2. **(Qual Summer 2017 #1)** Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable then the set $A = \{x \in \mathbb{R} : m(f^{-1}(x)) > 0\}$ has measure zero.

Let $A_{s,n} = \{x \in \mathbb{R} \mid m(f^{-1}(x) \cap [s, s+1)) > \frac{1}{n}\}$.

Let $\{x_1, \dots, x_K\}$ be K distinct members of $A_{s,n}$. As $f^{-1}(x_i)$ and $f^{-1}(x_j)$ are disjoint for $i \neq j$, each $f^{-1}(x_i) \cap [s, s+1]$ and $f^{-1}(x_j) \cap [s, s+1]$ are also disjoint for $i \neq j$. Thus we have:

$$1 = m([s, s+1))$$
$$\geq \sum_{j=1}^{K} m(f^{-1}(x_j) \cap [s, s+1))$$
$$> K \cdot \left(\frac{1}{n}\right)$$
$$\Rightarrow K < n$$

This implies $|A_{s,n}| < \infty$. Let $A_s = \{x \in \mathbb{R} \mid m(f^{-1}(x) \cap [s, s+1)) > 0\}$.

Note that $A_s = \bigcup_{n=1}^{\infty} E_{s,n}$, so A_s is a countable union of finite sets and is thus countable.

Let $x \in A$. Then as $m(f^{-1}(x)) > 0$, there exists some $k \in \mathbb{Z}$ such that $m(f^{-1}(x) \cap [k, k+1]) > 0$, or else $f^{-1}(x)$ would be a countable intersection of sets of measure zero, implying $m(f^{-1}(x)) = 0$ ↯. So x belongs to A_k for some k, meaning $A \subseteq \bigcup_{s \in \mathbb{Z}} A_s$ and is a countable union of countable sets, which means A is countable $\Rightarrow m(A) = 0$ //