3. **Spring2006.** Prove that a commutative ring with identity is a field if, and only if, it is simple.

R

only ideals of R are (0) and R

($\Rightarrow$) Assume R is a field. Let $I \subseteq R$ be a nonzero ideal. Then as all $a \in I$ are invertible, $\exists\, b \in R$ st $ab = 1$, so:

$$\text{for } r \in R, \quad \begin{aligned} r &= r \cdot 1 \\ &= r(ab) \\ &= (ra)b \end{aligned}$$

$\Rightarrow r \in I$ as $ra \in I$

$\Rightarrow I = R$ as r is arbitrary

$\therefore$ R is simple

($\Leftarrow$) Assume R is simple. Let $a \in R$ be nonzero. Then the ideal generated by a, $(a) = \{ar \mid r \in R\}$, is all of R. Then

$1 \in (a)$ (R has identity)

$\Rightarrow$ for some $r \in R$, $1 = ar$ (definition of (a))
$= ra$ (commutativity of R)

$\Rightarrow$ r is a multiplicative inverse for a

Since a was arbitrary we can conclude R is a field.

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