4. (a) Show that the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\phi(x, y, z) = (2y, -x, -xy + z)$$

is a diffeomorphism.

(b) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^3$. Compute $\phi_*(X)$ at $p = (x, y, z)$.

(c) Let $\alpha = dz - y dx$ be a 1-form on $\mathbb{R}^3$. Compute the pullback $\phi^*(\alpha)$ at $p = (x, y, z)$.

a) ϕ is smooth in each of its component functions, so it is smooth.

b) The differential of ϕ at p is

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ -y & -x & 1 \end{bmatrix}$$

Therefore

$$\phi_*(X) = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ -y & -x & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 2y \\ -x \\ -2xy \end{bmatrix}$$

$$= 2y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}$$

c) We have $\phi(u, v, w) = (2v, -u, -uv + w)$. Let

$x = 2v$, $y = -u$, and $z = -uv + w$. Then

$$\alpha = dz - y\,dx = -(v\,du + u\,dv) + dw + 2u\,dv$$

$$= -v\,du + u\,dv + dw$$

Thus $\phi^*(\alpha) = -y\,dx + x\,dy + dz$.