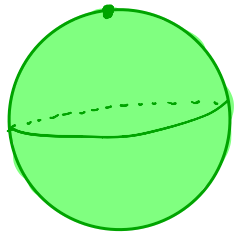
1. Let S be the unit sphere in $\mathbb{R}^3$. Find a C^∞ atlas on S that consists of two charts.

We'll cover the sphere with two charts:

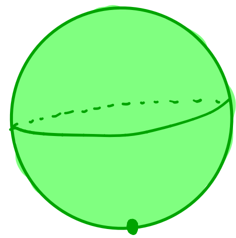
$U = S^2 \setminus (0,0,1)$

$V = S^2 \setminus (0,0,-1)$



U

(omits north pole)



V

(omits south pole)

Then define $\phi : S^2 \to \mathbb{R}^2$ by

$$(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

Then $\phi^{-1} : \mathbb{R}^2 \to S^3$ is

$$(x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$$

Similarly, define $\psi : S^2 \to \mathbb{R}^2$ by

$$(x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z} \right)$$

so then $\psi^{-1} : \mathbb{R}^2 \to S^2$ is

$$(x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{1+x^2+y^2} \right)$$

To see the transition map $\psi \circ \phi^{-1}$ is smooth:

$$\psi \circ \phi^{-1}(x,y,z) = \psi\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

$$= \left(\frac{\dfrac{2x}{x^2+y^2+1}}{\dfrac{2x^2+2y^2}{x^2+y^2+1}}, \frac{\dfrac{2y}{x^2+y^2+1}}{\dfrac{2x^2+2y^2}{x^2+y^2+1}}\right)$$

$$= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

This is clearly smooth. The other transition map is the same, so we are done.