4. **Fall2008.** Let $(R, +, \cdot)$ be a ring. Define a new ring $(S, \oplus, *)$ as follows. The elements of $S$ are those of $R$, $a \oplus b = a + b + 1$ and $a * b = a \cdot b + a + b$. Prove that $S$ is a ring and $S$ is isomorphic to $R$.

**abelian group under $\oplus$**

commutativity:

$$\begin{aligned} a \oplus b &= a + b + 1 \\ &= b + a + 1 \\ &= b \oplus a \end{aligned}$$

associativity:

$$\begin{aligned} (a \oplus b) \oplus c &= (a + b + 1) \oplus c \\ &= a + b + 1 + c + 1 \\ &= a + (b + c + 1) + 1 \\ &= a \oplus (b \oplus c) \end{aligned}$$

identity:

$$\begin{aligned} a \oplus (-1) &= a + 1 - 1 \\ &= a \end{aligned}$$

inverses:

$$\begin{aligned} a \oplus (-1-a) &= a - 1 - a + 1 \\ &= 0 \end{aligned}$$

**monoid under $*$**

associativity:

$$\begin{aligned} a * (b * c) &= a * (bc + b + c) \\ &= abc + ab + ac + a + bc + b + c \\ &= (ab + a + b)c + ab + a + b + c \\ &= (ab + a + b) * c \\ &= (a * b) * c \end{aligned}$$

identity:

$$\begin{aligned} 0 * a &= 0 + a \\ &= a + 0 \\ &= a * 0 \end{aligned}$$

**distributivity**

$$\begin{aligned} a * (b \oplus c) &= a(b \oplus c) + a + (b \oplus c) \\ &= a(b + c + 1) + a + b + c + 1 \\ &= ab + ac + a + a + b + c + 1 \\ &= (ab + a + b) + (ac + a + c) + 1 \\ &= (a * b) \oplus (a * c) \end{aligned}$$

the other side is exactly the same ok

Define $\phi : S \to R$ by $s \mapsto s+1$.

homomorphism

$$\phi(a \oplus b) = \phi(a+b+1)$$
$$= a+b+1+1$$
$$= (a+1)+(b+1)$$
$$= \phi(a)+\phi(b)$$

$$\phi(a * b) = \phi(ab+a+b)$$
$$= ab+a+b+1$$
$$= a(b+1)+b+1$$
$$= (a+1)(b+1)$$
$$= \phi(a)\phi(b)$$

bijective

Let $s \in \ker\phi = \{s \in S \mid s+1=0\}$. Easy to see $\ker S = -1$ which is the identity in $S$.

Let $r \in R$. Then $\phi(r-1) = r-1+1 = r$ (remember elements of $R$ are also elements of $S$).

we done