

3. **Fall2007.** Let T be a linear transformation on a complex vector space V , not necessarily finite dimensional. Let $\lambda_1, \dots, \lambda_s$ be distinct eigenvalues of T .

- (a) Suppose that for each j ($1 \leq j \leq s$), v_j is an eigenvector of T with eigenvalue λ_j . Prove that $\{v_1, \dots, v_s\}$ is linearly independent.
- (b) Now suppose that for each j , v_j is a generalized eigenvector of T with eigenvalue λ_j ; that is, there is some integer $m_j \geq 1$ such that

$$(T - \lambda_j)^{m_j} v_j = 0.$$

Again conclude that $\{v_1, \dots, v_s\}$ is linearly independent. (As a matter of notational convenience, assume each m_j is chosen to be minimal; $(T - \lambda_j)^{m_j-1} v_j \neq 0$.)

a) Assume $\{v_1, \dots, v_s\}$ is not LI. Let $k < s$ be the largest integer such that $\{v_1, \dots, v_k\}$ is LI. Then $v_{k+1} = \sum_{i=1}^k c_i v_i$, where at least one $c_i \neq 0$. Because all v_i are eigenvectors, we have

$$\begin{aligned} T v_{k+1} &= T \sum_{i=1}^k c_i v_i \\ &= \sum_{i=1}^k c_i T v_i \\ &= \sum_{i=1}^k c_i \lambda_i v_i \end{aligned}$$

and also,

$$\begin{aligned} T v_{k+1} &= \lambda_{k+1} v_{k+1} \\ &= \lambda_{k+1} \sum_{i=1}^k c_i v_i \\ &= \sum_{i=1}^k c_i \lambda_{k+1} v_i \end{aligned}$$

$$\begin{aligned}
 \text{So } T v_{k+1} - T v_{k+1} &= \sum_{i=1}^k c_i \lambda_{k+1} v_i - \sum_{i=1}^k c_i \lambda_i v_i \\
 &= \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) c_i v_i \\
 &= 0
 \end{aligned}$$

As all λ_i are distinct, $(\lambda_{k+1} - \lambda_i) \neq 0$. Thus as the v_i s are LI (not all the v_i s, just these ones), we must have that $c_i = 0 \quad \forall$

b) Recall a generalized eigenvector v_j of rank m_j satisfies

$$(T - \lambda_j)^{m_j} v_j = 0$$

for a minimal $m_j \in \mathbb{Z}^+$, i.e., $(T - \lambda_j)^{m_j-1} v_j \neq 0$.

Assume $\{v_1, \dots, v_s\}$ are not LI; i.e., $\sum_{i=1}^s c_i v_i = 0$ with at least

one nonzero c_i . We will show WLOG that $c_1 = 0$, and thus that all $c_i = 0$, which is a contradiction.

Let $w = (T - \lambda_1 I)^{m_1-1} v_1$. ^(nonzero by definition) Then $(T - \lambda_1 I)w = 0$, meaning w is an eigenvector of T with eigenvalue λ_1 .

Thus we know: $Tw = \lambda_1 w \Rightarrow Tw - \lambda_j w = \lambda_1 w - \lambda_j w \quad (\because)$

$$\Rightarrow (T - \lambda_j I)w = (\lambda_1 - \lambda_j)w$$

$$\Rightarrow (T - \lambda_j I)^n w = (\lambda_1 - \lambda_j)^n w \rightarrow$$

$$\begin{aligned}
 &(T - \lambda_j I)(\lambda_1 - \lambda_j)^n w \\
 &= (T - \lambda_j I)w (\lambda_1 - \lambda_j)^n \\
 &= (\lambda_1 - \lambda_j)w (\lambda_1 - \lambda_j)^n \\
 &= (\lambda_1 - \lambda_j)^{n+1} w
 \end{aligned}$$

Let $\lambda = \max_j \lambda_j$. Then we can knock out all but one of the general eigenvectors by applying a bunch of appropriate transformations:

$$0 = \sum_{i=1}^s c_i v_i$$

order doesn't matter:

$$\begin{aligned} (T - \lambda_i I)(T - \lambda_j I) &= T^2 - (\lambda_i + \lambda_j)T + \lambda_i \lambda_j I \\ &= T^2 - (\lambda_j + \lambda_i)T + \lambda_i \lambda_j I \\ &= (T - \lambda_j I)(T - \lambda_i I) \end{aligned}$$

and then I'll do that forever

also on this side but like cancel

$$\Rightarrow 0 = \left(\underbrace{(T - \lambda_1 I)^{m_1-1}}_{\text{is } w \text{ but no } v_1} \underbrace{\prod_{i=2}^s (T - \lambda_i I)^{n_i}}_{\text{kills all } v_j \text{ except } v_1} \right) \sum_{i=1}^s c_i v_i$$

$$= \left((T - \lambda_1 I)^{m_1-1} \prod_{i=2}^s (T - \lambda_i I)^{n_i} \right) c_1 v_1$$

$$= c_1 \left(\prod_{i=2}^s (T - \lambda_i I)^{n_i} \right) (T - \lambda_1 I)^{m_1-1} v_1$$

$$= c_1 \left(\prod_{i=2}^s (T - \lambda_i I)^{n_i} \right) w \quad \uparrow \text{eigenvector!}$$

$$= c_1 \prod_{i=2}^s (\lambda_1 - \lambda_i)^{n_i} w \quad (\text{do a lot of associativity})$$

$$= 0$$

But each $(\lambda_1 - \lambda_i)^{n_i}$ as well as w are nonzero, so $c_1 = 0$. The

result follows because I said WLOG before so I win