

Lecture 06:

Single-component model universes

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1 Matter-only and Radiation-only universes

- In the previous lecture, we solved the F.E. for our first model universe: an empty universe.
- An empty universe is the easiest to solve, but demonstrates how we use the F.E. to obtain an expression for the scale factor, $a(t)$, and then how we use that expression to determine proper distances.
- We will now turn to the next-easiest universes to solve: single component model universes.
- For now, we'll only consider flat (i.e., $\kappa = 0$) universes, but you should at least be aware of what we'd need to solve for a non-flat model universe.
- For a flat universe, the F.E. becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) \quad (1)$$

or:

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \varepsilon(t) a(t)^2 \quad (2)$$

- From Lecture 5, we know that, from the Fluid Equation, we can write ε in terms of a :

$$\varepsilon(a) = \varepsilon_0 a(t)^{-3(1+\omega)} \quad (3)$$

- And substituting this into the F.E. gives:

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \varepsilon_0 a(t)^{-3(1+\omega)} a(t)^2 \quad (4)$$

- And since:

$$a(t)^{-3(1+\omega)} a(t)^2 = a(t)^{-3(1+\omega)+2} = a(t)^{-3-3\omega+2} = a(t)^{-1-3\omega} \quad (5)$$

we get:

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \varepsilon_0 a(t)^{(-1-3\omega)} \quad (6)$$

- To solve this, we'll take a guess that $a \propto t^q$, i.e., that the expression for $a(t)$ is some kind of power-law of t .

- In that case, the LHS of Eq. 6 becomes, via differentiation wrt. time:

$$(qt^{q-1})^2 = q^2 t^{2q-2} \quad (7)$$

- And the RHS becomes:

$$\frac{8\pi G}{3c^2} \varepsilon_0 t^{(-1-3\omega)q} \quad (8)$$

giving:

$$q^2 t^{2q-2} = \frac{8\pi G}{3c^2} \varepsilon_0 t^{(-1-3\omega)q} \quad (9)$$

- For this equation to *always* hold true, then the exponents of t must be equal, i.e.,:

$$2q - 2 = (-1 - 3\omega)q \quad (10)$$

- Which we can rearrange to get q :

$$q = \frac{2}{3 + 3\omega} \quad (11)$$

- Substituting this into $a \propto t^q$ gives:

$$a \propto t^{\frac{2}{3+3\omega}} \quad (12)$$

or,

$$a = Ct^{\frac{2}{3+3\omega}} \quad (13)$$

- We find the constant of proportionality, C , by using $a(t_0) = 1$:

$$a(t_0) = Ct_0^{\frac{2}{3+3\omega}} = 1 \quad (14)$$

so

$$C = t_0^{\frac{-2}{3+3\omega}} \quad (15)$$

and Eq. 13 becomes:

$$a = t_0^{\frac{-2}{3+3\omega}} t^{\frac{2}{3+3\omega}} = \left(\frac{t}{t_0} \right)^{\frac{2}{3+3\omega}} \quad (16)$$

- t_0 is then found by substituting Eq. 16 back into the F.E., to give:

$$t_0 = \frac{1}{1 + \omega} \left(\frac{c^2}{6\pi G \varepsilon_0} \right)^{1/2} \quad (17)$$

it's a good idea for you to be able to demonstrate this yourself.

1.1 A proper distances

- Now that we've got a general expression for $a(t)$, we can use it to obtain a general expression for proper distance. Recall:

$$d_p(t_{\text{ob}}) = c \int_{t_{\text{em}}}^{t_{\text{ob}}} \frac{dt}{a(t)} = ct_0^{2/(3+3\omega)} \int_{t_{\text{em}}}^{t_{\text{ob}}} \frac{dt}{t^{-2/(3+3\omega)}} \quad (18)$$

- To perform this integral, we'll define $m = -2/(3 + 3\omega)$:

$$d_p(t_{\text{ob}}) = ct_0^{-m} \int_{t_{\text{em}}}^{t_{\text{ob}}} t^m dt = \frac{ct_0^{-m}}{m+1} (t_{\text{ob}}^{m+1} - t_{\text{em}}^{m+1}) \quad (19)$$

- Taking $t_{\text{ob}} = t_0$ (i.e., we're observing now), and taking a t_0^{m+1} out of the bracket, we get:

$$d_p(t_{\text{ob}}) = \frac{ct_0^{-m}}{m+1} t_0^{m+1} (1 - (t_{\text{em}}/t_0)^{m+1}) = \frac{ct_0}{m+1} (1 - (t_{\text{em}}/t_0)^{m+1}) \quad (20)$$

where

$$m+1 = \frac{-2}{3(1+\omega)} + 1 = \frac{-2+3+3\omega}{3(1+\omega)} = \frac{1+3\omega}{3(1+\omega)} \quad (21)$$

giving:

$$d_p(t_{\text{ob}}) = ct_0 \frac{3(1+\omega)}{1+3\omega} (1 - (t_{\text{em}}/t_0)^{(1+3\omega)/(3+3\omega)}) \quad (22)$$

- But, we don't typically know t_{em} . Instead, we measure redshift, z .
- We can use the relationship $1+z = 1/a(t_{\text{em}})$ to determine proper distance from redshift:

$$1+z = \frac{1}{a(t_{\text{em}})} = \left(\frac{t_{\text{em}}}{t_0} \right)^{-2/(3+3\omega)} = \left(\frac{t_{\text{em}}}{t_0} \right)^m \quad (23)$$

so

$$\frac{t_{\text{em}}}{t_0} = (1+z)^{1/m} \quad (24)$$

giving:

$$d_p(t_{\text{ob}}) = \frac{ct_0}{m+1} (1 - (1+z)^{(m+1)/m}) \quad (25)$$

or

$$d_p(t_{\text{ob}}) = ct_0 \frac{3(1+\omega)}{1+3\omega} (1 - (1+z)^{(-1+3\omega)/2}) \quad (26)$$

and, using Eq. 17 to write t_0 in terms of H_0 (you should attempt to do this yourself, by using the H_0 version of the F.E.), we get:

$$d_p(t_{\text{ob}}) = \frac{c}{H_0} \frac{2}{1+3\omega} (1 - (1+z)^{(-1+3\omega)/2}) \quad (27)$$

- Finally, the proper distance at the time of *emission* can be calculated by considering that - in an expanding universe - the universe was a factor of $1+z = 1/a(t_{\text{em}})$ *smaller* when the light was emitted, giving:

$$d_p(t_{\text{em}}) = \frac{d_p(t_{\text{ob}})}{1+z} \quad (28)$$