109. **(Harvard University Qual 2020)**

(i) Specify the range of $1 \le p < \infty$ for which

$$\varphi(f) = \int_0^1 \frac{f(t)}{\sqrt{t}} dt$$

defines a linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$.

(ii) For those values of p, calculate the norm of the linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$. The norm of the linear functional is defined as

$$||\varphi|| = \sup_{f \in L^p([0,1]) \setminus \{0\}} \frac{|\varphi(f)|}{||f||_p}.$$

i) We can identify the dual space of L^p with L^q, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus we can instead check when $\frac{1}{\sqrt{t}} \in L^q([0,1])$. Since $\int_0^1 \frac{1}{t^q} dt$ converges when $q < 1$, we need $q/2 < 1$ so $q \in (1,2)$ works. Then our desired range for p is $(2, \infty)$.

ii) We can again work with $q \in (1,2)$ instead of $p \in (2,\infty)$ and calculate

$$\left\| \frac{1}{\sqrt{t}} \right\|_{L^q([0,1])} = \left(\int_0^1 \left| \frac{1}{\sqrt{t}} \right|^q dt \right)^{1/q}$$

$$= \left(\int_0^1 t^{-q/2} dt \right)^{1/q} \quad \text{(note } \sqrt{t} \ge 0\text{)}$$

$$= \left(\frac{1}{1 - q/2} \right)^{1/q}$$

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