3. **Fall2007.** Let T be a linear transformation on a complex vector space V, not necessarily finite dimensional. Let $\lambda_1, \ldots, \lambda_s$ be distinct eigenvalues of T.

(a) Suppose that for each $j (1 \leq j \leq s)$, v_j s an eigenvector of T with eigenvalue λ_j. Prove that $\{v_1, \ldots, v_s\}$ is linearly independent.

(b) Now suppose that for each j, v_j is a generalized eigenvector of T with eigenvalue j; that is, there is some integer $m_j \geq 1$ such that

$$(T - \lambda_j)^{m_j} v_j = 0.$$

Again conclude that $\{v_1, \ldots, v_s\}$ g is linearly independent. (As a matter of notational convenience, assume each mj is chosen to be minimal; $(T - \lambda_j)^{m_j - 1} v_j \neq 0$.)

a) Assume $\{v_1, \ldots, v_s\}$ is not LI. Let $k < s$ be the largest integer such that $\{v_1, \ldots, v_k\}$ is LI. Then $v_{k+1} = \sum_{i=1}^{k} c_i v_i$, where at least one $c_i \neq 0$. Because all v_j are eigenvectors, we have

$$T v_{k+1} = T \sum_{i=1}^{k} c_i v_i = \sum_{i=1}^{k} c_i T v_i$$

$$= \sum_{i=1}^{k} c_i \lambda_i v_i$$

and OTOH,

$$T v_{k+1} = \lambda_{k+1} v_{k+1} = \lambda_{k+1} \sum_{i=1}^{k} c_i v_i$$

$$= \sum_{i=1}^{k} c_i \lambda_{k+1} v_i$$

So $$T v_{k+1} - T v_{k+1} = \sum_{i=1}^{k} c_i \lambda_{k+1} v_i - \sum_{i=1}^{k} c_i \lambda_i v_i$$

$$= \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) c_i v_i = 0$$

As all λ_i are distinct, $(\lambda_{k+1} - \lambda_i) \neq 0$. Thus as the v_is are LI (not all the v_is, just these ones), we must have that $c_i \equiv 0$ $\lightning$

b) Assume not, i.e., $\sum_{i=1}^{s} c_i v_i = 0$ with at least one nonzero c_i.

We will show WLOG that $c_1 = 0$, and thus that all $c_i = 0$, which is a contradiction.

Let $w = (T - \lambda_1 I)^{m_1 - 1} v_1$. Then $(T - \lambda_1 I) w = 0 \Rightarrow Tw = \lambda_1 w$.

So $(T - \lambda_i I) w = (\lambda_1 - \lambda_i) w$ and $(T - \lambda_i I)^n w = (\lambda_1 - \lambda_i)^n w$.

Let $n = \max_i \{m_i\}$. Then we can knock out all but one of the general eigenvectors by applying a bunch of appropriate transformations:

$$0 = \sum_{i=1}^{s} c_i v_i$$

order doesn't matter

$$\Rightarrow 0 = \left((T - \lambda_1 I)^{m_1 - 1} \prod_{i=2}^{s} (T - \lambda_i I)^n \right) \sum_{i=1}^{s} c_i v_i$$

$$= c_1 \left((T - \lambda_1 I)^{m_1 - 1} \prod_{i=2}^{s} (T - \lambda_i I)^n \right) v_1$$

$$= c_1 \left(\prod_{i=2}^{s} (T - \lambda_i I)^n \right) w$$

$$= c_1 \prod_{i=2}^{s} (\lambda_1 - \lambda_i) w \quad \text{(takes a hot minute)}$$

But each $(\lambda_1 - \lambda_i) \neq 0$, so $c_1 = 0$. The result follows as this works for every c_i.