Problem: Let $\mathbb{R}^\infty = \bigoplus_{n\geq 0} \mathbb{R}$, & equip it with the inner product $\langle x, y \rangle = \sum_{i\geq 0} x_i y_i$ & the corresponding metric topology. Show the unit sphere $S^\infty = \{ x \in \mathbb{R}^\infty \mid \|x\| = 1 \}$ is contractible.

Solution

We need a homotopy from the identity map of S^∞ to a constant map.

Let $\mathcal{S} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by $(x_1, x_2, \cdots) \mapsto (0, x_1, x_2, \cdots)$, "shifting" all coordinates. Consider the map $f_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$(x_1, x_2, \cdots) \mapsto (1-t)(x_1, x_2, \cdots) + t(0, x_1, x_2, \cdots) \quad (0 \leq t \leq 1).$$

Since f_t maps nonzero points to nonzero points, the map $x \mapsto \dfrac{f_t(x)}{|f_t(x)|}$ is well-defined on S^∞ and gives a homotopy from $id(S^\infty)$ to $\mathcal{S}|_{S^\infty}$.

Now let $g_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by

$$(x_1, x_2, \cdots) \mapsto (1-t)\mathcal{S}(x_1, x_2, \cdots) + t(1, 0, 0, \cdots) \quad (0 \leq t \leq 1).$$

As before, g_t maps nonzero points to nonzero points, so the map $x \mapsto \dfrac{g_t(x)}{|g_t(x)|}$ is well-defined on S^∞ and gives a homotopy from $\mathcal{S}|_{S^\infty}$ to a constant map.

The composition of these homotopies is a homotopy from the identity of S^∞ to a constant map, which shows S^∞ is contractible.