EFFECTIVE FIELD THEORY: INTRODUCTION AND APPLICATIONS

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Various ChPT: http://www.thep.lu.se/∼bijnens/chpt.html

Overview

- What is effective field theory?
- Introduction to ChPT including possible problems
- Results for two-flavour ChPT
- Results for three flavour ChPT
- Partial quenching and ChPT
- $\eta \to 3\pi$
- Some comments about Higgs sector and ChPT

Wikipedia

`http://en.wikipedia.org/wiki/Effective_field_theory`

In physics, an effective field theory is an approximate theory (usually a quantum field theory) that contains the appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale, but ignores the substructure and the degrees of freedom at shorter distances (or, equivalently, higher energies).

Wikipedia

`http://en.wikipedia.org/wiki/Chiral_perturbation_theory`

Chiral perturbation theory (ChPT) is an effective field theory constructed with a Lagrangian consistent with the (approximate) chiral symmetry of quantum chromodynamics (QCD), as well as the other symmetries of parity and charge conjugation. ChPT is a theory which allows one to study the low-energy dynamics of QCD. As QCD becomes non-perturbative at low energy, it is impossible to use perturbative methods to extract information from the partition function of QCD. Lattice QCD is one alternative method that has proved successful in extracting non-perturbative information.

Effective Field Theory

Main Ideas:

- Use right degrees of freedom : essence of (most) physics
- If mass-gap in the excitation spectrum: neglect degrees of freedom above the gap.

Examples:

- Solid state physics: conductors: neglect the empty bands above the partially filled one
- Atomic physics: Blue sky: neglect atomic structure

Power Counting

- gap in the spectrum $\Longrightarrow$ separation of scales
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$\Longrightarrow$ Need some ordering principle: power counting

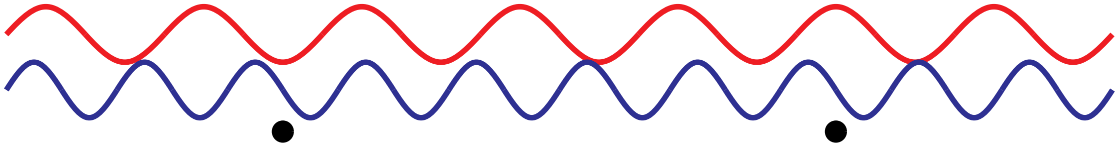
- Taylor series expansion does not work (convergence radius is zero)
- Continuum of excitation states need to be taken into account

Why is the sky blue ?

System: Photons of visible light and neutral atoms

Length scales: a few 1000 Å versus 1 Å

Atomic excitations suppressed by $\approx 10^{-3}$



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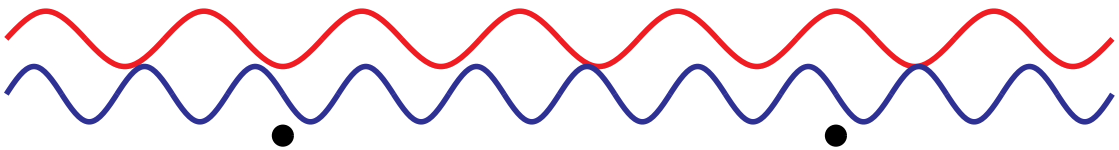

$$\mathcal{L}_A = \Phi_v^\dagger \partial_t \Phi_v + \dots \qquad \mathcal{L}_{\gamma A} = G F_{\mu\nu}^2 \Phi_v^\dagger \Phi_v + \dots$$

Units with $\hbar = c = 1$: G energy dimension -3:

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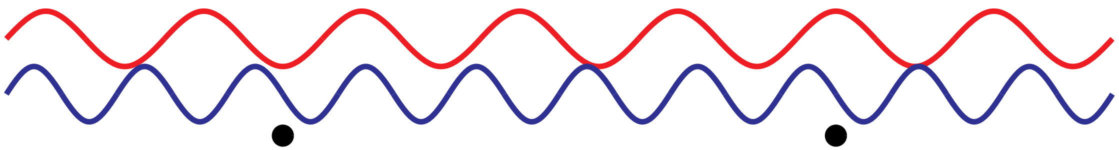

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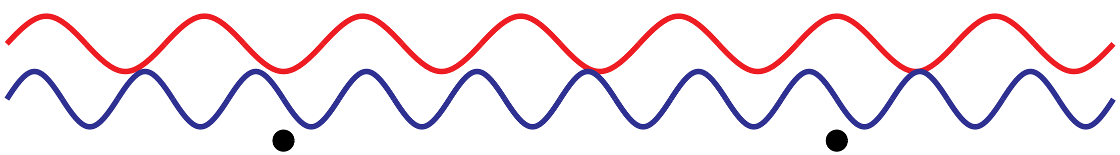

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blue light scatters a lot more than red $\left\{\begin{array}{l} \Longrightarrow \text{red sunsets} \\ \Longrightarrow \text{blue sky} \end{array}\right.$

Higher orders suppressed by 1 Å$/\lambda_\gamma$.

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  any model has few parameters but model-space is large
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Advantages

- Calculations are (relatively) simple
- It is general: model-independent
- Theory $\Longrightarrow$ errors can be estimated
- Systematic: ALL effects at a given order can be included
- Even if no convergence: classification of models often useful

Examples of EFT

- Fermi theory of the weak interaction
- Chiral Perturbation Theory: hadronic physics
- NRQCD
- SCET
- General relativity as an EFT
- 2,3,4 nucleon systems from EFT point of view

references

- A. Manohar, Effective Field Theories (Schladming lectures), hep-ph/9606222
- I. Rothstein, Lectures on Effective Field Theories (TASI lectures), hep-ph/0308266
- G. Ecker, Effective field theories, Encyclopedia of Mathematical Physics, hep-ph/0507056
- D.B. Kaplan, Five lectures on effective field theory, nucl-th/0510023
- A. Pich, Les Houches Lectures, hep-ph/9806303
- S. Scherer, Introduction to chiral perturbation theory, hep-ph/0210398
- J. Donoghue, Introduction to the Effective Field Theory Description of Gravity, gr-qc/9512024

School

PSI Zuoz Summer School on Particle Physics
Effective Theories in Particle Physics, July 16 - 22, 2006

`http://ltpth.web.psi.ch/zuoz_school/previous_summerschools/zuoz2006/index.html`

- R. Barbieri: Effective Theories for Physics beyond the Standard Model
- M. Beneke: Concept of Effective Theories, Heavy Quark Effective Theory and Soft-Collinear Effective Theory
- G. Colangelo: Chiral Perturbation Theory
- U. Langenegger: B Physics and Quarkonia
- H. Leutwyler: Historical and Other Remarks on Effective Theories
- A. Manohar: Nonrelativistic QCD
- L. Simons: Pion-Nucleon Interaction
- M. Sozzi: Kaon Physics

Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

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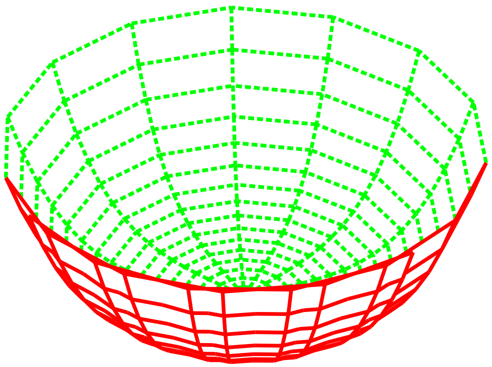
Derivation from QCD:

H. Leutwyler, *On The Foundations Of Chiral Perturbation Theory,* Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

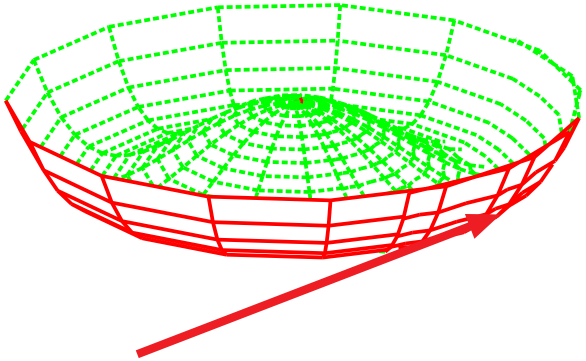
The mass gap: Goldstone Modes

UNBROKEN: $V(\phi)$



Only massive modes around lowest energy state (=vacuum)

BROKEN: $V(\phi)$



Need to pick a vacuum
$\langle\phi\rangle \neq 0$: Breaks symmetry
No parity doublets
Massless mode along bottom

For more complicated symmetries: need to describe the bottom mathematically: $G \to H \Longrightarrow G/H$ (explain)

The two symmetry modes compared

Wigner-Eckart mode	Nambu-Goldstone mode
Symmetry group G	G spontaneously broken to subgroup H
Vacuum state unique	Vacuum state degenerate
Massive Excitations	Existence of a massless mode
States fall in multiplets of G	States fall in multiplets of H
Wigner Eckart theorem for G	Wigner Eckart theorem for H
	Broken part leads to low-energy theorems
Symmetry linearly realized	Full Symmetry, G, nonlinearly realized
	unbroken part, H, linearly realized

Some clarifications

- $\phi(x)$: orientation of vacuum in every space-time point
- Examples: spin waves, phonons
- Nonlinear: acting by a broken symmetry operator changes the vacuum, $\phi(x) \to \phi(x) + \alpha$
- The precise form of ϕ is *not* important but it must describe the space of vacua (field transformations possible)
- In gauge theories: the *local* symmetry allows the vacua to be different in every point, hence the Goldstone Boson must not be observable as a massless degree of freedom.

The power counting

Very important:

Low energy theorems: Goldstone bosons do not interact at zero momentum

Heuristic proof:

- Which vacuum does not matter, choices related by symmetry
- $\phi(x) \to \phi(x) + \alpha$ should not matter
- Each term in $\mathcal{L}$ must contain at least one $\partial_\mu \phi$

Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral Symmetry Spontaneous Breakdown

Power counting: Dimensional counting

Expected breakdown scale: Resonances, so M_ρ or higher depending on the channel

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Chiral Symmetry

QCD: 3 light quarks: equal mass: interchange: $SU(3)_V$

But $\mathcal{L}_{QCD} = \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$

So if $m_q = 0$ then $SU(3)_L \times SU(3)_R$.

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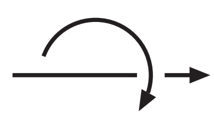
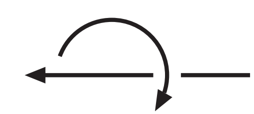
Can also see that via



$v < c$, $m_q \neq 0 \Longrightarrow$

$v = c$, $m_q = 0 \not\Longrightarrow$



Chiral Perturbation Theory

$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
$SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$

8 generators broken $\Longrightarrow$ 8 massless degrees of freedom
and interaction vanishes at zero momentum

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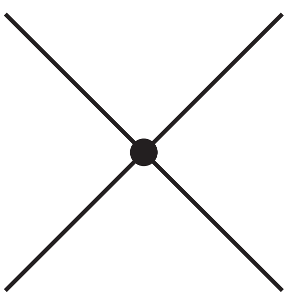
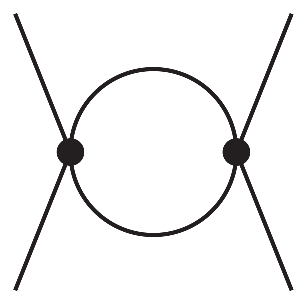
Power counting in momenta: (explain)



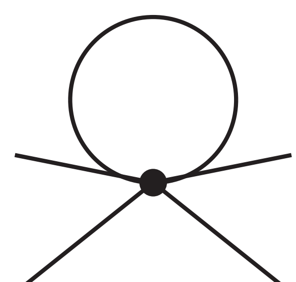
p^2



$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$

$1/p^2$

$\int d^4p$ $\qquad p^4$



$(p^2)\,(1/p^2)\,p^4 = p^4$

Chiral Perturbation Theory

Large subject:

- Steven Weinberg, Physica A96:327,1979: 1789 citations
- Juerg Gasser and Heiri Leutwyler, Nucl.Phys.B250:465,1985: 2290 citations
- Juerg Gasser and Heiri Leutwyler, Annals Phys.158:142,1984: 2254 citations
- Sum: 3777
- Checked on 18/2/2007 in SPIRES

For lectures, review articles: see
`http://www.thep.lu.se/~bijnens/chpt.html`

Chiral Perturbation Theories

- Baryons
- Heavy Quarks
- Vector Mesons (and other resonances)
- Structure Functions and Related Quantities
- Light Pseudoscalar Mesons
  - Two or Three (or even more) Flavours
  - Strong interaction and couplings to external currents/densities
  - Including electromagnetism
  - Including weak nonleptonic interactions
  - Treating kaon as heavy

Many similarities with strongly interacting Higgs

Lagrangians

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}.$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu\,,$
left and right external currents: $r(l)_\mu = v_\mu + (-) a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$
quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A \rangle = Tr_F\,(A)$

External currents?

- in QCD Green functions derived as functional derivatives w.r.t. external fields
- Green functions are the objects that satisfy Ward identities
- By introducing a local Chiral Symmetry, Ward identities are automatically satisfied (great improvement over current algebra)
- QCD Green functions form a connection QCD$\Leftrightarrow$ChPT
- $$\int [dG dq d\overline{q}] e^{i\int d^4x \mathcal{L}_{QCD}(q,\overline{q},G,l,r,s,p)} \approx$$
  $$\int [dU] e^{i\int d^4x \mathcal{L}_{ChPT}(U,l,r,s,p)}$$
  so also functional derivatives are equal

Lagrangians

$$
\begin{aligned}
\mathcal{L}_4 = {} & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
& + L_3\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle + L_4\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
& + L_5\langle D^\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
& + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
& - iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle \\
& + L_{10}\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu}\rangle + H_1\langle F^R_{\mu\nu} F^{R\mu\nu} + F^L_{\mu\nu} F^{L\mu\nu}\rangle + H_2\langle \chi^\dagger \chi\rangle
\end{aligned}
$$

L_i: Low-energy-constants (LECs)
H_i: Values depend on definition of currents/densities

These absorb the divergences of loop diagrams: $L_i \rightarrow L_i^r$
Renormalization: order by order in the powercounting

Lagrangians

Lagrangian Structure:

	2 flavour		3 flavour		3+3 PQChPT	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- replica method $\Longrightarrow$ PQ obtained from N_F flavour
- All infinities known
- 3 flavour special case of 3+3 PQ: $\hat{L}_i^r, K_i^r \to L_i^r, C_i^r$
- 53 $\to$ 52 arXiv:0705.0576 [hep-ph]

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$M^2 = 2B\hat{m}$
$B \neq B_0$, $F \neq F_0$ (two versus three-flavour)

LECs and μ

$l_3^r(\mu)$

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i}\, l_i^r(\mu) - \log\frac{M_\pi^2}{\mu^2}\,.$$

Independent of the scale μ.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

μ :

- m_π, m_K: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$ large N_c arguments????
- compromise: $\mu = m_\rho = 0.77$ GeV

Expand in what quantities?

- Expansion is in momenta and masses
- But is not unique: relations between masses (Gell-Mann–Okubo) exists
- Express orders in terms of physical masses and quantities (F_π, F_K)?
- Express orders in terms of lowest order masses?
- E.g. $s + t + u = 2m_\pi^2 + 2m_K^2$ in πK scattering

See e.g. Descotes-Genon talk

- I prefer physical masses
- Thresholds correct
- Chiral logs are from physical particles propagating

An example

$$m_\pi = \frac{m_0}{1 + a\frac{m0}{f0}} \qquad\qquad f_\pi = \frac{f_0}{1 + b\frac{m0}{f0}}$$

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$$m_\pi = m_0 - a\frac{m_0^2}{f_0} + a^2\frac{m_0^3}{f_0^2} + \cdots$$

$$f_\pi = f_0\left(1 - b\frac{m_0}{f_0} + b^2\frac{m_0^2}{f_0^2} + \cdots\right)$$

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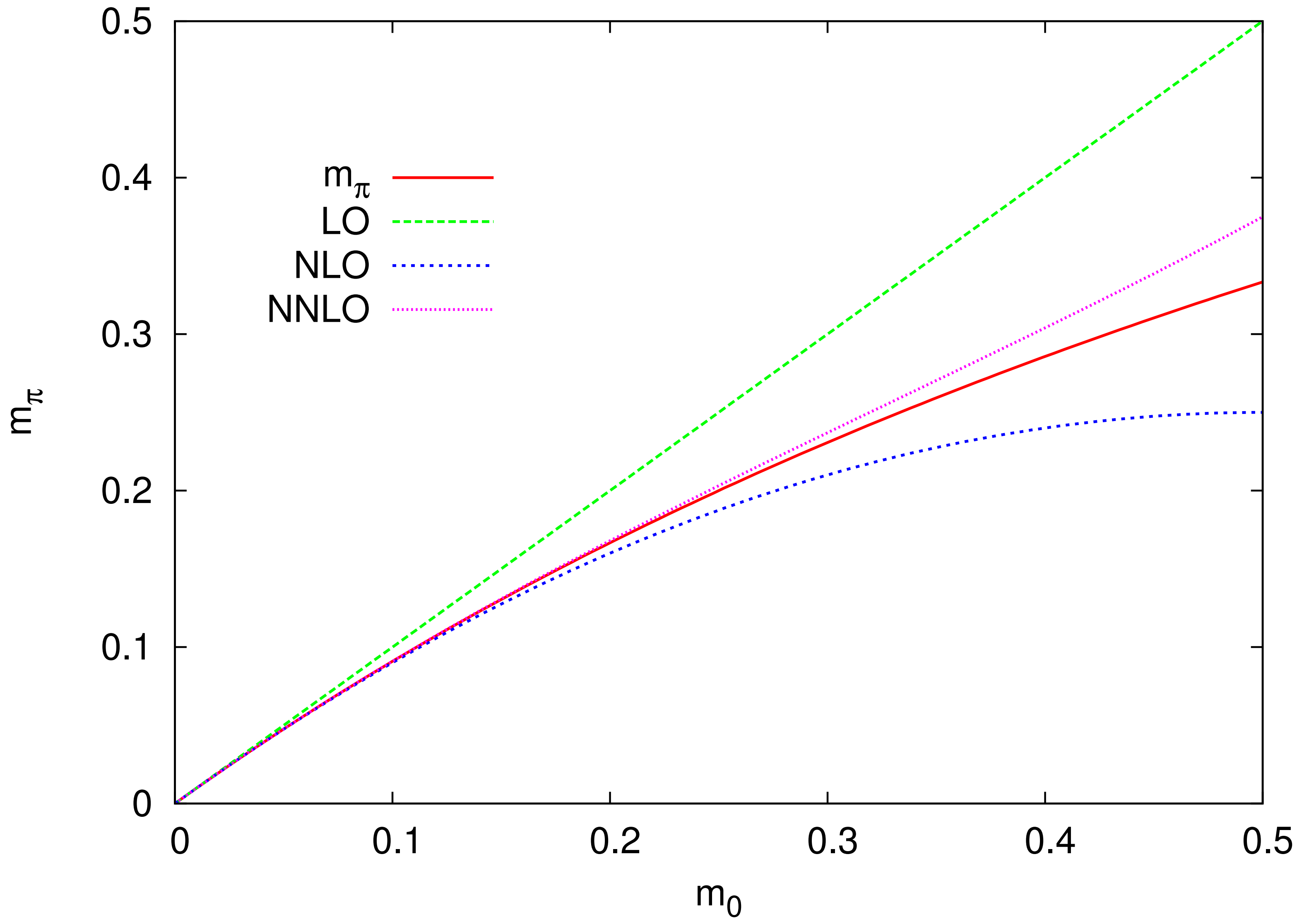
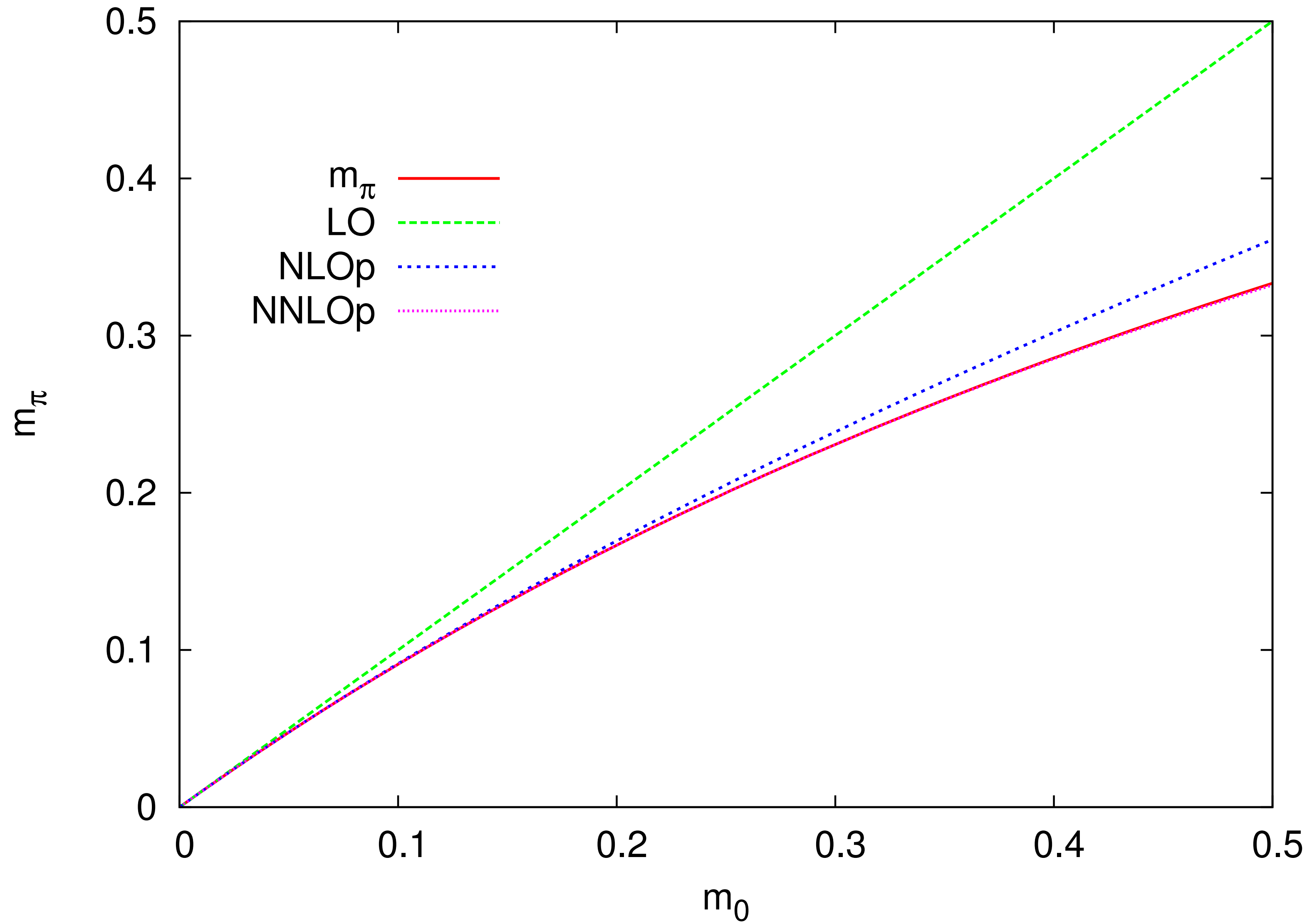
$$a = 1 \quad b = 0.5 \quad f_0 = 1$$

An example: m_0/f_0



An example: m_π/f_π



Two-loop Two-flavour

Review paper on Two-Loops: JB, hep-ph/0604043 Prog. Part. Nucl. Phys. 58 (2007) 521

Dispersive Calculation of the nonpolynomial part in q^2, s, t, u

- Gasser-Meißner: F_V, F_S: 1991 numerical
- Knecht-Moussallam-Stern-Fuchs: $\pi\pi$: 1995 analytical
- Colangelo-Finkemeier-Urech: F_V, F_S: 1996 analytical

Two-Loop Two-flavour

- Bellucci-Gasser-Sainio: $\gamma\gamma \to \pi^0\pi^0$: 1994
- Bürgi: $\gamma\gamma \to \pi^+\pi^-$, F_π, m_π: 1996
- JB-Colangelo-Ecker-Gasser-Sainio: $\pi\pi$, F_π, m_π: 1996-97
- JB-Colangelo-Talavera: $F_{V\pi}(t)$, $F_{S\pi}$: 1998
- JB-Talavera: $\pi \to \ell\nu\gamma$: 1997
- Gasser-Ivanov-Sainio: $\gamma\gamma \to \pi^0\pi^0$, $\gamma\gamma \to \pi^+\pi^-$: 2005-2006
- m_π, F_π, F_V, F_S, $\pi\pi$: simple analytical forms
- Colangelo-(Dürr-)Haefeli: Finite volume F_π, m_π 2005-2006

LECs

$\bar{l}_1$ to $\bar{l}_4$: ChPT at order p^6 and the Roy equation analysis in $\pi\pi$ and F_S Colangelo, Gasser and Leutwyler, *Nucl. Phys.* B 603 (2001) 125 [hep-ph/0103088]

$\bar{l}_5$ and $\bar{l}_6$: from F_V and $\pi \longrightarrow \ell\nu\gamma$ JB,(Colangelo,)Talavera

$$
\begin{aligned}
\bar{l}_1 &= -0.4 \pm 0.6\,, \\
\bar{l}_2 &= 4.3 \pm 0.1\,, \\
\bar{l}_3 &= 2.9 \pm 2.4\,, \\
\bar{l}_4 &= 4.4 \pm 0.2\,, \\
\bar{l}_6 - \bar{l}_5 &= 3.0 \pm 0.3\,, \\
\bar{l}_6 &= 16.0 \pm 0.5 \pm 0.7\,.
\end{aligned}
$$

$l_7 \sim 5 \cdot 10^{-3}$ from π^0-η mixing Gasser, Leutwyler 1984

LECs

Some combinations of order p^6 LECs are known as well: curvature of the scalar and vector formfactor, two more combinations from $\pi\pi$ scattering (implicit in b_5 and b_6)

Note: c_i^r for m_π, f_π, $\pi\pi$: small effect

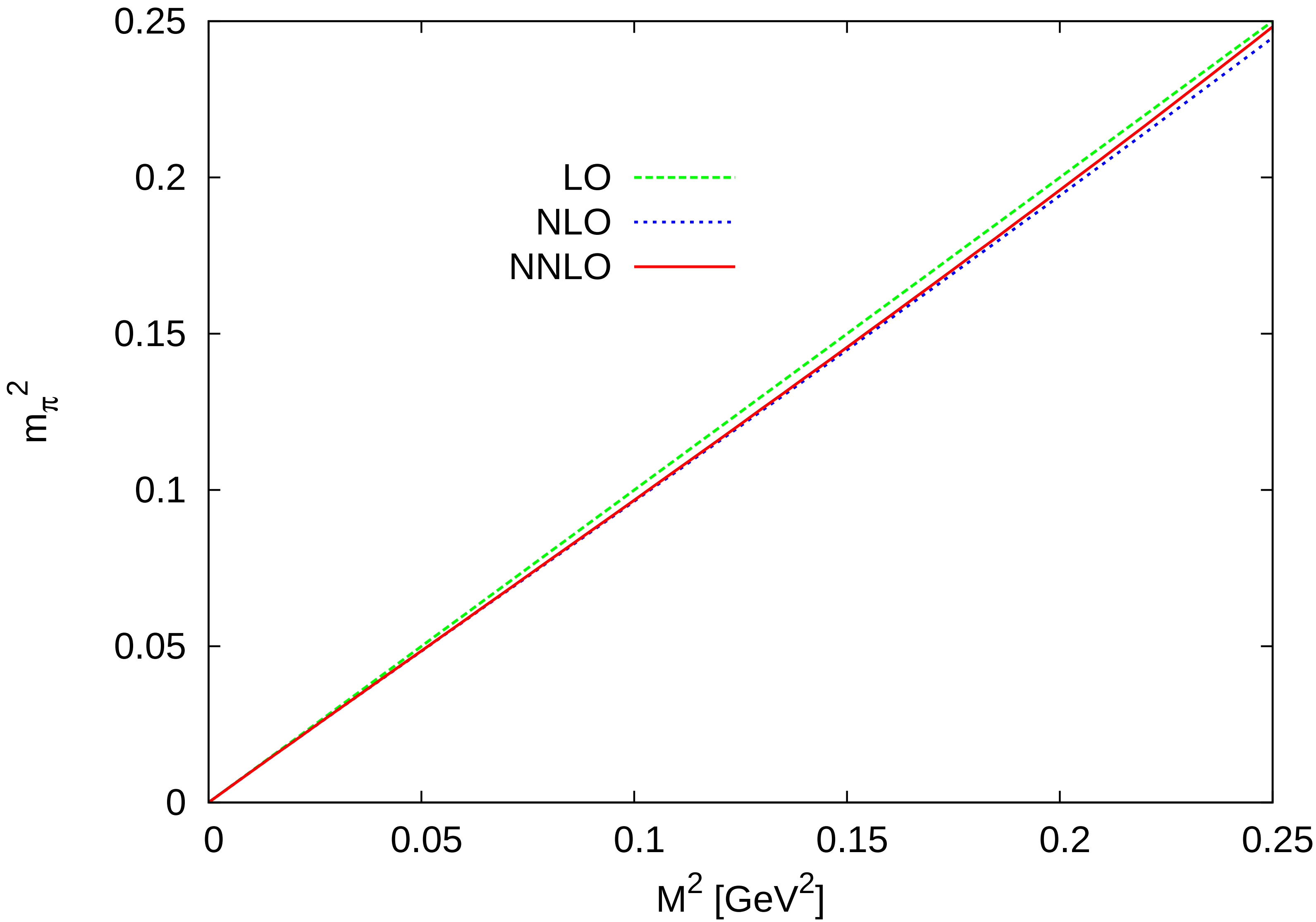
$c_i^r(770 MeV) = 0$ for plots shown

expansion in m_π^2/F_π^2 shown

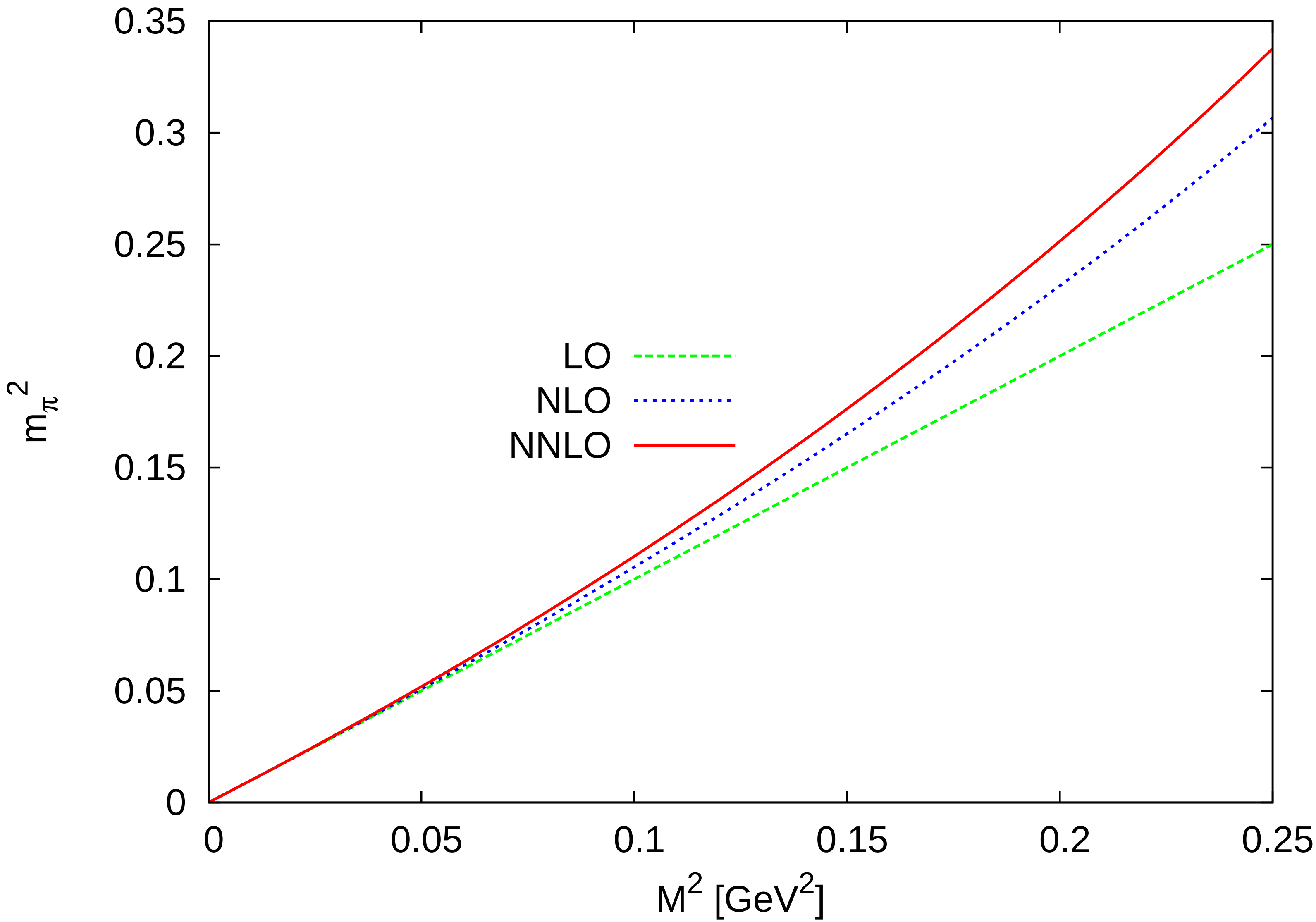
General observation:

- Obtainable from kinematical dependences: known
- Only via quark-mass dependence: poorely known

m_π^2

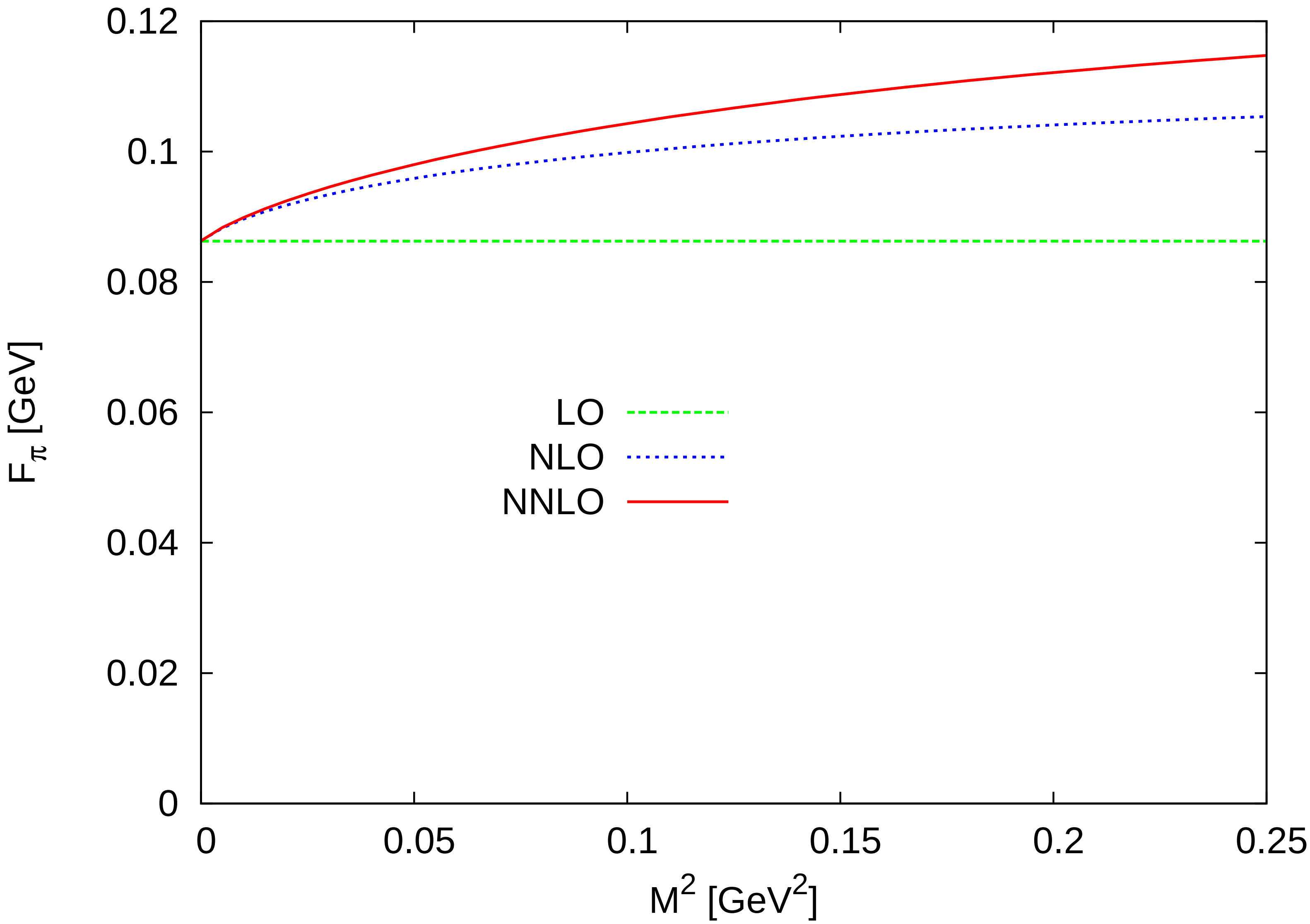


m_π^2 ($\bar{l}_3 = 0$)



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
m_π^2
0
0.05
0.1
0.15
0.2
0.25
M^2 [GeV^2]
LO
NLO
NNLO

F_π


0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.05
0.1
0.15
0.2
0.25
F_π [GeV]
M^2 [GeV^2]
LO
NLO
NNLO

Two-loop Three-flavour, $\leq$2001

- $\Pi_{VV\pi}$, $\Pi_{VV\eta}$, Π_{VVK} Kambor, Golowich; Kambor, Dürr; Amorós, JB, Talavera
- $\Pi_{VV\rho\omega}$ Maltman
- $\Pi_{AA\pi}$, $\Pi_{AA\eta}$, F_π, F_η, m_π, m_η Kambor, Golowich; Amorós, JB, Talavera
- Π_{SS} Moussallam $\boxed{L_4^r, L_6^r}$
- Π_{VVK}, Π_{AAK}, F_K, m_K Amorós, JB, Talavera
- $K_{\ell 4}$, $\langle \overline{q}q \rangle$ Amorós, JB, Talavera $\boxed{L_1^r, L_2^r, L_3^r}$
- F_M, m_M, $\langle \overline{q}q \rangle$ $(m_u \neq m_d)$ Amorós, JB, Talavera $\boxed{L_{5,7,8}^r, m_u/m_d}$

Two-loop Three-flavour, $\geq$2001

- $F_{V\pi}$, F_{VK^+}, F_{VK^0} Post, Schilcher; JB, Talavera $\boxed{L_9^r}$
- $K_{\ell 3}$ Post, Schilcher; JB, Talavera $\boxed{V_{us}}$
- $F_{S\pi}$, F_{SK} (includes σ-terms) JB, Dhonte $\boxed{L_4^r, L_6^r}$
- $K, \pi \to \ell\nu\gamma$ Geng, Ho, Wu $\boxed{L_{10}^r}$
- $\pi\pi$ JB,Dhonte,Talavera
- πK JB,Dhonte,Talavera
- relation l_i^r and L_i^r, C_i^r Gasser,Haefeli,Ivanov,Schmid
- Finite volume $\langle \overline{q}q \rangle$ JB,Ghorbani

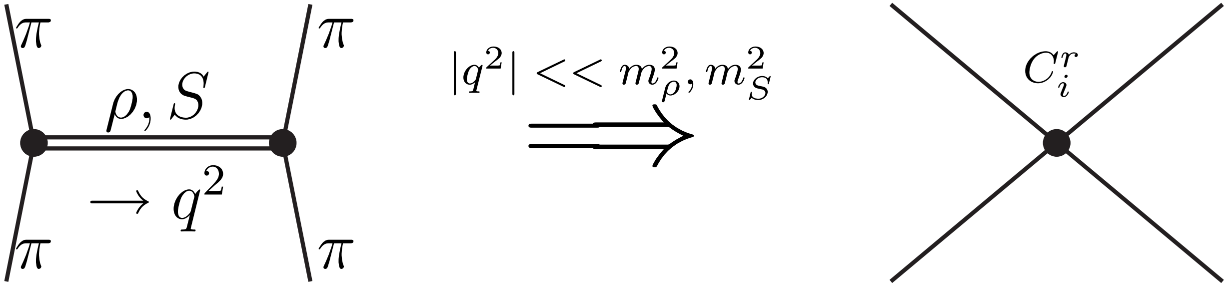
Two-loop Three-flavour

Known to be in progress

- $\eta \to 3\pi$: being written up JB,Ghorbani
- $K_{\ell 3}$ iso: preliminary results available JB,Ghorbani
- Finite Volume: sunsetintegrals being written up JB,Lähde
- relation c_i^r and L_i^r, C_i^r Gasser,Haefeli,Ivanov,Schmid

C_i^r

Most analysis use:
C_i^r from (single) resonance approximation



Motivated by large N_c: large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael,...

C_i^r

$$
\begin{aligned}
\mathcal{L}_V &= -\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu}\rangle + \frac{1}{2}m_V^2\langle V_\mu V^\mu\rangle - \frac{f_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle \\
&\quad -\frac{ig_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu,u^\nu]\rangle + f_\chi\langle V_\mu[u^\mu,\chi_-]\rangle \\
\mathcal{L}_A &= -\frac{1}{4}\langle A_{\mu\nu}A^{\mu\nu}\rangle + \frac{1}{2}m_A^2\langle A_\mu A^\mu\rangle - \frac{f_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle \\
\mathcal{L}_S &= \frac{1}{2}\langle \nabla^\mu S\nabla_\mu S - M_S^2 S^2\rangle + c_d\langle S u^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle \\
\mathcal{L}_{\eta'} &= \frac{1}{2}\partial_\mu P_1\partial^\mu P_1 - \frac{1}{2}M_{\eta'}^2 P_1^2 + i\tilde{d}_m P_1\langle\chi_-\rangle\,.
\end{aligned}
$$

$f_V = 0.20$, $\quad f_\chi = -0.025$, $\quad g_V = 0.09$, $\quad c_m = 42$ MeV, $\quad c_d = 32$ MeV, $\quad \tilde{d}_m = 20$ MeV,

$m_V = m_\rho = 0.77$ GeV, $\quad m_A = m_{a_1} = 1.23$ GeV, $\quad m_S = 0.98$ GeV, $\quad m_{P_1} = 0.958$ GeV

f_V, g_V, f_χ, f_A: experiment

c_m and c_d from resonance saturation at $\mathcal{O}(p^4)$

C_i^r

Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: C_i^r in the masses/decay constants and how these effects correlate into the rest
- No μ dependence: obviously only estimate

C_i^r

Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: C_i^r in the masses/decay constants and how these effects correlate into the rest
- No μ dependence: obviously only estimate

What we did about it:

- Vary resonance estimate by factor of two
- Vary the scale μ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult

Inputs

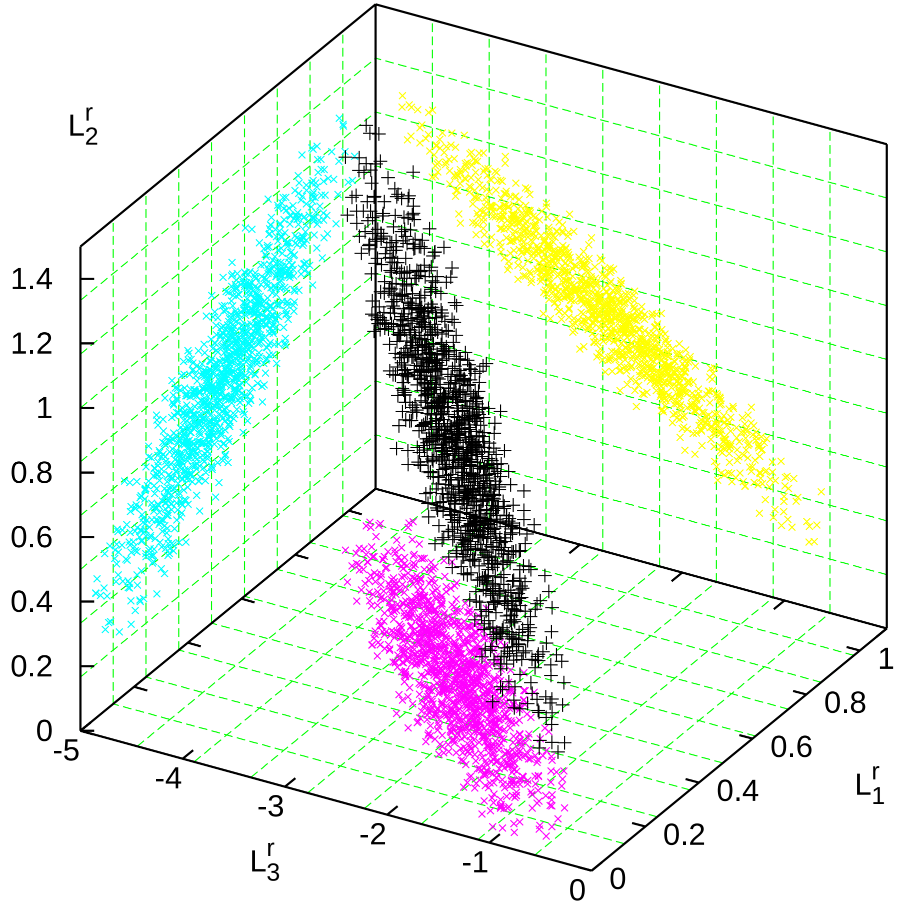
$K_{\ell 4}$: $F(0)$, $G(0)$, λ		E865 BNL
$m^2_{\pi^0}$, m^2_η, $m^2_{K^+}$, $m^2_{K^0}$		em with Dashen violation
F_{π^+}		
F_{K^+}/F_{π^+}		
$m_s/\hat{m}$	24 (26)	$\hat{m} = (m_u + m_d)/2$
L_4^r, L_6^r		

Outputs: I

	fit 10	same p^4	fit B	fit D
$10^3 L_1^r$	0.43 ± 0.12	0.38	0.44	0.44
$10^3 L_2^r$	0.73 ± 0.12	1.59	0.60	0.69
$10^3 L_3^r$	-2.35 ± 0.37	-2.91	-2.31	-2.33
$10^3 L_4^r$	$\equiv 0$	$\equiv 0$	$\equiv 0.5$	$\equiv 0.2$
$10^3 L_5^r$	0.97 ± 0.11	1.46	0.82	0.88
$10^3 L_6^r$	$\equiv 0$	$\equiv 0$	$\equiv 0.1$	$\equiv 0$
$10^3 L_7^r$	-0.31 ± 0.14	-0.49	-0.26	-0.28
$10^3 L_8^r$	0.60 ± 0.18	1.00	0.50	0.54

- errors are very correlated
- $\mu = 770$ MeV; 550 or 1000 within errors
- varying C_i^r factor 2 about errors
- $L_4^r, L_6^r \approx -0.3, \ldots, 0.6\ 10^{-3}$ OK
- fit B: small corrections to pion “sigma” term, fit scalar radius
- fit D: fit $\pi\pi$ and πK thresholds

Correlations



(older fit)

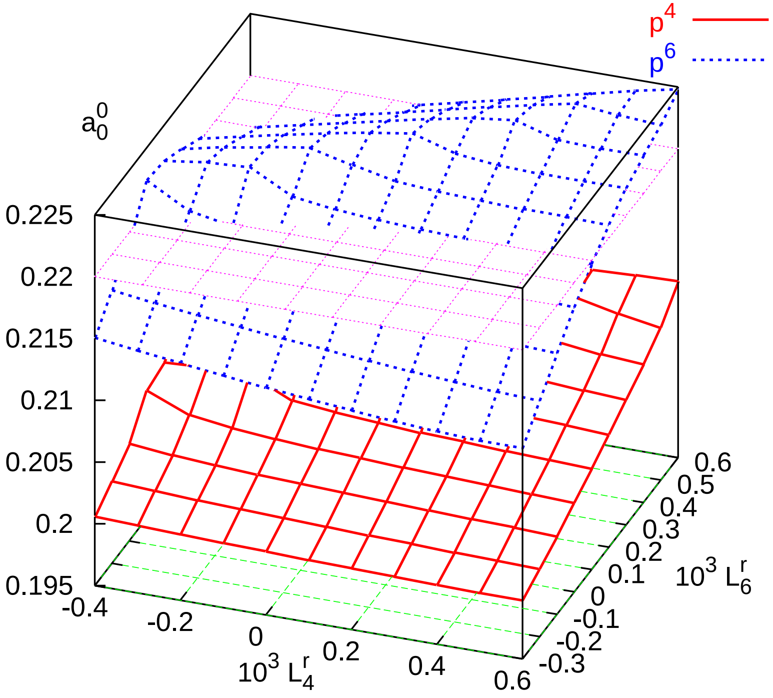
$$10^3\, L^r_1 = 0.52 \pm 0.23$$
$$10^3\, L^r_2 = 0.72 \pm 0.24$$
$$10^3\, L^r_3 = -2.70 \pm 0.99$$

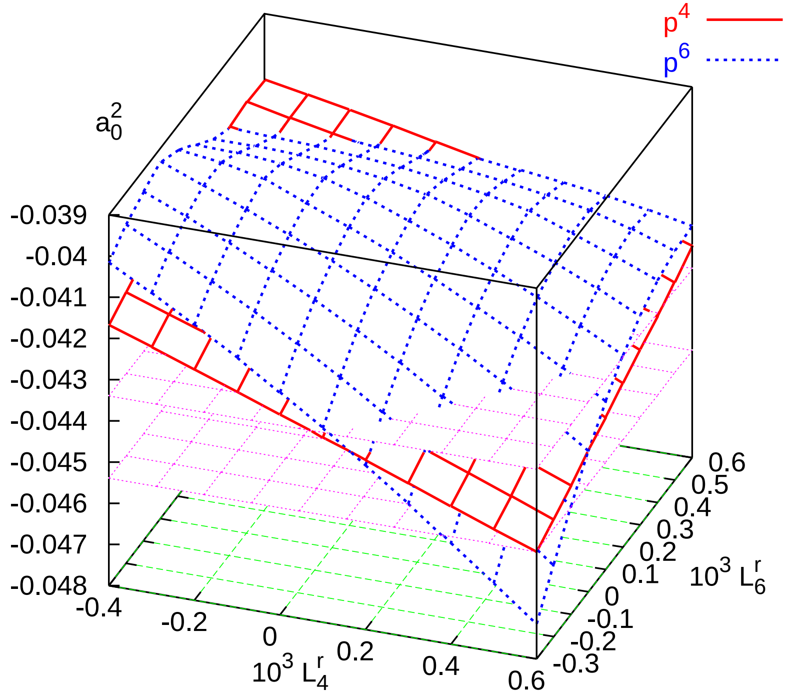
Outputs: II

	fit 10	same p^4	fit B	fit D
$2B_0\hat{m}/m_\pi^2$	0.736	0.991	1.129	0.958
m_π^2: p^4, p^6	0.006,0.258	0.009,$\equiv 0$	$-$0.138,0.009	$-$0.091,0.133
m_K^2: p^4, p^6	0.007,0.306	0.075,$\equiv 0$	$-$0.149,0.094	$-$0.096,0.201
m_η^2: p^4, p^6	$-$0.052,0.318	0.013,$\equiv 0$	$-$0.197,0.073	$-$0.151,0.197
m_u/m_d	0.45$\pm$0.05	0.52	0.52	0.50
F_0 [MeV]	87.7	81.1	70.4	80.4
$\frac{F_K}{F_\pi}$: p^4, p^6	0.169,0.051	0.22,$\equiv 0$	0.153,0.067	0.159,0.061

- $m_u = 0$ always very far from the fits
- F_0: pion decay constant in the chiral limit

$\pi\pi$



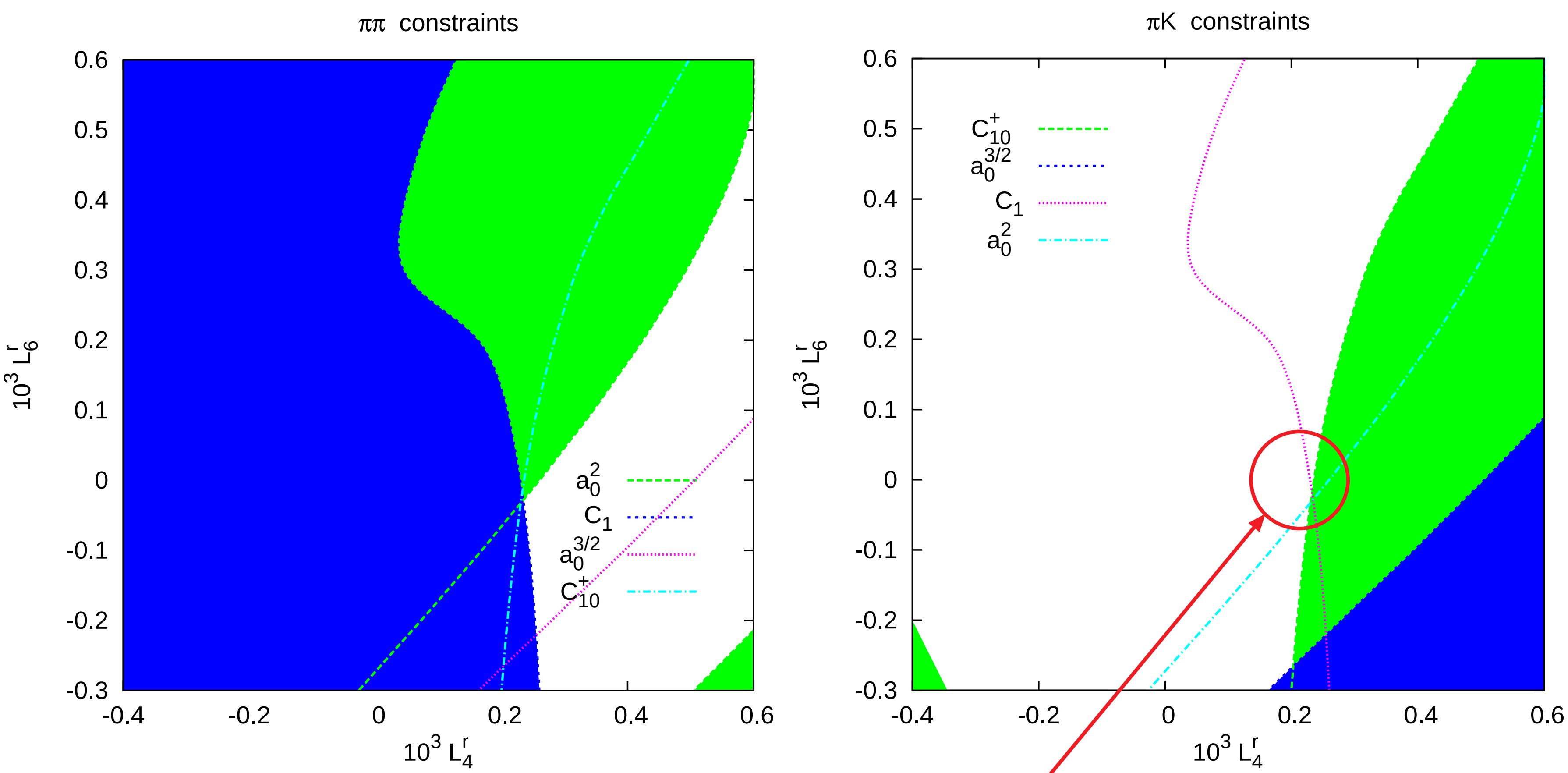


$a_0^0 = 0.220 \pm 0.005$, $a_0^2 = -0.0444 \pm 0.0010$

Colangelo, Gasser, Leutwyler

$a_0^0 = 0.159$ $a_0^2 = -0.0454$ at order p^2

$\pi\pi$ and πK



preferred region: fit D: $10^3 L_4^r \approx 0.2$, $10^3 L_6^r \approx 0.0$

General fitting needs more work and systematic studies

Quark mass dependences

Updates of plots in

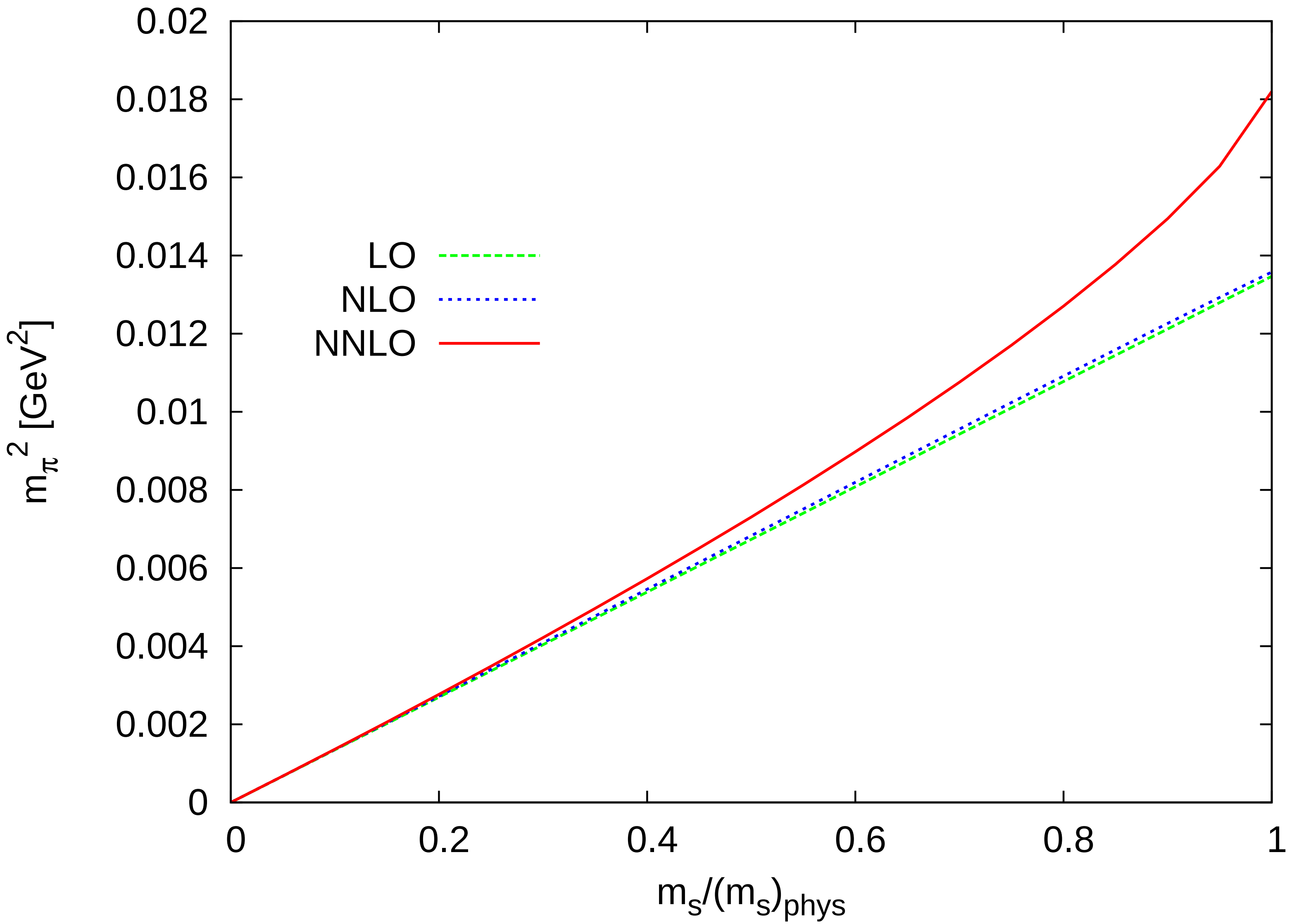
Amorós, JB and Talavera, hep-ph/0003258, Nucl. Phys. B585 (2000) 293

Some new ones

Procedure: calculate a consistent set of $m_\pi, m_K, m_\eta, f_\pi$ with the given input values (done iteratively)

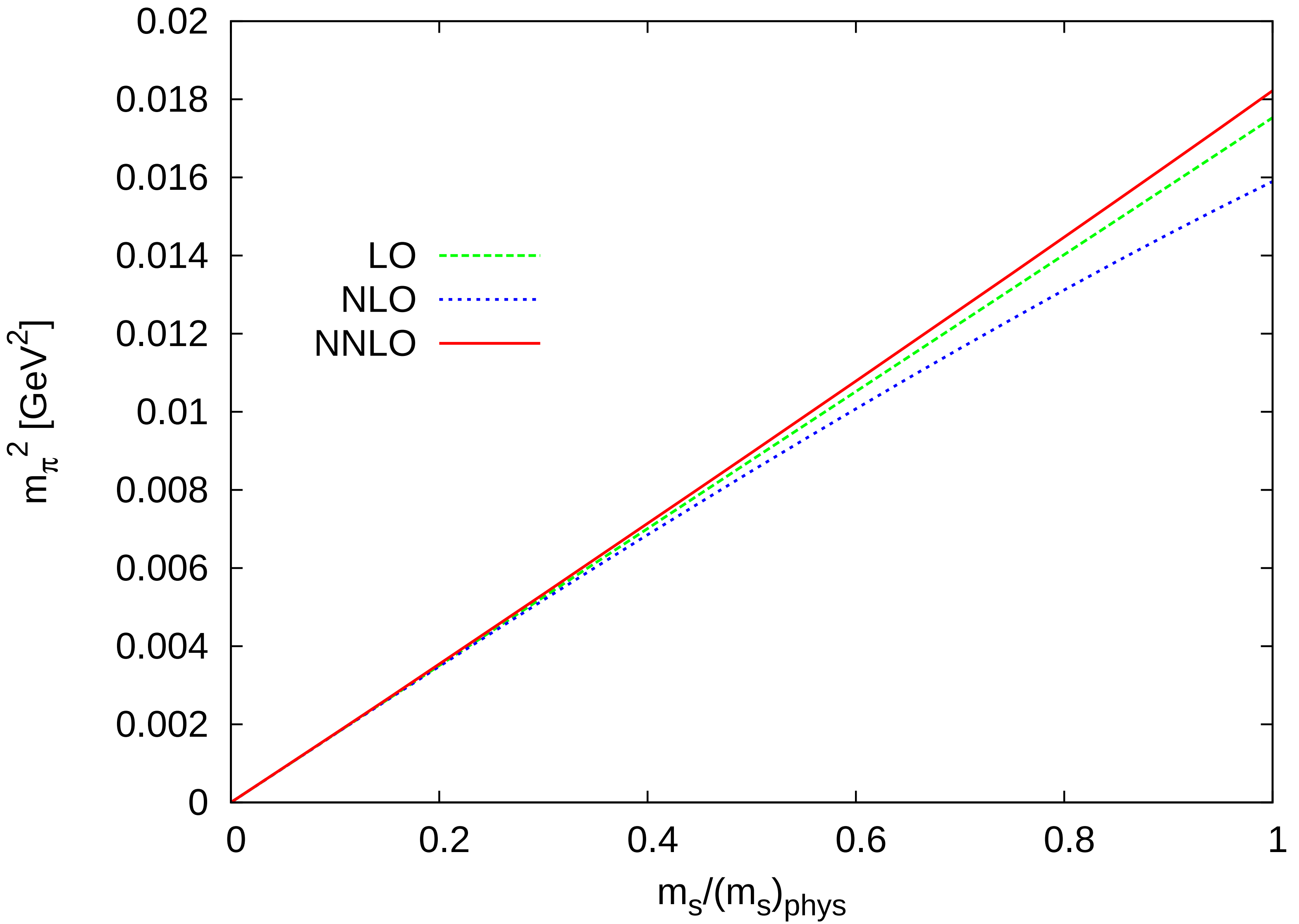
- vary $m_s/(m_s)_{phys}$, keep $m_s/\hat{m} = 24$
  m_π^2, m_K^2, F_π, F_K
- vary $m_s/(m_s)_{phys}$ keep $\hat{m}$ fixed
  m_π^2, F_π
- vary m_π, keep m_K fixed
  $f_+(0)$: the formfactor in $K_{\ell 3}$ decays
  $f_+(0)$,$f_+(0)/\left(m_K^2 - m_\pi^2\right)^2$

m_π^2 fit 10

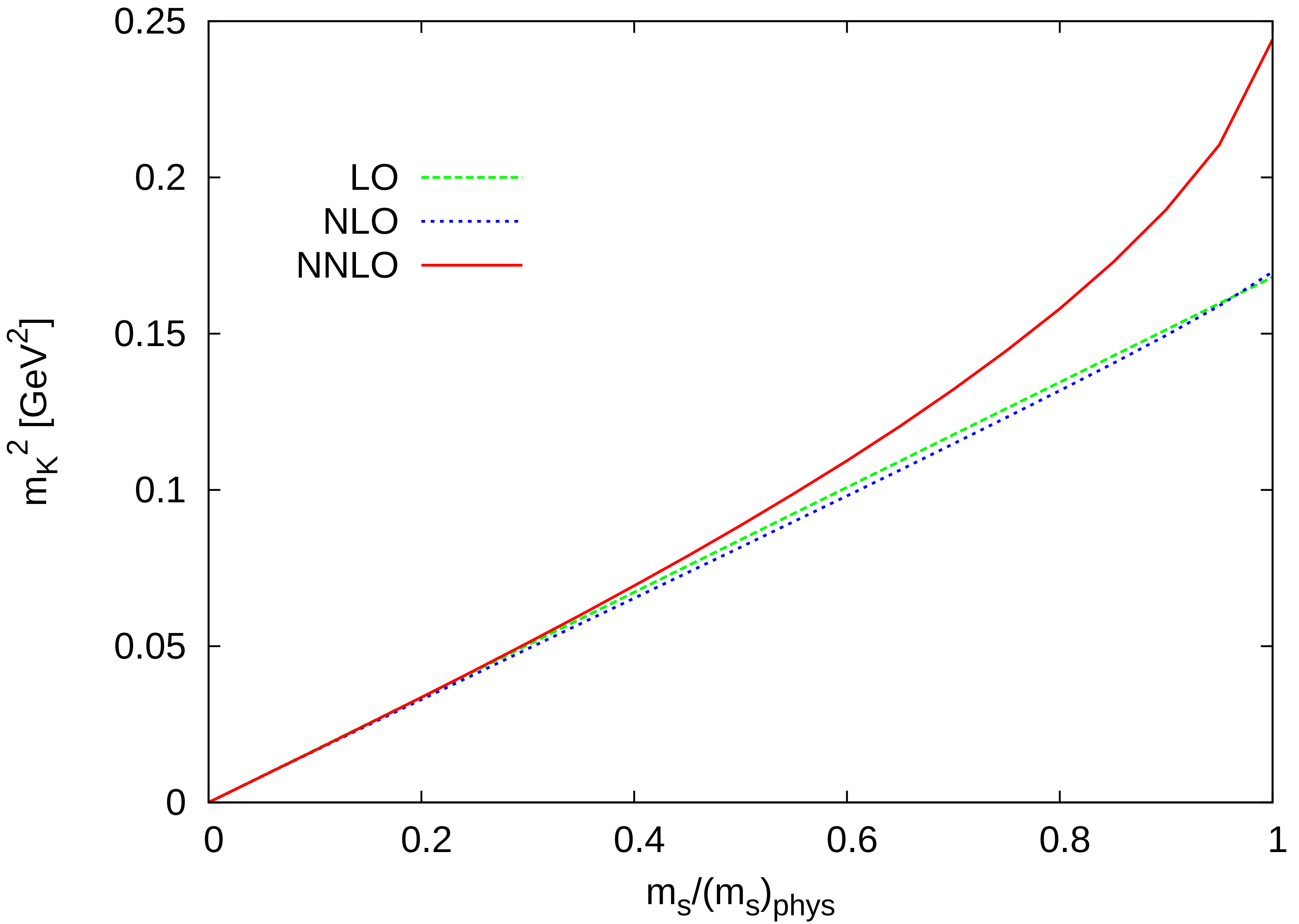


0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
0.2
0.4
0.6
0.8
1
LO
NLO
NNLO
m_π^2 [GeV^2]
$m_s/(m_s)_{phys}$

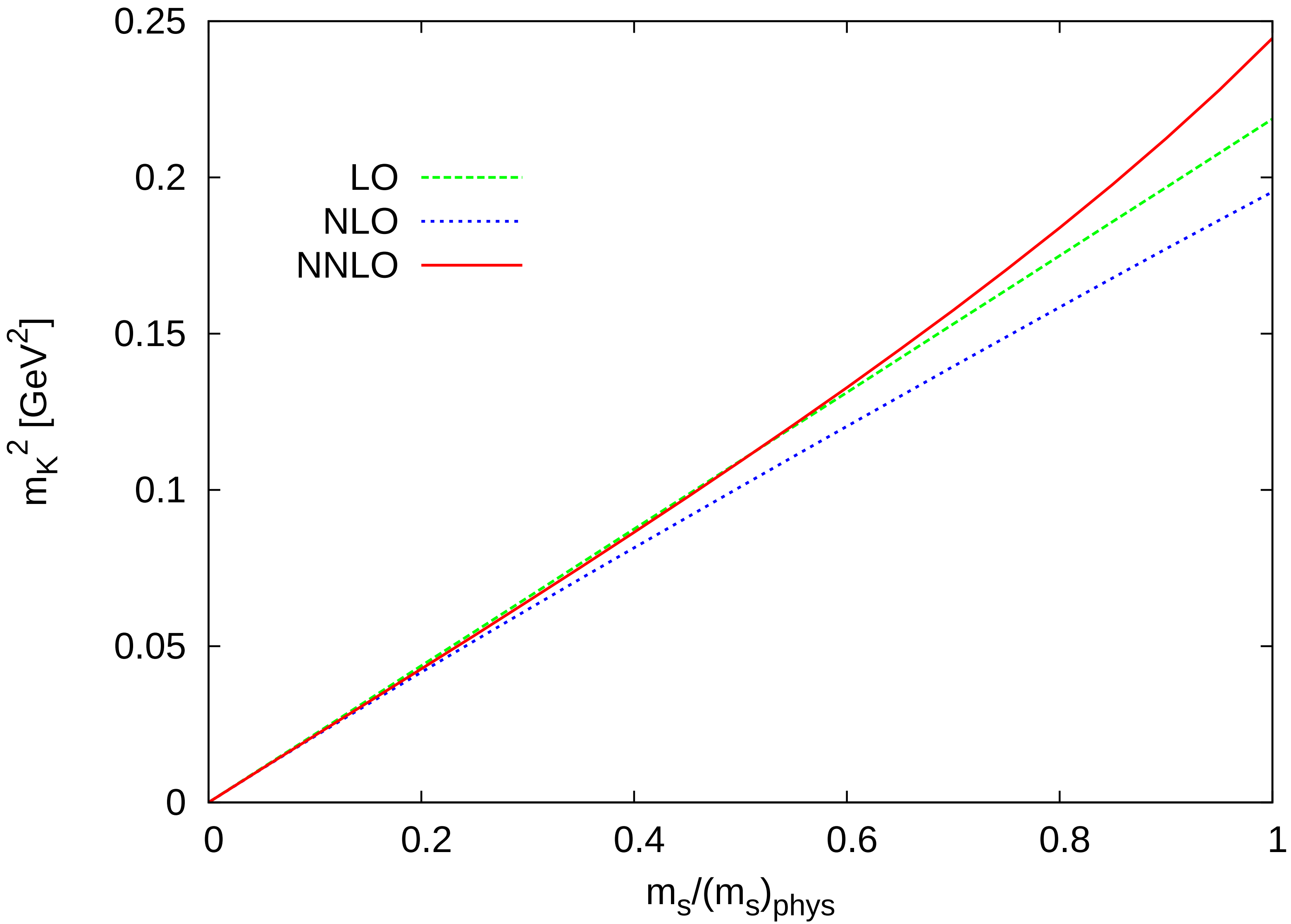
m_π^2 fit D


0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
0.2
0.4
0.6
0.8
1
LO
NLO
NNLO
m_π^2 [GeV^2]
$m_s/(m_s)_{phys}$

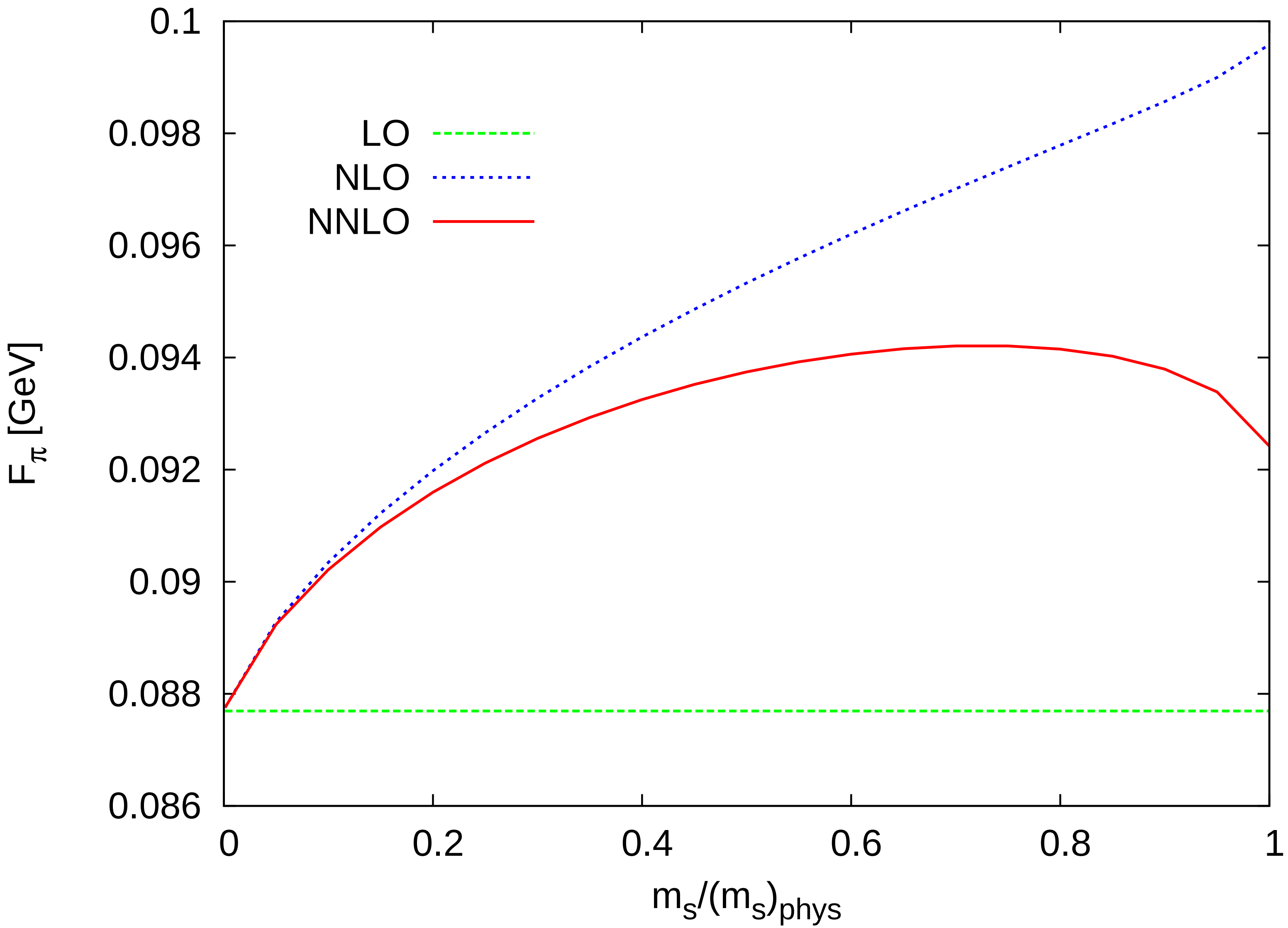
m_K^2 fit 10



m_K^2 fit D

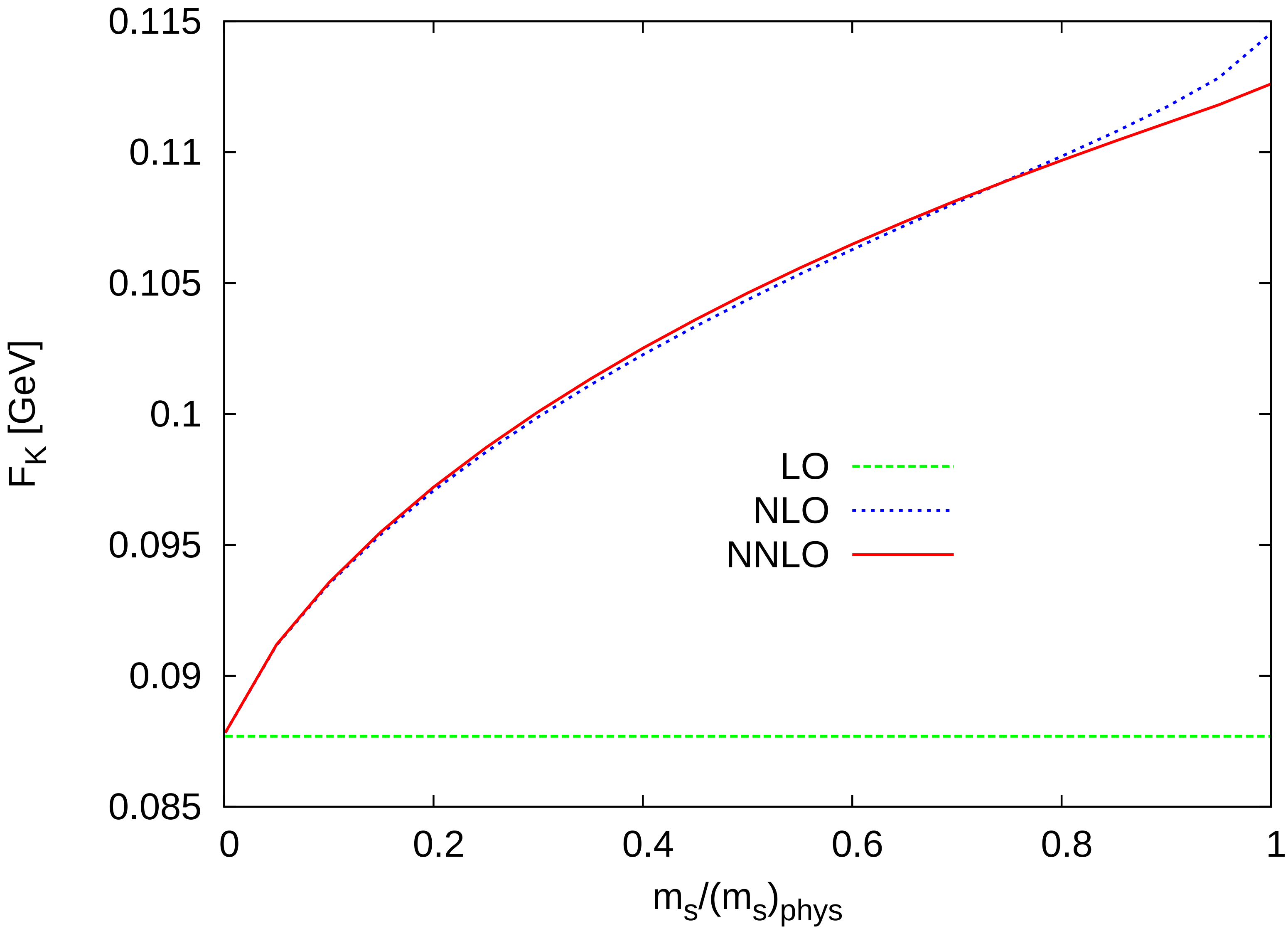


F_π fit 10

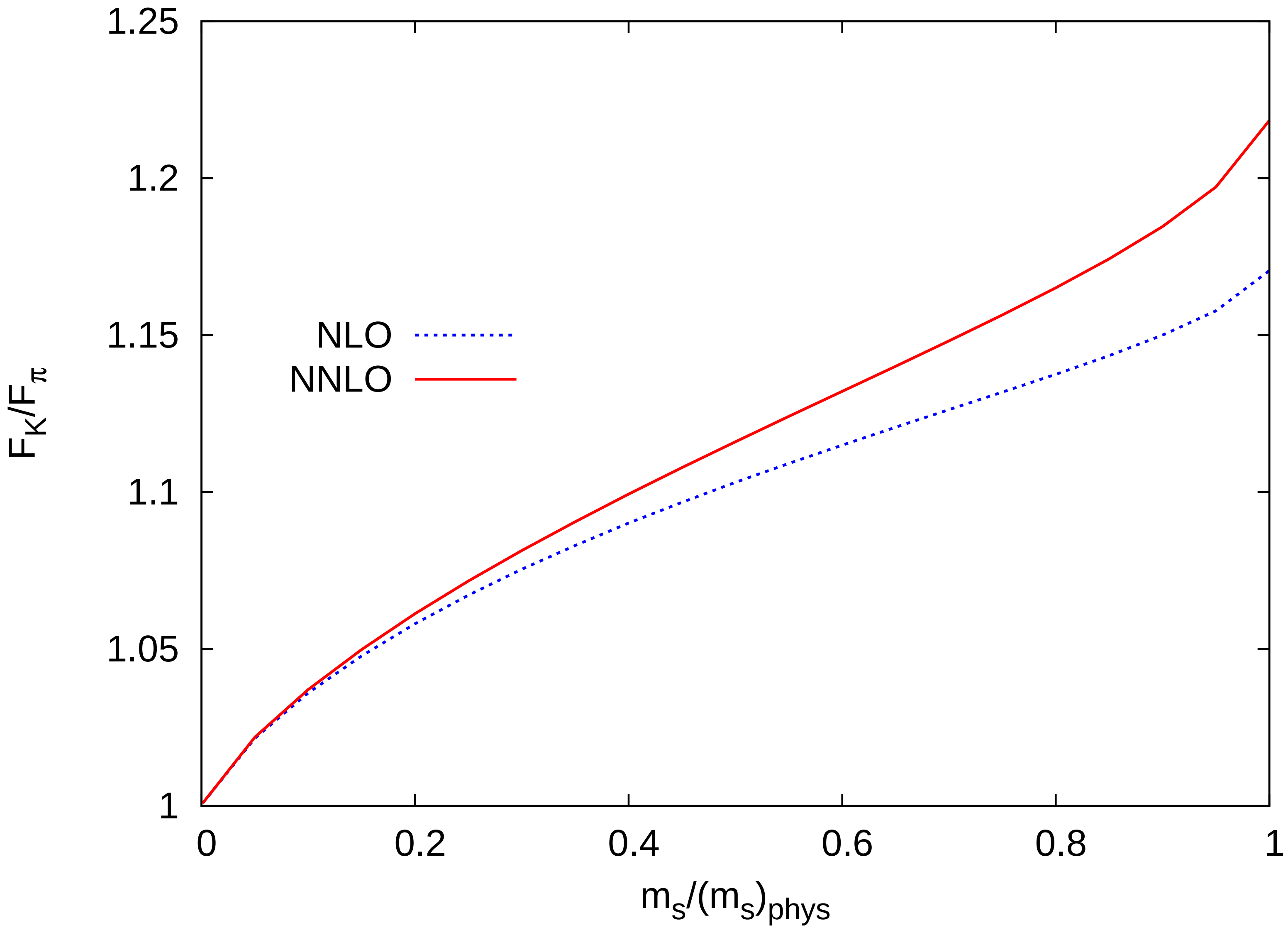


0.1
0.098
0.096
0.094
0.092
0.09
0.088
0.086
0
0.2
0.4
0.6
0.8
1
LO
NLO
NNLO
F_π [GeV]
$m_s/(m_s)_{phys}$

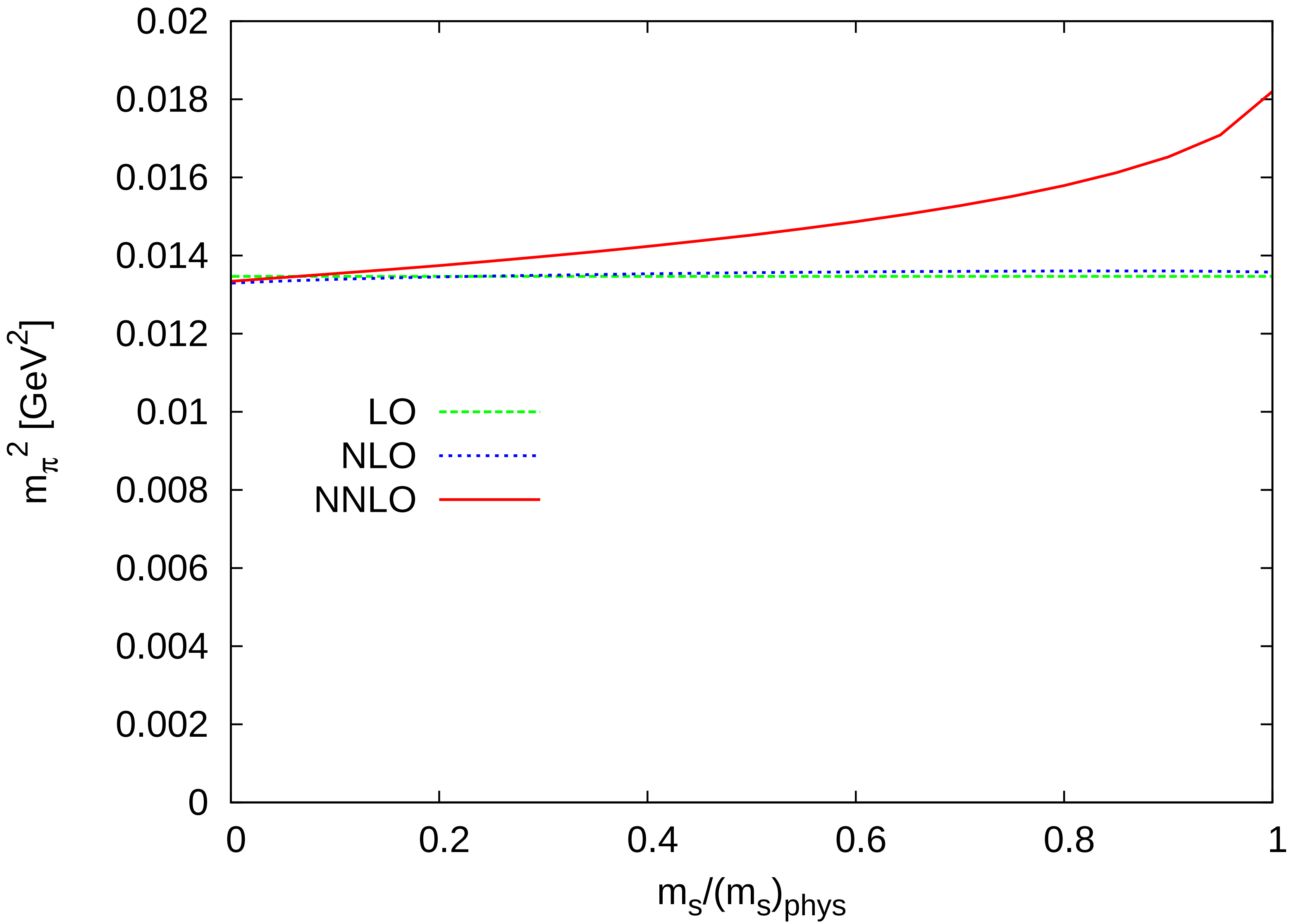
F_K fit 10


0.115
0.11
0.105
0.1
0.095
0.09
0.085
0
0.2
0.4
0.6
0.8
1
F_K [GeV]
$m_s/(m_s)_{phys}$
LO
NLO
NNLO

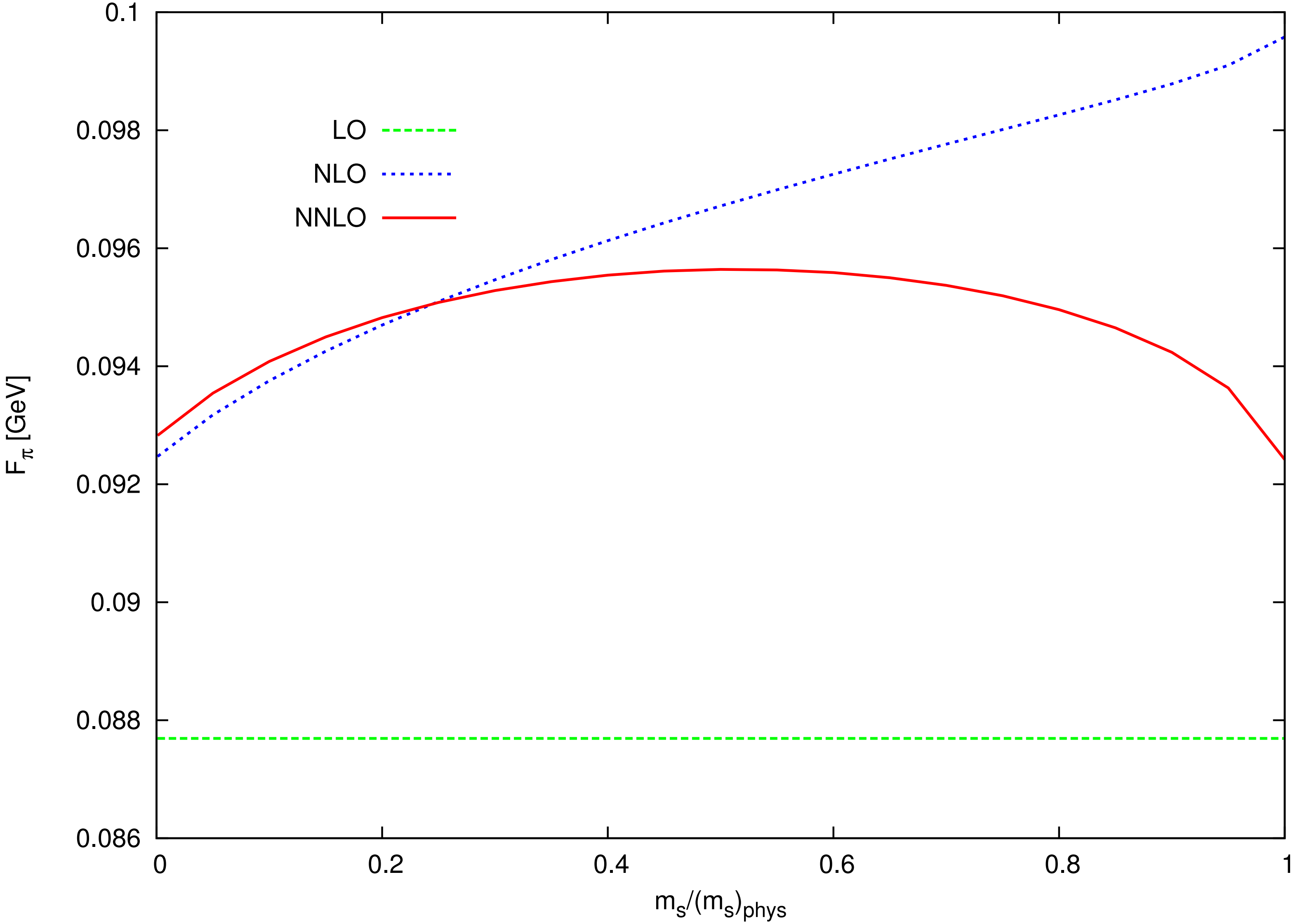
F_K/F_π fit 10



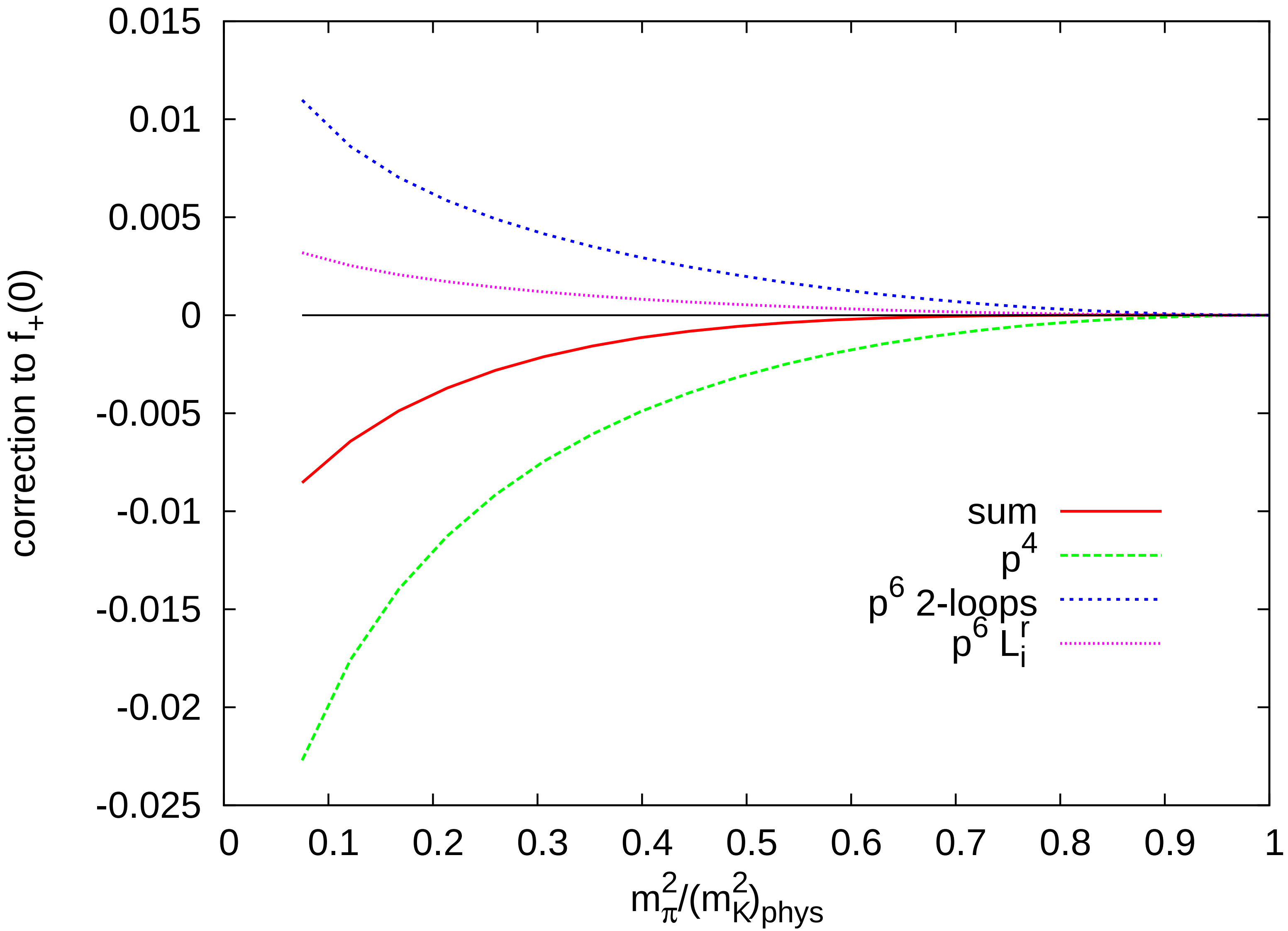
m_π^2 fit 10, fixed $\hat{m}$



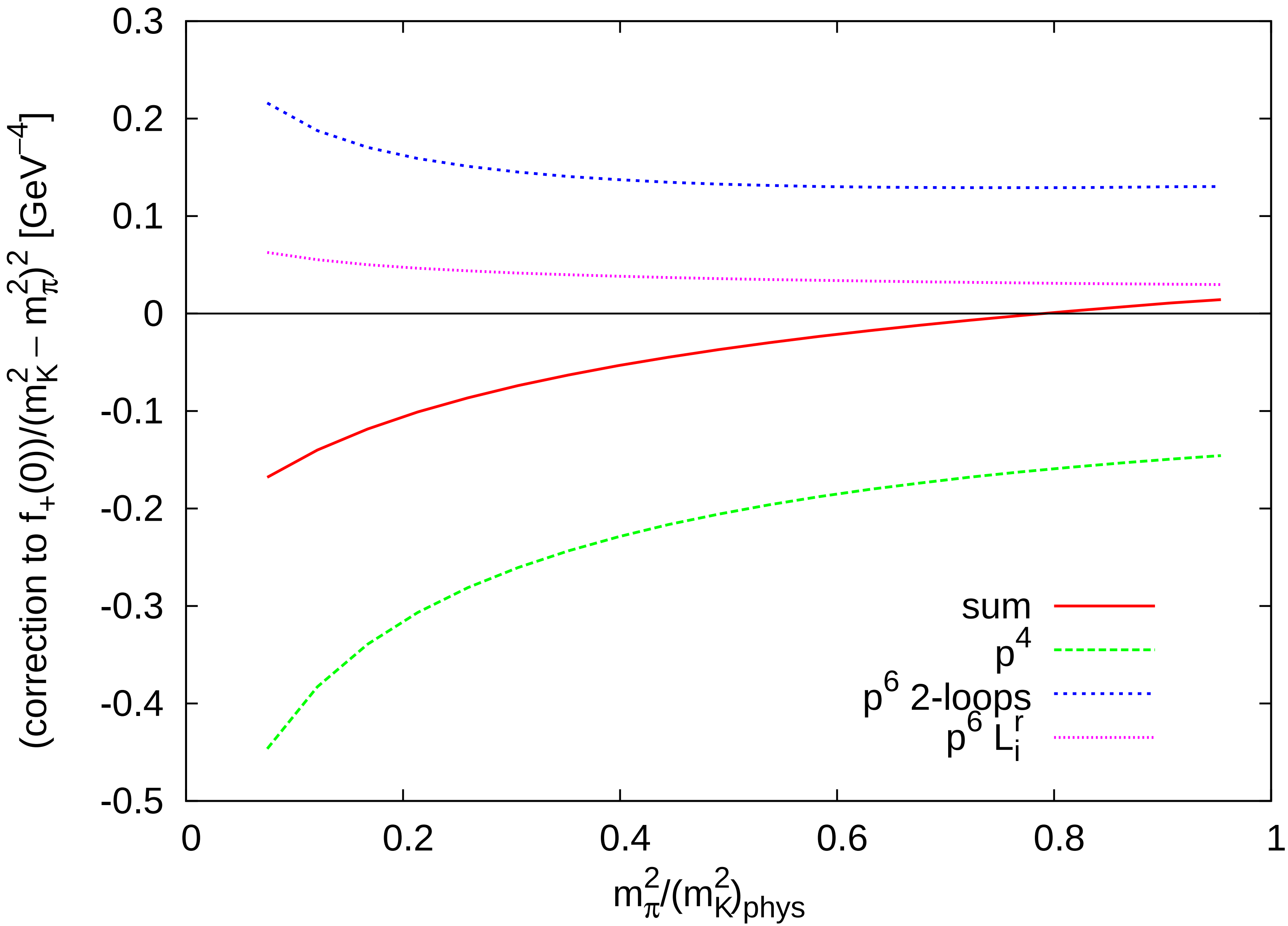
F_π fit 10, fixed $\hat{m}$



$f_+(0)$ fit 10, fixed m_K



$(f_+(0))/\left(m_K^2 - m_\pi^2\right)^2$ fit 10, fixed m_K



$f_0(t)$ in $K_{\ell 3}$

Main Result: JB,Talavera

$$
\begin{aligned}
f_0(t) \;=\;& 1 - \frac{8}{F_\pi^4}\left(C_{12}^r + C_{34}^r\right)\left(m_K^2 - m_\pi^2\right)^2 \\
& +8\frac{t}{F_\pi^4}\left(2C_{12}^r + C_{34}^r\right)\left(m_K^2 + m_\pi^2\right) + \frac{t}{m_K^2 - m_\pi^2}\left(F_K/F_\pi - 1\right) \\
& -\frac{8}{F_\pi^4} t^2 C_{12}^r + \overline{\Delta}(t) + \Delta(0)\,.
\end{aligned}
$$

$\overline{\Delta}(t)$ and $\Delta(0)$ contain NO C_i^r and only depend on the L_i^r at order p^6

$\Longrightarrow$

All needed parameters can be determined experimentally

$$\Delta(0) = -0.0080 \pm 0.0057[\text{loops}] \pm 0.0028[L_i^r]\,.$$

≥ 3-flavour: PQChPT

Essentially all manipulations from ChPT go through to PQChPT when changing trace to supertrace and adding fermionic variables

Exceptions: baryons and Cayley-Hamilton relations

So Luckily: can use the n flavour work in ChPT at two loop order to obtain for PQChPT: Lagrangians and infinities

Very important note: ChPT is a limit of PQChPT
$\Longrightarrow$ LECs from ChPT are linear combinations of LECs of PQChPT with the same number of sea quarks.

E.g. $L_1^r = L_0^{r(3pq)}/2 + L_1^{r(3pq)}$

PQChPT

One-loop: Bernard, Golterman, Sharpe, Shoresh, Pallante,. . .

with electromagnetism: JB,Danielsson, hep-lat/0610127

Two loops:

$m^2_{\pi^+}$ simplest mass case: JB,Danielsson,Lähde, hep-lat/0406017

F_{π^+}: JB,Lähde, hep-lat/0501014

F_{π^+}, $m^2_{\pi^+}$ two sea quarks: JB,Lähde, hep-lat/0506004

$m^2_{\pi^+}$: JB,Danielsson,Lähde, hep-lat/0602003

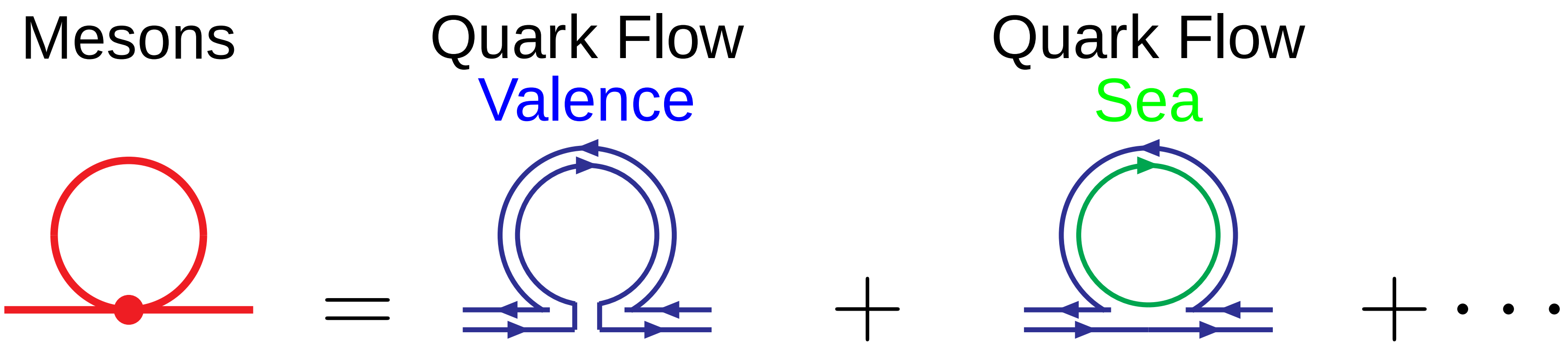
Neutral masses: JB,Danielsson, hep-lat/0606017

Lattice data: a and L extrapolations needed

Programs available from me (Fortran)

Formulas: `http://www.thep.lu.se/∼bijnens/chpt.html`

Partial Quenching and ChPT



Lattice QCD: Valence is easy to deal with, Sea very difficult

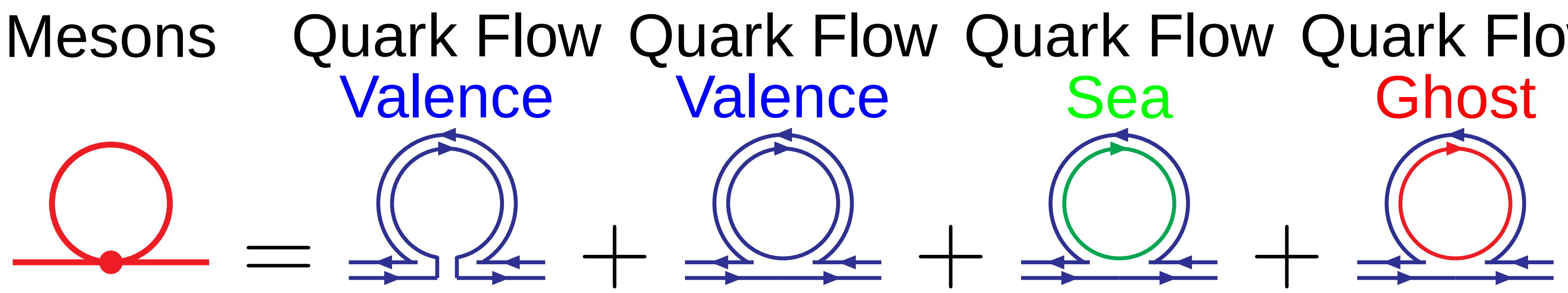
They can be treated separately: i.e. different quark masses
Partially Quenched QCD and ChPT (PQChPT)

One Loop or p^4: Bernard, Golterman, Pallante, Sharpe, Shoresh,...

Two Loops or p^6: This talk JB, Niclas Danielsson, Timo Lähde

PQChPT at Two Loops: General

Add ghost quarks: remove the unwanted free valence loops



Possible problem: QCD $\Longrightarrow$ ChPT relies heavily on unitarity

Partially quenched: at least one dynamical sea quark
$\Longrightarrow$ Φ_0 is heavy: remove from PQChPT

Symmetry group becomes $SU(n_v + n_s|n_v) \times SU(n_v + n_s|n_v)$ (approximately)

PQChPT at Two Loops: General

Essentially all manipulations from ChPT go through to PQChPT when changing trace to supertrace and adding fermionic variables

Exceptions: baryons and Cayley-Hamilton relations

So Luckily: can use the n flavour work in ChPT at two loop order to obtain for PQChPT: Lagrangians and infinities

Very important note: ChPT is a limit of PQChPT
$\Longrightarrow$ LECs from ChPT are linear combinations of LECs of PQChPT with the same number of sea quarks.

E.g. $L_1^r = L_0^{r(3pq)}/2 + L_1^{r(3pq)}$

PQChPT at Two Loop: Papers

valence equal mass, 3 sea equal mass:

$m^2_{\pi^+}$: JB,Danielsson,Lähde, hep-lat/0406017

Other mass combinations:

$F^2_{\pi^+}$: JB,Lähde, hep-lat/0501014

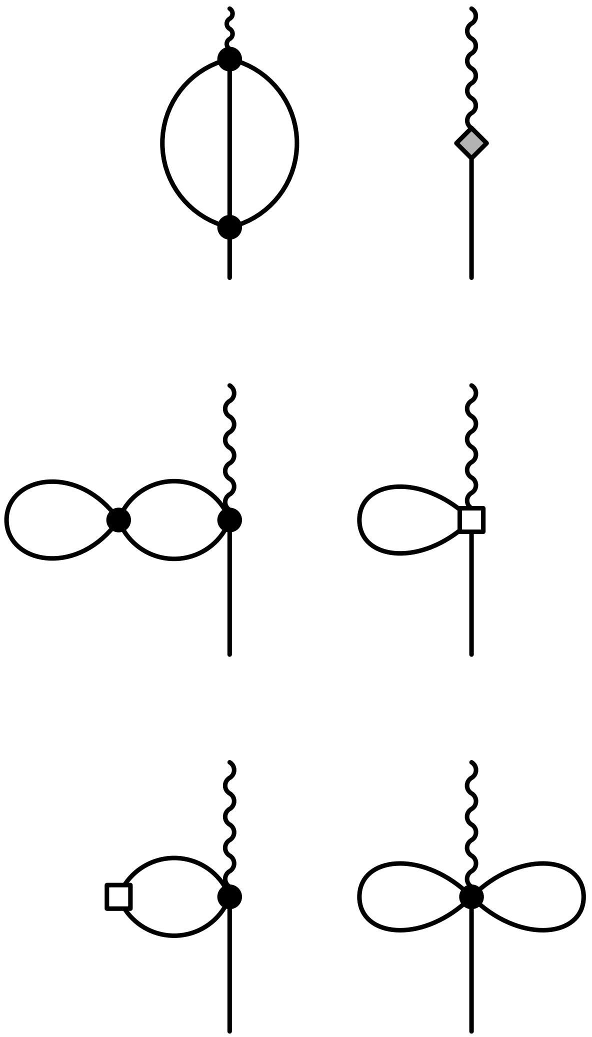
$F^2_{\pi^+}$, $m^2_{\pi^+}$ two sea quarks: JB,Lähde, hep-lat/0506004

In progress: the other charged masses

Actual Calculations:
- ➠ heavy use of FORM Vermaseren
- ➠ Main problem: sheer size of the expressions

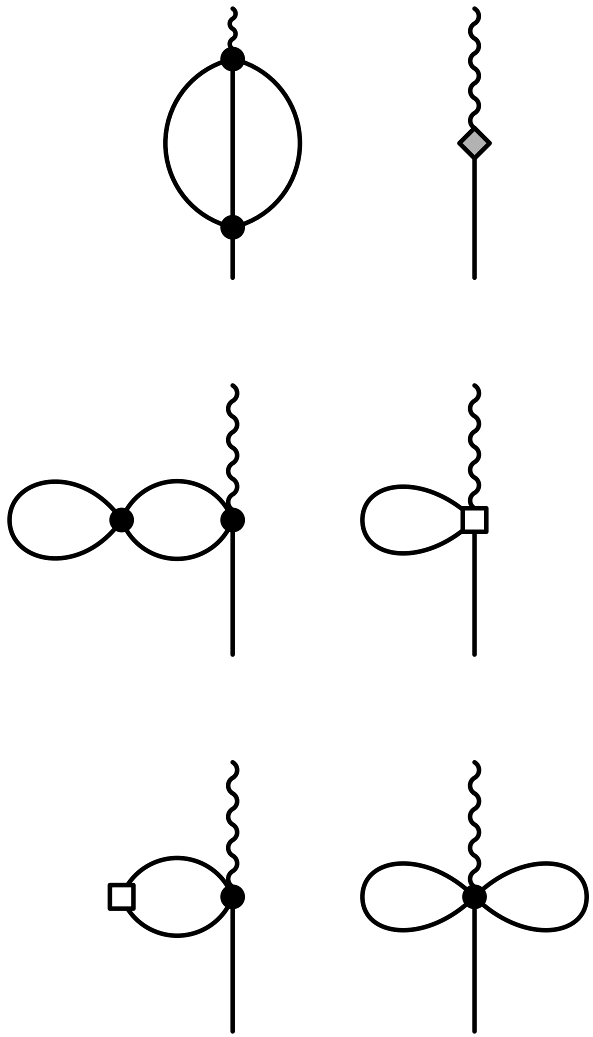
No fits to lattice data (yet): a and L extrapolations needed

Long Expressions



$\Longrightarrow$

Long Expressions



$$\Longrightarrow$$

$$
\begin{aligned}
\delta^{(6)22}_{\text{loops}} = \;& \pi_{16}\, L^r_0\; [4/9\,\chi_\eta\chi_4 - 1/2\,\chi_1\chi_3 + \chi_{13}^2 - 13/3\,\bar\chi_1\chi_{13} - 35/18\,\bar\chi_2] \;-\; 2\,\pi_{16}\, L^r_1\, \chi_{13}^2 \\
-\;& \pi_{16}\, L^r_2\; [11/3\,\chi_\eta\chi_4 + \chi_{13}^2 + 13/3\,\bar\chi_2] + \pi_{16}\, L^r_3\; [4/9\,\chi_\eta\chi_4 - 7/12\,\chi_1\chi_3 + 11/6\,\chi_{13}^2 - 17/6\,\bar\chi_1\chi_{13} - 43/36\,\bar\chi_2] \\
+\;& \pi_{16}^2\; [-15/64\,\chi_\eta\chi_4 - 59/384\,\chi_1\chi_3 + 65/384\,\chi_{13}^2 - 1/2\,\bar\chi_1\chi_{13} - 43/128\,\bar\chi_2] \;-\; 48\, L^r_4 L^r_5\, \bar\chi_1\chi_{13} \;-\; 72\, L^{r2}_4\, \bar\chi_1^2 \\
-\;& 8\, L^{r2}_5\, \chi_{13}^2 \;+\; \bar A(\chi_p)\,\pi_{16}\; [-1/24\,\chi_p + 1/48\,\bar\chi_1 - 1/8\,\bar\chi_1\, R^p_{q\eta} + 1/16\,\bar\chi_1\, R^c_p - 1/48\, R^p_{q\eta}\,\chi_p - 1/16\, R^p_{q\eta}\,\chi_q \\
+\;& 1/48\, R^\eta_{pp}\,\chi_\eta + 1/16\, R^c_p\,\chi_{13}] \;+\; \bar A(\chi_p)\, L^r_0\; [8/3\, R^p_{q\eta}\,\chi_p + 2/3\, R^c_p\,\chi_p + 2/3\, R^d_p] \;+\; \bar A(\chi_p)\, L^r_3\; [2/3\, R^p_{q\eta}\,\chi_p \\
+\;& 5/3\, R^c_p\,\chi_p + 5/3\, R^d_p] \;+\; \bar A(\chi_p)\, L^r_4\; [-2\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 0} - 2\,\bar\chi_1\, R^p_{q\eta} + 3\,\bar\chi_1\, R^c_p] \;+\; \bar A(\chi_p)\, L^r_5\; [-2/3\,\bar\chi^{pp}_{\eta\eta 1} - R^p_{q\eta}\,\chi_p \\
+\;& 1/3\, R^p_{q\eta}\,\chi_q + 1/2\, R^c_p\,\chi_p - 1/6\, R^c_p\,\chi_q] \;+\; \bar A(\chi_p)^2\; [1/16 + 1/72\,(R^p_{q\eta})^2 - 1/72\, R^p_{q\eta} R^c_p + 1/288\,(R^c_p)^2] \\
+\;& \bar A(\chi_p)\bar A(\chi_{ps})\; [-1/36\, R^p_{q\eta} - 5/72\, R^p_{s\eta} + 7/144\, R^c_p] \;-\; \bar A(\chi_p)\bar A(\chi_{qs})\; [1/36\, R^p_{q\eta} + 1/24\, R^p_{s\eta} + 1/48\, R^c_p] \\
+\;& \bar A(\chi_p)\bar A(\chi_\eta)\; [-1/72\, R^p_{q\eta} R^v_{\eta 13} + 1/144\, R^c_p R^v_{\eta 13}] \;+\; 1/8\,\bar A(\chi_p)\bar A(\chi_{13}) \;+\; 1/12\,\bar A(\chi_p)\bar A(\chi_{46})\, R^\eta_{pp} \\
+\;& \bar A(\chi_p)\bar B(\chi_p,\chi_p;0)\; [1/4\,\chi_p - 1/18\, R^p_{q\eta} R^c_p\,\chi_p - 1/72\, R^p_{q\eta} R^d_p + 1/18\,(R^c_p)^2\,\chi_p + 1/144\, R^c_p R^d_p] \\
+\;& \bar A(\chi_p)\bar B(\chi_p,\chi_\eta;0)\; [1/18\, R^\eta_{pp} R^c_p\,\chi_p - 1/18\, R^\eta_{13} R^c_p\,\chi_p] \;+\; \bar A(\chi_p)\bar B(\chi_q,\chi_q;0)\, [-1/72\, R^p_{q\eta} R^d_q + 1/144\, R^c_p R^d_q] \\
-\;& 1/12\,\bar A(\chi_p)\bar B(\chi_{ps},\chi_{ps};0)\, R^p_{s\eta}\,\chi_{ps} \;-\; 1/18\,\bar A(\chi_p)\bar B(\chi_1,\chi_3;0)\, R^q_{p\eta} R^c_p\,\chi_p \\
+\;& 1/18\,\bar A(\chi_p)\bar C(\chi_p,\chi_p,\chi_p;0)\, R^c_p R^d_p\,\chi_p \;+\; \bar A(\chi_p;\varepsilon)\,\pi_{16}\; [1/8\,\bar\chi_1 R^p_{q\eta} - 1/16\bar\chi_1\, R^c_p - 1/16\, R^c_p\,\chi_p - 1/16\, R^d_p] \\
+\;& \bar A(\chi_{ps})\,\pi_{16}\; [1/16\,\chi_{ps} - 3/16\,\chi_{qs} - 3/16\,\bar\chi_1] \;-\; 2\,\bar A(\chi_{ps})\, L^r_0\,\chi_{ps} \;-\; 5\,\bar A(\chi_{ps})\, L^r_3\,\chi_{ps} \;-\; 3\,\bar A(\chi_{ps})\, L^r_4\,\bar\chi_1 \\
+\;& \bar A(\chi_{ps})\, L^r_5\,\chi_{13} \;+\; \bar A(\chi_{ps})\bar A(\chi_\eta)\; [7/144\, R^\eta_{pp} - 5/72\, R^\eta_{ps} - 1/48\, R^\eta_{qq} + 5/72\, R^\eta_{qs} - 1/36\, R^\eta_{13}] \\
+\;& \bar A(\chi_{ps})\bar B(\chi_p,\chi_p;0)\; [1/24\, R^p_{s\eta}\,\chi_p - 5/24\, R^p_{s\eta}\,\chi_{ps}] \;+\; \bar A(\chi_{ps})\bar B(\chi_p,\chi_\eta;0)\; [-1/18\, R^\eta_{ps} R^z_{qp\eta}\,\chi_p \\
-\;& 1/9\, R^\eta_{ps} R^z_{qp\eta}\,\chi_{ps}] \;-\; 1/48\,\bar A(\chi_{ps})\bar B(\chi_q,\chi_q;0)\, R^d_q \;+\; 1/18\,\bar A(\chi_{ps})\bar B(\chi_1,\chi_3;0)\, R^q_{s\eta}\,\chi_s \\
+\;& 1/9\,\bar A(\chi_{ps})\bar B(\chi_1,\chi_3;0,k)\, R^q_{s\eta} \;+\; 3/16\,\bar A(\chi_{ps};\varepsilon)\,\pi_{16}\; [\chi_s + \bar\chi_1] \;-\; 1/8\,\bar A(\chi_{p4})^2 \;-\; 1/8\,\bar A(\chi_{p4})\bar A(\chi_{p6}) \\
+\;& 1/8\,\bar A(\chi_{p4})\bar A(\chi_{q6}) \;-\; 1/32\,\bar A(\chi_{p6})^2 \;+\; \bar A(\chi_\eta)\,\pi_{16}\; [1/16\,\bar\chi_1\, R^v_{\eta 13} - 1/48\, R^v_{\eta 13}\,\chi_\eta + 1/16\, R^v_{\eta 13}\,\chi_{13}] \\
+\;& \bar A(\chi_\eta)\, L^r_0\; [4 R^\eta_{13}\,\chi_\eta + 2/3\, R^v_{\eta 13}\,\chi_\eta] \;-\; 8\,\bar A(\chi_\eta)\, L^r_1\,\chi_\eta \;-\; 2\,\bar A(\chi_\eta)\, L^r_2\,\chi_\eta \;+\; \bar A(\chi_\eta)\, L^r_3\; [4 R^\eta_{13}\,\chi_\eta + 5/3\, R^v_{\eta 13}\chi_\eta] \\
+\;& \bar A(\chi_\eta)\, L^r_4\; [4\,\chi_\eta + \bar\chi_1\, R^v_{\eta 13}] \;-\; \bar A(\chi_\eta)\, L^r_5\; [1/6\, R^\eta_{pp}\,\chi_q + R^\eta_{13}\,\chi_{13} + 1/6\, R^v_{\eta 13}\,\chi_\eta] \;+\; 1/288\,\bar A(\chi_\eta)^2\,(R^v_{\eta 13})^2 \\
+\;& 1/12\,\bar A(\chi_\eta)\bar A(\chi_{46})\, R^v_{\eta 13} \;+\; \bar A(\chi_\eta)\bar B(\chi_p,\chi_p;0)\; [-1/36\,\bar\chi^{pp\eta}_{\eta\eta 1} - 1/18\, R^p_{q\eta} R^\eta_{pp}\,\chi_p + 1/18\, R^\eta_{pp} R^c_p\,\chi_p \\
+\;& 1/144\, R^d_p R^v_{\eta 13}] \;+\; \bar A(\chi_\eta)\bar B(\chi_p,\chi_\eta;0)\; [-1/18\,\bar\chi^{\eta p\eta}_{p\eta 1} + 1/18\,\bar\chi^{\eta p\eta}_{q\eta 1} + 1/18\,(R^\eta_{pp})^2 R^z_{qp\eta}\,\chi_p] \\
-\;& 1/12\,\bar A(\chi_\eta)\bar B(\chi_{ps},\chi_{ps};0)\, R^\eta_{ps}\,\chi_{ps} \;-\; \bar A(\chi_\eta)\bar B(\chi_\eta,\chi_\eta;0)\; [1/216\, R^v_{\eta 13}\,\chi_4 + 1/27\, R^v_{\eta 13}\,\chi_6] \\
-\;& 1/18\,\bar A(\chi_\eta)\bar B(\chi_1,\chi_3;0)\, R^1_{\eta\eta} R^3_{\eta\eta}\,\chi_\eta \;+\; 1/18\,\bar A(\chi_\eta)\bar C(\chi_p,\chi_p,\chi_p;0)\, R^\eta_{pp} R^d_p\,\chi_p \;+\; \bar A(\chi_\eta;\varepsilon)\,\pi_{16}\; [1/8\,\chi_\eta \\
-\;& 1/16\,\bar\chi_1\, R^v_{\eta 13} - 1/8\, R^\eta_{13}\,\chi_\eta - 1/16\, R^v_{\eta 13}\,\chi_\eta] \;+\; \bar A(\chi_1)\bar A(\chi_3)\; [-1/72\, R^p_{q\eta} R^c_q + 1/36\, R^1_{3\eta} R^3_{1\eta} + 1/144\, R^c_1 R^c_3] \\
-\;& 4\,\bar A(\chi_{13})\, L^r_1\,\chi_{13} \;-\; 10\,\bar A(\chi_{13})\, L^r_2\,\chi_{13} \;+\; 1/8\,\bar A(\chi_{13})^2 \;-\; 1/2\,\bar A(\chi_{13})\bar B(\chi_1,\chi_3;0,k) \\
+\;& 1/4\,\bar A(\chi_{13};\varepsilon)\,\pi_{16}\,\chi_{13} \;+\; 1/4\,\bar A(\chi_{14})\bar A(\chi_{34}) \;+\; 1/16\,\bar A(\chi_{16})\bar A(\chi_{36}) \;-\; 24\,\bar A(\chi_4)\, L^r_1\,\chi_4 \;-\; 6\,\bar A(\chi_4)\, L^r_2\,\chi_4 \\
+\;& 12\,\bar A(\chi_4)\, L^r_4\,\chi_4 \;+\; 1/12\,\bar A(\chi_4)\bar B(\chi_p,\chi_p;0)\,(R^p_{4\eta})^2\,\chi_4 \;+\; 1/6\,\bar A(\chi_4)\bar B(\chi_p,\chi_\eta;0)\; [R^p_{4\eta} R^\eta_{p4}\,\chi_4 - R^p_{4\eta} R^\eta_{q4}\,\chi_4] \\
-\;& 1/24\,\bar A(\chi_4)\bar B(\chi_\eta,\chi_\eta;0)\, R^v_{\eta 13}\,\chi_4 \;-\; 1/6\,\bar A(\chi_4)\bar B(\chi_1,\chi_3;0)\, R^1_{4\eta} R^3_{4\eta}\,\chi_4 \;+\; 3/8\,\bar A(\chi_4;\varepsilon)\,\pi_{16}\,\chi_4 \\
-\;& 32\,\bar A(\chi_{46})\, L^r_1\,\chi_{46} \;-\; 8\,\bar A(\chi_{46})\, L^r_2\,\chi_{46} \;+\; 16\,\bar A(\chi_{46})\, L^r_4\,\chi_{46} \;+\; \bar A(\chi_{46})\bar B(\chi_p,\chi_p;0)\; [1/9\,\chi_{46} + 1/12\, R^\eta_{pp}\,\chi_p \\
+\;& 1/36\, R^\eta_{pp}\,\chi_4 + 1/9\, R^\eta_{p4}\,\chi_6] \;+\; \bar A(\chi_{46})\bar B(\chi_p,\chi_\eta;0)\; [-1/18\, R^\eta_{pp}\,\chi_4 - 1/9\, R^\eta_{p4}\,\chi_6 + 1/9\, R^\eta_{q4}\,\chi_6 + 1/18\, R^\eta_{13}\,\chi_4] \\
-\;& 1/6\,\bar A(\chi_{46})\bar B(\chi_p,\chi_\eta;0,k)\; [R^\eta_{pp} - R^\eta_{13}] \;+\; 1/9\,\bar A(\chi_{46})\bar B(\chi_\eta,\chi_\eta;0)\, R^v_{\eta 13}\,\chi_{46} \;-\; \bar A(\chi_{46})\bar B(\chi_1,\chi_3;0)\; [2/9\,\chi_{46} \\
+\;& 1/9\, R^\eta_{p4}\,\chi_6 + 1/18\, R^\eta_{13}\,\chi_4] \;-\; 1/6\,\bar A(\chi_{46})\bar B(\chi_1,\chi_3;0,k)\, R^\eta_{13} \;+\; 1/2\,\bar A(\chi_{46};\varepsilon)\,\pi_{16}\,\chi_{46} \\
+\;& \bar B(\chi_p,\chi_p;0)\,\pi_{16}\; [1/16\,\bar\chi_1\, R^d_p + 1/96\, R^d_p\,\chi_p + 1/32\, R^d_p\,\chi_q] \;+\; 2/3\,\bar B(\chi_p,\chi_p;0)\, L^r_0\, R^d_p\,\chi_p \\
+\;& 5/3\,\bar B(\chi_p,\chi_p;0)\, L^r_3\, R^d_p\,\chi_p \;+\; \bar B(\chi_p,\chi_p;0)\, L^r_4\; [-2\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 0}\chi_p - 4\,\bar\chi_1 R^p_{q\eta}\,\chi_p + 4\,\bar\chi_1 R^c_p\,\chi_p + 3\,\bar\chi_1 R^d_p] \\
+\;& \bar B(\chi_p,\chi_p;0)\, L^r_5\; [-2/3\bar\chi^{pp}_{\eta\eta 1}\chi_p - 4/3\, R^p_{q\eta}\,\chi_p^2 + 4/3\, R^c_p\,\chi_p^2 + 1/2\, R^d_p\,\chi_p - 1/6\, R^d_p\,\chi_q] \\
+\;& \bar B(\chi_p,\chi_p;0)\, L^r_6\; [4\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 1} + 8\,\bar\chi_1 R^p_{q\eta}\,\chi_p - 8\,\bar\chi_1 R^c_p\,\chi_p] \;+\; 4\,\bar B(\chi_p,\chi_p;0)\, L^r_7\,(R^d_p)^2 \\
+\;& \bar B(\chi_p,\chi_p;0)\, L^r_8\; [4/3\,\bar\chi^{pp}_{\eta\eta 2} + 8/3\, R^p_{q\eta}\,\chi_p^2 - 8/3\, R^c_p\,\chi_p^2] \;+\; \bar B(\chi_p,\chi_p;0)^2\; [-1/18\, R^p_{q\eta} R^d_p\,\chi_p + 1/18\, R^c_p R^d_p\,\chi_p \\
+\;& 1/288\,(R^d_p)^2] \;+\; 1/18\,\bar B(\chi_p,\chi_p;0)\bar B(\chi_p,\chi_\eta;0)\; [R^\eta_{pp} R^d_p\,\chi_p - R^\eta_{13} R^d_p\,\chi_p]
\end{aligned}
$$

plus several more pages

Why so long expressions

- Many different quark and meson masses (χ_{ij})
- Charged propagators: $-i\, G^c_{ij}(k) = \frac{\epsilon_j}{k^2-\chi_{ij}+i\varepsilon} \quad (i \neq j)$
- Neutral propagators: $G^n_{ij}(k) = G^c_{ij}(k)\,\delta_{ij} - \frac{1}{n_{\text{sea}}}\,G^q_{ij}(k)$

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$$R^i_{jkl} = R^z_{i456jkl}, \qquad R^d_i = R^z_{i456\pi\eta},$$

$$R^c_i = R^i_{4\pi\eta} + R^i_{5\pi\eta} + R^i_{6\pi\eta} - R^i_{\pi\eta\eta} - R^i_{\pi\pi\eta}$$

$$R^z_{ab} = \chi_a - \chi_b, \qquad R^z_{abc} = \frac{\chi_a-\chi_b}{\chi_a-\chi_c}, \qquad R^z_{abcd} = \frac{(\chi_a-\chi_b)(\chi_a-\chi_c)}{\chi_a-\chi_d}$$

$$R^z_{abcdefg} = \frac{(\chi_a-\chi_b)(\chi_a-\chi_c)(\chi_a-\chi_d)}{(\chi_a-\chi_e)(\chi_a-\chi_f)(\chi_a-\chi_g)}$$

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- Relations $\Longrightarrow$ order of magnitude smaller

PQChPT at Two Loop

Problem: Plotting with many input parameters

Plot masses as a function of lowest order mass squared

I.e. of quark mass: $\boxed{\chi_i = 2B_0 m_i = m_M^{2(0)}}$

Remember: $\chi_i \approx 0.3\ GeV^2 \approx (550\ MeV)^2 \sim$ border ChPT

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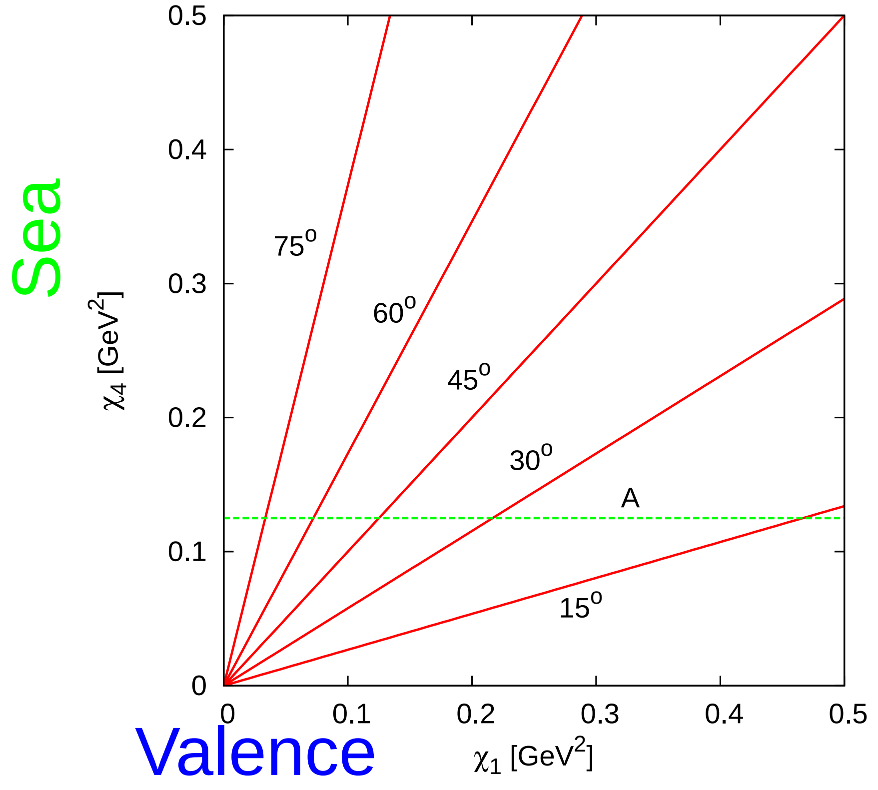
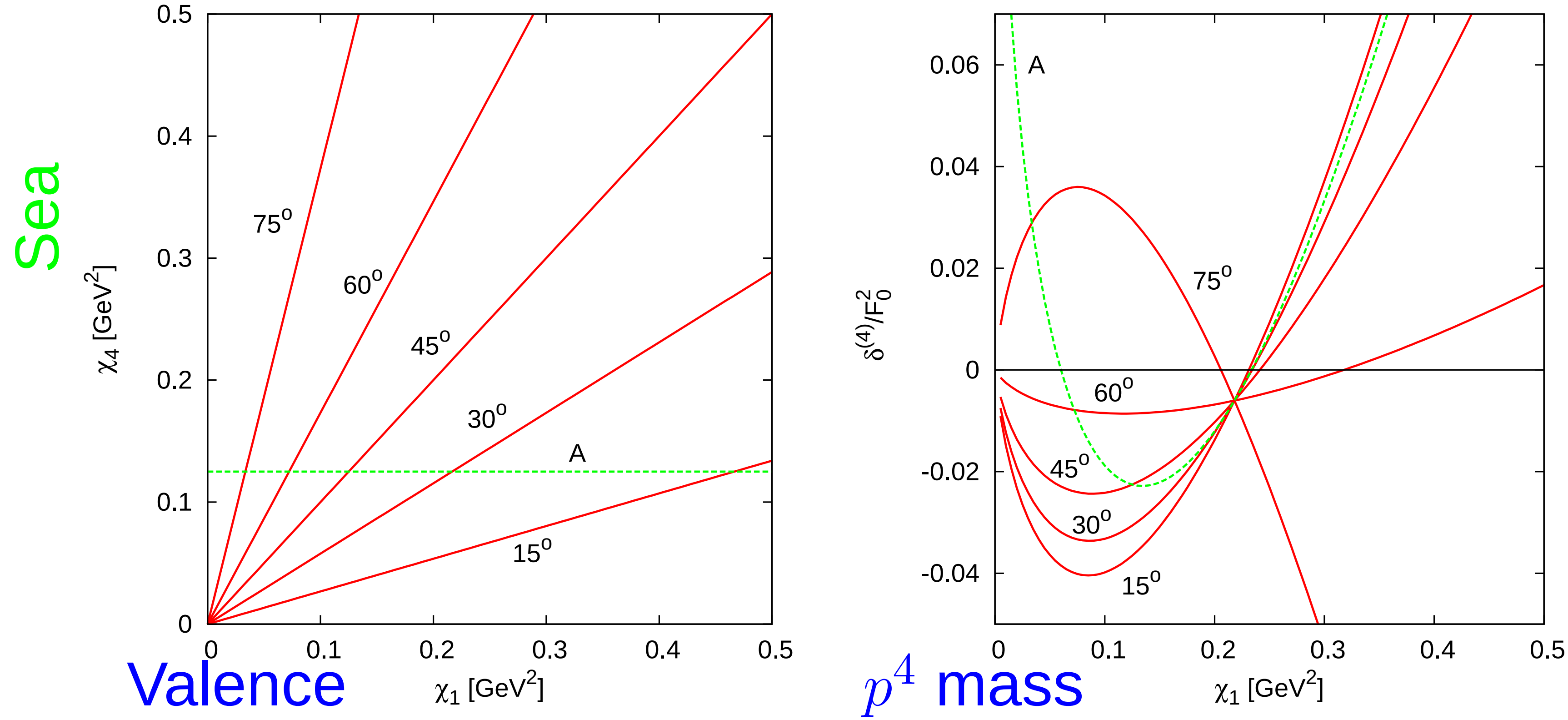
1+1 case: Valence: $\chi_1 = \chi_2 = \chi_3$
Sea: $\chi_4 = \chi_5 = \chi_6$

Plot along curves: $\chi_4 = \tan\theta\ \chi_1$ or χ_4 constant

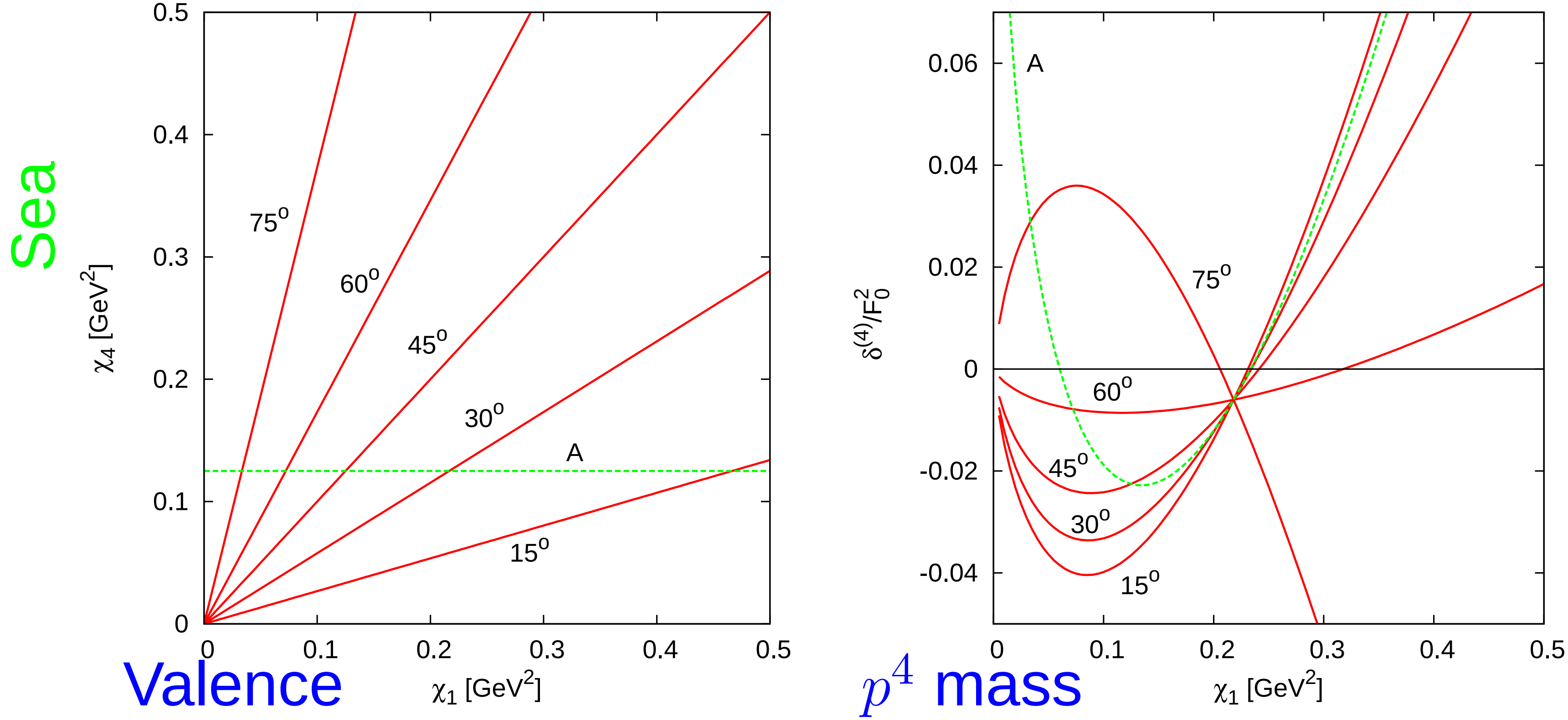
PQChPT: 1+1 case, 3 sea quarks



PQChPT: 1+1 case, 3 sea quarks

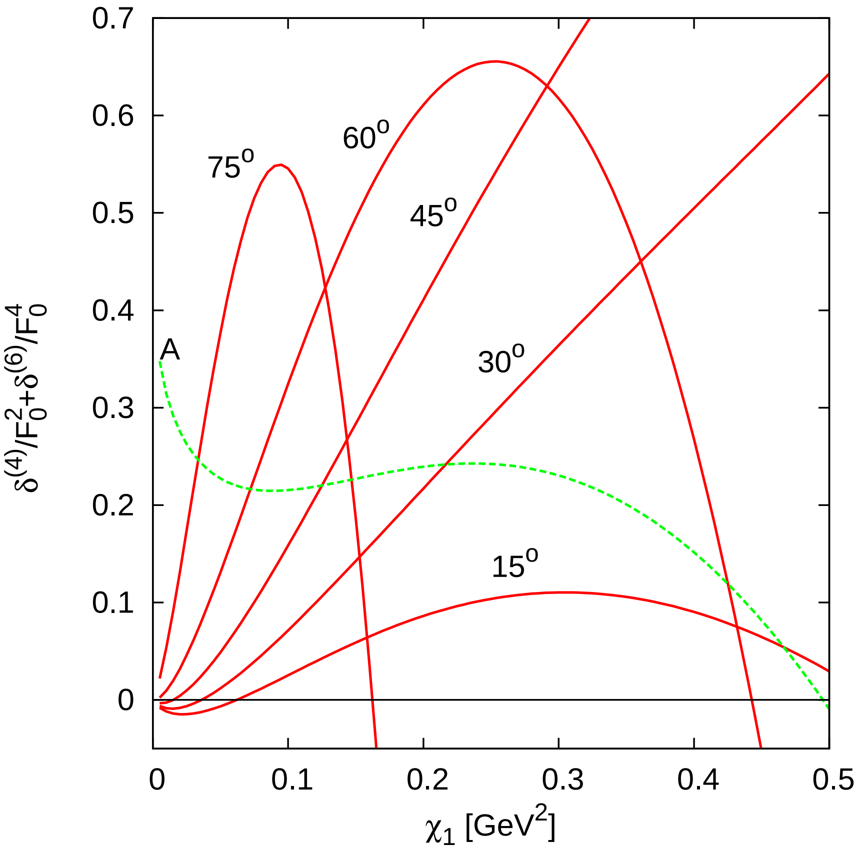


PQChPT: 1+1 case, 3 sea quarks



Notice the Quenched Chiral Logs: $$\frac{m_\pi^2}{\chi_1} = 1 + \frac{\alpha}{F^2}\chi_4 \log \chi_1 + \cdots$$

PQChPT: 1+1 case, 3 sea quarks



$p^4 + p^6$ relative correction mass

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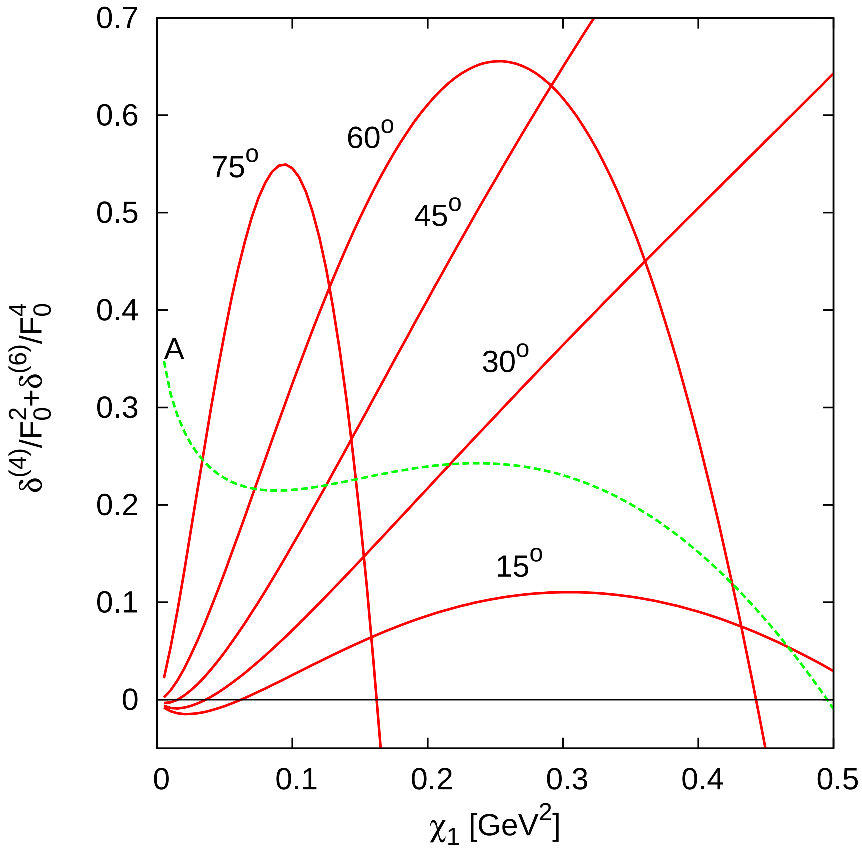
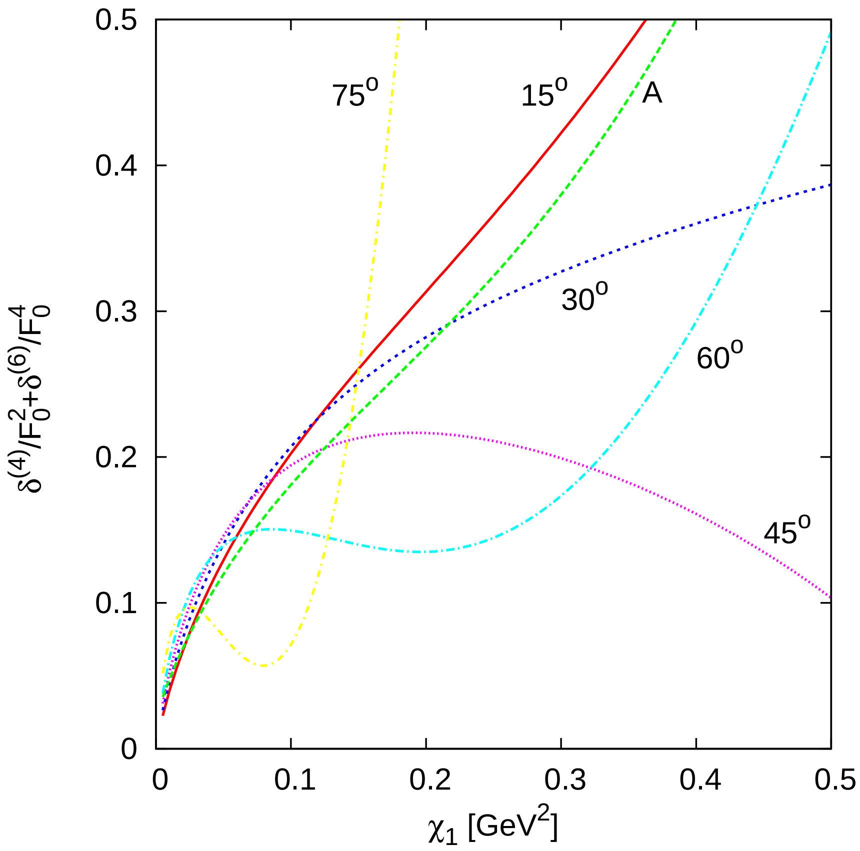
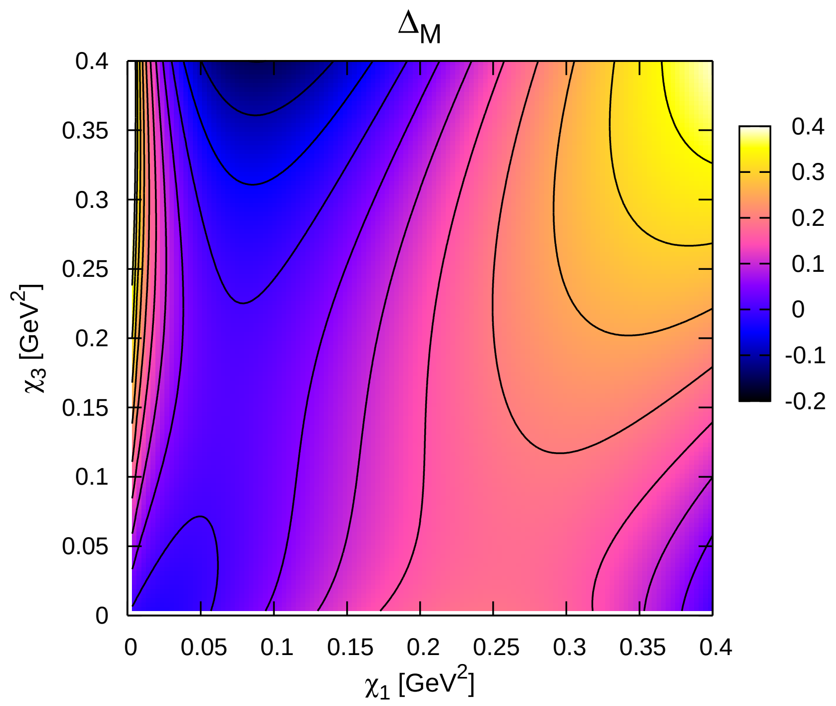




$p^4 + p^6$ relative correction mass decay constant

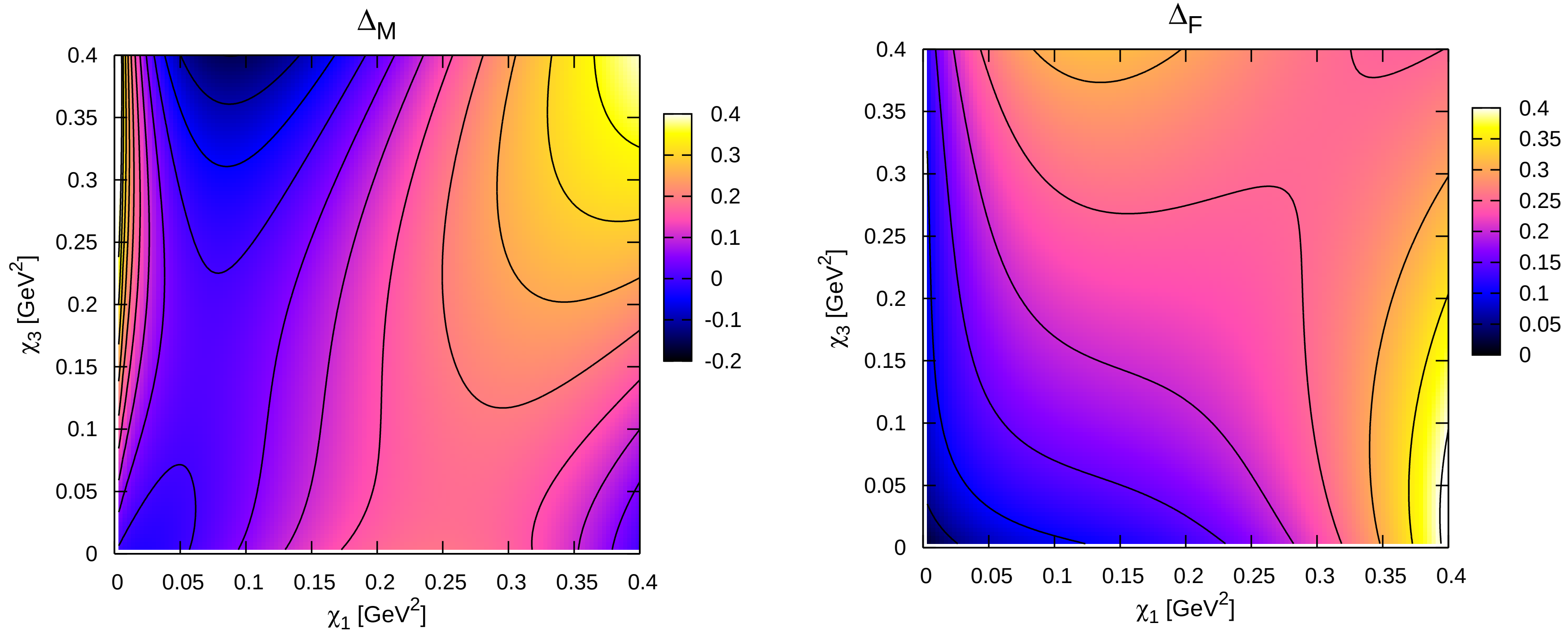
PQChPT: 1+1 case, 2 sea quarks



Relative Correction: Mass

χ_1: valence mass, χ_3: sea mass

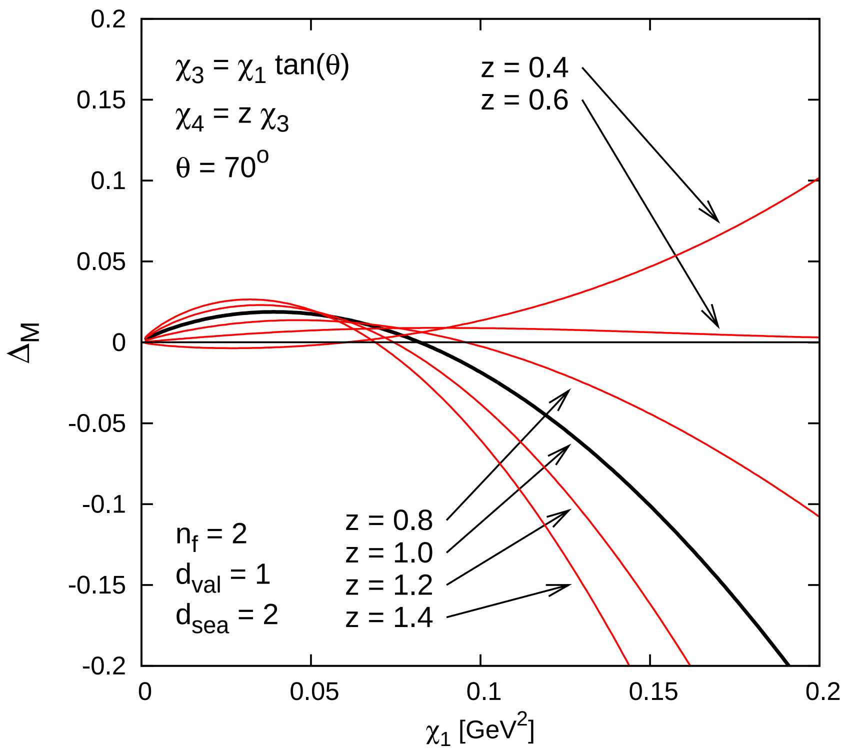
PQChPT: 1+1 case, 2 sea quarks



Relative Correction: Mass Decay Constant

χ_1: valence mass, χ_3: sea mass

PQChPT: 2 sea quarks



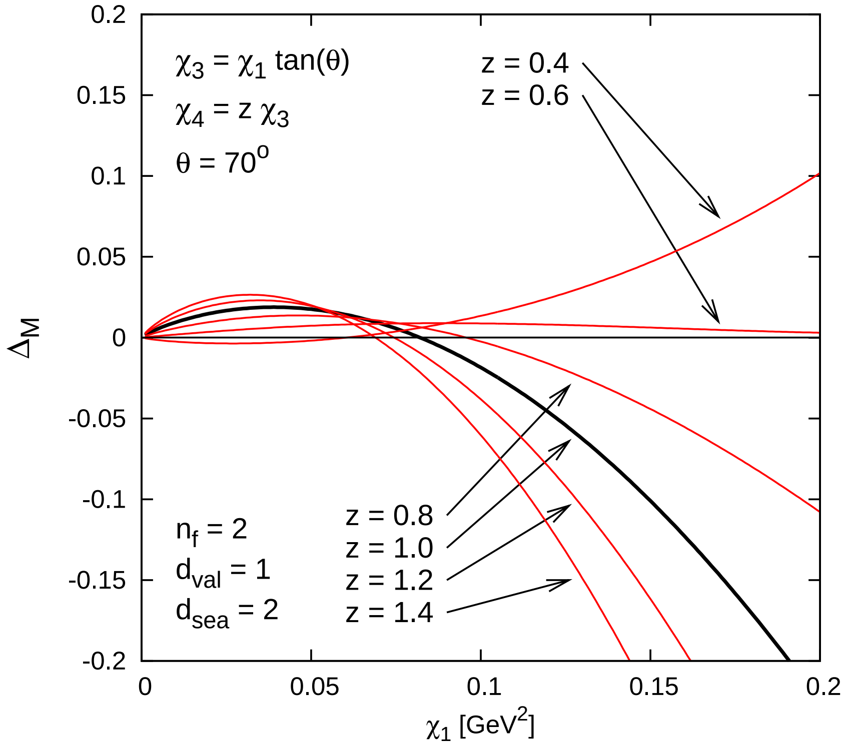
Relative Correction: Mass

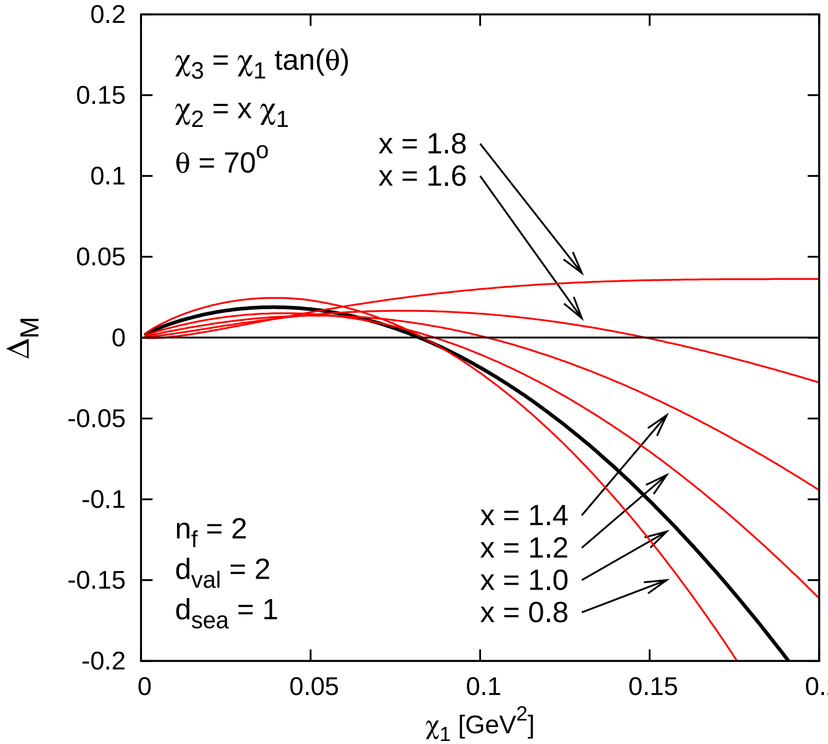
1+2 case

Valence: $\chi_1 \neq \chi_2$

Sea: $\chi_3 = \chi_4$

PQChPT: 2 sea quarks





Relative Correction: Mass
1+2 case
Valence: $\chi_1 \neq \chi_2$
Sea: $\chi_3 = \chi_4$

Mass
2+1 case
Valence: $\chi_1 = \chi_2$
Sea: $\chi_3 \neq \chi_4$

PQChPT: Fitting Strategy

For masses and Decay constants

- At order p^4: $L_4^r, L_5^r, L_6^r, L_8^r$
- At order p^6: all allowed quadratic quark mass combinations show up
- Problem: Need $L_0^r, L_1^r, L_2^r, L_3^r$ from $\pi\pi, \pi K$ scattering but only to order p^4
- Nonanalytic structure at p^6 given (only one included in numerics shown)

Conclusions PQChPT Part

- All relevant mass combinations for masses and decay constants for charged pseudoscalar mesons now known to two loops
- Decay constants and masses for two sea quarks converge nicely
- Masses for three sea quarks convergence slower
- Looking forward to getting lattice data to fit
- Expressions available from `http://www.thep.lu.se/~bijnens/chpt.html`
- For the numerical programs contact the authors

Wishing list

General:

- Quark-mass dependences everywhere
- Not only fits but also continuum infinite volume results at a given quark-mass (so we can also fit ourselves for studying other inputs)
- More use of the existing two-loop calculations
- Analytical NNLO only: not really fewer parameters
- Only more in LECs: p^4 *scattering* LECs also in masses/decay constants
- I.e. l_1^r, l_2^r ($n_F = 2$), L_1^r, L_2^r, L_3^r ($n_F = 3$), $\hat{L}_i^r$, $i = 0, 1, 2, 3$ (PQChPT)

Wishing list: Two-flavour

(with input from Gasser et al.)

- $\bar{l}_3$ and errors
- $\bar{l}_4$: from $F_\pi(m_q)$ and scalar radius: can lattice check this relation
- a_0^2 accurately predicted in terms of scalar radius: can lattice check this
- Isospin breaking in $\pi\pi$ scattering (important for CP violation in $K \to \pi\pi$)
- $\bar{l}_5 - \bar{l}_6$ Needed for $\pi \to \ell\nu\gamma$, can be had from Π_{AA}

Wishing list: ≥ 3-flavour

- Ideas on how to make all those calculations usable for you
- Three and more flavour: typically slow numerically
- large N_c suppressed couplings: i.e. m_s dependence of m_π, F_π
- L_4^r, L_6^r
- $sigma$ terms and scalar radii

Conclusions and final comments

- Lots of analytical work done in ChPT
- Use the correct ChPT
  - 2-flavour for varying $\hat{m}$ and possible for $N_f = 2$ and $N_f = 2+1$ at fixed m_s (but have different LECs)
  - otherwise 3-flavour
  - the various partially quenched versions
- Remember at which order in ChPT you compare things

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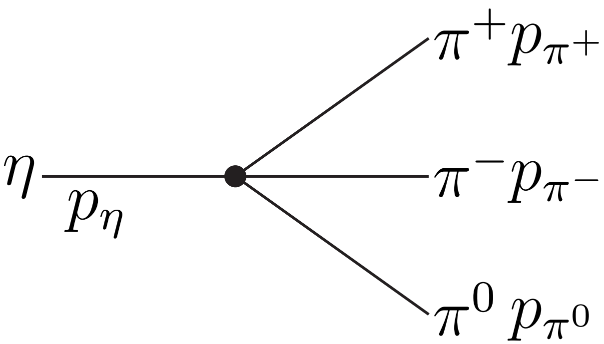
  - otherwise 3-flavour
  - the various partially quenched versions
- Remember at which order in ChPT you compare things
- Seen a lot of lattice work, looking forward to seeing more
- LECs in many talks: Boyle, Matsufuru, Urbach, Kuramashi and many parallel session talks

$\eta \to 3\pi$

Reviews: JB, Gasser, Phys.Scripta T99(2002)34 [hep-ph/0202242]

JB, Acta Phys. Slov. 56(2005)305 [hep-ph/0511076]



$$
\begin{aligned}
s &= (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \\
t &= (p_{\pi^-} + p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \\
u &= (p_{\pi^+} + p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2
\end{aligned}
$$

$$s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0\,.$$

$$\langle \pi^0\pi^+\pi^- \textbf{out}|\eta\rangle = i\,(2\pi)^4\;\delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right)\;A(s,t,u)\,.$$

$$\langle \pi^0\pi^0\pi^0 \textbf{out}|\eta\rangle = i\,(2\pi)^4\;\delta^4\left(p_\eta - p_1 - p_2 - p_3\right)\overline{A}(s_1,s_2,s_3)$$

$$\overline{A}(s_1,s_2,s_3) = A(s_1,s_2,s_3) + A(s_2,s_3,s_1) + A(s_3,s_1,s_2)\,,$$

$\eta \to 3\pi$: Lowest order (LO)

Pions are in $I = 1$ state $\Longrightarrow A \sim (m_u - m_d)$ or α_{em}

- α_{em} effect is small (but large via $m_{\pi^+} - m_{\pi^0}$)
- $\eta \to \pi^+\pi^-\pi^0\gamma$ needs to be included directly

$\eta \to 3\pi$: Lowest order (LO)

Pions are in $I = 1$ state $\Longrightarrow$ $A \sim (m_u - m_d)$ or α_{em}

ChPT:Cronin 67: $A(s,t,u) = \dfrac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \dfrac{3(s - s_0)}{m_\eta^2 - m_\pi^2}\right\}$

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with $Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ $\qquad \hat{m} = \frac{1}{2}(m_u + m_d)$

$$A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\,\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2}\,,$$

$$A(s,t,u) = \frac{\sqrt{3}}{4R}\,M(s,t,u)$$

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$$A(s,t,u) = \frac{\sqrt{3}}{4R}M(s,t,u)$$

LO: $\mathcal{M}(s,t,u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}$ $\qquad M(s,t,u) = \frac{1}{F_\pi^2}\left(\frac{4}{3}m_\pi^2 - s\right)$

$\eta \to 3\pi$ beyond p^4: p^2 and p^4

$\Gamma\,(\eta \to 3\pi) \propto |A|^2 \propto Q^{-4}$ allows a PRECISE measurement

$\boxed{Q \approx 24}$ gives lowest order $\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx 66$ eV .

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Other Source from $m_{K^+}^2 - m_{K^0}^2 \sim Q^{-2} \Longrightarrow \boxed{Q = 20.0 \pm 1.5}$

Lowest order prediction $\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx 140$ eV .

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At order p^4 Gasser-Leutwyler 1985: $$\frac{\int dLIPS|A_2 + A_4|^2}{\int dLIPS|A_2|^2} = 2.4\,,$$

($LIPS$=Lorentz invariant phase-space)

Major source: large S-wave final state rescattering

Experiment: 295 ± 17 eV (PDG 2006)

$\eta \to 3\pi$ beyond p^4: Dispersive

Try to resum the S-wave rescattering:

Anisovich-Leutwyler (AL), Kambor,Wiesendanger,Wyler (KWW)

Different method but similar approximations

Here: simplified version of AL

Up to p^8: No absorptive parts from $\ell \geq 2$

$\Longrightarrow M(s,t,u) =$

$$M_0(s) + (s-u)M_1(t) + (s-t)M_1(t) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

M_I: “roughly” contributions with isospin 0,1,2

$\eta \to 3\pi$ beyond p^4: Dispersive

3 body dispersive: difficult: keep only 2 body cuts

start from $\pi\eta \to \pi\pi$ ($m_\eta^2 < 3m_\pi^2$) standard dispersive analysis

analytically continue to physical m_η^2.

$$\boxed{M_I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \, \frac{\mathrm{Im} M_I(s')}{s' - s - i\varepsilon}}$$

$$\mathrm{Im} M_I(s') \longrightarrow \mathrm{disc} M_I(s) = \frac{1}{2i}\left(M_I(s+i\varepsilon) - M_I(s-i\varepsilon)\right)$$

$$\begin{aligned}
M_0(s) &= a_0 + b_0 s + c_0 s^2 + \frac{s^3}{\pi}\int \frac{ds'}{s'^3}\,\frac{\mathrm{disc} M_0(s')}{s' - s - i\varepsilon}\,,\\
M_1(s) &= a_1 + b_1 s + \frac{s^2}{\pi}\int \frac{ds'}{s'^2}\,\frac{\mathrm{disc} M_1(s')}{s' - s - i\varepsilon}\,,\\
M_2(s) &= a_2 + b_2 s + c_2 s^2 + \frac{s^3}{\pi}\int \frac{ds'}{s'^3}\,\frac{\mathrm{disc} M_2(s')}{s' - s - i\varepsilon}\,.
\end{aligned}$$

$\eta \to 3\pi$ beyond p^4

- Technical complications in solving
- Only 4 relevant constants:

$$M(s,t,u) = a + bs + cs^2 - d(s^2 + tu)$$

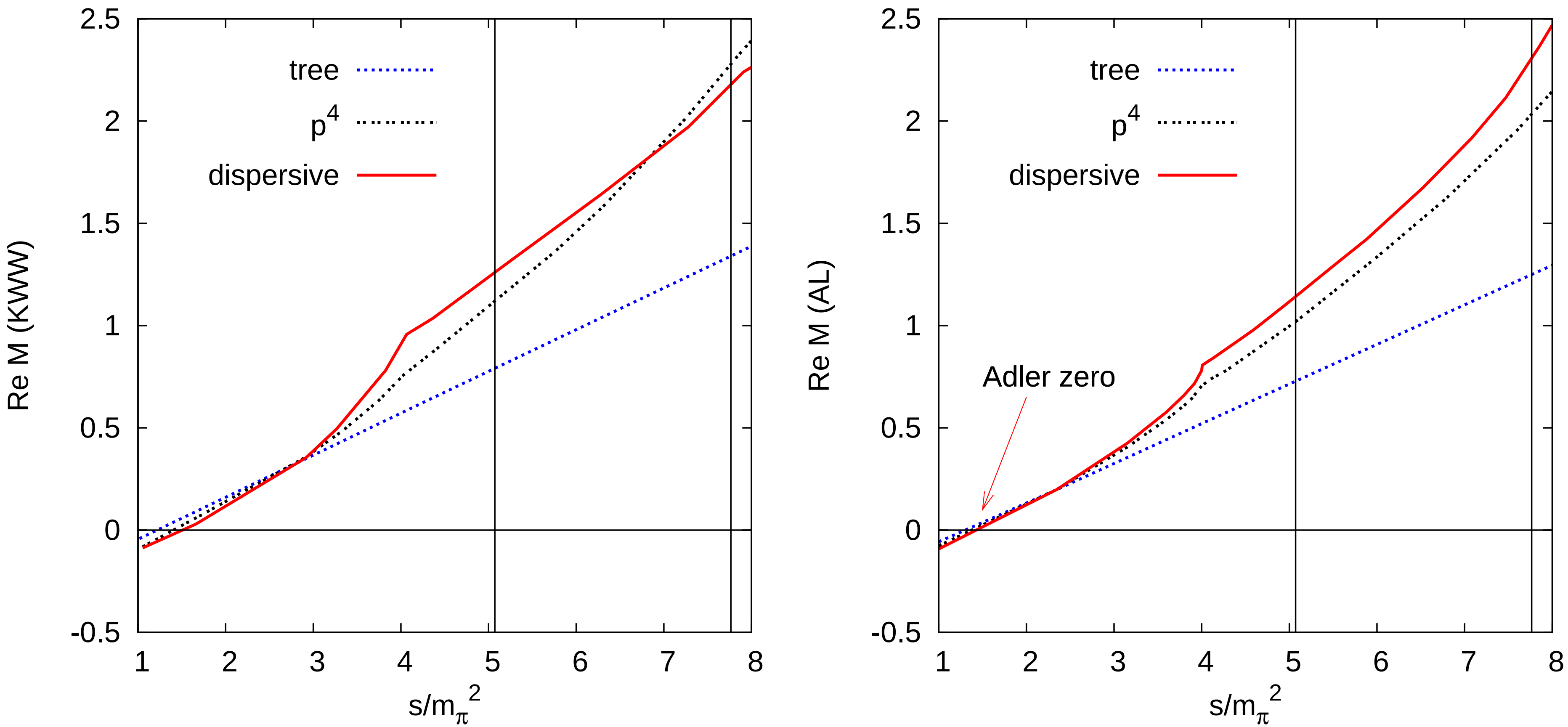
$$M_0(s) + \frac{4}{3}M_2(s) \;\; sM_1(s) + M_2(s) + s^2\frac{4L_3 - 1/(64\pi^2)}{F_\pi^2(m_\eta^2 - m_\pi^2)}$$

converge better

$$\begin{aligned} c &= c_0 + \frac{4}{3}c_2 = \frac{1}{\pi}\int \frac{ds'}{s'^3}\left\{\mathrm{disc}M_0(s') + \frac{4}{3}\mathrm{disc}M_2(s')\right\}, \\ d &= -\frac{4L_3 - 1/(64\pi^2)}{F_\pi^2(m_\eta^2 - m_\pi^2)} + \frac{1}{\pi}\int \frac{ds'}{s'^3}\left\{s'\mathrm{disc}M_1(s') + \mathrm{disc}M_2(s')\right\} \end{aligned}$$

Fix a, b by matching to tree level or p^4 amplitude

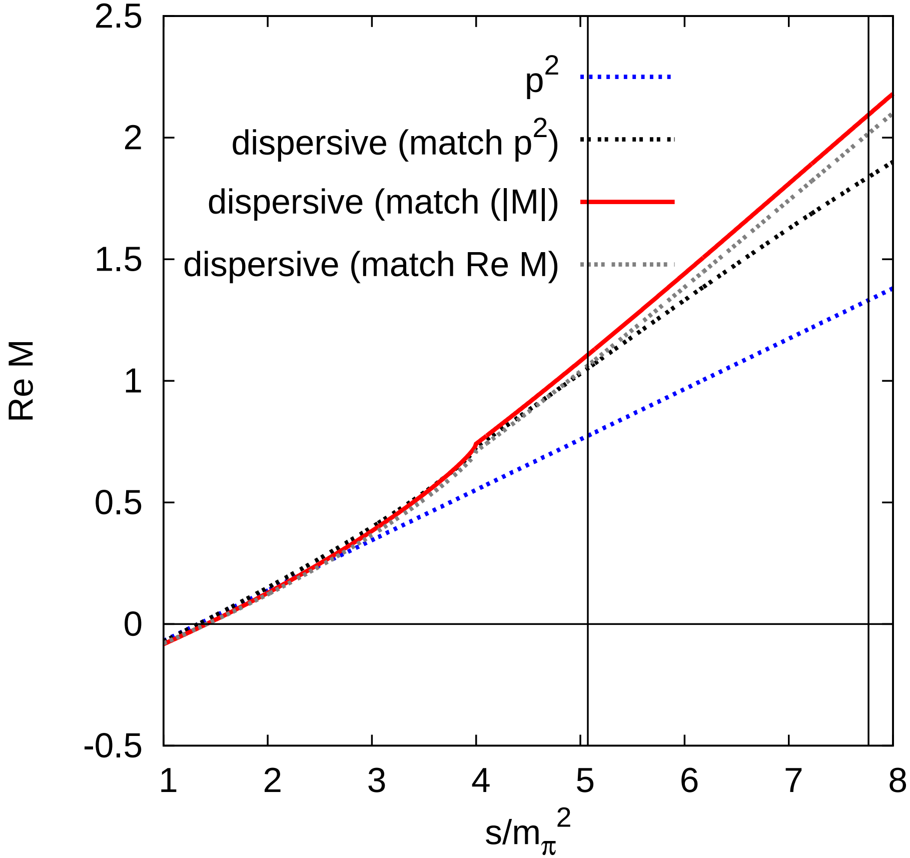
$\eta \to 3\pi$ beyond p^4



Along $s = u$ KWW

Along $s = u$ AL

$\eta \to 3\pi$ **beyond** p^4



Along $s = u$ BG

Very simplified analysis

JB, Gasser 2002

looks more like AL

Two Loop Calculation: why

- In $K_{\ell 4}$ dispersive gave about half of p^6 in amplitude
- Same order in ChPT as masses for consistency check on m_u/m_d
- Check size of 3 pion dispersive part
- At order p^4 unitarity about half of correction
- Technology exists:
  - Two-loops: Amorós,JB,Dhonte,Talavera,...
  - Dealing with the mixing π^0-η: Amorós,JB,Dhonte,Talavera 01

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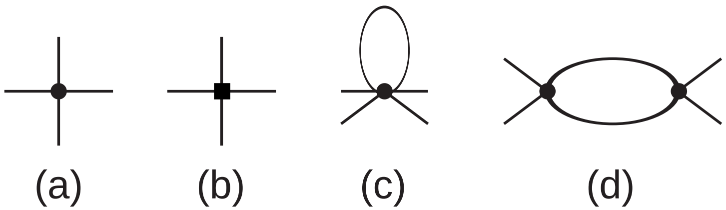
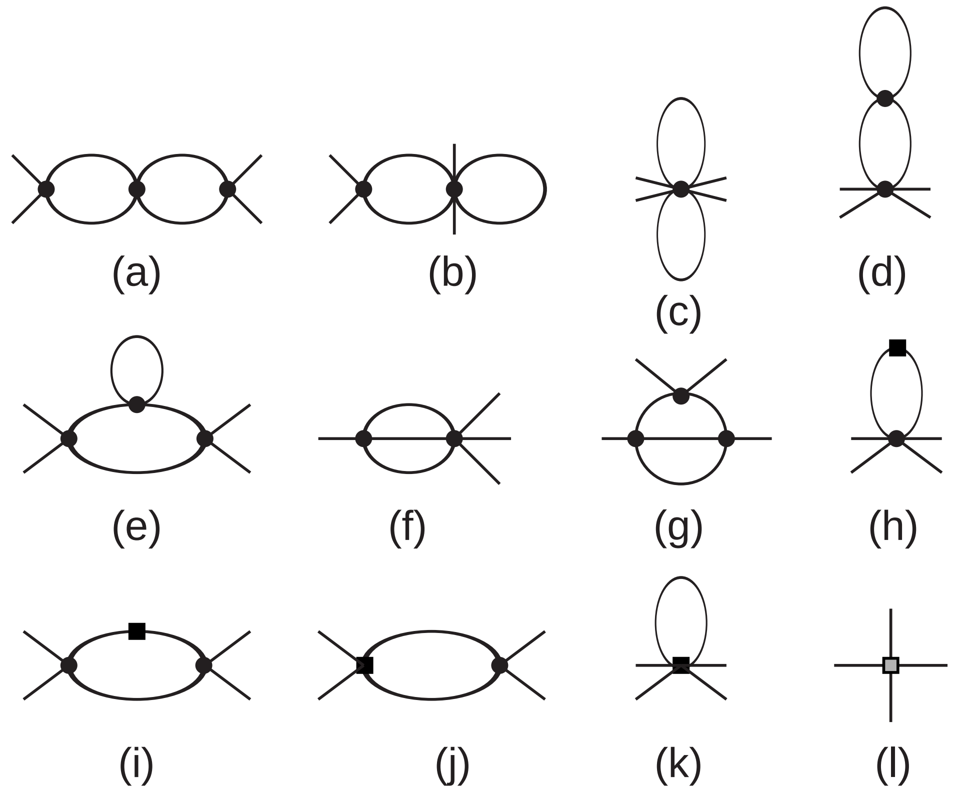
- Done: JB, Ghorbani, arXiv:0709.0230 [hep-ph]
  - Dealing with the mixing π^0-η: extended to $\eta \to 3\pi$

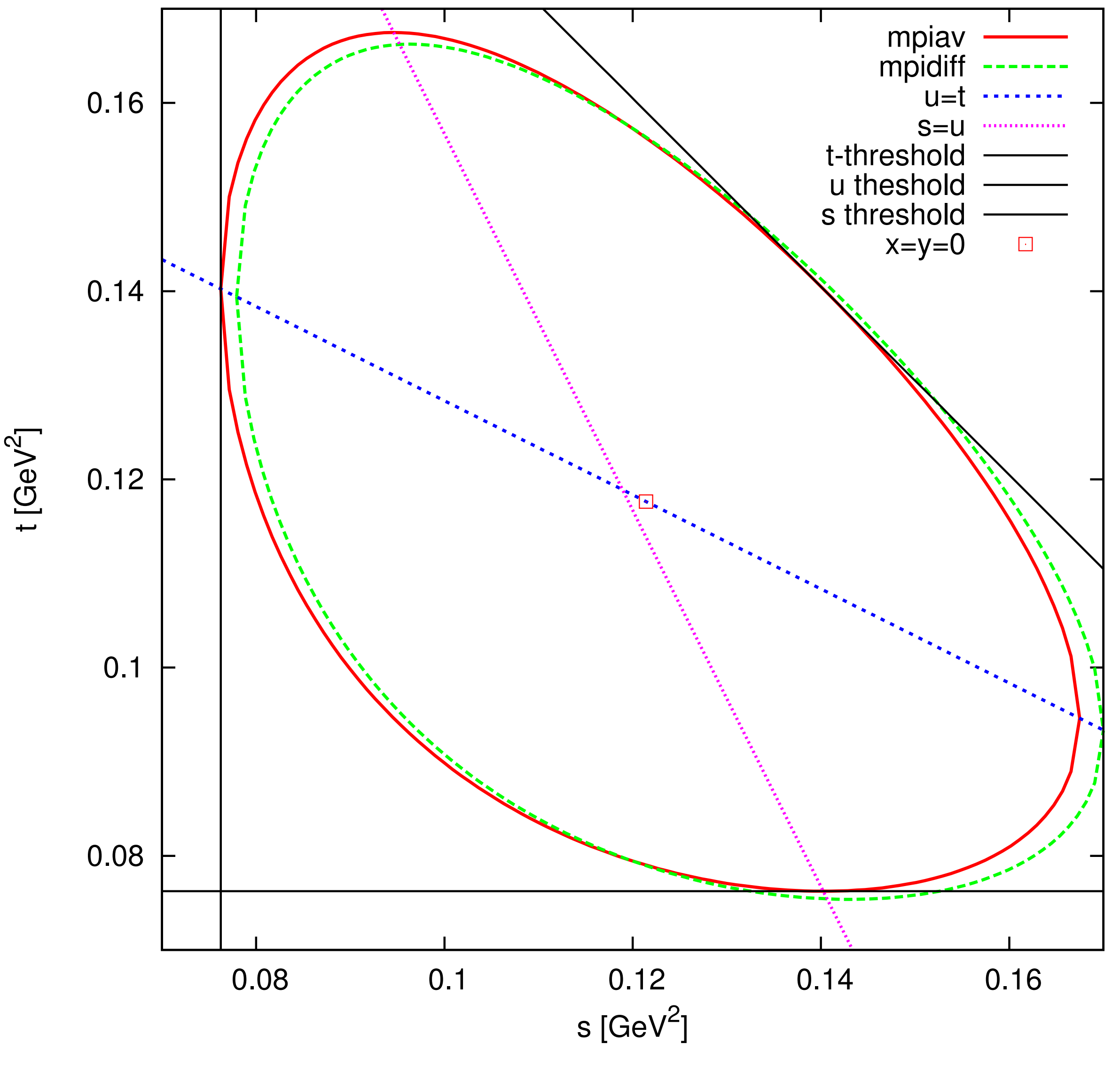
Diagrams





- Include mixing, renormalize, pull out factor $\frac{\sqrt{3}}{4R}$, ...
- Two independent calculations (comparison major amount of work)

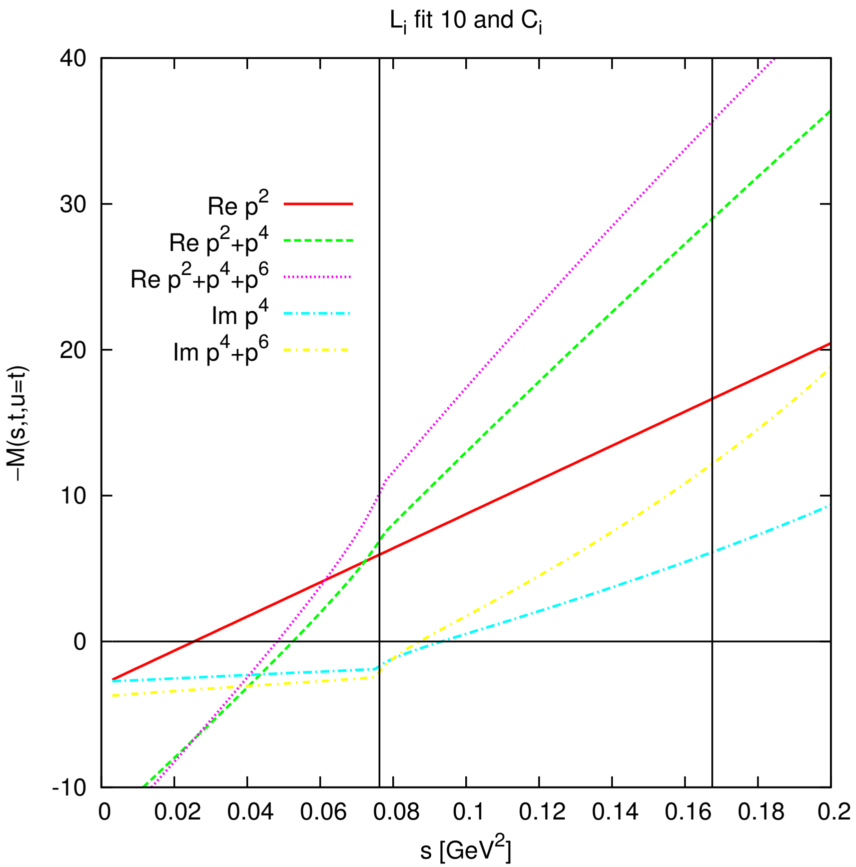
Dalitzplot



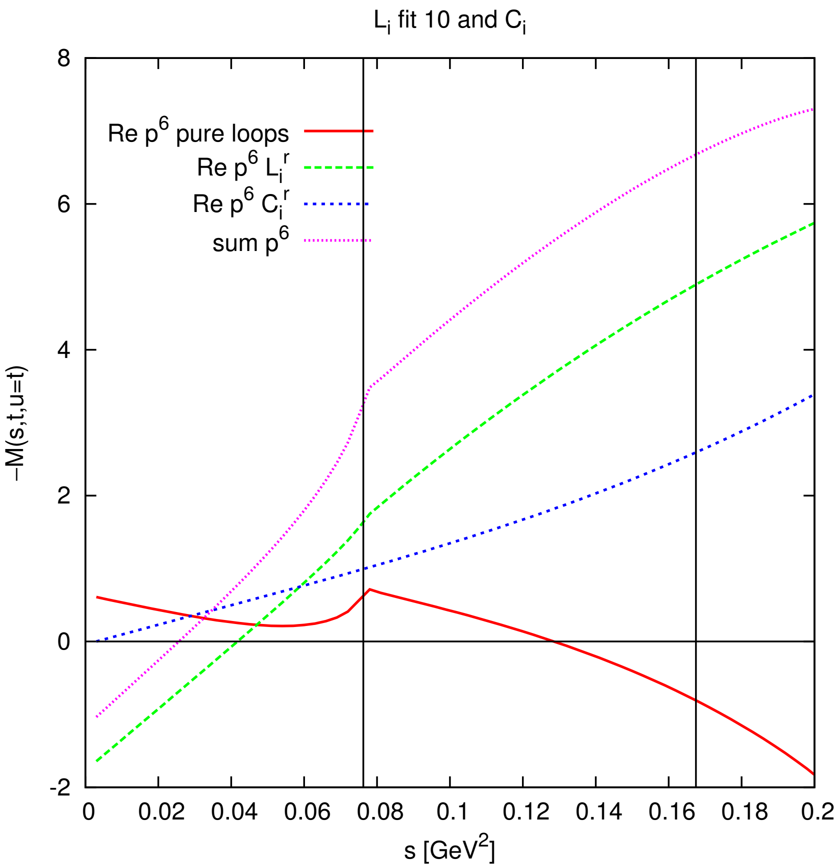
x variation: vertical

y variation: parallel to $t = u$

$\eta \to 3\pi$: $M(s, t = u)$

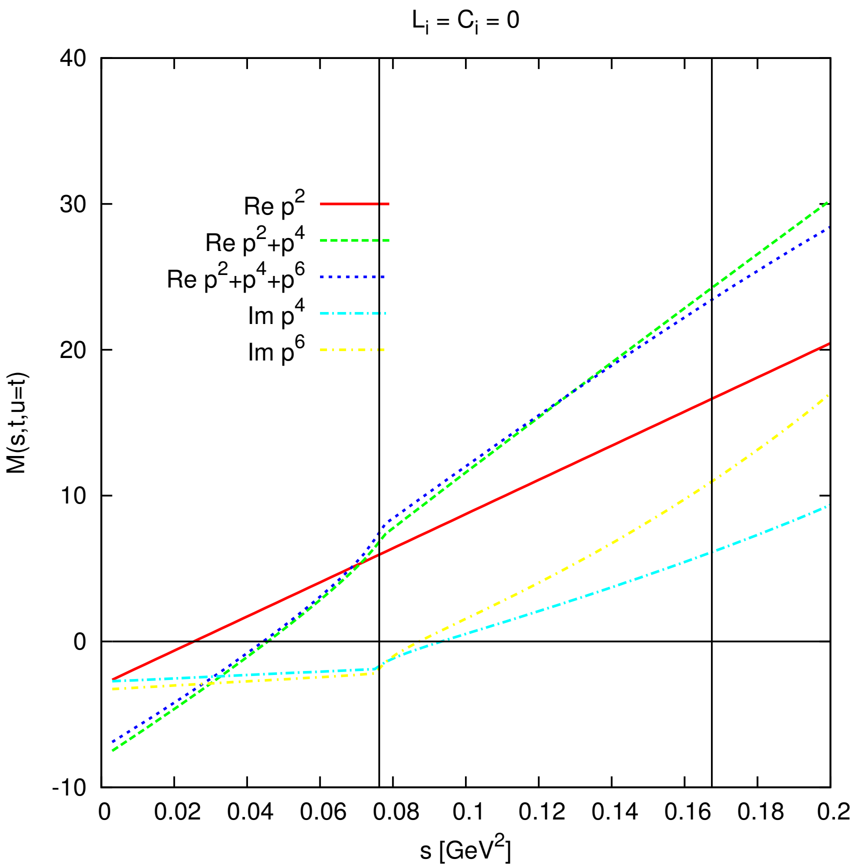


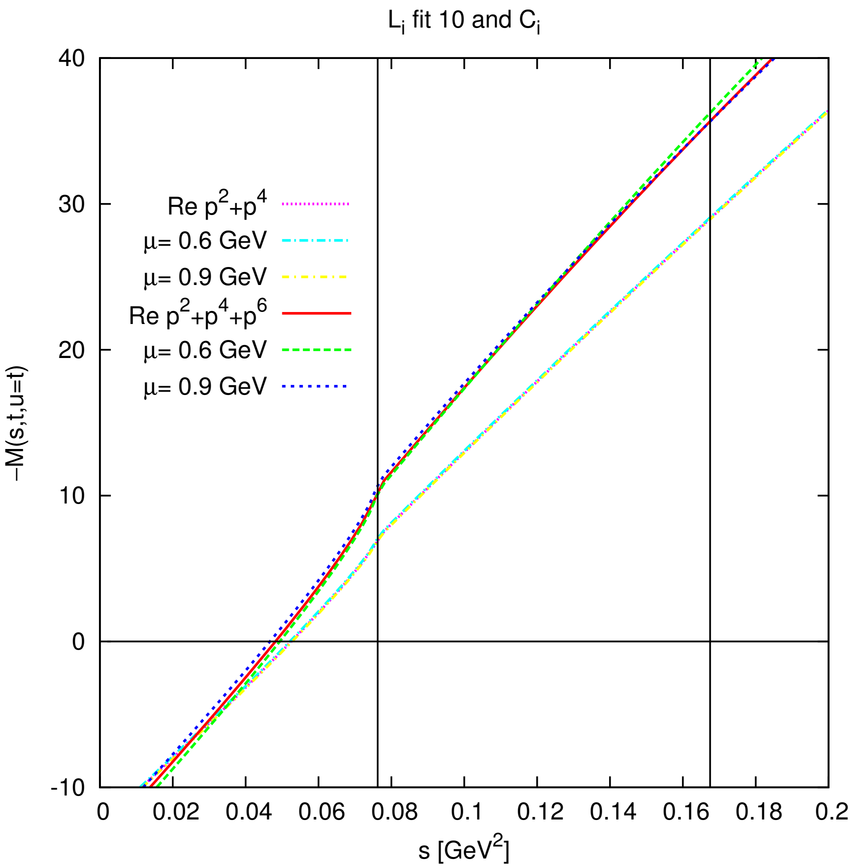
Along $t = u$



Along $t = u$ parts

$\eta \to 3\pi$: $M(s, t = u)$

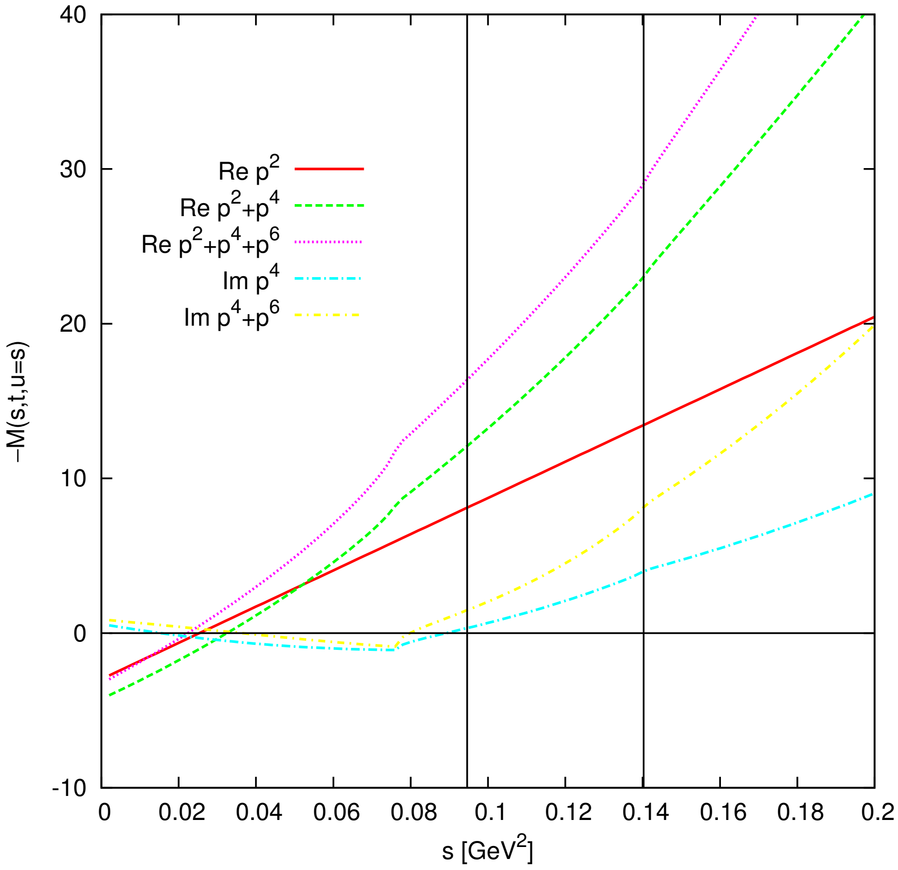




Along $t = u$
$L_i^r = C_i^r = 0$

Along $t = u$: μ dependence
I.e. where $C_i^r(\mu)$ estimated

$\eta \to 3\pi$: $M(s = u, t)$



Along $s = u$

Shape agrees with AL

Correction larger:
20-30% in amplitude

Dalitz plot

$$
\begin{aligned}
x &= \sqrt{3}\frac{T_+ - T_-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u-t) \\
y &= \frac{3T_0}{Q_\eta} - 1 = \frac{3\left((m_\eta - m_{\pi^o})^2 - s\right)}{2m_\eta Q_\eta} - 1 \stackrel{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta}\left(s_0 - s\right) \\
Q_\eta &= m_\eta - 2m_{\pi^+} - m_{\pi^0}
\end{aligned}
$$

T^i is the kinetic energy of pion π^i

$z = \frac{2}{3}\sum_{i=1,3}\left(\frac{3E_i - m_\eta}{m_\eta - 3m_\pi^0}\right)^2$ E_i is the energy of pion π^i

$$
\begin{aligned}
|M|^2 &= A_0^2\left(1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \cdots\right) \\
|\overline{M}|^2 &= \overline{A}_0^2\left(1 + 2\alpha z + \cdots\right)
\end{aligned}
$$

Experiment: charged

Exp.	a	b	d
KLOE	$-1.090 \pm 0.005^{+0.008}_{-0.019}$	$0.124 \pm 0.006 \pm 0.010$	$0.057 \pm 0.006^{+0.007}_{-0.016}$
Crystal Barrel	-1.22 ± 0.07	0.22 ± 0.11	0.06 ± 0.04 (input)
Layter et al.	-1.08 ± 0.014	0.034 ± 0.027	0.046 ± 0.031
Gormley et al.	-1.17 ± 0.02	0.21 ± 0.03	0.06 ± 0.04

KLOE has: $f = 0.14 \pm 0.01 \pm 0.02$.

Crystal Barrel: d input, but a and b insensitive to d

Theory: charged

	A_0^2	a	b	d	f
LO	120	-1.039	0.270	0.000	0.000
NLO	314	-1.371	0.452	0.053	0.027
NLO ($L_i^r = 0$)	235	-1.263	0.407	0.050	0.015
NNLO	538	-1.271	0.394	0.055	0.025
NNLOp (y from T^0)	574	-1.229	0.366	0.052	0.023
NNLOq (incl $(x,y)^4$)	535	-1.257	0.397	0.076	0.004
NNLO ($\mu = 0.6$ GeV)	543	-1.300	0.415	0.055	0.024
NNLO ($\mu = 0.9$ GeV)	548	-1.241	0.374	0.054	0.025
NNLO ($C_i^r = 0$)	465	-1.297	0.404	0.058	0.032
NNLO ($L_i^r = C_i^r = 0$)	251	-1.241	0.424	0.050	0.007
dispersive (KWW)	—	-1.33	0.26	0.10	—
tree dispersive	—	-1.10	0.33	0.001	—
absolute dispersive	—	-1.21	0.33	0.04	—
error	18	0.075	0.102	0.057	0.160

NLO to NNLO: Little change

Error on $|M(s,t,u)|^2$:

$|M^{(6)} M(s,t,u)|$

Experiment: neutral

Exp.	α
KLOE 2007	$-0.027 \pm 0.004^{+0.004}_{-0.006}$
KLOE (prel)	$-0.014 \pm 0.005 \pm 0.004$
Crystal Ball	-0.031 ± 0.004
WASA/CELSIUS	$-0.026 \pm 0.010 \pm 0.010$
Crystal Barrel	$-0.052 \pm 0.017 \pm 0.010$
GAMS2000	-0.022 ± 0.023
SND	$-0.010 \pm 0.021 \pm 0.010$

	$\overline{A}_0^2$	α
LO	1090	0.000
NLO	2810	0.013
NLO ($L_i^r = 0$)	2100	0.016
NNLO	4790	0.013
NNLOq	4790	0.014
NNLO ($C_i^r = 0$)	4140	0.011
NNLO ($L_i^r = C_i^r = 0$)	2220	0.016
dispersive (KWW)	—	$-(0.007$—$0.014)$
tree dispersive	—	-0.0065
absolute dispersive	—	-0.007
Borasoy	—	-0.031
error	160	0.032

Note: NNLO ChPT gets a_0^0 in $\pi\pi$ correct

α is difficult

Expand amplitudes and isospin:

$$
\begin{aligned}
M(s,t,u) &= A\left(1 + \tilde{a}(s-s_0) + \tilde{b}(s-s_0)^2 + \tilde{d}(u-t)^2 + \cdots\right) \\
\overline{M}(s,t,u) &= A\left(3 + \left(\tilde{b}+3\tilde{d}\right)\left((s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2\right) + \right.
\end{aligned}
$$

Gives relations ($R_\eta = (2m_\eta Q_\eta)/3$)

$$
\begin{aligned}
a &= -2R_\eta \,\mathrm{Re}(\tilde{a})\,, \quad b = R_\eta^2\left(|\tilde{a}|^2 + 2\mathrm{Re}(\tilde{b})\right)\,, \quad d = 6R_\eta^2\,\mathrm{Re}(\tilde{d})\,. \\
\alpha &= \frac{1}{2}R_\eta^2\,\mathrm{Re}\left(\tilde{b}+3\tilde{d}\right) = \frac{1}{4}\left(d + b - R_\eta^2|\tilde{a}|^2\right) \le \frac{1}{4}\left(d + b - \frac{1}{4}a^2\right)
\end{aligned}
$$

equality if Im$(\tilde{a}) = 0$

Large cancellation in α, overestimate of b likely the problem

r and decay rates

$$\sin\epsilon = \frac{\sqrt{3}}{4R} + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}
\Gamma(\eta\to\pi^+\pi^-\pi^0) &= \sin^2\epsilon\cdot 0.572\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon\cdot 1.59\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon\cdot 2.68\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon\cdot 2.33\ \mathrm{MeV} && \mathrm{NNLO}\,C_i^r=0\,,\\
\Gamma(\eta\to\pi^0\pi^0\pi^0) &= \sin^2\epsilon\cdot 0.884\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon\cdot 2.31\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon\cdot 3.94\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon\cdot 3.40\ \mathrm{MeV} && \mathrm{NNLO}\,C_i^r=0\,.
\end{aligned}$$

r and decay rates

$$r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

$$\begin{aligned} r_{\mathrm{LO}} &= 1.54 \\ r_{\mathrm{NLO}} &= 1.46 \\ r_{\mathrm{NNLO}} &= 1.47 \\ r_{\mathrm{NNLO}\,C_i^r=0} &= 1.46 \end{aligned}$$

PDG 2006

$$\begin{aligned} r &= 1.49 \pm 0.06 \quad \text{our average}\,. \\ r &= 1.43 \pm 0.04 \quad \text{our fit}\,, \end{aligned}$$

Good agreement

R and Q

	LO	NLO	NNLO	NNLO $(C_i^r = 0)$
R (η)	19.1	31.8	42.2	38.7
R (Dashen)	44	44	37	—
R (Dashen-violation)	36	37	32	—
Q (η)	15.6	20.1	23.2	22.2
Q (Dashen)	24	24	22	—
Q (Dashen-violation)	22	22	20	—

LO from $R = \dfrac{m_{K^0}^2 + m_{K^+}^2 - 2m_{\pi^0}^2}{2\left(m_{K^0}^2 - m_{K^+}^2\right)}$ (QCD part only)

NLO and NNLO from masses: Amorós, JB, Talavera 2001

$Q^2 = \dfrac{m_s + \hat{m}}{2\hat{m}} R = 12.7R$ $\quad$ $(m_s/\hat{m} = 24.4)$