4. **Fall2008.** Let V be a finitely generated F-vector space. Show that V has a basis and that two bases for V have the same number of elements.

Let $\{v_1, \dots, v_n\}$ generate V. If all the v_i are LI then we are done. If not, then we have

$$\sum_{i=1}^{n} c_i v_i = 0$$

where at least one c_i term is nonzero. Assume WLOG that $c_n \neq 0$. Then

$$v_n = -\sum_{i=1}^{n-1} \frac{c_i v_i}{c_n}$$

and we have a smaller spanning set $\{v_1, \dots, v_{n-1}\}$. Continue like this until we find a LI set and thus a basis. Note this will always happen because $\{v_1\}$ is LI.

Claim. For any LI set $\mathcal{L}$ and spanning set $\mathcal{S}$, $|\mathcal{L}| \leq |\mathcal{S}|$.

Proof. Let $|\mathcal{L}| = n$, $|\mathcal{S}| = m$. Assume $m < n$ for contradiction.

Since $\mathcal{S}$ spans, for $u_i \in \mathcal{L}$ we can write write $u_i = \sum_{j=1}^{m} a_{ij} v_j$.

Let A be the matrix with entries given by a_{ij}.

As $m < n$ (A is tall), the equation $A^T x = 0$ has a solution $b \neq 0$ (A^T is wide and has nontrivial nullspace). Then

$$\sum_{i=1}^{n} b_i u_i = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i a_{ij} v_j = \sum_{j=1}^{m} (A^T b)_j v_j = 0 \quad \lightning$$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases of V. As each is LI and spans, we can apply the previous claim to see $|\mathcal{B}_1| \leq |\mathcal{B}_2|$ and $|\mathcal{B}_2| \leq |\mathcal{B}_1|$, and thus $|\mathcal{B}_1| = |\mathcal{B}_2|$ //