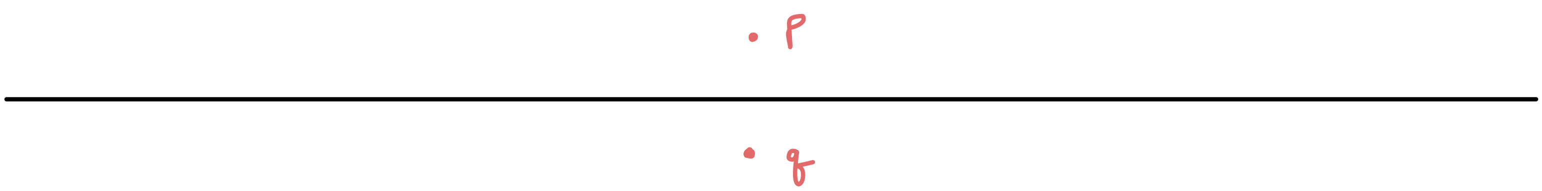
Problem 2.6.4. Show that a non-Hausdorff smooth manifold need not admit partitions of unity subordinate to arbitrary open covers.

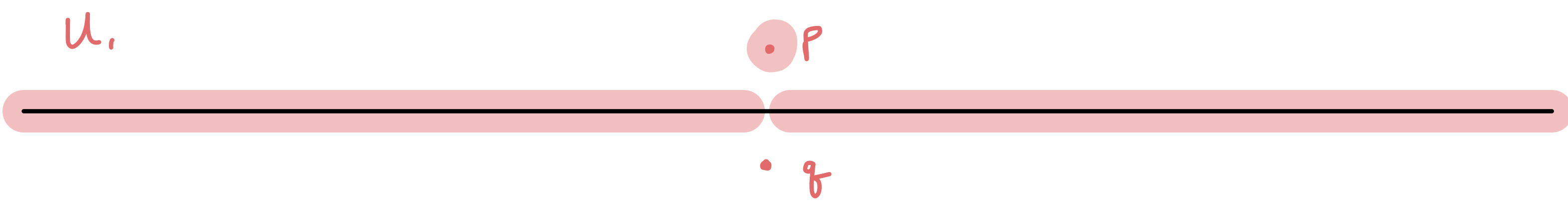
Consider the line with two origins:

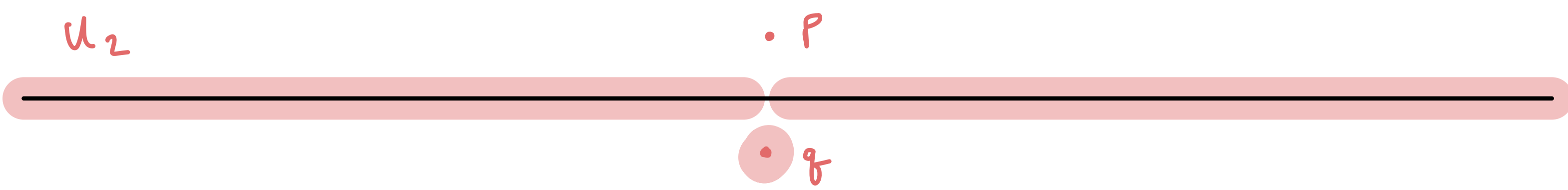


Cover the line with two open sets:

$$U_1 = (-\infty, 0) \cup \{p\} \cup (0, \infty)$$

$$U_2 = (-\infty, 0) \cup \{q\} \cup (0, \infty)$$





Then any partition of unity subordinate to this open cover contains some smooth ρ_1, ρ_2 such that $\rho_1(p) = \rho_2(q) = 1$ (as the two origins are covered by singleton sets). But then as ρ_1 and ρ_2 are continuous, we can find small enough ε so that $(\rho_1 + \rho_2)(\varepsilon) > 1$.

Thus no such partition of unity exists. As the line with two origins is a non-Hausdorff manifold, we are done.