89. **(Homework 5 - Chifan)** Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, $a \neq 0$. HINT: Start with $g(z) = f(1/z)$ and try to figure out what kind of singularity does g have at $z = 0$.

Let f be entire and injective, i.e., $f(z) = f(w) \Rightarrow z = w$.

Consider $g(z) := f\left(\frac{1}{z}\right)$. It is entire except when $z = 0$.

Case 1: $z = 0$ is a removable singularity of g.

$\Rightarrow$ g is bounded in a punctured disk centered at $z=0$

$\Rightarrow$ f is bounded outside a disk centered at $z=0$

But $f(z)$ is also bounded inside this disk, as it is continuous and thus bounded on compact sets. So f is entire and bounded, and thus constant by Liouville's. $\lightning$ as f is injective.

Case 2: $z = 0$ is an essential singularity of g.

By Casorati-Weierstrass, the image under g of any punctured disk centered at $z=0$ is dense in $\mathbb{C}$. So in turn the image under f of the complement of an open disk D is dense in $\mathbb{C}$.

As f is entire and nonconstant (it is injective), it is an open map (by the open mapping theorem, of course). Thus $f(D)$ is an open set. The intersection of a dense set and an open set is nonempty, so $f(D) \cap f(D^c) \neq \emptyset$. $\lightning$ because f is injective.

Thus $z=0$ is a pole. This means it has only finitely many terms of negative degree in its Laurent series expansion, which then means f has finitely many positive terms in its expansion. Thus f is a polynomial.

More cases.

Case 1: $\deg(f) = 0$

no, f injective means f nonconstant.

Case 2: $\deg(f) = n \geq 2$

We know f has n zeros. They cannot be distinct as f is injective. So let z_0 be a zero of f of multiplicity n.

Then $f(z) = \omega (z - z_0)^n$ for some $\omega \in \mathbb{C}$. But then

$f(z_0 + 1) = \omega = f(z_0 + e^{2\pi i/n})$ $\lightning$ f injective.

So $\deg(f) = 1$, i.e., $f(z) = az + b$ $a, b \in \mathbb{C}$ & $a \neq 0$ //