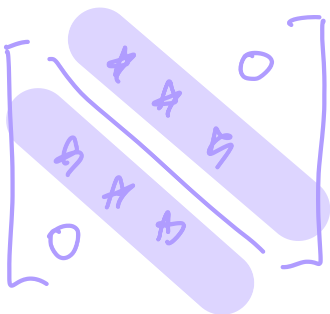
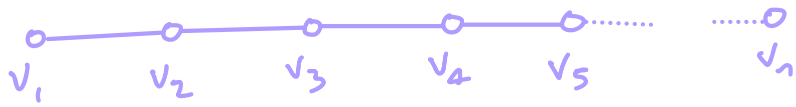
16. (***Claire and Yutian***) Show that the symmetric banded matrix ($a_{ij} \neq 0$ only if $|i-j| \leq b$ where b is the 'bandwidth') the Cholesky factorization can be done with no fill-in outside the band. Note that a tridiagonal matrix is the special case of a banded matrix with bandwidth $b = 1$.

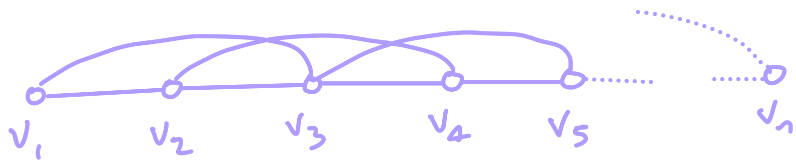
$A =$ 

$A_{ij} \neq 0 \Rightarrow |i-j| \leq b$

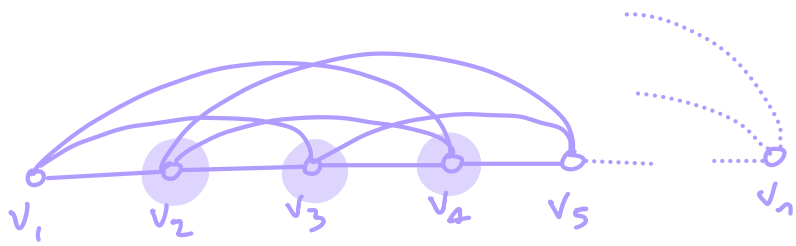
$b=1$



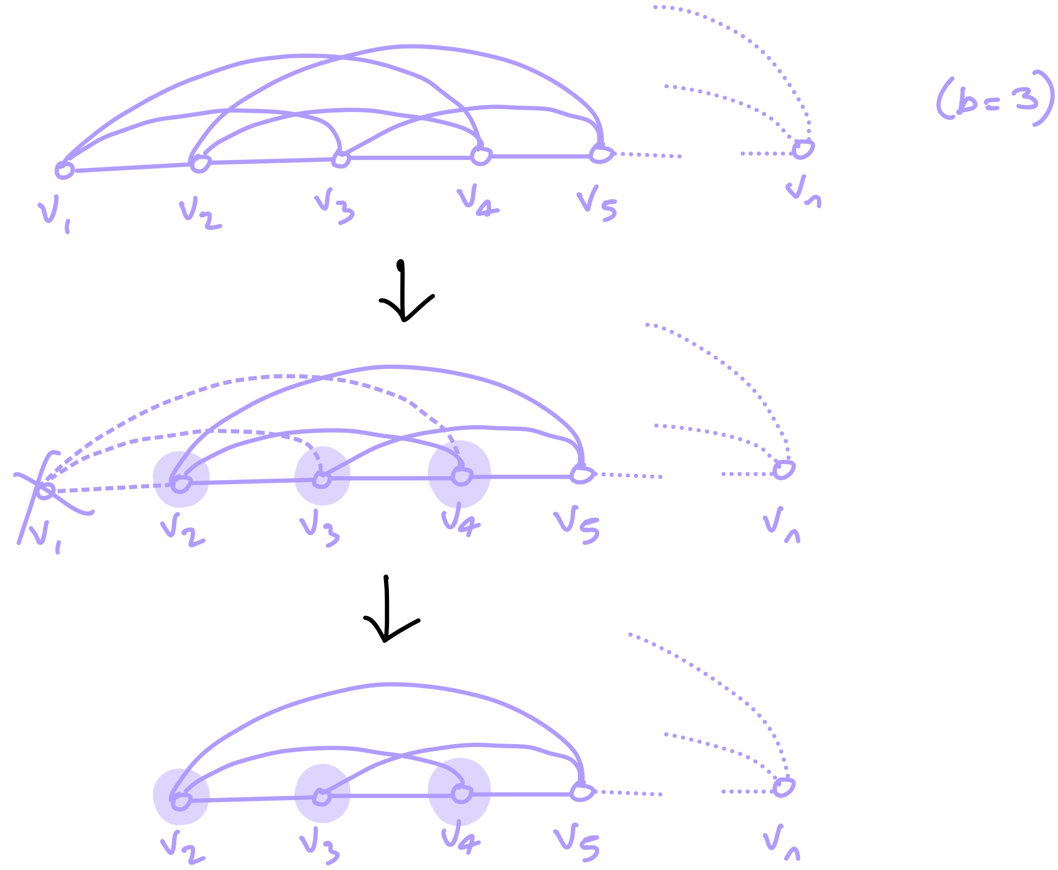
$b=2$



$b=3$



Let $P_n^b = (V, E)$ be defined by $V = \{v_1, \dots, v_n\}$ and $v_i v_j \in E \iff |i-j| \leq b$. Consider the neighbors of v_1, $N(v_1)$. By construction, they are exactly the vertices $\{v_2, \dots, v_{b+1}\}$. But for any $1 < i,j \leq b+1$, we have $|i-j| \leq b-1$. Thus all these vertices are adjacent, that is, $N(v_1)$ is a complete subgraph of P_n^b, and deleting v_1 will give no fill-in:



Finally, notice that $P_n^b - \{v_1\} \cong P_{n-1}^b$. Thus we may keep removing the ends of the underlying path to avoid fill-in. (specifically, remove an end of $P_n^1 \subseteq P_n^b$ at each iteration.)