21. (***George and Joe***) Show that the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 2 & & & \\ & & 1 & 3 & & \\ & & & \ddots & & \\ & & & & 1 & (n-1) \\ & & & & & 1 \end{bmatrix}$$

has one eigenvalue $\lambda = 1$ repeated n times. Symbolically compute the eigenvalues of $A + \epsilon e_n e_1^T$.

a) We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & & & & \\ & 1-\lambda & 2 & & & \\ & & 1-\lambda & 3 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & n-1 \\ & & & & & 1-\lambda \end{bmatrix}$$

performing cofactor expansion along the first row gives:

$$\det(A - \lambda I) = (1-\lambda) \det \begin{bmatrix} 1-\lambda & 2 & & & \\ & 1-\lambda & 3 & & \\ & & 1-\lambda & \ddots & \\ & & & \ddots & n-1 \\ & & & & 1-\lambda \end{bmatrix}$$

after $n-1$ more expansions along the first column...

$$= (1-\lambda)^n$$

$\Rightarrow$ eigenvalues of A satisfy

$$(1-\lambda)^n = 0$$

$\Rightarrow \lambda = 1$ with multiplicity n.

b) Now let $B = A + \varepsilon e_n e_1^T$, which is A with an ε in the lower left corner. Then we can do cofactor expansion along the first column again:

$$\det(\lambda I - B) = \det \begin{bmatrix} 1-\lambda & 1 & & & & \\ & 1-\lambda & 2 & & & \\ & & 1-\lambda & 3 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & n-1 \\ \varepsilon & & & & & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \det \begin{bmatrix} 1-\lambda & 2 & & & \\ & 1-\lambda & 3 & & \\ & & 1-\lambda & \ddots & \\ & & & \ddots & n-1 \\ & & & & 1-\lambda \end{bmatrix}$$

$$+ (-1)^{n+1} \varepsilon \det \begin{bmatrix} 1 & & & & \\ 1-\lambda & 2 & & & \\ & 1-\lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1-\lambda & n-1 \end{bmatrix}$$

$$= (1-\lambda)^n + (-1)^{n+1} \varepsilon (n-1)!$$

$\Rightarrow$ eigenvalues of B satisfy

$$(1-\lambda)^n + (-1)^{n+1} \varepsilon (n-1)! = 0$$

$$\Rightarrow \lambda = 1 - \left((-1)^{n+1} \varepsilon (n-1)! \right)^{1/n}$$