ODE Midterm I
Fall 2022
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1. (20 points) For the following one-dimensional autonomous sytem

$$x' = ax + x^3, \quad x \in \mathbb{R}$$

   (a) Find all the equilibrium points as a function of the parameter $a \in \mathbb{R}$.

   (b) Determine their stability.

   (c) Sketch the bifurcation diagram with respect to the parameter a in the (a, x)-plane.

2. (20 points) Consider the linear system $\dot{X} = AX$ where

$$A = \begin{bmatrix} -1 & 1 & -2 \\ -1 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Given that the complex eigenvalues of A are $-1 \pm i$ and the eigenvector associated with $-1 + i$ is $[1 \quad i \quad 0]^\intercal$, compute the third eigenvalue and the corresponding eigenvector.

   (b) Determine the stable, center, and unstable subspaces E^s, E^c, and E^u. Sketch the phase portrait on E^s.

   (c) Compute e^{tA} (write it as a single matrix).

3. (30 points) Consider a linear system $\dot{X} = A(t)X$ where $A(t)$ is a T-periodic matrix. Let ρ be a Floquet multiplier and r be the corresponding Floquet exponent (i.e., $\rho = e^{rT}$).

   (a) Show that there exists a solution $Y(t)$ such that $Y(t + T) = \rho Y(t)$.

   (b) $Y(t)$ can be written as $Y(t) = e^{rt}q(t)$. Prove that $q(t)$ is a T-periodic function.