3. Solve $u_t = u_{xx}$ in $[0,\pi]\times[0,\infty)$ with $u(0,t)=0$ and $u(\pi,t)=1$ for all t and $u(x,0)=1$ for $x\in(0,\pi)$. In what sense the solution takes the initial data and prove it is unique.

We want to homogenize the boundary conditions first; let $g(x,t) = \frac{x}{\pi}$.

Define $v = u - g$. Then:

$$\begin{cases} v_t = v_{xx} \quad (\text{as } g_{xx} = g_t = 0) \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = 1 - \frac{x}{\pi} \end{cases}$$

We know our solution takes the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2}{L}t\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx)\exp(-n^2\pi t)$$

where

initial condition

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \left(1-\frac{x}{\pi}\right)\sin(nx)\,dx$$

Then to get our actual solution, we just solve for u:

$$u = v + g = \frac{x}{\pi} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\left(\int_0^\pi \left(1-\frac{x}{\pi}\right)\sin(nx)\,dx\right)\sin(nx)\exp(-n^2\pi t)\right)$$

To show uniqueness, assume there is another solution μ. Let $w = u - \mu$.

Then

$$w_t = w_{xx}$$

$$\Rightarrow \quad w w_t = w w_{xx}$$

$$\Rightarrow \quad \int_0^\pi w w_t = \int_0^\pi w w_{xx}$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^\pi \frac{w^2}{2}\, dx = w w_x \Big|_0^\pi - \int_0^\pi w_x^2$$

$$= -\int_0^\pi w_x^2 \quad \text{as } w(0,t) = w(\pi,t) = 0$$

$$\leq 0$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^\pi \frac{w^2}{2}\, dx$$

and by the previously used ICs & $w(x,0) = 0$, we have $w \equiv 0$, and thus $\mu = u$, showing uniqueness.

The solution takes the data in the classical, or L^2, sense.