Lecture 3:
The Friedmann Equation

Dr. James Mullaney

February 10, 2018

1 The Robertson Walker metric

Slide 2

- In the last lecture, we saw how:

$$d_p(t_0) = a(t_0) \int_0^r dr = a(t_0)r = r \tag{1}$$

- But, how do we know r?
- We use the RW metric again. Over its travels through the universe from its emitting galaxy to us, a photon crosses a load of dr's as $a(t)$ changes. Just as when relating redshift to $a(t)$, we can say:

$$a(t)dr = cdt \tag{2}$$

- Giving:

$$dr = \frac{cdt}{a(t)} \tag{3}$$

- Integrating the LHS between 0 and r, corresponding to $t_{\rm em}$ to $t_{\rm ob}$ on the RHS gives:

$$r = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)} \tag{4}$$

- Meaning (from Eq. 1):

$$d_p(t_0) = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)} \tag{5}$$

Slide 3

- Saw in lecture 2 that an isotropic, homogeneous Universe can be fully described by just three numbers: κ, R_0, $a(t)$.
- $a(t)$ is paricularly important: it tells us how the Universe expands and contracts over time.
- $a(t)$ also enables us to relate redshifts (which are easily measured) to distances (which are much more difficult to measure).
- However, let's first focus on curvature, described by κ and R_0.

1.1 What is curvature in the context of a universe?

Slide 4

- Curvature in a universe would manifest itself in terms of the perceived sizes of distant objects.
- The *perceived* size of an object is the angular size it subtends.
- As such, it is convenient to think in terms of a triangle, with the object at one end, and the angular size at the other.
- In a flat universe, the angles of this triangle all add up to 180 degrees, and the objects subtends the angle that we have come to expect for flat geometry.
- In a negatively curved universe, the interior angles of the triangle add up to less than 180 degrees, and the object subtends a *smaller* angle that we would expect. This would be witnessed as distant galaxies appearing disproportionately smaller than nearby ones.
- By contrast, in a negatively curved universe, the interior angles of the triangle add up to more than 180 degrees, and the object subtends a *larger* angle that we would expect. This would be witnessed as distant galaxies appearing disproportionately larger than nearby ones.
- Such disproportionately large or small distant galaxies are not seen when we look to higher and higher distances (i.e., redshifts), so we can conclude that *if* the Universe is curved, then its radius of curvature, R_0 is much larger than the size of the observable Universe.

2 Relating curvature to content

- General relativity tells us that a universe's curvature is dictated by its content (whether mass or energy, since they are one and the same thing in relativity).
- The *Field Equation* links the two - it tells spacetime ($G_{\mu,\nu}$) how to curve in the presence of stress-energy ($T_{\mu,\nu}$).
- Unfortunately, both $G_{\mu,\nu}$ and $T_{\mu,\nu}$ are 4×4 tensors (i.e., matrices), and the equation as a whole represents ten non-linear second-order differential equations!
- Thus, in general, it can be extremely difficult to solve for $G_{\mu,\nu}$.

  Slide 6
- However, on large scales, we can make some sweeping simplifications.
- On very large scales, we can ignore the "clumpy" nature of the Universe and instead describe it as being filled with a uniform (i.e., homogeneous), isotropic gas or pressure $P(t)$ and (energy) density $\varepsilon(t)$.
- In such a case, then $T_{\mu,\nu}$ only depents on $P(t)$ and $\varepsilon(t)$, and the metric is given by the Robertson Walker metric.
- Our goal, therefore, is simply to relate $a(t)$, κ, and R_0 to $P(t)$ and $\varepsilon(t)$.

3 The Friedmann Equation

- To relate $a(t)$, κ, and R_0 to $P(t)$ and $\varepsilon(t)$, we'll consider the gravitational influence of the aforementioned perfect gas.
- The full General Relativistic approach to this is beyond the scope of this course, so we'll instead consider the Newtonian equivalent, which gives a good sense of the physics involved.
- We'll start with considering a large sphere of the universe, containing a perfect gas with energy density $\varepsilon(t)$.
- Since spacetime itself is affected by energy density, the surface of this sphere will expand or contract according to the gravitaional influence of the perfect gas.
- This expnasion/contraction is given by Newton's law of gravitation:

$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2} \tag{6}$$

- To solve this to get $R_s(t)$ is a bit tricky, since R_s is a function of time, so we can't immediately put all R_s terms on side and all t terms on the other.
- So, we have to be a bit clever. First, we'll multiply both sides by $\frac{dR_s}{dt}$:

$$\frac{d^2R_s}{dt^2}\frac{dR_s}{dt} = -\frac{GM_s}{R_s(t)^2}\frac{dR_s}{dt} \tag{7}$$

- How has that helped us to integrate this function to solve for R_s?
- First, for the LHS of Eq. 7, consider the product rule:

$$\frac{d}{dx}\left(\frac{dy}{dx}\frac{dy}{dx}\right) = \frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d^2y}{dx^2} = 2\frac{dy}{dx}\frac{d^2y}{dx^2} \tag{8}$$

- By comparing the RHS of Eq. 8 to the LHS of Eq. 7, we can see that:

$$\int \frac{d^2R_s}{dt^2}\frac{dR_s}{dt}dt = \frac{1}{2}\frac{dR_s}{dt}\frac{dR_s}{dt} + C = \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 + C \tag{9}$$

- So now, Eq. 7 has become:

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 + C = -\int \frac{GM_s}{R_s^2}\frac{dR_s}{dt}dt \tag{10}$$

  where C is a constant of integration.
- For the RHS of Eq. 10, we consider the chain rule:

$$\frac{d\frac{GM_s}{R_s}}{dt} = \frac{d\frac{GM_s}{R_s}}{dR_s}\frac{dR_s}{dt} = -\frac{GM_s}{R_s^2}\frac{dR_s}{dt} \tag{11}$$

- Meaning:

$$-\int \frac{GM_s}{R_s^2}\frac{dR_s}{dt}dt = \int \frac{d\frac{GM_s}{R_s}}{dt}dt = \frac{GM_s}{R_s} + U \tag{12}$$

where U is a constant of integration, which we can combine with C to give:

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s} + U \tag{13}$$

- Since the mass, M_s, within the sphere is constant (there's no net flow in or out of the sphere), we can say:

$$M_s = \frac{4}{3}\pi R_s(t)^3\rho(t) \tag{14}$$

and we can also say that $R_s = a(t)r_s$, where $a(t)$ is the scale factor and r is the radius of the sphere in spherical co-moving coordinates.

- Substituting for M_s and R_s (and since r_s is constant with time) gives:

$$\frac{1}{2}\left(r_s\dot{a}\right)^2 = \frac{4}{3}\pi\frac{Ga(t)^3r_s^3}{a(t)r_s} + U \tag{15}$$

- where $\dot{a} = da/dt$. When rearranged, this becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2} \tag{16}$$

- which is the Newtonian form of the Friedmann Equation.
- By contrast, the full GR Friedmann equation is given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\varepsilon(t)}{c^2} - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \tag{17}$$

- There are clear similarities between the two. Notice especially that mass density, $\rho(t)$, has become energy density, $\varepsilon(t)/c^2$, in the relativistic form of the Friedmann Equation. This is a direct result of the equivalence of mass and energy within relativity: $E = mc^2$.
- And the second term becomes associated with the curvature of the universe (κ and R_0).