

Backtesting

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A common practice in evaluating backtests of trading strategies is to discount the reported Sharpe ratios by 50%.¹ There are good economic and statistical reasons for reducing the Sharpe ratios. The discount is a result of data mining. This mining may manifest itself in academic researchers searching for asset-pricing factors that explain the behavior of equity returns, or by researchers at firms that specialize in quantitative equity strategies trying to develop profitable systematic strategies.

The 50% haircut is only a rule of thumb. Our article's goal is to develop an analytical way to determine the haircut's magnitude.

Our framework relies on the statistical concept of multiple testing. Suppose you have some new data, Y , and you propose that variable X explains Y . Your statistical analysis finds a significant relation between Y and X with a t -ratio of 2.0, which has a probability value of 0.05. We refer to this as a single test. Now consider the same researcher trying to explain Y with variables X_1, X_2, \dots, X_{100} . In this case, you cannot use the same criteria for significance. You expect that, by chance, some of these variables will produce t -ratios of 2.0 or higher. What is an appropriate cut off for statistical significance?

In this article, we present three approaches to multiple testing and the question in the previous example. The t -ratio is

generally higher as the number of tests (or X variables) increases.

In summary, any given strategy produces a Sharpe ratio. We transform the Sharpe ratio into a t -ratio. Suppose that t -ratio is 3.0. Although a t -ratio of 3.0 is highly significant in a single test, it may not be if we take multiple tests into account. We proceed to calculate a p -value that appropriately reflects multiple testing.

To do this, we must make an assumption on the number of previous tests. For example, Harvey, Liu, and Zhu [2015] (HLZ) document that at least 316 factors have been tested in the quest to explain the cross-sectional patterns in equity returns. Suppose the adjusted p -value is 0.05. We then calculate an adjusted t -ratio; in this case, it is 2.0. With this new t -ratio, we determine a new Sharpe ratio. The percentage difference between the original Sharpe ratio and the new Sharpe ratio is the haircut.

The haircut Sharpe ratio that is a result of multiple testing is the Sharpe ratio that would have resulted from a single test, that is, a single measured correlation of Y and X .

We argue that it is a serious mistake to use the usual 50% haircut. Our results show that the multiple testing haircut is nonlinear. The highest Sharpe ratios are only moderately penalized, while the marginal Sharpe ratios are heavily penalized. This makes economic sense. The marginal Sharpe ratio

strategies should be thrown out. The strategies with very high Sharpe ratios are probably true discoveries. In these cases, a 50% haircut is too punitive.

Our method does have a number of caveats, some of which apply to any use of the Sharpe ratio. First, high observed Sharpe ratios could be the results of non-normal returns, for instance, an option-like strategy with high ex ante negative skew. In this case, Sharpe ratios should be viewed in the context of the skew. Dealing with these non-normalities is the subject of future research.

Second, Sharpe ratios do not necessarily control for risk. That is, the strategy's volatility may not reflect the true risk. Our method also applies to information ratios, which use residuals from factor models.

Third, it is necessary in the multiple testing framework to take a stand on what qualifies as the appropriate significance level. Is it 0.10 or 0.05?

Fourth, we must make a choice of the multiple testing method. We present results for three methods, as well as the average of the methods. Finally, we need some judgment specifying the number of tests.

Given choices three to five, it is important to determine the haircuts' robustness to changes in these inputs. We provide a program that lets the user vary the key parameters to investigate the effect on the haircuts.² We also provide a program that determines the minimal level of profitability for a trading strategy to be considered significant.

METHOD

We start by describing the simplest situation in which a Sharpe ratio is used to evaluate the performance of a single investment strategy.

Let r_t denote the realized return for an investment strategy between time $t-1$ and t . The investment strategy involves zero initial investment, so that r_t measures the net gain/loss. Such a strategy can be a long-short strategy, i.e., $r_t = R_t^L - R_t^S$ where R_t^L and R_t^S are the gross investment returns for the long and short position, respectively. It can also be a traditional stock and bond strategy in which investors borrow and invest in a risky equity portfolio.

To evaluate whether an investment strategy can generate and maintain true profits, we form a statistical test to see if the expected excess returns are different from zero. Because investors can always switch their positions in the long-short strategy, we focus on a

two-sided alternative hypothesis. In other words, insofar as the long-short strategy can generate a mean return that is significantly different from zero (either positive or negative), we think of it as a profitable strategy. To test this hypothesis, we first construct key sample statistics. Given a sample of historical returns (r_1, r_2, \dots, r_T) , let $\hat{\mu}$ denote the mean and $\hat{\sigma}$ the standard deviation. A t -statistic is constructed to test the null hypothesis that the average return is zero:

$$t\text{-statistic} = \frac{\hat{\mu}}{\hat{\sigma} / \sqrt{T}} \quad (1)$$

Under the assumption that returns are i.i.d. normal,³ the t -statistic follows a t -distribution with $T-1$ degrees of freedom under the null hypothesis.

We can follow standard hypothesis-testing procedures to assess the statistical significance of the investment strategy.

The Sharpe ratio, one of the most commonly used summary statistics in finance, is linked to the t -statistic in a simple manner. Given $\hat{\mu}$ and $\hat{\sigma}$, the Sharpe ratio (\widehat{SR}) is defined as

$$\widehat{SR} = \frac{\hat{\mu}}{\hat{\sigma}} \quad (2)$$

which, based on Equation 1, is simply $t\text{-ratio} / \sqrt{T}$.⁴ Therefore, for a fixed T , a higher Sharpe ratio implies a higher t -statistic, which in turn implies a higher significance level (lower p -value) for the investment strategy. This equivalence between the Sharpe ratio and the t -statistic, among many other reasons, justifies the use of Sharpe ratio as an appropriate measure of an investment strategy's attractiveness under our assumptions.

Despite its widespread use, the Sharpe ratio for a particular investment strategy can be misleading.⁵ This is due to the finance profession's extensive data mining. Because academics, financial practitioners, and individual investors all have a keen interest in finding lucrative investment strategies from the limited historical data, it is not surprising for them to "discover" a few strategies that appear to be very profitable. Both the financial and the science literature recognize this data-mining issue. In finance, many well-established empirical abnormalities (e.g., certain technical trading rules, calendar effects, and the like) are overturned

once we take data mining biases into account.⁶ Profits from trading strategies that use cross-sectional equity characteristics involve substantial statistical biases.⁷ The return predictability of many previously documented variables is shown to be spurious once we perform appropriate statistical tests.⁸ In medical research, discoveries tend to be exaggerated.⁹ This phenomenon is called the winner's curse in medical science: the scientist who makes a discovery in a small study is cursed by finding an inflated effect.

Given the widespread use of the Sharpe ratio, we provide a probability-based multiple testing framework to adjust the conventional ratio for data mining. To illustrate the basic idea, we give a simple example in which we assume that all tests are independent. This example is closely related to the literature on data mining biases. However, we can generalize important quantities in this example using a multiple testing framework. This generalization is key to our approach, as it lets us study the more realistic case when different strategy returns are correlated.

To begin with, we calculate the p -value for the single test:

$$\begin{aligned} p^S &= \Pr(|r| > t\text{-ratio}) \\ &= \Pr(|r| > \widehat{SR} \cdot \sqrt{T}) \end{aligned} \quad (3)$$

where r denotes a random variable that follows a t -distribution with $T - 1$ degrees of freedom. This p -value might make sense if researchers are strongly motivated by an economic theory and directly construct empirical proxies to test the implications of the theory. It does not make sense if researchers have explored hundreds or even thousands of strategies and present only the most profitable one. In the latter case, the p -value for the single test may greatly overstate the true statistical significance.

To quantitatively evaluate this overstatement, we assume that researchers have tried N strategies and present the most profitable one (that is, the one with the largest Sharpe ratio). Additionally, we assume (for now) that the test statistics for these N strategies are independent. Under these simplifying assumptions and under the null hypothesis that none of these strategies can generate non-zero returns, the multiple testing p -value, p^M , for observing a maximal t -statistic that is at least as large as the observed t -ratio is

$$\begin{aligned} p^M &= \Pr(\max\{|r_i|, i = 1, \dots, N\} > t\text{-ratio}) \\ &= 1 - \prod_{i=1}^N \Pr(|r_i| \leq t\text{-ratio}) \\ &= 1 - (1 - p^S)^N \end{aligned} \quad (4)$$

When $N = 1$ (single test) and $p^S = 0.05$, $p^M = 0.05$, so there is no multiple testing adjustment. If $N = 10$ and we observe a strategy with $p^S = 0.05$, $p^M = 0.401$, implying a probability of about 40% in finding an investment strategy that generates a t -statistic that is at least as large as the observed t -ratio, much larger than the 5% probability for single test. Multiple testing greatly reduces the statistical significance of single test. Hence, p^M is the adjusted p -value after we take data mining into account. It reflects the likelihood of finding a strategy that is at least as profitable as the observed strategy after searching through N individual strategies.

By equating the p -value of a single test to p^M , we obtain the defining equation for the multiple testing adjusted (haircut) Sharpe ratio \widehat{HSR} :

$$p^M = \Pr(|r| > \widehat{HSR} \cdot \sqrt{T}) \quad (5)$$

Because p^M is larger than p^S , \widehat{HSR} will be smaller than \widehat{SR} . For instance, assuming there are twenty years of monthly returns ($T = 240$), an annual Sharpe ratio of 0.75 yields a p -value of 0.0008 for a single test. When $N = 200$, $p^M = 0.15$, implying an adjusted annual Sharpe ratio of 0.32 through equation 5. Hence, multiple testing with 200 tests reduces the original Sharpe ratio by approximately 60% ($= (0.75 - 0.32)/0.75$).

This simple example illustrates the gist of our approach. When there is multiple testing, the usual p -value p^S for single test no longer reflects the strategy's statistical significance. The multiple-testing-adjusted p -value p^M , on the other hand, is the more appropriate measure. When the test statistics are dependent, however, the approach in the example is no longer applicable, as p^M generally depends on the joint distribution of the N test statistics. For this more realistic case, we build on the work of HLZ to provide a multiple testing framework to find the appropriate p -value adjustment.

MULTIPLE TESTING FRAMEWORK

When more than one hypothesis is tested, false rejections of the null hypotheses are more likely to

occur; i.e., we incorrectly “discover” a profitable trading strategy. Multiple testing methods are designed to limit such occurrences. Multiple testing methods can be broadly divided into two categories: one controls the family-wise error rate and the other controls the false-discovery rate.¹⁰ Following HLZ, we present three multiple testing procedures.

Type I Error

We first introduce two definitions of Type I error in a multiple testing framework. Assume that M hypotheses are tested and their p -values are (p_1, p_2, \dots, p_M) . Among these M hypotheses, R are rejected. These R rejected hypotheses correspond to R discoveries, including both true discoveries and false discoveries (remember that the null hypothesis is no-skill). Let N_r denote the total number of false discoveries (also known as false positive), i.e., strategies incorrectly classified as profitable. Then the family-wise error rate (FWER) calculates the probability of making at least one false discovery:

$$\text{FWER} = \Pr(N_r \geq 1)$$

Instead of studying the total number of false rejections, i.e., profitable strategies that turn out to be unprofitable, an alternative definition—the false discovery rate—focuses on the proportion of false rejections. Let the false discovery proportion (FDP) be the proportion of false rejections (measured related to total number of rejections, R):

$$\text{FDP} = \begin{cases} \frac{N_r}{R} & \text{if } R > 0, \\ 0 & \text{if } R = 0. \end{cases}$$

Then the false discovery rate (FDR) is defined as:

$$\text{FDR} = E[\text{FDP}]$$

Both FWER and FDR are generalizations of the Type I error probability in a single test. Comparing the two definitions, procedures that control FDR allow the number of false discoveries to grow proportionally with the total number of tests and are thus more lenient than procedures that control FWER. Essentially,

FWER is designed to prevent even one error. FDR controls the error rate.¹¹

P-value Adjustment under FWER

We order the p -values in ascending orders, i.e., $p_{(1)} \leq p_{(2)} \leq \dots \leq p_{(M)}$ and let the associated null hypotheses be $H_{(1)}, H_{(2)}, \dots, H_{(M)}$.

Bonferroni's method adjusts each p -value equally. It inflates the original p -value by the number of tests M :¹²

$$\text{Bonferroni: } p_{(i)}^{\text{Bonferroni}} = \min[Mp_{(i)}, 1], i = 1, \dots, M$$

For example, if we observe $M = 10$ strategies and one of them has a p -value of 0.05, Bonferroni would say the more appropriate p -value is $Mp = 0.50$, and hence the strategy is not significant at 5%. For a more concrete example that we will use throughout this section, suppose we observe $M = 6$ strategies and the ordered p -value sequence is (0.005, 0.009, 0.0128, 0.0135, 0.045, 0.06). Five strategies would be deemed significant under single tests. Bonferroni suggests that the adjusted p -value sequence is (0.03, 0.054, 0.0768, 0.081, 0.270, 0.36). Therefore, only the first strategy is significant under Bonferroni.

Holm's method relies on the sequence of p -values and adjusts each p -value by:¹³

$$\text{Holm: } p_{(i)}^{\text{Holm}} = \min[\max_{j \leq i} \{(M - j + 1)p_{(j)}\}, 1], i = 1, \dots, M$$

Starting from the smallest p -value, Holm's method allows us to sequentially build up the adjusted p -value sequence. Using the previous example, the Holm adjusted p -value for the first strategy is $p_{(1)}^{\text{Holm}} = 6p_{(1)} = 0.03$, which is identical to the level Bonferroni prescribes. Under 5% significance, this strategy is significant.

The second strategy yields $p_{(2)}^{\text{Holm}} = \max[6p_{(1)}, 5p_{(2)}] = 5p_{(2)} = 0.045$, which is smaller than the Bonferroni-implied p -value. Given a cutoff of 5% and, in contrast to what Bonferroni concludes, this strategy is significant. Similarly, the next four adjusted p -values are calculated as $p_{(3)}^{\text{Holm}} = \max[6p_{(1)}, 5p_{(2)}, 4p_{(3)}] = 4p_{(3)} = 0.0512$, $p_{(4)}^{\text{Holm}} = \max[6p_{(1)}, 5p_{(2)}, 4p_{(3)}, 3p_{(4)}] = 4p_{(3)} = 0.0512$, $p_{(5)}^{\text{Holm}} = \max[6p_{(1)}, 5p_{(2)}, 4p_{(3)}, 3p_{(4)}, 2p_{(5)}] = 2p_{(5)} = 0.09$, $p_{(6)}^{\text{Holm}} = \max[6p_{(1)}, 5p_{(2)}, 4p_{(3)}, 3p_{(4)}, 2p_{(5)}, p_{(6)}] = 2p_{(5)} = 0.09$,

making none significant at 5% level. Therefore, the first two strategies are significant under Holm.

Comparing the multiple-testing-adjusted p -values to a given significance level, we can make a statistical inference for each of these hypotheses. If we made the mistake of assuming single tests, and given a 5% significance level, we would “discover” four factors. In multiple testing, both Bonferroni’s and Holm’s adjustment guarantee that the family-wise error rate (FWER) in making such inferences does not exceed the pre-specified significance level. Comparing these two adjustments, $p_{(i)}^{\text{Holm}} \leq p_{(i)}^{\text{Bonferroni}}$ for any i .¹⁴ Therefore, Bonferroni’s method is tougher, because it inflates the original p -values more than Holm’s method does. Consequently, the adjusted Sharpe ratios under Bonferroni will be smaller than those under Holm. Importantly, both of these procedures are designed to eliminate all false discoveries, no matter how many tests for a given significance level. Although this type of approach seems appropriate for a space mission (given the catastrophic consequence of a part failing), asset managers may be willing to accept the fact that the number of false discoveries will increase with the number of tests.

P-value Adjustment under FDR

Benjamini, Hochberg and Yekutieli (BHY)’s procedure defines the adjusted p -values sequentially:¹⁵

$$BHY: p_{(i)}^{\text{BHY}} = \begin{cases} p_{(M)} & \text{if } i = M, \\ \min\left[p_{(i+1)}^{\text{BHY}}, \frac{M \times c(M)}{i} p_{(i)}\right] & \text{if } i \leq M - 1, \end{cases}$$

where $c(M) = \sum_{j=1}^M \frac{1}{j}$. In contrast to Holm’s method, BHY starts from the largest p -value and defines the adjusted p -value sequence through pairwise comparisons. Again using the previous example, we first calculate the normalizing constant as $c(M) = \sum_{j=1}^6 \frac{1}{j} = 2.45$. To assess the significance of the six strategies, we start from the least significant one. BHY sets $p_{(6)}^{\text{BHY}}$ at 0.06, the same as the original value of $p_{(6)}$. For the fifth strategy, BHY yields $p_{(5)}^{\text{BHY}} = \min[p_{(6)}^{\text{BHY}}, \frac{6 \times 2.45}{5} p_{(5)}] = p_{(6)}^{\text{BHY}} = 0.06$. For the fourth strategy, BHY yields $p_{(4)}^{\text{BHY}} = \min[p_{(5)}^{\text{BHY}}, \frac{6 \times 2.45}{4} p_{(4)}] = \frac{6 \times 2.45}{4} p_{(4)} = 0.0496$. Similarly, the first three adjusted p -values are sequentially calculated as

$$p_{(3)}^{\text{BHY}} = \min[p_{(4)}^{\text{BHY}}, \frac{6 \times 2.45}{3} p_{(3)}] = p_{(4)}^{\text{BHY}} = 0.0496,$$

$$p_{(2)}^{\text{BHY}} = \min[p_{(3)}^{\text{BHY}}, \frac{6 \times 2.45}{2} p_{(2)}] = p_{(3)}^{\text{BHY}} = 0.0496, \text{ and}$$

$$p_{(1)}^{\text{BHY}} = \min[p_{(2)}^{\text{BHY}}, \frac{6 \times 2.45}{1} p_{(1)}] = p_{(2)}^{\text{BHY}} = 0.0496.$$

Therefore, the BHY-adjusted p -value sequence is (0.0496, 0.0496, 0.0496, 0.0496, 0.06, 0.06), making the first four strategies significant at 5% level. Based on our example, BHY leads to two more discoveries, compared to Holm, and Holm leads to one more discovery, compared to Bonferroni.

Hypothesis tests based on the adjusted p -values guarantee that the false discovery rate (FDR) does not exceed the pre-specified significance level. The constant $c(M)$ controls the test’s generality. In the original work by Benjamini and Hochberg [1995], $c(M)$ is set equal to one and the test works when p -values are independent or positively dependent. We adopt the choice in Benjamini and Yekutieli [2001] by setting $c(M)$ equal to $\sum_{j=1}^M \frac{1}{j}$. This allows our test to work under arbitrary dependency for the test statistics.

The three multiple testing procedures provide adjusted p -values that control for data mining. Based on these p -values, we transform the corresponding t -ratios into Sharpe ratios. In essence, our Sharpe ratio adjustment method aims to answer the following question: if the multiple-testing-adjusted p -value reflects the genuine statistical significance for an investment strategy, what is the equivalent single-test Sharpe ratio that one should assign to such a strategy as if there were no data mining?

For both Holm and BHY, we need the empirical distribution of p -values for strategies that have been tried so far. We use the structural model estimate from HLZ. The model is based on the performance data for more than 300 risk factors that the academic literature has documented. However, a direct multiple-testing adjustment based on these data is problematic for two reasons. First, we do not observe all the strategies that have been tried. Indeed, thousands more may have been tried, and ignoring these would materially affect our results on the haircut Sharpe ratio. Second, strategy returns are correlated. Correlation affects multiple testing, in that it effectively reduces the number of independent tests. Taking these two concerns into account, HLZ propose a new method to estimate the underlying distribution for factor returns. We use this distribution to make Sharpe ratio adjustments for a new strategy.

Multiple Testing and Cross-Validation

Recent important articles by López de Prado and his coauthors also consider the ex post data mining issue for standard backtests.¹⁶ Due to data mining, they show theoretically that only seven trials are needed to obtain a spurious two-year backtest that has an in-sample realized Sharpe ratio of more than 1.0, while the expected out-of-sample Sharpe ratio is zero. The phenomenon is analogous to the regression overfitting problem, where models found to be superior in in-sample tests often perform poorly out of sample, and is thus termed backtest overfitting. To quantify the degree of backtest overfitting, they propose calculating the probability of backtest overfitting (PBO) that measures the relative performance of a particular backtest among a basket of strategies, using cross-validation techniques.

Their research and our study share a common theme. We both attempt to evaluate the performance of an investment strategy in relation to other available strategies. Their method computes the chance for a particular strategy to outperform the median of the pool of alternative strategies. In contrast, our work adjusts the statistical significance for each individual strategy, so that the overall proportion of spurious strategies is controlled.

Despite these similar themes, our research is different in many ways. First, the objectives of analyses are different. Our research focuses on identifying the group of strategies that generate non-zero returns, while López de Prado evaluates the relative performance of a certain strategy that is fit in-sample. For example, consider a case with a group of factors that are all true. The one with the smallest t -ratio, although dominated by other factors in terms of t -ratios, may still be declared significant in our multiple-testing framework. In contrast, it will rarely be considered significant in the PBO framework, as it is dominated by other, more significant strategies. Second, our method is based on a single test statistic that summarizes a strategy's performance over the entire sample, whereas their method divides and joins the entire sample in numerous ways, with each way corresponding to an artificial hold-out period. Our method is therefore more in line with the statistics literature on multiple testing, while their work is more related to out-of-sample testing and cross-validation.

Third, the extended statistical framework in Harvey and Liu [2015] needs only test statistics. In

contrast, their work relies heavily on each individual strategy's time series. Although the López de Prado approach is data intensive, it does not need to make assumptions regarding the data-generating process for returns. As such, their approach is closer to the machine-learning literature and ours is closer to the econometrics literature. Finally, the PBO method assesses whether a strategy selection process is prone to over-fitting. It is not linked to any particular performance statistics. We primarily focus on Sharpe ratios, as they are directly linked to t -statistics and thus p -values, which are the required inputs for multiple-testing adjustment. Our framework can be easily generalized to incorporate other performance statistics, as long as they also have probabilistic interpretations.

In-Sample Multiple Testing versus Out-of-Sample Validation

Our multiple-testing adjustment is based on in-sample (IS) backtests. In practice, out-of-sample (OOS) tests are routinely used to select among many strategies.

Despite its popularity, OOS testing has several limitations. First, an OOS test may not be truly out of sample. A researcher tries a strategy. After running an OOS test, she finds that the strategy fails. She then revises the strategy and tries again, hoping it will work this time. This trial and error approach is not truly OOS, but the difference is hard for outsiders to see.

Second, an OOS test, like any other test in statistics, only works in a probabilistic sense. In other words, an OOS test's success can be due to luck for both the in-sample selection and the out-of-sample testing. Third, given that the researcher has experienced the data, there is no true OOS that uses historical data.¹⁷ This is especially the case when the trading strategy involves economic variables. No matter how we construct the OOS test, it is not truly OOS, because we know what happened in the economy.

Another important issue with the OOS method, which our multiple-testing procedure can potentially help solve, is the tradeoff between type I (false discoveries) and type II (missed discoveries) errors due to data splitting.¹⁸ In holding out some data, researchers increase the chance of missing true discoveries for the shortened in-sample data. For instance, suppose we have 1,000 observations. Splitting the sample in half

and estimating 100 different strategies in-sample, i.e., based on 500 observations, suppose we identify 10 strategies that look promising (in-sample tests). We then take these 10 strategies to the OOS tests and find that two strategies work. Note that, in this process, we might have missed perhaps three strategies after the first-step IS tests, due to bad luck in the short IS period. These true discoveries are lost because they never get to the second-step OOS tests.

Instead of the 50-50 split, now suppose we use a 90-10 data split. Suppose we identify 15 promising strategies. Among the strategies are two of the three true discoveries that we missed when we had a shorter in-sample period. Although this is good, we unfortunately have only 100 observations held out for the OOS exercise; it will be difficult to separate the good from the bad. At its core, the OOS exercise faces a tradeoff between in-sample and out-of-sample testing power. Although a longer in-sample period leads to a more powerful test, which reduces the chance of committing a type II error (i.e., missing true discoveries), the shorter out-of-sample period provides too little information to truly discriminate among the factors that are found significant in-sample.

So how does our research fit? First, one should be very cautious of OOS tests, because it is hard to construct a true OOS test. The alternative is to apply our multiple-testing framework to the full data, to identify the true discoveries. This involves making a more stringent cutoff for test statistics.

Another and (in our opinion) more promising framework involves merging the two methods. Ideally, we want the strategies to pass both the OOS test on split data and the multiple test on the entire data. The problem is how to deal with the true discoveries that we miss if the in-sample data is too short. As a tentative solution, we can first run the IS tests with a lenient cutoff (e.g., p -value = 0.2) and use the OOS tests to see which strategy survives. At the same time, we can run multiple testing for the full data. We then combine the IS/OOS test and the multiple tests by looking at the intersection of survivors. We leave the details of this approach to future research.

APPLICATIONS

To show how to adjust Sharpe ratios for multiple testing, we first use an example to illustrate how Bonfer-

roni's adjustment works under the assumption that test statistics are independent. We next relax the independence assumption and use the HLZ model to adjust the Sharpe ratio for a new strategy. One salient feature of the HLZ model is that it allows dependency in test statistics. The appendix shows how to apply the framework in HLZ to Sharpe ratio adjustment.

Three Strategies

To illustrate how the Sharpe ratio adjustment works, we begin with three investment strategies that have appeared in the literature. All of these strategies are zero-cost hedge portfolios that simultaneously take long and short positions on the cross-section of U.S. equities. The strategies are the earnings-to-price ratio (E/P), momentum (MOM), and betting against beta factor (BAB , Frazzini and Pedersen [2014]). These strategies cover three distinct types of investment styles (i.e., value (E/P), trend-following (MOM), and potential distortions induced by leverage (BAB)) and generate a range of Sharpe ratios.¹⁹ None of these strategies reflect transaction costs and as such, the Sharpe ratios (and t -statistics) are overstated and should be considered "before-costs" Sharpe ratios.

Two important ingredients in the Sharpe ratio adjustment are the Sharpe ratios' initial values and the number of trials. To highlight the effect of these two inputs, we focus on the simplest independent case. With independence, the multiple testing p -value p^M and the single test p -value p^S are linked through equation 4. When p^S is small, this relation is approximately the same as in Bonferroni's adjustment. Hence, the multiple-testing adjustment we use for this example can be thought of as a special case of Bonferroni's adjustment.

Exhibit 1 shows the summary statistics for these strategies. Among these strategies, the strategy based on E/P is the least profitable, as measured by the Sharpe ratio. It has an average monthly return of 0.43% and a monthly standard deviation of 3.47%, the multiple testing p -value increases to 0.029. The haircut (hc), which captures the percentage change in the Sharpe ratio, is about 27%. When there are more trials, the haircut is even larger.

Sharpe ratio adjustment depends on the Sharpe ratio's initial value. Across the three investment strategies, the Sharpe ratio ranges from 0.43 (E/P) to 0.78 (BAB). The haircut is not uniform across different initial

EXHIBIT 1

Multiple Testing Adjustment for Three Investment Strategies

Summary statistics for three investment strategies: *E/P*, *MOM*, and *BAB* (betting against beta, Frazzini and Pedersen [2014]). “Mean” and “Stdev.” report the monthly mean and standard deviation of returns, respectively; \widehat{SR} reports the annualized Sharpe ratio; “*t*-stat” reports the *t*-statistic for the single-hypothesis test that the mean strategy return is zero ($t\text{-stat} = \widehat{SR} \times \sqrt{T/12}$); p^S and p^M report the *p*-value for single and multiple tests, respectively; \widehat{HSR} reports the Bonferroni-adjusted Sharpe ratio; and \widehat{hc} reports the percentage haircut for the adjusted Sharpe ratio ($\widehat{hc} = (\widehat{SR} - \widehat{HSR}) / \widehat{SR}$).

Strategy/Number of Tests (N)	Mean(%) (monthly)	Stdev.(%) (monthly)	\widehat{SR} (annual)	<i>t</i> -stat	p^S (single)	p^M (multiple)	\widehat{HSR} (annual)	\widehat{hc} (haircut)
Panel A: N = 10								
E/P	0.43	3.47	0.43	2.99	2.88×10^{-3}	2.85×10^{-2}	0.31	26.6%
MOM	1.36	7.03	0.67	4.70	3.20×10^{-6}	3.20×10^{-5}	0.60	10.9%
BAB	0.70	3.09	0.78	7.29	6.29×10^{-13}	6.29×10^{-12}	0.74	4.6%
Panel B: N = 50								
E/P	0.43	3.47	0.43	2.99	2.88×10^{-3}	1.35×10^{-1}	0.21	50.0%
MOM	1.36	7.03	0.67	4.70	3.20×10^{-6}	1.60×10^{-5}	0.54	19.2%
BAB	0.70	3.09	0.78	7.29	6.29×10^{-13}	3.14×10^{-11}	0.72	7.9%
Panel C: N = 100								
E/P	0.43	3.47	0.43	2.99	2.88×10^{-3}	2.51×10^{-1}	0.16	61.6%
MOM	1.36	7.03	0.67	4.70	3.20×10^{-6}	1.60×10^{-5}	0.51	23.0%
BAB	0.70	3.09	0.78	7.29	6.29×10^{-13}	6.29×10^{-11}	0.71	9.3%

Sharpe ratio levels. For instance, when the number of trials is 50, the haircut is almost 50% for the least profitable *E/P* strategy, but only 7.9% for the most profitable *BAB* strategy.²⁰ We believe this non-uniform feature of our Sharpe ratio adjustment procedure is economically sensible, as it lets us discount mediocre Sharpe ratios harshly while keeping the exceptional ones relatively intact. It is sufficient to make up for the loss in statistical significance due to data snooping. This phenomenon is not unique to the three strategies or the Bonferroni method that we present. As we will discuss further, this is a general property of our Sharpe ratio adjustment method.

Sharpe Ratio Adjustment for a New Strategy

Given the population of investment strategies that have been published, we now show how to adjust a new investment strategy's Sharpe ratio. Consider a new strategy that generates a Sharpe ratio of \widehat{SR} in *T* periods,²¹ or, equivalently, the *p*-value p^S . Assuming that *N* other strategies have been tried, we draw *N* *t*-statistics from the model in HLZ. Additional details are described in the appendix. These *N* + 1 *p*-values are then adjusted using the aforementioned three multiple testing procedures. In particular, we obtain the adjusted *p*-value p^M for p^S . To take the uncertainty in drawing *N* *t*-statistics

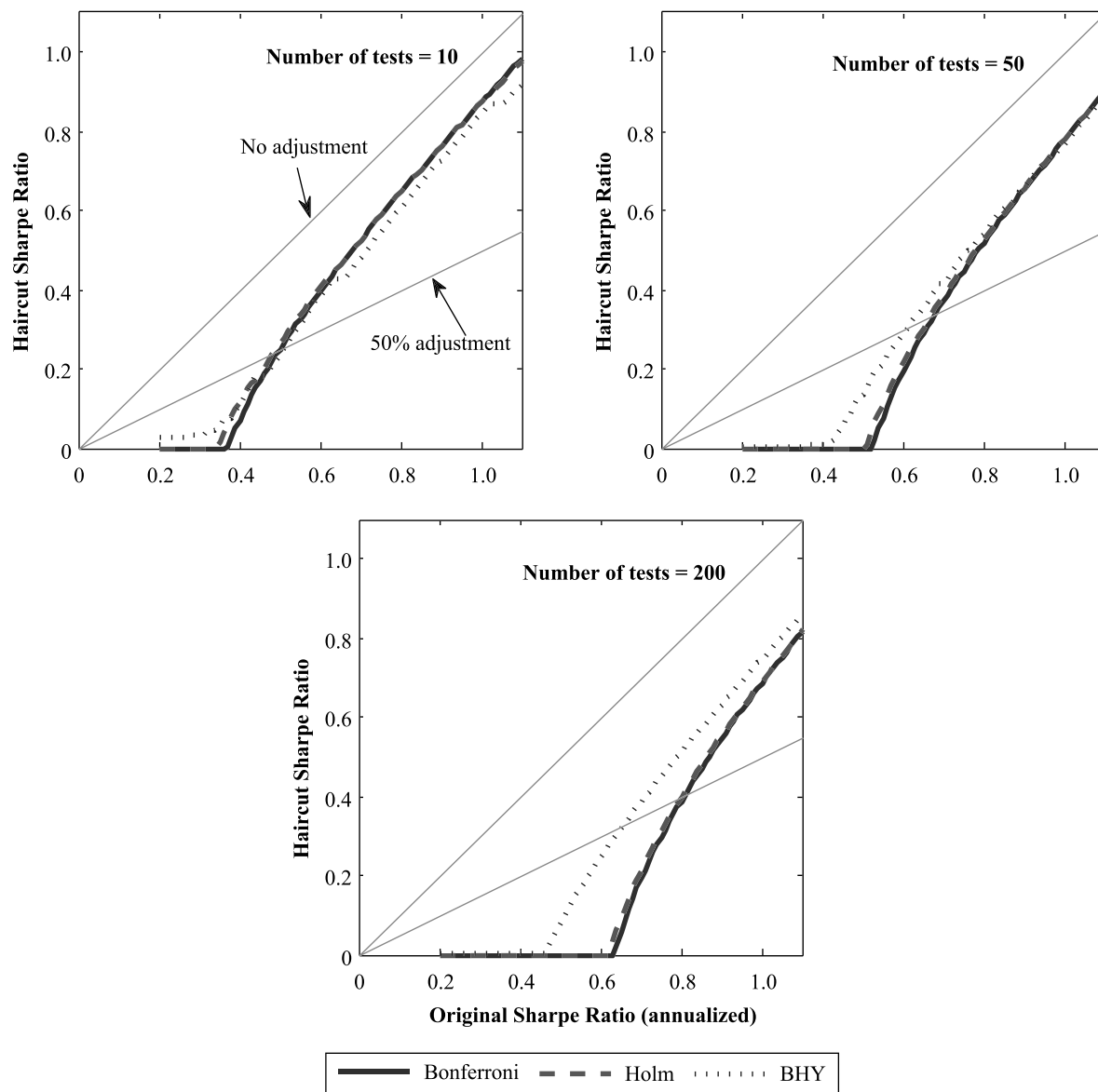
into account, we repeat the procedure many times to generate a sample of p^M s. The median of this sample is taken as the final multiple-testing-adjusted *p*-value. This *p*-value is then transformed back into a Sharpe ratio—the multiple-testing-adjusted Sharpe ratio. Exhibit 2 shows the original vs. haircut Sharpe ratios, and Exhibit 3 shows the corresponding haircut. With a user-specified correlation level, we linearly interpolate among the five sets of parameter estimates in HLZ to find a new set of parameter estimates that exactly achieves the assumed correlation level.

First, as previously discussed, the haircuts depend on the Sharpe ratios' levels. Across the three types of multiple testing adjustment and different numbers of tests, the haircut is almost always more than and sometimes much larger than 50% when the annualized Sharpe ratio is less than 0.4. On the other hand, when the Sharpe ratio is greater than 1.0, the haircut is at most 25%. This shows the 50% rule of thumb discount for the Sharpe ratio is inappropriate: 50% is too lenient for relatively small Sharpe ratios (< 0.4) and too harsh for large ones (> 1.0). This nonlinear feature of the Sharpe ratio adjustment makes economic sense. Marginal strategies are heavily penalized because they are likely false discoveries.

Second, the three adjustment methods imply different haircut magnitudes. Given the theoretical

EXHIBIT 2

Original vs. Haircut Sharpe Ratios



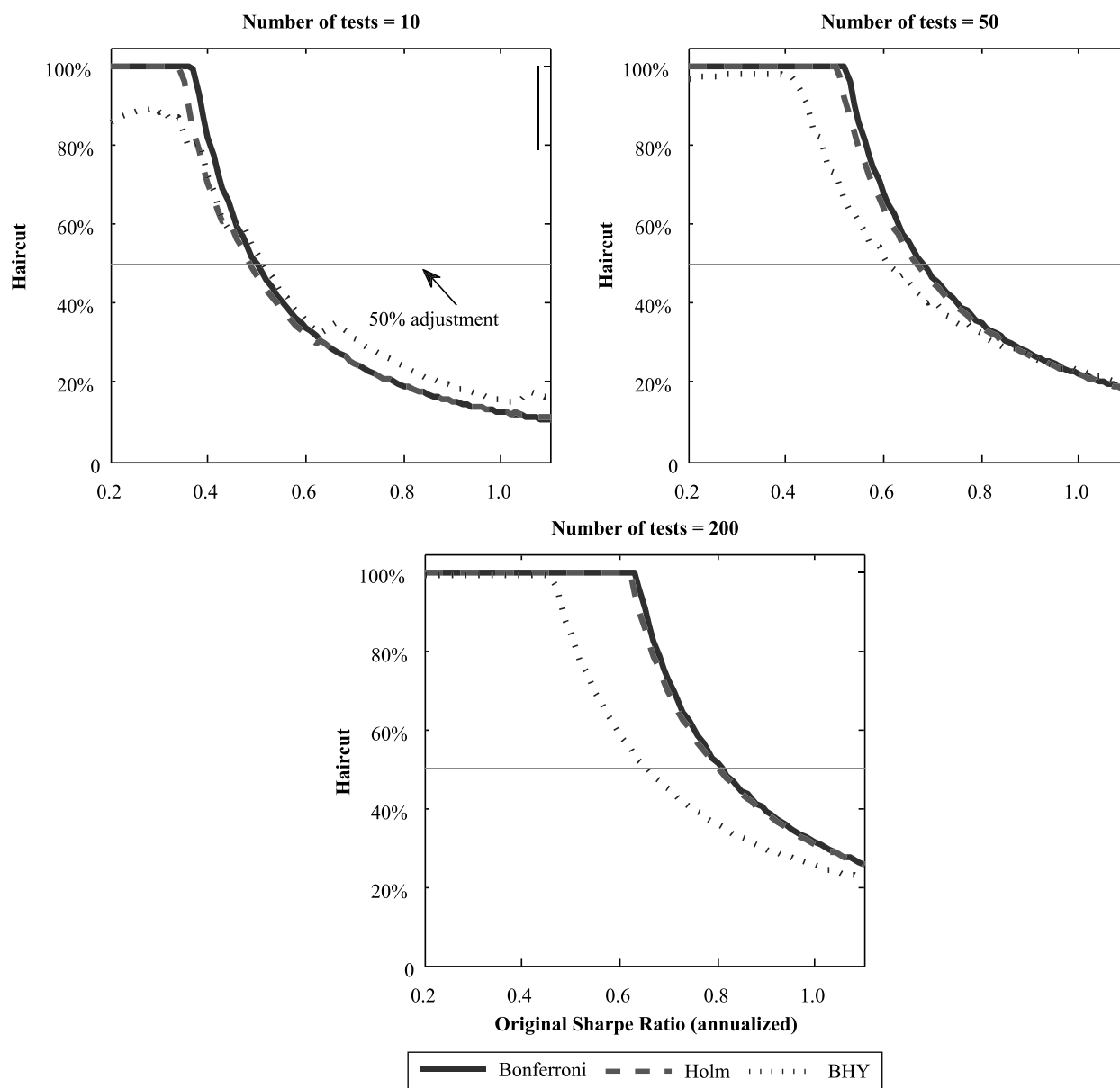
objectives that these methods try to control (i.e., family-wise error rate (FWER) vs. false discovery rate (FDR)), we divide the three adjustments into two groups: Bonferroni and Holm in one group and BHY in the other group. Comparing Bonferroni and Holm's method, we see that Holm's method implies a smaller haircut than does Bonferroni's method. This is consistent with our previous discussion about Holm's adjustment being less aggressive than Bonferroni's adjustment. However, the difference is relatively small (compared to the differ-

ence between Bonferroni and BHY), especially when the number of tests is large. The haircuts under BHY, on the other hand, are usually a lot smaller than those under Bonferroni and Holm when the Sharpe ratio is small (< 0.4). For large Sharpe ratios (> 1.0), however, the haircuts under BHY are consistent with those under Bonferroni and Holm.

In the end, we advocate the BHY method. The FWER seems appropriate for applications where a false discovery brings a severe consequence. In financial

EXHIBIT 3

Haircuts



applications, it seems reasonable to control for the rate of false discoveries, rather than the absolute number.

Minimum Profitability for Proposed Trading Strategies

There is another way to pose the problem. Given an agreed-on significance level, such as 0.05, what is

the minimum average monthly return that a proposed strategy must exceed? Our framework is ideally suited to answer this question.

The answer to the question depends on a number of inputs. We need to measure the strategy's volatility. The number of observations is also a critical input.²² Finally, we need to take a stand on the number of tests that have been conducted.

EXHIBIT 4

Minimum Profitability Hurdles

Average monthly return hurdles under single and multiple tests. At 5% significance, the table shows the minimum average monthly return for a strategy to be significant at 5% with 300 tests. All numbers are in percentage terms. See the appendix for the link to the program.

	Annualized Volatility		
	$\sigma = 5\%$	$\sigma = 10\%$	$\sigma = 15\%$
Panel A: Observations = 120			
Single	0.258	0.516	0.775
Bonferroni	0.496	0.992	1.488
Holm	0.486	0.972	1.459
BHY	0.435	0.871	1.305
Panel B: Observations = 240			
Single	0.183	0.365	0.548
Bonferroni	0.351	0.702	1.052
Holm	0.344	0.688	1.031
BHY	0.307	0.616	0.923
Panel C: Observations = 480			
Single	0.129	0.258	0.387
Bonferroni	0.248	0.496	0.744
Holm	0.243	0.486	0.729
BHY	0.217	0.435	0.651
Panel D: Observations = 1000			
Single	0.089	0.179	0.268
Bonferroni	0.172	0.344	0.516
Holm	0.169	0.337	0.505
BHY	0.151	0.302	0.452

Exhibit 4 presents an example. Here we consider four different sample sizes: 120, 240, 480, and 1,000 and three different levels of annualized volatility: 5%, 10%, and 15%. We then assume that the total number of tests is 300. To generate the table, we first find the threshold t -ratios based on the multiple-testing-adjustment methods provided in the previous section and then transform these t -ratios into mean returns based on the formula in equation (1).

Exhibit 4 shows the large differences between the return hurdles for single testing and multiple testing. For example, in panel B (240 observations) and 10% volatility, the minimum required average monthly return for a single test is 0.365% per month, or 4.4% annually. However, for BHY, the return hurdle is much higher: 0.616% per month, or 7.4% annually. The Programs section (following the conclusions) details the program that we use to generate these return hurdles and provides an Internet address to download the program.

CONCLUSIONS

There are many considerations involved in evaluating a trading strategy. The criteria may include the strategy's economic foundation, Sharpe ratio, significance level, drawdown, consistency, diversification, recent performance, etc. We provide a real-time evaluation method for determining the significance of a candidate trading strategy. Our method explicitly takes into account that hundreds, if not thousands of strategies have been proposed and tested in the past. Given these multiple tests, inference must be recalibrated.

Our method follows the following steps. First, we transform the Sharpe ratio into a t -ratio and determine its probability value, e.g., 0.05. Second, we determine the appropriate p -value, explicitly recognizing the multiple tests that preceded the discovery of this particular investment strategy. Third, based on this new p -value, we transform the corresponding t -ratio back to a Sharpe ratio. The new measure, which we call the haircut Sharpe ratio, takes multiple testing or data mining into account. Our method is readily applied to other popular risk metrics, such as value at risk (VaR).²³

Our method is ideally suited to determine minimum profitability hurdles for proposed strategies. We provide open access code where the inputs are the desired significance level, the number of observations, the strategy volatility, and the assumed number of tests. The output is the minimum average monthly return that the proposed strategy must exceed.

There are many caveats to our method. We do not observe the entire history of tests and, as such, we need to use judgment on an important input—the number of tests—for our method. In addition, we use Sharpe ratios as our starting point. Our method is not applicable insofar as the Sharpe ratio is not the appropriate measure (e.g., non-linearities in the trading strategy or the variance not being a complete measure of risk).

Of course, true out-of-sample tests of a particular strategy (not a holdout sample of historical data) is a cleaner way to evaluate a strategy's viability. For some strategies, models can be tested on previously unpublished data or even on uncorrelated markets. However, for the majority of trading strategies, true out-of-sample tests are not available. Our method lets investors make decisions in real time on a proposed strategy's viability.

PROGRAMS

We make the code and data for our calculations publicly available at: <http://faculty.fuqua.duke.edu/~charvey/backtesting>

Haircut Sharpe Ratios

The Matlab function *Haircut_SR* lets the user specify key parameters to make Sharpe ratio adjustments and calculate the corresponding haircuts. It has eight inputs that provide summary statistics for a return series of an investment strategy and the number of tests allowed for. The first input is the sampling frequency for the return series. Five options (daily, weekly, monthly, quarterly, and annually) are available.²⁴ The second input is the number of observations, in terms of the sampling frequency provided in the first step. The third input is the strategy returns' Sharpe ratio. It can either be annualized or based on the sampling frequency provided in the first step; it can be autocorrelation-corrected or not. Subsequently, the fourth input asks if the Sharpe ratio is annualized and the fifth input asks if the Sharpe ratio has been corrected for autocorrelation.²⁵ The sixth input asks for the autocorrelation of the returns if the Sharpe ratio has not been corrected for autocorrelation.²⁶ The seventh input is the number of tests assumed. Lastly, the eighth input is the assumed average level of correlation among strategy returns.

To give an example of how the program works, suppose that we have an investment strategy that generates an annualized Sharpe ratio of 1.0 over 120 months. The Sharpe ratio is not autocorrelation corrected and the monthly autocorrelation coefficient is 0.1. We allow for 100 tests in multiple testing and assume the average level of correlation is 0.4 among strategy returns. With this information, the input vector for the program is

Input vector

$$= \begin{bmatrix} \text{D / W / M / Q / A} = 1, 2, 3, 4, 5 \\ \text{\# of obs} \\ \text{Sharpe ratio} \\ \text{SR annualized? (1 = Yes)} \\ \text{AC correction needed? (0 = Yes)} \\ \text{AC level} \\ \text{\# of tests assumed} \\ \text{Average correlation assumed} \end{bmatrix} = \begin{bmatrix} 3 \\ 120 \\ 1 \\ 1 \\ 1 \\ 0.1 \\ 100 \\ 0.4 \end{bmatrix}$$

EXHIBIT 5

Program Outputs

Inputs:
Frequency = Monthly;
Number of Observations = 120;
Initial Sharpe Ratio = 1.000;
Sharpe Ratio Annualized = Yes;
Autocorrelation = 0.100;
A/C Corrected Annualized Sharpe Ratio = 0.912
Assumed Number of Tests = 100;
Assumed Average Correlation = 0.400.

Outputs:
Bonferroni Adjustment:
Adjusted P-value = 0.465;
Haircut Sharpe Ratio = 0.232;
Percentage Haircut = 74.6%.

Holm Adjustment:
Adjusted P-value = 0.409;
Haircut Sharpe Ratio = 0.262;
Percentage Haircut = 71.3%.

BHY Adjustment:
Adjusted P-value = 0.169;
Haircut Sharpe Ratio = 0.438;
Percentage Haircut = 52.0%.

Average Adjustment:
Adjusted P-value = 0.348;
Haircut Sharpe Ratio = 0.298;
Percentage Haircut = 67.3%.

Passing this input vector to *Haircut_SR*, the function generates a sequence of outputs, as shown in Exhibit 5. The program summarizes the return characteristics by showing an annualized, autocorrelation-corrected Sharpe ratio of 0.912, as well as the other data the user provides. The program output includes adjusted *p*-values, haircut Sharpe ratios, and the haircuts involved for these adjustments under a variety of adjustment methods. For instance, under BHY, the adjusted, annualized Sharpe ratio is 0.438 and the associated haircut is 52.0%.

Profit Hurdles

The Matlab function *Profit_Hurdle* lets the user calculate the required mean return for a strategy at a given level of significance. It has five inputs. The first input is the user-specified significance level. The second input is the number of monthly observations for the strategy.

EXHIBIT 6

Program Outputs

Inputs:	
Significance Level = 5.0%;	
Number of Observations = 240;	
Annualized Return Volatility = 10.0%;	
Assumed Number of Tests = 300;	
Assumed Average Correlation = 0.400.	
Outputs:	
Minimum Average Monthly Return:	
Independent = 0.365%;	
Bonferroni = 0.702%;	
Holm = 0.686%;	
BHY = 0.621%;	
Average for Multiple Tests = 0.670%.	

The third input is the strategy's annualized return volatility. The fourth input is the number of tests assumed. Lastly, the fifth input is the assumed average level of correlation among strategy returns. The program does not allow for any autocorrelation in the strategy returns.

To give an example of how the program works, suppose we are interested in the required return for a strategy that covers 20 years and has an annual volatility of 10%. In addition, we allow for 300 tests and specify a significance level of 5%. Finally, we assume that the average correlation among strategy returns is 0.4. With these specifications, the input vector for the program is

Input vector

$$= \begin{bmatrix} \text{Significance level} \\ \text{\# of obs} \\ \text{Annualized return volatility} \\ \text{\# of tests assumed} \\ \text{Average correlation assumed} \end{bmatrix} = \begin{bmatrix} 0.05 \\ 240 \\ 0.1 \\ 300 \\ 0.4 \end{bmatrix}$$

Passing the input vector to *Profit_Hurdle*, the function generates a sequence of outputs, as shown in Exhibit 6. The program summarizes the data the user provides. The program output includes return hurdles for a variety of adjustment methods. For instance, the adjusted return hurdle under BHY is 0.621% per month and the average multiple testing return hurdle is 0.670% per month.

Correlation Adjustment

We use the model estimated in HLZ to provide correlation adjustment when tests are correlated.

HLZ study 316 strategies that have been documented by the academic literature. They propose a structural model to capture trading strategies' underlying distribution. Two key features mark their model. First, there is publication bias, so not all tried factors make it to publication. Second, tests may be correlated and this affects multiple testing adjustment. Taking these two concerns into account, HLZ postulate a mixture distribution for strategy returns. With probability p_0 , a strategy has a mean return of zero and therefore comes from the null distribution. With probability $1 - p_0$, a strategy has a nonzero mean and therefore comes from the alternative distribution. To capture the heterogeneity in strategy mean returns, HLZ assume that the mean returns for true strategies are drawn from an exponential distribution with a mean of λ . After fixing the mean returns, HLZ assume that the innovations in returns follow a normal distribution with a mean of zero and a standard deviation of $ma = 15\%$. (Heterogeneity in return standard deviations are captured by the heterogeneity in return means.) Importantly, return innovations are correlated in the cross-section and are captured by the pairwise correlation ρ . At a certain level of correlation (ρ) between strategy returns, and by matching the model-implied t-ratio quantiles with the observed t-ratio quantiles, HLZ estimate the probability (p_0), the total number of trials (M), and (λ). They show that both p_0 and M are increasing as the level of correlation rises.

We use the model estimates in HLZ to approximate strategy returns' underlying distribution. The relevant parameters for our application are ρ , p_0 , and λ . HLZ provide five sets of estimates, corresponding to five

EXHIBIT 7

Model Parameter Estimates in HLZ

The HLZ model estimates that ρ is the correlation coefficient between two strategy returns in the same period. p_0 is the probability of having a strategy that has a mean of zero. λ is the mean parameter of the exponential distribution for the monthly means of the true factors.

ρ	p_0	$\lambda(\%)$
0	0.396	0.550
0.2	0.444	0.555
0.4	0.485	0.554
0.6	0.601	0.555
0.8	0.840	0.560

levels of correlation (i.e., $\rho = 0, 0.2, 0.4, 0.6$ and 0.8). Exhibit 7 shows HLZ's model estimates.

For our application, at a user-specified level of ρ , we use linear interpolation to generate the parameter estimates for p_0 and λ . For example, if the user specifies $\rho = 0.3$, then the parameter estimates for p_0 and λ would be:

$$\begin{aligned} p_0(0.3) &= 0.5 \times p_0(0.2) + 0.5 \times p_0(0.4) \\ &= 0.5 \times 0.444 + 0.5 \times 0.485 = 0.465 \end{aligned}$$

$$\begin{aligned} \lambda(0.3) &= 0.5 \times \lambda(0.2) + 0.5 \times \lambda(0.4) \\ &= 0.5 \times 0.555 + 0.5 \times 0.554 = 0.555 \end{aligned}$$

where $p_0(\rho)$ and $\lambda(\rho)$ denote the estimate for p_0 and λ when the correlation is set at ρ , respectively. When ρ is greater than 0.8 , we interpolate based on $\rho = 0.6$ and $\rho = 0.8$, that is:

$$\begin{aligned} p_0(\rho) &= \frac{0.8 - \rho}{0.2} \times p_0(0.6) + \frac{\rho - 0.6}{0.2} \times p_0(0.8) \\ \lambda(\rho) &= \frac{0.8 - \rho}{0.2} \times \lambda(0.6) + \frac{\rho - 0.6}{0.2} \times \lambda(0.8) \end{aligned}$$

When ρ is not specified, we use the preferred estimates in HLZ, i.e., $\rho = 0.2$.

The user specifies the value for ρ . We take the following steps to obtain the multiple-testing-adjusted Sharpe ratios:

- We obtain the estimate for p_0 and λ using the aforementioned linear interpolation.
- The user calculates the Sharpe ratio (\widehat{SR}) of the new strategy under consideration and specifies how many alternative strategies have been tried (N).
- With these parameter specifications ($\rho, p_0, \lambda, \widehat{SR}, N$), we run B ($=5,000$) sets of simulations to find the haircut Sharpe ratio. The following steps describe the steps of the simulations:
 - a. For each set of simulation, we draw N strategies based on the model in HLZ that is parameterized by ρ, p_0 , and λ . In particular, with probability p_0 , the strategy mean is drawn as zero; with probability $1 - p_0$, the strategy mean is drawn from an exponential distribution with mean λ . The return innovations are contemporaneously correlated with a correlation coefficient of ρ and are assumed to be uncorrelated over time. All strategy returns have a volatility of $\sigma = 15\%$.

- b. We calculate the p -values for the N simulated return series and use the three multiple-testing-adjustment procedures described in the main text to calculate the adjusted p -value for the new strategy.
- c. We take the median p -value across the B sets of simulations as the final adjusted p -value. Lastly, we convert this p -value into the haircut Sharpe ratio \widehat{HSR} .

Intuitively, a larger p_0 implies more flukes among the strategies and thus a higher haircut. A larger λ means that true strategies have higher means and are thus more significant. As a result, the haircut is smaller. Our model lets a user calculate exactly the haircut level needed for a specification of p_0 and λ .

ENDNOTES

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¹While practices may vary across different investment management companies, the 50% haircut is generally considered the industry standard. In presenting this article to numerous practitioners of finance, such as at Inquire Europe-UK, CQA, and SQA, no one challenged the 50% assumption.

²Our program is available at

<http://faculty.fuqua.duke.edu/~charvey/backtesting>

³Without the normality assumption, the t -statistic becomes asymptotically normally distributed, based on the central limit theorem.

⁴Lower-frequency Sharpe ratios can be calculated straightforwardly, assuming higher-frequency returns are independent. For instance, if $\hat{\mu}$ and $\hat{\sigma}$ denote the mean and volatility of monthly returns, respectively, then the annual Sharpe ratio equals $12\hat{\mu} / \sqrt{12}\hat{\sigma} = \sqrt{12}\hat{\mu} / \hat{\sigma}$.

⁵It can also be misleading if returns are not i.i.d. (for example, non-normality and/or autocorrelation), or if the volatility does not reflect the risk.

⁶See Sullivan, Timmermann, and White [1999, 2001] and White [2000].

⁷See Leamer [1978], Lo and MacKinlay [1990], Fama [1991], and Schwert [2003]. A recent article by McLean and Pontiff [2015] shows a significant degradation of performance of identified anomalies after publication.

⁸See Welch and Goyal [2008].

⁹See Button et al. [2013].

¹⁰For the literature on the family-wise error rate, see Holm [1979], Hochberg [1988], and Hommel [1988]. For the literature on the false-discovery rate, see Benjamini and Hochberg [1995],

Benjamini and Liu [1999], Benjamini and Yekutieli [2001], Storey [2003], and Sarkar and Guo [2009].

¹¹For more details on FWER and FDR, see HLZ.

¹²For the statistics literature on Bonferroni's method, see Schweder and Spjøtvoll [1982] and Hochberg and Benjamini [1990]. For the applications of Bonferroni's method in finance, see Shanken [1990], Ferson and Harvey [1999], Boudoukh et al. [2007], and Patton and Timmermann [2010].

¹³For the literature on Holm's procedure and its extensions, see Holm [1979] and Hochberg [1988]. Holland, Basu, and Sun [2010] emphasize the importance of Holm's method in accounting research.

¹⁴See Holm [1979] for the proof.

¹⁵For the statistical literature on BHY's method, see Benjamini and Hochberg [1995], Benjamini and Yekutieli [2001], Sarkar [2002], and Storey [2003]. For the applications of methods that control the *false discovery rate* in finance, see Barras, Scaillet, and Wermers [2010], Bajgrowicz and Scaillet [2012], and Kosowski et al. [2006].

¹⁶See Bailey et al. [2014, 2015] and López de Prado [2013].

¹⁷See López de Prado [2013] for a similar argument.

¹⁸See Hansen and Timmermann [2012] for a discussion on sample splitting for univariate tests.

¹⁹For *E/P*, we construct an investment strategy that takes a long position in the top decile (highest *E/P*) and a short position in the bottom decile (lowest *E/P*) of the cross-section of *E/P* sorted portfolios. For *MOM*, we construct an investment strategy that takes a long position in the top decile (past winners) and a short position in the bottom decile (past losers) of the cross-section of portfolios sorted by past returns. Both the data for *E/P* and *MOM* are obtained from Ken French's on-line data library for the period from July 1963 to December 2012. For *BAB*, return statistics are extracted from table IV of Frazzini and Pedersen [2013].

²⁰Mathematically, this happens because the *p*-value is very sensitive to the *t*-statistic when the *t*-statistic is large. In our example, when *N* = 50 and for *BAB*, the *p*-value for a *t*-statistic of 7.29 (single test) is one-50th of the *p*-value for a *t*-statistic of 6.64 (multiple-testing-adjusted *t*-statistic), i.e., $p^M/p^S \approx 50$.

²¹Assuming *T* is in months, if \widehat{SR} is an annualized Sharpe ratio, $t\text{-stat} = \widehat{SR} \times \sqrt{T/12}$; if \widehat{SR} is a monthly Sharpe ratio, $t\text{-stat} = \widehat{SR} \times \sqrt{T}$.

²²The number of observations is also central to converting a Sharpe ratio to a *t*-statistic.

²³Let $VaR(\alpha)$ of a return series to be the α -th percentile of the return distribution. Assuming that returns are approximately normally distributed, it can be shown that *VaR* is related to Sharpe ratio by $\frac{VaR(\alpha)}{\sigma} = SR - z_\alpha$, where z_α is the *z*-score for the $(1 - \alpha)$ -th percentile of a standard normal distribution and σ is the standard deviation of the return. Multiple-testing-adjusted Sharpe ratios can then be used to adjust *VaR*s. As with the Sharpe ratio, if non-normalities exist, these features need to be reflected in the *VaR*.

²⁴We use number one, two, three, four, and five to indicate daily, weekly, monthly, quarterly, and annually sampled returns, respectively.

²⁵For the fourth input, "1" denotes a Sharpe ratio that is annualized and "0" denotes otherwise. For the fifth input, "1" denotes a Sharpe ratio that is not autocorrelation corrected and "0" denotes otherwise.

²⁶We follow Lo [2002] to adjust Sharpe ratios for autocorrelation.

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