Consider the system

$$\begin{cases} \dot{x} = -y + x(r^4 - 3r^2 + 1) \\ \dot{y} = x + y(r^4 - 3r^2 + 1) \end{cases} \qquad (r = x^2 + y^2)$$

a) Use the Poincare-Bendixson theorem to show there exists a periodic solution inside $r = 1$. You may use the fact that $(0,0)$ is a source without proof.

b) Use the Poincare-Bendixson theorem to show there exists another periodic solution inside the annular region $1 < r < 2$.

We first find our equilibrium points:

$$\dot{x} = -y + x(r^4 - 3r^2 + 1) = 0 \qquad \dot{y} = x + y(r^4 - 3r^2 + 1) = 0$$

$$\Rightarrow y = x(r^4 - 3r^2 + 1) \qquad \Rightarrow x = -y(r^4 - 3r^2 + 1)$$

$$\Rightarrow x = -x(r^4 - 3r^2 + 1)^2$$

$$\Rightarrow x\Big(\underbrace{1 + (r^4 - 3r^2 + 1)^2}_{>0}\Big) = 0$$

$$\Rightarrow x = 0$$

$$\Rightarrow y = 0$$

So the origin is our only equilibrium point. We are given it is a source, so it is unstable.

Let (x,y) be a normal vector on $r=1$. We'll take its dot product
pointing out
with the gradient vector:

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -y + x(r^4 - 3r^2 + 1) \\ x + y(r^4 - 3r^2 + 1) \end{bmatrix}$$

$$= -xy + x^2(r^4 - 3r^2 + 1) + xy + y^2(r^4 - 3r^2 + 1)$$

$$= \underbrace{(x^2 + y^2)}_{=1}(r^4 - 3r^2 + 1)$$

$$= 1 - 3 + 1$$

$$= -1 < 0$$

Thus the gradient vector on the boundary is pointing <u>inside</u> the region $r < 1$.

Since we have the forward orbit of vectors thus being contained in an annular region (the origin is a source) with no equilibria (see: the origin) P-B applies & we must have a periodic orbit.

b) If we consider a change of variables, namely $t \to -t$, we can work out the same math as before with $\begin{bmatrix} -x \\ -y \end{bmatrix}$ being our outward normal. We have:

$$\begin{bmatrix} -x \\ -y \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \begin{bmatrix} -y + x(r^4 - 3r^2 + 1) \\ x + y(r^4 - 3r^2 + 1) \end{bmatrix}$$

$\Rightarrow$

$$= xy - x^2(r^4 - 3r^2 + 1) - xy - y^2(r^4 - 3r^2 + 1)$$

$$= -(x^2 + y^2)(r^4 - 3r^2 + 1)$$

so on $r = 1$, this is $-(1)(1 - 3 + 1) > 0$ & the gradient points out.

but on $r = 2$, this is $-(2)(16 - 12 + 1) < 0$ & we have inward pointage. So we have a periodic solution by Poincare - Bendixson, but it is only stable in backward time.