Problem 2.2 (HW06-2020). Suppose that f is globally Lipschitz on $\mathbb{R}^n$. Prove that $\dot{x} = f(x)$ has a global solution.

Suppose we have a local solution $x(t)$ for $0 \leq t < \beta$. Then by Picard iteration, we can write

$$x(t) = x(0) + \int_0^t f(x(s))\,ds$$

$$x(t) - x(0) = \int_0^t f(x(s)) + f(x(0)) - f(x(0))\,ds$$

$$\|x(t) - x(0)\| = \left\| \int_0^t f(x(s)) + f(x(0)) - f(x(0))\,ds \right\|$$

$$\leq \int_0^t \|f(x(s)) - f(x(0))\| + \|f(x(0))\|\,ds$$

$$= \int_0^t \|f(x(s)) - f(x(0))\| + \int_0^t \|f(x(0))\|\,ds$$

Since f is globally Lipschitz (say w/ constant K), we have

$$\|x(s) - x(0)\| \leq \int_0^t \|f(x(s)) - f(x(0))\| + \int_0^t \|f(x(0))\|\,ds$$

$$\leq \int_0^t K\|x(s) - x(0)\|\,ds + \beta\|f(x(0))\|$$

$$\leq \beta\|f(x(0))\| e^{Kt} \quad \text{(Gronwall's)}$$

$$\leq \beta\|f(x(0))\| e^{K\beta}$$

Thus $x(t)$ is bounded & contained in a compact set. By p 146?

We can then extend the domain of $x(t)$ to $(-\infty, \infty)$, which is global.

need C^1?