(10 points) **(R-4)** Let g be a Lebesgue measurable function on $\mathbb{R}$ such that

$$\|fg\|_1 \le \|f\|_1 \qquad (\text{ for all } f \in L^1(\mathbb{R})).$$

Let $c > 1$ be a real number. Prove that

$$m(\{x \in \mathbb{R} : |g(x)| > c\}) = 0.$$

Let $E = \{x \in \mathbb{R} \mid |g(x)| > c\}$. In the case that $m(E) = \infty$, we will consider $E_k = \{E \cap [k, k+1] \mid k \in \mathbb{Z}\}$. Assume for contradiction that $m(E) > 0$ (or $m(E_k) > 0$, but we'll stick to the finite case).

Let $f(x) = \chi_E(x)$. Then f is also L^1, and so

$$\int_E |g| = \int_{\mathbb{R}} \chi_E |g| = \|fg\|_1 \overbrace{\le}^{\text{given}} \|f\|_1 = \int_{\mathbb{R}} \chi_E = m(E)$$

Therefore $\int_E |g| \le m(E)$.

OTOH, $$\int_E |g| = \int_{\mathbb{R}} \chi_E |g| \overbrace{\ge}^{\text{Chebyshev's inequality}} c\, m(E) \overbrace{>}^{c>1 \text{ and } m(E)>0} m(E)$$

Therefore $\int_E |g| > m(E)$ $\lightning$. We conclude $m(E) = 0$.