7. (***Liz and James***) The stopping criterion in the bisection method is $|b-a|<\epsilon$. For most other algorithms, the stopping criterion is $|f(x)|<\epsilon$. Show that if f is smooth and ϵ is small, then these two stopping criteria are within a factor of approximately $|f'(x^*)|$.

Suppose we use the bisection method to find a root x^* of a smooth function f. Then we have as its output $x_k = \frac{b_k - a_k}{2}$, with $|b_k - a_k| < \varepsilon$ for some small ε. Expanding about x^* in the Taylor series, we have

$$f(x_k) = f(x^*) + f'(x^*)(x_k - x^*) + \frac{f''(\xi)}{2}(x_k - x^*)^2$$

where $\xi \in [x_k, x^*]$. As $x_k, x^* \in [a_k, b_k]$, and $|b_k - a_k| < \varepsilon$, it follows that $|x_k - x^*| < \varepsilon$. Then by the Taylor expansion,

$$|f(x_k)| \leq |f'(x^*)||x_k - x^*| \quad \text{and}$$

$$|f(x_k)| \leq |f'(x^*)||b_k - a_k|$$

This proves the claim.