Question 3. Consider

$$\cancel{\dot{x} = y - x} \qquad \dot{x} = -x^3 - x^2 y$$
$$\cancel{\dot{y} = x - y - xz} \qquad \dot{y} = -y + x^3$$
$$\cancel{\dot{z} = xy - z.}$$

(a) State the definitions of a (Lyapunov) stable equilibrium point and an asymptotically stable equilibrium point.

(b) Is the origin $(0, 0, 0)^\top$ a (Lyapunov) stable equilibrium point or asymptotically stable equilibrium point?

(c) What is the basin of attraction?

Hint: Find an appropriate Lyapunov function and use LaSalle's invariance principle.

a) An equilibrium point x^* of a flow ϕ_t is Lyapunov stable if for every neighborhood N of x^*, there is another neighborhood $M \subseteq N$ such that if $x \in M$, $\phi_t(x) \in N$ for all $t \geq 0$.

x^* is called asymptotically stable if there exists a neighborhood N of x^* such that for all $x \in N$, $\lim_{t \to \infty} \phi_t(x) = x^*$.

b) Let $L : E \to \mathbb{R}$ for $E \subseteq \mathbb{R}^2$ be defined by $(x, y) \mapsto x^2 + y^2$.

Then $L(0) = 0$, and $L > 0$ otherwise. To apply Lasalle's invariance principle we need to calculate $\dot{L}$:

$$\begin{aligned}
\dot{L}((x,y)) &= 2x\dot{x} + 2y\dot{y} \\
&= 2x(-x^3 - x^2 y) + 2y(-y + x^3) \\
&= -2x^4 - 2x^3 y - 2y^2 + 2x^3 y \\
&= -2x^4 - 2y^2 \\
&< 0 \text{ for nonzero } (x,y)
\end{aligned}$$

We can also see that the origin is the largest forward invariant subset of the set $\{(x,y) \mid \dot{L}((x,y)) = 0\}$, as:

$\Rightarrow$

$$\dot{L}((x,y)) = 0$$

$$\Rightarrow -2x^4 - 2y^2 = 0$$

$$\Rightarrow x^4 + 2y = 0$$

$$\Rightarrow x = y = 0 \text{ or } 2y = -x^4$$

$$\Rightarrow \text{(if latter) } \dot{y} = -y + x^3$$

but as x^4 is positive, y is negative and $\dot{y} \neq 0$.

Thus we conclude the origin is asymptotically stable.

c) We can show that the origin is the only equilibrium point, & so its basin of attraction is $\mathbb{R}^2$.

$$\dot{x} = -x^3 - x^2 y = 0$$

$$-x^2(x+y) = 0$$

$$\Rightarrow \underbrace{x = 0}_{\text{If this then } y=0 \text{ immediately}} \text{ or } \underbrace{x = -y}_{\text{not so immediate}}$$

$$\dot{y} = -y + x^3 = 0$$

$$= -y - y^3 = 0$$

$$= -y(1+y^2) = 0$$

$\Rightarrow y = 0$ or we are in complex land and let's not

so $x = y = 0$ and we get what we want.