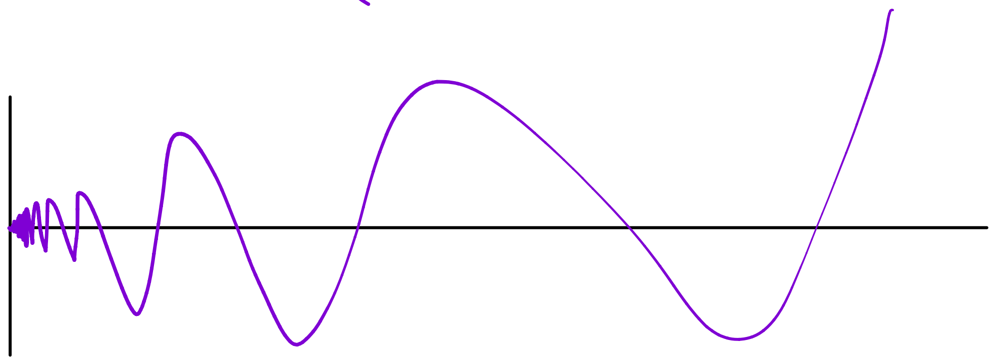
A function that is continuous on $[0,1]$, AC on $[\varepsilon,1]$, but not AC on $[0,1]$

Consider the function $f:[0,1]\to\mathbb{R}$, defined by

$$f(x)=\begin{cases} x\cos(\pi/2x) & 0<x\le 1 \\ 0 & x=0 \end{cases}$$



Note f is continuous over $[0,1]$, and in fact has bounded first derivative over $[\varepsilon,1]$ for $\varepsilon>0$. Thus it is Lipschitz on $[\varepsilon,1]$. Lipschitz functions are absolutely continuous, so $f\in AC([\varepsilon,1])$.

We can show that f is not of bounded variation on $[0,1]$, so is not absolutely continuous there either.

Indeed, for a natural number n, define a partition of $[0,1]$ by $P_n=\{x_0,\dots,x_{2n-1}\}$ where $x_0=0$ and $x_k=\dfrac{1}{2k+1-k}$, $1\le k<2n$, i.e., $\{0,1/2n,1/2n-1,1/2n-2,\dots,1/3,1/2,1\}$. Then

$$f(x_k)=\frac{1}{2n-k}\cos\left(\frac{(2n-k)\pi}{2}\right) \quad \text{so} \quad f(x_k)=\begin{cases} 1/(2n-k) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Then

$$|f(x_k) - f(x_{k-1})| = \begin{cases} \frac{1}{2n-k-1} & k \text{ odd} \\ \frac{1}{2n-k} & k \text{ even} \end{cases}$$

This means we will sum every nonzero term twice in the total variation:

$$TV(f) = \sum_{k=1}^{2n-1} |f(x_k) - f(x_{k-1})|$$

$$= 1 + \sum_{k=0}^{n-1} 2\left(\frac{1}{2n-2k}\right)$$

$$= \sum_{k=0}^{n-1} \frac{1}{n-k}$$

$$= \sum_{k=1}^{n} \frac{1}{k}$$

This is the harmonic series, which diverges. So $f \notin BV([0,1])$ and is therefore not in $AC([0,1])$.