2. **Fall2005.** Let R be any ring and I any ideal. Let n be a natural number. Denote n-by-n matrices over R by $\mathrm{Mat}_n(R)$. Show that $\mathrm{Mat}_n(I)$ is an ideal in $\mathrm{Mat}_n(R)$, and

$$\mathrm{Mat}_n(R)/\mathrm{Mat}_n(I) \cong \mathrm{Mat}_n(R/I).$$

$\mathrm{Mat}_n(I)$ is an ideal

group under addition: yes b/c matrix addition happens component-wise

absorbs products: yes b/c matrix multiplication is a big sum of products (all on the left or right individually)

Let $\phi: \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(R/I)$ be given by

$$M \longmapsto M + I$$

homomorphism

$$\begin{aligned}\phi(M+N) &= M + N + I \\ &= M + I + N + I \\ &= \phi(M) + \phi(N)\end{aligned}$$

$$\begin{aligned}\phi(MN) &= \left(\sum M_{ij} N_{ij}\right) + I \\ &= \sum M_{ij} N_{ij} + I \\ &= \sum (M_{ij} + I)(N_{ij} + I) \\ &= \phi(M)\phi(N)\end{aligned}$$

surjective

Let $M + I \in M_n(I)$. Then $\phi(M) = M + I$ ~~wow~~

Let $N \in \ker\phi$. Then $\phi(N) = 0 \Rightarrow N + I = 0$

$$\begin{aligned}&\Rightarrow N_{ij} + I = I \ \forall\ i,j \\ &\Rightarrow N_{ij} \in I \\ &\Rightarrow \ker\phi = \mathrm{Mat}_n(I)\end{aligned}$$

By first iso, $\mathrm{Mat}_n(R)/\ker\phi = \mathrm{Mat}_n(R)/\mathrm{Mat}_n(I)$

$$\cong \mathrm{Mat}_n(R/I)$$