n+j})\right) \\
&\quad = aq^n \sum_{j=0}^{n-1} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}+j}\beta_j}{(q)_{n-j-1}} \\
&\quad = \frac{-aq^{2n-1}(1-aq)}{(1-aq^{2n-1})}\alpha(aq,n-1).
\end{aligned}$$

□

Corollary 2. *For $n \geq 1$*

$$\alpha(a,n) - A(a,n) = -aq^{2n-1}A(a,n-1) \tag{2.10}$$

$$\alpha_1(n) - \alpha_0(n) = -q^{2n-1}\alpha_0(n-1). \tag{2.11}$$

Proof. Equation (2.10) is merely a restatement of (2.9), and (2.11) is (2.10) with $a = 1$. □

Theorem 3. *For $n \geq 0$,*

$$aq^n\bar{\alpha}(a,n) = \alpha(a,n) - \frac{(1-aq^{2n})(1-aq)}{1-aq^{2n+1}}\alpha(aq,n). \tag{2.12}$$

Proof.

$$\begin{aligned}
aq^n\bar{\alpha}(a,n) &= aq^n(1-aq^{2n}) \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}+j}\beta_j}{(q)_{n-j}} \\
&= (1-aq^{2n}) \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j\,(1-(1-aq^{n+j}))}{(q)_{n-j}}
\end{aligned}$$

$$= \alpha(a,n) - (1-aq^{2n}) \sum_{j=0}^{n} \frac{(aq)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q)_{n-j}}$$
$$= \alpha(a,n) - \frac{(1-aq^{2n})(1-aq)}{(1-aq^{2n+1})} \alpha(aq,n).$$

□

Corollary 4.

$$q^n \bar{\alpha}(1,n) = \alpha_1(n) - (1-q^{2n})\alpha_0(n) \tag{2.13}$$

Proof. Set $a = 1$ in (2.10). □

Theorem 5.

$$\alpha(a,n+1) = (1-aq^{2n+2}) \Bigg((1-aq)\alpha^*(aq,n) + \frac{aq^{2n}\alpha(a,n)}{(1-aq^{2n})} - \frac{(aq)_{n-1}(-1)^n q^{\binom{n+1}{2}} (1-aq^{2n+1})\beta_0}{(q)_{n+1}} \Bigg). \tag{2.14}$$

Proof.

$$\alpha(a,n+1) - (1-aq^{2n+2})(1-aq)\alpha^*(aq,n)$$
$$= \left(1-aq^{2n+2}\right) \Bigg\{ \sum_{j\geq 0} \frac{(aq)_{n+j}(-1)^{n+1-j} q^{\binom{n+1-j}{2}} \beta_j}{(q)_{n+1-j}} - \left(1-aq^{2n+1}\right) \sum_{j\geq 0} \frac{(aq)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_{j+1}}{(q)_{n-j}} \Bigg\}$$
$$= \left(1-aq^{2n+2}\right) \Bigg(\frac{-(aq)_n(-1)^n q^{\binom{n+1}{2}} \beta_0}{(q)_{n+1}} + \sum_{j\geq 0} \frac{(aq)_{n+j+1}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_{j+1}}{(q)_{n-j}} - \left(1-aq^{2n+1}\right) \sum_{j\geq 0} \frac{(aq)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_{j+1}}{(q)_{n-j}} \Bigg)$$
$$= \left(1-aq^{2n+2}\right) \Bigg(\frac{-(aq)_n(-1)^n q^{\binom{n+1}{2}} \beta_0}{(q)_{n+1}} + \sum_{j\geq 0} \frac{(aq)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_{j+1}}{(q)_{n-j}} \left(\left(1-aq^{n+j+1}\right) - \left(1-aq^{2n+1}\right) \right) \Bigg)$$

$$
\begin{aligned}
&= \left(1 - aq^{2n+2}\right)\left(\frac{-(aq)_n(-1)^n q^{\binom{n+1}{2}}\beta_0}{(q)_{n+1}}\right.\\
&\qquad \left. - a\sum_{j\geq 0}\frac{(aq)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}+n+j+1}\beta_{j+1}}{(q)_{n-j-1}}\right)\\
&= \left(1 - aq^{2n+2}\right)\left(\frac{-(aq)_n(-1)^n q^{\binom{n+1}{2}}\beta_0}{(q)_{n+1}}\right.\\
&\qquad \left. + aq^{2n}\sum_{j\geq 1}\frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q)_{n-j}}\right)\\
&= \left(1 - aq^{2n+2}\right)\left(\frac{-(aq)_n(-1)^n q^{\binom{n+1}{2}}\beta_0}{(q)_{n+1}} - \frac{aq^{2n}(aq)_{n-1}(-1)^n q^{\binom{n}{2}}\beta_0}{(q)_n}\right.\\
&\qquad \left. + \frac{aq^{2n}\alpha(a,n)}{(1-aq^{2n})}\right)\\
&= \frac{aq^{2n}\left(1-aq^{2n+2}\right)}{(1-aq^{2n})}\alpha(a,n)\\
&\qquad - \frac{(aq)_{n-1}(-1)^n q^{\binom{n+1}{2}}\left(1-aq^{2n+2}\right)\left(1-aq^{2n+1}\right)\beta_0}{(q)_{n+1}}.
\end{aligned}
$$

□

Until we examine applications in the next three sections, it is difficult to discern the object of the theorems just completed. In practice, we shall study $\alpha_0(n) = A(1,n)$.

If we know a closed form for $\alpha_0(n)$, then by Corollaries 2 and 4 we have closed forms for $\alpha_1(n)$ and $\bar{\alpha}(1,n)$. When we consider applications we will find that we do not require a closed form for $\alpha(a,n)$ in order to deduce, via recurrences, a nice closed form for $\alpha_1(n) = \alpha(1,n)$.

3 The Fifth order Mock Theta Function $\mathcal{F}_0(q)$

At the end of [6, §12], the following identity was proved that

$$\sum_{n\geq 0}\frac{q^{n^2}a^n(bq)_n}{(q^2;q^2)_n}=\frac{1}{(aq;q)_\infty}\sum_{n\geq 0}\frac{(-1)^n a^n q^{2n^2}(a^2;q^2)_n(1-aq^{2n})p_n(-b;-\frac{a}{q},-1:q)}{(q^2;q^2)_n(1-a)}, \tag{3.1}$$

where the little q-Jacobi polynomial [10, p. 27] is:

$$p_n(x;A,B:q)=\sum_{j=0}^{n}\frac{(q^{-n})_j\,(ABq^{n+1})_j\,(xq)^j}{(q)_j(Aq)_j}. \tag{3.2}$$

The original object in [6] was to provide an identity which reduced to the first Rogers-Ramanujan identity [10, p. 36] when $b=1$, and to the fifth order mock theta function identity [4, p. 114]

$$f_0(q):=\sum_{j\geq 0}\frac{q^{j^2}}{(-q;q)_j}=\frac{1}{(q)_\infty}\sum_{\substack{n=0\\|j|\leq n}}^{\infty}(-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}) \tag{3.3}$$

when $b=-1$.

The proof of (3.1) was a standard Bailey pair argument, where

$$\beta_n=\frac{(bq)_n}{(q^2;q^2)_n}, \tag{3.4}$$

and

$$\alpha_n(a,q)=(-1)^n q^{n^2}(a^2;q^2)_n(1-aq^{2n})p_n(-b;-\frac{a}{q},-1:q). \tag{3.5}$$

The fact that α_n is a multiplicative factor times a q-orthogonal polynomial led to the work presented in [7]. In fact, the classic three term recurrence formula for p_n can be recast as a recurrence for α_n. Namely, in the notation of section 2, for $n\geq 1$,

$$\alpha_0(n)+bq^n\alpha_0(n-1)=bq^{3n-1}\alpha_0(n-1)+q^{4n-4}\alpha_0(n-2), \tag{3.6}$$

where $\alpha_0(0)=1$ and $\alpha_0(-1)=-q$.

Furthermore this recurrence is easily established because by (2.1) and (2.3)

$$\alpha_0(n) = \sum_{j=0}^{n} \frac{(q)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} (bq)_j}{(q)_{n-j}(q^2;q^2)_j}, \tag{3.7}$$

and the sequence $(bq)_j$, $j = 0, 1, 2, \ldots$ forms a basis of the polynomials in b. In addition

$$\begin{aligned} b(bq)_n &= (bq)_n(-q^{-n-1}(1 - bq^{n+1}) + q^{-n-1}) \\ &= q^{-n-1}(-(bq)_{n+1} + (bq)_n). \end{aligned} \tag{3.8}$$

Hence (3.6) can be reduced to a linear combination of $(bq)_0, (bq)_1, \ldots, (bq)_n$ which of necessity has to be valid for the coefficients of each $(bq)_j$.

Now when $b = 1$, it is a simple exercise to prove by mathematical induction that

$$\alpha_0(n) = q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}.$$

Then by (2.9), we obtain the closed form for $\alpha(1, n)$ that immediately yields (3.3).

This section provides a prototype for all that follows. *The role of the free parameter b is crucial in showing that the expression given in* (3.7) *satisfies the recurrence* (3.6).

4 The Seventh Order Mock Theta Functions $\mathcal{F}_0(q)$

Theorem 6. *If $\beta_n = (bq)_n/(q)_{2n}$, then the related $\alpha_0(n)$ satisfies for $n \geq 1$,*

$$\alpha_0(n) + bq^n \alpha_n(n-1) = bq^{3n-3}\alpha_0(n-2) + q^{4n-7}\alpha_0(n-3), \tag{4.1}$$

with initial values $\alpha_0(0) = 1, \alpha_0(-1) = -q, \alpha_0(-2) = bq^4$.

Proof. In this case,

$$\alpha_0(n) = \sum_{j=0}^{n} \frac{(q)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} (bq)_j}{(q)_{n-j}(q)_{2j}}.$$

Recalling (3.8), we see that (4.1) can be rewritten as a massive combination of the $(bq)_j$; namely if we write

$$c(n,j) = \frac{(q)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}(q)_{2j}},$$

then (4.1) becomes

$$\sum_{j=0}^{n} c(n,j)(bq)_j + \sum_{j=0}^{n-1} q^{n-j-1}c(n-1,j)\left(-(bq)_{j+1} + (bq)_j\right)$$
$$= \sum_{j=0}^{n-2} q^{3n-j-4}c(n-2,j)\left(-(bq)_{j+1} + (bq)_j\right) + \sum_{j=0}^{n-3} q^{4n-j-8}c(n-3,j)\left(-(bq)_{j+1} + (bq)_j\right).$$

The proof is now reduced to a purely algebraic exercise comparing the coefficients of $(bq)_j$ on each side of this identity. This takes care of $n \geq 3$. The cases for smaller n are then treated by inspection. The values $\alpha_0(0), \alpha_0(-1)$, and $\alpha_0(-2)$ are reverse engineered to make the recurrence valid for $n \geq 1$. $\square$

Now we do not require a closed form for general b. In each of the following special cases, the form of $\alpha_0(n)$ can easily be proved by mathematical induction using the recurrence (4.1).

Case 1. $b = 1$,

$$\alpha_0(2n) = 2q^{3n^2}\sum_{j=0}^{n-1} q^{-j^2-j} - q^{3n^2-n}\sum_{|j|<n} q^{-j^2}$$

$$\alpha_0(2n-1) = -q^{3n(n-1)+1}\sum_{|j|<n} q^{-j^2} + 2q^{3n^2-4n+1}\sum_{j=0}^{n-2} q^{-j^2-j}$$

Case 2. $b = -1$,

$$\alpha_0(2n) = (-1)^n q^{3n^2-n}$$
$$\alpha_0(2n+1) = (-1)^n q^{3n^2+3n+1}$$

Case 3. $b = 0$,

$$\begin{aligned} \alpha_0(3n) &= q^{6n^2-n} \\ \alpha_0(3n+1) &= 0 \\ \alpha_0(3n+2) &= -q^{6n^2+7n+2} \end{aligned}$$

Case 4. $q \to q^2$, then $b = q^{-1}$.

$$\alpha_0(n) = (-1)^n q^{n(3n-1/2)}$$

These were the results mentioned in [4, p. 133]. From Case 1, one can deduce (1.7) and (1.8) and consequently (1.3). Case 2 yields a formula of Rogers (cf. [15, p. 158, eq.(61)]). Case 3 yields [15, p. 160, eq. (79)], and Case 4 yields [15, p. 155, eq. (33)].

We note that

$$\frac{1}{(q^{n+1})_n} + \frac{q^n}{(q^{2n+1})_n} = \frac{(1-q^{2n})}{(1-q^n)(q^{n+1})_n} = \frac{1}{(q)_n},$$

and if we replace n by $n+1$, then $1/(q^n)_n$ becomes $1/(q^{n+1})_{n+1}$. Thus the other two Bailey pairs for seventh order mock theta functions can be treated using Theorem 3 (which considers $q^j\beta_j$) and Theorem 5 (which considers β_{j+1}). The computations are tedious, but everything has been included in section 2 to make the results routine.

5 First Extension

The last two sections considered two fruitful recurrences: (3.6) and (4.1). it is thus reasonable to expect that these two recurrences might be the first two in a family of recurrences. If so, the next plausible recurrence would be

$$\alpha_0(n) + bq^n\alpha_0(n-1) = bq^{3n-5}\alpha_0(n-3) + q^{4n-10}\alpha_0(n-4), \tag{5.1}$$

with $\alpha_0(0) = 1, \alpha_0(-1) = -q, \alpha_0(-2) = bq^4$, and $\alpha_0(-3) = -b^2q^8$.

Theorem 7. *With $\alpha_0(n)$ given in (5.1), the corresponding β_n (for $a = 1$) is given by $\beta_0 = 1$, and for $n > 0$*

$$\beta_n = \frac{(1-bq)}{(q)_{2n}} \prod_{j=2}^{n} \left(1 - bq^j + q^{2j-2}\right).$$

Proof. Now $\alpha_0(n)$ is easily seen to be a polynomial in b of degree n. Furthermore the β_n are a basis for the polynomials in b of degree n, and finally for $n \geq 2$,

$$
\begin{aligned}
b\beta_{n-1} &= \left((-q^{-n})\left(1 - bq^n + q^{2n-2}\right) + q^{-n} + q^{n-2}\right)\beta_{n-1} && (5.2)\\
&= -q^{-n}\beta_n(1-q^{2n})(1-q^{2n-1}) + (q^{-n}+q^{n-2})\beta_{n-1}. && (5.3)
\end{aligned}
$$

Thus we can easily rewrite (5.1) as a polynomial identity in the β_n and coefficient comparison reveals that these β_n fulfill the defining recurrence (5.1). □

Corollary 8. *Identities* (1.10) *and* (1.11) *hold.*

Remark. It is possible to deduce [15, p. 156, eq.(47)] and [15, p. 165, eq.(121)] from (1.11). Indeed (1.11) is equivalent to each of these.

Proof. One can prove by induction that if $b = -\frac{2}{q}$, then

$$\alpha_0(n) = \left\lfloor \frac{3n+2}{2} \right\rfloor q^{\binom{n}{2}}.$$

Identity (1.10) then follows by appropriately applying (1.4) with $a = 1$ and obtaining $\alpha(1,n)$ from (2.2) and (2.9).

One can prove by induction that if $b = -\frac{1}{q}$, then

$$\alpha_0(n) = (-1)^{\lfloor \frac{n}{3} \rfloor} q^{\binom{n}{2}}.$$

Identity (1.11) then follows by appropriately applying (1.4) with $a = 1$ and obtaining $\alpha(1,n)$ from (2.2) and (2.9). □

6 Second Extension

We now have in hand three defining recurrences for $\alpha_0(n)$, namely (3.6), (4.1), and (5.1). The natural next recurrence is

$$\alpha_0(n) + bq^n\alpha_0(n-1) = bq^{3n-7}\alpha_0(n-4) + q^{4n-13}\alpha_0(n-5), \tag{6.1}$$

with $\alpha_0(0) = 1, \alpha_0(-1) = -q, \alpha_0(-2) = bq^4, \alpha_0(-3) = -b^2q^8, \alpha_0(-4) = b^3q^{13}$.

Theorem 9. *Identity* (1.12) *is true.*

Proof. Given the recurrence (6.1), we contend the related β_j appropriate for (2.1) and (2.3) is given by

$$\beta_j = \frac{B(b,n)}{(q)_{2n}}, \tag{6.2}$$

where $B(z,0) = 1$, and for $n > 0$,

$$B(z,n) = (q^{2n-2} - zq^n)B(z,n-1) + \sum_{j=1}^{n-1} q^{(j-1)(2n-j-1)} B(z,n-j). \tag{6.3}$$

To establish this we follow the exact same argument used in Theorems 5 and 6. Namely, $\alpha_0(n)$ is a polynomial in b of degree n (easily established by induction using (6.1)). Furthermore it is clear by (2.1) and (2.3) that the related β_j are uniquely determined once the $\alpha_0(n)$ sequence is given. On the other hand, if we replace the β_j in $\alpha_0(n)$ by the expression given in (6.2), and if we note that by (6.3)

$$bB(b,n) = -q^{-n-1}\big(B(b,n+1) - q^{2n-2}B(b,n) - \sum_{j=1}^{n} q^{(j-1)(2n-j+1)} B(b,n+1-j)\big) \tag{6.4}$$

we see that (6.1) becomes an identity about a linear combination in the $B(b,n)$. Proving this assertion is only an exercise in coefficient comparison. This establishes (6.2).

Next we note $B(0,0) = B(0,1) = 1$, and for $n > 1$

$$B(0,n) = \left(1 + q^{(n-1)+(n-1)}\right) B(0,n-1) + q^{(n-1)+(n-2)} B(0,n-2) \tag{6.5}$$

$$+ \sum_{j=3}^{n-1} q^{(n-1)+2(n-2)+\cdots+2(n-j+1)+(n-j)} B(0,n-j). \tag{6.6}$$

Inspection of the partitions generated reveals that

$$B(0,n) = t_n(q). \tag{6.7}$$

Also with $b = 0$, we can establish by induction that

$$\alpha_0(n) = \begin{cases} -q^{10\nu^2-3\nu} & \text{if } n = 5\nu \\ q^{10\nu^2+13\nu+4} & \text{if } n = 5\nu+4 \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

From (6.8) one can easily construct $\alpha(1, n)$ using (2.9) and (2.2). Filling all this into (1.4), we obtain

$$\sum_{n\geq 0} \frac{t_n(q)q^{n^2}}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} q^{35n^2-3n} \left(1 - q^{2n+2}\right), \tag{6.9}$$

which is (1.12). □

Before treating (1.13), some definitions are in order.

Definition. *The positive integer sequences* $\{n\}$, $\{n, n\}$, $\{n, n-1\}$, $\{n, n-1, n-1, n-2\}$, $\{n, n-1, n-1, n-2, n-2, n-3\}, \ldots,$ $\{n, n-1, n-1, \ldots, n-j+1, n-j+1, n-j\}$ *(and in addition* $\{0\}$*) are called augmented tubes (or a-tubes).*

Definition. *An augmented tubular partition of* n *is a partition of* n *(into non-negative parts) whose parts form a disjoint union of a-tubes. If there are* j *different disjoint unions of a-tubes for a given partition, then the partition is counted* j *times.* $\tau_m(q)$ *is the generating function for augmented tubular partitions whose parts are all* $< m$.

The possibility that a given partition is counted more than once often arises. For example, (3)+(2)+(1)+(0), (3+2)+(1)+(0) and (3)+(2+1)+(0).

Now referring to (6.3), we see that $B\left(-\frac{-z}{q}, 0\right) = 1$, $B\left(-\frac{z}{q}, 1\right) = 1 + zq^0$ and for $n > 1$

$$B\left(-\frac{z}{q}, n\right) = \left(zq^{n-1} + q^{2n-2}\right) B\left(-\frac{z}{q}, n-1\right) \tag{6.10}$$

$$+ \sum_{j=2}^{n-1} q^{(n-1)+(n-2)+(n-2)+\cdots+(n-j+1)+(n-j+1)+(n-j)} B\left(-\frac{z}{q}, n-j\right). \tag{6.11}$$

Inspection of the partitions generated reveals that

$$B\left(-\frac{1}{q}, n\right) = \tau_n(q). \tag{6.12}$$

Also with $b = -\frac{1}{q}$, we can establish by induction that

$$\alpha_0(n) = q^{\binom{n}{2}} \sum_{|4j|\leq n} (-1)^j q^{-2j^2}. \tag{6.13}$$

Theorem 10. *Identity* (1.13) *is true.*

Proof. From (6.13) one can easily construct $\alpha(1,n)$ using (2.9) and (2.2). Filling all this into (1.4), we obtain

$$\sum_{n\geq 0}\frac{\tau_n(q)q^{n^2}}{(q)_{2n}}=\frac{1}{(q)_\infty}\left\{1+\sum_{n\geq 1}q^{n(3n-1)/2}(1-q^n)\sum_{|4j|<n}(-1)^jq^{-2j^2}+\sum_{n\geq 1}(-1)^nq^{22n^2-2n}\right\}, \tag{6.14}$$

which is (1.13). □

7 Tubular Partitions and Rogers-Ramanujan Partitions

If in the tubular partitions we replace every appearance of $n+n$ by $2n$ and every appearance of $n+(n-1)+(n-1)+\cdots+(n-j+1)+(n-j+1)+(n-j)$ by $(2n-1)+(2n-3)+\cdots+(2n-2j+1)$, we see that there is a bijection between the tubular partitions with all parts $< m$ and partitions with all parts $< 2m$, difference at least 2 between parts and no 1's. Thus, we see that the Rogers-Ramanujan partitions have appeared. Consequently

$$t_m(q)=\sum_{j=0}^{m-1}\begin{bmatrix}2m-2-j\\ j\end{bmatrix}q^{j^2+j}, \tag{7.1}$$

where

$$\begin{bmatrix}R\\ S\end{bmatrix}=\begin{cases}0 & \text{if} S<0, \text{ or } S>R\\ \frac{(q)_R}{(q)_S(q)_{R-S}} & \text{otherwise.}\end{cases}$$

MacMahon [12, p.42, §289] was the first to discern the right-hand side of (7.1).

Consequently, by Theorem 8 and (7.1), we see that

$$\begin{aligned}\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{35n^2-3n}\left(1-q^{20n+2}\right)&=1+\sum_{n=1}^{\infty}\frac{t_n(q)q^{n^2}}{(q)_{2n}}\\ &=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_{2n}}\sum_{j=0}^{n-1}\begin{bmatrix}2n-2-j\\ j\end{bmatrix}q^{j^2+j}\end{aligned}$$

$$= 1 + \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q)_{2n+2}} \sum_{j=0}^{n} \frac{(q)_{2n-j} q^{j^2+j}}{(q)_j (q)_{2n-2j}}$$

$$= 1 + \sum_{n,j\geq 0} \frac{q^{j^2+j+(n+j+1)^2}}{(q^{2n+j+1};q)_{j+2}(q)_j(q)_{2n}},$$

which is (1.14).

8 And Beyond

We have studied in this paper the first four (i.e. $k = 1,2,3,4$) $\alpha_0(n)$ sequences defined by

$$\alpha_0(n) + bq^n \alpha_0(n-1) = bq^{3n-2k+1}\alpha_0(n-k) + q^{4n-3k-1}\alpha_0(n-k-1), \quad (8.1)$$

with initial values, $\alpha_0(0) = 1$ and $\alpha_0(-n) = (-1)^n b^{n-1} q^{\binom{n+2}{2}-2}$ for $1 \leq n \leq k$.

The variety of results discovered in these first four cases clearly suggests that the cases with $k > 4$ merit investigation. It is worthwhile to note that the $\alpha_0(n)$, which we treated for $k \leq 4$ when $b = 0$ and $b = -\frac{1}{q}$, have nice representations in general.

Theorem 11. *Let $k \geq 4$ and let $\alpha_0(n)$ be defined by* (8.1) *and the given initial conditions. Then if $b = 0$, and $n \geq 0$,*

$$\alpha_0(n) = \begin{cases} (-1)^{k-1} q^{(2k+2)\nu^2-(k-1)\nu} & \text{if } n = (k+1)\nu \\ (-1)^k q^{(2k+2)\nu^2+(3k+1)\nu+\nu} & \text{if } n = (k+1)\nu + k + 1 \\ 0 & \text{otherwise} \end{cases} \quad (8.2)$$

and if $b = -\frac{1}{q}$, and $n > 0$

$$\alpha_0(n) = q^{\binom{n}{2}} \sum_{|kj|\leq n} (-1)^j q^{-k(k-3)n^2/2} \quad (8.3)$$

Proof. Both (8.2) and (8.3) are proved by induction using (8.1). □

9 Conclusion

It is my hope that this paper will generate studies is several directions. Theorem 10 makes clear that the general sequence defined in (8.1) has immediate ties to both classical theta series via (8.2) and to the Hecke-type series associated with the mock theta functions via (8.3). What is still mysterious is the nature of the corresponding β_n. For $k = 1, 2, 3$, and 4, we see a variety of connections with both classical work on theta and mock theta functions when $k < 4$ and with tubular partitions for $k = 4$. What is the nature of β_n for $k > 4$?

Apart from the specific sequences considered in this paper, one should consider the recurrence approach to Bailey pairs in general. The fundamental results proved in section 2 can be widely applied. They were originally developed here to provide the resolution of the question asked at the end of [4], and answered in section 4. Surely this is just the beginning.

References

[1] G.E. Andrews, *On the Alder polynomials and a new generalization of the Rogers-Ramanujan identities*, Trans. Amer. Math. Soc., **204** (1975), 40-64.

[2] G.E. Andrews, *Connection coefficient problems and partitions*, Proc. Symp. in Pure Math., **34** (1979), 1-24.

[3] G.E. Andrews, *Multiple series Rogers-Ramanujan type identities*, Pac. J. Math., **114** (1984), 267-283.

[4] G.E. Andrews, *q-Series: Their Development and Application ...,* C.B.M.S. Regional Conference Series in Math. No. 66, Amer. Math. Soc., Providence, 1986.

[5] G.E. Andrews, *The fifth and seventh order mock theta functions*, Trans. Amer. Math. Soc., **293** (1986), 113-134.

[6] G.E. Andrews, *Parity in partition identities*, Ramanujan J., **23** (2010), 45-90.

[7] G.E. Andrews, *q-Orthogonal polynomials, Rogers-Ramanujan identities and mock theta functions*, Proc. of the Steklov Inst. Dedicated to the 75^{th} Birthday of A.A. Karatsuba, **276** (2012), 21-32.

[8] W.N. Bailey, *Some identities in combinatory analysis*, Proc. London Math. Soc., **49** (1947), 421-435.

[9] W.V. Bailey, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2), **50** (1949), 1-10.

[10] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encycl. Math. and Appl., Vol. 35, Cambridge University Press, Cambridge, 1990.

[11] D. Hickerson, *On the seventh order mock theta functions*, Invent. Math., **94** (1988), 661-677.

[12] P.A. MacMahon, *Cominatory Analysis*, Vol. 2, Cambridge University Press, London, 1916 (Reprinted: Chelsea, New York, 1960).

[13] A. Selberg, *Über die Mock-Thetafunktionen siebenter Ordnung*, Arch. Math. og Naturvidenskab, **41** (1938), 1-15.

[14] L.J. Slater, *A new proof of Rogers' transformations of infinite series*, Proc. London Math. Soc. (2), **53** (1951), 460-475.

[15] L.J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2), **54**, (1952), 147-167.

[16] L.J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., **25** (1893), 318-343.

[17] L.J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (2), **16** (1917), 315-336.

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