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A METHOD FOR SIMULATING NON-NORMAL DISTRIBUTIONS

ALLEN I. FLEISHMAN

STEVENS INSTITUTE OF TECHNOLOGY

A method of introducing a controlled degree of skew and kurtosis for Monte Carlo studies was derived. The form of such a transformation on normal deviates $[X \sim N(0, 1)]$ is $Y = a + bX + cX^2 + dX^3$. Analytic and empirical validation of the method is demonstrated.

Key words: computer simulation, departures from normality, kurtosis, Monte Carlo study, skew.

Many of the probability statements yielded by methods of statistical inference are based on the assumption of normality. When doing statistical testing, for example, the rejection of the null hypothesis is equivalent to the rejection of at least one of the underlying assumptions, e.g., distributional assumption, null hypothesis, homoscedasticity. For example, in testing the equality of variances coming from non-normal populations, rejection of the null hypothesis can be due to either non-equality of variances and/or non-normality. As concluded by Scheffé, "... the effect of violation of the normality assumption is slight on inferences about means (i.e., fixed effect ANOVA only − AF), but dangerous about inferences about variances [Scheffé, 1959, p. 337]."

In analyzing data, the data analyst, by using these statistics, implicitly makes the assumption that his data are sampled from a normal population. On the other hand, many of our data sets have outliers (which at least increase the kurtosis), have scores which are limited by the floor and/or ceiling effects of the test, or have been truncated due to either implicit or explicit selection (tending to cause the kurtosis to decrease and affecting the skew, if the selection was done only at one end of the distribution). In other words, many, if not most of the psychological variables seen are skewed and/or kurtotic to various degrees.

When investigating the effects of non-normality, it is extremely important to consider non-normality in the degree seen in the literature. In other words, if a statistic is not robust (or biased) given an unusual distribution seldom seen in psychological data (such as a chi square distribution with one degree of freedom), the statistic still may be totally valid and useful with more "typical" psychological data. The suggestion to use "typical" non-normality also has been made by Pearson and Please [1975]. After observing various empirical distributions, they focused on "typical" non-normality, with skew less than 0.8 and kurtosis between −0.6 and +0.6 in testing robustness (or lack of it). They related various levels of skew and kurtosis to Type I (alpha) error using: the single sample t-test—affected especially by skew; the single sample variance test ($\chi^2 = s^2/\sigma^2$)—affected especially by kurtosis; the two sample variance ratio test ($F = s_1^2/s_2^2$)—affected by positive values of kurtosis; and the two sample t-test (with equal n and identical non-normal distributions)—not affected by non-normality.

In the literature, there appear to be four methods to generate non-normality for Monte Carlo studies. (1) Outliers. The creation of this special type of non-normality involves adding a small proportion of data having a variance much greater (e.g., nine

This work was done while the author was at the University of Illinois at Champaign-Urbana.

Requests for reprints should be sent to Allen I. Fleishman, Department of Managerial Science, Stevens Institute of Science, Castle Point Station, Hoboken, NJ 07030.

times greater) than the remainder of the population. This procedure tends to create highly kurtotic but symmetrical distributions. Outliers might be produced when there are occasional (large) keypunching errors, unobserved but highly uncooperative subjects, or, in general, large exceptions to the rule. Tukey and his colleagues have frequently used this method [Tukey, 1960; Andrews, et al., 1972; Devlin, et al., 1975]. (2) Extreme non-normality. In this method, one samples from extreme distributions (e.g., chi square, rectangular, lognormal, exponential, t, Cauchy distributions). This is usually accomplished by an inverse probability function on uniform deviates [Hammersley & Handscomb, 1965, p. 37; Knuth, 1969, p. 102; Ramberg & Schmeiser, 1974]. (3) Transformation to unknown non-normality. The methodologist would transform a random normal deviate by some type of skewing function. One typically wouldn't know the degree of non-normality prior to sampling. The methodology of Brewer and Hills [1968; 1969] is of this latter type. This study investigated the effects of non-normality on the Pearson-Lawley equation. (4) Tabular. One could generate a large table which contains an entire non-normal population to be sampled from within the study. Brown and Lathrop [Note 1] used this method to discover the lack of robustness of the Fisher Z transformation to non-normality. Perhaps the earliest work on non-normality i.e., the work of Pearson and his students, also falls into this class. They used (manually) a random number table and referred to tables of various Pearson type distributions ["Sophister", 1928; Church, 1926].

What is needed is a method to generate distributions which fulfill all of the following requirements: they should have a priori known parameters, enable the researcher to change distributions with the least amount of difficulty, be realistic simulations of empirical distributions, capable of generating widely different distributions, and should operate as efficiently as possible. The method to be discussed in this paper uses a polynomial transformation, and will be called the power method. The transformation will be of the form:

$$Y = a + bX + cX^2 + dX^3 \tag{1}$$

or

$$Y = ((dX + c)X + b)X + a. \tag{2}$$

This transformation, therefore, needs three multiplications, three sums, and the memory for only four constants (actually only three constants, as will be seen in (5) below)—operations which the computer can perform extremely efficiently. Nor would different distributions need different sub-programs to generate their data, further maximizing the efficiency of the program. In the above transformation, X is a random variate distributed normally with zero mean and unit variance, ($N(0, 1)$); Y will have a distribution dependent upon the constants. The constants needed for various skew and kurtosis levels are presented in Table 1. The value of the constant a of (2) will be equal to $-c$, as will be seen in (5) below.

In order to solve for the constants, one needs to find the various moments of Y. The first moment (mean) can be found by:

$$\mathcal{E}(Y) = a + b\mathcal{E}(X) + c\mathcal{E}(X^2) + d\mathcal{E}(X^3). \tag{3}$$

Given that $X \approx N(0, 1)$, then $\mathcal{E}(X) = 0$, $\mathcal{E}(X^2) = 1$, $\mathcal{E}(X^3) = 0$, $\mathcal{E}(X^4) = 3$, $\mathcal{E}(X^5) = 0$, $\mathcal{E}(X^6) = 15$, $\mathcal{E}(X^7) = 0$, $\mathcal{E}(X^8) = 105$, $\mathcal{E}(X^9) = 0$, $\mathcal{E}(X^{10}) = 945$, $\mathcal{E}(X^{11}) = 0$, $\mathcal{E}(X^{12}) = 10395$, $\mathcal{E}(X^{13}) = 0$, $\mathcal{E}(X^{14}) = 135135$ [Kendall & Stewart, 1969, (3.16)]. Upon substitution into (3) we find

$$\mathcal{E}(Y) = a + c. \tag{4}$$

For the distribution to have a mean of zero, which is convenient, it is necessary and sufficient to have

$$a = -c. \tag{5}$$

To find the variance of Y one needs to know the second moment of Y:

$$\begin{aligned}\mathcal{E}(Y^2) = a^2 + 2ab\mathcal{E}(X) + (2ac + b^2)\mathcal{E}(X^2) \\ + 2(ad + bc)\mathcal{E}(X^3) + (2bd + c^2)\mathcal{E}(X^4) + 2cd\mathcal{E}(X^5) + d^2\mathcal{E}(X^6);\end{aligned} \tag{6}$$

substituting the various expected values yields

$$\mathcal{E}(Y^2) = a^2 + 2ac + b^2 + 6bd + 3c^2 + 15d^2. \tag{7}$$

Solving and substituting the expected values for the third and fourth moments gives:

$$\begin{aligned}\mathcal{E}(Y^3) = a^3 + 3a^2c + 3ab^2 + 18abd + 9ac^2 + 45ad^2 \\ + 9b^2c + 90bcd + 15c^3 + 315cd^2\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{E}(Y^4) = a^4 + 4a^3c + 6a^2b^2 + 36a^2bd + 18a^2c^2 + 90a^2d^2 + 36ab^2c \\ + 360abcd + 60ac^3 + 1260acd^2 + 3b^4 + 60b^3d + 90b^2c^2 \\ + 630b^2d^2 + 1260bc^2d + 3780bd^3 + 105c^4 + 5670c^2d^2 + 10395d^4.\end{aligned} \tag{9}$$

The variance is defined to be:

$$\text{Var}(Y) = \mathcal{E}(Y^2) - (\mathcal{E}(Y))^2. \tag{10}$$

Substitution from (5) and (7) yields

$$\text{Var}(Y) = b^2 + 6bd + 2c^2 + 15d^2. \tag{11}$$

In Table 1, the variance of Y was arbitrarily set to be 1.0. As measures of skew and kurtosis, respectively Kendall and Stewart [1969, Vol. I, (3.89) and (3.90), p. 85] recommend:

$$\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}, \tag{12}$$

and

$$\gamma_2 = \frac{\kappa_4}{\kappa_2^{2}}. \tag{13}$$

Where (κ_i) are the cumulants given by [Kendall & Stewart, 1969, Vol. I, (3.41)]:

$$\kappa_2 = \mathcal{E}(Y^2) - [\mathcal{E}(Y)]^2, \tag{14}$$

$$\kappa_3 = \mathcal{E}(Y^3) - 3\mathcal{E}(Y^2)\mathcal{E}(Y) + 2[\mathcal{E}(Y)]^3, \tag{15}$$

and

$$\kappa_4 = \mathcal{E}(Y^4) - 4\mathcal{E}(Y^3)\mathcal{E}(Y) - 3[\mathcal{E}(Y^2)]^2 + 12\mathcal{E}(Y^2)[\mathcal{E}(Y)]^2 + 6[\mathcal{E}(Y)]^4. \tag{16}$$

Solving (12) and (13) in terms of the coefficients and setting the mean to be zero ($c = -a$) gives:

$$\gamma_1 = 2c(b^2 + 24bd + 105d^2 + 2) \tag{17}$$

and

$$\gamma_2 = 24(bd + c^2[1 + b^2 + 28bd] + d^2[12 + 48bd + 141c^2 + 225d^2]). \tag{18}$$

Solving (17) for c, gives

$$c = \frac{\gamma_1}{2(b^2 + 24bd + 105d^2 + 2)}. \tag{19}$$

TABLE 1

Power Method Weights

Skew	Kurtosis	B	C	D
1.75	3.75	0.92966052480111	0.39949667453766	-0.03646699281275
1.50	3.75	0.86588620352314	0.22102762101262	0.02722069915809
1.50	3.50	0.88690855456083	0.23272187792846	0.01875401444244
1.50	3.25	0.91023877496903	0.24780864411835	0.00869952997029
1.50	3.00	0.93620992090360	0.26831868322542	-0.00368190099903
1.50	2.75	0.96443747224458	0.29807621191230	-0.01963521430303
1.50	2.50	0.99209856718687	0.34526935903177	-0.04181526211241
1.25	3.75	0.81888156132542	0.16064255561731	0.04916517172492
1.25	3.50	0.83472669039047	0.16546665419634	0.04385221308384
1.25	3.25	0.85174062710067	0.17101073821620	0.03803066692496
1.25	3.00	0.87016387886005	0.17749222807992	0.03157509494526
1.25	2.75	0.89031839050274	0.18523508277808	0.02430713561023
1.25	2.50	0.91264314105424	0.19474622768576	0.01596248199126
1.25	2.25	0.93774043576005	0.20686671601473	0.00613024990315
1.25	2.00	0.96640616806420	0.22308878847471	-0.00586255218100
1.25	1.75	0.99949784644724	0.24624842887675	-0.02117724378041
1.25	1.50	1.03732397122554	0.28227102596774	-0.04209052633812
1.00	3.75	0.78942074416451	0.11942383662867	0.06153961924505
1.00	3.50	0.80290583376385	0.12210992461489	0.05733551785617
1.00	3.25	0.81713276543078	0.12508112045759	0.05284904949443
1.00	3.00	0.83221632289426	0.12839670935047	0.04803205907079
1.00	2.75	0.84830145553715	0.13213547065430	0.04282253816015
1.00	2.50	0.86557488958491	0.13640488393449	0.03713875125893
1.00	2.25	0.88428277118735	0.14135625914686	0.03086993391983
1.00	2.00	0.90475830311225	0.14721081863342	0.02386092280190
1.00	1.75	0.92746633976156	0.15430725098288	0.01588548300086
1.00	1.50	0.95307689770618	0.16319427626410	0.00659736974453
1.00	1.25	0.98258511915167	0.17482469452982	-0.00456507744552
1.00	1.00	1.01748518639311	0.19099508385633	-0.01857699796908
1.00	0.75	1.05993380621160	0.21543408088777	-0.03728846051332
1.00	0.50	1.11465523356736	0.25852489125964	-0.06601339414569
0.75	3.75	0.76995202064185	0.08563059561704	0.06934855449019
0.75	3.50	0.78217273051806	0.08725259129727	0.06568498605230
0.75	3.25	0.79495262685357	0.08901550782281	0.06182200554749
0.75	3.00	0.80836339881256	0.09094289213403	0.05773230755921
0.75	2.75	0.82249224466377	0.09306441724945	0.05338234897766
0.75	2.50	0.83744678260912	0.09541813650752	0.04873027147556
0.75	2.25	0.85336207930094	0.09805385102577	0.04372288092319
0.75	2.00	0.87041098768531	0.10103830525054	0.03829112262516
0.75	1.75	0.88881983583405	0.10446351079607	0.03234306422362
0.75	1.50	0.90889310938952	0.10846068760906	0.02575256208705
0.75	1.25	0.93105392309623	0.11322488108796	0.01834005494883
0.75	1.00	0.95591357125244	0.11906128313604	0.00983810049833
0.75	0.75	0.98439732894675	0.12647935041464	-0.00017482979206
0.75	0.50	1.01798354640471	0.13640251351290	-0.01241224193515
0.75	0.25	1.05917362852414	0.15068875788687	-0.02819626089809
0.75	0.0	1.11251460048528	0.17363001955694	-0.05033444870926
0.75	-0.25	1.20392340617686	0.22758947506748	-0.09549567396576
0.50	3.75	0.75739984777977	0.05552444121576	0.07425915142054
0.50	3.50	0.76890587541111	0.05647215540722	0.07088677148643

0.50	3.25	0.78088173005011	0.05749287097856	0.06735271683459
0.50	3.00	0.79338100476375	0.05859728796468	0.06363759352080
0.50	2.75	0.80646754404870	0.05979852701132	0.05971815189213
0.50	2.50	0.82021829990300	0.06111289719250	0.05556617644075
0.50	2.25	0.83472726718530	0.06256098771565	0.05114694309845
0.50	2.00	0.85011102914029	0.06416925946524	0.04641702467833
0.50	1.75	0.86651677519629	0.06597243296920	0.04132108400060
0.50	1.50	0.88413424213468	0.06801719309367	0.03578703942948
0.50	1.25	0.90321412393338	0.07036816659914	0.02971850575197
0.50	1.00	0.92409763318404	0.07311802793159	0.02298245181387
0.50	0.75	0.94726632241948	0.07640557409735	0.01538797196646
0.50	0.50	0.97343106918044	0.08045036185716	0.00664738328997
0.50	0.25	1.00370252335312	0.08562503291528	-0.00370088000554
0.50	0.0	1.03994603972583	0.09262357406250	-0.01646085654705
0.50	-0.25	1.08559667905205	0.10290996902235	-0.03319706659066
0.50	-0.50	1.14784905722603	0.12015606910630	-0.05750353451604
0.25	3.75	0.75031534111078	0.02734119591845	0.07699282409939
0.25	3.50	0.76144830727079	0.02778212551548	0.07376857545917
0.25	3.25	0.77300829583485	0.02825487458003	0.07040005844916
0.25	3.00	0.78504099113665	0.02876378789438	0.06687116600052
0.25	2.75	0.79760024256974	0.02931412174287	0.06316282101938
0.25	2.50	0.81075018336126	0.02991231084290	0.05925218604949
0.25	2.25	0.82456809276114	0.03056633874422	0.05511158940232
0.25	2.00	0.83914834011794	0.03128626308577	0.05070703595619
0.25	1.75	0.85460794420601	0.03208497913365	0.04599609338072
0.25	1.50	0.87109461567493	0.03297936179585	0.04092481046466
0.25	1.25	0.88879874777889	0.03399203130579	0.03542308246001
0.25	1.00	0.90797193683084	0.03515419180007	0.02939742137986
0.25	0.75	0.92895681403887	0.03651041219964	0.02271917644022
0.25	0.50	0.95223758733324	0.03812714596039	0.01520430356261
0.25	0.25	0.97853113001303	0.04010900967596	0.00657629354597
0.25	0.0	1.00896426283423	0.04263274479965	-0.00360752773660
0.25	-0.25	1.04545395821482	0.04602657996297	-0.01611868374910
0.25	-0.50	1.09162984652106	0.05098546424880	-0.03246963121043
0.25	-0.75	1.15546858231190	0.05928145029513	-0.05617881116691
0.25	-1.00	1.26341280092760	0.07746243900117	-0.10003604502301
0.0	3.75	0.74802080799221	0.0	0.07787271610187
0.0	3.50	0.75903729021108	0.0	0.07469419122736
0.0	3.25	0.77046795694613	0.0	0.07137653241549
0.0	3.00	0.78235622045349	0.0	0.06790455640586
0.0	2.75	0.79475308530197	0.0	0.06426034643397
0.0	2.50	0.80771907418732	0.0	0.06042254280525
0.0	2.25	0.82132681354781	0.0	0.05636538554628
0.0	2.00	0.83566457198565	0.0	0.05205739701455
0.0	1.75	0.85084120886649	0.0	0.04745952834774
0.0	1.50	0.86699326941512	0.0	0.04252248423852
0.0	1.25	0.88429545439108	0.0	0.03718274611280
0.0	1.00	0.90297659829926	0.0	0.03135645239684
0.0	0.75	0.92334504635701	0.0	0.02492958648521
0.0	0.50	0.94583093702434	0.0	0.01774144664586
0.0	0.25	0.97106090002478	0.0	0.00955505507423
0.0	0.0	1.00000000000000	0.0	0.0
0.0	-0.25	1.03424763182041	0.0	-0.01154929007313
0.0	-0.50	1.07673274256343	0.0	-0.02626832123859
0.0	-0.75	1.13362194989244	0.0	-0.04673170311060
0.0	-1.00	1.22100956933052	0.0	-0.08015837236135
-0.25	3.75	0.75031534111078	-0.02734119591845	0.07699282409939

Setting the variance to be 1.0 and using (11) and (19), gives

$$2 = 2b^2 + 12bd + \frac{\gamma_1^2}{(b^2 + 24bd + 105d^2 + 2)^2} + 30d^2 \cdot \tag{20}$$

Therefore, all four coefficients (a, b, c, and d) can be found with the restriction that the mean, variance, skew and kurtosis are 0.0, 1.0, γ_1 and γ_2, respectively. This is done by solving (18) and (20) obtaining b and d, with c and a given by (19) and (5), respectively. Since (18) and (20) are rather complicated non-linear expressions, an iterative solution was used based on a modification of Newton's method by Brown and Conte [1967]*. A solution was accepted if the absolute difference between the expected and the obtained mean, variance, skew and kurtosis was less than 10^{-15}. Inasmuch as the solution was dependent on the starting values of b and d, the initial values were varied for as much as six times with one hundred iterations each. If no solution was obtained then it was assumed that no solution was possible. A table of solved weights for skew less than or equal to 1.75 and kurtosis of less than or equal to 3.75 both decreasing in steps of 0.25 is given in Table 1 (where constant a is equal to $-c$). A table where the skew was less than 2.5 and the kurtosis was less than or equal to 9.0 in steps of 0.05 was developed (but shall not be presented due to lack of space). By inspection of (17), negative values of γ_1 can be obtained by reversing the sign of c and a. As these changes will have no effect on the mean, variance, or kurtosis, it is necessary only to consider positive values of skewness within Table 1. For example, for skew = +0.25 and kurtosis = −1.00, a = +0.0775, b = +1.2634, c = −0.0775, and d = −0.1000; to obtain skew = −0.25 and kurtosis = −1.00, a = −0.0775, b = +1.2634, c = +0.0775, and d = −0.1000. Not all values within the parameter space were obtained. Those which were obtained are presented in Figure 1. It can be seen that those values which were found follow a very regular pattern. No kurtosis was found at or below −1.20, which is the kurtosis for a rectangular distribution, although for a symmetric distribution the kurtosis did approach this value (−1.15). Another point to be noted is that given a certain value of skew, for all points above a particular value of kurtosis all solutions were found, and below this value no solution was obtained. This space can be described by a parabola defined by

$$\text{skew}^2 < 0.0629576{*}\text{kurtosis} + 0.0717247. \tag{21}$$

This is a limitation of the power method. Alternative procedures are needed for more extreme combinations of parameters.

Alternative Solutions to "Controllable" Transformations

One approach that might be attempted is the generation of skewed and kurtotic distributions via central or non-central chi square distributions with one or more degrees of freedom. Let us assume that coefficient d = 0.0. Then (1) simplifies to

$$Y = a + bX + cX^2 \tag{22}$$

Completing the square

$$Y = c\left((X + \frac{b}{2c}\right)^2 + a - \frac{b^2}{4c} \cdot \tag{23}$$

This equation is the same as a non-central chi square with one degree of freedom, which can be scaled to have a different mean and variance. That is,

$$Y = g(X + \delta)^2 + e \tag{24}$$

* The computer program used was called BROWNZ; it is described in *FORTUOI Writeups*, available from the Computing Services Office, University of Illinois, Champaign, Illinois.

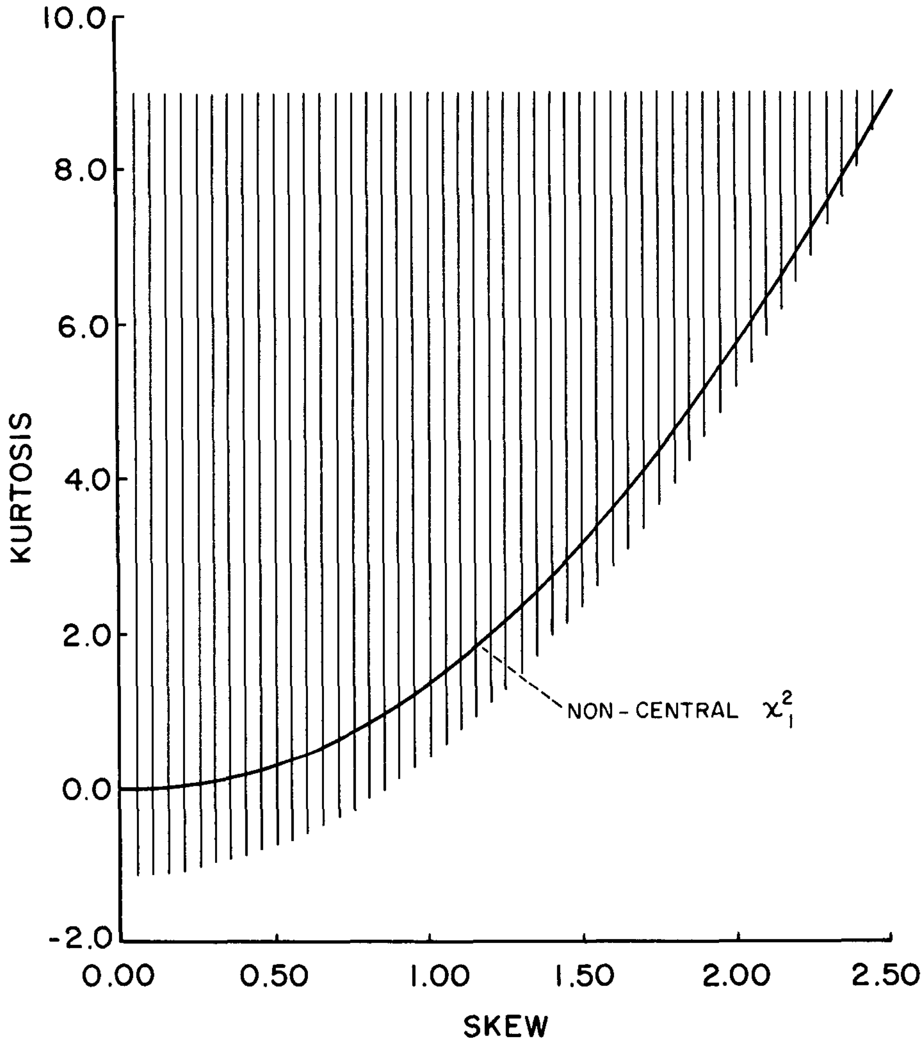

FIGURE 1
Values of Skewness and Kurtosis possible by Power Transformation

is a non-central chi square variable, with one degree of freedom, δ being the non-centrality parameter, e and g are parameters rescaling the mean and variance respectively. Therefore, an exact solution to the chi square family with one degree of freedom is available using the power method. Patnaik [1949] had derived the cumulants of the central and non-central chi square with n degrees of freedom and non-centrality, δ. The first four cumulants are:

$$\kappa_1 = n + \delta \tag{25}$$

$$\kappa_2 = 2(n + 2\delta), \tag{26}$$

$$\kappa_3 = 8(n + 3\delta), \tag{27}$$

and

$$\kappa_4 = 48(n + 4\delta). \tag{28}$$

Therefore, one is able to solve for the skew and kurtosis, which are:

$$\text{skew} = \frac{8(n + 3\delta)}{[2(n + 2\delta)]^{3/2}} \tag{29}$$

and

$$\text{kurtosis} = \frac{48(n + 4\delta)}{[2(n + 2\delta)]^{2}}\,. \tag{30}$$

It is easily demonstrated that the non-normality is greatest when the degrees of freedom = 1. The skew and kurtosis values of the non-central chi square with one degree of freedom have been superimposed on Figure 1. Therefore, the power technique subsumes in terms of the first four moments, all chi square distributions for producing non-normality.

Recently, Ramberg and Schmeiser [1974] have derived a method of simulating non-normal deviates which they called the Generalized Lambda Distribution (GLD). This method may be faster than the power technique (as GLD utilizes a uniform as opposed to a normal deviate, but GLD raises it to a fractional power, as opposed to an integer power, which may make GLD slower in some computers). However the main problem with this paper is that their table is in terms of their constants, as opposed to skew and kurtosis. Therefore, the selection of constants for specific levels of skew and kurtosis (e.g. 0.0 & 0.0) is extremely difficult.

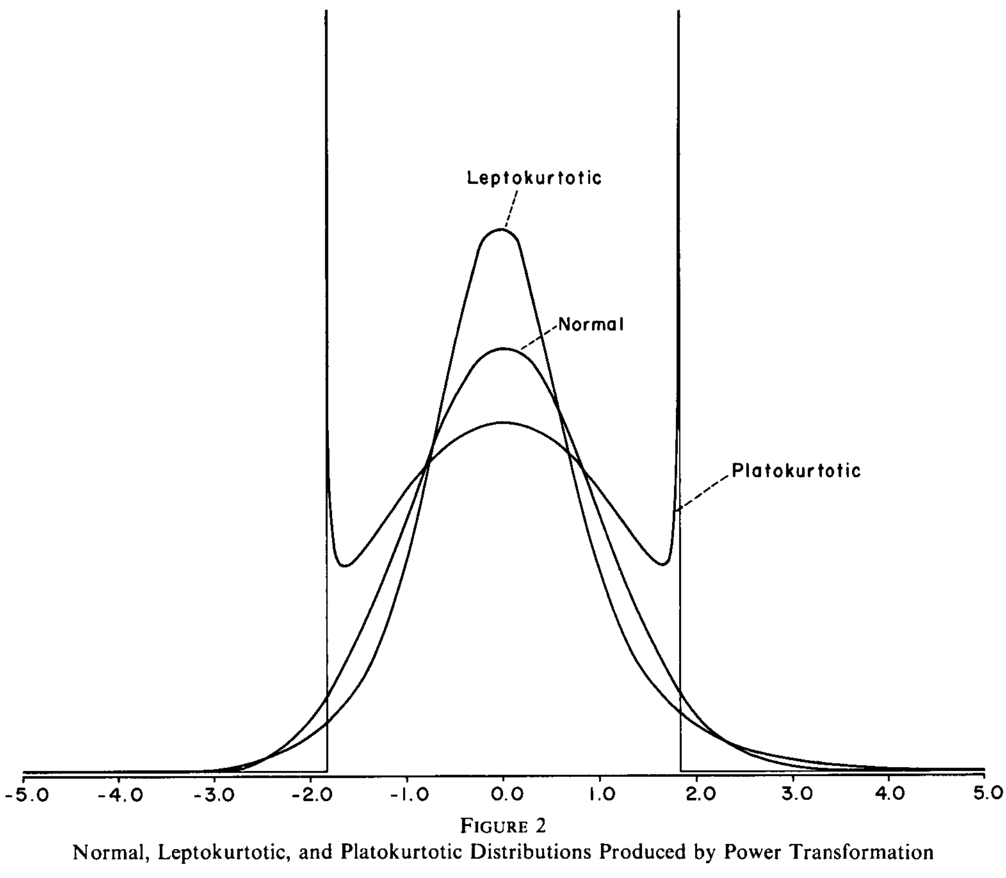

FIGURE 2
Normal, Leptokurtotic, and Platokurtotic Distributions Produced by Power Transformation

TABLE 2

Numerical Integration Results

Expected Skew	Expected Kurtosis	Obtained Area	Obtained Mean	Obtained Variance	Obtained Skew	Obtained Kurtosis
0.00	0.00	1.000007	-0.00006	1.000022	-0.00001	-0.00005
0.00	1.00	0.999976	0.00000	0.999941	0.00000	1.00063
0.25	3.00	1.000034	0.00007	0.999912	0.24908	2.97978
-0.25	3.00	1.000034	-0.00007	0.999912	-0.24908	2.97978
0.00	-1.00	0.998844	0.00000	0.998217	0.00000	-1.00024

Evaluation of Technique

As a check on the accuracy of the formulae and the efficiency of the technique, two validation methods were used. As a mathematical check of the distributions, a numerical integration was performed to check the shape of various distributions and the first four moments. A Monte Carlo study was conducted to illustrate the accuracy and time requirements of the algorithm.

Numerical integration

Five distributions were utilized: (1) Normal—skew = 0.0 and kurtosis = 0.0; (2) skew = 0.0 and kurtosis = +1.00; (3) skew = +0.25 and kurtosis = 3.00; (4) skew = −0.25 and kurtosis = 3.00; and (5) skew = 0.0 and kurtosis = −1.0. About 240,000 normal scores were created in the interval −12.0 to +12.0 in steps of 0.0001. Each normal score was then transformed by (2), and then grouped into about 4800 intervals from −12.00 to +12.0 in steps of 0.005. The ordinates of the transformed normal scores were summed per interval. This technique will numerically approximate the shape of the five distributions. Figure 2 presents the normal, the positively skewed leptokurtic (skew =0.25, kurtosis = 3.0) and the platykurtic (skew = 0.0, kurtosis = −1.0) distrigution. The areas and the first four

TABLE 3

Monte Carlo Simulated Distributions

Expected Skew	Expected Kurtosis	Obtained Mean	Standard Error of Mean	Obtained Variance	Standard Error of Variance	Obtained Skew	Obtained Kurtosis
0.00	1.00	-0.00453	0.00316	1.00145	0.00548	-0.02199	1.00194
0.25	3.00	0.00107	0.00316	0.99543	0.00707	0.25243	2.88297
-0.25	3.00	-0.00162	0.00316	0.99451	0.00707	-0.18629	3.04341
0.00	-1.00	-0.00424	0.00316	1.00489	0.00316	0.00009	-1.00207

moments of each distribution were obtained by integration of the above distributions. The area, mean, variance, skew and kurtosis are presented for each distribution in Table 2. As can be seen, within machine accuracy and rounding error, all areas, means and variances equalled 1.0, 0.0 and 1.0, respectively. The skew and kurtosis were also approximately equal to the expected values.

As can be seen in Figure 2, some functions may not be unimodal, hence *may* be undesirable. (On the other hand, this figure could be a perfectly valid depiction of a floor and ceiling effect or a Winsorized distribution.) Such non-unimodal distributions could occur when the transformation function has two zero first derivatives (a saddle point will give a unimodal distribution). For this to occur, the power function must have coefficients such that $4c^2 - 12bd$ is positive. This is the case for the power transformation for skew = 0.00 and kurtosis = -1.00 but not for the other four distributions presented.

Given the obvious differences between this distribution and the rectangular distribution, a technique may be desired that controls for more than the first four moments. However, as researchers seldom report either skew or kurtosis, it is felt that control of the higher moments must therefore be based solely on non-empirical information and would only introduce unnecessary complications.

However if one could theoretically expect that the variable should be 'somewhat' normally distributed (or a transformation of normal variates) then the power method should be appropriate.

Monte Carlo check

For the above non-normal distributions, 100,000 normal deviates were generated by GGNRF [*International Mathematical and Statistical Libraries: Computer Subroutine Libraries in Mathematics and Statistics,* 1974], and transformed by the power transformation. The mean, variance, skew, and kurtosis were calculated for each empirical distribution. These are presented in Table 3 along with the standard error of the mean and variance [Evans, 1951]. The obtained statistics performed, within sampling error, to the expected values. To generate these deviates took 0.000100 seconds per deviate or about 1.7 minutes to produce 1,000,000 deviates (all computations were done on a IBM 360/75 computer).

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