Problem 2.1 (Miess 3.9). Suppose where $g, k\colon [0,a] \to \mathbb{R}$ are continuous, $a > 0$, $k(t) \geq 0$ and that $g(t)$ obeys the inequality

$$g(t) \leq c(t) + \int_0^t k(s)g(s)\,ds$$

for all $0 \leq t \leq a$. Suppose additionally that $\dot{c} \geq 0$. Prove that

$$g(t) \leq c(t)e^{\int_0^t k(s)\,ds}$$

Let $G(t) = c(t) + \int_0^t k(s)g(s)\,ds$. Then

$$\begin{aligned} G'(t) &= c'(t) + k(t)g(t) \\ &\leq c'(t) + k(t)\left(c(t) + \int_0^t k(s)g(s)ds\right) \\ &= c'(t) + k(t)G(t) \end{aligned}$$

So we have an ODE $G'(t) - k(t)G(t) \leq c'(t)$, which we can solve with an integrating factor $e^{-\int_0^t k(s)ds}$:

$$G'(t)e^{-\int_0^t k(s)ds} - G(t)k(t)e^{-\int_0^t k(s)ds} \leq c'(t) \quad (\dot{c} \geq 0)$$

$$\Rightarrow \left(G(t)e^{-\int_0^t k(s)ds}\right)' \leq c'(t)$$

$$\Rightarrow G(t)e^{-\int_0^t k(s)ds} \leq \int_0^t c'(s)\,ds$$

$$\Rightarrow G(t) \leq c(t)e^{\int_0^t k(s)ds}$$

$$\Rightarrow g(t) \leq c(t)e^{\int_0^t k(s)ds} \quad (g \leq G)$$