36. (**Dr. Ayati: F12 Final** – ***Claire and Yutian***) Derive the two-point Gaussian quadrature formula on the interval [-1,1] using the method of undetermined coefficients.

We want to find $\{x_1, x_2\}$ and $\{w_1, w_2\}$ such that

$$\int_{-1}^{1} p(x)\,dx = w_1 p(x_1) + w_2 p(x_2)$$

is exact for polynomials of degree 3 or less. So we have the equalities

$$\text{I}: \quad \int_{-1}^{1} dx = 2 = w_1 + w_2$$

$$\text{II}: \quad \int_{-1}^{1} x\,dx = 0 = w_1 x_1 + w_2 x_2$$

$$\text{III}: \quad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2$$

$$\text{IV}: \quad \int_{-1}^{1} x^3\,dx = 0 = w_1 x_1^3 + w_2 x_2^3$$

If we multiply both sides of II by $-x_1^2$ and subtract it from IV, we get the equation

$$\begin{aligned} 0 &= w_1 x_1^3 + w_2 x_2^3 - w_1 x_1^3 - w_2 x_1^2 x_2 \\ &= w_2 x_2 \left(x_2^2 - x_1^2 \right) \end{aligned}$$

We now have cases: i) $w_2 = 0$ ii) $x_2 = 0$ iii) $x_1 = x_2$ iv) $x_1 = -x_2$

i) $w_2 = 0$

$$\mathrm{I} \implies w_1 = 2$$

$$\mathrm{II} \implies 2x_1 = 0$$

$$\implies x_1 = 0$$

$$\mathrm{III} \implies 2x_1^2 = \frac{2}{3} \quad \lightning$$

ii) $x_2 = 0$

$$\mathrm{II} \implies w_1 x_1 = 0$$

$$\implies w_1 \text{ or } x_1 = 0$$

$$\mathrm{III} \implies w_1 x_1^2 = \frac{2}{3} \quad \lightning$$

iii) $x_1 = x_2$

$$\mathrm{II} \implies w_1 x_1 + w_2 x_2 = 0$$

if $x_1 \neq 0$, then $w_1 = w_2 = 0$

$$\mathrm{I} \implies w_1 + w_2 = 2 \quad \lightning$$

if $x_1 = 0$, then $w_2 x_2 = 0$

$$\implies w_2 \text{ or } x_2 = 0 \quad \lightning$$

So we must have $x_1 = -x_2$. Then

$$\mathrm{II} \implies w_1 x_1 = w_2 x_1 \implies w_1 = w_2$$

$$\mathrm{I} \implies w_1 = w_2 = 1$$

$$\mathrm{III} \implies 2x_1^2 = \frac{2}{3} \implies x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}.$$

So our method is

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$