Problem 6.2 (8.5 HSD; HW08-2019). Consider the system

$$\dot{x} = x^2 + y$$
$$\dot{y} = x - y + a$$

where $a \in \mathbb{R}$.

(a) Find all equilibrium points.

(b) Describe the behavior of the linearized system at each equilibrium point.

(c) Describe any bifurcations that occur.

a) $\dot{x} = 0$

$x^2 + y = 0$

$x^2 = -y$

$\dot{y} = 0$

$x + x^2 + a = 0$

$x^2 + x + \frac{1}{4} = \frac{1}{4} - a$

$\left(x + \frac{1}{2}\right)^2 = \frac{1}{4} - a$

$x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - a}$

If $a = \frac{1}{4}$, then we have one equilibrium at $\left(-\frac{1}{2}, -\frac{1}{4}\right)$.

If $a < \frac{1}{4}$, then we have two equilibria:

$$\left(-\frac{1}{2} + \sqrt{1/4 - a},\ -\left(-\frac{1}{2} + \sqrt{1/4 - a}\right)^2\right) \text{ and}$$

$$\left(-\frac{1}{2} - \sqrt{1/4 - a},\ -\left(-\frac{1}{2} - \sqrt{1/4 - a}\right)^2\right)$$

If $a > \frac{1}{4}$ we have no equilibria.

b) The linearization is:

$$J(x,y) = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}$$

eq. point #1: $J(-1/2, -1/4) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

But $\det \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda = \lambda(\lambda+2)$

So we are dealing with a non-hyperbolic equilibrium. To desribe its behavior we look at our nullclines:

x-nullcline: $y = -x^2$

y-nullcline: $y = x + 1/4$

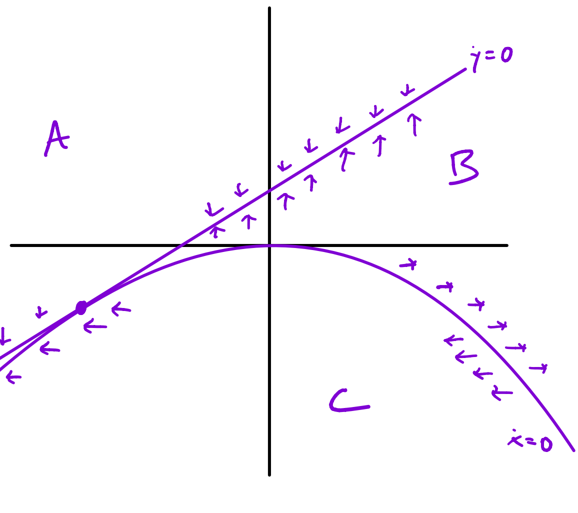
$(1,0)$: $\dot{x} = 1$ $\dot{y} = 1 + \frac{1}{4}$

A: $\dot{x} > 0$ $\dot{y} < 0$

B: $\dot{x} > 0$ $\dot{y} > 0$

C: $\dot{x} < 0$ $\dot{y} > 0$

D: $\dot{x} > 0$ $\dot{y} > 0$



We can see our equilibrium point is not stable, as we can escape through the B region.

eq. point #2: $J(x,y) = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}$

$$J\left(-\frac{1}{2} + \sqrt{1/4 - a},\ -\left(-\frac{1}{2} + \sqrt{1/4 - a}\right)^2\right) = \begin{bmatrix} -1 + 2\sqrt{1/4-a} & 1 \\ 1 & -1 \end{bmatrix}$$

As $\det \begin{bmatrix} -1 + 2\sqrt{1/4-a} & 1 \\ 1 & -1 \end{bmatrix} = 1 - 2\sqrt{1/4-a} - 1 < 0$, we

know our two eigenvalues have opposite signs (because the determinant is the product of the eigenvalues), so this point is a saddle.

eq. point #3: Now $\det \begin{bmatrix} -1 - 2\sqrt{1/4-a} & 1 \\ 1 & -1 \end{bmatrix} = 2\sqrt{1/4-a} > 0$

and $\operatorname{tr} \begin{bmatrix} -1 + 2\sqrt{1/4-a} & 1 \\ 1 & -1 \end{bmatrix} = -2 - 2\sqrt{1/4-a} < 0$. Thus

as the trace is the sum of the two eigenvalues, we can conclude from the sign of the determinant that they are both negative; i.e, we have a sink.

c) As a changes from $< \frac{1}{4} \rightarrow = \frac{1}{4} \rightarrow > \frac{1}{4}$, the number of equilibria go from $2 \rightarrow 1 \rightarrow 0$, coinciding in the middle case & annihilating after. Thus we have a saddle-node bifurcation at $a = \frac{1}{4}$.