16. **(aTm-F11-04)** Suppose f is a continuous function on $\{z \in \mathbb{C} : |z| \leq 1\}$ and f is holomorphic on the open unit disc. Prove that if $f(z)$ is real when $|z| = 1$, then f is a constant function.

If f is constant there is nothing to show, so assume f is nonconstant.

Write $f = u(x,y) + iv(x,y)$ (restricting to the disk). Then u and v are harmonic, and $v \equiv 0$ on the boundary of the disk. Then $v \equiv 0$ on the entire disk as well, as it must achieve its minimum and maximum on the boundary. So $f = u$ is real-valued.

Now if f is nonconstant, the open mapping theorem applies and f maps open sets to open sets. But f maps the open unit disk to a subset $E \subseteq \mathbb{R}$, which is not open in the usual topology on $\mathbb{C}$. $\lightning$ Thus we conclude $f \equiv \lambda$ for some $\lambda \in \mathbb{R}$.