ODE MIDTERM II
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1. (20 points) Consider

$$\begin{aligned} \dot{x} &= x^2 - y^2 \\ \dot{y} &= y \end{aligned} \tag{1}$$

(a) Use the power series method to find the quadratic approximation for U, the unstable manifold of the equilibrium $(0,0)$.

(b) Sketch U and a few other solutions of this system in the xy-plane. (Hint: $y = 0$ is invariant under the flow.)

(c) Write down the linearization of (1) at $(0,0)$. Find its equilibrium points and draw the phase portrait.

(d) Compare solutions of the nonlinear system and its linearization. Is there a conjugacy between the two systems in a neighborhood of $(0,0)$? If yes, construct a conjugacy. If no, explain why this does not violate the Hartman-Grobman theorem.

2. (20 points) Let

$$g(x) := \frac{f(x)}{1 + |f(x)|},$$

where $f : E \to \mathbb{R}^n$ is locally Lipschitz and E is an open subset in $\mathbb{R}^n$ containing x_0. Prove the solution of

$$x' = g(x),\ x(0) = x_0 \tag{2}$$

exists on $(-\infty, \infty)$ by showing the following:

(a) For $g(x)$ locally Lipschitz, if there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in E$, then the solution to $x' = g(x)$, $x(0) = x_0$ exists on $(-\infty, \infty)$.

(b) Show $g(x) = \frac{f(x)}{1+|f(x)|}$ is locally Lipschitz.

(c) Combine (a) and (b) to prove the solution to (2) exists for all t in $(-\infty, \infty)$.

3. (10 points) Consider

$$\begin{aligned} \dot{x} &= -x + 2y^3 - 2y^4 \\ \dot{y} &= -x - y + xy \end{aligned}$$

Determine the stability (not asymptotic stability, instability, or asymptotic instability) of the equilibrium point (0,0) by finding an appropriate Liapunov function. Hint: Consider $L(x,y) = x^m + y^n$. Determine m and n based on the conditions of the Liapunov theorem.

4. (20 points) The following system is a Hamiltonian system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^2 - x \end{aligned}$$

(a) Find a Hamiltonian function $H(x,y)$ and its critical points.

(b) Consider

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial x} \\ \dot{y} &= -\frac{\partial H}{\partial y} \end{aligned}$$

where H is the function you found in part (a). Use an appropriate Liapunov function to determine the stability of the origin in the above system. (Hint: You can't directly use the properties of the Hamiltonian or gradient systems.)