Question 1. Consider the motion of an undamped harmonic oscillator, given by the equation

$$\ddot{x} = -4x,$$

where $x(t)$ represents the location of the oscillator, and $\ddot{x}$ denotes the second derivative of x with respect to t.

(a) Introduce a new variable y for the velocity of the oscillator and formulate the motion of the harmonic oscillator as a two dimensional linear system of the form $\dot{X} = AX$, where $X = (x, y)^\top$ and A is a 2×2 matrix.

(b) Find the eigenvalues and eigenvectors of A.

(c) Show there exists an invertible matrix T so that $T^{-1}AT = B$ where B is in Jordan canonical form.

(d) Use the Jordan canonical form to find the general solution of the system $\dot{X} = AX$.

a) Let $y = \dot{x}$. Then our system becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = -4x \end{cases}$$

or, letting $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$, we have the system $\dot{X} = AX$.

b) eigenvalues: $\det \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} = 0$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda = \pm 2i$$

eigenvectors: $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2i \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow y = 2ix$$

$\Rightarrow \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ is an eigenvector, & similarly so is $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$.

c) Let $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ and

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

This is in canonical form so T works.

d) The matrix $T^{-1}AT$ has eigenvalues $\pm 2i$, so the system

$\dot{Y} = T^{-1}ATY$ is solved by

$$Y(t) = c_1 \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

Then our general solution for $\dot{X} = AX$ is

$$X(t) = TY = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ -c_1\sin(2t) + c_2\cos(2t) \end{bmatrix}$$

$$= \begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ -2c_1\sin(2t) + 2c_2\cos(2t) \end{bmatrix}$$