Problem 3.3 (Exam 1 Fall 2019). For the following system compute the stable and unstable curves of the origin. Sketch the curves and corresponding eigenspaces.

$$\dot{x} = 2x + y^3$$
$$\dot{y} = -y$$

equilibria: $\dot{y} = -y = 0 \Rightarrow y = 0$ $\qquad$ $\dot{x} = 2x + y^3 = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

so the origin is our only equilibrium. We have

$$J(x,y) = \begin{bmatrix} 2 & 3y \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

our eigenvalues & eigenvectors here are $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\lambda_2 = -1,\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus E^s is the y-axis & E^u is the x-axis.

For the stable curve, assume it looks like $x = ay + by^2 + cy^3 + h.o.t.$

Then $\dot{x} = a\dot{y} + 2by\dot{y} + 3cy^2\dot{y} + h.o.t.$

$= -ay - 2by^2 - 3cy^3 + h.o.t.$

$$\Rightarrow 2ay + 2by^2 + (2c+1)y^3 + h.o.t = -ay - 2by^2 - 3cy^3 + h.o.t.$$

Matching coefficients,

$$2a = -a \qquad 2b = -2b \qquad 2c+1 = -3c$$

$$\Rightarrow a = 0 \qquad \Rightarrow b = 0 \qquad \Rightarrow c = -\frac{1}{5}$$

Thus our stable curve is $x = -\frac{1}{5}y^3 + \text{h.o.t.}$

For the unstable curve, we do the same thing, but we assume our curve takes the form $y = ax + bx^2 + cx^3 + \text{h.o.t.}$ Then

$$\dot{y} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} + \text{h.o.t.}$$

$$= a(2x+y^3) + 2bx(2x+y^3) + 3cx^2(2x+y^3) + \text{h.o.t.}$$

$$= 2ax + ay^3 + 4bx^2 + 2bxy^3 + 6cx^3 + 3cx^2y^3 + \text{h.o.t.}$$

$$= 2ax + 4bx^2 + 6cx^3 + \text{h.o.t.}$$

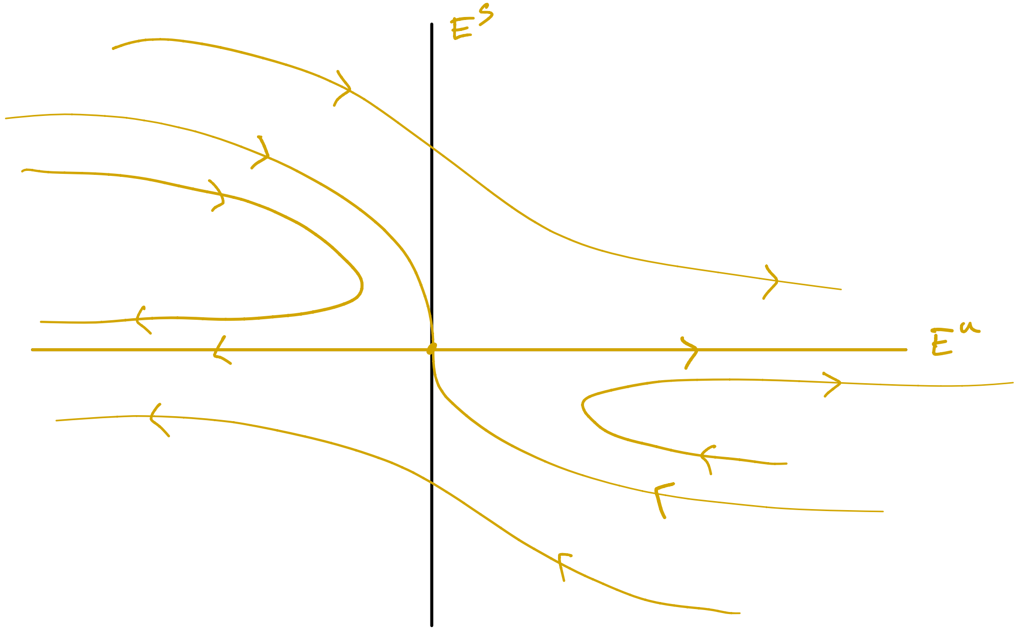
$$\Rightarrow -ax - bx^2 - cx^3 + \text{h.o.t.} = 2ax + 4bx^2 + 6cx^3 + \text{h.o.t.}$$

Matching coefficients,

$$-a = 2a \qquad -b = 4b \qquad -c = 6c$$

$$\Rightarrow a = 0 \qquad \Rightarrow b = 0 \qquad \Rightarrow c = 0$$

Thus our unstable curve is $y = 0 + \text{h.o.t.}$


E^s
E^u