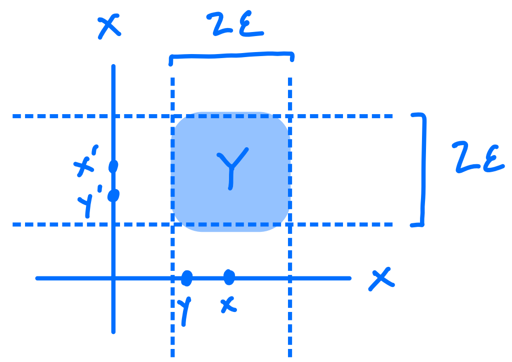
125. **(General Topology HW - Dr. Tehrani)** If d is a metric on X, and X is given the metric topology, show that $d : X \times X \to \cancel{X}\ \mathbb{R}$ is continuous.

Let (a,b) be an open interval in $\mathbb{R}$ with $a<b$ WLOG. To show d is continuous, we need to show the set

$$d^{-1}((a,b)) = \{(x,x') \in X \times X \mid a < d(x,x') < b\}$$

contains an open set about each $(x,x') \in d^{-1}((a,b))$, where "open" means open in the product topology on $X \times X$. In shorter words, we need to find $B_\varepsilon(x) \times B_\varepsilon(x') \subseteq d^{-1}((a,b))$.

Let $(x,x') \in d^{-1}((a,b))$. Let ε be small enough so that $B_{2\varepsilon}(d(x,x'))$ is properly contained in (a,b). (for later!)



draw triangles (pretend $X=\mathbb{R}$)

Claim. $Y = B_\varepsilon(x) \times B_\varepsilon(x') \subseteq d^{-1}((a,b))$.

Proof. Let $(y,y') \in Y$. WTS $(y,y') \in d^{-1}((a,b))$.

We have:

(1)
$$\begin{aligned} d(x,x') &\le d(x,y) + d(y,x') \\ &\le \underbrace{d(x,y)}_{<\varepsilon} + d(y,y') + \underbrace{d(y',x')}_{<\varepsilon} \\ &< d(y,y') + 2\varepsilon \end{aligned}$$

and

(2)
$$\begin{aligned} d(y,y') &\le d(y,x) + d(x,y') \\ &\le \underbrace{d(y,x)}_{<\varepsilon} + d(x,x') + \underbrace{d(x',y')}_{<\varepsilon} \\ &< d(x,x') + 2\varepsilon \end{aligned}$$

(Δ inequality)

Now by our choice of ε,

$$\begin{aligned} a &< d(x,x') - 2\varepsilon \\ &< d(y,y') \quad \text{(by ①)} \\ &< d(x,x') + 2\varepsilon \quad \text{(by ②)} \\ &< b \quad \text{(again by our choice of } \varepsilon) \end{aligned}$$

Therefore $(y,y') \in d^{-1}((a,b))$ and the claim is proved.