3. **Fall2007.** Let T be a linear transformation on a complex vector space V, not necessarily finite dimensional. Let $\lambda_1, \dots, \lambda_s$ be distinct eigenvalues of T.

(a) Suppose that for each $j (1 \leq j \leq s)$, v_j s an eigenvector of T with eigenvalue λ_j. Prove that $\{v_1, \dots, v_s\}$ is linearly independent.

(b) Now suppose that for each j, v_j is a generalized eigenvector of T with eigenvalue j; that is, there is some integer $m_j \geq 1$ such that

$$(T - \lambda_j)^{m_j} v_j = 0.$$

Again conclude that $\{v_1, \dots, v_s\}$ g is linearly independent. (As a matter of notational convenience, assume each mj is chosen to be minimal; $(T - \lambda_j)^{m_j - 1} v_j \neq 0$.)

a) Assume $\{v_1, \dots, v_s\}$ is not LI. Let $k < s$ be the largest integer such that $\{v_1, \dots, v_k\}$ is LI. Then $v_{k+1} = \sum_{i=1}^{k} c_i v_i$, where at least one $c_i \neq 0$. Because all v_i are eigenvectors, we have

$$T v_{k+1} = T \sum_{i=1}^{k} c_i v_i$$

$$= \sum_{i=1}^{k} c_i T v_i$$

$$= \sum_{i=1}^{k} c_i \lambda_i v_i$$

and OTOH,

$$T v_{k+1} = \lambda_{k+1} v_{k+1}$$

$$= \lambda_{k+1} \sum_{i=1}^{k} c_i v_i$$

$$= \sum_{i=1}^{k} c_i \lambda_{k+1} v_i$$

So $Tv_{k+1} - Tv_{k+1} = \sum_{i=1}^{k} c_i \lambda_{k+1} v_i - \sum_{i=1}^{k} c_i \lambda_i v_i$

$$= \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) c_i v_i$$

$$= 0$$

As all λ_i are distinct, $(\lambda_{k+1} - \lambda_i) \neq 0$. Thus as the v_is are LI (not all the v_is, just these ones), we must have that $c_i \equiv 0$ $\lightning$

b) Recall a generalized eigenvector v_j of rank m_j satisfies

$$(T - \lambda_j)^{m_j} v_j = 0$$

for a minimal $m_j \in \mathbb{Z}^+$, i.e., $(T - \lambda_j)^{m_j - 1} v_j \neq 0$.

Assume $\{v_1, \dots, v_s\}$ are not LI; i.e., $\sum_{i=1}^{s} c_i v_i = 0$ with at least one nonzero c_i. We will show WLOG that $c_1 = 0$, and thus that all $c_i = 0$, which is a contradiction.

Let $w = (T - \lambda_1 I)^{m_1 - 1} v_1$. Then $(T - \lambda_1 I) w = 0$, meaning w is an eigenvector of T with eigenvalue λ_1.

Thus we know: $Tw = \lambda_1 w \Rightarrow Tw - \lambda_j w = \lambda_1 w - \lambda_j w$

$$\Rightarrow (T - \lambda_j I) w = (\lambda_1 - \lambda_j) w$$

$$\Rightarrow (T - \lambda_j I)^n w = (\lambda_1 - \lambda_j)^n w$$

$(T - \lambda_j I)(\lambda_1 - \lambda_j)^n w$
$= (T - \lambda_j I) w (\lambda_1 - \lambda_j)^n$
$= (\lambda_1 - \lambda_j) w (\lambda_1 - \lambda_j)^n$
$= (\lambda_1 - \lambda_j)^{n+1} w$

Let $n = \max\{m_i\}$. Then we can knock out all but one of the general eigenvectors by applying a bunch of appropriate transformations:

$$0 = \sum_{i=1}^{s} c_i v_i$$

order doesn't matter:

$$(T-\lambda_i I)(T-\lambda_j I) = T^2 - (\lambda_i + \lambda_j)T + \lambda_i \lambda_j I$$
$$= T^2 - (\lambda_j + \lambda_i)T + \lambda_i \lambda_j I$$
$$= (T-\lambda_j I)(T-\lambda_i I)$$

and then like do that forever

also on this side but who cares

$$\Rightarrow 0 = \left(\underbrace{(T-\lambda_1 I)^{m_1-1}}_{\text{is } w \text{ but no } v_1} \underbrace{\prod_{i=2}^{s} (T-\lambda_i I)^{n}}_{\text{kills all } v_i \text{ except } v_1} \right) \sum_{i=1}^{s} c_i v_i$$

$$= \left((T-\lambda_1 I)^{m_1-1} \prod_{i=2}^{s} (T-\lambda_i I)^{n} \right) c_1 v_1$$

$$= c_1 \left(\prod_{i=2}^{s} (T-\lambda_i I)^{n} \right) (T-\lambda_1 I)^{m_1-1} v_1$$

$$= c_1 \left(\prod_{i=2}^{s} (T-\lambda_i I)^{n} \right) \underset{\substack{\uparrow \\ \text{eigenvector!}}}{w}$$

$$= c_1 \prod_{i=2}^{s} (\lambda_1 - \lambda_i) w \quad \text{(do a lot of associativity)}$$

$$= 0$$

But each $(\lambda_1 - \lambda_i) \neq 0$, so $c_1 = 0$. The result follows because I said WLOG before so I win