49. **(Homework 5 - Chifan)** If f is uniformly continuous and integrable on $\mathbb{R}$, then $\lim_{|x|\to\infty} f(x) = 0$. True or False?

It suffices to show the result on $\mathbb{R}^+$.

Note that f being integrable means $\lim_{a,b\to\infty} \int_a^b f(x)\,dx = 0$.

Assume $\lim_{x\to\infty} f(x) \neq 0$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $f(x_n) > \varepsilon$ for some $\varepsilon > 0$ and so that $|x_n - x_{n-1}| > 1$.

Since f is uniformly continuous, there is a $\delta > 0$ such that if $|x - x_n| < \delta$, then $|f(x) - f(x_n)| < \varepsilon/2$. So on the interval $(x_n - \delta, x_n + \delta)$, we have that $|f(x)| > \varepsilon/2$ (as the value of f will be at most $\varepsilon/2$ away from $f(x_n) > \varepsilon$). Thus

$$\left| \int_{x_n-\delta}^{x_n+\delta} f(x)\,dx \right| > \frac{\varepsilon}{2} \int \chi_{(x_n-\delta,\, x_n+\delta)}(x)\,dx$$

$$= \frac{\varepsilon}{2} \cdot 2\delta$$

$$= \varepsilon\delta$$

But $x_n \to \infty$, so $\lim_{a,b\to\infty} \int_a^b f(x)\,dx \neq 0$ $\lightning$.