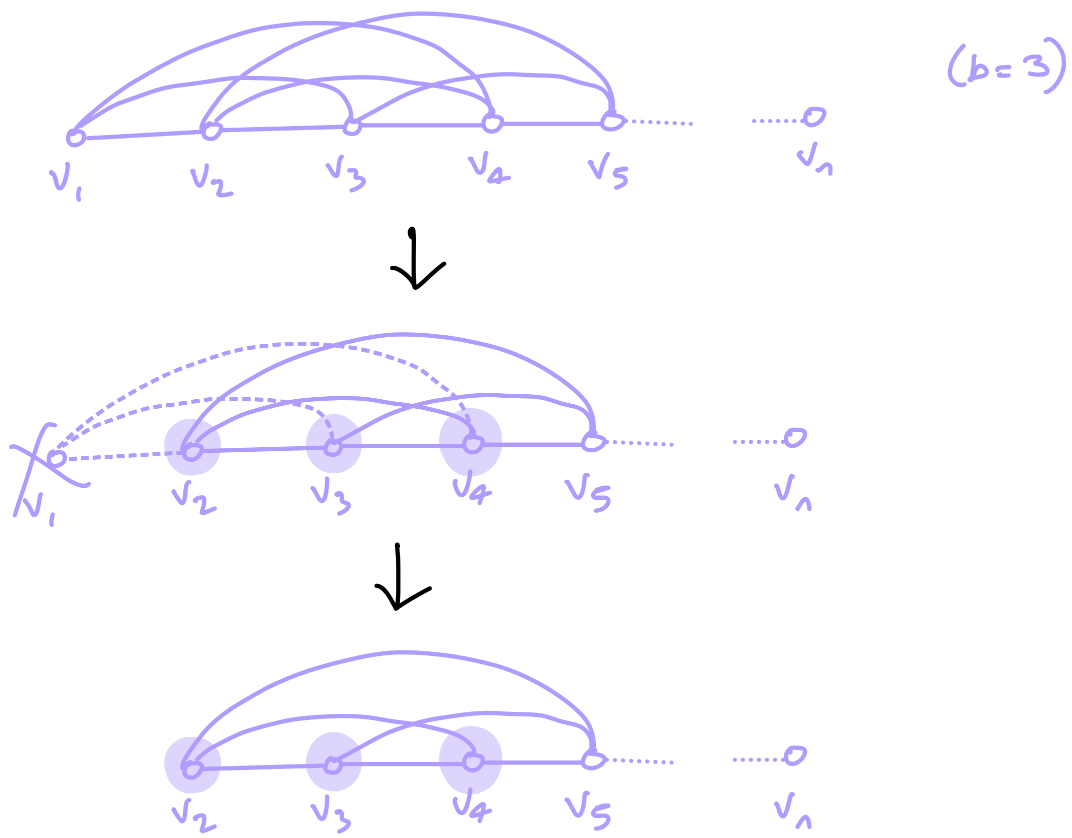
16. (***Claire and Yutian***) Show that the symmetric banded matrix ($a_{ij} \neq 0$ only if $|i-j| \leq b$ where b is the 'bandwidth') the Cholesky factorization can be done with no fill-in outside the band. Note that a tridiagonal matrix is the special case of a banded matrix with bandwidth $b = 1$.

Let $P_n^b = (V, E)$ be defined by $V = \{v_1, \dots, v_n\}$ where $v_i v_j \in E \iff |i-j| \leq b$. Consider the neighbors of v_1; by construction, they are exactly the vertices $\{v_2, \dots, v_{b+1}\}$. But for any $1 < k \leq b+1$, we have $|k-(b+1)| < b$, meaning all such v_k are already adjacent to each other, and so $N(v_1)$ is a complete subgraph of P_n^b. Therefore deleting v_1 gives no fill-in.



Finally, notice that $P_n^b - \{v_1\} = P_{n-1}^b$. Thus an ordering that gives no fill-in is any that always removes the ends of the underlying path in P_n^b (specifically the subgraph $P_n^1 \in P_n^b$).