1. **(Review Final - Chifan)** Let $f$ be an analytic function on the open disk $\mathbb{D}$. Assume that for every $z \in \mathbb{D}$ the power series expansion around $z$ has a vanishing coefficient. Show that $f$ is a polynomial function.

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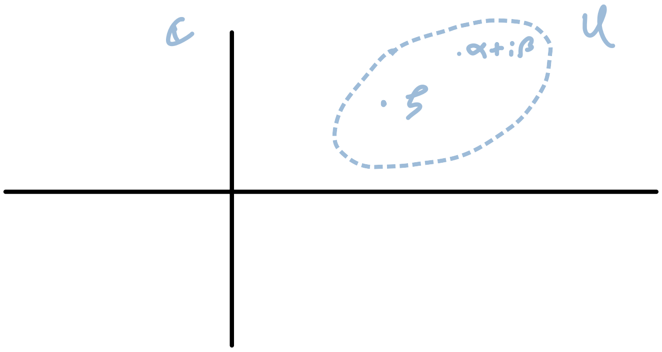
$$\Downarrow$$

$$\forall z \in \mathbb{D},\ f^{(n)}(z) = 0 \text{ for some } n \in \mathbb{Z}^+$$

Let $Z_n := \{ z \in \mathbb{D} \mid f^{(n)}(z) = 0 \}$, the set of points in the disk vanishing at the $n$th derivative of $f$. Note $\bigcup_{n=0}^{\infty} Z_n$ is not countable as it contains every point in $\mathbb{D}$. Thus one of the $Z_n$ must be uncountable; $\bigcup_{n=0}^{\infty} Z_n$ is a countable union of sets, which would be countable if every $Z_n$ were countable.

Let $Z_k$ be this uncountable set; then $f^{(k)} \equiv 0$. Assume not. Note $f^{(k)}$ is analytic, so its zeros are isolated, that is, for each $\xi \in Z_k$, there is a neighborhood $U$ such that $\xi \in U$ and there are no other zeros in $U$. But for each such $U$ we can associate to it a complex number $\alpha + i\beta$, where $\alpha, \beta \in \mathbb{Q}$. Thus we have a bijection between a subset of $\mathbb{Q} \times \mathbb{Q}$ and $Z_k$, $\lightning$ as $\mathbb{Q} \times \mathbb{Q}$ is countable.




$\Rightarrow$ a derivative of $f$ vanishes everywhere on $\mathbb{D}$

$\therefore$ $f$ is a polynomial $//$