87. Prove that there is no function f that is analytic on the punctured disc $\mathbb{D} \setminus \{0\}$ and f' has a simple pole at $z = 0$.

Because f is analytic in $\mathbb{D} \setminus \{0\}$, we can write it as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

So

$$f'(z) = \sum_{k=-\infty}^{\infty} k a_k z^{k-1}$$

$$= \sum_{k=-\infty}^{\infty} (k+1)\, a_{k+1} z^k$$

$$= \sum_{k=-\infty}^{\infty} b_k z^k \qquad b_k = (k+1) a_k$$

The absence of a simple pole is then equivalent to $b_{-1} = 0$. And indeed,

$$b_{-1} = (-1+1)\, a_{-1}$$

$$= 0$$

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