Problem 4.1 (9.11 HSD). Prove that the linearization at an equilibrium point of a planar Hamiltonian system has eigenvalues that are either $\pm\lambda$ or $\pm i\lambda$ where $\lambda \in \mathbb{R}$.

Hamiltonian system: $\dot{x} = \frac{\partial H}{\partial y}(x,y)$

$$\dot{y} = -\frac{\partial H}{\partial x}(x,y)$$

Say we have an equilibrium point (irrelevant to name it). Then the linearized system is

$$J(x,y) = \begin{bmatrix} \frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial y} \end{bmatrix}$$

Then $\det\left(J(x,y) - \lambda I\right) = \det \begin{bmatrix} H_{xy} - \lambda & H_{yy} \\ -H_{xx} & -H_{xy} - \lambda \end{bmatrix}$

$$= (H_{xy} - \lambda)(-H_{xy} - \lambda) + H_{xx} H_{yy}$$

$$= \lambda^2 - H_{xy}^2 + H_{xx} H_{yy}$$

$$\Rightarrow \lambda^2 = H_{xy}^2 - H_{xx} H_{yy}$$

$$\Rightarrow \lambda = \pm\sqrt{H_{xy}^2 - H_{xx} H_{yy}}$$

Then the result follows, based on the sign of the radicand.