(10 points) **(C-3)** Consider the fractional linear map

$$F(z) = \frac{z-i}{z+i} \qquad (z \in \mathbb{C}),$$

and let L be the horizontal line in $\mathbb{C}$ containing the point $z = i$. What is the image of L under F? Give a concrete and detailed description.

This is a Möbius transformation because it takes the form $\frac{az+b}{cz+d}$ (with $a=1$, $b=-i$, $c=1$, & $d=i$), and we have $ad-bc = 2i \neq 0$.

Thus lines are mapped to lines/circles, & we can plug in three points to see where our line goes.

$$i \mapsto 0$$

$$1+i \mapsto \frac{1}{1+2i} = \frac{1-2i}{5} = \frac{1}{5} - i\frac{2}{5}$$

$$2+i \mapsto \frac{2}{2+2i} = \frac{1}{1+i} = \frac{1-i}{2} = \frac{1}{2} - i\frac{1}{2}$$

These aren't colinear, so they go to a circle. Let's torture ourselves and do some high school geometry to find it.

three points on a circle: $(0,0)$ $(1/5, -2/5)$ $(1/2, -1/2)$

this circle has a radius r & a center (h,k) so these hold:

$$h^2 + k^2 = r^2$$

$$(1/2 - h)^2 + (-1/2 - k)^2 = r^2$$

$$(1/5 - h)^2 + (-2/5 - k)^2 = r^2$$

or equivalently:

$$h^2 + k^2 = (1/2 - h)^2 + (-1/2 - k)^2$$

$$h^2 + k^2 = (1/5 - h)^2 + (-2/5 - k)^2$$

keep going:

$$h^2 + k^2 = h^2 - h + \frac{1}{4} + k^2 + k + \frac{1}{4}$$

$$h^2 + k^2 = h^2 - \frac{2}{5}h + \frac{1}{25} + k^2 + \frac{4}{5}k + \frac{4}{25}$$

oh yes:

$$h = k + \frac{1}{2}$$

$$\cancel{\frac{2}{5}h = \frac{4}{5}k + \frac{1}{5}}$$

$$h = 2k + \frac{1}{2}$$

now substitute:

$$k + \frac{1}{2} = 2k + \frac{1}{2}$$

$$k = 2k$$

$$k = 0$$

$$\Rightarrow h = 1/2$$

what is the radius? $r^2 = h^2 + k^2 = \frac{1}{4}$

$$\Rightarrow r = \frac{1}{2}$$

So our circle is of radius $1/2$ and centered at $(1/2, 0)$.

Going back to complex land, this means

$$F(L) = \left\{ z \;\middle|\; |z - 1/2| = \frac{1}{2} \right\}$$