Problem 1.3 (Discussion, unknown date). Consider the system $\dot{x} = Ax$, $x(0) = x_0$ where

$$A = \begin{bmatrix} -2 & -1 & -2 \\ -2 & -2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

We can write A in Jordan form:

$$A = PJP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1 & 0 & -1 \end{bmatrix}$$

(a) What are the eigenvalues of A?

(b) What is the algebraic multiplicity of each eigenvalue?

(c) What is the geometric multiplicity of each eigenvalue?

(d) Write down the fundamental solution $\Phi(t)$ to $\dot{x} = Ax$.

(e) Is the linear system *stable*? If so, say why. If not give an example of a solution that is unbounded.

a) We can get the eigenvalues by reading off the diagonal of J: they are $\lambda_1, \lambda_2 = -2, 0$.

b) We just need to sum the sizes of the corresponding Jordan blocks for each eigenvalue; thus the algebraic multiplicity of λ_1 is 1 & that of λ_2 is 2.

c) Now we count the number of blocks instead, getting geometric multiplicities of 1 for both eigenvalues.

d) The fundamental solution is given by

$$\Phi(t) = P \exp(Jt) P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 1 & t \\ 2e^{-2t} & 0 & 1 \\ -e^{-2t} & -1 & -t-1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-2t} - t + 1/2 & \frac{1}{2}e^{-2t} - 1/2 & \frac{1}{2}e^{-2t} - t - 1/2 \\ e^{-2t} - 1 & e^{-2t} & e^{-2t} - 1 \\ -\frac{1}{2}e^{-2t} + t + 1/2 & -\frac{1}{2}e^{-2t} + 1/2 & -\frac{1}{2}e^{-2t} + t + \frac{3}{2} \end{bmatrix}$$

e) No, this is not stable. An unbounded solution is given by (for example)

$$x(t) = \begin{bmatrix} \frac{1}{2}e^{-2t} - t + 1/2 \\ e^{-2t} - 1 \\ -\frac{1}{2}e^{-2t} + t + 1/2 \end{bmatrix}$$