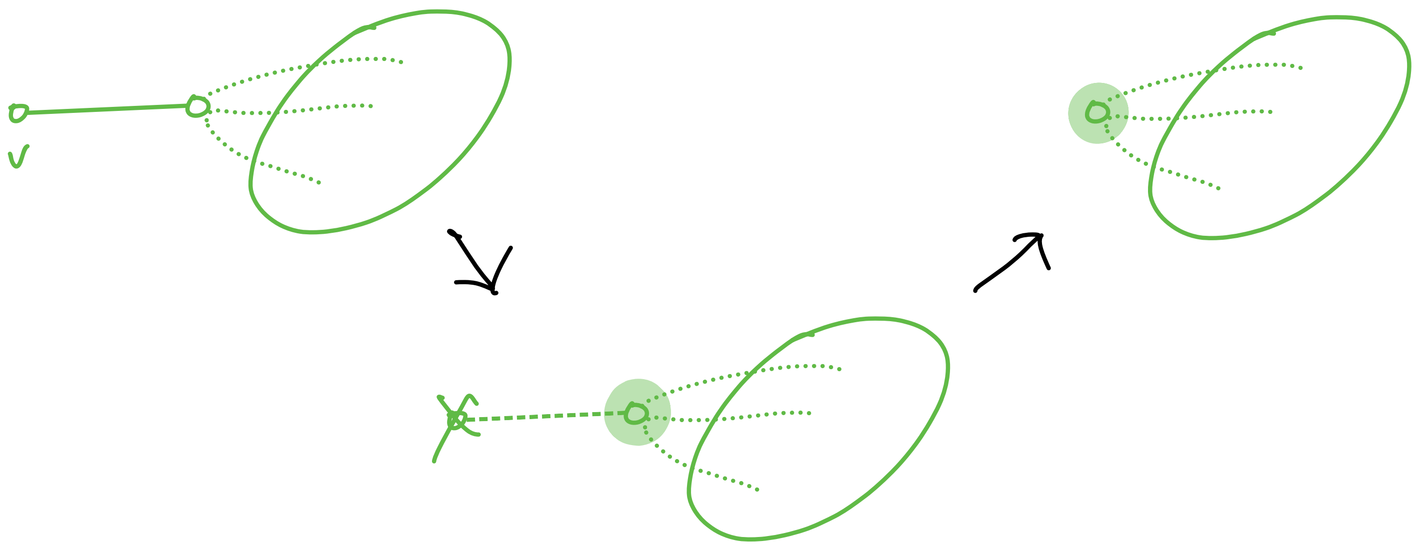
15. (***Abdul and Bakhtiar***) Show that the graph of a symmetric matrix is a tree (a connected undirected graph with no cycles), then the matrix can be re-ordered so that the Cholesky factorization gives no fill-in.

Note that deleting a vertex with one neighbor gives no fill-in, as there are no other neighbors to connect:



So all we need to do to avoid fill-in is to keep deleting these kinds of vertices (<u>leaves</u>). This will always be possible if:

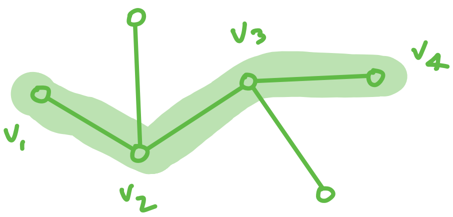
1. every tree has a leaf, and
2. deleting a leaf from a tree gives another tree.

Let T be a tree (acyclic and connected).

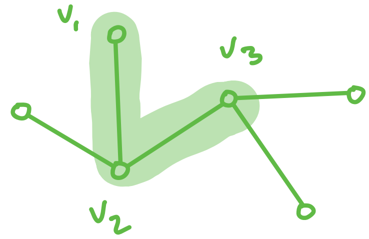
Proof of 2: Let v be a leaf in T. As v has only one neighbor, $T - v$ is still connected. Since vertex/edge deletion cannot form a new cycle, $T - v$ is acyclic and thus $T - v$ is a tree.

Proof of ①: Assume for contradiction that T has no leaves, i.e., each vertex has at least two neighbors. Consider a maximal path $P = v_1 v_2 \cdots v_n$ in T (maximal path: all v_i are distinct, and P is not contained in a longer path)

maximal path:



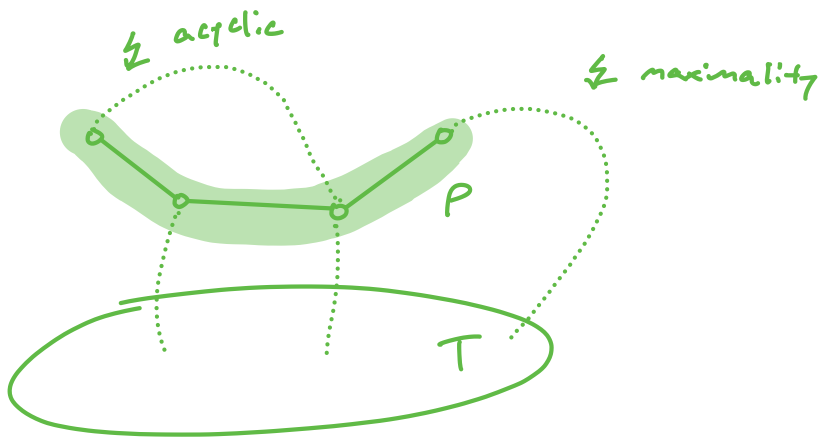
not a maximal path:



Consider an end of the path, say v_1 WLOG. By assumption v_1 has at least one other neighbor besides v_2, say w.

If $w \in P$, then $w = v_i$ for some $i \leq n$, and $v_1 v_2 \cdots w v_1$ is a cycle ↯

So $w \notin P$; but then $w v_1 v_2 \cdots v_n$ extends P to a longer path ↯



Since ① and ② are true, we can always progress through Cholesky if the associated graph is a tree, without fill-in, simply by deleting leaves at each step. The classic example of an ordering that will do this is depth first search with post-order traversal.