4. **Spring2020.** Let R be a commutative ring with 1. Call ideals I, J *relatively prime* if $I + J = R$. Prove the following two statements independently (i.e., statement (a) is not used to prove statement (b)).

(a) Assume I, J are relatively prime and $I \cap J = 0$. Prove that $R \cong R/I \times R/J$.

(b) Prove that if I and J are relatively prime, so are I^m and J^n for any positive integers m, n.

a) Consider the map

$$\phi : R \to R/I \times R/J$$
$$r \mapsto (r + I, r + J)$$

We will show ϕ is an isomorphism.

<u>homomorphism</u>

ϕ is a ring homomorphism because it is the canonical projection map in each of its coordinates.

<u>injective</u>

Note $\ker\phi = I \cap J$, and $I \cap J = \emptyset$ by assumption. So ϕ is injective.

<u>surjective</u>

Since $I + J = R$, there exist some $i \in I, j \in J$ such that $x + y = 1$. Then $i = 1 - j \Rightarrow \phi(i) = (0, 1)$ and similarly $\phi(j) = (1, 0)$.

Let $(r_1 + I, r_2 + J) \in R/I \times R/J$. Then

$$\begin{aligned} \phi(jr_1 + ir_2) &= \phi(jr_1) + \phi(ir_2) \\ &= \phi(j)\phi(r_1) + \phi(i)\phi(r_2) \\ &= (1, 0)(r_1 + I, r_1 + J) + (0, 1)(r_2 + I, r_2 + J) \\ &= (r_1 + I, 0) + (0, r_2 + J) \\ &= (r_1 + I, r_2 + J) \end{aligned}$$

So ϕ is surjective and we have shown the isomorphism.