Problem R-3B. Let f be a nonnegative, integrable function on $[0,1]$ and let $0 < c < 1$ be a constant. Assume that

$$\int_E f \leq c \cdot m(E)$$

for each Lebesgue measurable set $E \subseteq [0,1]$. Prove that $f \leq 1$ a.e. on $[0,1]$.

Let $G = \{x \in [0,1] \mid f(x) > 1\}$. Assume for contradiction that $m(G) > 0$. Then by Chebyshev's inequality,

$$m(G) \leq \int_{[0,1]} f$$

$$\leq \int_G f \quad \text{(as } f \text{ is nonnegative)}$$

$$\leq c \cdot m(G) \quad \text{(given)}$$

$$< m(G) \quad \text{(as } 0 < c < 1\text{)} \quad \lightning$$

Therefore $m(G) = 0$ and we have shown the result.