4) **[George]**

(a) Suppose A is an $n \times n$ matrix such that $A^2 = -I$ (where I is an $n \times n$ identity matrix). Find an explicit formula for e^{tA}.

(b) Solve

$$\dot{x} = \begin{bmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

a) Observe

$$A^k = \begin{cases} I & k \equiv 0 \pmod 4 \\ A & k \equiv 0 \pmod 4 \\ -I & k \equiv 0 \pmod 4 \\ -A & k \equiv 0 \pmod 4 \end{cases}$$

Thus $\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$

$$= I + tA - \frac{t^2}{2!} I - \frac{t^3}{3!} A + \cdots$$

(we can rearrange as the series converges absolutely)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} A$$

$$= \cos(t) I + \sin(t) A$$

b) Let A be the matrix provided. Then $x(t) = \exp(tA)\, x(0)$. Note $A^2 = -I$; then by the above,

$$x(t) = \exp(tA)\, x(0)$$

$$= (\cos(t) I + \sin(t) A)\, x(0)$$

$$= \begin{bmatrix} \cos(t) - 10\sin(t) \\ -7\sin(t) \\ -5\sin(t) \\ \cos(t) - 2\sin(t) \end{bmatrix} = \cos(t) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} 10 \\ 7 \\ 5 \\ 2 \end{bmatrix}$$