5. (***George and Joe***) Show that if A is real, square and invertible, then the QR factorization is unique apart from a diagonal scaling by factors of ± 1. That is, if $A = Q_1R_1 = Q_2R_2$ with Q_1, Q_2 orthogonal, R_1, R_2 upper triangular, then there is a diagonal matrix D with the diagonal entries ± 1, where $Q_2 = Q_1D$ and $DR_2 = R_1$. In particular, show that if the diagonal entries of R_1, R_2 are all positive, then $Q_1 = Q_2$ and $R_1 = R_2$.

We have

$$Q_1 R_1 = Q_2 R_2$$
$$\Rightarrow \quad R_1 = Q_1^T Q_2 R_2$$
$$\Rightarrow \quad Q_1^T Q_2 = R_1 R_2^{-1}$$

Since the product of upper triangular matrices is upper triangular, $\tilde{D} := Q_1^T Q_2$ is upper triangular. We also have

$$\tilde{D}\tilde{D}^T = Q_1^T Q_2 \left(Q_1^T Q_2\right)^T$$
$$= Q_1^T Q_2 Q_2^T Q_1$$
$$= I$$

This means $\tilde{D}$ is also orthogonal, and $\tilde{D}^T = \tilde{D}^{-1}$. The LHS is lower triangular. The RHS is upper triangular (as the inverse of an upper triangular matrix is upper triangular). This forces $\tilde{D}^T$, and therefore $\tilde{D}$, to be diagonal. The combination of being real, diagonal, and orthogonal forces the entries of $\tilde{D}$ to be ± 1.

Now all of the desired equations hold if $D := \tilde{D}$. If everything is positive then $\tilde{D} = I$ and the equalities $Q_1 = Q_2$ and $R_1 = R_2$ follow immediately.