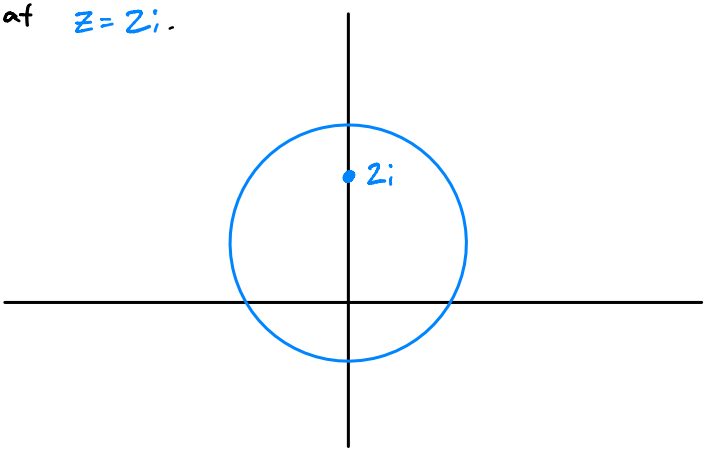
Problem C-4A. In $\mathbb{C}$, let C be the circle of radius 2 centered at $z = i$. Prove that

$$\int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}.$$

The zeros of $(z^2+4)^2$ are $\pm 2i$, with multiplicity 2 each.

Thus we are dealing with one pole of order 2 inside our contour, at $z = 2i$.



Let's find the residue. Calling the integrand f now.

$$\operatorname{Res}(f, 2i) = \lim_{z \to 2i} \left((z-2i)^2 f(z) \right)'$$

$$= \lim_{z \to 2i} \left(\frac{(z-2i)^2}{(z+2i)^2 (z-2i)^2} \right)'$$

$$= \lim_{z \to 2i} \left(\frac{1}{(z+2i)^2} \right)'$$

$$= -\lim_{z \to 2i} \frac{2}{(z+2i)^3}$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{64i^3} = \frac{1}{32i}$$

Thus by the residue theorem,

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}(f, 2i)$$

$$= 2\pi i \cdot \left(\frac{1}{32i}\right)$$

$$= \frac{\pi}{16}$$