(10 points) **(R-4)** Let g be a Lebesgue measurable function on $\mathbb{R}$ such that

$$\|fg\|_1 \leq \|f\|_1 \qquad (\text{ for all } f \in L^1(\mathbb{R})).$$

Let $c > 1$ be a real number. Prove that

$$m(\{x \in X : |g(x)| > c\}) = 0.$$

Let $E = \{x \in \mathbb{R} \mid |g(x)| > c\}$. Note $m(E) < \infty$ as g is L^1.

So assume for contradiction that $m(E) > 0$.

Let $f(x) = \chi_E(x)$. Then f is also L^1, and

$$\int_E |g| = \int_{\mathbb{R}} \chi_E |g| = \|fg\|_1 \leq \|f\|_1 = \int_{\mathbb{R}} \chi_E = m(E)$$

Therefore $\int_{\mathbb{R}} \chi_E |g| \leq m(E)$.

But $\int_{\mathbb{R}} \chi_E |g| \geq c\, m(E) > m(E)$ as $c > 1$. Therefore

$\int_{\mathbb{R}} \chi_E |g| > m(E)$ $\lightning$. We conclude $m(E) = 0$.