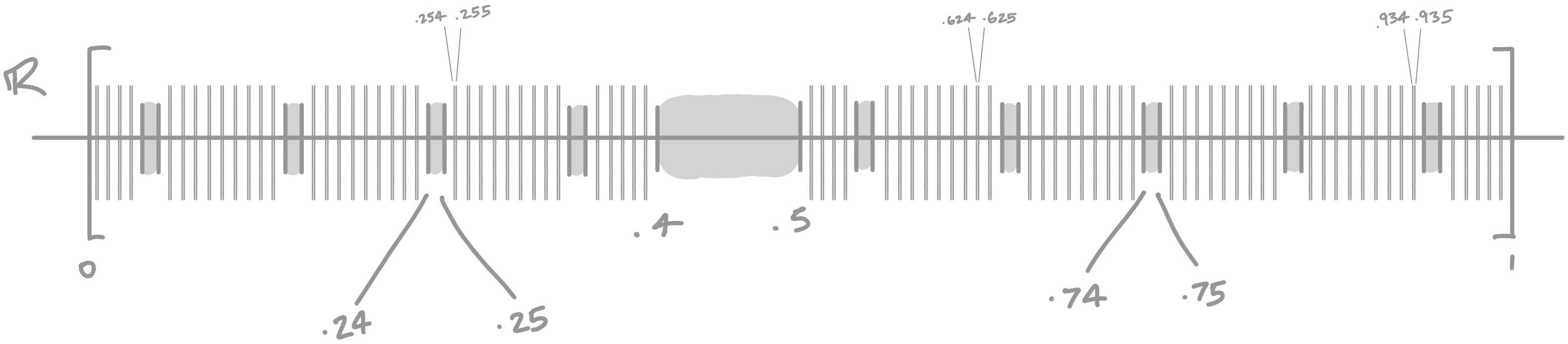
13. **(Homework 3 - Chifan)** Let $A \subset [0,1]$ be a set containing all numbers which do not have the digit 4 appearing in their decimal expression. Find $m(A)$.



Let $B_1 = [.4, .5]$, and let B_n' be the set of $x \in [0,1]$ containing a 4 in the n^{th} place of the decimal expansion of x. Inductively define $B_n = B_n' \setminus \bigcup_{k=1}^{n-1} B_k$. Then B_n consists of closed intervals of the form $[.a_1 a_2 \cdots a_{n-1} 4, \ .a_1 a_2 \cdots a_{n-1} 5]$. Note each a_i cannot be a 4 by construction, and so there are 9^{n-1} such intervals, each having length $\frac{1}{10^n}$. Thus:

$$m(A) = m\left(\overline{\bigcup_{n=1}^{\infty} B_n}\right) = m\left([0,1] \setminus \bigcup_{n=1}^{\infty} B_n\right)$$

$$= 1 - \sum_{n=1}^{\infty} \frac{9^{n-1}}{10^n} \quad \text{(as } B_i \cap B_j = \emptyset \text{ for } i \neq j\text{)}$$

$$= 1 - \sum_{n=0}^{\infty} \frac{9^n}{10^{n+1}}$$

$$= 1 - \sum_{n=0}^{\infty} \frac{1}{10}\left(\frac{9}{10}\right)^n$$

$$= 1 - \frac{1/10}{1 - 9/10} = 1 - 1 = 0 \quad //$$

Each of these intervals has measure $\frac{1}{10^n}$. It can also be seen that there are 10^{n-1} such intervals in B_n. Note that $B_n \cap B_{n+1}$ is exactly the closed interval

$$[.a_1 a_2 \cdots a_{n-1} 44 , \; .a_1 a_2 \cdots a_{n-1} 55]$$

a closed interval of length $1/10^n$. Thus $\overline{B_n}$ is a set of measure $1 - 1/10^n = \frac{10^n - 1}{10^n}$.

$$m(A) = m\left(\bigcup_{n=1}^{\infty} \overline{B_n} \right)$$

$$= m\left(\overline{\bigcap_{n=1}^{\infty} B_n} \right)$$

$$1 - \sum_{k=0}^{\infty} \frac{9^k}{10^{k+1}} \qquad A_n = \overline{\bigcup_{n=1}^{\infty} B_n}$$

$$= 1 - \sum_{k=0}^{\infty} \frac{1}{10} \left(\frac{9}{10}\right)^k$$

$$= 1 - \frac{1/10}{1 - 9/10}$$

$.4 - .5 \qquad .x4 - .x5 \qquad .xx4 - .xx5$

$$m(A) = 1 - \frac{1}{10} - 9\left(\frac{1}{10}\right) - 9^2$$

$$= 1 - 1$$

$$= 0$$

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