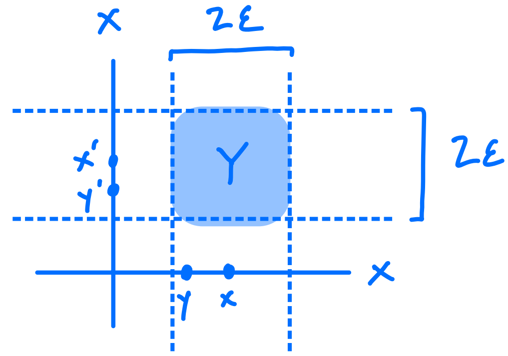
125. **(General Topology HW - Dr. Tehrani)** If d is a metric on X, and X is given the metric topology, show that $d : X \times X \to \cancel{X}\ \mathbb{R}$ is continuous.

Let (a,b) be an open interval in $\mathbb{R}$ with $a<b$ WLOG. To show d is continuous, we need to show the set

$$A = d^{-1}((a,b)) = \{(x,x') \in X \times X \mid a < d(x,x') < b\}$$

is open in the product topology on $X \times X$. To do this, we will find an open set $Y = B_\varepsilon(x) \times B_\varepsilon(x')$ about each $(x,x') \in A$ such that $Y \subseteq A$.

Let $(x,x') \in A$. Let ε be small enough so that $B_{2\varepsilon}(d(x,x'))$ is properly contained in (a,b). (for later!)



draw triangles (pretend $X=\mathbb{R}$)

Claim. $Y = B_\varepsilon(x) \times B_\varepsilon(x') \subseteq A$.

Proof. Let $(y,y') \in Y$. WTS $(y,y') \in A$.

We have:

(1) $$d(x,x') \le d(x,y) + d(y,x') \le \underbrace{d(x,y)}_{<\varepsilon} + d(y,y') + \underbrace{d(y',x')}_{<\varepsilon} < d(y,y') + 2\varepsilon$$

and

(2) $$d(y,y') \le d(y,x) + d(x,y') \le \underbrace{d(y,x)}_{<\varepsilon} + d(x,x') + \underbrace{d(x',y')}_{<\varepsilon} < d(x,x') + 2\varepsilon$$

(Δ inequality)

Now by our choice of ε,

$$
\begin{aligned}
a &< d(x,x') - 2\varepsilon \\
&< d(y,y') \quad \text{(by ①)} \\
&< d(x,x') + 2\varepsilon \quad \text{(by ②)} \\
&< b \quad \text{(again by our choice of } \varepsilon\text{)}
\end{aligned}
$$

Therefore $(y,y') \in A$ by definition and the claim, ergo the whole question, is proved.