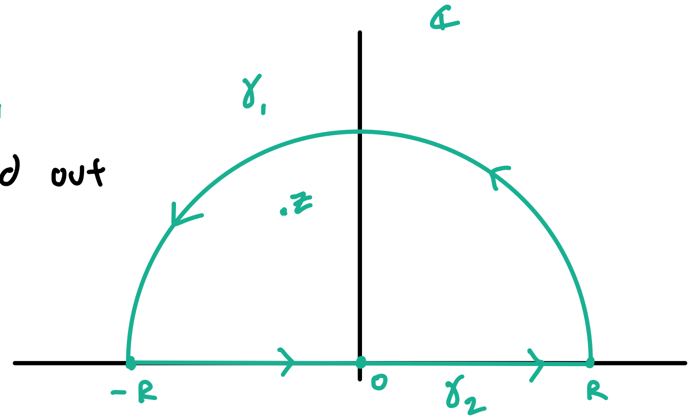
71. **(Practice Midterm 2 - Curto)** Let f be a function which is analytic on the upper half-plane, and on the real line. Assume there exists numbers $B > 0$ and $c > 0$ such that

$$|f(\xi)| \leq \frac{B}{|\xi|^c}$$

for all ξ. Prove that for any z in the upper half plane, we have the integral formula

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}dt.$$



Consider the curve $\gamma = \gamma_2 \circ \gamma_1$, where γ_1 is the upper semicircle of radius R traced out counterclockwise and γ_2 is the line segment from $-R$ to R. Let z be a point in the upper half plane, and let R be large enough so that z is in the interior of γ.

Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)}{t-z}dt$$

$$= \frac{1}{2\pi i}\left(\int_{\gamma_1} \frac{f(t)}{t-z}dt + \int_{-R}^{R} \frac{f(t)}{t-z}dt\right)$$

So if the first integral is 0 as $R \to \infty$ we get what we want.

We have:

$$\left|\int_{\gamma_1} \frac{f(t)}{t-z}dt\right| \leq \int_{\gamma_1} \left|\frac{f(t)}{t-z}\right| dt \leq \int_{\gamma_1} \frac{B}{|t|^c|t-z|}dt \quad \text{(from problem)}$$

(if R large, then $|t-z| \geq R/2$)

$$\leq \int_{\gamma_1} \frac{B}{|t|^c R/2}dt$$

R

wow what a large R

$$= \frac{2B}{R}\int_{\gamma_1} \frac{1}{|t|^c}\, dt$$

$$= \frac{2B}{R}\int_0^{\pi} \frac{1}{R^c}\, R\, d\theta \qquad \text{(for large } R\text{)}$$

$$\le \frac{2B}{R}\,\frac{\pi R}{R^c}$$

$$= \frac{2B\pi}{R^c}$$

$$\Rightarrow \lim_{R\to\infty} \left| \int_{\gamma_1} \frac{f(t)}{t-z}\, dt \right| \le \lim_{R\to\infty} \frac{2B\pi}{R^c}$$

$$= 0$$

//