

CONGRUENCES FOR THE FISHBURN NUMBERS

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ABSTRACT. The Fishburn numbers, $\xi(n)$, are defined by a formal power series expansion

$$\sum_{n=0}^{\infty} \xi(n)q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^n (1 - (1-q)^j).$$

For half of the primes p , there is a non-empty set of numbers $T(p)$ lying in $[0, p-1]$ such that if $j \in T(p)$, then for all $n \geq 0$,

$$\xi(pn + j) \equiv 0 \pmod{p}.$$

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1. INTRODUCTION

The Fishburn numbers $\xi(n)$ are defined by the formal power series

$$(1) \quad \sum_{n=0}^{\infty} \xi(n)q^n = \sum_{n=0}^{\infty} (1-q; 1-q)_n$$

where

$$(2) \quad (A; q)_n = (1-A)(1-Aq) \dots (1-Aq^{n-1}).$$

The Fishburn numbers have arisen in a wide variety of combinatorial settings. One can gain some sense of the extent of their applications in [9, Sequence A022493]. Namely, these numbers arise in such combinatorial settings as linearized chord diagrams, Stoimenow diagrams, nonisomorphic interval orders, unlabeled $(2+2)$ -free posets, and ascent sequences. They were first defined in the work of Fishburn (cf. [6, 7, 8]), and have recently found a connection with mock modular forms [4].

It turns out that the Fishburn numbers satisfy congruences reminiscent of those for the partition function $p(n)$ [2, Chapter 1]. Surprisingly, in contrast to $p(n)$, we shall see in Section 4 that there are congruences of the form $\xi(pn + b) \equiv 0 \pmod{p}$ for half of all the primes p . For example, for all $n \geq 0$,

$$(3) \quad \xi(5n + 3) \equiv \xi(5n + 4) \equiv 0 \pmod{5},$$

$$(4) \quad \xi(7n + 6) \equiv 0 \pmod{7},$$

$$(5) \quad \xi(11 + 8) \equiv \xi(11n + 9) \equiv \xi(11n + 10) \equiv 0 \pmod{11},$$

$$(6) \quad \xi(17n + 16) \equiv 0 \pmod{17}, \text{ and}$$

$$(7) \quad \xi(19n + 17) \equiv \xi(19n + 18) \equiv 0 \pmod{19}.$$

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These results all follow from a general result stated as Theorem 3.1 in Section 3. The next section is devoted to background lemmas. Theorem 3.1 is then proved in Section 3. In Section 4 we discuss an infinite family of primes p for which these congruences hold. We conclude with some open problems.

2. BACKGROUND LEMMAS

The sequence of pentagonal numbers is given by

$$(8) \quad \{n(3n-1)/2\}_{n=-\infty}^{\infty} = \{0, 1, 2, 5, 7, 12, 15, 22, \dots\}.$$

Throughout this work the symbol λ will be used to designate a pentagonal number.

In our first lemma, $f(q)$ will denote an arbitrary polynomial in $\mathbb{Z}[q]$, and p will be a fixed prime. Then we separate the terms in $f(q)$ according to the residue of the exponent modulo p . Thus,

$$(9) \quad f(q) = \sum_{i=0}^{p-1} q^i \phi_i(q^p).$$

We also suppose that for every p^{th} root of unity ζ (including $\zeta = 1$),

$$f(\zeta) = \sum_{\lambda} c_{\lambda} \zeta^{\lambda}$$

where the λ 's sum over some set of pentagonal numbers that includes 0. The c 's are thus defined to be 0 outside this prescribed set of pentagonal numbers, and the c 's are independent of the choice of ζ .

Lemma 2.1. *Under the above conditions, $\phi_j(1) = 0$ if j is not a pentagonal number.*

Proof. The assertion is not immediate because the p^{th} roots of unity are not linearly independent. In particular, if ζ is a primitive p^{th} root of unity, then

$$1 + \zeta + \zeta^2 + \dots + \zeta^{p-1} = 0.$$

However, we know that the ring of integers in $\mathbb{Q}(\zeta)$ has $1, \zeta, \zeta^2, \dots, \zeta^{p-2}$ as a basis [1, page 187]. Hence,

$$\phi_0(1)(-\zeta - \zeta^2 - \dots - \zeta^{p-1}) + \sum_{j=1}^{p-1} \zeta^j \phi_j(1) = c_0(-\zeta - \zeta^2 - \dots - \zeta^{p-1}) + \sum_{\lambda \neq 0} c_{\lambda} \zeta^{\lambda}.$$

Therefore, if $1 \leq j \leq p-1$,

$$\phi_j(1) - \phi_0(1) = \begin{cases} c_{\lambda} - c_0 & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ -c_0 & \text{otherwise} \end{cases}$$

is a linear system of $p-1$ equations in p variables $\phi_j(1)$, $0 \leq j \leq p-1$. However, the $\zeta = 1$ case adds one further equation

$$\phi_0(1) + \phi_1(1) + \dots + \phi_{p-1}(1) = \sum_{\lambda} c_{\lambda}.$$

We now have a linear system of p equations in p variables, and the determinant of the system is p . Hence, there is a unique solution which is the obvious solution

$$\phi_j(1) = \begin{cases} c_{\lambda} & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

■

In the next three lemmas, we require some variations on Leibniz's rule for taking the n^{th} derivative of a product. Each is probably in the literature, but is included here for completeness.

Lemma 2.2.

$$\left(q \frac{d}{dq}\right)^n (A(q)B(q)) = \sum_{j=1}^n q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)),$$

where the $c_{n,j}$ are the Stirling numbers of the second kind given by $c_{n,0} = c_{n,n+1} = 0$, $c_{1,1} = 1$, and $c_{n+1,j} = jc_{n,j} + c_{n,j-1}$ for $1 \leq j \leq n+1$.

Proof. The result is a tautology when $n = 1$. To pass from n to $n+1$, we note

$$\begin{aligned} \left(q \frac{d}{dq}\right)^{n+1} (A(q)B(q)) &= q \frac{d}{dq} \left(\left(q \frac{d}{dq}\right)^n (A(q)B(q)) \right) \\ &= q \frac{d}{dq} \sum_{j=1}^n q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &= q \frac{d}{dq} \sum_{j=1}^n q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &= \sum_{j=1}^n j q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &\quad + \sum_{j=1}^n q^{j+1} c_{n,j} \left(\frac{d}{dq}\right)^{j+1} (A(q)B(q)) \\ &= \sum_{j=1}^{n+1} q^j (j c_{n,j} + c_{n,j-1}) \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &= \sum_{j=1}^{n+1} q^j c_{n+1,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)). \end{aligned}$$

■

Lemma 2.3.

$$\left(\frac{d}{dt}\right)^n f(qe^t) \Big|_{t=0} = \left(q \frac{d}{dq}\right)^n f(q).$$

Proof. By Lemma 2.2 with $A(q) = f(q)$ and $B(q) = 1$, we see that

$$(10) \quad \left(q \frac{d}{dq}\right)^n f(q) = \sum_{j=1}^n q^j c_{n,j} f^{(j)}(q).$$

On the other hand, we claim

$$(11) \quad \left(\frac{d}{dt}\right)^n f(qe^t) = \sum_{j=1}^n q^j e^{jt} c_{n,j} f^{(j)}(qe^t).$$

When $n = 1$, this is just the chain rule applied to $f(qe^t)$. To pass from n to $n + 1$, we note

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{n+1} f(qe^t) &= \frac{d}{dt} \left(\frac{d}{dt}\right)^n f(qe^t) \\
&= \frac{d}{dt} \sum_{j=1}^n q^j e^{jt} c_{n,j} f^{(j)}(qe^t) \\
&= \sum_{j=1}^n j q^j e^{jt} c_{n,j} f^{(j)}(qe^t) \\
&\quad + \sum_{j=1}^n q^{j+1} e^{(j+1)t} c_{n,j} f^{(j+1)}(qe^t) \\
&= \sum_{j=1}^{n+1} (j c_{n,j} + c_{n,j-1}) q^j e^{jt} f^{(j)}(qe^t) \\
&= \sum_{j=1}^{n+1} c_{n+1,j} q^j e^{jt} f^{(j)}(qe^t).
\end{aligned}$$

Comparing (11) with $t = 0$ to (10), we see that our lemma is established. \blacksquare

We now turn to the generating function for the Fishburn numbers as given by Zagier [10, page 946]. Namely,

$$(12) \quad F(1 - q) = \sum_{n=0}^{\infty} \xi(n) q^n = \sum_{n=0}^{\infty} (1 - q; 1 - q)_n.$$

To facilitate the study, we concentrate on

$$(13) \quad F(q) = \sum_{n=0}^{\infty} (q; q)_n$$

and

$$(14) \quad F(q, N) = \sum_{n=0}^N (q; q)_n = \sum_{i=0}^{p-1} q^i A_p(N, i, q^p),$$

where $A_p(N, i, q^p)$ is a polynomial in q^p . We note that if ζ is a p^{th} root of unity

$$(15) \quad F(\zeta) = F(\zeta, m) = F(\zeta, p - 1)$$

for all $m \geq p$. Furthermore,

$$(16) \quad \left(q \frac{d}{dq} \right)^r F(q) \Big|_{q=\zeta} = \left(q \frac{d}{dq} \right)^r F(q, m) \Big|_{q=\zeta} = \left(q \frac{d}{dq} \right)^r F(q, (r+1)p - 1) \Big|_{q=\zeta}$$

for all $m \geq (r+1)p$ because $(1 - q^p)^{r+1}$ divides $(q; q)_j$ for all $j \geq (r+1)p$.

Similarly, for all $m \geq (r+1)p$,

$$(17) \quad F^{(r)}(q) \Big|_{q=\zeta} = F^{(r)}(q, m) \Big|_{q=\zeta} = F^{(r)}(q, (r+1)p - 1) \Big|_{q=\zeta}.$$

In the next lemma, we require a Stirling-like array of numbers $C_{N,i,j}(p)$ given by $C_{N,i,0}(p) = i^N$ ($C_{0,0,0}(p) = 1$), $C_{N,i,N+1}(p) = 0$, and for $1 \leq j \leq N$,

$$(18) \quad C_{N+1,i,j}(p) = (i + jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p).$$

Lemma 2.4.

$$\left(q \frac{d}{dq}\right)^N F(q, n) = \sum_{j=0}^N \sum_{i=0}^{p-1} C_{N,i,j}(p) q^{i+jp} A_p^{(j)}(n, i, q^p).$$

Proof. In light of the fact that $C_{0,i,0}(p) = 1$ for all i , the $N = 0$ assertion is

$$F(q, n) = \sum_{i=0}^{p-1} q^i A_p(n, i, q^p),$$

which is just the definition of the A 's given in (14). To pass from N to $N + 1$, we note

$$\begin{aligned} \left(q \frac{d}{dq}\right)^{N+1} F(q, n) &= q \frac{d}{dq} \sum_{j=0}^N \sum_{i=0}^{p-1} C_{N,i,j}(p) q^{i+jp} A_p^{(j)}(n, i, q^p) \\ &= \sum_{j=0}^N \sum_{i=0}^{p-1} C_{N,i,j}(p) (i + jp) q^{i+jp} A_p^{(j)}(n, i, q^p) \\ &\quad + \sum_{j=0}^N \sum_{i=0}^{p-1} C_{N,i,j}(p) q^{i+jp} p q^p A_p^{(j+1)}(n, i, q^p) \\ &= \sum_{j=0}^{N+1} \sum_{i=0}^{p-1} ((i + jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p)) q^{i+jp} A_p^{(j)}(n, i, q^p) \\ &= \sum_{j=0}^{N+1} \sum_{i=0}^{p-1} C_{N+1,i,j}(p) q^{i+jp} A_p^{(j)}(n, i, q^p). \end{aligned}$$

■

We now define, for any positive integer p , two special sets of integers:

$$(19) \quad S(p) = \{j \mid 0 \leq j \leq p-1 \text{ such that } n(3n-1)/2 \equiv j \pmod{p} \text{ for some } n\}$$

and

$$(20) \quad T(p) = \{k \mid 0 \leq k \leq p-1 \text{ such that } k \text{ is larger than every element of } S(p)\}.$$

For example, for $p = 11$, we have

$$S(11) = \{0, 1, 2, 4, 5, 7\} \quad \text{and} \quad T(11) = \{8, 9, 10\}.$$

Lemma 2.5. *If $i \notin S(p)$, then*

$$A_p(pn - 1, i, q) = (1 - q)^n \alpha_p(n, i, q)$$

where the $\alpha_p(n, i, q)$ are polynomials in $\mathbb{Z}[q]$.

Proof. This result is equivalent to the assertion that for $0 \leq j < n$,

$$A_p^{(j)}(pn - 1, i, 1) = 0,$$

and by (17) we need only prove for $j \geq 0$,

$$(21) \quad A_p^{(j)}((j+1)p - 1, i, 1) = 0$$

because $n \geq (j+1)$.

We proceed to prove (21) by induction on j . When $j = 0$, we only need show that if $i \notin S(p)$,

$$A_p(p - 1, i, 1) = 0.$$

Following [10, Section 5], we define (where ζ is now an N^{th} root of unity)

$$(22) \quad \begin{aligned} F(\zeta e^t) &= \sum_{n=0}^{\infty} \frac{b_n(\zeta) t^n}{n!} \\ &= e^{t/24} \sum_{n=0}^{\infty} \frac{c_n(\zeta) t^n}{24^n n!} \\ &= \sum_{M=0}^{\infty} \frac{t^M}{24^M M!} \sum_{n=0}^M \binom{M}{n} c_n(\zeta), \end{aligned}$$

where we have replaced Zagier's ξ with ζ to avoid confusion with $\xi(n)$. In [10, Section 5], we see that

$$(23) \quad c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n+2} \sum_{m=1}^{N/2} \chi(m) \zeta^{(m^2-1)/24} B_{2n+2} \left(\frac{m}{N} \right),$$

where the B 's are Bernoulli polynomials and $\chi(m) = \left(\frac{12}{m} \right)$. Note that the only non-zero terms in the sum in (23) have

$$(24) \quad \zeta^{((6m \pm 1)^2 - 1)/24} \chi(6m \pm 1) = (-1)^m \zeta^{m(3m \pm 1)/2},$$

i.e., $c_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number. Hence, by (22) we see that $b_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number.

Hence, if ζ is now a p^{th} root of unity,

$$\begin{aligned} F(\zeta) &= F(\zeta, p-1) \\ &= b_0(\zeta) \\ &= \sum_{\lambda} c_{\lambda} \zeta^{\lambda}, \end{aligned}$$

where the sum over λ is restricted to a subset of the pentagonal numbers. On the other hand,

$$\begin{aligned} F(\zeta) &= F(\zeta, p-1) \\ &= \sum_{i=0}^{p-1} \zeta^i A_p(p-1, i, 1). \end{aligned}$$

Hence, by Lemma 2.1, for $i \notin S(p)$,

$$A_p(p-1, i, 1) = 0$$

which is (21) when $j = 0$. Now let us assume that

$$(25) \quad A_p^{(j)}(p(j+1)-1, i, 1) = 0$$

for $0 \leq j < \nu < n$. By Lemma 2.4,

$$(26) \quad \left(q \frac{d}{dq}\right)^\nu F(q, p(\nu+1)-1) = \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu, i, j}(p) \zeta^i A_p^{(j)}(p(\nu+1)-1, i, 1).$$

But for $j < \nu$,

$$A_p^{(j)}(p(\nu+1), i, 1) = A_p^{(j)}(p(j+1)-1, i, 1) = 0.$$

Hence the only terms in the sum in (26) where ζ is raised to a non-pentagonal power, i , arise from the terms with $j = \nu$, namely

$$(27) \quad C_{\nu, i, \nu}(p) \zeta^i A_p^{(\nu)}(p(\nu+1)-1, i, 1),$$

and we note that $C_{\nu, i, \nu}(p) \neq 0$.

Applying Lemma 2.3 to the left side of (26), we see that by (22)

$$(28) \quad \begin{aligned} b_\nu(\zeta) &= \left(q \frac{d}{dq}\right)^\nu F(q) \Big|_{q=\zeta} \\ &= \left(q \frac{d}{dq}\right)^\nu F(q, (\nu+1)p-1) \\ &= \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu, i, j}(p) \zeta^i A_p^{(j)}(p(\nu+1)-1, i, 1). \end{aligned}$$

Recall that $b_\nu(\zeta)$ is a linear combination of powers of ζ where the exponents are pentagonal numbers. Hence the expression given in (27) must be zero by Lemma 2.1. Therefore,

$$A_p^{(\nu)}(p(\nu+1)-1, i, 1) = 0,$$

and this proves (21) and thus proves Lemma 2.5. ■

3. THE MAIN THEOREM

We recall from (12) that

$$\begin{aligned} \sum_{n=0}^{\infty} \xi(n) q^n &= \sum_{j=0}^{\infty} (1-q; 1-q)_j \\ &= 1 + \sum_{j=1}^{\infty} \prod_{i=1}^j \sum_{h=1}^i (-1)^{h-1} q^h \binom{i}{h} \\ &= 1 + \sum_{j=1}^{\infty} (q^j + O(q^{j+1})). \end{aligned}$$

Hence,

$$(29) \quad \sum_{n=0}^{\infty} \xi(n) q^n = F(1-q, N) + O(q^{N+1}).$$

We are now in a position to state and prove the main theorem of this paper.

Theorem 3.1. *If p is a prime and $i \in T(p)$ (as defined in (20)), then for all $n \geq 0$,*

$$\xi(pn + i) \equiv 0 \pmod{p}.$$

Remark 3.2. *Congruences (3)–(7) are the cases $p = 5, 7, 11, 17$ and 19 of Theorem 3.1.*

Proof. We begin with a simple observation derived from Lucas's theorem for the congruence class of binomial coefficients modulo p [5, page 271]. Namely if π is any integer congruent to a pentagonal number modulo p , and $i \in T(p)$, then

$$(30) \quad \binom{\pi}{i} \equiv 0 \pmod{p},$$

because the final digit in the p -ary expansion of π is smaller than i because i is in $T(p)$.

Now by Lemma 2.5, we may write

$$\begin{aligned} F(q, pn - 1) &= \sum_{i=0}^{p-1} q^i A_p(pn - 1, i, q^p) \\ &= \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} q^i A_p(pn - 1, i, q^p) + \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} q^i (1 - q^p)^n \alpha_p(n, i, q^p). \end{aligned}$$

So

$$\begin{aligned} F(1 - q, pn - 1) &= \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} (1 - q)^i A_p(pn - 1, i, (1 - q)^p) \\ &\quad + \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} (1 - q)^i (1 - (1 - q)^p)^n \alpha_p(n, i, (1 - q)^p) \\ &:= \Sigma_1 + \Sigma_2. \end{aligned}$$

Now modulo p ,

$$\begin{aligned} \Sigma_2 &\equiv \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} (1 - q)^i q^{pn} \alpha_p(n, i, 1) \\ &= O(q^{pn}). \end{aligned}$$

Therefore, modulo p ,

$$F(1 - q, pn - 1) \equiv \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} (1 - q)^i A_p(pn - 1, i, 1 - q^p) \pmod{p}.$$

Let us look at the terms in this sum where q is raised to a power that is congruent to an element of $T(p)$. Such a term must arise from the expansion of some $(1 - q)^i$ where $i \in S(p)$ because $A_p(pn - 1, 1, 1 - q^p)$ is a polynomial in q^p .

By (30) all such terms have a coefficient congruent to 0 modulo p . Therefore, every term q^j in $F(1 - q, pn - 1)$ where j is congruent to an element of $T(p)$ must have a coefficient congruent to 0 modulo p .

To conclude the proof, we let $n \rightarrow \infty$. ■

4. AN INFINITE SET OF PRIMES WITH CONGRUENCES

At this stage, one might ask whether one can identify an infinite set of primes p for which congruences such as those described in Theorem 3.1 are found. The answer to this question can be answered affirmatively.

Theorem 4.1. *Let $R = \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$. (The elements of R are those numbers r , $0 < r < 23$, such that $\left(\frac{r}{23}\right) = -1$.) Let p be a prime of the form $p = 23k + r$ for some nonnegative integer k and some $r \in R$. Then $T(p)$ is not empty, i.e., at least one congruence such as those described in Theorem 3.1 must hold modulo p .*

Remark 4.2. *From the Prime Number Theorem for primes in arithmetic progression, we see that, asymptotically, $T(p)$ is not empty for half of the primes and $T(p)$ equals the empty set for half of the primes.*

Proof. Assume p is a prime for which $T(p)$ is empty. That means there is a pentagonal number which is congruent to -1 modulo p . Then $n(3n-1)/2 \equiv -1 \pmod{p}$ for some integer n . By completing the square we then obtain $(6n-1)^2 \equiv -23 \pmod{p}$. Thus, by contrapositive, if we know that -23 is a quadratic nonresidue modulo p , then we know that such a pentagonal number does not exist (which means $T(p)$ is not empty).

Thus, if $\left(\frac{-23}{p}\right) = -1$, then $T(p)$ is not empty. But thanks to properties of the Legendre symbol, we know

$$\begin{aligned} \left(\frac{-23}{p}\right) &= \left(\frac{-1}{p}\right) \left(\frac{23}{p}\right) \\ &= (-1)^{\frac{p-1}{2}} (-1)^{\frac{23-1}{2} \frac{p-1}{2}} \left(\frac{p}{23}\right) \text{ by quadratic reciprocity} \\ &= (-1)^{\frac{12(p-1)}{2}} \left(\frac{p}{23}\right) \text{ since } p = 23k + r \\ &= \left(\frac{r}{23}\right) \end{aligned}$$

and we want this value to be -1 . The theorem then follows by the nature of the construction of R . ■

Thus, we clearly have infinitely many primes p for which the Fishburn numbers will exhibit at least one congruence modulo p .

5. CONCLUSION

There are many natural open questions that could be answered at this point.

- First, we believe that Theorem 3.1 lists all the congruences of the form $\xi(pn+b) \equiv 0 \pmod{p}$, but we have not proved this at this time.
- Numerical evidence seems to indicate that Theorem 3.1 can be strengthened. Namely, for certain values of $j > 1$ and certain primes p , it appears that

$$\xi(p^j n + b) \equiv 0 \pmod{p^j}$$

for certain values b and all n .

- Numerical evidence suggests that Lemma 2.5 could be strengthened as follows: If $i \notin S(p)$, then

$$A_p(pn - 1, i, q) = (q; q)_n \beta_p(n, i, q)$$

for some polynomial $\beta_p(n, i, q)$. That is to say, in Lemma 5, it was proved that $(1 - q)^n$ divides $A_p(pn - 1, i, q)$; it appears that the factor $(1 - q)^n$ can be strengthened to $(q; q)_n$.

- With an eye towards the recent work of Andrews and Jelínek [3], consider the power series given by

$$\sum_{n=0}^{\infty} a(n)q^n \quad := \quad \sum_{n=0}^{\infty} \left(\frac{1}{1-q}, \frac{1}{1-q} \right)_n$$

which begins

$$1 - q + q^2 - 2q^3 + 5q^4 - 16q^5 + 61q^6 - 271q^7 + 1372q^8 - 7795q^9 + \dots$$

We conjecture that, for all $n \geq 0$, $a(5n + 4) \equiv 0 \pmod{5}$.

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