Problem: Let X, Y, Z be path connected and locally path connected. Assume the following is a commutative diagram of continuous maps:

$$\begin{tikzcd} Y \arrow[rr, "t"] \arrow[dr, "r"'] & & Z \arrow[dl, "s"] \\ & X & \end{tikzcd}$$

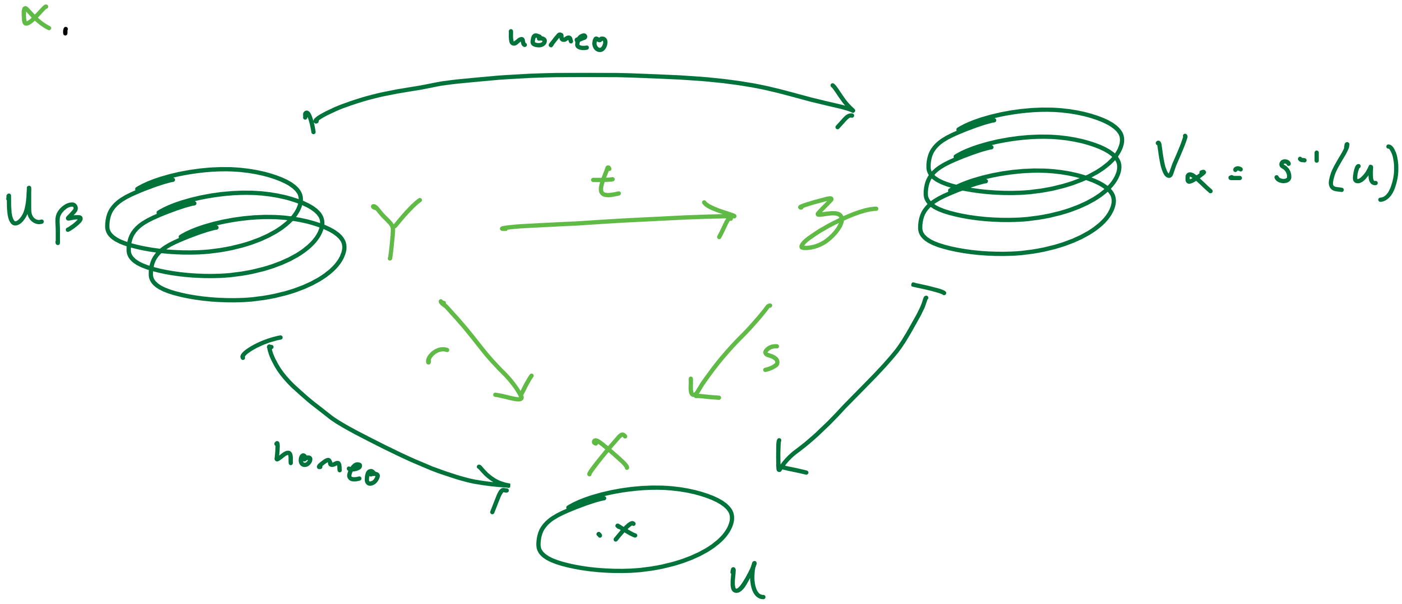
Show if either r & s, or r & t, are covering maps, then the third is also.

<u>Solution</u>

Assume r and t are covering maps. WTS s is a covering map.

Let $x \in X$. Let $U \subseteq X$ be an open neighborhood of x evenly covered by r, and let $\{V_\alpha\} \subseteq Z$ be the connected components of $s^{-1}(U)$.

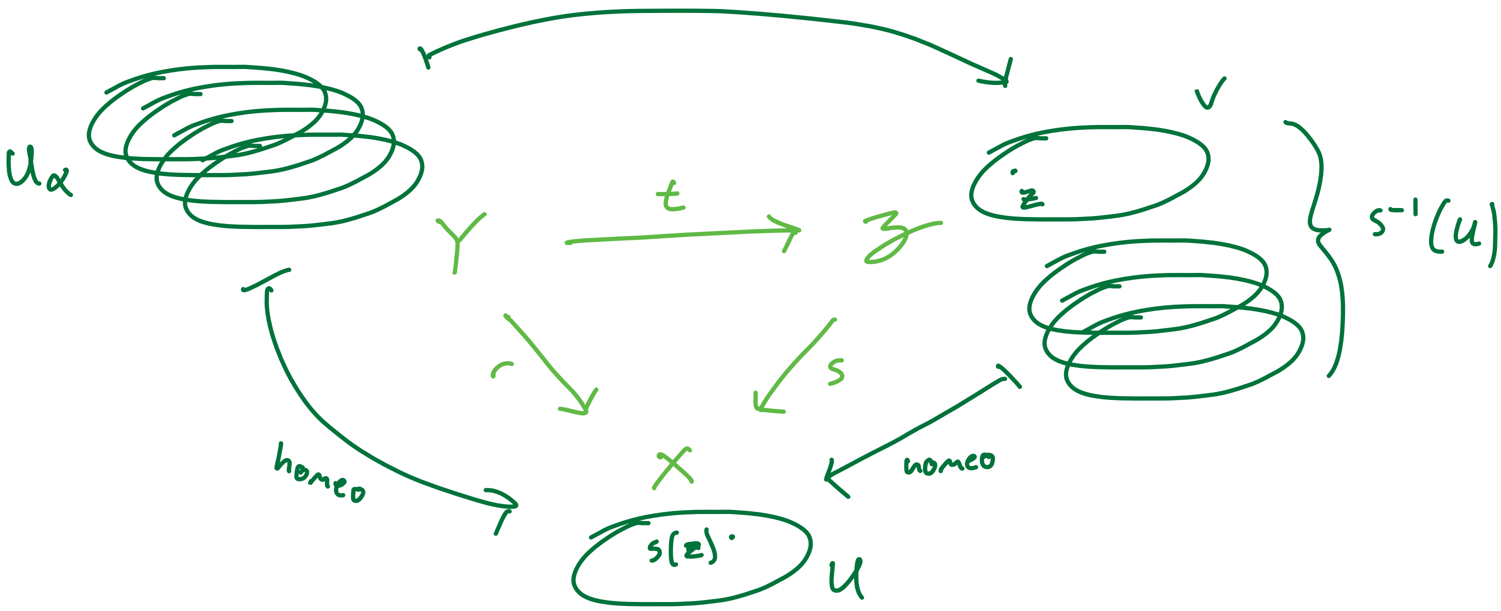
Let $\{U_\beta\} \subseteq Y$ be the slices of $r^{-1}(U)$. Since the diagram commutes and t is continuous, t maps each U_β onto V_α for some α.



The restriction of t to $\{U_\beta\}$ is a covering map, so each slice is mapped homeomorphically onto some V_α. As the diagram commutes, $s|_{V_\alpha} = (r \circ t^{-1})(V_\alpha)$ is also a homeomorphism, so s is a covering map.

Now assume r & s are covering maps. WTS t is a covering map.

Let $z \in Z$. Then $\exists$ some open neighborhood $U \subseteq X$ about $s(z)$ that is evenly covered by both r and s (take an intersection). Let $V \subseteq Z$ be a slice of $s^{-1}(U)$ such that $z \in V$. Let $\{U_\alpha\} = r^{-1}(U)$.



Since the diagram commutes, t maps each U_α into some slice of $s^{-1}(U)$, and in fact a single slice each as U_α is connected & t is continuous. In particular $t^{-1}(V)$ is some disjoint union of U_α slices. They are mapped homeomorphically onto V as r and s are homeomorphisms on these slices & $t = r \circ s^{-1}$. Thus t is a covering map and we are done.