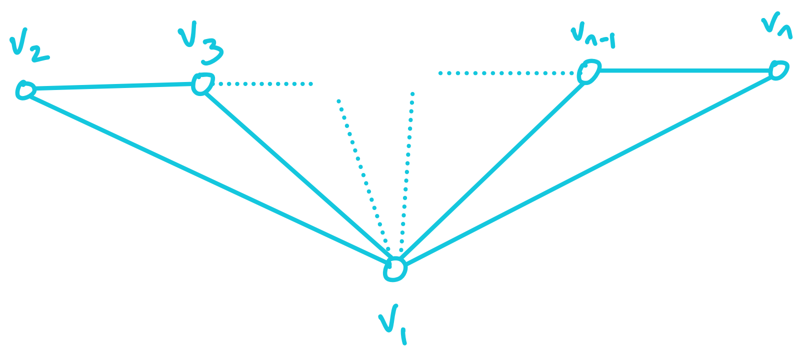
14. (***Nandita and Paria***) The graph of an arrowhead matrix is a spokes graph. If we have a symmetric matrix that is the sum of a tridiagonal matrix and a matrix $\mathbf{u}\mathbf{e}_1^T + \mathbf{e}_1\mathbf{u}^T$ that fills in the first row and column, show that the corresponding graph is a wheel graph: a cycle of $n-1$ vertices together with a "hub vertex" that is connected to every node in the cycle. Show that there is an ordering that gives no fill-in for this graph.

Tridiagonal :
$$\begin{bmatrix} * & * & & & \\ * & * & * & * & \\ & * & * & * & \\ & & * & * & * \\ & & & * & \ddots \end{bmatrix} \rightarrow A_{ij} = \begin{cases} * & |i-j| \leq 1 \\ 0 & |i-j| < 1 \end{cases}$$

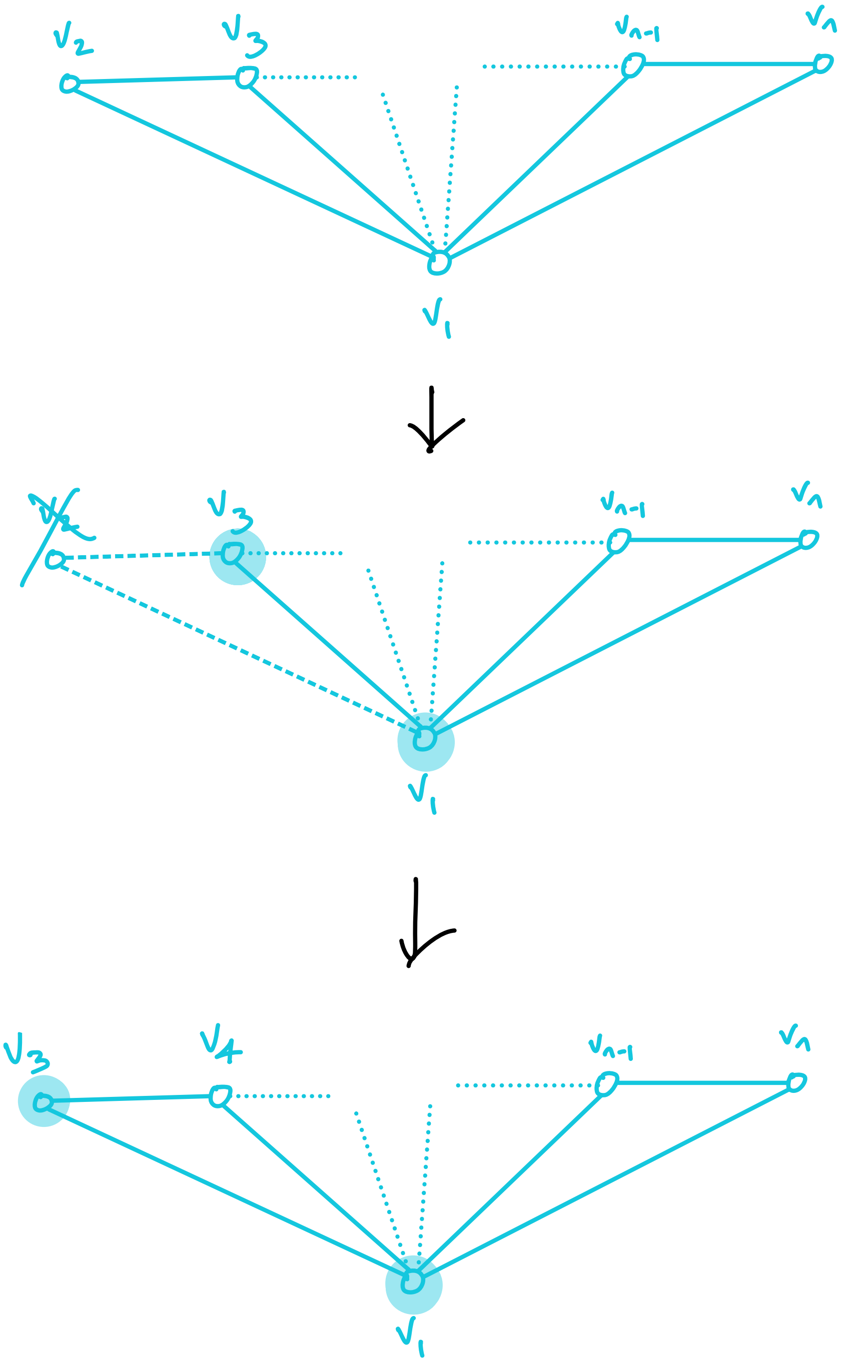
This matrix corresponds to the path graph P_n :

By definition, filling up the first row and column of A corresponds to adding an edge between v_1 and every other vertex :



(Note this is <u>not</u> the wheel graph W_n, as the $v_2 v_n$ edge is missing)

Consider the ends of the path $v_2 \cdots v_n$. They have two neighbors: v_1 and the next/previous vertex in the path. Since v_1 is adjacent to every vertex already, deleting either of them gives no fill-in:



Note the resulting graph is the path $v_3 \cdots v_n$ with v_1 adjacent to all v_i in the path, which is what we started with (only less one vertex). Therefore continuing to delete the end of the path will give no fill-in.