Real Analysis Final
Fall 2021
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1. (12 points) Define:
   (a) (2 points) Uniform convergence of a sequence of functions on a set $E \subseteq \mathbb{R}$.
   (b) (3 points) Variation of a function f on a closed, bounded interval $[a, b]$, with respect to a partition P of $[a, b]$.
   (c) (2 points) Point of closure of a subset E of a metric space X.
   (d) (2 points) Equicontinuity for a colleciton $\mathcal{F}$ of real-valued functions on a metric space X.
   (e) (3 points) State the Cantor intersection theorem.

2. (8 points) Let f be a continuous function on a closed, bounded, nondegenerate interval $[a, b]$ such that

   (i) f is of bounded variation on $[a, b]$; and

   (ii) f maps sets of measure zero to sets of measure zero; that is, for E a measurable subset of $[a, b]$, $m(E) = 0 \implies m(f(E)) = 0$.

   Prove that f is absolutely continuous on $[a, b]$.

3. (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, and define

$$\mu(E) \coloneqq m(f^{-1}(E)) \quad (E \subseteq \mathbb{R}, E \text{ Borel}).$$

   Prove:
   (a) $\mu(E) \geq 0$ for all E Borel;
   (b) μ is monotone;
   (c) μ is countably additive;
   (d) μ is not translation invariant; e.g., find a counterexample of a Borel function f and a set E such that $\mu(E + 1) \neq \mu(E)$, where $E + 1 \coloneqq \{x + 1 \mid x \in E\}$.