Problem 3.2 (HW05-2020). (a) Find the equilibrium point of the following system and calculate the stable an unstable curves of the equilibrium point. Sketch these curves and corresponding eigenspaces (E^s, E^u).

$$\dot{x} = -x$$
$$\dot{y} = y + x^2$$

(b) Solve the following nonlinear system and show that $z = -y^2/3 - x^2y/6 - x^4/30$ and $x = y = 0$ are the stable surface and unstable curve of its equilibrium point, respectively.

$$\dot{x} = -x$$
$$\dot{y} = -y + x^2$$
$$\dot{z} = z + y^2$$

a) $\dot{x} = -x = 0$ $\qquad$ $\dot{y} = y + x^2 = 0$

$\Rightarrow x = 0$ $\qquad$ $\Rightarrow y + 0^2 = 0$

$\Rightarrow y = 0$

The only equilibrium point is the origin. To find E^s and E^u, we find the Jacobian:

$$J(x,y) = \begin{bmatrix} -1 & 0 \\ 2x & 1 \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow E^s = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad E^u = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

To calculate the stable/unstable curves, we use a power series approximation. E^s is tangent to the x-axis, so we assume our curve takes the form $y = ax + bx^2 + cx^3 + \text{h.o.t.}$

Then $\dot{y} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} + \text{h.o.t.}$

$= -ax - 2bx^2 - 3cx^3 + \text{h.o.t.}$

$\Rightarrow ax + bx^2 + cx^3 + x^2 = -ax - 2bx^2 - 3cx^3$

$\Rightarrow ax + (b+1)x^2 + cx^3 = -ax - 2bx^2 - 3cx^3$

equating coefficients,

$a = -a \qquad b+1 = -2b \qquad c = -3c$

$\Rightarrow a = 0 \qquad \Rightarrow b = -1/3 \qquad \Rightarrow c = 0$

so our stable curve is $y = -\frac{1}{3}x^2 + \text{h.o.t.}$

Let's do it all again for E^u. This time assume the curve is given by $x = ay + by^2 + cy^3 + \text{h.o.t.}$ Then

$\dot{x} = a\dot{y} + 2by\dot{y} + 3cy^2\dot{y} + \text{h.o.t.}$

$= a(y + x^2) + 2by(y + x^2) + 3cy^2(y + x^2) + \text{h.o.t.}$

$= ay + ax^2 + 2by^2 + 2bx^2y + 3cy^3 + 3cx^2y^2 + \text{h.o.t.}$

$= ay + 2by^2 + 3cy^3 + \text{h.o.t.}$

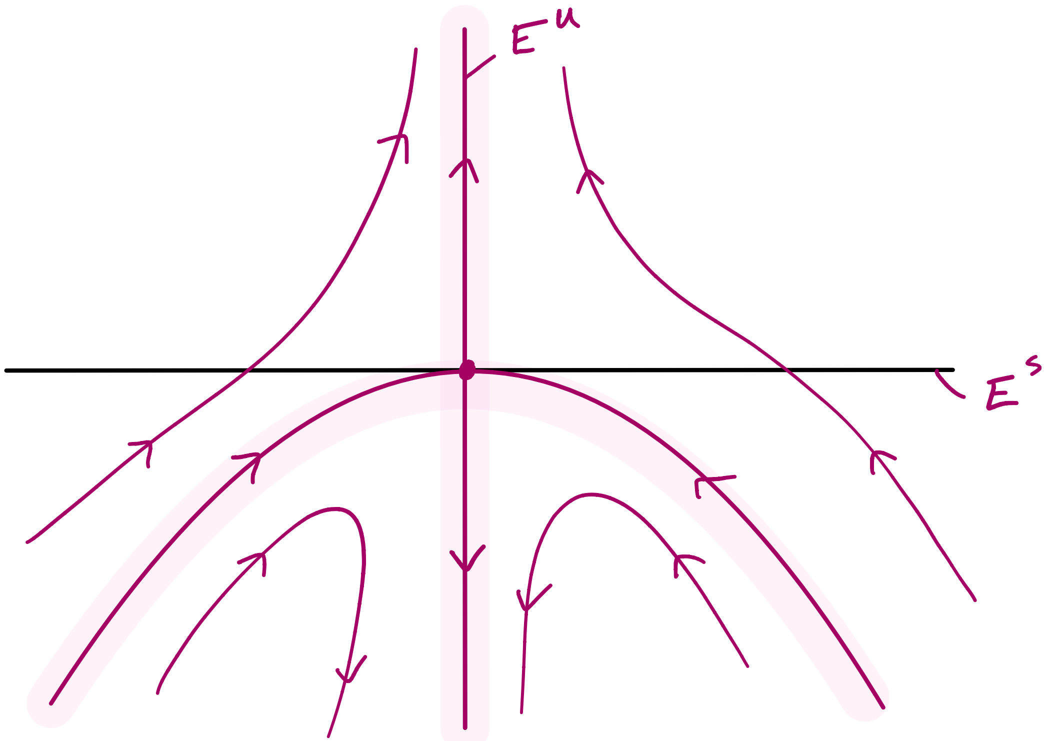
$\Rightarrow -ay - by^2 - cy^3 + \text{h.o.t.} = ay + 2by^2 + 3cy^3 + \text{h.o.t.}$

matching coefficients,

$-a = a \qquad -b = 2b \qquad -c = 3c$

$\Rightarrow a = 0 \qquad \Rightarrow b = 0 \qquad \Rightarrow c = 0$

the unstable curve is thus $x = 0 + \text{h.o.t.}$



E^u
E^s