# Lecture 2:
# The Shape of the Universe and Cosmological Distances

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## 1 Understanding Gravity

### Slide 2

- The only signiciant force acting on large (i.e., Universe-sized) scales is gravity.
- Therefore to understand how the Universe has evolved, we must first have a good understanding of gravity.
- Newton was the first to attempt this. His laws are still a very good approximation for most circumstances.
- In Newton's laws, the strength of gravitational force between two objects is dictated by a property known as their "gravitational mass", $m_g$.
- This force acts on an object's "inertial mass", $m_i$, telling that object how to move.
- It just so happens that, to within the errors of our best measurements, $m_g = m_i$. This is known as the "equivalence principle".
- The equivalence principle implies that there is a unique acceleration due to gravity everywhere in the universe that is independent of $m$.
- But Newton's theories didn't really tell us what gravity actually is.
- **Slide 3** Based on the Equivalence Principle, Einstein realised that there was no experiment that could tell the difference between constant acceleration and gravity.
- **Slide 4** This means that a beam of light behaves the same within a gravitational field as it does within a rocket undergoing constant acceleration.
- The beam of light will appear to bend.
- **Slide 5** But Fermat's theorem states that light travels via the shortest possible route, which Einstein knew as via straight lines in *spacetime.*
- So if the presence of mass causes light to *appear* to travel through a curve, but is actually travelling in straight lines in spacetime, it must mean that *spacetime is curved* by the presence of mass.
- Therefore, spacetime *must be curved.*

## 2 Describing curvature

- On realising that spacetime – the very fabric of the Universe – was curved, the few early cosmologists started investigating means of describing curvature.
- For our purposes, this is usually done in terms of measuring distances between points on a curved (or flat) surface.
- Isotropic, homogeneous objects (i.e., like the Universe) can only have three different types of curvature (and one of them isn't even curved):
  - Flat
  - Positively curved everywhere - like a sphere.
  - Negativelty curved everywhere - like a saddle-shape.
- **Slide 6** shows how to calculate the distance between two points on each of these different curved 2D surfaces.
- But for curved spacetime, we need to consider distances between "events" in four dimensions.
- **Slide 7** The equation for calculating distances between two events is known as the **Robertson-Walker metric**.

## 3 What is "distance"?

- Before going further, we should define what we mean by "distance" within an expanding or contracting Universe, especially when the messenger – light – can take a very long time to reach us.
- The first – and most important – distance we'll consider is the *Proper Distance*.
- Proper distance is the distance you would measure with a hypothetical, infintely long, inflexible ruler.
- As we'll see later in the course, there are other types of distance that differ from proper distance.
- **Slide 8** The *current* Proper Distance to a galaxy is the distance to that galaxy *right now, at this instant*. This is *not* the distance at which the galaxy *"appears"* to be since, in an expanding universe, the galaxy has moved further away in the intervening period between when the light was emitted and the time when we observe it.
- **Slide 9** The current proper distance to a galaxy is the integral of the Robertson-Walker metric.
- Because it is at *this instance*, then $dt = 0$, and because we're measuring distance along the radial direction $d\theta = 0$ and $d\phi = 0$.
- The RW metric then reduces to:

$$ds = a(t)dr \tag{1}$$

- And integrating over $r$ gives the proper distance:

$$d_p = a(t) \int_0^r dr = a(t) r \tag{2}$$

## 4 Redshifts and distances

- On cosmological scales, pretty much the only things we can measure for a galaxy are its flux (maybe in different filters), angular size, and redshift.
- This would present serious difficulties for measuring cosmological distances if it weren't for the fact that redshift is directly related to the scale factor.
- An interesting property of light is that it travels along the null geodesic. In other words, it travels along lines of *constant spacetime*.
- **Slide 10** This means that, for light, $ds = 0$.
- Therefore, for light travelling along the radial direction (i.e., $d\theta = 0$ and $d\phi = 0$), the RW metrtic reduces to:

$$0 = -c^2 dt^2 + a(t)^2 dr^2 \tag{3}$$

  or

$$\frac{cdt}{a(t)} = dr \tag{4}$$

- To see how this can be related to redshift, we'll consider a single photon emitted by a galaxy at time $t_{\mathrm{em}}$ and observed at time $t_{\mathrm{ob}}$.
- Integrating both sides of Eq. 4 between $t_{\mathrm{em}}$ and $t_{\mathrm{ob}}$ gives:

$$c \int_{t_{\mathrm{em}}}^{t_{\mathrm{ob}}} \frac{dt}{a(t)} = \int_0^r dr = r \tag{5}$$

  since in the intervening time, the light has travelled across the distance $r$ in comoving coordinates.

- The next crest of the photon's wavelength is emitted at $t_{\mathrm{em}} + \lambda_{\mathrm{em}}/c$ and observed at $t_{\mathrm{ob}} + \lambda_{\mathrm{ob}}/c$, giving:

$$c \int_{t_{\mathrm{em}}+\lambda_{\mathrm{em}}/c}^{t_{\mathrm{ob}}+\lambda_{\mathrm{ob}}/c} \frac{dt}{a(t)} = \int_0^r dr = r \tag{6}$$

- Since the RHS of both Eqs. 5 and 6 are the same, we can say:

$$\int_{t_{\mathrm{em}}+\lambda_{\mathrm{em}}/c}^{t_{\mathrm{ob}}+\lambda_{\mathrm{ob}}/c} \frac{dt}{a(t)} = \int_{t_{\mathrm{em}}}^{t_{\mathrm{ob}}} \frac{dt}{a(t)} \tag{7}$$

  where the $c$s have been cancelled.

- From both sides, we then subtract:

$$\int_{t_{\rm em}+\lambda_{\rm em}/c}^{t_{\rm ob}} \frac{dt}{a(t)} \tag{8}$$

which, for the LHS of Eq. 7, corresponds to:

$$\int \frac{dt}{a(t)}\bigg|_{t_{\rm ob}+\lambda_{\rm ob}/c} - \int \frac{dt}{a(t)}\bigg|_{t_{\rm em}+\lambda_{\rm em}/c} - \int \frac{dt}{a(t)}\bigg|_{t_{\rm ob}} + \int \frac{dt}{a(t)}\bigg|_{t_{\rm em}+\lambda_{\rm em}/c} \tag{9}$$

with the second and fourth term cancelling.

- Doing the same for the RHS of Eq. 7, it then becomes:

$$\int_{t_{\rm em}}^{t_{\rm em}+\lambda_{\rm em}/c} \frac{dt}{a(t)} = \int_{t_{\rm ob}}^{t_{\rm ob}+\lambda_{\rm ob}/c} \frac{dt}{a(t)} \tag{10}$$

- Over the time between the emission of the first and the second crest of the photon's wave, $a(t)$ is effectively constant, so can come out of both integrals:

$$\frac{1}{a(t_{\rm em})} \int_{t_{\rm em}}^{t_{\rm em}+\lambda_{\rm em}/c} dt = \frac{1}{a(t_{\rm ob})} \int_{t_{\rm ob}}^{t_{\rm ob}+\lambda_{\rm ob}/c} dt \tag{11}$$

- Performing the integral then gives:

$$\frac{1}{a(t_{\rm em})} \left[t_{\rm em} + \lambda_{\rm em}/c - t_{\rm em}\right] = \frac{1}{a(t_{\rm ob})} \left[t_{\rm ob} + \lambda_{\rm ob}/c - t_{\rm ob}\right] \tag{12}$$

- Cancelling $t_{\rm em}$, $t_{\rm ob}$, and $c$ gives:

$$\frac{\lambda_{\rm em}}{a(t_{\rm em})} = \frac{\lambda_{\rm ob}}{a(t_{\rm ob})} \tag{13}$$

- or

$$\frac{a(t_{\rm ob})}{a(t_{\rm em})} = \frac{\lambda_{\rm ob}}{\lambda_{\rm em}} \tag{14}$$

- and using

$$z = \frac{\lambda_{\rm ob} - \lambda_{\rm em}}{\lambda_{\rm em}} \tag{15}$$

gives

$$\frac{a(t_{\rm ob})}{a(t_{\rm em})} = \frac{1}{a(t_{\rm em})} = 1 + z \tag{16}$$

- Meaning that by measuring the redshift of a galaxy immediately tells us the relative scale factor of the Universe when the light from that galaxy was emitted.

- This is the second **Important Equation** of the course.