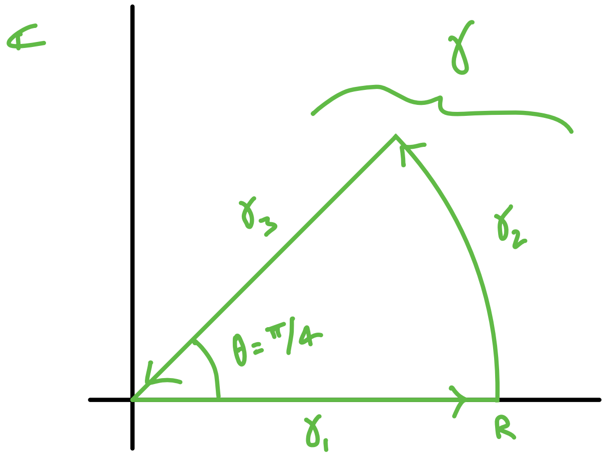
57. **(Homework 3 - Chifan)** Prove that

$$\int_0^\infty \sin\left(x^2\right)dx = \int_0^\infty \cos\left(x^2\right)dx = \frac{\sqrt{2\pi}}{4}$$

we will work with the function $f(z) = e^{-z^2}$ & consider $\int_\gamma f(z)\,dz$.
where $\gamma = \gamma_3 \circ \gamma_2 \circ \gamma_1$, where $\gamma_1(t) = t \quad 0 \le t < R$

$\gamma_2(t) = Re^{i\frac{\pi}{4}t} \quad 0 \le t < 1$

$\gamma_3(t) = e^{i\frac{\pi}{4}}(R-t) \quad 0 \le t < R$



Note $\int_\gamma f(z)\,dz = 0$ b/c analytic closed curve yada yada

Then $$\int_\gamma f(z)\,dz = \underbrace{\int_0^R f(t)\,\gamma_1'(t)\,dt}_{1} + \underbrace{\int_0^1 f(t)\,\gamma_2'(t)\,dt}_{2} + \underbrace{\int_0^R f(t)\,\gamma_3'(t)\,dt}_{3}$$

we will be interested in these as $R \to \infty$.

$$1 = \int_0^R e^{-t^2}\,dt$$

$\vdots$

then $$\lim_{R\to\infty} \int_0^R e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$ (half the Gaussian integral, proof available if you give me $\$5$)

② We have $\left| \int_{\gamma_2} f(t)\,\gamma_2'(t)\,dt \right| = \left| \int_0^1 e^{-(Re^{i\frac{\pi}{4}t})^2} R i \frac{\pi}{4} e^{i\frac{\pi}{4}t}\,dt \right|$

$$\leq \int_0^1 \left| e^{-R^2 e^{i\frac{\pi}{2}t}} \right| \underbrace{\left| Ri\frac{\pi}{4} \right|}_{=\frac{\pi}{4}R} \underbrace{\left| e^{i\frac{\pi}{4}t} \right|}_{=1} dt$$

note $e^{-R^2 e^{i\frac{\pi}{2}t}} = e^{-R^2\left(\cos\frac{\pi}{2}t + i\sin\frac{\pi}{2}t\right)}$

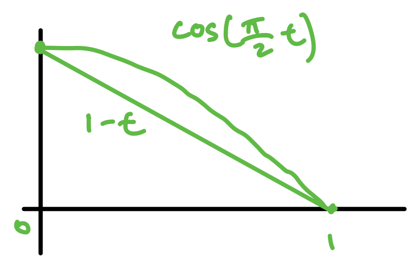
$$= e^{-R^2\cos\frac{\pi}{2}t}\, e^{i\sin\frac{\pi}{2}t}$$

$$\Rightarrow \left| e^{-R^2 e^{i\frac{\pi}{2}t}} \right| = \left| e^{-R^2\cos\frac{\pi}{2}t} \right| \underbrace{\left| e^{i\sin\frac{\pi}{2}t} \right|}_{=1}$$

$$= e^{-R^2\cos\frac{\pi}{2}t}$$

$$\therefore \frac{\pi}{4}R \int_0^1 \left| e^{-R^2 e^{i\frac{\pi}{2}t}} \right| dt \leq \frac{\pi}{4}R \int_0^1 e^{-R^2\cos\frac{\pi}{2}t}\,dt$$

note that $\cos\frac{\pi}{2}t \geq 1-t$:



proof by graph

so $\frac{\pi}{4}R \int_0^1 e^{-R^2\cos\frac{\pi}{2}t}\,dt \leq \frac{\pi}{4}R \int_0^1 e^{-R^2(1-t)}\,dt$

(more negative makes the integral smaller) let $u = 1 - t$, $du = -dt$:

$$\frac{\pi}{4} R \int_0^1 e^{-R^2(1-t)}\, dt = -\frac{\pi}{4} R \int_1^0 e^{-R^2 u}\, du$$

$$= \frac{\pi}{4} R \int_0^1 e^{-R^2 u}\, du$$

$$= \frac{\pi}{4} R \int_0^1 e^{-R^2 u}\, du$$

$$= \frac{\pi}{4} R \left(-\frac{1}{R^2} e^{-R^2 u} \Big|_0^1 \right)$$

$$= \frac{\pi}{4} R \left(-\frac{1}{R^2} e^{-R^2} + \frac{1}{R^2} \right)$$

$$= -\frac{\pi}{4R} e^{-R^2} + \frac{\pi}{4R}$$

$$= -\frac{\pi}{4R} \left(e^{-R^2} - 1 \right)$$

and $\lim\limits_{R \to \infty} -\frac{\pi}{4R} \left(e^{-R^2} - 1 \right)$

$$= \left(\lim_{R \to \infty} -\frac{\pi}{4R} \right) \left(\lim_{R \to \infty} \left(e^{-R^2} - 1 \right) \right)$$

$$= 0 \cdot (1 - 1)$$

$$= 0$$

so ② $= 0$

$$③ \int_0^1 f(t)\gamma_3'(t)\,dt = \int_0^R \left(e^{-e^{i\frac{\pi}{2}}(R-t)^2}\right)\left(-e^{i\frac{\pi}{4}}\right)dt$$

$$= -e^{i\frac{\pi}{4}}\int_0^R e^{-i(R-t)^2}\,dt$$

let $u = R-t,\ du = -dt$

$$= e^{i\frac{\pi}{4}}\int_R^0 e^{-iu^2}\,du$$

$$= -e^{i\frac{\pi}{4}}\int_0^R e^{-iu^2}\,du$$

then $\lim\limits_{R\to\infty} -e^{i\frac{\pi}{4}}\int_0^R e^{-iu^2}\,du = -e^{i\frac{\pi}{4}}\int_0^\infty e^{-iu^2}\,du$

Now ① + ② + ③ = 0 b/c Cauchy integral theorem

$$\hookrightarrow \frac{\sqrt{\pi}}{2} + 0 - e^{i\frac{\pi}{4}}\int_0^\infty e^{-iu^2}\,du = 0$$

$$\Rightarrow \int_0^\infty e^{-iu^2}\,du = \left(\frac{\sqrt{\pi}}{2}\right)\left(e^{-i\frac{\pi}{4}}\right)$$

$$= \left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \quad \text{(do the trig)}$$

$$= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$$

finally:
$$\int_0^\infty e^{-ix^2}\,dx = \int_0^\infty \cos x^2\,dx - i\int_0^\infty \sin x^2\,dx$$

$$= \frac{\sqrt{2\pi}}{4} - i\,\frac{\sqrt{2\pi}}{4}$$

$$\Rightarrow \int_0^\infty \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}$$ and

$$\int_0^\infty \sin x^2\,dx = \frac{\sqrt{2\pi}}{4} \quad //$$