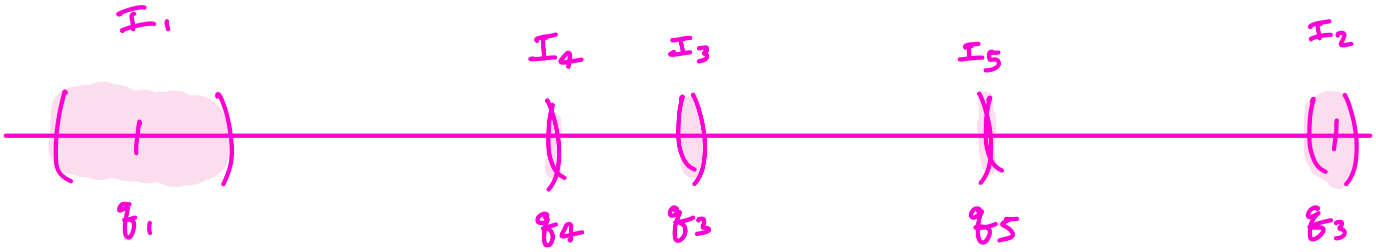
23. **(Midterm 1 - Chifan)** Is there an uncountable closed set E with $\mathbb{Q} \subset E^c$?

Enumerate $\mathbb{Q}$ as $\{q_1, q_2, \dots\}$. Let $I_k = \left(q_k - \frac{1}{2^k}, q_k + \frac{1}{2^k}\right)$.



Then $F = \bigcup_{k=1}^{\infty} I_k$ contains $\mathbb{Q}$ and

$$m(F) \leq \sum_{k=1}^{\infty} m(I_k)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - 1/2} = 2$$

Then $E = F^c$ is a set of infinite measure and is thus uncountable, and by construction contains no rational numbers. //