12. (**Dr. Ayati: F22 Midterm** – ***Claire and Yutian***) Suppose that $f(\xi) = f'(\xi) = 0$, $f''(\xi) \neq 0$ so that f has a double root at ξ, and that f'' is defined and continuous in a neighborhood of ξ. If $\{x_k\}$ is a sequence obtained by Newton's method, show that $\xi - x_{k+1} = \frac{1}{2}(\xi - x_k)\frac{f''(\eta_k)}{f''(\chi_k)}$ where both η_k and χ_k lie between ξ and x_k.

Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \xi - x_{k+1} = \xi - x_k + \frac{f(x_k)}{f'(x_k)} \quad ①$$

By Taylor's theorem with remainder, we have (about x_k):

$$0 = f(\xi) = f(x_k) + f'(x_k)(\xi - x_k) + \frac{f''(\eta_k)}{2}(\xi - x_k)^2$$

$$= \frac{f(x_k)}{f'(x_k)} + (\xi - x_k) + \frac{1}{2}\frac{f''(\eta_k)}{f'(x_k)}(\xi - x_k)^2$$

$$= \xi - x_{k+1} + \frac{1}{2}\frac{f''(\eta_k)}{f'(x_k)}(\xi - x_k)^2 \quad (\text{by } ①)$$

$$\Rightarrow \xi - x_{k+1} = -\frac{1}{2}\frac{f''(\eta_k)}{f'(x_k)}(\xi - x_k)^2 \quad ②$$

By the mean value theorem (applied to f'):

$$f''(\chi_k)(\xi - x_k) + f'(x_k) = f'(\xi) = 0$$

$$\Rightarrow f'(x_k) = -f''(\chi_k)(\xi - x_k)$$

Substituting into 2, we obtain

$$\xi - x_{k+1} = -\frac{1}{2}\frac{f''(\eta_k)}{f'(x_k)}(\xi - x_k)^2$$

$$\Rightarrow \xi - x_{k+1} = \frac{1}{2}\frac{f''(\eta_k)}{f''(x_k)}(\xi - x_k)$$