Bounds on Fluctuations of First Passage Times for Counting Observables in Classical and Quantum Markov Processes

George Bakewell-Smith, Federico Girotti, Madalin Guta, Juan P. Garrahan

Quantum Markov Processes

- Quantum open system dynamics: Lindblad generator

$$\mathcal{L} = \mathcal{J} + \mathcal{L}_0, \quad \mathcal{J}(x) := \sum_i L_i x L_i^*, \quad \mathcal{L}_0(x) := -i[H,x] - \frac{1}{2} L_i^* L_i x + x L_i^* L_i$$

  where H is the system Hamiltonian and L_i is the jump operator for jump type $i \in I$.

- Counting process (e.g. photon counts): outcome of continuous-time monitoring via counting measurement

$$\omega = \{(i_0, t_0), (i_1, t_1), \ldots, (i_k, t_k)\}$$

  where (i_j, t_j) corresponds to the jump type of the j^{th} jump, which occurs at time t_j

- Statistic 1: fixed time. Total number of counts of type $\mathcal{A} \subseteq I$ up to time t, denoted $N_{\mathcal{A}}(t)$
- Statistic 2: fixed number of counts. Time taken to reach k counts of type $\mathcal{A}$, called the *first passage time* (FPT), denoted $T_{\mathcal{A}}(k)$

Thermodynamic uncertainty relations (TURs) *lower bound* the probability of observing fluctuations of such statistics in terms of quantities of the system. Such lower bounds exist for classical and quantum systems.

Upper Bounds on FPT Fluctuations

In our previous work (GBS-Girotti-Guta-Garrahan Phys. Rev. Lett. **131** 197101) we provided *upper bounds* on the probability of fluctuations for classical systems in the fixed time ensemble. Upper bounds for the fixed time ensemble for quantum Markov processes have been found in (Girotti, Garrahan, and Guta 2023). For finite time such upper bounds take the form of concentration inequalities. In the asymptotic regime the fluctuation probability obeys a large deviation principle and the probability decays exponentially according to some rate function. The bounds correspond to a lower bound on this rate function.
Here we extend this work and derive upper bounds on the probability of fluctuations for first passage times of counting processes, for both classical and quantum systems.

Results

We provide a series of concentration bounds for i) Total counts of a quantum Markov process, ii) Counts of a subset of jump types (e.g. counting one frequency of photon but not others). We also provide the equivalent classical upper bounds for the same statistics. For conciseness we state just the bound for total counts of a quantum process below, and the large deviation principle:

$$\mathbb{P}\left(\frac{T_I(k)}{k} \geq \langle t_I \rangle + \Delta t\right) \leq C \exp\left(-k \frac{\Delta t^2 \varepsilon}{8 c_q^2} h\left(\frac{5\Delta t}{2 c_q}\right)\right), \quad k \geq 0$$
$$\asymp \exp\left(-k I\left(\frac{T_I(k)}{k}\right)\right), \quad k \to \infty \tag{1}$$

where $h : x \mapsto (\sqrt{1+x} + \frac{x}{2} + 1)^{-1}$ and $I(T_I(k)/k)$ is the large deviation rate function, which is typically difficult to calculate. This bound states that the probability of the empirical process $T_I(k)/k$ taking a value greater than its average $\langle t_I \rangle$ plus some deviation Δt is limited by the right hand side of the above equation. The upper bound contains simpler quantities of the system, an initial state factor C, the absolute spectral gap ε, and the quantity $c_q := \|\mathcal{L}_0^{-1}\|$. The exponent in equation 1 corresponds to a lower bound on $I(T_I(k)/k)$.

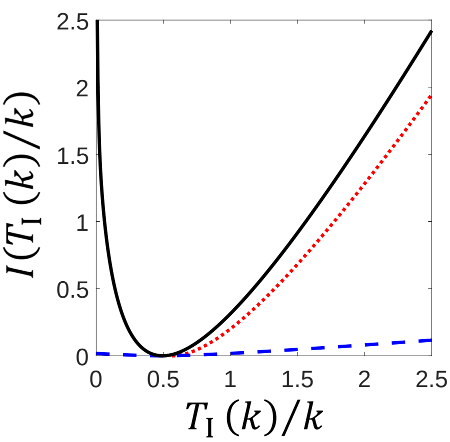

Figure 1: **Bounds on the rate function of the FPT for total number of jumps of a three-level quantum system.** Exact rate function $I(T_I(k)/k)$ (full/black) of the FPT for the total number of quantum jumps, for a simple three-level system. Equation 1 gives a lower bound on the entire rate function (dashed/blue). Another bound we derive is a bound for more general counting processes which bounds the right tails (dotted/red).

Applications

Quantum: Confidence intervals for parameter estimation in metrology; bounding accuracy of quantum clocks
Classically: Glasses, molecular motors, finance