

5. (*George and Joe*) Show that if A is real, square and invertible, then the QR factorization is unique apart from a diagonal scaling by factors of ± 1 . That is, if $A = Q_1 R_1 = Q_2 R_2$ with Q_1, Q_2 orthogonal, R_1, R_2 upper triangular, then there is a diagonal matrix D with the diagonal entries ± 1 , where $Q_2 = Q_1 D$ and $D R_2 = R_1$. In particular, show that if the diagonal entries of R_1, R_2 are all positive, then $Q_1 = Q_2$ and $R_1 = R_2$.

We have

$$Q_1 R_1 = Q_2 R_2$$

$$\Rightarrow R_1 = Q_1^T Q_2 R_2$$

$$\Rightarrow Q_1^T Q_2 = R_1 R_2^{-1}$$

Since the product of upper triangular matrices is upper triangular,

$\tilde{D} := Q_1^T Q_2$ is upper triangular. We also have

$$\begin{aligned} \tilde{D} \tilde{D}^T &= Q_1^T Q_2 (Q_1^T Q_2)^T \\ &= Q_1^T Q_2 Q_2^T Q_1 \\ &= I \end{aligned}$$

This means \tilde{D} is also orthogonal, i.e.,

$$\tilde{D}^T = \tilde{D}^{-1}$$

The LHS is lower triangular. The RHS is upper triangular (as the inverse of an upper triangular matrix is upper triangular).

This forces \tilde{D}^T , and therefore \tilde{D} , to be diagonal. The combination of being real, diagonal, and orthogonal forces the entries of \tilde{D} to be ± 1 .

Now all of the desired equations hold if $D := \tilde{D}$. If everything is positive then $\tilde{D} = I$ and the equalities $Q_1 = Q_2$ and $R_1 = R_2$ follow immediately.