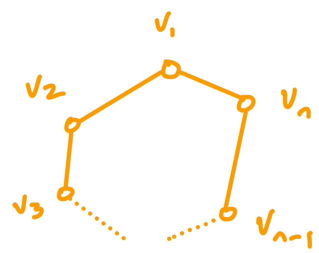
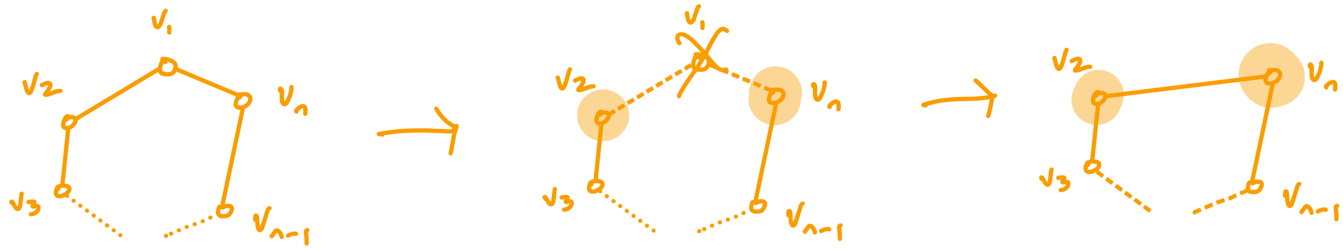
13. (***George and Joe***) Suppose we add a single symmetric entry to the top right and the bottom left corners to A. What is the graph of the resulting matrix? How much fill-in will be generated by Cholesky factorization? Assume that the resulting matrix is positive definite so that the Cholesky factorization exists.

In the original problem, $A_{ij} = \begin{cases} 1 & |i-j| \leq 1 \\ 0 & |i-j| > 1 \end{cases}$ is a "tridiagonal" matrix. Note this corresponds to the graph $G=(V,E)$ where $V=\{v_1, \dots, v_n\}$ and $v_{ij} \in E \iff |i-j|=1$, which is the graph P_n, the path graph on n vertices:

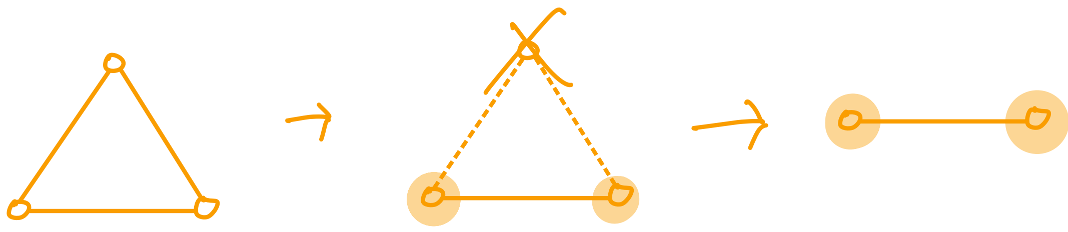
Adding an entry to the top right and bottom corners of A adds an edge between v_1 and v_n, resulting in the cycle graph C_n:



Note that performing our "delete and connect neighbors" operation on any vertex in C_n leaves the graph C_{n-1}:



Thus any ordering gives the same amount of fill-in; 1 new edge is created at each stage until we get to $n=3$, at which point the graph is complete (fully connected)



Thus the total process creates $n-3$ new edges, meaning $2(n-3)$ is our total amount of fill-in.