Problem R-5A. Recall that a subset E of $\mathbb{R}$ is said to be *measurable* if whenever A is an arbitrary subset of $\mathbb{R}$, one has

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C),$$

where m^* denotes Lebesgue outer measure and ^{C} denotes set complement.

Prove that a subset E of $\mathbb{R}$ is measurable if and only if for every $\epsilon > 0$ there exists an open set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ and $m^*(G \setminus E) < \epsilon$.

$(\Rightarrow)$ Assume E is measurable. Let $\varepsilon > 0$.

First assume $m(E) < \infty$. Then we can find a countable collection of open intervals $\{I_k\}$ so that $E \subseteq \bigcup I_k$ and

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon$$

Let $G = \bigcup I_k$. Then G is open & $E \subseteq G$ so

$$m^*(G) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon$$

$$\Rightarrow m^*(G) - m^*(E) < \varepsilon$$

and by the excision property of outer measure,

$$m^*(G \setminus E) = m^*(G) - m^*(E) < \varepsilon$$

as desired.

If $m^*(E) = \infty$, then we can express E as the disjoint union of a countable collection of measurable sets $\{E_k\}$, so that $m^*(E_k) < \infty$ for each E_k. From before, we can find an open set G_k for each E_k such that

$$m^*(\ _k) - m^*(E_k) < \varepsilon/2^k$$

Let $G = \bigcup G_k$. Then $E \subseteq G$ and

$$\setminus E = \left(\bigcup G_k\right) \setminus E$$

$$\subseteq \bigcup (G_k \setminus E_k)$$

$$\Rightarrow \quad m^*(G \setminus E) \leq \sum_{k=1}^{\infty} m^*(G_k \setminus E_k)$$

$$< \sum_{k=1}^{\infty} \varepsilon/2^k$$

$$= \varepsilon$$

($\Leftarrow$) Assume that for any $\varepsilon > 0$, there is an open set G such that $E \subseteq G$ and $m^*(G \setminus E) < \varepsilon$.

For each $k \in \mathbb{N}$, pick G_k such that $E \subseteq G_k$ and $m^*(G_k \setminus E) < 1/k$. Let $G = \bigcap G_k$. Then G is open and $E \subseteq G_k$.

As $G \setminus E \subseteq G_k \setminus E$ for any k, by monotonicity

$$m^*(G \setminus E) \leq m^*(G_k \setminus E)$$
$$< 1/k$$

$$\Longrightarrow m^*(G \setminus E) = 0$$

Since G is measurable & sets of measure zero are measurable,

$$E = G \cap (G \setminus E)^c$$

is an intersection of measurable sets & is measurable.