Lecture 05:
Solving the Friedmann Equation II:
Our first model universe

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1 Three equations

- The Friedmann Equation (F.E.) gives us the means to determine how, given a set energy densities, the scale factor changes over the history (and future) of the universe:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0}\frac{1}{a(t)} \tag{1}$$

- Since the scale factor quite literally describes the expansion and contraction of the universe, the F.E. provides the means to work out the dynamic evolution of the universe.
- What we now need to do is solve the F.E. to obtain an expression for $a(t)$ for a given set of energy densities.
- However, in its standalone form, the F.E. is not enough to work out how the universe expands or contracts.
- We also need expressions for the various energy densities, $\varepsilon(t)$ (for matter, radiation and dark energy) which, from the First Law of Thermodynamics, are given by:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \tag{2}$$

- While the pressure, P, is given by the Equation of State:

$$P = \varepsilon\omega \tag{3}$$

- Of course, things are slightly complicated by the fact that we've got more than one type of energy density.
- Thankfully, however, the energy densities can be added linearly, meaning the total energy density is given by:

$$\varepsilon_{\mathrm{Tot}} = \sum_i \varepsilon_i = \varepsilon_{\mathrm{m}} + \varepsilon_{\mathrm{p}} + \varepsilon_{\mathrm{d}} \tag{4}$$

  and the total pressure is given by:

$$P_{\mathrm{Tot}} = \sum_i P_i = \sum_i \varepsilon_i\omega_i = \varepsilon_{\mathrm{m}}\omega_{\mathrm{m}} + \varepsilon_{\mathrm{p}}\omega_{\mathrm{p}} + \varepsilon_{\mathrm{d}}\omega_{\mathrm{d}} \tag{5}$$

- As such, the Fluid Equation must hold for all three types of energy density separately:

$$\dot{\varepsilon_i} + 3\frac{\dot{a}}{a}(\varepsilon_i + P_i) = 0 \tag{6}$$

2 The evolving energy densities

- To solve the F.E. we require expressions for $\varepsilon(t)$ for the various types of energy density.
- But, what would be *even more useful* would be to have equivalent espressions for ε in terms of scale factor, a. In other words $\varepsilon(a)$.
- This would tell us how energy density changes as the universe expands or contracts.
- For this, we can write the Fluid Equation as:

$$\frac{d\varepsilon_i}{dt} + 3\frac{da}{dt}\frac{1}{a}\varepsilon_i(1+\omega_i) = 0 \tag{7}$$

- Cancellling the dts and dividing both sides by ε gives:

$$\frac{d\varepsilon_i}{\varepsilon_i} + 3\frac{da}{a}(1+\omega_i) = 0 \tag{8}$$

- Since ω is independent of time and scale factor, integrating then gives:

$$\ln(\varepsilon_i) = -3(1+\omega_i)\ln(a) + b \tag{9}$$

  where b is a constant of integration. Equivalently:

$$\varepsilon_i(a) = Ba^{-3(1+\omega_i)} \tag{10}$$

  where $B = e^b$.

- To determine B, we recall that the current scale factor is defined to be $a(t_0) = 1$, and the current energy density is $\varepsilon_{i,0}$, so:

$$\varepsilon_{i,0} = B \times 1^{-3(1+\omega_i)} \tag{11}$$

  meaning $B = \varepsilon_{i,0}$, and

$$\varepsilon_i(a) = \varepsilon_{i,0}a^{-3(1+\omega_i)} \tag{12}$$

- Substituting the values for ω given in the previous lecture gives:
  - Matter: $\omega = 0$, giving $\varepsilon_{\mathrm{m}} = \varepsilon_{\mathrm{m},0}a^{-3} = \varepsilon_{\mathrm{m},0}/a^3$
  - Radiation: $\omega = 1/3$, giving $\varepsilon_{\mathrm{p}} = \varepsilon_{\mathrm{p},0}a^{-4} = \varepsilon_{\mathrm{p},0}/a^4$
  - Dark Energy: $\omega = -1$, giving $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{d},0}a^0 = \varepsilon_{\mathrm{m},0}$
- The first (i.e., matter) makes sense: as the universe expands, matter gets diluted as the volume of the universe increases.

- The second (i.e., radiation) at first doesn't makes sense. Why would the energy density of photons decrease faster that that due to volume dilution? It's because as well as volume dilution, the wavelengths of the photons expand as the universe expands, meaning each indivual photon's energy also falls as the scale factor increases (due to $E = hf$).

- But the last one (i.e., Dark Energy) makes least sense of all – the energy density of Dark Energy is constant with respect to the scale factor. This means that the Dark Energy density does not get diluted as the universe expands. Each "new" cubic meter of the universe is "produced" with its own allocation of Dark Energy!

3 Solving our first model universe: A universe containing nothing!

- So, we've got everything we need to determine how the universe expands or contracts with time (encapsulated in $a(t)$) given a set of energy densities, $\varepsilon(t)$.

- Doing this for the real Universe involves solving:

$$\int_0^a \frac{da}{[\Omega_{p,0}/a^2 + \Omega_{m,0}/a + \Omega_{d,0}a^2 + (1 - \Omega_0)]^{1/2}} = H_0 t \tag{13}$$

- But this is too tricky for our first attempt at a universe (indeed, it can't be solved analytically), so we'll desmonstrate the principles using something much simpler...

- A model universe containing no energy density.

- In this case $\varepsilon(t) = 0$, meaning the F.E. is:

$$\frac{\dot{a}}{a} = \sqrt{\frac{-\kappa c^2}{R_0}\frac{1}{a^2}} \tag{14}$$

- Clearly, the flat, $\kappa = 0$ solution is trivial, with a solution of $\dot{a} = 0$. This would be an entirely static, infinite-age universe, neither expanding nor contracting.

- But, more interestingly are the non-flat empty universes, where $\kappa \neq 0$.

- The positively curved solution is unphysical, since it give an imaginary expression for $\dot{a}$.

- But the negatively curved, $\kappa = -1$ solution is viable. Cancelling the a^2 on both sides of Eq. 14, and taking $\kappa = -1$ gives:

$$\frac{da}{dt} = \pm\frac{c}{R_0} \tag{15}$$

- Integrating both sides, and taking the postive solution gives:

$$a(t) = \frac{ct}{R_0} \tag{16}$$

- Using the definition that t_0 corresponds to *now*, and $a(t_0) = 1$, we get:

$$1 = \frac{ct_0}{R_0} \tag{17}$$

or

$$t_0 = \frac{R_0}{c} \tag{18}$$

- And substituting back into Eq. 16 gives:

$$a(t) = \frac{t}{t_0} \tag{19}$$

- And, since $H_0 = \dot{a}$ (again, because $a = 1$ when evaluated at t_0), then $H_0 = 1/t_0$.
- Since the rate of expansion is constant (there's no energy density to either speed up or slow down the expansion), the age of the univerese is *exactly* the inverse of the Hubble constant.

3.1 From $a(t)$ to proper distances

- With an expression for $a(t)$, we can now attempt to determine the current proper distance, $d_p(t_{\rm ob})$, between two points in this empty universe using (last seen in Lecture 2):

$$d_p(t_{\rm ob}) = c\int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)} \tag{20}$$

- Substituting our expression for $a(t)$ into the above gives:

$$d_p(t_{\rm ob}) = ct_0\int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{t} = ct_0(\ln(t_{\rm ob}) - \ln(t_{\rm em})) = \mathrm{ct_0 ln}\left(\frac{\mathrm{t_{ob}}}{\mathrm{t_{em}}}\right) \tag{21}$$

- But, taking $t_{\rm ob} = t_0$ (i.e., we're observing *now*), and $a(t_{\rm em}) = t_{\rm em}/t_0$, we get:

$$d_p(t_{\rm ob}) = ct_0\ln\left(\frac{1}{a(t_{\rm em})}\right) \tag{22}$$

- However, it's quite tricky to measure $a(t_{\rm em})$ directly, so instead we use the relation:

$$\frac{1}{a(t_{\rm em})} = 1 + z \tag{23}$$

to get:

$$d_p(t_{\rm ob}) = ct_0\ln(1+z) \tag{24}$$

- Which means we can *easily* calculate a distance given a readily-measurable redshift.
- Finally, the relationship:

$$\frac{a_(t_{\rm ob})}{a(t_{\rm em})} = \frac{1}{1+z} \tag{25}$$

tells us that the distances between two points in the universe was a factor of $1 + z$ *smaller* when the light was emitted (in this expanding universe), meaning we can readily calculate the distance between two points when the light was emitted:

$$d_p(t_{\rm em}) = ct_0\frac{\ln(1+z)}{1+z} \tag{26}$$