6. (***Nandita and Paria***) We can assign weights to the least square problem to emphasize the importance of certain components. Doing so can be generalized to minimizing $\|A\mathbf{x}-b\|_C$, where the "C"-norm corresponding to a symmetric positive definite matrix "C" is given by $\|\mathbf{x}\|_C = \sqrt{\mathbf{x}^T C\mathbf{x}}$. Derive the normal equations for this problem.

Since C is symmetric positive definite, we can write $C = LL^T$, i.e., as its Cholesky decomposition.

The key is we can go back & forth between the C-norm and the 2-norm using the Cholesky decomp:

$$\|x\|_C^2 = x^T C x$$

$$= x^T L L^T x$$

$$= (L^T x)^T L^T x$$

$$= \|L^T x\|_2^2$$

$$\Rightarrow \|x\|_C = \|L^T x\|_2$$

Using this, we have

$$\|Ax - b\|_C = \|L^T(Ax - b)\|_2$$

$$= \|L^T A x - L^T b\|_2$$

We know what the normal equations are for the 2-norm: $A^T A x = A^T b$. So we substitute $L^T A$ for A and $L^T b$ for b, and rewrite the normal equations:

$$A^TAx = A^Tb$$

$$\Rightarrow (L^TA)^T L^TAx = (L^TA)^T L^Tb$$

$$\Rightarrow A^TLL^TAx = A^TLL^Tb$$

$$\Rightarrow \underbrace{ACAx = A^TCb}$$

these are our normal equations.