ON q-DIFFERENCE EQUATIONS FOR CERTAIN WELL-POISED BASIC HYPERGEOMETRIC SERIES

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1. Introduction

In this paper we shall study the following q-series.

$$C_{k,i}(a_1,\ldots,a_\lambda;x;q) = C_{k,i}((a);x;q)_\lambda$$
$$= \sum_{n=0}^{\infty} \frac{M_n(x)q^{-in}(1-x^iq^{2ni})\Pi_n(-x,q)P_n(1;a_1,\ldots,a_\lambda)}{(1-x)\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}; \quad (1.1)$$

$$D_{k,i}(a_1,\ldots,a_\lambda;x;q) = D_{k,i}((a);x;q)_\lambda$$
$$= \sum_{n=0}^{\infty} \frac{M_n(xq)q^{-in}\Pi_n(-xq,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})} \times$$
$$\times \left\{1+\frac{(-1)^{\lambda+1}x^iq^{(2n+1)i-\lambda n}(a_1\ldots a_\lambda)^{-1}p_{n+1}(1;a_1,\ldots,a_\lambda)}{p_{n+1}(xq;a_1^{-1},\ldots,a_\lambda^{-1})}\right\}; \quad (1.2)$$

$$H_{k,i}(a_1,\ldots,a_\lambda;x;q) = H_{k,i}((a);x;q)_\lambda$$
$$= P_\infty(xq;a_1^{-1},\ldots,a_\lambda^{-1})C_{k,i}((a);x;q)_\lambda/\Pi_\infty(-xq,q); \quad (1.3)$$

$$J_{k,i}(a_1,\ldots,a_\lambda;x;q) = J_{k,i}((a);x;q)_\lambda$$
$$= P_\infty(xq;a_1^{-1},\ldots,a_\lambda^{-1})D_{k,i}((a);x;q)_\lambda/\Pi_\infty(-xq,q), \quad (1.4)$$

where

$$\Pi_n(b,q) = \prod_{j=0}^{n-1}(1+bq^j), \qquad \Pi_\infty(b,q) = \prod_{j=0}^{\infty}(1+bq^j),$$

$$p_n(b;y_1,\ldots,y_\lambda) = \prod_{j=1}^{\lambda}(1-by_jq^{n-1}),$$

$$P_n(b;y_1,\ldots,y_\lambda) = \prod_{j=1}^{n}p_j(b;y_1,\ldots,y_\lambda),$$

$$P_\infty(b;y_1,\ldots,y_\lambda) = \prod_{j=1}^{\infty}p_j(b;y_1,\ldots,y_\lambda),$$

$$M_n(x) = M_n((a);\lambda;k;x;q) = (-1)^{n(\lambda+1)}x^{kn}q^{\frac{1}{2}(2k-\lambda+1)n^2+\frac{1}{2}(\lambda+1)n}(a_1\ldots a_\lambda)^{-n}.$$

$C_{k,i}((a);x;q)_\lambda$ generalizes a certain well-poised basic hypergeometric

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series that appears in several famous theorems: namely,

$$C_{k,1}((a);x;q)_\lambda$$
$$= {}_{\lambda+3}\Phi_{\lambda+2}\left[\begin{matrix} x, qx^{\frac12}, -qx^{\frac12}, a_1, \ldots, a_\lambda \\ x^{\frac12}, -x^{\frac12}, xq/a_1, \ldots, xq/a_\lambda \end{matrix}; q, \frac{q^{\frac12(2k-\lambda+1)n+\frac12(\lambda-1)}x^k(-1)^{\lambda-1}}{a_1 a_2 \ldots a_\lambda}\right]. \qquad (1.5)$$

The notation is that of Slater [(**10**) 91]. In particular, when $\lambda = 3$, $k = 1$, this function appears in the limiting form of Jackson's theorem [(**10**) 96, equation (3.3.1.3)]; when $\lambda = 5$, $k = 2$, the resulting series appears in Watson's q-analogue of Whipple's theorem [(**10**) 100, equation (3.4.1.5)]. Selberg (**9**) has studied these series in the case $\lambda = 0$; the case $\lambda = 1$ was treated in (**2**).

Our main object is to obtain q-difference equations for these series. Originally these results were to appear in a future work on partitions (**1**); however, once the results were derived, it was found that, apart from their intrinsic interest, they were quite useful in proving theorems on infinite products, continued fractions, and q-series. Hence it was felt that a separate exhibition of these theorems would be worth while.

In § 3, we shall derive some theorems on infinite products. For example,

$${}_4\Phi_3\left[\begin{matrix} x, qx^{\frac12}, -qx^{\frac12}, a; q, x^{\frac12}q^{\frac12 n}a^{-1} \\ x^{\frac12}, -x^{\frac12}, xqa^{-1} \end{matrix}\right] = \frac{\Pi_\infty(-xq,q)\,\Pi_\infty((xq)^{\frac12}a^{-1},q)}{\Pi_\infty(-xqa^{-1},q)\,\Pi_\infty((xq)^{\frac12},q)}. \qquad (1.6)$$

We shall also obtain known results like the limiting form of Jackson's theorem [(**10**) 96, equation (3.3.1.3)].

In § 4, we obtain new theorems on continued fractions. The most interesting result here seems to be Theorem 6, which may well be the 'general theorem' on continued fractions referred to by Ramanujan in one of his letters to Hardy [(7) xxviii].

THEOREM 6. *If* $a_1^{-1}+a_2^{-1} = -b$, *and* $a_1^{-1}a_2^{-1} = -a$, *then*

$$\frac{H_{2,1}(a_1,a_2;x;q)}{H_{2,1}(a_1,a_2;xq;q)} = 1+bxq-\frac{xq(1+axq^2)}{1+bxq^2+}\,\frac{xq^2(1+axq^3)}{1-bxq^3+}\cdots.$$

As we shall see, this theorem contains many of the most elegant known results on continued fractions of the Rogers–Ramanujan type.

In § 5, we shall apply our results to the study of q-series. Much work has been done on problems of this type by Starcher (**12**) and Fine (**5**) for q-difference equations of first and second order. We shall therefore restrict ourselves to equations of orders 3 and 4.

2. Main theorems

Theorem 1 will contain the principal results; in Theorem 2, we shall collect some useful information for later use.

THEOREM 1. *Let k and i be real with $2k-1 \geqslant \lambda \geqslant 0$ where λ is an integer; $|q| < 1$. Furthermore, if $2k-1 = \lambda$, then*

$$|x^k(a_1 \ldots a_\lambda)^{-1}q^{\frac{1}{2}(\lambda+1)-i}| < 1.$$

Under these conditions,

$$H_{k,i}((a);x;q)_\lambda - H_{k,i-1}((a);x;q)_\lambda = x^{i-1}J_{k,k-i+1}((a);x;q)_\lambda; \tag{2.1}$$

$$J_{k,i}((a);x;q)_\lambda = \sum_{j=0}^{\lambda}(-1)^j\sigma_j(a_1^{-1},\ldots,a_\lambda^{-1})x^jq^jH_{k,i-j}((a);xq;q)_\lambda; \tag{2.2}$$

$$H_{k,-i}((a);x;q)_\lambda = -x^{-i}H_{k,i}((a);x;q)_\lambda; \tag{2.3}$$

$$H_{k,0}((a);x;q)_\lambda = 0. \tag{2.4}$$

$\sigma_j(y_1,\ldots,y_\lambda)$ is the j-th elementary symmetric function of $y_1,\ldots,y_\lambda$; $\sigma_0(y_1,\ldots,y_\lambda) = 1$.

Proof. We start with (2.1).

$$C_{k,i}((a);x;q)_\lambda - C_{k,i-1}((a);x;q)_\lambda$$
$$= \sum_{n=0}^{\infty} M_n(x)(1-x)^{-1}\{q^{-in}(1-q^n)+x^{i-1}q^{n(i-1)}(1-xq^n)\}\times$$
$$\times\frac{\Pi_n(-x,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}.$$

Consequently splitting the series into two parts by means of the terms enclosed in $\{\}$, we have

$$C_{k,i}((a);x;q)_\lambda - C_{k,i-1}((a);x;q)_\lambda$$
$$= \sum_{n=1}^{\infty} M_n(x)q^{-in}\frac{\Pi_{n-1}(-xq,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_{n-1}(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}+$$
$$+x^{i-1}\sum_{n=0}^{\infty} M_n(x)q^{n(i-1)}\frac{\Pi_n(-xq,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}$$
$$= (-1)^{\lambda+1}x^kq^{k+1-i}(a_1\ldots a_\lambda)^{-1}\sum_{n=0}^{\infty} M_n(x)q^{(2k-\lambda-i+1)n}\times$$
$$\times\frac{\Pi_n(-xq,q)P_{n+1}(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_{n+1}(xq;a_1^{-1},\ldots,a_\lambda^{-1})}+$$
$$+x^{i-1}\sum_{n=0}^{\infty} M_n(x)q^{n(i-1)}\frac{\Pi_n(-xq,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}$$
$$= x^{i-1}\sum_{n=0}^{\infty} M_n(xq)q^{-(k-i+1)n}\frac{\Pi_n(-xq,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(xq;a_1^{-1},\ldots,a_\lambda^{-1})}\times$$
$$\times\left\{1+\frac{(-1)^{\lambda+1}(xq)^{k-i+1}q^{2n(k-i+1)-\lambda n}(a_1\ldots a_\lambda)^{-1}p_{n+1}(1;a_1,\ldots,a_\lambda)}{p_{n+1}(xq;a_1^{-1},\ldots,a_\lambda^{-1})}\right\}$$
$$= x^{i-1}D_{k,k-i-1}((a);x;q)_\lambda.$$

We now deduce (2.1) utilizing (1.3) and (1.4).

We prove (2.2) as follows. In (1.2) we combine fractions in the term enclosed in $\{\}$. This yields

$$D_{k,i}((a);x;q)_\lambda = \sum_{n=0}^{\infty} M_n(xq)q^{-in}\frac{\Pi_n(-xq,q)P_n(1;a_1,\dots,a_\lambda)}{\Pi_n(-q,q)P_{n+1}(xq;a_1^{-1},\dots,a_\lambda^{-1})}\times$$

$$\times\Big\{\sum_{j=0}^{\lambda}(-1)^j x^j\sigma_j(a_1^{-1},\dots,a_\lambda^{-1})q^{j(n+1)}+$$

$$+(-1)^{\lambda-1}x^i q^{(2n+1)i-\lambda n}\sum_{j=0}^{\lambda}(-1)^{-j}\sigma_j(a_1^{-1},\dots,a^{-1})q^{n(\lambda-j)}\Big\}$$

$$= \sum_{n=0}^{\infty} M_n(xq)q^{-in}\frac{\Pi_n(-xq,q)P_n(1;a_1,\dots,a_\lambda)}{\Pi_n(-q,q)P_{n+1}(xq;a_1^{-1},\dots,a_\lambda^{-1})}\times$$

$$\times\sum_{j=0}^{\lambda}(-1)^j\sigma_j(a_1^{-1},\dots,a_\lambda^{-1})x^j q^{j(n+1)}(1-x^{i-j}q^{(2n+1)(i-j)})$$

$$= (1-xq)\big(p_1(xq;a_1^{-1},\dots,a_\lambda^{-1})\big)^{-1}\sum_{j=0}^{\lambda}(-1)^j\sigma_j(a_1^{-1},\dots,a_\lambda^{-1})x^j q^j\times$$

$$\times\sum_{n=0}^{\infty}M_n(xq)q^{-(i-j)n}(1-xq)^{-1}(1-(xq)^{i-j}q^{2n(i-j)})\times$$

$$\times\frac{\Pi_n(-(xq),q)P_n(1;a_1,\dots,a_\lambda)}{\Pi_n(-q,q)P_n((xq)q;a_1^{-1},\dots,a_\lambda^{-1})}$$

$$= (1-xq)\big(p_1(xq;a_1^{-1},\dots,a_\lambda^{-1})\big)^{-1}\sum_{j=0}^{\lambda}(-1)^j\sigma_j(a_1^{-1},\dots,a_\lambda^{-1})x^j q^j\times$$

$$\times C_{k,i-j}((a);xq;q)_\lambda.$$

We now deduce (2.2), utilizing (1.3) and (1.4).

(2.3) follows directly from the identity

$$-x^{-i}q^{-in}(1-x^iq^{2ni}) = q^{-n(-i)}(1-x^{-i}q^{2n(-i)}).$$

(2.4) follows directly from the fact that $1-x^iq^{2ni}=0$ for all n when $i=0$.

We now proceed to Theorem 2.

THEOREM 2.

$$C_{k,1}(a_1,\dots,a_\lambda,-x;x^2;q) = (1+x)C_{k-\frac{1}{2},1}(a_1,\dots,a_\lambda;x^2;q); \tag{2.5}$$

$$\lim_{a_\lambda\to\infty} H_{k,i}(a_1,\dots,a_\lambda;x;q) = H_{k,i}(a_1,\dots,a_{\lambda-1};x;q); \tag{2.6}$$

$$\lim_{a_\lambda\to\infty} J_{k,i}(a_1,\dots,a_\lambda;x;q) = J_{k,i}(a_1,\dots,a_{\lambda-1};x;q); \tag{2.7}$$

$$H_{k,1}((a);x;q)_\lambda = J_{k,k}((a);x;q)_\lambda = J_{k,k+1}((a);x;q)_\lambda; \tag{2.8}$$

if the two sets $\{a_1,\dots,a_\lambda\}$ and $\{qa_1^{-1},\dots,qa_\lambda^{-1}\}$ are equal, then

$$D_{k,i}((a);1;q)_\lambda = \Pi_\infty(-(-1)^{c-\lambda}q^{2k-1-i-\frac{1}{2}\lambda},q^{2k-\lambda+1})\times$$

$$\times\Pi_\infty(-(-1)^{c-\lambda}q^{i-\frac{1}{2}\lambda},q^{2k-\lambda+1})\Pi_\infty(-q^{2k-\lambda+1},q^{2k-\lambda-1}), \tag{2.9}$$

where $a_1\dots a_\lambda = (-1)^c q^{\frac{1}{2}\lambda}$.

Proof. We start with (2.5).

$$C_{k,1}(a_1,\ldots,a_\lambda,-x;x^2;q) = \sum_{n=0}^{\infty}(-1)^{n(\lambda-1)}x^{(2k-\frac{1}{2})n}(a_1\ldots a_\lambda)^{-n}q^{\frac{1}{2}(2(k-\frac{1}{2})-\lambda-1)n^2}\times$$
$$\times q^{\frac{1}{2}\lambda n}\frac{(1-xq^n)(1+xq^n)}{(1-x^2)}\frac{\Pi_n(-x^2,q)P_n(1;a_1,\ldots,a_\lambda)}{\Pi_n(-q,q)P_n(x^2q;a_1^{-1},\ldots,a_\lambda^{-1})}\frac{(1+x)}{(1+xq^n)}$$
$$= (1+x)C_{k-\frac{1}{2},\frac{1}{2}}(a_1,\ldots,a_\lambda;x^2;q).$$

(2.6) and (2.7) are deduced from an appeal to Tannery's theorem [(**14**) 371] and the fact that

$$\lim_{a_\lambda\to\infty} a_\lambda^{-n}\Pi_n(-a_\lambda,q) = \lim_{a_\lambda\to\infty}(a_\lambda^{-1}-1)(a_\lambda^{-1}-q)\ldots(a_\lambda^{-1}-q^{n-1})$$
$$= (-1)^n q^{\frac{1}{2}n(n-1)}.$$

We obtain the first equation in (2.8) from (2.1) in the case $i = 1$ utilizing (2.4). To obtain the remainder of (2.8), we set $i = 0$ in (2.1) and then apply (2.3) and (2.4).

Finally we treat (2.9). Under the conditions stated there, we have immediately that $a_1a_2\ldots a_\lambda = (qa_1^{-1})(qa_2^{-1})\ldots(qa)_\lambda^{-1}$ and therefore

$$(a_1a_2\ldots a_\lambda)^2 = q^\lambda.$$

Consequently $a_1a_2\ldots a_\lambda = (-1)^c q^{\frac{1}{2}\lambda}$. Thus

$$D_{k,i}((a);1;q)_\lambda$$
$$= \sum_{n=0}^{\infty}(-1)^{n(\lambda-c+1)}q^{\frac{1}{2}(2k-\lambda+1)n^2+\frac{1}{2}n+(k-i)n}\{1+(-1)^{\lambda-c+1}q^{(2n+1)i-\lambda n-\frac{1}{2}\lambda}\}$$
$$= \sum_{n=-\infty}^{\infty}(-1)^{n(\lambda-c+1)}q^{\frac{1}{2}(2k-\lambda+1)n^2-\frac{1}{2}(2k-2i-1)n}.$$

(2.9) now follows from Jacobi's identity [(**10**) 105, equation (3.5.8)].

3. Infinite products

By means of equations (2.1)–(2.4) and (2.8), we may derive first-order difference equations for $H_{k,1}((a);x;q)_\lambda$ in the cases $k = 0$, $\lambda = 1$; $k = \frac{1}{2}$, $\lambda = 1$; $k = 1$, $\lambda = 3$. Formulae for infinite products are then easily deduced.

THEOREM 3.

$$_4\Phi_3\begin{bmatrix} x, qx^{\frac{1}{2}}, -qx^{\frac{1}{2}}, a; q, a^{-1} \\ x^{\frac{1}{2}}, -x^{\frac{1}{2}}, xq/a \end{bmatrix} = 0.$$

Proof.

$$H_{0,1}(a;x;q) = J_{0,0}(a;x;q) - J_{0,1}(a;x;q) \quad \text{(by (2.8) } k = 0\text{)}.$$
$$J_{0,1}(a;x;q) = H_{0,1}(a;xq;q) \quad \text{(by (2.2) } k = 0,\ i = 1, \text{ and (2.4))}.$$

Therefore $\qquad H_{0,1}(a;x;q) = H_{0,1}(a;xq;q).$

This implies $$H_{0,1}(a;x;q) = H_{0,1}(a;xq^N;q)$$

for any integer $N \geqslant 0$. Consequently since $|q| < 1$ and $H_{0,1}(a;x;q)$ is continuous in x at $x = 0$, we have, by (**10**) 92, equation (3.2.2.11),

$$H_{0,1}(a;x;q) = \sum_{n=0}^{\infty} \frac{\Pi_n(-a,q)a^{-n}}{\Pi_n(-q,q)} = 0.$$

Therefore, by (1.3), $$C_{0,1}(a;x;q) = 0.$$

This reduces to Theorem 3 by (1.5) with $k = 0$, $\lambda = 1$.

THEOREM 4. *Equation* (1.6) *is valid.*

Proof.

$$H_{\frac12,1}(a;x;q) = J_{\frac12,\frac12}(a;x;q) = J_{\frac12,\frac12}(a;x;q) \quad (\text{by } (2.8)\ k = \tfrac12).$$

$$(1-x^{\frac12})H_{\frac12,\frac12}(a;x;q) = J_{\frac12,1}(a;x;q) \quad (\text{by } (2.1)\ k = \tfrac12, \text{ and } (2.3)).$$

$$J_{\frac12,1}(a;x;q) = H_{\frac12,1}(a;xq;q) \quad (\text{by } (2.2)\ k = \tfrac12,\ i = 1,\ \lambda = 1, \text{ and } (2.4)).$$

$$J_{\frac12,\frac12}(a;x;q) = (1+(xq)^{\frac12}a^{-1})H_{\frac12,\frac12}(a;xq;q)$$
$$(\text{by } (2.2)\ k = \tfrac12,\ i = \tfrac12,\ \lambda = 1,\ (2.3), \text{ and } (2.4)).$$

Hence

$$\begin{aligned} H_{\frac12,1}(a;x;q) &= (1+(xq)^{\frac12}a^{-1})H_{\frac12,\frac12}(a;xq;q) \\ &= (1+(xq)^{\frac12}a^{-1})(1+(xq)^{\frac12})^{-1}J_{\frac12,1}(a;xq;q) \\ &= (1+(xq)^{\frac12}a^{-1})(1+(xq)^{\frac12})^{-1}H_{\frac12,1}(a;xq^2;q). \end{aligned}$$

We may now deduce

$$H_{\frac12,1}(a;x;q) = \frac{\Pi_\infty((xq)^{\frac12}/a,q)}{\Pi_\infty((xq)^{\frac12},q)}$$

by the technique employed in Theorem 3.

Consequently, by (1.3),

$$C_{\frac12,1}(a;x;q) = \frac{\Pi_\infty(-xq,q)\Pi_\infty((xq)^{\frac12}/a,q)}{\Pi_\infty(-xq/a,q)\Pi_\infty((xq)^{\frac12},q)}.$$

This reduces to Theorem 4 by (1.5) with $k = \frac12$, $\lambda = 1$.

THEOREM 5.

$${}_6\Phi_5\left[\begin{matrix} x, qx^{\frac12}, -qx^{\frac12}, a_1, a_2, a_3;\ q, xq/a_1a_2a_3 \\ x^{\frac12}, -x^{\frac12}, xq/a_1, xq/a_2, xq/a_3 \end{matrix}\right]$$
$$= \frac{\Pi_\infty(-xq,q)\Pi_\infty(-xq/a_1a_2,q)\Pi_\infty(-xq/a_1a_3,q)\Pi_\infty(-xq/a_2a_3,q)}{\Pi_\infty(-xq/a_1,q)\Pi_\infty(-xq/a_2,q)\Pi_\infty(-xq/a_3,q)\Pi_\infty(-xq/a_1a_2a_3,q)}.$$

Remark. This theorem is known as the limiting form of Jackson's theorem [(**10**) 96, equation (3.3.1.3)].

Proof. We let $\sigma_j = \sigma_j(a_1^{-1}, a_2^{-1}, a_3^{-1})$.

$$H_{1,1}((a);x;q)_3 = J_{1,1}((a);x;q)_3 = J_{1,2}((a);x;q)_3 \quad \text{(by (2.8) } k = 1\text{)}. \tag{3.1}$$

$$J_{1,1}((a);x;q)_3 = (1-\sigma_2 xq)H_{1,1}((a);xq;q)_3+\sigma_3 xqH_{1,2}((a);xq;q)_3$$
$$\text{(by (2.2) } k = 1,\ i = 1,\ \lambda = 3, \text{ (2.3), and (2.4))}. \tag{3.2}$$

$$J_{1,2}((a);x;q)_3 = H_{1,2}((a);xq;q)_3-xq(\sigma_1-xq\sigma_3)H_{1,1}((a);xq;q)_3$$
$$\text{(by (2.2) } k = 1,\ i = 2,\ \lambda = 3, \text{ (2.3), and (2.4))}. \tag{3.3}$$

Hence, by (3.3) and (3.1),

$$H_{1,2}((a);xq;q)_3 = H_{1,1}((a);x;q)_3+xq(\sigma_1-xq\sigma_3)H_{1,1}((a);xq;q)_3. \tag{3.4}$$

Substituting (3.4) into (3.2) and utilizing (3.1), we obtain

$$(1-xq\sigma_3)H_{1,1}((a);x;q)_3 = (1-\sigma_2 xq-x^2q^2\sigma_1\sigma_3-x^3q^3\sigma_3^2)H_{1,1}((a);xq;q)_3. \tag{3.5}$$

We may rewrite this last formula as

$$H_{1,1}((a);x;q)_3 = \frac{(1-xq/a_1a_2)(1-xq/a_1a_3)(1-xq/a_2a_3)H_{1,1}((a);xq;q)_3}{(1-xq/a_1a_2a_3)}. \tag{3.6}$$

We may now deduce

$$H_{1,1}((a);x;q)_3 = \frac{\Pi_\infty(-xq/a_1a_2,q)\Pi_\infty(-xq/a_1a_3,q)\Pi_\infty(-xq/a_2a_3,q)}{\Pi_\infty(-xq/a_1a_2a_3,q)} \tag{3.7}$$

by the technique employed in Theorem 3.

Consequently, by (1.3),

$$C_{1,1}((a);x;q)_3$$
$$= \frac{\Pi_\infty(-xq,q)\Pi_\infty(-xq/a_1a_2,q)\Pi_\infty(-xq/a_1a_3,q)\Pi_\infty(-xq/a_2a_3,q)}{\Pi_\infty(-xq/a_1,q)\Pi_\infty(-xq/a_2,q)\Pi_\infty(-xq/a_3,q)\Pi_\infty(-xq/a_1a_2a_3,q)}. \tag{3.8}$$

This reduces to Theorem 5 by (1.5) with $k = 1$, $\lambda = 3$.

4. Continued fractions of the Rogers–Ramanujan type

Our results here depend for the most part on the derivation of second-order q-difference equations. We shall study the cases $k = 2$, $i = 1$, $\lambda = 2$; $k = 1$, $i = \frac{1}{2}$, $\lambda = 1$.

We start with the result stated in § 1, as Theorem 6.

Proof of Theorem 6. Here we let $\sigma_1(a_1^{-1}, a_2^{-1}) = -b$, $\sigma_2(a_1^{-1}, a_2^{-1}) = -a$.

$$H_{2,2}((a);x;q)_2-H_{2,1}((a);x;q)_2 = xJ_{2,1}((a);x;q)_2 \text{ (by (2.1) } k = i = 2\text{)}. \tag{4.1}$$

$$H_{2,1}((a);x;q)_2 = J_{2,2}((a);x;q)_2 = J_{2,3}((a);x;q)_2 \text{ (by (2.8) } k = 2\text{)}. \tag{4.2}$$

$$J_{2,2}((a);x;q)_2 = H_{2,2}((a);xq;q)_2 + bxqH_{2,1}((a);xq;q)_2$$
$$\text{(by (2.2) } k-i=\lambda=2\text{)}. \quad (4.3)$$

$$J_{2,1}((a);x;q)_2 = (1+axq)H_{2,1}((a);xq;q)_2$$
$$\text{(by (2.2) } k=2,\ i=1,\ \lambda=2,\ \text{(2.3), and (2.4))}. \quad (4.4)$$

Now (4.2) and (4.3) imply that

$$H_{2,1}((a);x;q)_2 - bxqH_{2,1}((a);xq;q)_2 = H_{2,2}((a);xq;q)_2. \quad (4.5)$$

(4.1), (4.4), and (4.5) imply that

$$H_{2,1}((a);x;q)_2 - (bxq+1)H_{2,1}((a);xq;q)_2 = xq(1+axq^2)H_{2,1}((a);xq^2;q)_2. \quad (4.6)$$

Consequently, rewriting (4.6), we obtain

$$H_{2,1}((a);x;q)_2 = (1+bxq)H_{2,1}((a);xq;q)_2 + xq(1+axq^2)H_{2,1}((a);xq^2;q)_2. \quad (4.7)$$

Thus

$$\frac{H_{2,1}((a);x;q)_2}{H_{2,1}((a);xq;q)_2} = 1+bxq+\frac{xq(1+axq^2)}{H_{2,1}((a);xq;q)_2/H_{2,1}((a);xq^2;q)_2}. \quad (4.8)$$

By iterating (4.8) we obtain Theorem 6.

Corollary 1. (Ramanujan [(**7**) 215])

$$\frac{\Pi_\infty(-q^2,q^5)\Pi_\infty(-q^3,q^5)}{\Pi_\infty(-q,q^5)\Pi_\infty(-q^4,q^5)} = 1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots.$$

Proof. In Theorem 6, let $a_1 \to \infty$, $a_2 \to \infty$. This yields $b = a = 0$. Utilizing (4.2), (2.6), (2.9), and (2.2) for $\lambda = 0$, we find

$$H_{2,1}(\ ;1;q) = J_{2,2}(\ ;1;q) = \frac{\Pi_\infty(-q^2,q^5)\Pi_\infty(-q^3,q^5)\Pi_\infty(-q^3,q^5)}{\Pi_\infty(-q,q)};$$

$$H_{2,1}(\ ;q;q) = J_{2,1}(\ ;1;q) = \frac{\Pi_\infty(-q,q^5)\Pi_\infty(-q^4,q^5)\Pi_\infty(-q^5,q^5)}{\Pi_\infty(-q,q)}.$$

The desired result now follows directly from Theorem 6.

The next result is equivalent to one stated by Ramanujan [(**7**) xxviii] and was proved by G. N. Watson [(**13**) 236].

Corollary 2.

$$\frac{\Pi_\infty(-q^3,q^6)\Pi_\infty(-q^3,q^6)}{\Pi_\infty(-q,q^6)\Pi_\infty(-q^5,q^6)} = 1+\frac{q(1-q)}{1+}\,\frac{q^2(1-q^2)}{1+}\,\frac{q^3(1-q^3)}{1+}\cdots.$$

Proof. In Theorem 6, let $a_1 = -a_2 = q^{\frac{1}{2}}$. Hence $b = 0$, $a = -q^{-1}$. The desired result now follows as in Corollary 1.

COROLLARY 3. (Gordon [(6) 741])

$$\frac{\Pi_\infty(-q^3,q^8)\Pi_\infty(-q^5,q^8)}{\Pi_\infty(-q,q^8)\Pi_\infty(-q^7,q^8)} = 1+q+\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^6}{1+q^7+}\cdots.$$

Proof. In Theorem 6, first replace q by q^2, then set $a_1 = -q$, and let $a_2 \to \infty$. Hence $a = 0$, $b = q^{-1}$. The desired result now follows as in Corollary 1.

Our next theorem generalizes a continued fraction expansion of A. Selberg [(9) 19].

THEOREM 7. *Let* $a_1^{-1} = -a$. *Then*

$$\frac{(1-x)H_{1,\frac{3}{2}}(a_1;x^2;q^2)}{H_{1,\frac{3}{2}}(a_1;x^2q^2;q^2)} = 1+ax^2q^2+\frac{xq(1-axq^2)(1-xq)}{1+ax^2q^4+}\,\frac{xq^2(1-axq^3)(1+xq^2)}{1-ax^2q^6+}\cdots.$$

Proof. Here $\sigma_1(a_1^{-1}) = -a$.

$$H_{1,\frac{1}{2}}(a_1;x;q)(1+x^{\frac{1}{2}}) = J_{1,\frac{3}{2}}(a_1;x;q) \quad \text{(by (2.1) with } k = 1,\ i = \tfrac{1}{2}). \tag{4.9}$$

$$H_{1,\frac{3}{2}}(a_1;x;q)-H_{1,\frac{1}{2}}(a_1;x;q) = x^{\frac{1}{2}}J_{1,\frac{1}{2}}(a_1;x;q)$$
$$\text{(by (2.1) with } k = 1,\ i = \tfrac{3}{2}). \tag{4.10}$$

$$J_{1,\frac{1}{2}}(a_1;x;q) = H_{1,\frac{1}{2}}(a_1;xq;q)-ax^{\frac{1}{2}}q^{\frac{1}{2}}H_{1,\frac{3}{2}}(a_1;xq;q)$$
$$\text{(by (2.2) with } k = 1,\ i = \tfrac{1}{2}, \text{ and (2.3))}. \tag{4.11}$$

$$J_{1,\frac{3}{2}}(a_1;x;q) = H_{1,\frac{3}{2}}(a_1;xq;q)+axqH_{1,\frac{1}{2}}(a_1;xq;q)$$
$$\text{(by (2.2) with } k = 1,\ i = \tfrac{3}{2}). \tag{4.12}$$

Hence, by (4.9), (4.10), and (4.12),

$$(1+x^{\frac{1}{2}})H_{1,\frac{3}{2}}(a_1;x;q) = (1+axq)H_{1,\frac{1}{2}}(a_1;xq;q)+x^{\frac{1}{2}}q^{\frac{1}{2}}J_{1,\frac{3}{2}}(a_1;xq;q). \tag{4.13}$$

Thus, by (4.11) and (4.13),

$$(1+x^{\frac{1}{2}})H_{1,\frac{3}{2}}(a_1;x;q)$$
$$= (1+axq)H_{1,\frac{3}{2}}(a_1;xq;q)+x^{\frac{1}{2}}q^{\frac{1}{2}}(1-ax^{\frac{1}{2}}q)H_{1,\frac{3}{2}}(a_1;xq^2;q). \tag{4.14}$$

If we let $\quad f(x) = \Pi_\infty(x,q)H_{1,\frac{3}{2}}(a_1;x^2;q^2),$

then $\quad f(x) = (1+ax^2q^2)f(xq)+xq(1-axq^2)(1+xq)f(xq^2).$

Consequently

$$\frac{f(x)}{f(xq)} = 1+ax^2q^2+\frac{xq(1-axq^2)(1+xq)}{f(xq)/f(xq^2)}. \tag{4.15}$$

By iterating (4.15), we obtain Theorem 7.

5. q-series identities

Intricate q-series identities seem to be somewhat difficult to obtain by the method of q-difference equations. Although the series form of the Rogers–Ramanujan identities is relatively easily derived by the method of q-difference equations [(9) 11], Selberg deduced the corresponding identities of modulus 7 from the related q-difference equation by involved and ingenious use of certain recurrence formulae. We shall show in Theorem 8 that a much simpler, though possibly less revealing, derivation of the identities of modulus 7 may be given. In Theorem 9, we shall give a similar proof of Bailey's identities related to the modulus 9.

LEMMA 1. *Suppose, for $|q| < 1$, there exists a function $f(x)$ which is analytic at $x = 0$, and satisfies*

$$\sum_{i=0}^{s} p_i(x,q) f(xq^i) = 0, \qquad f(0) = 1 \tag{5.1}$$

(where $p_i(x,q)$ is a polynomial in x and q, $p_i(x,q) = \sum_{j=0}^{d_i} \xi_j^{(i)}(q) x^j$), and that $\sum_{i=0}^{s} \xi_0^{(i)}(q) y^i \neq 0$ for $|y| < 1$. Then $f(x)$ is the only function analytic at 0 for which (5.1) *is fulfilled.*

Proof. Suppose $g(x)$ is analytic at 0 and fulfils (5.1). Let

$$g(x) = \sum_{n=0}^{\infty} B_n(q) x^n; \qquad f(x) = \sum_{n=0}^{\infty} A_n(q) x^n.$$

Now $B_0(q) = A_0(q) = 1$, since $g(0) = f(0) = 1$. Assume $B_n(q) = A_n(q)$ for $n < N$. Then, by (5.1),

$$0 = \sum_{i=0}^{s} \sum_{j=0}^{d_i} \xi_j^{(i)}(q) x^j \sum_{n=0}^{\infty} A_n(q) x^n q^{in} = \sum_{m=0}^{\infty} x^m \sum\nolimits_1 \xi_j^{(i)}(q) A_n(q) q^{in},$$

where $\sum_1$ is over all j and n such that $j+n = m$, $0 \leqslant n$, $0 \leqslant j \leqslant d_i$, $0 \leqslant i \leqslant s$. Hence the inner sum in the last series is zero for all m, and the same is true if $A_n(q)$ is replaced by $B_n(q)$.

Thus

$$A_N(q) \sum_{i=0}^{s} \xi_0^{(i)}(q) q^{in} + \sum_{\substack{j+n=N \\ 0 \leqslant n < N,\, 0 < j \leqslant d_i \\ 0 \leqslant i \leqslant s}} \xi_j^{(i)}(q) A_n(q) q^{in} = 0. \tag{5.2}$$

But, by hypothesis, $\sum_{i=0}^{s} \xi_0^{(i)}(q) q^{in} \neq 0$. Hence

$$\begin{aligned} A_N(q) &= \Big(\sum_{i=0}^{s} \xi_0^{(i)}(q) q^{in}\Big)^{-1} \Big(-\sum\nolimits_2 \xi_j^{(i)}(q) A_n(q) q^{in}\Big) \\ &= \Big(\sum_{i=0}^{s} \xi_0^{(i)}(q) q^{in}\Big)^{-1} \Big(-\sum\nolimits_2 \xi_j^{(i)}(q) B_n(q) q^{in}\Big) \\ &= B_N(q), \end{aligned}$$

where $\sum_2$ is over all j, n, and i such that

$$j+n = N,\ 0 \leqslant n < N,\ 0 < j \leqslant d_i,\ 0 \leqslant i \leqslant s.$$

Hence by induction, $A_n(q) = B_n(q)$ for all n. Therefore $f(x) = g(x)$.

The next theorem is originally due to Rogers [(**8**) 331]; however, it was rediscovered by Selberg [(**9**) 15], and was subsequently proved by Dyson [(**4**) 35], Bailey [(**3**) 421], and Slater [(**10**) 155].

THEOREM 8.

$$\prod_{\substack{n=1\\ n\not\equiv 0,\ \pm 3 \pmod 7}}^{\infty} (1-q^n)^{-1} = \Pi_\infty(q,q) \sum_{n=0}^{\infty} \frac{q^{2n^2}}{\Pi_n(-q^2,q^2)\Pi_{2n}(q,q)}; \quad (5.3)$$

$$\prod_{\substack{n=1\\ n\not\equiv 0,\ \pm 2 \pmod 7}}^{\infty} (1-q^n)^{-1} = \Pi_\infty(q,q) \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{\Pi_n(-q^2,q^2)\Pi_{2n}(q,q)}; \quad (5.4)$$

$$\prod_{\substack{n=1\\ n\not\equiv 0,\ \pm 1 \pmod 7}}^{\infty} (1-q^n)^{-1} = \Pi_\infty(q,q) \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{\Pi_n(-q^2,q^2)\Pi_{2n+1}(q,q)}. \quad (5.5)$$

Proof. From (2.1) and (2.2), we see that if

$$Q_i(x) = J_{3,i}(0;x;q),$$

then

$$Q_3(x)-Q_2(x) = x^2q^2Q_1(xq); \quad (5.6)$$

$$Q_2(x)-Q_1(x) = xqQ_2(xq); \quad (5.7)$$

$$Q_1(x) = Q_3(xq). \quad (5.8)$$

These equations may be solved for $Q_3(x)$ to yield

$$Q_3(x)-(1+xq)Q_3(xq)-x^2q^2Q_3(xq^2)+x^3q^5Q_3(xq^3) = 0. \quad (5.9)$$

Hence by Lemma 1, (5.9) has a unique solution (since it has one solution) with $Q_3(0) = 1$. Therefore the system (5.6)-(5.8) has unique solutions with $Q_1(0) = Q_2(0) = Q_3(0) = 1$.

Consider now

$$R_3(x) = \sum_{m=0}^{\infty} \frac{x^{2m}q^{2m^2}\Pi_\infty(xq^{2m+1},q)}{\Pi_m(-q^2,q^2)}; \quad (5.10)$$

$$R_2(x) = \sum_{m=0}^{\infty} \frac{x^{2m}q^{2m^2-2m}\Pi_\infty(xq^{2m+1},q)}{\Pi_m(-q^2,q^2)}; \quad (5.11)$$

$$R_1(x) = \sum_{m=0}^{\infty} \frac{x^{2m}q^{2m^2-2m}\Pi_\infty(xq^{2m+2},q)}{\Pi_m(-q^2,q^2)}. \quad (5.12)$$

Clearly

$$R_1(x) = R_3(xq).$$

Also

$$\begin{aligned} R_3(x)-R_2(x) &= \sum_{m=0}^{\infty} \frac{x^{2m}q^{2m^2}\Pi_\infty(xq^{2m+1},q)(1-q^{2m})}{\Pi_m(-q^2,q^2)} \\ &= \sum_{m=0}^{\infty} \frac{x^{2(m+1)}q^{2(m+1)^2}\Pi_\infty(xq^{2m+3},q)}{\Pi_m(-q^2,q^2)} \\ &= x^2q^2 \sum_{m=0}^{\infty} \frac{(xq)^{2m}q^{2m^2+2m}\Pi_\infty((xq)q^{2m+2},q)}{\Pi_m(-q^2,q^2)} \\ &= x^2q^2R_1(xq). \end{aligned}$$

Finally

$$\begin{aligned} R_2(x)-R_1(x) &= \sum_{m=0}^{\infty} \frac{x^{2m}q^{2m^2+2m}\Pi_\infty(xq^{2m+2},q)(1+xq^{2m+1}-1)}{\Pi_m(-q^2,q^2)} \\ &= xq \sum_{m=0}^{\infty} \frac{(xq)^{2m}q^{2m^2+2m}\Pi_\infty((xq)q^{2m+1},q)}{\Pi_m(-q^2,q^2)} \\ &= xqR_2(xq). \end{aligned}$$

Thus we see that the functions $R_a(x)$ $(a = 1, 2, 3)$ fulfil (5.6), (5.7), and (5.8); therefore by the remark following (5.9), $R_a(x) = Q_a(x)$ $(a = 1, 2, 3)$. Taking $x = 1$, $a = 1, 2, 3$, and utilizing (2.9), we obtain (5.3), (5.4), and (5.5).

Our last theorem is originally due to Bailey [(**3**) 422]; it has subsequently been proved by L. J. Slater [(**11**) 156].

THEOREM 9.

$$\prod_{\substack{n=1 \\ n\not\equiv 0,\, \pm 4 \pmod 9}}^{\infty} (1-q^n)^{-1} = \frac{\Pi_\infty(-q^3,q^3)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{q^{3m^2}\Pi_{3m}(-q,q)}{\Pi_m(-q^3,q^3)\Pi_{2m}(-q^3,q^3)}; \tag{5.13}$$

$$\prod_{\substack{n=1 \\ n\not\equiv 0,\, \pm 2 \pmod 9}}^{\infty} (1-q^n)^{-1} = \frac{\Pi_\infty(-q^3,q^3)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{q^{3m^2+3m}\Pi_{3m}(-q,q)(1-q^{3m+2})}{\Pi_m(-q^3,q^3)\Pi_{2m+1}(-q^3,q^3)}; \tag{5.14}$$

$$\prod_{\substack{n=1 \\ n\not\equiv 0,\, \pm 1 \pmod 9}}^{\infty} (1-q^n)^{-1} = \frac{\Pi_\infty(-q^3,q^3)}{\Pi_\infty(-q,q)} \sum_{m=0}^{\infty} \frac{q^{3m^2+3m}\Pi_{3m+1}(-q,q)}{\Pi_m(-q^3,q^3)\Pi_{2m+1}(-q^3,q^3)}. \tag{5.15}$$

Proof. From (2.1) and (2.2), we see that if

$$S_i(x) = J_{4,i}(\ ;x;q),$$

then

$$S_4(x)-S_3(x) = x^3q^3S_1(xq); \tag{5.16}$$

$$S_3(x)-S_2(x) = x^2q^2S_2(xq); \tag{5.17}$$

$$S_2(x)-S_1(x) = xqS_3(xq); \tag{5.18}$$

$$S_1(x) = S_4(xq). \tag{5.19}$$

These equations may be solved for $S_4(x)$ to yield

$$S_4(x)-(1+xq)S_4(xq)-x^2q^2(1+xq+xq^2)S_4(xq^2)+ \\ +x^4q^7S_4(xq^3)-x^6q^{13}S_4(xq^4) = 0. \tag{5.20}$$

Hence by Lemma 1, (5.20) has a unique solution (since it has one) with $S_4(0) = 1$. Therefore the system (5.16)–(5.19) has unique solutions with $S_1(0) = S_2(0) = S_3(0) = S_4(0) = 1$.

Consider now

$$T_4(x) = \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2}\Pi_\infty(-x^3q^{6m+3}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+1}, q)}; \tag{5.21}$$

$$T_3(x) = \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}(1-x)\Pi_\infty(-x^3q^{6m+3}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)}; \tag{5.22}$$

$$T_2(x) = \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}(1-x^2q^{3m+2})\Pi_\infty(-x^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+1}, q)}; \tag{5.23}$$

$$T_1(x) = \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+2}, q)}. \tag{5.24}$$

Clearly

$$T_1(x) = T_4(xq).$$

Furthermore

$$\begin{aligned} T_2(x)-T_1(x) &= \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+6}, q^3)(1-x^2q^{3m+2}-1+xq^{3m+1})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+1}, q)} \\ &= \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+6}, q^3)xq^{3m+1}(1-xq)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+1}, q)} \\ &= xq\sum_{m=0}^{\infty} \frac{(xq)^{3m}q^{3m^2+3m}\Pi_\infty(-(xq)^3q^{6m+3}, q^3)(1-xq)}{\Pi_m(-q^3, q^3)\Pi_\infty(-(xq)q^{3m}, q)} \\ &= xqT_3(xq). \end{aligned}$$

Also

$$\begin{aligned} T_4(x)-T_3(x) &= \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2}\Pi_\infty(-x^3q^{6m+3}, q^3)(1-xq^{3m}-q^{3m}+xq^{3m})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)} \\ &= x^3q^3\sum_{m=0}^{\infty} \frac{(xq)^{3m}q^{3m^2+3m}\Pi_\infty(-(xq)^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-(xq)q^{3m+2}, q)} \\ &= x^3q^3T_1(xq). \end{aligned}$$

In order to verify (5.17) for the $T_a(x)$, we must make use of the following auxiliary function, $h(x)$.

$$h(x) = \sum_{m=0}^{\infty} \frac{x^{3m+4}q^{3(m+1)(m+2)}\Pi_\infty(-x^3q^{6m+9}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+3}, q)}$$

$$= \sum_{m=0}^{\infty} \frac{x^{3m+1}q^{3m(m+1)}\Pi_\infty(-x^3q^{6m+3}, q^3)(1+q^{3m})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)}.$$

Thus

$$T_3(x)-T_2(x)+h(x)$$

$$= \sum_{m=0}^{\infty} \left\{\frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+3}, q^3)(1-x)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)} - \right.$$

$$-\frac{x^{3m}q^{3m^2+3m}(1-x^2q^{3m+2})\Pi_\infty(-x^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+1}, q)} +$$

$$\left. +\frac{x^{3m+1}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+3}, q^3)(1+q^{3m})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)} \times$$

$$\times\{(1-x)(1+x^3q^{6m+3})-(1-x^2q^{3m+2})(1+xq^{3m})+x(1+q^{3m})(1+x^3q^{6m+3})\}$$

$$= \sum_{m=0}^{\infty} \frac{x^{3m}q^{3m^2+3m}\Pi_\infty(-x^3q^{6m+6}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m}, q)} x^2q^{3m+2}(1+xq^{3m})(1+xq^{3m+1})$$

$$= \sum_{m=0}^{\infty} \frac{x^2q^2x^{3m}q^{3m^2+6m}\Pi_\infty(-x^3q^{6m+9}, q^3)(1+x^3q^{6m+6})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+2}, q)}$$

$$= \sum_{m=0}^{\infty} \frac{x^2q^2x^{3m}q^{3m^2+6m}\Pi_\infty(-x^3q^{6m+9}, q^3)(1-x^2q^{3m+4}+x^2q^{3m+4}(1+xq^{3m+2}))}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+2}, q)}$$

$$= x^2q^2 \sum_{m=0}^{\infty} \frac{(xq)^{3m}q^{3m^2+3m}\Pi_\infty(-(xq)^3q^{6m+6}, q^3)(1-(xq)^2q^{3m+2})}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+2}, q)} +$$

$$+ \sum_{m=0}^{\infty} \frac{x^{3m+4}q^{3(m+1)(m+2)}\Pi_\infty(-x^3q^{6m+9}, q^3)}{\Pi_m(-q^3, q^3)\Pi_\infty(-xq^{3m+3}, q)}$$

$$= x^2q^2T_2(xq)+h(x).$$

Therefore $$T_3(x)-T_2(x) = x^2q^2T_2(xq).$$

Thus we see that the functions $T_a(x)$ $(a = 1, 2, 3, 4)$ fulfil (5.16)–(5.19); therefore by the remark following (5.20), $T_a(x) = S_a(x)$ $(a = 1, 2, 3, 4)$. Taking $x = 1$, $a = 1, 2, 4$ and utilizing (2.9), we obtain (5.13), (5.14), and (5.15).

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Remark (added in proof). Theorem 3 is a limiting case of an identity due to R. P. Agarwal, *Proc. Camb. Phil. Soc.* 49 (1953) 441–5, last equation.