3. Solve $u_t = u_{xx}$ in $[0, \pi] \times [0, \infty)$ with $u(0,t) = 0$ and $u(\pi, t) = 1$ for all t and $u(x, 0) = 1$ for $x \in (0, \pi)$. In what sense the solution takes the initial data and prove it is unique.

We want to homogenize the boundary conditions first; let $g(x,t) = \frac{x}{\pi}$.

Define $v = u - g$. Then:

$$\begin{cases} v_t = v_{xx} \quad (\text{as } g_{xx} = g_t = 0) \\ v(0,t) = v(\pi, t) = 0 \\ v(x,0) = 1 - \frac{x}{\pi} \end{cases}$$

We know our solution takes the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{n^2\pi^2}{L} t\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx) \exp(n^2 \pi t)$$

where

initial condition

$$a_n = \frac{2}{L} \int_0^L f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \sin(nx)\, dx$$

Then to get our actual solution, we just solve for u:

$$u = v + g = \frac{x}{\pi} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \left(\int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \sin(nx)\, dx \right) \sin(nx) \exp(n^2 \pi t) \right)$$

To show uniqueness, assume there is another solution μ. Let $W = u - \mu$.

Then

$$w_t = w_{xx}$$

$$\Rightarrow \quad ww_t = ww_{xx}$$

$$\Rightarrow \quad \int_0^\pi ww_t = \int_0^\pi ww_{xx}$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi \frac{w^2}{2}\,dx = ww_x\Big|_0^\pi - \int_0^\pi w_x^2$$

$$= -\int_0^\pi w_x^2 \quad \text{as } w(0,t) = w(\pi,t) = 0$$

$$\le 0$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi \frac{w^2}{2}\,dx$$

and by the previously used ICs & $w(x,0) = 0$, we have $W \equiv 0$, and thus $\mu = u$, showing uniqueness.

The solution takes the data in the classical, or L^2, sense.