THE DUAL OF GÖLLNITZ'S (BIG) PARTITION THEOREM*

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Dedicated to the memory of Basil Gordon

ABSTRACT: We present a new companion to the deep partition theorem of Göllnitz and discuss it in the context of a generalization of Göllnitz's theorem by Alladi-Andrews-Gordon that was obtained by the method of weighted words. After providing a q-theoretic proof of the new companion theorem, we discuss its analytic representation and its link to the key identity of Alladi-Andrews-Gordon.

Keywords: partitions, Göllnitz theorem, method of weighted words, key identity

2000 Subject Classification: 05A15, 05A17, 05A19, 11B65, 33D15

1. Introduction

A Rogers-Ramanujan (R-R) type identity is a q-hypergeometric identity in the form of an infinite (possibly multiple) series equals an infinite product. The series is the generating function of partitions whose parts satisfy certain difference conditions, whereas the product is the generating function of partitions whose parts usually satisfy certain congruence conditions. For a discussion of a variety of R-R type identities, see Andrews [16], Ch.9.

The partition theorem which is the combinatorial interpretation of a q-hypergeometric identity, is called a Rogers-Ramanujan type partition identity. A q-hypergeometric R-R type identity is usually discovered first and then its combinatorial interpretation as a partition theorem is given. There are important instances of Rogers-Ramanujan type partition identities being discovered first and their q-hypergeometric versions given later. Perhaps the first such significant example is the 1926 partition theorem of Schur [24]:

Theorem S:

Let $T(n)$ denote the number of partitions of an integer n into parts $\equiv \pm 1 \, (mod\, 6)$.

Let $S(n)$ denote the number of partitions of n into distinct parts $\equiv \pm 1 \, (mod\, 3)$.

Let $S_1(n)$ denote the number of partitions of n into parts that differ by ≥ 3, where the inequality is strict if a part is a multiple of 3.

Then

$$T(n) = S(n) = S_1(n).$$

In the early 1990s, Alladi and Gordon [9] obtained a generalization and two parameter refinement of the equality $S(n) = S_1(n)$, and in that process derived a *key identity*, namely a q-hypergeometric identity in two free parameters that yielded Theorem S as a special case. Thus the R-R type identity for Schur's theorem came half a century later.

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*Research of both authors supported by grants from the National Security Agency.
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One of the deepest R-R type partition identities is a 1967 theorem of Göllnitz [23]:

Theorem G:

Let $B(n)$ denote the number of partitions of n into parts $\equiv 2, 5,$ or $11 \,(mod\, 12)$.

Let $C(n)$ denote the number of partitions of n into distinct parts $\equiv 2, 4,$ or $5 \,(mod\, 6)$.

Let $D(n)$ denote the number of partitions of n into parts that differ by ≥ 6, where the inequality is strict if a part is $\equiv 0, 1,$ or $3 \,(mod\, 6)$, and with 1 and 3 not occurring as parts. Then

$$B(n) = C(n) = D(n).$$

Göllnitz's succeeded in proving Theorem G in the refined form

$$C(n; k) = D(n; k), \tag{1.1}$$

where $C(n; k)$ and $D(n; k)$ denote the number of partitions of the type counted by $C(n)$ and $D(n)$ respectively, with the extra condition that the number of parts is k, and with the convention that parts $\equiv$ 0, 1, or 3 (mod 6) are counted twice.

The equality $B(n) = C(n)$ is easy because

$$\begin{aligned}\sum_{n=0}^{\infty} B(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{12m-10})(1-q^{12m-7})(1-q^{12m-1})} \\ &= \prod_{m=1}^{\infty}(1+q^{6m-4})(1+q^{6m-2})(1+q^{6m-1}) = \sum_{n=0}^{\infty} C(n)q^n.\end{aligned} \tag{1.2}$$

This is one reason that we focus on the deeper equality $C(n) = D(n)$, the second reason being that it is this equality which can be refined as in (1.1).

Göllnitz's proof of Theorem G and of (1.1) is very intricate and difficult. Andrews [14] subsequently provided a simpler proof. Theorem G is a good example of a Rogers-Ramanujan type partition theorem for which its q-hypergeometric representation came much later. In 1995, Alladi, Andrews, and Gordon [8] established the following *key identity*

$$\begin{aligned}&\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^{\alpha}(1-q^{\phi}))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} \\ &= \sum_{i,j,k} \frac{a^i b^j c^k q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k} = (-aq)_\infty(-bq)_\infty(-cq)_\infty\ ,\end{aligned} \tag{1.3}$$

and viewed a three parameter refinement of Theorem G and (1.1) as emerging from the R-R type identity (1.3) under the substituions

$$\text{(dilation)} \quad q \mapsto q^6, \quad \text{and (translations)} \quad a \mapsto aq^{-4}, b \mapsto bq^{-2}, c \mapsto cq^{-1}. \tag{1.4}$$

In (1.3) and throughout, $T_m = m(m+1)/2$ is the m-th triangular number. Also in (1.3) and in what follows, we have made use of the standard notation

$$(z)_n = (z; q)_n = \prod_{j=0}^{n-1}(1 - zq^j), \tag{1.5}$$

$$(1.6)\qquad (z)_\infty = (z;q)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{j=0}^{\infty}(1-zq^j), \quad \text{when} \quad |q|<1.$$

When the base is q, then as in (1.5) and (1.6) we simply write $(z)_n$ and $(z)_\infty$, but when the base in anything other than q, it will always be displayed.

The partition interpretation of (1.3) in terms of parts which are colored integers satisfying certain difference conditions and a specific lexicographic ordering, and the way in which these difference conditions under the substitutions (1.4) translate to the conditions governing $D(n)$ in Theorem G, will be discussed in Section 3. In [8] it was noted that by changing the ordering of the colored integers, companion identities to Theorem G can be produced and an example was given. Alladi [1], [3], considered reformulations of Theorem G by changing the substitutions in (1.4); it was shown in [5] and [6] that these reformulations led to other key identities which were simpler in structure compared to (1.3).

Historically, many of the most famous Rogers-Ramanujan type identities came in pairs, such as the two prototype Rogers-Ramanujan identities to the modulus 5 and the two Göllnitz-Gordon identities to the modulus 8. There are also the two Little Gollnitz identities to the modulus 8. Schur's classic partition theorem involving parts $\equiv \pm 1\,(mod\, 6)$ (or its version into distinct parts $\equiv \pm 1\,(mod\, 3)$), does not have a companion as in the above cases, but companions in a different sense were found by Andrews [15] using a computer search, and by Alladi-Gordon [10] by the method of weighted words by changing the lexicographic ordering of the colored integers. The companions found in [8] and in [10] by changing the lexicographic ordering, had values the same as the original partition functions. So let us call such companions as equi-valued companions or *mates*, with the term *companion* used here only when the values are <u>not</u> identical with the original.

It appears that Göllnitz regarded his Theorem G as a three dimensional extension of Schur's theorem because his proof of Schur's theorem is patterned along Gleissberg's proof [22] of Schur's theorem. Thus in analogy with Schur's theorem which has only mates but no companions, subsequent researchers did not seek companions for Göllnitz' theorem. However, in 1999, Alladi [4] found a companion to Gollnitz' theorem by replacing the residue classes 2, 4, 5 (mod 6) of $C(n)$ in Theorem G by 1, 3, 5 (mod 6), which lent itself to a nice combinatorial treatment using 2-modular Ferrers graphs because $C(n)$ then relates to partitions into distinct odd parts. What we produce here in Theorem A below is not just a companion to Theorem G but its dual.

The discovery of Theorem A was due to a fortuitous circumstance. We were attending a seminar at the University of Florida in which graduate student Keith Grizzell was reporting his joint work with Alexander Berkovich on partition function differences [19]. During questions at the end of the talk, we realized that a specialization of their result was the proof that for each n, the coefficient of q^n in the power series expansion of

$$(1.7)\qquad \prod_{m=0}^{\infty}\frac{1}{(1-q^{12m+1})(1-q^{12m+4})(1-q^{12m+7})}$$

was at least the size of $B(n)$, the coefficient of q^n in the expansion of the product in (1.2). (For the reason as to why we wrote the products in (1.4) as starting from $m=1$, and the product in (1.7) as starting from $m = 0$, see Section 5). This led us to wonder whether one could construct a new companion to Theorem G. This is what led us to

Theorem A:
Let $B^*(n)$ denote the number of partitions of n into parts $\equiv 1, 7,$ or $10\,(mod\,12)$.
Let $C^*(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2,$ or $4\,(mod\,6)$.
Let $D^*(n)$ denote the number of partitions of n into parts that differ by at least 6, where the inequality is strict if the larger part is $\equiv 0, 3,$ or $5\,(mod\,6)$, with the exception that 6+1 may appear in the partition. Then

$$B^*(n) = C^*(n) = D^*(n).$$

In the next section we provide a q-theoretic proof of Theorem A similar in spirit to the proof of Theorem G in [14]. In Section 4, we compare Göllnitz' Theorem G and our Theorem A with the two hierarchies of theorems due to Andrews [12], [13] for the moduli $2^k - 1$ starting from Schur's theorem to the modulus 3. This comparison helps us construct the key identity that represents Theorem A and this is done in Section 5; it will be seen that even though it is constructed very differently, it is actually the same as (1.3). What is interesting is that Theorem A is not a mate but a companion to Theorem G although the key identity from which it is born is the same! Notice that Theorems A and G are duals because the residues 1,2,4 (mod 6) defining $C^*(n)$ in Theorem A are replaced by residues -1, -2, -4 (mod 6) defining $C(n)$ in Theorem G. Similarly the residues the residues 1, 7, 10 (mod 12) defining $B^*(n)$ in Theorem A are replaced by the residues -1, -7, -10 (mod 12) in Theorem G. In view of the key identity, there is a three parameter refinement of Theorem A. This stated in Section 5 and proved q-theoretically in Section 6. Finally in Section 7 we discuss the role of key identities in producing companions to R-R type partition theorems.

It was Basil Gordon who drew Alladi's attention to the fact the method of weighted words introduced in [9] to study Schur's theorem could also be used to generalize Göllnitz' theorem. Thus we feel honored to dedicate this paper to the memory of Professor Gordon.

2. Proof of Theorem A

Since the equality $B^*(n) = C^*(n)$ is trivial (see (1.2)), we focus on the deeper equality $C^*(n) = D^*(n)$. Just as Göllnitz established the refined equality (1.1), our proof here will establish the refinement of Theorem A given by

$$C^*(n, k) = D^*(n, k), \tag{2.1}$$

where k is the number of parts with the convention that parts $\equiv$ 0, 3, 5 (mod 6) are counted twice.

Proof: Let $d(m; t, q)$ denote the polynomial generating function of the partitions enumerated by $D^*(n)$ with the added conditions that

(i) all parts are $\leq m$,

(ii) the exponent of t is the number of parts with the convention that parts $\equiv$ 0, 3, or 5 (mod 6) are counted twice, and

(iii) the exponent of q is the number being partitioned.

We claim that

$$d(6m;t,q)+t^2q^{6m+3}d(6m-6;t,q)=(1+tq)(1+tq^2)(1+tq^4)d(6m-8;tq^6,q). \tag{2.2}$$

By iterating (2.2) and letting $m\to\infty$, we get

$$d(\infty;t,q)=\prod_{m=0}^{\infty}(1+tq^{6m+1})(1+tq^{6m+2})(1+tq^{6m+4}), \tag{2.3}$$

which yields (2.1) because the coefficients of t^kq^n on the left and right hand sides of (2.3) are $D^*(n;k)$ and $C^*(n;k)$ respectively. Thus our objective here is to prove (2.2).

We define

$$\chi(m)=0, \quad \text{if} \quad m\equiv 1,2,4\,(mod\,6), \quad \text{and} \quad 1 \quad \text{if} \quad m\equiv 0,3,5\,(mod\,6). \tag{2.4}$$

Also let

$$d(m;t,q)=1 \quad \text{for} \quad -6\le m\le 0, \quad \text{and} \quad d(m;t,q)=0 \quad \text{for} \quad n<-7. \tag{2.5}$$

Then (2.4), (2.5) and the definition of $d(m;t,q)$ yield

$$d(m;t,q)=d(m-1;t,q)+t^{1+\chi(m)}q^m d(m-6-\chi(m);t,q)+\epsilon(m), \tag{2.6}$$

where

$$\epsilon(m)=t^3q^7 \quad \text{if} \quad m=6, \quad \text{and} \quad 0 \quad \text{otherwise}. \tag{2.7}$$

To realize (2.6), note that the difference

$$d(m;t,q)-d(m-1;t,q)$$

is the generating function when the largest part is m. This accounts for the $t^{1+\chi(m)}q^m$ on the right in (2.6). If this part m is removed from the partition, then by the definition of the difference conditions governing the partitions enumerated by D^*, the resulting partition has largest part $\le m-6-\chi(m)$. The generating function of such partitions is $d(m-6-\chi(m))$ as on the right in (2.6) with one exception which is accounted by the $\epsilon(m)$ term in (2.6) as given by (2.7); this term corresponds to the exceptional partition 6+1 of the number 7 having the largest part $m=6$, with 6 being counted twice. This justifies (2.6).

For the sake of clarity in our calculations, we will actually list the six cases of (2.6) corresponding to the six residue classes modulo 6, namely:

$$d(6m;t,q)=d(6m-1;t,q)+t^2q^{6m}d(6m-7;t,q) \tag{2.8.1}$$

$$d(6m-1;t,q)=d(6m-2;t,q)+t^2q^{6m-1}d(6m-8;t,q) \tag{2.8.2}$$

$$(2.8.3) \qquad d(6m-2;t,q) = d(6m-3;t,q) + tq^{6m-2} d(6m-8;t,q)$$

$$(2.8.4) \qquad d(6m-3;t,q) = d(6m-4;t,q) + t^2 q^{6m-3} d(6m-10;t,q)$$

$$(2.8.5) \qquad d(6m-4;t,q) = d(6m-5;t,q) + tq^{6m-4} d(6m-10;t,q)$$

$$(2.8.6) \qquad d(6m-5;t,q) = d(6m-6;t,q) + tq^{6m-5} d(6m-11;t,q)$$

It is to be noted that the anomalous $\epsilon(m)$ in (2.6) requires that $m > 1$ for the above six equations to hold. For $m \leq 6$ we calculate $d(m;t,q)$ from (2.6).

For convenience, we set

$$(2.9) \qquad h(m;t) = d(6m-2;t,q)$$

and

$$(2.10) \qquad k(m;t) = d(6m-4;t,q).$$

By combining (2.8.1) and (2.8.2) we see that

$$(2.11) \qquad d(6m;t,q) = h(m;t) + t^2 q^{6m-1}(1+q) h(m-1;t) + t^4 q^{12m-7} h(m-2;t).$$

On the other hand, (2.8.4) asserts that

$$(2.12) \qquad d(6m-3;t,q) = k(n;t) + t^2 q^{6m-3} k(m-1;t),$$

whereas by (2.8.5)

$$(2.13) \qquad d(6m-5;t,q) = k(n;t) - tq^{6m-4} k(m-1;t).$$

Hence by (2.8.6) and (2.13) we get

$$(2.14) \qquad d(6m-6;t,q) = k(n;t) - tq^{6m-5}(1+q) k(m-1;t) + t^2 q^{12m-5} k(m-2;t).$$

Note that by (2.8.3)

$$(2.15) \qquad d(6m-3;t,q) = h(n;t) - tq^{6m-2} h(m-1;t).$$

If we set

$$(2.16) \qquad S_1(m) := h(m;t) - tq^{6m-2} h(m-1;t) - k(m-1;t) - t^2 q^{6m-3} k(m-1;t),$$

then (2.12) and (2.15) show that

$$S_1(m) = 0, \quad \text{for} \quad m \geq 2. \tag{2.17}$$

Similarly, by setting

$$S_2(m) := h(m-1;t) + t^2 q^{6m-7}(1+q)h(m-2;t) + t^4 q^{12m-19} h(m-3;t)$$

$$-k(m;t) + tq^{6m-5}(1+q)k(m-1;t) - t^2 q^{12m-5} k(m-2;t), \tag{2.18}$$

we see that (2.12) and (2.15) yield

$$S_2(m) = 0, \quad \text{for all} \quad m. \tag{2.19}$$

What we want now is a linear recurrence for the $h(n)$. This is a pure linear algebra problem. To this end, let us regard the six equations

$$S_1(m-i) = 0, \quad \text{for} \quad 0 \leq i \leq 2, \tag{2.20.1}$$

and

$$S_2(m-i) = 0, \quad \text{for} \quad 0 \leq i \leq 2, \tag{2.20.2}$$

as linear equations in the variables $h(m;t), k(m;t), k(m-1;t), k(m-2;t), k(m-3;t)$ and $k(m-4;t)$. Solving this system will yield $h(m;t)$ in terms of $h(m-i;t)$ for $1 \leq i \leq 4$. The result is that

$$J(m) := F(m, t, h(m;t), h(m-1;t), h(m-2;t), h(m-3;t), h(m-4;t)) = 0, \tag{2.21}$$

where

$$F(m,t,X,Y,Z,W,V) = X - (t^3 q^{18m-29} + t^4 q^{12m}(q^{-15} + q^{-16} + q^{19}))W - t^6 q^{18m-34} V \tag{2.22}$$

$$-(1 + tq^{6m}(q^{-2} + q^{-4} + q^{-5}))Y - t^2(q^{6m}(q^{-3} + q^{-6} + q^{-7}) - q^{12m}(q^{-12} + q^{-13} + q^{-15}))Z.$$

Now clearly by the definition of $J(m)$, the right hand side of (2.2) satisfies

$$F(m-1, tq^6, h(m-1;tq^6), h(m-2;tQ^6), h(m-3;tq^6, h(m-4;tq^6, h(m-5;tq^6)) = 0. \tag{2.23}$$

As for the left hand side of (2.2), we note that

$$G(n) := d(6m;t,q) + t^2 q^{6m+3} d(6m-6;t,q) = \tag{2.24}$$

$$h(m;t) + t^2 q^{6m-1} h(m-1;t)(1+q+q^4) + t^4 q^{12m-7}(1+q^3+q^4)h(m-2;t) + t^6 q^{18m-16} h(m-3,t),$$

by (2.11). Thus

$$F(m-1, tq^6, G(m), G(m-1), G(m-2), G(m-3), G(m-4)) \tag{2.25}$$

$$= J(m)+t^2q^{6m-1}(1+q+q^4)J(m-1)+t^4q^{12m-7}(1+q^3+q^4)J(m-2)+t^6q^{18m-16}J(m-3) = 0,$$

by (2.21), provided $m \geq 8$.

Finally, comparing (2.24) and (2.25) we see that both sides of (2.2) satisfy the same fourth order recurrence provided $m \geq 8$. The validity of (2.2) for $2 \leq m \leq 8$ was checked using Macsyma. Hence the two sides of (2.2) are identical. That proves (2.1) which is a refined version of Theorem A.

3. Theorem G via the method of weighted words

In order to derive the *key identity* for Theorem A, we will first describe here how the key identity (1.3) for Theorem G was obtained, and the generalization and refinement of Theorem G that it provided. That is, we will now sketch the main ideas of *the method of weighted words* approach to Theorem G.

The method was initiated by Alladi-Gordon [9] to obtain generalizations and refinements of Schur's celebrated 1926 partition theorem [24]. The main idea in [9] was to establish the *key identity*

$$\sum_{i,j} a^i b^j \sum_m \frac{q^{T_{i+j-m}+T_m}}{(q)_{i-m}(q)_{j-m}(q)_m} = (-aq)_\infty(-bq)_\infty, \tag{3.1}$$

and to view a two parameter refinement of Schur's theorem involving partitions into distinct parts $\equiv 1, 2 (mod\, 3)$ as emerging from (3.1) under the transformations

$$(dilation) \quad q \mapsto q^3, \quad \text{and} \quad translations \quad a \mapsto aq^{-2}, b \mapsto bq^{-1}. \tag{3.2}$$

Two proofs of (3.1) were given in [9], one involving the q-Chu-Vandermonde summation, and another by a combinatorial interpretation of an equivalent version of (3.1). Both proofs were much easier compared to the proof of (1.3) in [8].

The interpretation of the product in (3.1) as the generating function of bi-partitions into distinct parts in two colors is clear. In [] it was shown that the series in (3.1) is the generating function of partitions (= words with weights attached) into distinct parts occurring in three colors - two primary colors a and b, and one secondary color ab, and satisfying certain gap conditions. It was Gordon's insight that one ought to start with two primary colors (in undilated form) as in the product in (3.1), and derive Schur's theorem as a dilated version.

With Schur's theorem having been successfully generalized by the method of weighted words, Gordon suggested to Alladi that the method ought to be applied to generalize and refine Theorem G. This would require three primary colors a, b, c and three secondary colors $ab, ac.bc$. We now describe the principal ideas of this method for Theorem G.

We consider the integer 1 as occuring only in three primary colors a, b and c, but the integers $n \geq 2$ as occuring in all six colors - a, b, c as well as ab, ac and bc. An integer n in color a is denoted by symbol a_n, with similar interpretation for the symbols b_n, c_n, ab_n, ac_n, and bc_n. The alphabets a, b, c play a dual role; on the one hand they represent colors and on the other they are free parameters.

To define partitions we need an ordering of the symbols and the one we choose is

$$a_1 < b_1 < c_1 < ab_2 < ac_2 < a_2 < bc_2 < b_2 < c_2 < ab_3 < ... \tag{3.3}$$

The effect of the substitutions (1.4) is to convert the symbols to

$$\begin{cases} a_m \mapsto 6m-4,\ b_m \mapsto 6m-2,\ c_n \mapsto 6m-1, \text{for } m \geq 1, \\ ab_m \mapsto 6m-6,\ ac_m \mapsto 6m-5,\ bc_n \mapsto 6m-3, \text{for } m \geq 2. \end{cases} \tag{3.4}$$

so that the ordering (3.3) becomes

$$2 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12 < ..., \tag{3.5}$$

This is one reason for the choice of the ordering of symbols in (3.3), because they convert to the natural ordering of the integers in (3.5) under the transformations (3.4).

To view Theorem G in this context, we think of the primary colors a, b, c as corresponding to the residue classes 2,4 and 5(mod 6) and so the secondary colors ab, ac, bc correspond to the residue classes $2 + 4 \equiv 6$, $2 + 5 \equiv 7$ and $4 + 5 \equiv 9 \pmod 6$. Note that integers of secondary color occur only when $n \geq 2$ and so ab_1, ac_1 and bc_1 are missing in (3.3). This is why integers $ac_1 = 1$ and $bc_1 = 3$ do not appear in (3.5). This explains the absence of 1 and 3 among the parts enumerated by $D(n)$ in Theorem G. Note that ab_1 corresponds to the integer 0, which is not counted as a part in ordinary partitions anyway.

In (3.3) for a given subscript, the ordering of the colors is

$$ab < ac < a < bc < b < c. \tag{3.5}$$

We use (3.5) to say for instance that ab is of *lower order* compared to a, or equivalently that a is of *higher order* than ab. We also use the term *Level* for the subscript of a symbol when referring to a collection of symbols with the same subscript. For example, in the ordering (3.3), we have listed all symbols at Level 1, followed by symbols at Level 2, etc.

By a partition (word) π we mean a collection of symbols in non–increasing order as given by a prescribed ordering of the symbols. In this section we use the ordering as in (3.3). By $\sigma(\pi)$ we mean the sum of the subscripts of the symbols π. For example $ab_3c_2c_2b_2a_2c_1a_1a_1$ is a partition π of the integer $\sigma(\pi) = 14$. Also the gap between any two symbols is the absolute value of the difference between their subscripts. In discussing partitions, we think of the subscripts as weights or parts.

By a vector partition $\pi = (\pi_1; \pi_2; \pi_3)$ of n, we mean that π_j are partitions (words), and that

$$\sigma(\pi) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3) = n.$$

The product in (1.4) is clearly the generating function of vector partitions in which each of the components π_1, π_2 and π_3 are partitions into distinct parts in colors a, b, c respectively.

To understand the partitions whose generating function is the series in (1.4), we consider Type 1 partitions π which are of the form $x_1 + x_2 + \ldots$, where the x_i are symbols from (3.3) with the condition that the gap between x_i and x_{i+1} is ≥ 1 with strict inequality if x_i of a lower order (color) compared x_{i+1} or if x_i and x_{i+1} are of the same secondary color. The principal result of Alladi-Andrews-Gordon [] which is a substantial extension of Theorem G is the following:

Theorem 1: *Let $C(n;i,j,k)$ denote the number of vector partitions $(\pi_1;\pi_2;\pi_3)$ of n such that π_1 has i distinct parts all in color a, π_2 has j distinct parts all in color b, and π_3 has k distinct parts all in color c.*

Let $D(n;\alpha,\beta,\gamma,\delta,\varepsilon,\phi)$ denote the number of Type 1 partitions of n having α $a-parts$, β $b-parts$, $\dots$, and ϕ $bc-parts$.

Then

$$C(n;i,j,k) = \sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} D(n;\alpha,\beta,\gamma,\delta,\varepsilon,\phi).$$

Clearly

$$\sum_{i,j,k,n} C(n;i,j,k)a^i b^j c^k q^n = \sum_{i,j,k}^{\infty} \frac{a^i b^j c^k q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k} = (-aq)_\infty(-bq)_\infty(-cq)_\infty. \tag{3.6}$$

It turns out that

$$\sum_n D(n;\alpha,\beta,\gamma,\delta,\varepsilon,\phi)q^n = \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}. \tag{3.7}$$

The proof of (3.7) in [8] is quite involved and goes by induction on $s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi$, the number of parts of the Type-1 partitions, and also appeals to *minimal partitions* whose generating functions are given by multinomial coefficients. Instead of giving the details of the derivation of (3.7) here, we shall in the next section, use the same ideas to derive a similar identity for a three parameter generalization of Theorem A.

Indeed the proof of (3.7) is one of the main aspects of [8]. The second main feature in [8] is the proof of the key identity (1.3) which relies on the Watson's q analogue ${}_8\phi_7$ of Whipple's transformation and the ${}_6\psi_6$ summation of Bailey. For the proof of (1.4), we refer the reader to [].

4. Two infinite hierarchies from Schur's theorem

Why is it that Göllnitz' Theorem G is so much more complicated to prove than Schur's Theorem S. The reason is that in the case of Schur's theorem, the method of weighted words shows that one starts with two primary colors in the product, and in the series one considers the *complete alphabet* of colors generated by the expansion, namely a, b, and ab. In contrast, in the case of Göllnitz' theorem, we start with three primary colors a, b, c in the product, but in its series expansion, we do not consider the complete alphabet of colors generated by a, b, c, because we consider only the secondary colors ab, ac, bc and not the ternary color abc. Thus we are dealing with an *incomplete alphabet* of colors and this is what causes the significant increase in depth and difficulty.

Extending the ideas in his 1967 proof of Schur's theorem [11], Andrews [12], [13] soon after obtained two hierarchies of partition theorems from Schur's theorem making use of the completeness of the alphabets. We now describe his results:

For a given integer $r \geq 2$, let $a_1, a_2, .., a_r$ be r distinct positive integers such that

$$\sum_{i=1}^{k-1} a_i < a_k, \quad 1 \leq k \leq r. \tag{4.1}$$

Condition (4.1) ensures that $2^r - 1$ sums $\sum \varepsilon_i a_i$, where $\varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0$, are all distinct. Let these sums in increasing order be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_{2^r-1}$.

Next let $N \geq \sum_{i=1}^{r} a_i \geq 2^r - 1$ be a modulus, and A_N denote the set of all positive integers congruent to some $a_i (\text{mod } N)$. Similarly, letA'_N denote the set of all positive integers congruent to some $\alpha_i (\text{mod } N)$ Also let $\beta_N(m)$ denote the least positive residue of $m (\text{mod } N)$. Finally, if $m = \alpha_j$ for some j, let $\phi(m)$ denote the number of terms appearing in the defining sum of m and $\psi(m)$ the smallest a_i appearing in this sum. Then the first general theorem of Andrews [12] is:

Theorem A1: *Let $C^*(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N.*

Let $D^*(A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, ..., b_\nu$ from A'_N such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(b_{i+1})) + \psi(\beta_N(b_{i+1})) - \beta_N(b_{i+1}). \tag{4.2}$$

Then

$$C^*(A_N; n) = D^*(A'_N; n).$$

To describe the second general theorem of Andrews (1969), let a_i, α_i and N be as above. Now let $-A_N$ denote the set of all positive integers congruent to some $-a_i (\text{mod } N)$, and $-A'_N$ the set of all positive integers congruent to some $-\alpha_i (\text{mod } N)$. The quantities $\beta_N(m), \phi(m), \psi(m)$ are also as above. We then have (Andrews [13])

Theorem A2: *Let $C(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$.*

Let $D(-A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, ..., b_\nu$, taken from $-A'_N$ such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(-b_i)) + \psi(\beta_N(-b_i)) - \beta_N(-b_i) \tag{4.3}$$

and also

$$b_\nu \geq N(\phi(\beta_N(-b_s) - 1)). \tag{4.4}$$

Then

$$C(-A_N; n) = D(-A'_N; n).$$

Theorems A1 and A2 may be viewed as duals by comparing the functions C and C^*, but there are two essential differences between them. One is that the gap conditions for $b_i - b_{i+1}$ in (4.2) are given in terms of b_{i+1}, whereas in (4.3) they are given in terms of b_i. Also, Theorem A2 has condition (4.4) on the smallest part b_ν, while there is no such condition in Theorem A1.

When $r = 2$, $a_1 = 1$, $a_2 = 2$, $N = 3 = 2^r - 1$, Theorems A1 and A2 both become Theorem S. Thus the two hierarchies emanate from Theorem S, and it is only when $r = 2$ that the hierarchies coincide. Thus Theorem S is its own dual. Conditions (4.2) and (7.3) can be understood better by classifying b_{i+1} (in Theorem A1) and b_i (in Theorem A2) in terms of their residue classes (mod N). In particular, with $r = 3$, $a_1 = 1$, a_2, $a_3 = 4$ and $N = 7 = 2^3 - 1$, Theorems A1 and A2 yield the following corollaries.

Corollary 1: *Let $C^*(n)$ denote the number of partitions of n into distinct parts* $\equiv 1, 2$ or 4 (mod 7).

Let $D^*(n)$ denote the number of partitions of n in the form $b_1 + b_2 + ..., b_\nu$ such that $b_i - b_{i+1} \geq 7, 7, 12, 7, 10, 10$ or 15 if $b_{i+1} \equiv 1, 2, 3, 4, 5, 6$ or $7 (\text{mod } 7)$. Then

$$C^*(n) = D^*(n).$$

Corollary 2: *Let $C(n)$ denote the number of partitions ofn into distinct parts* $\equiv 3, 5$ or 6 (mod 7).

Let $D(n)$ denote the number of partitions of n in the form $b_1 + b_2 + ... + b_\nu$ such that $b_i - b_{i+1} \geq 10, 10, 7, 12, 7, 7$ or 15 if $b_i \equiv 8, 9, 3, 11, 5, 6$ or 14 (mod 7) and $b_\nu \neq 1, 2, 4$ or 7. Then

$$C(n) = D(n).$$

Andrews' proofs of Theorems A1 and A2 are extensions of his proof of Theorem S [] and not as difficult as the proof Göllnitz' theorem.

It is to be noted that the residue classes $2, 4, 5\,(mod\,6)$ in Theorem G could be viewed as $-4, -2, -1\,(mod\,6)$, and so Theorem G is akin to Corollary 2. Similarly the residue classes $1, 2, 4\,(mod\,6)$ in Theorem A make Theorem A akin to Corollary 1.

Although similar in appearance, Theorems A1 and A2 are combinatorially very different. It was noticed by Andrews and Olsson [18], that Theorem A1 could be reformulated in a more convenient form, and this was exploited by Bessenrodt [20] to provide a nice combinatorial proof by using N-modular Ferrers graphs. Such an approach did not work for Theorem A2.

Alladi and Gordon observed (see Theorem 15 of [2] for a discussion of this) that the two Andrews hierarchies could be combined into a single infinite chain of partition theorems by considering a method of weighted words approach in the study of the product

$$(-z_1 q)_\infty (-z_2 q)_\infty ... (-z_r q)_\infty. \tag{4.5}$$

Theorem A1 would emerge from this by using the transformations

$$\begin{cases} \text{(dilation) } q \mapsto q^N, \\ \text{(translations)} z_t \mapsto z_t q^{a_t - N}, \end{cases} \tag{4.6}$$

Theorem A2 would emerge using the same dilation as in (4.6) but with the translations $z_t \mapsto z_t q^{a_t}$ instead. Subsequently, Corteel and Lovejoy [21] provided a combinatorial proof of Theorem 15 of [] and other results by an iteration of the bijective correspondence in the Alladi-Gordon [9] combinatorial proof of the generalization of Schur's theorem.

Although the method of weighted words provides a "unifying approach" to the two hierarchies as pointed out in [2], Gordon had told Alladi in 1994 that if one were to directly approach Theorem A1 by the method of weighted words, it is more convenient to have the symbols representing the colored integers starting at Level 0 instead of at Level 1. For Theorem A2, one would use the symbols representing the colored integers starting at Level 1. This is because, if the primary colors start at Level 1, then the secondary colors start only at Level 2, the ternary from Level 3, and so on. This corresponds to the lower bound given by (4.4) in Theorem A2. On the other hand, no such lower bound condition on the parts is needed in Theorem A1, and so it is best to start with primary colors from Level 0, so that all colors start at Level 0. Since our new companion Theorem A corresponds to Corollary 1 of Theorem A1 as noted above, we will construct the key identity for Theorem A in the next section by starting the colored integers (parts) from Level 0, instead of from Level 1.

5. Construction of the key identity for Theorem A

As before, we will consider integers in six colors, three of which are primary colors, namely a, b, c, and three are secondary colors ab, ac, bc. As before, we will let a_n denote the integer n in color a, with similar interpretation for $b_n, ..., bc_n$. The main difference here is that all integers $n \geq 0$ will occur in all six colors.

For the partitions enumerated by $D^*(n)$ in Theorem A, there is the unusual condition that 6+1 could occur in the partition. This could be replaced by the equivalent condition that if in a partition π counted by $D^*(n)$, we have 7 as the smallest part, then instead of replacing 7 by 6+1, we replace it by 7+0. In other words, a partition π enumerated by $D^*(n)$ with smallest part 7, has a mate which a partition π_0 with 7 replaced by 7+0. The partition π_0 is not an ordinary partition because it has 0 as a part; it is a *special* partition which will replace the the corresponding partition having 6+1 as the two smallest parts.

The ordering of the colored integers that we choose is

$$bc_{-1} < a_0 < b_0 < ab_0 < c_0 < ac_0 < bc_0 < a_1 < b_1 < ab_1 < c_1 < ac_1 < bc_0 < ... \tag{5.1}$$

Using certain partitions called Type 1* partitions defined below which are formed by symbols from (5.1), we will determine their generating function as a series which has the product representation (see () below)

$$\prod_{m=0}^{\infty}(1+aq^m)(1+bq^m)(1+cq^m). \tag{5.2}$$

A three parameter refinement of the generating function of $D^*(n)$ emerges from the product in (5.2) by the substitutions

$$\text{(dilation)} \quad q \mapsto q^6, \quad \text{and (translations)} \quad a \mapsto aq, b \mapsto bq^2, c \mapsto cq^4. \tag{5.3}$$

Under the substitutions in (5.3), the symbols in (5.1) become

$$\begin{cases} a_m \mapsto 6m+1,\ b_m \mapsto 6m+2,\ c_m \mapsto 6m+4, \\ ab_m \mapsto 6m+3,\ ac_m \mapsto 6m+5,\ bc_m \mapsto 6m+6, \end{cases} \tag{5.4}$$

and so the ordering in (5.1) becomes

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12 < ...,$$

the natural ordering among the non-negative integers. This is the reason for choosing the ordering in (5.1). Note that $bc_{-1} \mapsto 0$ under these substitutions.

For a given *level* m (subscript), the ordering of the colors is

$$a < b < ab < c < ac < bc. \tag{5.5}$$

Given a pair of colors, we will use (5.5) to determine which is of higher order and which is of lower order. With this notion of order, the gap conditions governing $D^*(n)$ translate to Type 1* partitions which are partitions (words) whose parts are the symbols in (5.1) satisfying the following conditions:

(5.6)
$$\begin{cases} \text{parts are } \geq 0, \text{ and the gap between them is } \geq 1, \text{with strict inequality if either} \\ \text{the larger part is of lower order or consecutive parts are of same secondary color,} \\ \text{and if the smallest part is}\, a_1, \text{then it can be replaced by}\, a_1 + bc_{-1}. \end{cases}$$

NOTE: In the undilated case, the replacement of a_1 by $a_1 + bc_{-1}$ yields a special partition of an integer one smaller than the original. In the dilated case of $D^*(n)$, the replacement of 7 by 7+0 yields a special partition of the same integer. If we had chosen to retain 6+1 as the replacement of 7, then in the undilated case, we would have $bc_0 + a_0$ as the replacement of a_1, and this too would have yielded a partition of an integer one smaller than the original. We prefer the replacement by $a_1 + bc_{-1}$ because this satisfies the same gap condition as the other Type 1* partitions do.

Notation and conventions: Given a partition π, we denote by $\nu(\pi)$ the number of parts of π. The number of parts of π in a given color will be denoted by a subscript, such as $\nu_a(\pi)$. The least part of π is denoted by $\ell(\pi)$. Also $\sigma(\pi)$ is the sum of the parts (subscripts of the symbols), namely the integer being partitioned. By a *minimal partition* we mean a partition for which $\sigma(\pi)$ is minimal for a specified ordering of the colors. For example, among the partitions whose parts occur in colors

$$a\,(ab)\,b\,c\,c\,b\,(ac)\,c \tag{5.7}$$

the minimal partition π is

$$\pi : a_{10} + ab_8 + b_7 + c_5 + c_4 + b_3 + ac_1 + c_0, \tag{5.8}$$

and for this minimal partition, $\sigma(\pi) = 31$. Sometimes, as in (5.7), for the sake of clarity, we may put parenthesis around the secondary colors. Also, <u>occasionally</u>, for convenience, we <u>may write</u> the parts of a minimal partition in <u>ascending</u> order instead of the standard descending order for partitions. This will be indicated suitably.

We will also be using the q-multinomial coefficient of order 6, namely

$$\begin{bmatrix} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \\ n_1, n_2, n_3, n_4, n_5, n_6 \end{bmatrix} := \frac{(q)_{n_1+n_2+n_3+n_4+n_5+n_6}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}}. \tag{5.9}$$

Since the n_6 in the bottom row of the q-multinomial coefficient on the left in (5.9) is superfluous, most authors omit highlighting it in the notation of the q-multinomial coefficient. But it is convenient and useful to display all six integers n_i in the second row of the multinomial coefficient as we shall see in the sequel. With these notations and conventions, we are ready to derive the key identity for Theorem A.

The main idea in our derivation of the key identity (as it was in []) is that the generating function of all partitions satisfying certain conditions can be obtained from the minimal partition satisfying those conditions. For example, the generating function of all partitions whose colored parts are in the order as specified by (5.7) is

$$\frac{q^{\sigma(\pi)}}{(q)_{\nu(\pi)}} = \frac{q^{38}}{(q)_8},$$

where π is the minimal partition as in (5.8).

In what follows, we will typically consider a Type 1* partition π with

$$\nu_a(\pi) = \alpha,\ \nu_b(\pi) = \beta,\ \nu_c(\pi) = \gamma,\ \nu_{ab}(\pi) = \delta,\ \nu_{ac}(\pi) = \varepsilon,\ \text{and}\, \nu_{bc}(\pi) = \phi. \tag{5.10}$$

With the number of parts in different colors as specified by (5.10), we let

$$s = \alpha + \beta + \gamma + \delta + \epsilon + \phi. \tag{5.11}$$

We now consider the generating function $H(\alpha, \beta, \gamma, \delta, \varepsilon, \phi)$ of all minimal Type 1* partitions *excluding the special partitions* that have the number of colored parts as specified by (5.10). We <u>claim</u> that

$$H = H(\alpha, \beta, \gamma, \delta, \varepsilon, \phi) = q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}} \begin{bmatrix} s \\ \alpha, \beta, \gamma, \delta, \epsilon, \phi \end{bmatrix}. \tag{5.12}$$

To establish (5.12), we consider the generating functions $H_a = H_a(\alpha, \beta, ..., \phi)$, $H_b, ...,$ H_{bc} of all minimal partitions counted by H, but with smallest part $a_0, b_0, ..., bc_0$ respectively. Clearly

$$H_a + H_b + H_{ab} + H_c + H_{ac} + H_{bc} = H. \tag{5.13}$$

We call a generating functions of such minimal partitions with a specified smallest part as a *local generating function.* In (5.13), the local generating functions on the left have been listed according to the order of the colors at each level as given by (5.5), and this will be useful in their computation which we take up now.

The local generating functions are given by:

Lemma: *Let s be as in (5.11) and $\sigma = T_{s-1} + T_{\delta-1} + T_{\varepsilon-1} + T_{\phi-1}$. Then*

$$H_a(\alpha, \beta, \gamma, \delta, \varepsilon, \phi) = q^{\sigma} \begin{bmatrix} s-1 \\ \alpha-1, \beta, \gamma, \delta, \varepsilon, \phi \end{bmatrix} \tag{5.14.a}$$

(5.14.b) $$H_b(\alpha,\beta,\gamma,\delta,\varepsilon,\phi) = q^{\sigma+\alpha}\begin{bmatrix} s-1 \\ \alpha,\beta-1,\gamma,\delta,\varepsilon,\phi \end{bmatrix}$$

(5.14.ab) $$H_{ab}(\alpha,\beta,\gamma,\delta,\varepsilon,\phi) = q^{\sigma+\alpha+\beta}\begin{bmatrix} s-1 \\ \alpha,\beta,\gamma,\delta-1,\varepsilon,\phi \end{bmatrix}$$

(5.14.c) $$H_c(\alpha,\beta,\gamma,\delta,\varepsilon,\phi) = q^{\sigma+\alpha+\beta+\delta}\begin{bmatrix} s-1 \\ \alpha,\beta,\gamma-1,\delta,\varepsilon,\phi \end{bmatrix}$$

(5.14.ac) $$H_{ac}(\alpha,\beta,\gamma,\delta,\varepsilon,\phi) = q^{\sigma+\alpha+\beta+\delta+\gamma}\begin{bmatrix} s-1 \\ \alpha,\beta,\gamma,\delta,\varepsilon-1,\phi \end{bmatrix}$$

(5.14.bc) $$H_{bc}(\alpha,\beta,\gamma,\delta,\varepsilon,\phi) = q^{\sigma+s+\alpha+\beta+\delta+\gamma+\varepsilon}\begin{bmatrix} s-1 \\ \alpha,\beta,\gamma,\delta,\varepsilon,\phi-1 \end{bmatrix}.$$

The claim (5.12) clearly follows from the Lemma and (5.13) because of the well known recurrence for the q multinomial coefficients given by

(5.15) $$\begin{bmatrix} n_1+n_2+\cdots+n_t \\ n_1,n_2,\cdots,n_t \end{bmatrix} = \begin{bmatrix} n_1+n_2+\cdots+n_t-1 \\ n_1-1,n_2,n_3,\cdots,n_t \end{bmatrix} + q^{n_1}\begin{bmatrix} n_1+n_2+\cdots+n_t-1 \\ n_1,n_2-1,n_3,\cdots,n_t \end{bmatrix}$$
$$+\, q^{n_1+n_2}\begin{bmatrix} n_1+n_2+\cdots+n_t-1 \\ n_1,n_2,n_3-1,n_4,\cdots,n_t \end{bmatrix} + \cdots + q^{n_1+\cdots+n_{t-1}}\begin{bmatrix} n_1+n_2+\cdots+n_t-1 \\ n_1,n_2,\cdots,n_{t-1},n_t-1 \end{bmatrix}$$

in the case $t = 6$ with

(5.16) $$n_1 = \alpha,\ n_2 = \beta,\ n_3 = \delta,\ n_4 = \gamma,\ n_5 = \varepsilon,\ n_6 = \phi$$

which means $s = n_1 + n_2 + ... + n_6$.

So we now focus our attention on the Lemma which can be proved in two ways.

First proof of the Lemma: We proceed by induction on s following the method in []. For this we note that the Lemma is valid when $s = 0, 1$. For $s \geq 2$, to derive (5.14a), we classify the minimal partitions enumerated by H_a in six possible cases according to the smallest two parts and apply the induction hypothesis.

Case (i): π ends with $a_1 + a_0$,

For the minimal partitions π ending in $a_1 + a_0$, if we delete a_0, we get a partition π' into $s-1$ parts with $\nu_a(\pi') = \alpha-1$, but where each part is one above what one would have in a minimal partition enumerated by $H_a(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$. By the induction hypothesis the generating function of the partitions in Case (i) is

(5.17) $$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1}\begin{bmatrix} s-2 \\ \alpha-2,\beta,\gamma,\delta,\varepsilon,\phi \end{bmatrix} = q^{\sigma}\begin{bmatrix} s-2 \\ \alpha-2,\beta,\gamma,\delta,\varepsilon,\phi \end{bmatrix}.$$

Case (ii): π ends with $b_1 + a_0$.

Delete a_0 from π to get a partition enumerated by $H_b(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$, with 1 added to each part. So by the induction hypothesis, the generating function for Case (ii) is

(5.18)
$$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1+\alpha-1}\left[\begin{matrix} s-2 \\ \alpha-1,\beta-1,\gamma,\delta,\varepsilon,\phi \end{matrix}\right] = q^{\sigma+\alpha-1}\left[\begin{matrix} s-2 \\ \alpha-1,\beta-1,\gamma,\delta,\varepsilon,\phi \end{matrix}\right].$$

Case (iii): π ends with $ab_1 + a_0$.

Deleting a_0 from π, we have a partition π' enumerated by $H_{ab}(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$ with each part larger by 1. So using (5.14ab) with α replaced by $\alpha-1$, we see that the generating function for Case (iii) is

$$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1+\alpha-1+\beta}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma,\delta-1,\varepsilon,\phi \end{matrix}\right]$$

$$= q^{\sigma+\alpha-1+\beta}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma,\delta-1,\varepsilon,\phi \end{matrix}\right]. \tag{5.19}$$

Case (iv): π ends with $c_1 + a_0$.

As before we delete a_0 from π to get a partition enumerated by $H_c(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$ with each of its $s-1$ parts larger by 1. So using (5.14c) with $\alpha-1$ in place of α, we see that the generating function for Case (iv) is

$$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1+\alpha-1+\beta+\delta}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma-1,\delta,\varepsilon,\phi \end{matrix}\right]$$

$$= q^{\sigma+\alpha-1+\beta+\delta}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma-1,\delta,\varepsilon,\phi \end{matrix}\right]. \tag{5.20}$$

Case (v): π ends with $ac_1 + a_0$.

By deleting a_0 from π, we have a partition counted by $H_{ac}(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$ with each of its $s-1$ parts larger by 1. Thus from (5.14ac) with $\alpha-1$ in place of α, we see that the generating function for Case (v) is

$$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1+\alpha-1+\beta+\delta+\gamma}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma,\delta,\varepsilon-1,\phi \end{matrix}\right]$$

$$= q^{\sigma+\alpha-1+\beta+\delta+\gamma}\left[\begin{matrix} s-2 \\ \alpha-1,\beta,\gamma-1,\delta,\varepsilon-1,\phi \end{matrix}\right]. \tag{5.21}$$

Case (vi): π ends with $bc_1 + a_0$

Once again, deleting a_0 from π, we get a partition counted by $H_{bc}(\alpha-1,\beta,\gamma,\delta,\varepsilon,\phi)$ with each of its $s-1$ parts larger by 1. So (5.14bc) with $\alpha-1$ in place of α gives the generating function for Case (vi) as

$$q^{T_{s-2}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}+s-1+\alpha-1+\beta+\delta+\gamma+\varepsilon}\begin{bmatrix} s-2 \\ \alpha-1,\beta,\gamma,\delta,\varepsilon,\phi-1\end{bmatrix}$$

$$= q^{\sigma+\alpha-1+\beta+\delta+\gamma+\varepsilon}\begin{bmatrix} s-2 \\ \alpha-1,\beta,\gamma-1,\delta,\varepsilon,\phi-1\end{bmatrix}. \tag{5.22}$$

Thus by adding the generating functions on the right in (5.17) - (5.22), and using (5.15) with

$$t=6, n_1=\alpha,\ n_2=\beta,\ n_3=\delta,\ n_4=\gamma,\ n_5=\varepsilon,\ n_6=\phi$$

we get (5.14a).

The proofs of (5.14b) - (5.14bc) are similar and so we omit them. We just point out here that each of these is mildly more complex. In the case of (5.14a), since a_0 is the smallest symbol, the second smallest parts all came from Level 1. In the case of (5.14b), if the second smallest part is in color a, then the partition π will end with $a_2 + b_0$. Thus by deleting b_0 we would end up with a partition enumerated by $H_a(\alpha,\beta-1,\gamma,\delta,\varepsilon,\phi)$ with each part larger by 2 compared to the minimal partition. This does not affect the proof of (5.14b) by induction on s.

Rule of cycles: To prove (5.14a), in discussing cases (i) - (vi) above, we considered the colors of the second smallest part in the sequence as given by the ordering of colors (5.5). To derive any of the identities in the Lemma with a smallest part in a specified color, we would discuss the second smallest part by starting with the same color and moving cyclically to the right. For instance, for (5.14b), we would discuss the cases involving the second smallest part by starting with color b, followed by ab, c, ac, bc and a. This would address the mild complexity mentioned above. This rule of cycles was noticed already in [].

Remark: The induction proof of the Lemma, and therefore of the claim (5.12), worked because we were able to guess the correct formula for H. This is true for any theorem to be proved by induction. So the question may be raised as to how we conjectured (5.12) in the first place? The answer is that we constructed the *absolutely minimal partition* first and from this deduced (5.12). This leads us to the second proof of (5.12) which we just sketch:

Sketch of the second proof of (5.12): Given the number of parts in each of the colors, by the *absolutely minimal partition*, we mean the the partition π for which $\sigma(\pi)$ is minimal among ALL the orderings of the colors. Thus if the number of parts in the various colors are specified as in (5.10), the absolutely minimal partition is obtained by listing the parts in increasing order as follows: First write a with repetition α, followed by b with repetition β, followed by ab if $\delta \geq 1$, followed by c with repetition γ, followed by an ac if $\varepsilon \geq 1$, followed by bc if $\phi \geq 1$, followed by triples ab, ac, bc, in succession, dropping whichever color runs out. This ordering is dictated by the gap conditions (5.6) which we use to attach the weights to the sequence of colors listed. This will yield the absolutely minimal partition π with $\sigma(\pi) = \sigma$.

Once this absolutely minimal partition is constructed, the collection of ALL minimal partitions is obtained by permuting the colors of the absolutely minimal partition. Each non-identity permutation of the colors increases the sum of the weights. At this point the fundamental property of the q-multinomial coefficients is invoked, namely that they actually keep tract of these increases caused by the permutations. This then yields (5.12). Similarly one can derive (5.14a) - (5.14bc). We omit giving further details of this proof because they are cumbersome and also because we have given the induction proof.

Remarks: As early as 1992 when [8] was being written, Gordon told Alladi that the way to construct the key identity via local generating functions is to compute the absolutely minimal partitions. This is what led to the formulae for the local generating functions in [8] and [9]. Since the induction proof worked so conveniently, the approach via absolutely minimal partitions was not presented either in [8] or in [9] because it was painstaking. Since this paper is dedicated to the memory of Gordon, we wish to point out this idea of Gordon and how it helped us determine the formulae in (5.12) and in the Lemma.

With (5.12) established, it follows that the generating function of all Type 1* partitions *excluding the special partitions* is

$$\Sigma_1 := \sum_{i,j,k\geq 0} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}}}{(q)_\alpha (q)_\beta (q)_\gamma (q)_\delta (q)_\varepsilon (q)_\phi}. \tag{5.23}$$

Now the special partitions are those with smallest part a_1, and with bc_{-1} added to the partition. To get the generating function of the special partitions we proceed as follows:

<u>Step 1:</u> Start with all Type 1* non-special minimal partitions π with the number of parts as specified by (5.10) and with $\ell(\pi) = a_0$, and add one to each of the s parts of π to get partitions π_1. From (5.14a) we see that the generating function of the π_1 with number of parts as given by (5.10) is

$$q^{s+T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}} \begin{bmatrix} s-1 \\ \alpha-1, \beta, \gamma, \delta, \varepsilon, \phi \end{bmatrix} \tag{5.24}$$

<u>Step 2:</u> Draw the Ferrers graph of these π_1, and imbed colums of lengths up to $s-1$ in the graph to increase the gaps between the parts arbitrarily, but keeping the smallest part as a_1. These yield all Type 1* non-special partitions with π_2 with $\ell(\pi_2) = a_1$. In terms of generating functions, this means we divide the expression in (5.24) by $(q)_{s-1}$ to get the generating function of the π_2 satisfying (5.10). We need to multiply this generating function by $a^i b^j c^k$ and sum over all i, j, k to get the generating function of all π_2 which is

$$\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}}}{(q)_{\alpha-1} (q)_\beta (q)_\gamma (q)_\delta (q)_\varepsilon (q)_\phi}. \tag{5.25}$$

Step 3: Finally, add bc_{-1} as a part to all the π_2 to get all the special partitions π_3. This means the expression in (5.25) is to be multiplied by bcq^{-1}. So the generating function of all special partitions is

$$\Sigma_2 := \sum_{i\geq 1, j,k\geq 0} a^i b^{j+1} c^{k+1} \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}-1}}{(q)_{\alpha-1}(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}. \tag{5.26}$$

Thus the *key identity* for Theorem A is

$$\Sigma_1 + \Sigma_2 = \sum_{i,j,k\geq 0} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}$$

$$+ \sum_{i\geq 1, j,k\geq 0} a^i b^{j+1} c^{k+1} \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}-1}}{(q)_{\alpha-1}(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}$$

$$= (-a)_\infty(-b)_\infty(-c)_\infty. \tag{5.27}$$

Just as Theorem 1 was the combinatorial version of (1.3), the combinatorial version of of key identity (5.27) is the following partition theorem:

Theorem 1*:

Let $C^*(n;i,j,k)$ denote the number of vector partitions $(\pi_1;\pi_2;\pi_3)$ of n such that π_1 has i non-negative distinct parts all in color a, π_2 has j non-negative distinct parts all in color b, and π_3 has k non-negative distinct parts all in color c.

Let $D^*(n;\alpha,\beta,\gamma,\delta,\varepsilon,\phi)$ denote the number of Type 1 partitions of n with $\nu_a(\pi)=\alpha$, $\nu_b(\pi)=\beta$, $\ldots$, and $\nu_{bc}(\pi)=\phi$.*

Then

$$C^*(n;i,j,k) = \sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} D^*(n;\alpha,\beta,\gamma,\delta,\varepsilon,\phi).$$

Upon seeing (5.27), Alexander Berkovich asked what the coefficient of $a^ib^jc^k$ in the combined series on the left of (5.27) is? To determine this, we need to replace ϕ by $\phi-1$ in Σ_2. . Thus the coefficient of $a^ib^jc^k$ in (5.7) is

$$\sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}$$

$$+ \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-2}-1}}{(q)_{\alpha-1}(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_{\phi-1}}. \tag{5.28}$$

Note that

$$\frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-1}}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}+\frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-2}-1}}{(q)_{\alpha-1}(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_{\phi-1}}.$$

$$=\frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-2}-1}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}\times\{q^\phi+(1-q^\alpha)(1-q^\phi)\}. \tag{5.29}$$

Since

$$q^\phi+(1-q^\alpha)(1-q^\phi)=1-q^\alpha(1-q^\phi),$$

we see from (5.29) that the two sums in (5.27) amalgamate to yield

$$\sum_{i,j,k\geq 0}a^ib^jc^k\sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi\\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}}\frac{q^{T_{s-1}+T_{\delta-1}+T_{\varepsilon-1}+T_{\phi-2}-1}\{1-q^\alpha(1-q^\phi)\}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}$$

$$=(-a)_\infty(-b)_\infty(-c)_\infty. \tag{5.30}$$

Observe that the subsitutions

$$a\mapsto aq,\quad b\mapsto bq,\quad c\mapsto cq, \tag{5.31}$$

converts (5.30) to (1.3) because these substitutions increase the power of q in the inner sum in (5.30) by $i+j+k$, and

$$i+j+k=s+\alpha+\varepsilon+\phi.$$

Thus the key identity for Theorem A, namely (5.27), is equivalent to (1.3) which is the key identity for Theorem G.

Since we have shown that (5.27) is the same as (1.3), we do not prove (5.27) here, but refer the reader to [8] for the proof of (1.3) instead. But then, in the next section, we will provide two proofs of Theorem 1* without appeal to the key identity to demonstrate how the proof of Theorem A extends to the proof of its three parameter refinement.

We conclude this section by stressing that Theorem 1* and Theorem 1 are combinatorially different, as are Theorems A and G, because in showing (5.27) to be equivalent to (1.3), we had to do the amalgamation and the subsitutions in (5.31), and these change the underlying combinatorics. When the substitutions (5.31) are made, the order of the colors (5.5) changes to (3.5). This change can be viewed as follows: Reverse the ordering of the colors in (5.10) and interchange a and c. This converts (5.5) to (3.5) and vice-versa.

6. Two proofs of Theorem 1*

We will now prove Theorem 1*, not via its key identity (5.27), but directly. The first proof is patterned along the method of Section 2.

First proof of Theorem 1*: Let κ denote any of the six colors a, b, c, ab, ac, bc. For $n \geq 1$, we denote by $d_\kappa(m) = d_\kappa(m; a, b, c)$ the polynomial in a, b, c representing the generating function of all Type 1* partitions with parts $\leq \kappa_{n-1}$. In this section, for the special Type 1* partitions, we think of the smallest two parts as $bc_0 + a_0$ instead of $a_1 + bc_{-1}$.

As initial conditions, we let for all κ,

$$d_\kappa(m) = 0 \quad \text{if} \quad m < -1, \quad \text{and} \quad d_\kappa(m) = 1 \quad \text{if} \quad m = 0 \,\text{or}\, 1. \tag{6.1}$$

For example, $d_{ab}(1) = 1 + (a + b + ab)q^0 = 1 + a + b + ab$.

The conditions (5.6) imply that the $d_\kappa(m)$ satisfy the following recurrences:

$$d_{bc}(m) = d_{ac}(m) + bcq^{m-1} d_{ac}(m-1) + \delta_{m,1} abc, \tag{6.2.1}$$

$$d_{ac}(m) = d_c(m) + acq^{m-1} d_c(m-1) \tag{6.2.2}$$

$$d_c(m) = d_{ab}(m) + cq^{m-1} d_c(m-1) \tag{6.2.3}$$

$$d_{ab}(m) = d_b(m) + abq^{m-1} d_b(m-1) \tag{6.2.4}$$

$$d_b(m) = d_a(m) + bq^{m-1} d_b(m-1) \tag{6.2.5}$$

$$d_a(m) = d_{bc}(m-1) + aq^{m-1} d_a(m-1) \tag{6.2.6}$$

In (6.2.1), the $\delta_{m,1}$ is the Kronecker delta, and this term corresponds to the $\epsilon(m)$ correction term introduced in (2.6).

These recurrences can all be established using the definition of Type 1* partitions. We prove only (6.2.1) since it is the only one with the correction term.

The difference

$$d_{bc}(m) - d_{ac}(m)$$

is the generating function of all Type 1* partitions with largest part $(bc)_{m-1}$. After subtracting $(bc)_{m-1}$ from such partitions, we are left with Type 1* partitions with largest part $\leq (ac)_{m-2}$ because $(bc)_{m-2}$ cannot be a part when $(bc)_{m-1}$ is present. This accounts for the term $bcq^{m-1} d_{ac}(m-1)$ on the right in (5.2.1). However, when $m = 1$, the special partition $bc_0 + a_0$ is counted by $d_{bc}(1)$, but will not be present on the right unless the $\delta_{m,1} abc$ is added to the right hand side of (6.2.1). We need this correction term just at the start, that is when $m = 1$. The proofs of (6.2.2) - (6.2.5) are similar.

Our main objective is to prove

Theorem 2: *For all* κ,

$$\lim_{m \to \infty} d_\kappa(m) = (-a)_\infty (-b)_\infty (-c)_\infty.$$

We will deduce Theorem 2 from

Lemma 2: *For* $n \geq 2$

$$d_{bc}(m) + abq^m d_{bc}(m-1) = (1+a)(1+b)(1+c)d_c(m-1; aq, bq, cq).$$

Iteration of the q-difference equation in Lemma 2 directly yields Theorem 2. So we foucus our attention on the proof of Lemma 2.

Combining (6.2.1) and (6.2.2) we get

$$d_{bc}(m) = d_c(m) + q^{m-1}(ac+bc)d_c(m-1) + abc^2q^{2m-3}d_c(m-2). \tag{6.3}$$

Next rewrite (6.2.6) as

$$d_{bc}(m-1) = d_a(m) - aq^{m-1}d_a(m-1). \tag{6.4}$$

Then by (6.2.5) and (6.4) we get

$$d_{bc}(m-1) = d_b(m) - (a+b)q^{m-1}d_b(m-1) + abq^{2m-3}d_b(m-2). \tag{6.5}$$

Similarly, if we eliminate $d_{ab}(m)$ using (6.2.3) and (6.2.4), we find that

$$S_1(m) = 0, \tag{6.6}$$

where

$$S_1(m) = d_c(m) - cq^{m-1}d_c(m-1) - d_b(m) - abq^{m-1}d_b(m-1). \tag{6.7}$$

Next by (6.2.1) and (6.2.2)

$$d_{bc}(m-1) = d_c(m-1) + (ac+bc)q^{m-2}d_c(m-2) + acq^{m-3}d_c(m-3). \tag{6.8}$$

Hence by (6.5) and (6.8) we get

$$S_2(m) = 0, \tag{6.9}$$

where

$$S_2(m) = d_c(m-1) + (ac+bc)q^{m-2}d_c(m-2) + abc^2q^{2m-5}d_c(m-3)$$

$$-d_b(m) + (a+b)q^{m-1}d_b(m-1) - abq^{2m-3}d_b(m-2). \tag{6.10}$$

What we want is a linear recurrence for the $d_c(m)$. This is pure linear algebra problem. To this end, we regard the six expressions

$$S_j(m-i) = 0, \quad \text{for} \quad 0 \leq i \leq 2, \quad 1 \leq j \leq 2,$$

as linear equations in the variables $d_c(m), d_b(m), d_b(m-1), d_b(m-2), d_b(m-3)$, and $d_b(m-4)$. Solving this system will yield $d_c(m)$ as a linear combination of $d_c(m-i)$, for $1 \le i \le 4$. The result is that for $m \ge 5$,

$$J(m) := F(m; a, b, c, d_c(m), d_c(m-1), d_c(m-2), d_c(m-3), d_c(m-4)) = 0, \tag{6.11}$$

where

$$F(m; a, b, c, X, Y, Z, W, V) = X - (1 + q^{m-1}(a+b+c))Y - q^{3m-8}a^2b^2c^2V \tag{6.12}$$

$$-\{q^{m-2}(abq+ac+bc) - q^{2m-3}(ab+ac+bc)\}Z - \{q^{3m-6}abc + q^{2m-5}(a^2bcq + ab^2cq + abc^2)\}W.$$

Clearly by (6.12), the right hand side of the expression in Lemma 2 satisfies

$$F(m-1; aq, bq, cq, d_c^*(m-1), d_c^*(m-2), d_c^*(m-3), d_c^*(m-4), d_c^*(m-5)) = 0, \tag{6.13}$$

where

$$d_c^*(m) = d_c(m; aq, bq, cq).$$

Setting $G(m)$ for the left hand side of Lemma 2, note that

$$G(m) := d_{bc}(m) + abq^m d_{bc}(m-1) \tag{6.14}$$

$$= d_c(m) + (ac+bc)q^{m-1}d_c(m-1) + abc^2q^{2m-3}d_c(m-2)$$

$$+abq^n(d_c(m-1) + (ac+bc)q^{m-2}d_c(m-2) + abc^2q^{2m-5}d_c(m-3))$$

$$= d_c(m) + (abq+ac+bc)q^{m-1}d_c(m-1)$$

$$+(abc^2 + a^2bcq + ab^2cq)q^{2m-3}d_c(m-2) + a^2b^2c^2q^{3m-5}d_c(m-3).$$

Thus

$$F(m-1, aq, bq, cq, G(m), G(m-1), G(m-2), G(m-3), G(m-4)) \tag{6.15}$$

$$= J(m) + q^{m-1}(bc+ac+abq)J(m-1)$$

$$+q^{2m-3}(abc^2 + ab^2cq + a^2bcq)J(m-2) + q^{3m-5}a^2b^2c^2J(m-3) = 0,$$

by (6.11) provided $m \ge 8$.

On comparing (6.13) and (6.15), we see that both sides of Lemma 2 satisfy identical fourth order recurrences provided $n \ge 8$. The truth of Lemma 2 for $2 \le m < 8$ was checked using Macsyma. This completes the proof of Lemma 2 which implies Theorem 2 (and Theorem 1*).

Second proof of Theorem 2: This proof relies on the proof of Theorem G given in Andrews [14]. We will rephrase Theorem 2' of [14] in a manner that will make it more easily applicable here.

First we define

$$\lambda_m = \sum_{r=0}^{m}\sum_{s=0}^{m-r}\sum_{t=0}^{m-r-s} \begin{bmatrix} m \\ r,s,t,m-r-s-t \end{bmatrix} a^r b^s c^t q^{T_{r-1}+T_{s-1}+T_{t-1}}, \tag{6.16}$$

where

$$\begin{bmatrix} m \\ r,s,t,m-r-s-t \end{bmatrix} = \frac{(q)_m}{(q)_r(q)_s(q)_t(q)_{m-r-s-t}}. \tag{6.17}$$

The q-multinomial coefficients of order 4 in (6.17) can be written as a product of three q-binomial coefficents, namely

$$\begin{bmatrix} m \\ r,s,t,m-r-s-t \end{bmatrix} = \begin{bmatrix} m \\ r,m-r \end{bmatrix}\begin{bmatrix} m-r \\ s,m-r-s \end{bmatrix}\begin{bmatrix} m-r-s \\ t,m-r-s-t \end{bmatrix}, \tag{6.18}$$

where

$$\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i \\ j,i-j \end{bmatrix} = \frac{(q)_i}{(q)_j(q)_{i-j}}. \tag{6.19}$$

We note that

$$\sum_{m=0}^{\infty} \frac{\lambda_m z^m}{(q)_m} = \sum_{m,r,s,t\geq 0} \frac{z^{m+r+s+t} a^r b^s c^t q^{T_{r-1}+T_{s-1}+T_{t-1}}}{(q)_r(q)_s(q)_t(q)_m} = \frac{(-az)_\infty(-bz)_\infty(-cz)_\infty}{(z)_\infty}. \tag{6.20}$$

It follows from (6.16) that

$$\lim_{m\to\infty} \lambda_m = (-a)_\infty(-b)_\infty(-c)_\infty. \tag{6.21}$$

We now restate Theorem 2' of [14] in the form of Lemma 3 below, but to keep things succinct, we let

$$\sigma_1 = a+b+c, \quad \sigma_2 = ab+ac+bc, \quad \text{and} \quad \sigma_3 = abc.$$

Lemma 3: *Let* $D_{-1}=0, D_0=1, D_1=1+\sigma_1,$ *and*

$$D_2 = 1+\sigma_1(1+q)+{\sigma_1}^2 q+\sigma_2(1-q)+\sigma_3,$$

and for $m\geq 3$

$$\begin{aligned} D_m &= (1+\sigma_1 q^{m-1})D_{m-1} + \sigma_2 q^{m-2}(1-q^{m-1})D_{m-2} \\ &\quad +\sigma_3 q^{2m-5}(\sigma_1+q^{m-1})D_{m-3} + {\sigma_3}^2 q^{3m-9} D_{m-4}. \end{aligned} \tag{6.22}$$

Then

$$D_m = \sum_{0\le 2s\le m} q^{ms-s(s+3)/2}{\sigma_3}^2 \begin{bmatrix} m-s \\ s \end{bmatrix} \lambda_{m-2s}. \tag{6.23}$$

Remark: We note by (6.21) and (6.23) that

$$\lim_{m\to\infty} D_m = \lim_{m\to\infty} \lambda_m = (-a)_\infty(-b)_\infty(-c)_\infty. \tag{6.24}$$

Now for $m \ge 2$ we define six new polynomial sequences

$$\delta_{bc}(m) = D_m + q^{m-1}\sigma_2 D_{m-1} + \sigma_1\sigma_3 q^{2m-3} D_{m-2} + {\sigma_3}^2 q^{3m-6} D_{m-3}, \tag{6.25.1}$$

$$\delta_{ac}(m) = D_m + (ab+ac)q^{m-1} D_{m-1} + a\sigma_3 q^{2m-3} D_{m-2}, \tag{6.25.2}$$

$$\delta_c(m) = D_m + abq^{m-1} D_{m-1}, \tag{6.25.3}$$

$$\delta_{ab}(m) = D_m + (ab-c)q^{m-1} D_{m-1} - \sigma_3 q^{2m-3} D_{m-2}, \tag{6.25.4}$$

$$\delta_b(m) = D_m - cq^{m-1} D_{m-1}, \tag{6.25.5}$$

$$\delta_a(m) = D_m - (b+c)q^{m-1} D_{m-1} + bcq^{2m-3} D_{m-2}. \tag{6.25.6}$$

Next we want to show that the $\delta_\kappa(m)$ for each color κ will satify the same recurrence as $d_\kappa(m)$. Observe that for $n \ge 4$

$$\delta_{bc}(m) = \delta_{ac}(m) + bcq^{m-1}(D_{m-1} + (ab+ac)q^{m-2}D_{m-2}) + \sigma_3 q^{2m-5} D_{m-3}$$

$$= \delta_{ac}(m) + bcq^{m-1}\delta_{ac}(m-1). \tag{6.26.1}$$

$$\delta_{ac}(m) = \delta_c(m) + acq^{m-1}(D_{m-1} + abq^{m-2}D_{m-2})$$

$$= \delta_c(m) + acq^{m-1}\delta_c(m-1). \tag{6.26.2}$$

$$\delta_c(m) = \delta_{ab}(m) + acq^{m-1}(D_{m-1} + abq^{m-2}D_{m-2}).$$

$$= \delta_{ab}(m) + acq^{m-1}\delta_c(m-1). \tag{6.26.3}$$

$$\delta_{ab}(m) = \delta_b(m) + abq^{m-1}(D_{m-1} - cq^{m-2}D_{m-2})$$

$$\text{(6.26.4)} \qquad = \delta_b(m) + abq^{m-1}\delta_b(m-1).$$

$$\delta_b(m) = \delta_a(m) + bq^{m-1}(D_{m-1} - cq^{m-2}D_{m-2})$$

$$\text{(6.26.5)} \qquad = \delta_a(m) + bq^{m-1}\delta_b(m-1).$$

Finally,

$$\delta_a(m) - \delta_{bc}(m-1) - aq^{m-1}\delta_a(m-1)$$

$$= D_m - (1+\sigma_1 q^{m-1})D_{m-1} - \sigma_2 q^{m-2}(1-q^{m-1})D_{m-2}$$

$$\text{(6.26.6)} \qquad -\sigma_3 q^{2m-5}(\sigma_1 + q^{m-1})D_{m-3} - {\sigma_3}^2 q^{3m-9} D_{m-4} = 0,$$

by (6.22).

The recurrences (6.26.1)-(6.26.6) are the same as (6.2.1)-(6.2.6) for $n \geq 4$. The equality

$$d_\kappa(3) = \delta_\kappa(3),$$

for all κ, can be checked using any computer algebra system (in this case Macsyma was used). Thus we conclude that

$$\text{(6.27)} \qquad d_\kappa(m) = \delta_\kappa(m), \quad \text{for} \quad m \geq 3.$$

In conclusion we have

$$\lim_{m\to\infty} d_\kappa(m) = \lim_{m\to\infty} \delta_\kappa(m) = \lim_{m\to\infty} D_m = (-a)_\infty(-b)_\infty(-c)_\infty,$$

which is Theorem 2.

7. Key identities and companions

By a *key identity* for a partition theorem, we mean a q-hypergeometric identity with one or more parameters, such that under a transformations $q \mapsto q^M$ and certain choices of the parameter(s), the partition theorem emerges. The same key identity could be used to generate a companion. For example, the key identity

$$\text{(7.1)} \qquad 1 + \sum_{k=0}^{\infty} \frac{(aq)_{k-1}(1-aq^{2k})(-1)^k a^{2k} q^{k(5k-1)/2}}{(q)_k} = (aq)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k},$$

may be viewed as a key identity for the two Rogers-Ramanujan identities, because they emerge from (7.1) by the choices $a = 1$ and $a = q$. It is only for these two choices of a the left hand side of (7.1) has a product representation. The key identities discussed in this paper are those, which unlike (7.1), have product representations for all values of the parameters. The two Little Göllnitz theorems, which may be viewed as duals, emerge from the Lebesgue identity

$$\sum_{k=0}^{\infty} \frac{(-bq)_k q^{k(k+1)/2}}{(q)_k} = (-bq^2; q^2)_\infty (-q)_\infty, \tag{7.2}$$

under the dilation $q \mapsto q^2$, and the translations $b \mapsto bq$, or $b \mapsto bq^{-1}$. The Lebesgue identity (7.2), enjoys a product representation for all values of b.

The reformulations of Theorem G obtained in [1] and [3] led to key identities (see [5], [6]) that were different from (1.3) and simpler in structure. In contrast, we have demonstrated here that the key identity for Theorem A, namely (5.27), is equivalent to (1.3), the key identity for Theorem G. But the companion Theorem A is combinatorially very different from Theorem G, just as Theorems A1 and A2 for $r \geq 3$ are combinatorially different.

Alladi-Andrews-Berkovich [7] obtained a deep four parameter key identity extending (1.3). From this key identity, under the transformations

$$\text{(dilation)} \quad q \mapsto q^{15}, \quad \text{and (translations)} \quad a \mapsto aq^{-8},\ b \mapsto bq^{-4},\ c \mapsto cq^{-2},\ d \mapsto dq^{-1},$$

they stated a mod 15 partition theorem analogous to, but deeper than, Theorem A2 in the case $r = 4$. In view of Theorem A, it might be worthwhile to study the partition theorem that emerges from the four parameter key identity in [7] by first replacing a by aq^{-1}, b by bq^{-1}, c by cq^{-1}, and d by dq^{-1}, to go from a Level 1 start to a Level 0 start, and then applying the transformations

$$\text{(dilation)} \quad q \mapsto q^{15}, \quad \text{and (translations)} \quad a \mapsto aq^{1},\ b \mapsto bq^{2},\ c \mapsto cq^{4},\ d \mapsto dq^{8}.$$

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