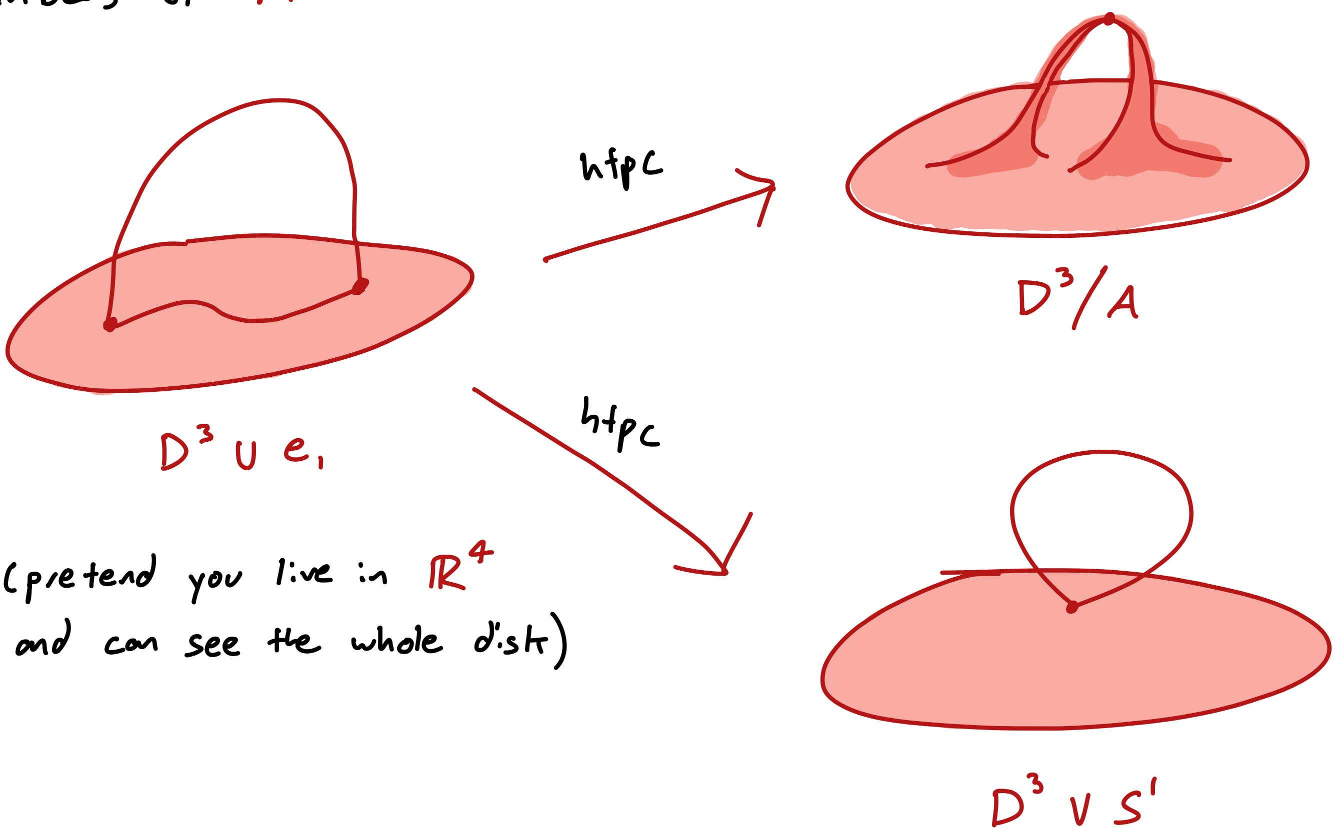
Problem 1.2.23 (Iowa, Winter 2016). Find the fundamental group of the space obtained from the three disk by identifying a finite collection of points. That is, find $\pi_1(D_2/A)$ where $A \subseteq D_3$ is finite.

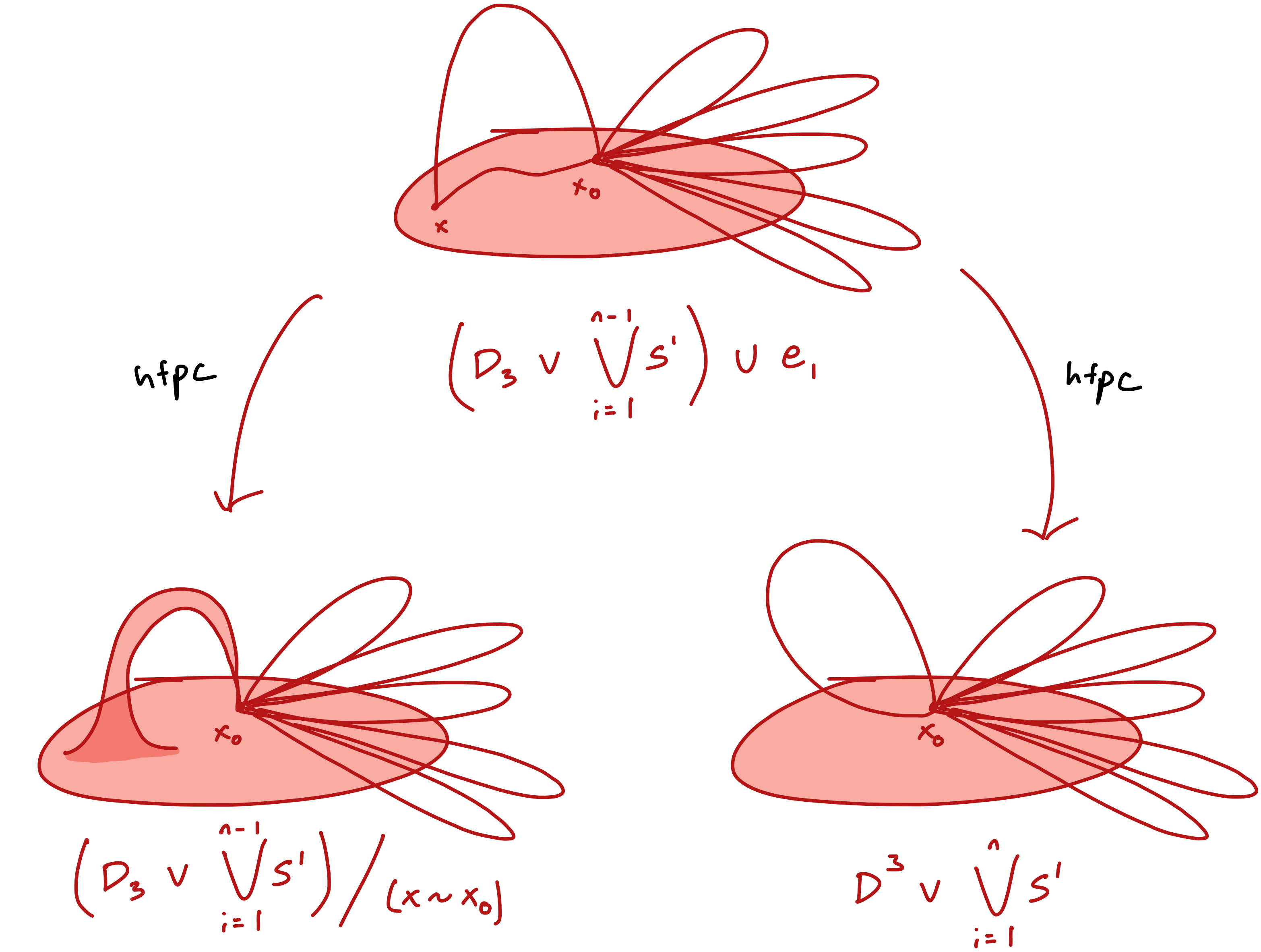
First assume $|A| = 2$. Consider the space made by attaching a 1-cell to the points of A; it is homotopic to both D_3/A via contracting the added 1-cell, and it is also homotopic to $D_3 \vee S^1$, contracting along a path between the two members of A.



(pretend you live in $\mathbb{R}^4$ and can see the whole disk)

Thus $\pi_1(D^3/A) \cong \pi_1(D^3 \vee S^1) = \mathbb{Z}$.

Inductively assume D^3/A is homotopic to $D^3 \vee \bigvee_{i=1}^{n-1} S^1$ for some n. Now add one $x \in D^3$ to A. As before, consider the space obtained by attaching a 1-cell to x and the wedge point x_0. Then we can contract along this 1-cell or a path connecting x and x_0 to get the desired homotopy equivalence:



Thus D^3/A is homotopic to $D^3 \vee \bigvee_{i=1}^{|A|-1} S^1$, and therefore $\pi_1(D^3/A)$ is isomorphic to the free group on $|A|-1$ generators (i.e., the $(|A|-1)$-fold free product of $\mathbb{Z}$).