1. (***Victoria and Fatemeh***) An $n \times n$ matrix is diagonally dominant $|a_{ii}| > \sum_{j:j\neq i} |a_{ij}|$ for all i. Show that strictly row dominant matrices are invertible. Give an example of a row dominant (but not strictly dominant) matrix that is not invertible: $|a_{ii}| \geq \sum_{j:j\neq i} |a_{ij}|$ for all i.

Let A be a strictly diagonally dominant matrix. Assume for contradiction that A is not invertible; then $\exists$ a nonzero vector x such that $Ax = 0$. Let $x_k = \|x\|_\infty$. Then

$$Ax = 0 \Rightarrow \sum_{j=1}^{n} a_{kj} x_j = 0$$

$$\Rightarrow a_{kk} x_k = -\sum_{j \neq k} a_{kj} x_j$$

$$\Rightarrow a_{kk} = -\sum_{j \neq k} a_{kj} \frac{x_j}{x_k}$$

Taking absolute values,

$$|a_{kk}| = \left| \sum_{j \neq k} a_{kj} \frac{x_j}{x_k} \right|$$

$$\leq \sum_{j \neq k} \left| a_{kj} \frac{x_j}{x_k} \right|$$

$$= \sum_{j \neq k} |a_{kj}| \underbrace{\left| \frac{x_j}{x_k} \right|}_{\leq 1 \text{ as } x_k = \|x\|_\infty}$$

$$\leq \sum_{j \neq k} |a_{kj}| \quad \lightning \quad (A \text{ is strictly diagonally dominant})$$

We thus conclude A is invertible.

Now consider the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. B is clearly diagonally dominant, but $\det B = 0$ so it is singular.