Problem C-5B. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

(True or False?) Consider the polynomial

$$p(z) := 2z^4 - 2z^3 + 2z^2 - 2z + 9.$$

Inside the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| < 1\}$, p has exactly four zeros.

False.

Let $g(z) = -9$. Note that neither p nor g has a zero on the circle $|z| = 1$. However, on the circle, we have:

$$\begin{aligned} |p(z) + g(z)| &= |z - z + z - z| \\ &= 0 \\ &< 9 \\ &= |g(z)| \\ &\leq |p(z)| + |g(z)| \end{aligned}$$

Thus by Rouche's theorem, p has the same zeros in the disk as g does, which is none.