Problem 7.2 (HW11-2019). Consider the nonlinear system

$$\begin{aligned} \dot{x} &= -y + xz^2 \\ \dot{y} &= x + yz^2 \\ \dot{z} &= -z(x^2 + y^2) \end{aligned}$$

(a) Show that the nonlinear system has a periodic solution $\gamma(t) = (\cos t, \sin t, 0)$.

(b) Find the linearization of the system about $\gamma(t)$, which is an autonomous system.

(c) Find the fundamental matrix $\Phi(t)$ of the linearization with $\Phi(0) = I$.

(d) Find the characteristic exponents and characteristic multipliers of $\gamma(t)$.

(e) What can you say about the stability of $\gamma(t)$?

a) $\gamma(t)$ is obviously periodic with period 2π. We need to show it solves the ODE.

$$x(t) = \cos t \qquad y(t) = \sin t \qquad z(t) = 0$$

$$\dot{x}(t) = -\sin(t) = -y + xz^2 \checkmark$$

$$\dot{y}(t) = \cos(t) = x + yz^2 \checkmark$$

$$\dot{z}(t) = 0 = -z(x^2+y^2) \checkmark$$

b)

$$J(x,y,z) = \begin{bmatrix} z^2 & -1 & 2xz \\ 1 & z^2 & 2yz \\ -2xz & -2yz & -(x^2+y^2) \end{bmatrix}$$

$$J(\gamma(t)) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So the linearized system is $\dot{X} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} X$

c) Since the matrix in the linearized system is in canonical form, its eigenvalues are $\pm i$ & -1, so its solution is given by

$$x(t) = c_1 \begin{bmatrix} \cos(-t) \\ -\sin(-t) \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \sin(-t) \\ \cos(-t) \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then our fundamental solution is

$$\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

and indeed,

$$\Phi(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

d) We construct the monodromy matrix $M = \Phi(T + t_0)$ where $T = 2\pi$ & $t_0 = 0$:

$$\Phi(2\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi} \end{bmatrix} \Rightarrow \text{Floquet multipliers are } 1, e^{-2\pi}$$

characteristic exponents are

$$e^{\rho_1 2\pi} = 1 \Rightarrow \rho_1 = 0$$

$$e^{\rho_2 2\pi} = e^{-2\pi} \Rightarrow \rho_2 = -1$$

e) Since the characteristic exponents have nonpositive real part, $\gamma(t)$ is stable.