Problem C-2A. Let
$$u(z) := \operatorname{Re}\frac{i+z}{i-z} \qquad (\text{for } z \neq i),$$
and let $u(i) := 0$.

Show that u is harmonic on the unit disc, u is 0 on the unit circle, and u is continuous on the closed unit disc except at the point $z = i$.

harmonic

Note $\frac{i+z}{i-z}$ is holomorphic on $\mathbb{C} \setminus \{i\}$, meaning its real (and imaginary) part is harmonic. At $z=i$, we have $u(z)=0$ so trivially $\Delta u = 0$.

vanishes on boundary

The function $\frac{i+z}{i-z}$ is a Möbius transformation, so it sends circles to circles (or lines which are also circles). We can plug three points on the unit circle to see where they go:

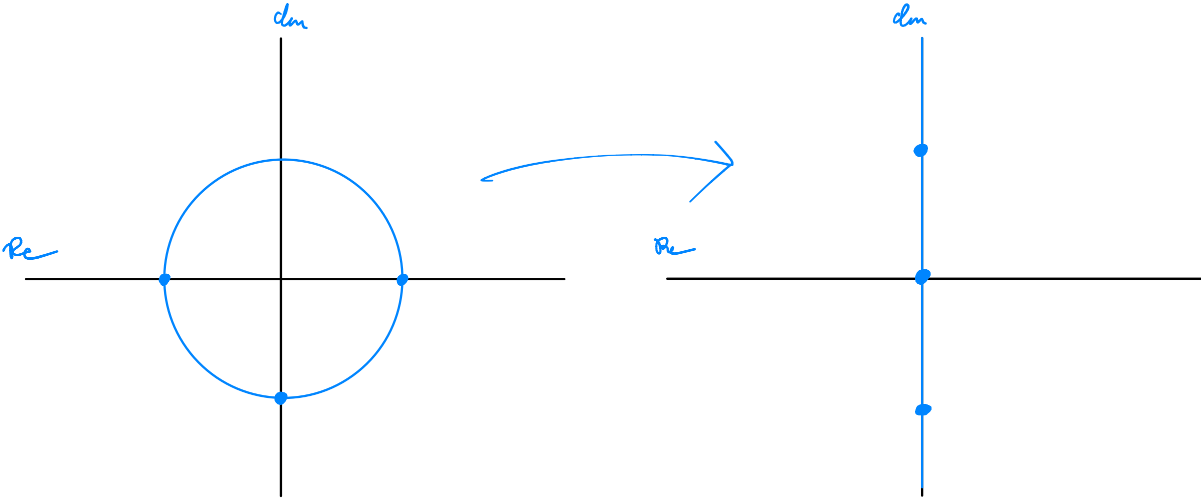
$$1 \longmapsto \frac{i+1}{i-1} = \frac{(i+1)(i+1)}{(i+1)(i-1)} = \frac{i^2+2i+1}{i^2-1} = \frac{2i}{-2} = -i$$

real conjugate should have a notation

I propose $\overset{\scriptscriptstyle \backslash}{z}$

$$-1 \longmapsto \frac{i-1}{i+1} = \frac{(i-1)(i-1)}{(i-1)(i+1)} = \frac{i^2-2i+1}{i^2-1} = \frac{-2i}{2} = i$$

$$-i \longmapsto 0$$



We see the Möbius transform takes the unit circle to the imaginary axis. Thus

$$\mathrm{Re}\left(\frac{i+z}{i-z}\right)=0 \qquad |z|=1$$

from which it follows that u also vanishes on the boundary.

<u>continuous except at i</u>

Since holomorphic functions are infinitely differentiable, so are its real & imaginary parts; i.e., they are C^∞. So u is very continuous except at i.