**Problem 2.5.** (a) Show that the function $x(t) = 3e^t$ is a the solution to the following differential equation

$$\dot{x} = x^2 - 3e^t x + 3e^t$$

with initial condition $x(0) = 3$.

(b) Give the variational equaions associated to this above initial value problem.

(c) Explicitly give the sensitivity function defined by

$$S(t) = \frac{\partial \varphi}{\partial x}(t, x_0)$$

evaluated at $x_0 = 3$. Here $\varphi(t, x_0)$ represents the flow map satisfying $\varphi(0, x_0) = x_0$.

a) We have $(3e^t)' = 3e^t$ and

$$\dot{x} = (3e^t)^2 - 3e^t(3e^t) + 3e^t$$

$$= 9e^{2t} - 9e^{2t} + 3e^t$$

$$= 3e^t \checkmark$$

initial condition: $3e^{(0)} = 3 \checkmark$

b) the variational equations are given by $u'(t) = A(t)u$, where $A = J(t, \phi(t, x_0))$ is the Jacobian. Here, this is just the same as taking a derivative. So

$$J(t,x) = D_x(x^2 - 3e^t x + 3e^t)$$
$$= 2x - 3e^t$$

$$\Rightarrow J(t, \phi(t,x_0)) = 2\phi(t,x_0) - 3e^t$$

For this problem, $\phi(t,x_0) = \phi(t,3) = 3e^t$ by part a. So the above equates to $3e^t$, & our variational equations are given by

$$u'(t) = 3e^t u \qquad u(0) = u_0 \text{ for } u_0 \text{ small}$$

c) We first solve the variational equations:

$$\Rightarrow \frac{du}{dt} = 3e^t u$$

$$\Rightarrow \frac{1}{u}\, du = 3e^t\, dt$$

$$\Rightarrow \int \frac{1}{u}\, du = \int 3e^t\, dt$$

$$\Rightarrow \ln u = 3e^t + \ln C$$ (because I can)

$$\Rightarrow u = Ce^{3e^t}$$

Solve for $C$:

$$u(0) = Ce^3$$

$$\Rightarrow C = u_0 e^{-3}$$

$$\Rightarrow u = u_0 e^{3(e^t - 1)}$$

Then

$$S(t) = \frac{\partial \phi}{\partial x}(t, x_0)$$

$$= \frac{\partial \phi}{\partial x}(t, 3)$$

$$= \lim_{u_0 \to 0} \frac{\phi(t, u_0 + 3) - \phi(t, 3)}{u_0}$$

$$= \lim_{u_0 \to 0} \frac{u_0 e^{3(e^t - 1)}}{u_0} = e^{3(e^t - 1)}$$