Question 2. Consider the system

$$\dot{x} = y - ax$$
$$\dot{y} = -ay + \frac{x}{1+x},$$

where a is a positive parameter. Answer the following questions.

(a) For each qualitatively different value of $a > 0$, find all equilibrium points. When the Hartman-Grobman theorem applies, classify each equilibrium point.

(b) Describe the bifurcation that occurs as a varies and find the critical value of a (call it a^*) at which the bifurcation occurs. (You do not need to compute the center manifold at $a = a^*$.)

(c) Sketch the phase plane (phase portrait) for $a > a^*$ which qualitatively describes the full dynamics of the system. *Hint: You should indicate the equilibrium points, a heteroclitic orbit, the stable and unstable curves, and six trajectories.*

a) To find equilibrium points, we set each component to zero:

$$\dot{x} = y - ax = 0 \qquad \dot{y} = -ay + \frac{x}{1+x} = 0$$

$$\Rightarrow y = ax \qquad \Rightarrow -a^2x + \frac{x}{1+x} = 0$$

$$\Rightarrow x\left(-a^2 + \frac{1}{1+x}\right) = 0$$

$$\frac{x}{1+x} \qquad \frac{(1+x) - x}{(1+x)^2} \qquad \Rightarrow x = 0 \text{ or } a^2 = \frac{1}{1+x}$$

$$\Rightarrow a^2 + a^2x = 1$$

$$\frac{-3}{4}, \frac{-6}{4} \qquad \Rightarrow x = \frac{1-a^2}{a^2}$$

Thus our equilibrium points are $(0,0)$ and $\left(\frac{1-a^2}{a^2}, \frac{a-a^3}{a^2}\right)$.

To classify these, we plug them into

$$DF(x,y) = \begin{bmatrix} -a & 1 \\ \frac{1}{(1+x)^2} & -a \end{bmatrix}$$

So: $DF_{(0,0)} = \begin{bmatrix} -a & 1 \\ 1 & -a \end{bmatrix}$

eigenvalues: $(-a-\lambda)(-a-\lambda) - 1 = 0$

$$\Rightarrow \lambda^2 + 2\lambda a + a^2 - 1 = 0$$

$$\Rightarrow (\lambda + a + 1)(\lambda + a - 1) = 0$$

Since $a>0$, we must have $\lambda_1 = -a+1$, $\lambda_2 = -a-1$.

If $0<a<1$, then $\lambda_2 < 0 < \lambda_1$ & we have a <u>saddle</u> equilibrium.

If $a=1$ then $\lambda_1 = 0$ & H-G does not apply.

If $a>1$ then $\lambda_2 < \lambda_1 < 0$ & we have a <u>sink</u>.

Now

$$DF_{\left(\frac{1-a^2}{a^2}, \frac{a-a^3}{a^2}\right)} = \begin{bmatrix} -a & 1 \\ \dfrac{1}{\left(1+\frac{1-a^2}{a^2}\right)^2} & -a \end{bmatrix}$$

$$= \begin{bmatrix} -a & 1 \\ a^4 & -a \end{bmatrix}$$

$-ax + y = -ax + a^2 x$

$y = a^2 x$

eigenvalues: $(-a-\lambda)(-a-\lambda) - a^4 = 0$

$$\Rightarrow \lambda^2 + 2a\lambda + a^2 - a^4 = 0$$

$$\Rightarrow \lambda^2 + 2a\lambda + a^2 = a^4$$

$$\Rightarrow (\lambda + a)^2 = a^4$$

$$\Rightarrow \lambda_1, \lambda_2 = -a \pm a^2$$

If $0 < a < 1$, then $a^2 < a$ & $\lambda_2 < \lambda_1 < 0$, so we have a sink.

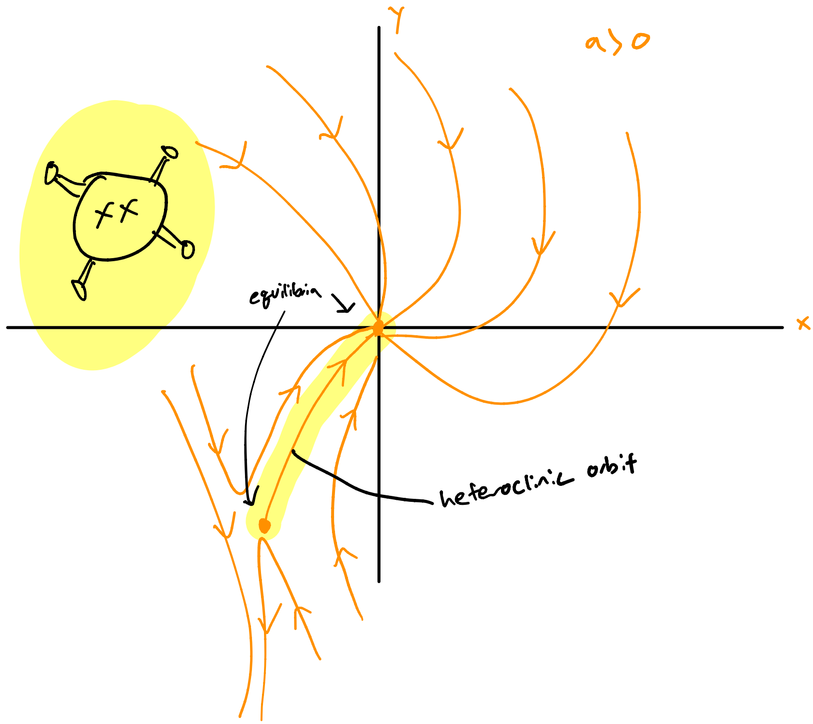
If $a = 1$, then $\lambda_1 = 0$ and H-G does not apply.

If $a > 1$, then $a^2 > a$ & $\lambda_2 < 0 < \lambda_1$, so we have a saddle.

b) Note the stability of both equilibria change at $a=1$, so our critical value of a is $a^* = 1$.

The point $(0,0)$ switches from a saddle (unstable) to a sink (stable).
The point $\left(\frac{1-a^2}{a^2}, \frac{a-a^3}{a^2}\right)$ does the opposite.

c)



stable subspace of $(0,0)$ is xy plane

unstable subspace of $\left(\frac{1-a^2}{a^2}, \frac{a-a^3}{a^2}\right)$ is $\text{span}\left\{\begin{bmatrix} 1 \\ a^2 \end{bmatrix}\right\}$

stable subspace of $\left(\frac{1-a^2}{a^2}, \frac{a-a^3}{a+2}\right)$

is $\text{span}\left\{\begin{bmatrix} 1 \\ -a^2 \end{bmatrix}\right\}$

$$\begin{bmatrix} -a & 1 \\ a^4 & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(-a+a^2) \\ y(-a+a^2) \end{bmatrix}$$

$$\Rightarrow -ax + y = -ax + a^2 x$$

$$\Rightarrow y = a^2 x$$