

2. Solve $u_t = u_{xx}$ in $[0, \pi] \times [0, \infty)$ with $u(0, t) = 0$ and $u(\pi, t) = 0$ for all t and $u(x, 0) = 1$ for $x \in (0, \pi)$. In what sense the solution takes the initial data and prove it is unique.

Our solution is given by

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2 \pi^2}{L} t\right) \\ &= \sum_{n=1}^{\infty} a_n \sin(nx) \exp(-n^2 \pi t) \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x) dx \quad \text{initial condition} \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx \\ &= \frac{2}{\pi} \left(-\frac{1}{n} \cos(nx) \right) \Big|_0^{\pi} = -\frac{2}{\pi n} (\cos(\pi n) - 1) \\ &= -\frac{2}{\pi n} ((-1)^n - 1) \end{aligned}$$

We need to show uniqueness. Let v be another solution, & let

$w = u - v$. Then $w_t = w_{xx}$. Multiply both sides by w and integrate over $(0, \pi)$:

$$w_t = w_{xx}$$

$$\int_0^\pi w w_t dx = \int_0^\pi w w_{xx} dx$$

$$\int_0^\pi \left(\frac{w^2}{2} \right)_t dx = w w_x \Big|_0^\pi - \int_0^\pi w_x^2 dx$$

$$= - \int_0^\pi w_x^2 dx$$

$$\leq 0$$

$$\Rightarrow \frac{d}{dt} \int_0^\pi \frac{w^2}{2} dx \leq 0$$

and we must have equality because the integrand is positive. It follows by the initial conditions $w(x, 0) = 0$ and $w(0, t) = w(\pi, t) = 0$ that $w \equiv 0$; thus $u = v$ & we are done. 