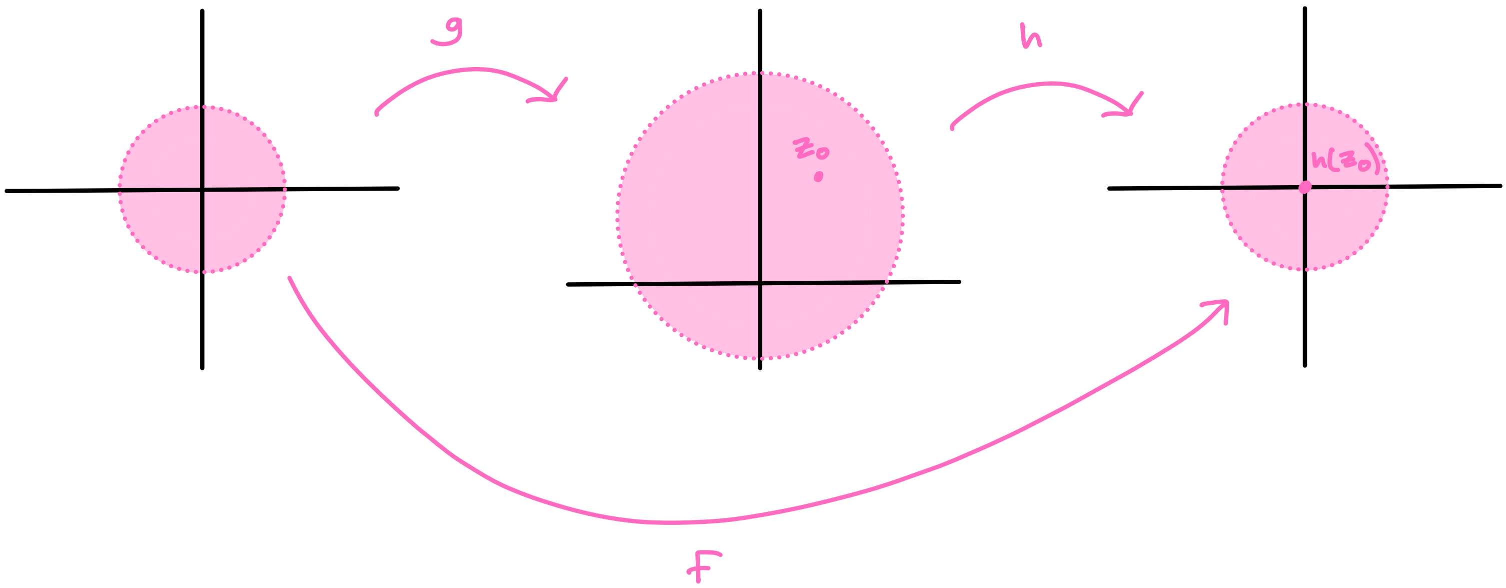
148. **(Practice Final - Dr. Curto)** Let h be an isomorphism of the disc $B(i;2)$ with $\mathbb{D}$, and assume that $h(z_0) = 0$ for a (necessarily unique) point $z_0 \in B(i;2)$. Show that

$$h(z) = \frac{2(z - z_0)}{4 - (z-i)(\overline{z_0} + i)} e^{i\theta}$$

?

for some real number θ and for all $z \in B(i;2)$.

Let $g(z) = 2z + i$. Then $f = h \circ g$ is an automorphism of the unit disk:



We know then that f takes the form $f(z) = \dfrac{z_1 - z}{1 - \overline{z_1} z} e^{i\theta}$, where $\theta \in \mathbb{R}$ and $f(z_1) = 0$. (Here, $f(g^{-1}(z_0)) = 0$.)

We want to find $h(z)$. Note $h = f \circ g^{-1}$, so we just need to calculate g^{-1} and plug everything in. To do the former, let $w = g(z)$. Then

$$w = 2z + i$$

$$\Longrightarrow z = \frac{w - i}{2}$$

$$\Longrightarrow g^{-1}(w) = \frac{w - i}{2}$$

In particular, $g^{-1}(z_0) = \dfrac{z_0 - i}{2}$ and $\overline{g^{-1}(z_0)} = \dfrac{\overline{z_0} + i}{2}$.

Putting everything together,

$$h(z) = (f \circ g^{-1})(z) = f\left(\frac{z-i}{2}\right)$$

$$= \frac{g^{-1}(z_0) - \frac{z-i}{2}}{1 - \overline{g^{-1}(z_0)}\,\frac{z-i}{2}} e^{i\theta}$$

$$= \frac{\frac{z_0 - i - z + i}{2}}{1 - \left(\frac{\overline{z_0}+i}{2}\right)\left(\frac{z-i}{2}\right)} e^{i\theta}$$

$$= \frac{2(z_0 - z)}{4 - (\overline{z_0}+i)(z-i)} e^{i\theta}$$ (multiply by $4/4$)