Question 3. Consider

$$\dot{x} = y - x$$
$$\dot{y} = x - y - xz$$
$$\dot{z} = xy - z.$$

(a) State the definitions of a (Lyapunov) stable equilibrium point and an asymptotically stable equilibrium point.

(b) Is the origin $(0, 0, 0)^\top$ a (Lyapunov) stable equilibrium point or asymptotically stable equilibrium point?

(c) What is the basin of attraction?

Hint: Find an appropriate Lyapunov function and use LaSalle's invariance principle.

a) An equilibrium x^* of a flow ϕ_t is Lyapunov stable if for every neighborhood N of x^*, there is another neighborhood $M \subseteq N$ such that if $x \in M$, then $\phi_t(x) \in N \;\forall\, t \geq 0$.

x^* is asymptotically stable if it is stable and has a neighborhood N such that if $x \in N$, $\lim_{t \to \infty} \phi_t(x) = x^*$.

b) Define $L : E \to \mathbb{R}$ for $E \subseteq \mathbb{R}^3$ by $(x, y, z) \mapsto x^2 + y^2 + z^2$.

Note $L((0,0,0)) = 0$, and $L(x) > 0 \;\forall\, x \neq 0$. Now

$$\dot{L}((x,y,z)) = 2x\dot{x} + 2y\dot{y} + 2z\dot{z}$$
$$= 2x(y-x) + 2y(x-y-xz) + 2z(xy-z)$$
$$= 2xy - 2x^2 + 2xy - 2y^2 - 2xyz + 2xyz - 2z^2$$
$$= 4xy - 2x^2 - 2y^2 - 2z^2$$
$$= -2(x^2 - 2xy + y^2) - 2z^2$$
$$= -2(x-y)^2 - 2z^2$$

This is strictly negative for all $x \in \mathbb{R}^3$. Now note that the set $\{(x,y,z) \mid \dot{L} = 0\} = \{(x,y,z) \mid -2(x-y)^2 - 2z^2 = 0\}$

$$= \{(x,y,z) \mid (x-y)^2 + z^2 = 0\}$$

$$= \{(x,y,z) \mid x = y \text{ and } z = 0\}$$

Assume (x,y,z) are in this set and $x, y \neq 0$. Then $\dot{z}(x,y,z) \neq 0$, which means only $(0,0,0)$ is a positively invariant subspace, and so it is asymptotically stable by the weak Lyapunov function & Lasalle's invariance principle.

c) Since $(0,0,0)$ is the only equilibrium:

$$\dot{x} = 0 \Rightarrow x = y$$

$$\dot{y} = 0 \Rightarrow x \text{ or } z \text{ zero}$$

$$\dot{z} = 0 \Rightarrow x^2 = z \text{ or } x = y = z = 0$$

if $x = 0$, then $z = 0^2 = 0$ &

if $z = 0$, then $x^2 = 0 \Rightarrow$

its basin of attraction is all of $\mathbb{R}^3$.