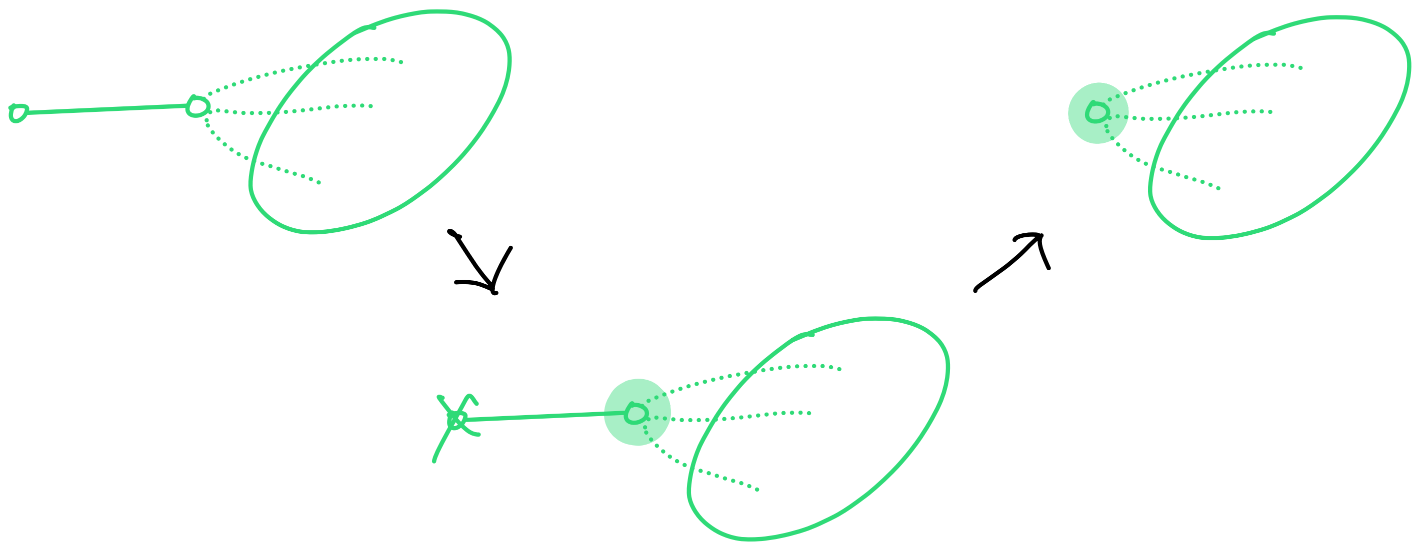
15. (***Abdul and Bakhtiar***) Show that the graph of a symmetric matrix is a tree (a connected undirected graph with no cycles), then the matrix can be re-ordered so that the Cholesky factorization gives no fill-in.

Note that deleting a vertex with one neighbor gives no fill-in, as there are no other neighbors to connect:



So all we need to do to avoid fill-in is to keep deleting degree 1 vertices (or <u>leaves</u>). This will always be possible if

① every tree has a leaf, and

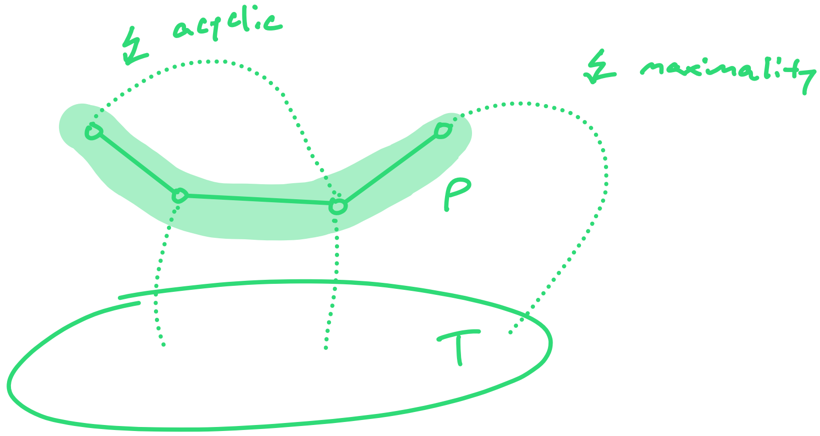
② deleting a leaf from a tree gives another tree.

Let $T$ be a tree.

Proof of ②: Let $v$ be a leaf in $T$. As $v$ has only one neighbor, $T-v$ is still connected. Since deleting edges cannot form a new cycle, $T-v$ is acyclic and thus $T-v$ is a tree.

Proof of 1: Assume for contradiction that $T$ has no leaves, i.e, each vertex has at least two neighbors. Consider a maximal path $P = v_1 v_2 \cdots v_n$ in $T$ (maximal path: all $v_i$ are distinct, and $P$ cannot be made any longer by adding some $v_{n+1}$ to the path).

Consider the ends of the path, say $v_1$ WLOG. By assumption $v_1$ has at least one other neighbor besides $v_2$. If this vertex lies in $P$, then we will have formed a cycle $\lightning$ $T$ is a tree. So $v_1$ is adjacent to some $w \notin P$, making a longer path $w v_1 \cdots v_n$ $\lightning$ $P$ is maximal.




Since 1 and 2 are true, we can always progress through Cholesky if the associated graph is a tree without fill-in, simply by deleting leaves at each step. The classic example of an ordering that will do this is depth first search with post-order traversal.