24. (**Dr. Ayati: F12 Final** – ***Javier and Zhihua***) Show that the Chebyshev polynomials on [-1,1], $T_n = cos(n * arccos x)$, $n = 0, ...,$ have the recurrence relationship
$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, n=1,...
Use this to argue that the Chebyshev polynomials are indeed polynomials.
Note: $cos((n+1)\lambda) + cos((n-1)\lambda) = 2cos(\lambda)cos(n\lambda)$.

We have

$$T_{n+1}(x) + T_{n-1}(x) = \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x))$$

$$= 2\cos(\arccos(x))\cos(n\cdot\arccos(x))$$

$$= 2x\cos(n\cdot\arccos(x))$$

$$= 2x\,T_n(x)$$

$$\Rightarrow T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

And since

$$T_0(x) = \cos(0) = 1 \quad \text{and}$$

$$T_1(x) = \cos(\arccos(x)) = x$$

we can see that all of the T_n will be a polynomial, as by the recurrence relation T_{n+1} is a polynomial in lower order terms.