Problem 2.4. (a) State the existence and uniqueness theorem for autonomous systems $\dot{x} = f(x)$ about a point x_0.

(b) Find two solutions for

$$\dot{x} = |x|^{1/3}, \quad x(0) = 0.$$

Does this example contradict the existence and uniqueness theorem?

a) Consider the IVP $\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$ for $f \in C^1(\mathbb{R}^n)$. Then

a solution exists & is unique; i.e., $\exists\ a > 0$ such that

$x : (t_0 - a, t_0 + a) \to \mathbb{R}^n$ is unique & satisfies the IVP.

b) One solution is $x \equiv 0$. We can find another by splitting into left & right solutions:

<u>$x \geq 0$</u>

$$\frac{dx}{dt} = x^{1/3}$$

$$x^{-1/3}\, dx = dt$$

$$\int x^{-1/3}\, dx = \int dt$$

$$\frac{3}{2} x^{2/3} = t + C$$

The initial condition forces $C = 0$, so

$$x^{2/3} = \frac{2}{3} t \implies x = \left(\frac{2}{3}\right)^{3/2} t^{3/2}$$

We can get a solution on the negative x-axis similarly.

There is no ↯ to E&U because $|x|^{1/3}$ is not C^1.