Problem R-3A. Let $\{f_n\}$ be a sequence of nonnegative. integrable functions on $[0,1]$, and assume that

$$\lim_n \int f_n = 0.$$

For each $n \geq 1$, let

$$A_n := \{x \in [0,1] : f_n(x) \geq 1\},$$

and let $a_n := m(A_n)$, the Lebesgue measure of the measurable set A_n. Prove that $a_n \to 0$.

Chebyshev's inequality gives

$$m(A_n) \leq \int_0^1 f_n$$

So

$$\varlimsup_{n\to\infty} m(A_n) \leq \varlimsup_{n\to\infty} \int_0^1 f_n$$

$$= 0$$

$$\Rightarrow a_n \to 0$$