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The product of partial theta functions

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Abstract

In this paper, we prove a new identity for the product of two partial theta functions. An immediate corollary is Warnaar's generalization of the Jacobi triple product identity.

Keywords: Partial theta functions; Jacobi triple product identity; The Lost Notebook

1. Introduction

In [5, Eq. (1.7)], one of the authors proved the following generalization of Jacobi's triple product identity:

$$1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\bigl(a^n+b^n\bigr)=(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(ab/q)_{2n}q^n}{(q)_n(a)_n(b)_n(ab)_n}, \tag{1.1}$$

where

$$(a;q)_n=(a)_n=(1-a)(1-aq)\cdots\bigl(1-aq^{n-1}\bigr). \tag{1.2}$$

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The celebrated Jacobi triple product identity [3, p. 12, Eq. (1.6.1)]

$$\sum_{n=-\infty}^{\infty}(-1)^n a^n q^{n(n-1)/2} = (q)_\infty (a)_\infty (q/a)_\infty \tag{1.3}$$

follows immediately from (1.1) upon setting $b = q/a$ and noting that the sum on the right-hand side of (1.1) reduces to 1 in this instance.

Sums of the form

$$\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n-1)/2}$$

are called partial theta functions owing to the fact that the sum in (1.3) is usually referred to as a complete theta function or just a theta function. Partial theta functions appear often in Ramanujan's The Lost Notebook [4]. An extensive explication of Ramanujan's discoveries was given in [2]. This was further elaborated on in [5], with (1.1) playing a central role.

Our object here is to prove the following theorem for the product of partial theta functions.

Theorem 1.1. *We have*

$$\left(\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n-1)/2}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{n(n-1)/2}\right) = (q)_\infty (a)_\infty (b)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n-1})_n q^n}{(q)_n (a)_n (b)_n}.$$

Section 2 will be devoted to the proof of Theorem 1.1. In Section 3, we deduce the generalized triple product identity (1.1) from Theorem 1.1, and in the conclusion we discuss the relationship of Theorem 1.1 to other theorems in q-hypergeometric series.

2. Proof of Theorem 1.1

In the following we employ standard notation for basic hypergeometric series, see e.g. [3, p. 4, Eq. (1.2.22)].

We begin by noting that the substitutions and limits $z \to z/a$, $b \to q$ followed by $a \to \infty$, $c \to 0$ followed by $z \to a$ in Heine's first and second transformations [1, p. 19, Cor. 2.3 and p. 39]

$$\begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1(c/b, z; az; q, b) \\ &= \frac{(c/a, az)_\infty}{(c, z)_\infty} {}_2\phi_1(abz/c, a; az; q, c/a) \end{aligned}$$

imply the identities

$$\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}} = (q)_\infty (a)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q)_n (a)_n} \tag{2.1a}$$

$$= (a)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m^2}}{(q)_m (a)_m}. \tag{2.1b}$$

Therefore,

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{n,m=0}^{\infty}\frac{b^m q^{n+m^2}}{(q)_n(a)_n(q)_m(b)_m}\qquad\text{(by (2.1a) and (2.1b))}\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{b^{m-n}q^{n+(m-n)^2}}{(q)_n(a)_n(q)_{m-n}(b)_{m-n}}\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\frac{b^m q^{m^2}}{(q)_m(b)_m}\sum_{n=0}^{m}\frac{(q^{1-m}/b)_n(q^{-m})_n}{(q)_n(a)_n}q^n.
\end{aligned}$$

The sum over n may be performed by the q-Chu–Vandermonde sum [3, Eq. (1.5.3)]

$${}_2\phi_1\big(a,q^{-n};b;q,q\big)=\frac{(b/a)_n}{(b)_n}\,a^n,$$

resulting in

$$\left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)=(q)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\frac{(abq^{m-1})_m q^m}{(q)_m(a)_m(b)_m}.$$

3. Proof of identity (1.1)

Theorem 3.1. *Identity* (1.1) *is valid.*

Proof. Define

$$L(a,b):=(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(abq^{n-1})_n q^n}{(q)_n(a)_n(b)_n}.$$

Then

$$\begin{aligned}
&(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(ab/q)_{2n}q^n}{(q)_n(a)_n(b)_n(ab)_n}\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(1-ab/q)(abq^n)_{n-1}q^n}{(q)_n(a)_n(b)_n}\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{((1-abq^{n-1})-(ab/q)(1-q^n))(abq^n)_{n-1}q^n}{(q)_n(a)_n(b)_n}\\
&\quad=(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(abq^{n-1})_n q^n}{(q)_n(a)_n(b)_n}
\end{aligned}$$

$$
\begin{aligned}
&-ab(q)_\infty(a)_\infty(b)_\infty\sum_{n=1}^{\infty}\frac{(abq^n)_{n-1}q^{n-1}}{(q)_n(a)_n(b)_n}\\
&= L(a,b) - ab\,(q)_\infty(aq)_\infty(bq)_\infty\sum_{n=0}^{\infty}\frac{(abq^{n+1})_n q^n}{(q)_n(aq)_n(bq)_n}\\
&= L(a,b) - ab\,L(aq,bq)\\
&= \left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)\\
&\quad -\left(\sum_{n=0}^{\infty}(-1)^n a^{n+1} q^{\binom{n+1}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^{n+1} q^{\binom{n+1}{2}}\right) \quad \text{(by Theorem 1.1)}\\
&= \left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)\\
&\quad -\left(\sum_{n=1}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=1}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)\\
&= 1 + \sum_{n=1}^{\infty}(-1)^n a^n q^{\binom{n}{2}} + \sum_{n=1}^{\infty}(-1)^n b^n q^{\binom{n}{2}}. \qquad \square
\end{aligned}
$$

4. Conclusion

There are numerous corollaries that follow from Theorem 1.1. Most important is Theorem 3.1 of Section 3. As shown in [5], there are extensive implications of Theorem 3.1.

In light of the fact that (1.3) may be rewritten as

$$
(q)_\infty(a)_\infty(q/a)_\infty = \sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}} - (q/a)\sum_{n=0}^{\infty}(-1)^n (q^2/a)^n q^{\binom{n}{2}}, \tag{4.1}
$$

we see that

$$
\begin{aligned}
&(q)_\infty^2(a)_\infty(q/a)_\infty(b)_\infty(q/b)_\infty\\
&\quad = \left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}} - (q/a)\sum_{n=0}^{\infty}(-1)^n (q^2/a)^n q^{\binom{n}{2}}\right)\\
&\qquad \times\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}} - (q/b)\sum_{n=0}^{\infty}(-1)^n (q^2/b)^n q^{\binom{n}{2}}\right)\\
&\quad = L(a,b) - (q/a)L\bigl(q^2/a,b\bigr) - (q/b)L\bigl(a,q^2/b\bigr) + \bigl(q^2/ab\bigr)L\bigl(q^2/a,q^2/b\bigr).
\end{aligned} \tag{4.2}
$$

Noting that in q-hypergeometric series notation

$$L(a,b)=(q)_\infty(a)_\infty(b)_\infty\times{}_4\phi_3\left(\begin{matrix}(ab/q)^{1/2},-(ab/q)^{1/2},(ab)^{1/2},-(ab)^{1/2}\\a,b,ab/q\end{matrix};q,q\right),$$

we see that we have an identity between four ${}_4\phi_3$'s and the infinite product on the left-hand side of (4.2).

Also, one can make use of Gauss' formula [1, p. 23, Eq. (2.2.12)]

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and $(q;q^2)_\infty(-q;q)_\infty=1$ [1, p. 5, Eq. (1.2.5)] to deduce from Theorem 1.1 with $b=-q$, that

$$\sum_{n=0}^{\infty}\frac{(-aq^n;q)_nq^n}{(q^2;q^2)_n(a)_n}=\frac{(-q;q)_\infty}{(a)_\infty}\sum_{n=0}^{\infty}(-1)^na^nq^{\binom{n}{2}}.$$

Or, we can use a further instance of Jacobi's identity [1, p. 23, Eq. (2.2.12)]

$$1+2\sum_{n=1}^{\infty}(-1)^nq^{n^2}=\frac{(q)_\infty}{(-q;q)_\infty}$$

to infer that

$$\sum_{n=0}^{\infty}\frac{(aq^{2n-1};q^2)_nq^{2n}}{(q)_{2n}(a;q^2)_n}=\frac{1}{2(a;q^2)_\infty}\left(\frac{1}{(-q;q)_\infty}+\frac{1}{(q)_\infty}\right)\sum_{n=0}^{\infty}(-1)^na^nq^{n^2-n}.$$

Finally we point out that Theorem 1.1 may also be deduced from Theorem 3.1. To achieve this one merely iterates the functional equation

$$L(a,b)-ab\,L(aq,bq)=1+\sum_{n=1}^{\infty}(-1)^nq^{\binom{n}{2}}\bigl(a^n+b^n\bigr),$$

which we see from the proof of Theorem 3.1 is an assertion equivalent to Theorem 3.1.

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