20. (**Dr. Ayati: F21 Midterm** – ***Claire and Yutian***) Find the Lagrange interpolation polynomial $p_2(x)$ through the points $(-1,1)$, $(0,2)$, $(1,0)$. For an arbitrary function $f \in C^3[-1,1]$ that goes through these three points, find a (reasonably sharp) constant K such that $|f(x) - p_2(x)| \leq K \cdot \max_{\xi\in[-1,1]} |f'''(\xi)|$.

For the interpolating polynomial: $p_n(x) = \sum_{i=1}^{n-1} y_i \mathcal{L}_i(x)$

$x_1 = -1 \quad x_2 = 0 \quad x_3 = 1$

$y_1 = 1 \quad y_2 = 2 \quad y_3 = 0$

$$\mathcal{L}_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$\mathcal{L}_1(x) = \left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right) = (-x)\left(\frac{1-x}{2}\right) = \frac{x^2-x}{2}$$

$$\mathcal{L}_2(x) = \left(\frac{x+1}{0+1}\right)\left(\frac{x-1}{0-1}\right) = (x+1)(1-x) = 1-x^2$$

$$\mathcal{L}_3(x) = \left(\frac{x+1}{1+1}\right)\left(\frac{x-0}{1-0}\right) = x\left(\frac{x+1}{2}\right) = \frac{x^2+x}{2}$$

So

$$p_2(x) = \frac{x^2-x}{2} + 2 - 2x^2$$

$$= \frac{4 - x - 3x^2}{2}$$

For the second part, note $f(x) - p_2(x) = \frac{f'''(c)}{3!}(x-x_1)(x-x_2)(x-x_3)$

where $\min_i x_i \leq c \leq \max_i x_i$. Then

$$|f(x)-p(x)| \leq \max_{\xi\in[-1,1]} \left| \frac{f'''(\xi)}{3!} (\xi-1)(\xi)(\xi+1) \right|$$

$$\leq \frac{1}{6} \max_{\xi\in[-1,1]} \left| f'''(\xi)\,(\xi^3-\xi) \right|$$

$$= \frac{1}{6} \max_{\xi\in[-1,1]} |f'''(\xi)| \max_{\xi\in[-1,1]} |\xi^3-\xi|$$

So we need to find the maximum of $g(x) = x^3 - x$ on $[-1,1]$. Note $g'(x) = 3x^2-1$, and $g''(x) = 6x$. $g'(x)=0$ at $\pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$, and $g''\left(-\frac{\sqrt{3}}{3}\right) < 0$ so $x = -\frac{\sqrt{3}}{3}$ is our maximum.

Now $$|f(x)-p(x)| \leq \frac{1}{6} \max_{\xi\in[-1,1]} |f'''(\xi)| \max_{\xi\in[-1,1]} |\xi^3-\xi|$$

$$= \frac{1}{6} \max_{\xi\in[-1,1]} |f'''(\xi)| \; |g(-\sqrt{3}/3)|$$

$$= \frac{1}{6} \max_{\xi\in[-1,1]} |f'''(\xi)| \left(\frac{2\sqrt{3}}{9}\right)$$

$$= \frac{\sqrt{3}}{27} \max_{\xi\in[-1,1]} |f'''(\xi)|$$

Thus $K = \sqrt{3}/27$ is our desired constant.