31. (**Dr. Stewart: F14 Final** ***Liz and James***) Find the least squares approximation to $f(x) = 1/(1+x)$ on the interval [0,1] by linear functions $p(x) = ax + b$.

We start by generating a set of orthogonal polynomials using G-S. With weight function $w(x) = 1$ and inner product $(f,g)_w = \int_0^1 fgw$:

$$P_k = \{1, x\} \qquad q_0 = 1$$

$$q_1 = x - \frac{\int_0^1 x \, dx}{\int_0^1 dx} = x - \frac{1}{2}$$

Then we find our least squares approximation, which is

$$\ell(x) = \sum_{k=0}^{n-1} \frac{\int_0^1 f(x) q_k(x) \, dx}{\int_0^1 q_k^2(x) \, dx} q_k(x)$$

$$= \underbrace{\int_0^1 \frac{1}{1+x} dx}_{\mathrm{I}} + \frac{\left.\int_0^1 \frac{x - 1/2}{1+x} dx \right] \mathrm{II}}{\left.\int_0^1 (x - 1/2)^2 dx \right] \mathrm{III}} (x - 1/2)$$

$$\mathrm{I}: \int_0^1 \frac{1}{1+x} dx = \int_1^2 \frac{1}{u} du = \ln 2$$

$$\mathrm{II}: \int_0^1 \frac{x - 1/2}{1+x} dx = \int_0^1 \frac{x}{1+x} dx - \frac{1}{2} \int_0^1 \frac{1}{1+x} dx$$

$$= \left(\int_0^1 1 - \frac{1}{1+x} dx \right) - \frac{\ln 2}{2}$$

$$= 1 - \ln 2 - \frac{\ln 2}{2}$$

$$= 1 - \frac{3\ln 2}{2}$$

III: $$\int_0^1 (x - 1/2)^2 \, dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{4 - 6 + 3}{12}$$

$$= \frac{1}{12}$$

Putting these together,

$$\ell(x) = \ln 2 + 12\left(1 - \frac{3\ln 2}{2}\right)\left(x - \frac{1}{2}\right)$$

$$= \ln 2 + (12 - 18\ln 2)\left(x - \frac{1}{2}\right)$$

$$= \ln 2 + 12x - 6 - 18\ln 2 \, x + 9\ln 2$$

$$= 10\ln 2 - 6 + x(12 - 18\ln 2)$$