Problem 3.1 (HW04-2020)**.** Consider the system

$$\dot{x} = y$$
$$\dot{y} = -f(x) - g(y)$$

where f and g are locally Lipschitz and satisfy

$$f(0) = 0, \quad xf(x) > 0 \text{ for } x \neq 0, \quad x \in (-c, c)$$
$$g(0) = 0, \quad yf(y) > 0 \text{ for } y \neq 0, \quad y \in (-c, c)$$

Show that the origin is stable (asymptotically). *Hint*: Consider $V(x,y) = \int_0^x f(z)dz + y^2/2$ as a Lyapunov function.

Let's first make sure the origin is the only equilibrium.

$$\dot{x} = 0 \Rightarrow y = 0$$

$$\dot{y} = 0 \Rightarrow -f(x) - g(y) = 0$$
$$\Rightarrow -f(x) - g(0) = 0$$
$$\Rightarrow f(x) = 0$$
$$\Rightarrow x = 0 \quad (\text{as } xf(x) > 0 \text{ if } x \neq 0)$$

We will consider $V(x,y) = \int_0^x f(t)\,dt + y^2/2$ as a Lyapunov function.

We WTS: i) $V(0,0) = 0$ (obvious?)

ii) $V(x,y) > 0 \ \forall (x,y) \neq (0,0)$

iii) $\dot{V}(x,y) < 0 \ \forall (x,y) \neq (0,0)$

ii) Firstly, $y^2/2 > 0$. Then since $xf(x) > 0$ for nonzero x, x & $f(x)$ must match sign. Thus since the integral is over $(0,x)$ (nondegenerate), we have $f(x) > 0 \Rightarrow V(x,y) > 0$.

iii) We have $\dot{V}(x,y) = f(x)\dot{x} + y\dot{y}$

$$= f(x)y - f(x)y - g(y)y$$
$$= -g(y)y < 0$$

Thus V is a strict Lyapunov function & we are finished.