Lecture 06:
Single-component model universes

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1 Matter-only and Radiation-only universes

- In the previous lecture, we solved the F.E. for our first model universe: an empty universe.
- An empty universe is the easiest to solve, but demonstrates how we use the F.E. to obtain an expression for the scale factor, $a(t)$, and then how we use that expression to determine proper distances.
- We will now turn to the next-easiest universes to solve: single component model universes.
- For now, we'll only consider flat (i.e., $\kappa = 0$) universes, but you should at least be aware of what we'd need to solve for a non-flat model universe.
- For a flat universe, the F.E. becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) \tag{1}$$

or:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon(t)a(t)^2 \tag{2}$$

- From Lecture 5, we know that, from the Fluid Equation, we can write ε in terms of a:

$$\varepsilon(a) = \varepsilon_0 a(t)^{-3(1+\omega)} \tag{3}$$

- And substituting this into the F.E. gives:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon_0 a(t)^{-3(1+\omega)}a(t)^2 \tag{4}$$

- And since:

$$a(t)^{-3(1+\omega)}a(t)^2 = a(t)^{-3(1+\omega)+2} = a(t)^{-3-3\omega+2} = a(t)^{-1-3\omega} \tag{5}$$

we get:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon_0 a(t)^{(-1-3\omega)} \tag{6}$$

- To solve this, we'll take a guess that $a \propto t^q$, i.e., that the expression for $a(t)$ is some kind of power-law of t.

- In that case, the LHS of Eq. 6 becomes, via differentiation wrt. time:

$$(qt^{q-1})^2 = qt^{2q-2} \tag{7}$$

- And the RHS becomes:

$$\frac{8\pi G}{3c^2}\varepsilon_0 t^{(-1-3\omega)q} \tag{8}$$

giving:

$$qt^{2q-2} = \frac{8\pi G}{3c^2}\varepsilon_0 t^{(-1-3\omega)q} \tag{9}$$

- For this equation to *always* hold true, then the exponents of t must be equal, i.e,:

$$2q - 2 = (-1 - 3\omega)q \tag{10}$$

- Which we can rearrange to get q:

$$q = \frac{2}{3 + 3\omega} \tag{11}$$

- Substituting this into $a \propto t^q$ gives:

$$a \propto t^{\frac{2}{3+3\omega}} \tag{12}$$

or,

$$a = Ct^{\frac{2}{3+3\omega}} \tag{13}$$

- We find the constant of proportionality, C, by using $a(t_0) = 1$:

$$a(t_0) = Ct_0^{\frac{2}{3+3\omega}} = 1 \tag{14}$$

so

$$C = t_0^{\frac{-2}{3+3\omega}} \tag{15}$$

and Eq. 13 becomes:

$$a = t_0^{\frac{-2}{3+3\omega}} t^{\frac{2}{3+3\omega}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3+3\omega}} \tag{16}$$

- t_0 is then found by substituting Eq. 16 back into the F.E., to give:

$$t_0 = \frac{1}{1+\omega}\left(\frac{c^2}{6\pi G\varepsilon_0}\right)^{1/2} \tag{17}$$

it's a good idea for you to be able to demonstrate this yourself.

1.1 A proper distances

- Now that we've got a general expression for $a(t)$, we can use it to obtain a general expression for proper distance. Recall:

$$d_p(t_{\text{ob}}) = c\int_{t_{\text{em}}}^{t_{\text{ob}}} \frac{dt}{a(t)} = ct_0^{2/(3+3\omega)} \int_{t_{\text{em}}}^{t_{\text{ob}}} \frac{dt}{t^{-2/(3+3\omega)}} \tag{18}$$

- To perform this integral, we'll define $m = -2/(3+3\omega)$:

$$d_p(t_{\rm ob}) = ct_0^{-m} \int_{t_{\rm em}}^{t_{\rm ob}} t^m dt = \frac{ct_0^{-m}}{m+1}(t_{\rm ob}^{m+1} - t_{\rm em}^{m+1}) \tag{19}$$

- Taking $t_{\rm ob}= t_0$ (i.e., we're observing now), and taking a t_0^{m+1} out of the bracket, we get:

$$d_p(t_{\rm ob}) = \frac{ct_0^{-m}}{m+1} t_0^{m+1}(1-(t_{\rm em}/t_0)^{m+1}) = \frac{ct_0}{m+1}(1-(t_{\rm em}/t_0)^{m+1}) \tag{20}$$

where

$$m+1 = \frac{-2}{3(1+\omega)} + 1 = \frac{-2+3+3\omega}{3(1+\omega)} = \frac{1+3\omega}{3(1+\omega)} \tag{21}$$

giving:

$$d_p(t_{\rm ob}) = ct_0 \frac{3(1+\omega)}{1+3\omega}(1-(t_{\rm em}/t_0)^{(1+3\omega)/(3+3\omega)}) \tag{22}$$

- But, we don't typically know $t_{\rm em}$. Instead, we measure redshift, z.
- We can use the relationship $1+z = 1/a(t_{\rm em})$ to determine proper distance from redshift:

$$1+z = \frac{1}{a(t_{\rm em})} = \left(\frac{t_{\rm em}}{t_0}\right)^{-2/(3+3\omega)} = \left(\frac{t_{\rm em}}{t_0}\right)^m \tag{23}$$

so

$$\frac{t_{\rm em}}{t_0} = (1+z)^{1/m} \tag{24}$$

giving:

$$d_p(t_{\rm ob}) = \frac{ct_0}{m+1}(1-(1+z)^{(m+1)/m}) \tag{25}$$

or

$$d_p(t_{\rm ob}) = ct_0 \frac{3(1+\omega)}{1+3\omega}(1-(1+z)^{(-1+3\omega)/2}) \tag{26}$$

and, using Eq. 17 to write t_0 in terms of H_0 (you should attempt to do this yourself, by using the H_0 version of the F.E.), we get:

$$d_p(t_{\rm ob}) = \frac{c}{H_0}\frac{2}{1+3\omega}(1-(1+z)^{(-1+3\omega)/2}) \tag{27}$$

- Finally, the proper distance at the time of *emission* can be calculated by considering that - in an expanding universe - the universe was a factor of $1+z = 1/a(t_{\rm em})$ *smaller* when the light was emitted, giving:

$$d_p(t_{\rm em}) = \frac{d_p(t_{\rm ob})}{1+z} \tag{28}$$