Problem: Find two different smooth structures on a topological space which are still diffeomorphic.

Solution

Our topological space will be $\mathbb{R}$.

smooth structure #1: $\{U, \phi\}$ given by $U = \mathbb{R}$ & $\phi(t) = t$.

smooth structure #2: $\{V, \psi\}$ given by $V = \mathbb{R}$ & $\psi(t) = t^{1/3}$.

Denote $\mathbb{R}$ with these structures as R_1 & R_2, respectively.

These are not compatible as the transition map $(\psi \circ \phi^{-1})(t) = t^{1/3}$ is not smooth. Thus R_1, R_2 are indeed distinct.

To show they are diffeomorphic, define $F: R_1 \to R_2$ by $t \mapsto t^3$. Then $(\psi \circ F \circ \phi^{-1})(t) = t$, which is smooth. Therefore F is the desired diffeomorphism.