2. Solve $u_t = u_{xx}$ in $[0,\pi] \times [0,\infty)$ with $u(0,t) = 0$ and $u(\pi,t) = 0$ for all t and $u(x,0) = 1$ for $x \in (0,\pi)$. In what sense the solution takes the initial data and prove it is unique.

Our solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L} t\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx) \exp(-n^2\pi t)$$

where

initial condition

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx) f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\cos(nx)\right)\Big|_0^{\pi} = -\frac{2}{\pi n}\left(\cos(\pi n) - 1\right)$$

$$= -\frac{2}{\pi n}\left((-1)^n - 1\right)$$

We need to show uniqueness. Let v be another solution, & let $w = u - v$. Then $w_t = w_{xx}$. Multiply both sides by w and integrate over $(0,\pi)$:

$$w_t = w_{xx}$$

$$\int_0^\pi w w_t \, dx = \int_0^\pi w w_{xx} \, dx$$

$$\int_0^\pi \left(\frac{w^2}{2}\right)_t dx = w w_x \Big|_0^\pi - \int_0^\pi w_x^2 \, dx$$

$$= - \int_0^\pi w_x^2 \, dx$$

$$\leq 0$$

$$\Rightarrow \frac{d}{dt} \int_0^\pi \frac{w^2}{2} dx \leq 0$$

and we must have equality because the integrand is positive. It follows by the initial conditions $w(x,0) = 0$ and $w(0,t) = w(\pi,t) = 0$ that $w \equiv 0$; thus $u = v$ & we are done.

This solution takes the data in the classical, or L^2 sense; if u_n is the n^{th} partial sum then for any $t > 0$:

$$\lim_{n \to \infty} \| u_n - u \|_{L^1([0,\pi])} = \lim_{n \to \infty} \int_0^\pi |u_n - u| \, dx = 0$$