71. **(Homework 8 - Chifan)** Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then

(a) f maps sets of measure zero to sets of measure zero, and

(b) f maps measurable sets to measurable sets.

a)

Let $\varepsilon > 0$. As f is absolutely continuous, there exists $\delta > 0$ such that if a finite collection of disjoint open intervals $\{(c_k, d_k)\}_{k=1}^{n}$ satisfies

$$\sum_{k=1}^{n} c_k - d_k < \delta, \text{ then } \sum_{k=1}^{n} |f(c_k) - f(d_k)| < \varepsilon.$$

Let N be a set of measure zero. Then we can find a countable open cover $\mathcal{O}$ of N such that $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$ for disjoint open intervals $I_k = (a_k, b_k)$ and so that $m(\mathcal{O}) < \delta$.

As f is continuous on $\mathbb{R}$, the extreme value theorem applies and for each $I_k \in \mathcal{O}$, f takes a minimum α_k and a maximum β_k on $\overline{I_k} = [a_k, b_k]$.

Now $$\sum_{k=1}^{\infty} I_k = \sum_{k=1}^{\infty} b_k - a_k \leq \sum_{k=1}^{\infty} \beta_k - \alpha_k \Rightarrow \sum_{k=1}^{\infty} \beta_k - \alpha_k < \delta,$$

so by the absolute continuity of f, for any finite collection of $I_k \in \mathcal{O}$,

$$\sum_{k=1}^{n} \beta_k - \alpha_k < \delta \Rightarrow \sum_{k=1}^{n} |f(\beta_k) - f(\alpha_k)| < \varepsilon$$

$$\Rightarrow \sum_{k=1}^{\infty} |f(\beta_k) - f(\alpha_k)| \leq \varepsilon$$

and as α_k and β_k are extrema,

$$m(f(\mathcal{O})) = \sum_{k=1}^{\infty} f(I_k) \leq \sum_{k=1}^{\infty} |f(\beta_k) - f(\alpha_k)| \leq \varepsilon$$

$$\Rightarrow m(f(N)) = 0 \quad //$$

b)

Let E be a measurable set; then we can write $E = F \cup N$, where F is an F_σ set and N is a set of measure zero. By the previous work, we just need to show $f(F)$ is an F_σ set.

Write $F = \bigcup_{k=1}^{\infty} C_k$, where C_k is a closed set. Note $C_k \cap [-n, n]$ is compact, and $F = \bigcup_{\substack{n=1 \\ k=1}}^{\infty} C_k \cap [-n,n]$. Since f is continuous it maps compact sets to compact sets, so $f(F) = \bigcup_{\substack{n=1 \\ k=1}}^{\infty} f(C_k \cap [-n,n])$ is a countable union of closed sets, and is thus an F_σ set. //