85. **(Rudin Chapter 10 #3)** Suppose f and g are entire functions, and $|f(z)| \le |g(z)|$ for every z. What conclusion can we draw?

Let Z_g denote the set of zeros of g, and let $h(z) = \frac{f(z)}{g(z)}$. Then $|h(z)| \le 1$ for all $z \notin Z_g$. Note h is bounded in punctured neighborhoods about each $\zeta \in Z_g$, so each such ζ is a removable singularity of h. Then the function $\tilde{h}$ defined by

$$\tilde{h}(\zeta) = \begin{cases} h(\zeta) & \zeta \notin Z_g \\ \lim_{z \to \zeta} h(z) & \zeta \in Z_g \end{cases}$$

is bounded and entire.

Thus by Liouville's theorem $\tilde{h}(z) = w$ for some $w \in \mathbb{C}$.

$\Longrightarrow f(z) = w\,g(z)$, so we conclude f & g are scalar multiples of each other.