29. (**Dr. Ayati: F12 Final** – ***Victoria and Fatemeh***) Construct orthogonal polynomials (in the L^2 sense) of degrees 0, 1, and 2 on the interval (0,1) with weight function $w(x) = -ln(x)$. Note that $\int_0^1 -ln(x)x^k dx = \frac{1}{(k+1)^2}$.
Recall that Gram-Schmidt orthogonalization is an inductive process,
$q(x) = x^{n+1} - a_0\phi_0(x) - ... - a_n\phi_n(x)$, where $a_j = \frac{\int_0^1 w(x)x^{n+1}\phi_j(x)dx}{\int_0^1 w(x)\phi_j^2(x)dx}$.

We will orthogonalize $\{1, x, x^2\}$ with G-S.

$$q_0 = 1$$

$$q_1 = x - \frac{\int_0^1 -x\ln x \, dx}{\int_0^1 -\ln x \, dx} = x - \frac{1}{4}$$

$$q_2 = x^2 - \frac{\int_0^1 -x^2(x-\frac{1}{4})\ln x \, dx}{\int_0^1 -(x-\frac{1}{4})^2 \ln x \, dx}(x - \tfrac{1}{4}) - \frac{\int_0^1 -x^2\ln x \, dx}{\int_0^1 -\ln x \, dx}$$

Breaking this into parts:

$$\int_0^1 -x^2(x-\tfrac{1}{4})\ln x \, dx = \int_0^1 -x^3\ln x + \tfrac{1}{4}x^2\ln x \, dx$$

$$= \frac{1}{16} - \left(\frac{1}{4}\cdot\frac{1}{9}\right)$$

$$= \frac{1}{16} - \frac{1}{36}$$

$$= \frac{5}{144}$$

$$\int_0^1 -(x-\tfrac{1}{4})^2 \ln x \, dx = \int_0^1 \left(-x^2 + \tfrac{1}{2}x - \tfrac{1}{16}\right)\ln x \, dx$$

$$= \frac{1}{9} - \frac{1}{8} + \frac{1}{16} = \frac{7}{144}$$

Putting back together:

$$q_2(x) = x^2 - \frac{5}{7}\left(x - \frac{1}{4}\right) - \frac{1}{9}$$

$$= x^2 - \frac{5}{7}x + \frac{17}{252}$$

We have our orthogonal polynomials:

$$q_0(x) = 1$$

$$q_1(x) = x - \frac{1}{4}$$

$$q_2(x) = x^2 - \frac{5}{7}x + \frac{17}{252}$$