Cosmology
Lecture 2

The shape of the Universe and cosmological distances

Learning objectives

- How General Relativity implies that spacetime is curved.
- The three possible types of homogeneous, isotropic curvature: flat, positive, negative.
- Measuring distances in curved spacetime; the Robertson-Walker metric.
- Defining distances in an expanding/contracting universe.
- Relating redshift, z, to the scale factor, $a(t)$.

Understanding gravity:

Newtonian dynamics:

$$F_g = -\frac{GM_g m_g}{r^2}$$

$$F_g = m_i a$$

and we assume:

$$m_i = m_g$$

which is called the **Equivalence Principle**

The equivalence principle implies that there is a unique acceleration everywhere in the Universe that is due to gravity and independent of *m*.

This acceleration can be calculated using Poisson's equation:

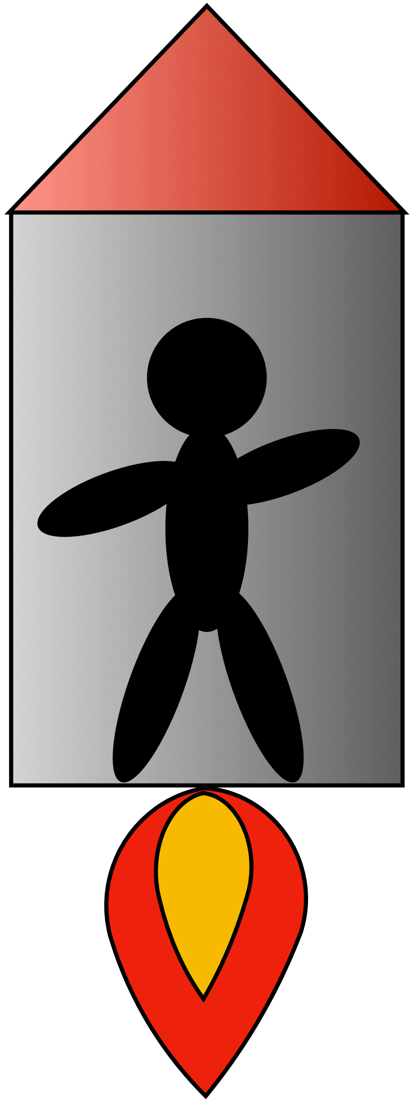
$$\nabla^2 \phi = 4\pi G \rho$$

$$a = -\nabla \phi$$

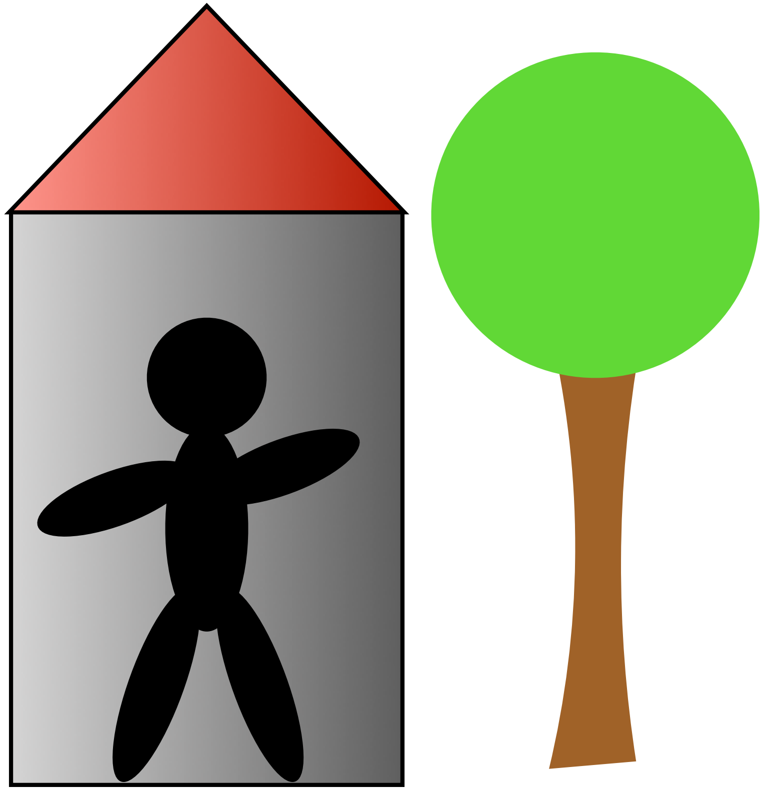
where ϕ is gravitational potential

Understanding gravity:

General relativity



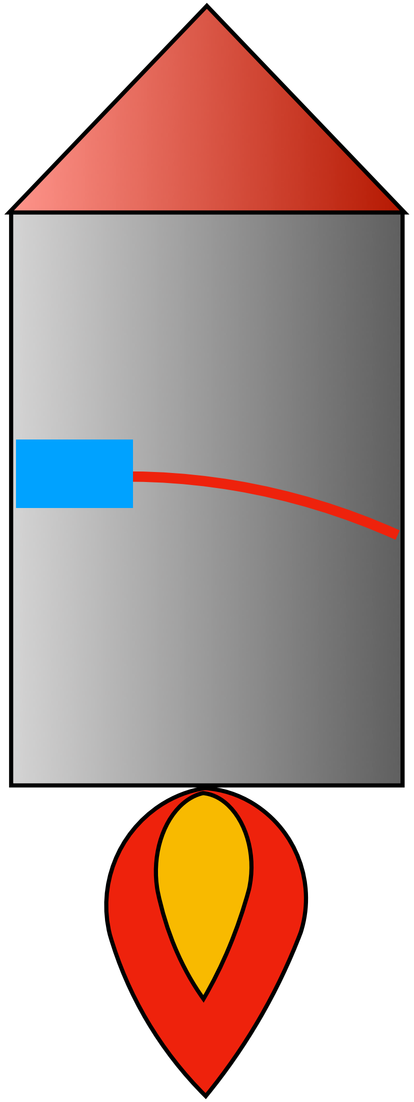
$$F = ma$$



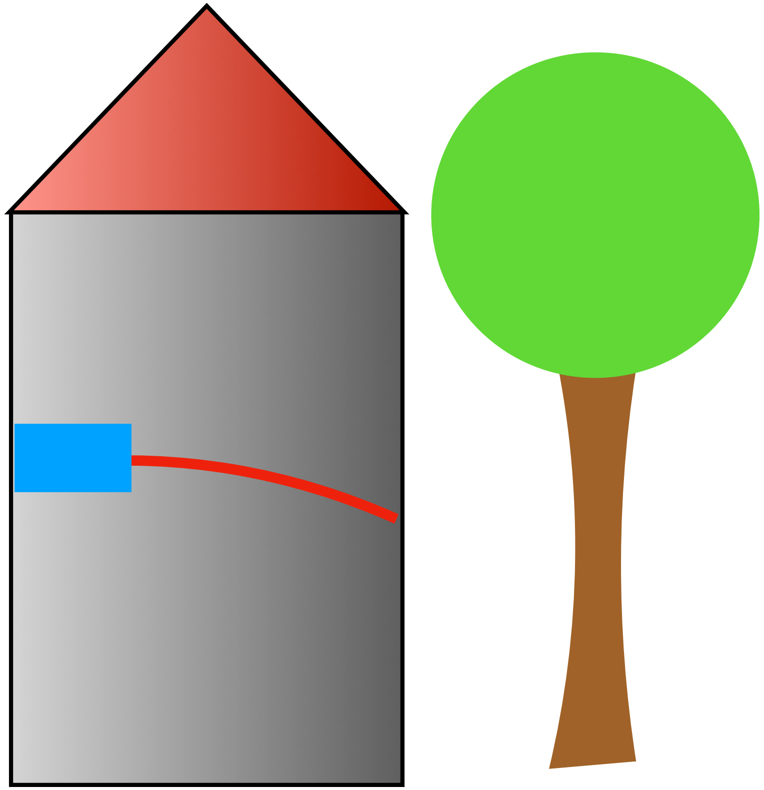
$$F = -\frac{GMm}{r^2}$$

Understanding gravity

General relativity



$$F = ma$$



$$F = -\frac{GMm}{r^2}$$

Fermat’s principle

“Light takes the shortest route between two points”

In Euclidian (i.e., flat) space, that’s a straight line.

In GR, light still follows a straight line in space, but that space may be curved and, indeed, *is* curved by mass.

The fact that light bends within a gravitational field implies that space is not flat, but curved.

Describing curvature

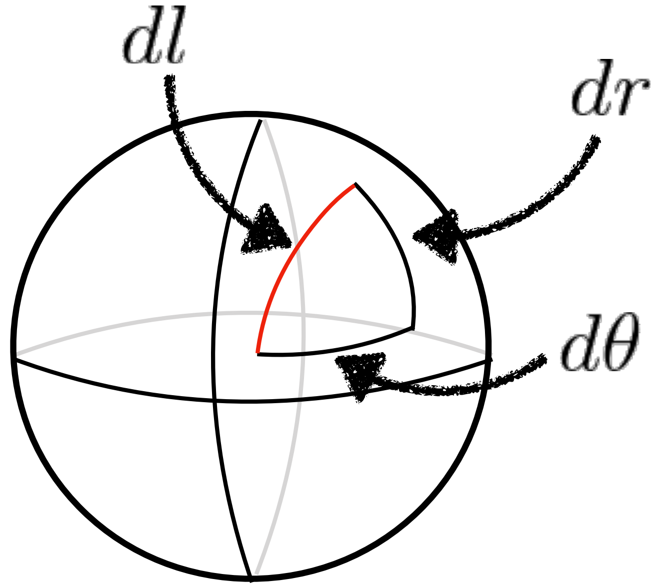
Distances between two points on a 2D surface:

1. On a flat surface:

$$dl^2 = dx^2 + dy^2$$

2. On a sphere (+ve curvature):

$$dl^2 = dr^2 + R^2 sin^2\left(\frac{r}{R}\right) d\theta^2$$



3. On a "saddle" (-ve curvature):

$$dl^2 = dr^2 + R^2 sinh^2\left(\frac{r}{R}\right) d\theta^2$$

Describing curvature

Distances between two points in 4D (spacetime)
The **Robertson-Walker metric**:

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

where

$$S_\kappa(r) = \begin{cases} R_0 sin(r/R_0) \text{ if } \kappa > 0 & \text{+ve curvature} \\ r \text{ if } \kappa = 0 & \text{flat} \\ R_0 sinh(r/R_0) \text{ if } \kappa < 0 & \text{-ve curvature} \end{cases}$$

and

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

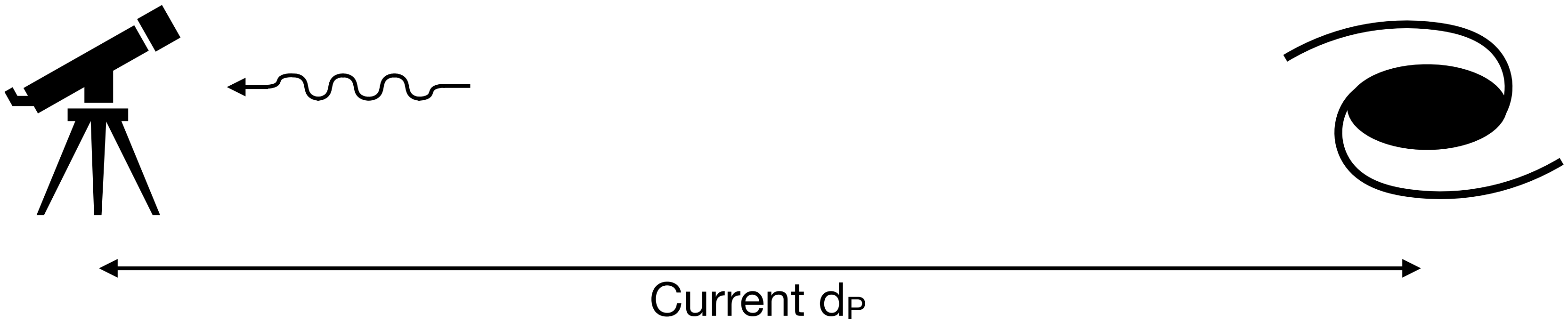
and r θ ϕ are co-moving coordinates, with $a(t)$ the scale factor

Proper distance at the current time

The *current* proper distance to a galaxy is the distance *now,* i.e.

A galaxy emits photon at time t_e, where t_e is some time before now:

In the time it takes for the photon to reach us, the galaxy moves away from us due to the expansion of the universe:



We observe the photon at time to. The *current* proper distance, d_P, is the distance at t_o.

Distances in the Robertson-Walker metric

RWM gives the distance between two points in **spacetime**:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

Proper distance is the distance between two points at an instant in time, t:

$$dt = 0$$

$$ds^2 = a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

Along the radial direction, this gives:

$$ds = a(t) dr$$

Integrating gives (since a(t) is constant):

$$d_p = a(t) \int_0^r dr = a(t) r$$

Redshifts and distances

Light travels along a null geodesic in spacetime, where ds = 0.
It also travels along a radial path, so $d\theta = 0$ and $d\phi = 0$

For light, the RW metric becomes:

$$c^2 dt^2 = a(t)^2 dr^2 \quad \text{or} \quad c\frac{dt}{a(t)} = dr$$

If we consider the light emitted and observed, when we integrate we get:

$$\frac{\lambda_{\rm em}}{a(t_{\rm em})} = \frac{\lambda_{\rm ob}}{a(t_{\rm ob})}$$

And since $z = (\lambda_{\rm ob} - \lambda_{\rm em})/\lambda_{\rm em}$

$$\boxed{1 + z = \frac{a(t_{\rm ob})}{a(t_{\rm em})} = \frac{1}{a(t_{\rm em})}}$$

Getting the feel for it

- On cosmological scales, there are only three types of curvature: flat, positive, and negative.

- Distances can be a bit of a fuzzy concept in cosmology and we have to define them carefully.

- The most important distance is the current proper distance: the distance between two points (e.g., galaxies) *right now*, which is **not** the distance they appear to be.

- While you may think redshift is due to relative velocities, cosmological redshift is, in fact, a measure of the scale factor of the universe when the light was emitted, relative to today.