Problem 1.2 (Discussion, unknown data). (a) Compute the solution to $\dot{x} = Ax$, $x(0) = x_0$ where A is given by

$$A = \begin{bmatrix} -2 & 6 \\ -3 & -4 \end{bmatrix}$$

(b) Do the same where A is given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a) eigenvalues:

$$\det \begin{bmatrix} -2-\lambda & 6 \\ -3 & -4-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 26$$

I'm very cool so we complete the square:

$$\lambda^2 + 6\lambda = -26$$

$$\lambda^2 + 6\lambda + 9 = -17$$

$$(\lambda + 3)^2 = -17$$

$$\lambda = -3 \pm i\sqrt{17}$$

eigenvectors:

$$\begin{bmatrix} -2 & 6 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (-3 + i\sqrt{17}) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-2x + 6y = x(-3 + i\sqrt{17})$$

$$6y = x(-1 + i\sqrt{17})$$

$$y = \frac{x(-1 + i\sqrt{17})}{6}$$

letting $x = 1$, we have a complex eigenvector $\begin{bmatrix} 1 \\ -1/6 \end{bmatrix} + i \begin{bmatrix} 0 \\ \sqrt{17}/6 \end{bmatrix}$.

Let $T = \begin{bmatrix} 1 & 0 \\ -1/6 & \sqrt{17}/6 \end{bmatrix}$. Then $T^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{17}}{17} & \frac{6\sqrt{17}}{17} \end{bmatrix}$ and

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{17}}{17} & \frac{6\sqrt{17}}{17} \end{bmatrix} \begin{bmatrix} -2 & 6 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/6 & \sqrt{17}/6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ -\frac{20\sqrt{17}}{17} & -\frac{18\sqrt{17}}{17} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/6 & \sqrt{17}/6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & \sqrt{17} \\ -\sqrt{17} & -3 \end{bmatrix}$$

This is in canonical form, so the solution to the system $\dot{y} = T^{-1}ATy$ is given by

$$Y(t) = c_1 e^{-3t} \begin{bmatrix} \cos(t\sqrt{17}) \\ -\sin(t\sqrt{17}) \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} \sin(t\sqrt{17}) \\ \cos(t\sqrt{17}) \end{bmatrix}$$

Then we can find our full solution by computing $X = TY$:

$\Rightarrow$

$$Tr = \begin{bmatrix} 1 & 0 \\ -1/6 & \sqrt{17}/6 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t}\cos(t\sqrt{17}) + c_2 e^{-3t}\sin(t\sqrt{17}) \\ -c_1 e^{-3t}\sin(t\sqrt{17}) + c_2 e^{-3t}\cos(t\sqrt{17}) \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-3t}\cos(t\sqrt{17}) + c_2 e^{-3t}\sin(t\sqrt{17}) \\ -c_1 e^{-3t}\cos(t\sqrt{17}) - c_2 e^{-3t}\sin(t\sqrt{17}) \\ +\sqrt{17}\left(-c_1 e^{-3t}\sin(t\sqrt{17}) + c_2 e^{-3t}\cos(t\sqrt{17})\right) \end{bmatrix}$$

$$= c_1 e^{-3t} \begin{bmatrix} \cos(t\sqrt{17}) \\ -\cos(t\sqrt{17}) - \sqrt{17}\sin(t\sqrt{17}) \end{bmatrix}$$

$$+ c_2 e^{-3t} \begin{bmatrix} \sin(t\sqrt{17}) \\ -\sin(t\sqrt{17}) + \sqrt{17}\cos(t\sqrt{17}) \end{bmatrix}$$

Then $x(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \sqrt{17} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_0$

$$\Rightarrow c_1 = x_1 \quad \& \quad -c_1 + c_2\sqrt{17} = x_2$$

$$-x_1 + c_2\sqrt{17} = x_2$$

$$c_2 = \frac{(x_1 + x_2)\sqrt{17}}{17}$$

and we have our particular solution.

b)

B_1

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

B_2

We can consider the highlighted matrices B_1 and B_2 separately.

B_1 is in JCF, & its eigenvalues are $\lambda = 1$ with algebraic multiplicity 3. The ODE $\dot{x} = B_1 x$ is then solved by

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}$$

Similarly, B_2 is in canonical form and $\dot{x} = B_2 x$ is solved by

$$x(t) = c_4 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_5 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

So our solution to the 5×5 is $x(t) =$

$$c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} t^2/2 \\ t \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos(t) \\ -\sin(t) \end{bmatrix} + c_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(t) \\ \cos(t) \end{bmatrix}$$

Then

$$x(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = x_0$$

So our constants correspond to the coefficients of x_0.