109. **(Practice Midterm 2 - Chifan)** Let $f_n \in H(\Omega)$ where Ω is a region. Assume f_n has no zeros in Ω and f_n converges uniformly to f on compact subsets of Ω. Prove that either f has no zeros or $f(z) = 0$ for all $z \in \Omega$.

Assume $f \not\equiv 0$. As f is holomorphic, its zeros are isolated. So let γ be a closed contour about a zero ζ of f such that no other zeros of f lie on $\partial\gamma$. By the argument principle,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = 1$$ (f is holomorphic and has no poles)

and

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)} = 0$$ (f_n has no zeros or poles)

Because we have uniform convergence, $\lim_{n\to\infty}\int_\gamma \frac{f_n'(z)}{f_n(z)} = \int_\gamma \frac{f'(z)}{f(z)}$

and we have a contradiction. Thus either $f \equiv 0$ (to avoid a $\lightning$ in the first place) or f has no zeros (to avoid a $\lightning$ in the second place), as desired.