Analysis PhD Qualifying Exam
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Part I: Real Analysis

Choose either problem **R-1A** or **R-1B** below and solve it.

R-1.A. (10 points) Let $E := (0, +\infty)$, $g \in L^1(E)$, and define $f : E \to \mathbb{R}$ by

$$f(x) := \int_{(0,x)} g \ (x \in E).$$

(a) Prove that f is absolutely continuous on E.

(b) Prove that f may fail to be Lipschitz on E, that is, find $g \in L^1(E)$ such that the associated f is not Lipschitz on E.

(c) Prove that when $g \in L^\infty(E)$, then f is Lipschitz.

(d) For $g \in L^\infty$, find the best Lipschitz constant for f.

R-1B. (10 points) Let $\{E_k\}_{k=1}^n$ be a finite family of measurable subsets of $[0,1]$. Assume that every $x \in [0,1]$ belongs to at least three sets in the family. Prove that there exists $k = 1, \ldots, n$ such that

$$m(E_k) \geq \frac{3}{n},$$

where m denotes Lebesgue measure.

R-2. (10 points) Consider the sequence of real valued functions on $[0,1]$ given by

$$f_n(x) := \begin{cases} 2n & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & x \in \left[0, \frac{1}{2n}\right) \cup \left(\frac{1}{n}, \right] . \end{cases}$$

(a) Find

$$\int_0^1 \lim_n f_n.$$

(b) Find

$$\lim_n \int_0^1 f_n.$$

(c) Does Fatou's Lemma apply to the sequence $\{f_n\}$? Why or why not?

(d) Does the Lebesgue Dominated Convergence Theorem apply to the sequence $\{f_n\}$? Why or why not?