Cosmology
Lecture 4

Solving the Friedmann Equation
Part 1:
Thermodynamics and
The Equation of State

Key learning objectives

- A recap of the relativistic Friedmann Equation
- The critical density: the energy density that would ensure a flat universe.
- Obtaining an expression that describes how energy density changes with time.
- The energy density of the three main types of energy:
  - Radiation;
  - Matter;
  - Dark Energy.

The relativistic Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

so:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

we define:

$$H(t = t_0) = H_0, \quad a(t = t_0) = 1 \quad \text{and} \quad \varepsilon(t = t_0) = \varepsilon_0$$

giving:

$$H_0 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

The Critical Density

In a flat universe: $\kappa = 0$

giving:

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2$$

if: $\varepsilon(t) > \varepsilon_c(t)$ then $\kappa > 0$ and universe is +vely curved

$\varepsilon(t) < \varepsilon_c(t)$ then $\kappa < 0$ and universe is -vely curved

define: $$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$ as the density parameter

from observations: $0.995 < \Omega(t) < 1.005$

The Critical Density

From

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \quad ,$$

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2$$ and $$\varepsilon(t) = \Omega(t)\varepsilon_c(t)$$

get:

$$1 - \Omega(t) = -\kappa\frac{c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\Omega(t) > 1$$

If: $$\Omega(t) = 1$$ then it remain so at all times

$$\Omega(t) < 1$$

Solving the Friedmann Equation

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

The F.E. has too many unknowns to solve for $a(t)$ and $\varepsilon(t)$

We need more equations…

$$dQ = dE + PdV$$

First law of thermodynamics

$$P = \omega\varepsilon$$

Equation of state

Solving the Friedmann Equation

From First Law of Thermodynamics, get:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

Fluid Equation

which can be combined with the F.E. to get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

Acceleration Equation

The Equation of State simply relates P to ε

For a perfect "gas": $P = \varepsilon\omega$ where $\omega = \frac{\langle v^2 \rangle}{3c^2}$

which is effectively 0 for non-relativistic (e.g., Baryonic) material, and 1/3 for photons

$\omega < -1/3$ results in a positive acceleration: a **"Dark Energy"**

Getting the feel of it…

- The Friedmann Equation relates the Hubble parameter to the scale factor, curvature and energy density of the Universe.
- Using it, we can define a critical density for the Universe. If the true energy density is greater or less than this critical density, the Universe is curved.
- Current measurements are consistent with the Universe being flat; that, or the radius of curvature is very large.
- The Friedmann equation is *not* enough to determine how the scale factor and energy densities evolve over time; it's a single equation with two unknowns: $a(t)$ and $\varepsilon(t)$
- For that, we need to also include:
  - The First Law of Thermodynamics, which relates the scale factor to the energy density and pressure.
  - The Equation of State, which relates pressure to energy density.