17. (**Dr. Ayati: S13 Midterm** – ***Victoria and Fatemeh***) Given $x \in \mathbb{R}^n$, prove the equivalence relation for the vector norm

(a) $\|x\|_1 \le \|x\|_2 \le \sqrt{n}\|x\|_1$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$ This is a repetition. Problem 2 of 5800 Module 1.

(c) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

a) This is not true. For example, if $x = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, then

$$\|x\|_1 = 2 + 3 + 6 = 11$$ and

$$\|x\|_2 = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

So $\|x\|_1 > \|x\|_2$ which contradicts the left inequality.

However, we can prove $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$.

LHS: We have

$$\|x\|_1^2 = \left(\sum_i |x_i|\right)^2$$

$$= \sum_i |x_i|^2 + \sum_{\substack{i,j \\ i<j}} 2|x_i||x_j|$$

$$\ge \sum_i |x_i|^2$$

$$= \|x\|_2^2$$

So $\|x\|_2 \le \|x\|_1$ after taking the square root of both sides.

RHS: We have

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$= \sum_{i=1}^{n} |x_i| \cdot 1$$

$$\leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2}$$

$$= \sqrt{n}\, \|x\|_2$$

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b) LHS: $\|x\|_\infty = \max_i |x_i| = |x_k| \quad k < n$

$$= \sqrt{|x_k|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$= \|x\|_2$$

RHS: $\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$

$$\leq \sum_{i=1}^{n} |x_k|^2 \quad \text{where } |x_k| = \max_i |x_i|$$

$$= n \|x\|_\infty \Rightarrow \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty$$

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c) LHS: $\|x\|_\infty = \max_i |x_i| = |x_k| \quad k < n$

$$\leq \sum_i |x_i|$$

$$= \|x\|_1$$

RHS: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$$\leq \sum_{i=1}^{n} |x_k|$$ where $|x_k| = \max_i |x_i|$

$$= n\|x\|_\infty$$