Problem 2.3 (Midterm 1 Fall 2019). Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and satisfies

$$|f(x)| \le K|x| + B, \quad x \in \mathbb{R}^n, K \ge 0, B \ge 0.$$

Show that the solution of $\dot{x} = f(x)$, $x(0) = x_0$ exists for all time $-\infty < t < \infty$ and moreover

$$|x(t)| \le x_0 e^{K|t|} + \frac{B}{K}\left(e^{K|t|} - 1\right).$$

You need only show this inequality for $0 \le t < \infty$.

Assume we have our solution $x(t)$ for $0 \le t \le T$. Rewriting,

$$x(t) = x_0 + \int_0^t f(x(s))\, ds$$

Then

$$|x(t)| = \left| x_0 + \int_0^t f(x(s))\, ds \right|$$

$$\le |x_0| + \int_0^t |f(x(s))|\, ds$$

$$\le |x_0| + \int_0^t (K|x(s)| + B)\, ds$$

Now apply Gronwall's to get

$$|x(t)| \le x_0 e^{Kt} + \frac{B}{K}\left(e^{Kt} - 1\right)$$

$$\le x_0 e^{KT} + \frac{B}{K}\left(e^{KT} - 1\right)$$

Thus $x(t)$ is trapped in a compact set, meaning it can be extended to all of $0 \le t \le \infty$.