Problem 7. Assume an ODE has true solution denoted by $Y(t)$. A first-order-correct numerical method returns an approximate solution $y_h(t)$, where h is the step length parameter, with error

$$Y(t) - y_h(t) = hD(t) + \mathcal{O}(h^2). \quad (1)$$

Show that $Y(t) - (2y_h(t) - y_{2h}(t)) = \mathcal{O}(h^2)$.

Substitute $h \mapsto 2h$ in (1):

$$Y(t) - y_{2h}(t) = 2hD(t) + O(h^2) \quad (2)$$

Multiply (1) by -2 and add it to (2):

$$-2\left(Y(t) - y_h(t)\right) = -2\left(hD(t) + O(h^2)\right)$$

$$+ \quad Y(t) - y_{2h}(t) = 2hD(t) + O(h^2)$$

$$-Y(t) + 2y_h(t) - y_{2h}(t) = -O(h^2)$$

Multiply by -1 on both sides:

$$Y(t) - 2y_h(t) + y_{2h}(t) = O(h^2)$$

$$\Rightarrow Y(t) - \left(2y_h(t) - y_{2h}(t)\right) = O(h^2)$$