Real Analysis Final
Fall 2021
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1. (12 points) Define:
   (a) (2 points) Uniform convergence of a sequence of functions on a set $E \subseteq \mathbb{R}$.
   (b) (3 points) Variation of a function f on a closed, bounded interval $[a, b]$, with respect to a partition P of $[a, b]$.
   (c) (2 points) Point of closure of a subset E of a metric space X.
   (d) (2 points) Equicontinuity for a colleciton $\mathcal{F}$ of real-valued functions on a metric space X.
   (e) (3 points) State the Cantor intersection theorem.

2. (8 points) Let f be a continuous function on a closed, bounded, nondegenerate interval $[a, b]$ such that

   (i) f is of bounded variation on $[a, b]$; and

   (ii) f maps sets of measure zero to sets of measure zero; that is, for E a measurable subset of $[a, b]$, $m(E) = 0 \implies m(f(E)) = 0$.

   Prove that f is absolutely continuous on $[a, b]$.

3. (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, and define

$$\mu(E) \coloneqq m(f^{-1}(E)) \quad (E \subseteq \mathbb{R}, E \text{ Borel}).$$

   Prove:
   (a) $\mu(E) \geq 0$ for all E Borel;
   (b) μ is monotone;
   (c) μ is countably additive;
   (d) μ is not translation invariant; e.g., find a counterexample of a Borel function f and a set E such that $\mu(E + 1) \neq \mu(E)$, where $E + 1 \coloneqq \{x + 1 \mid x \in E\}$.

4. (20 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Royden-Fitzpatrick."
   (a) (4 points) True or false? Let $\mathcal{F}$ be a collection of measurable functions on $\mathbb{R}$, and let

$$g \coloneqq \sup_{f \in \mathcal{F}} f.$$

   Then g is measurable on $\mathbb{R}$.
   (b) (6 points) True or false? Consider the normed linear space X of Riemann integrable functions on $[0, 1]$, with the norm $\|f\|_R \coloneqq (R) \int_0^1 |f(x)|dx$. Then X is a Banach space. (Hint: the sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions given by

$$f_n(x) \coloneqq \begin{cases} 1 & x = \frac{i}{k}, \text{ if } 1 \leq k \leq n \text{ and } 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

   The sequence $\{f_n\}$ is increasing and $f_n \to \chi_{\mathbb{Q}}$ as $n \to \infty$.)
   (c) (4 points) True or false? Recall that ℓ^∞ is the Banach space of real bounded sequences, equipped with the supremum norm. The space ℓ^∞ is separable. (Hint: $2^{\mathbb{N}}$ is not countable.)

(d) (6 points) True or false? Let g be strictly increasing and absolutely continuous on a closed, bounded, nondegenerate interval $[a, b]$, and let $\mathcal{O}$ be an open subset of (a, b). Then

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'.$$

5. (16 points) (a) (6 points) A Hilbert space $\mathcal{H}$ may be characterized as a Banach space in which the so-called parallelogram law holds; that is, for each pair of vectors x and y in $\mathcal{H}$, we have

$$\|x+y\|_{\mathcal{H}}^2 + \|x-y\|_{\mathcal{H}}^2 = 2 \cdot \|x\|_2^{\mathcal{H}} + 2 \cdot \|y\|_{\mathcal{H}}^2.$$

Using only characteristic functions of measurable subsets of $[0, 1]$, prove that $L^1([0, 1])$ is not a Hilbert space.

(b) (10 points) (Embedding of ℓ^p in $L^p[1, \infty)$.) Let $1 \le p < \infty$. For $a = (a_1, a_2, \dots) \in \ell^p$, define $f_a : [1, \infty) \to \mathbb{R}$ by

$$f_a(x) \coloneqq \sum_{k=1}^{\infty} a_k \chi_{[k,k+1)}(x) \quad (x \in [1, \infty)).$$

i. (2 points) Prove that $f_a \in L^p[1, \infty)$.
ii. (2 points) Prove that $\|f_a\|_p = \|a\|_p$.
iii. (3 points) Prove that the map $T : \ell^p \to L^p[1, \infty)$ given by $T(a) \coloneqq f_a$ is linear and injective, but not surjective.
iv. (3 points) True or false? The range of T is dense in $L^p[1, \infty)$.

6. (6 points) Let f be a Lipschitz function on $[0, 1]$ and let g an absolutely continuous function on $[0, 1]$. Prove that $f \circ g$ is absolutely continuous on $[0, 1]$.

7. (16 points) On a closed, bounded, nondegenerate interval $[a, b]$, consider a sequence $\{f_n\}$ of increasing, absolutely continuous functions on $[a, b]$. Assume:

(i) $f_n(a) = 0$ for all $n \in \mathbb{N}$;

(ii) $f_n' \le f_{n+1}'$ a.e. on $[a, b]$, for all $n \in \mathbb{N}$;

(iii) The sequence $\{f_n(b)\}$ is bounded.

Prove:

(a) (4 points) There exists $f : [a, b] \to \mathbb{R}$ such that $\{f_n\} \to f$ pointwise on $[a, b]$;
(b) (6 points) f is absolutely continuous on $[a, b]$; and
(c) (6 points) $\{f_n\}$ converges to f uniformly on $[a, b]$.

8. (14 points) (a) (7 points) For a nonempty subset E of a metric space (X, ρ) and a point $x \in X$, define the distance from x to E by

$$\operatorname{dist}(x, E) \coloneqq \inf\{\rho(x, y) \mid y \in E\}$$

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i. (4 points) Show that f is continuous on X.
ii. (3 points) Show that $\overline{E} = f^{-1}(\{0\})$.

(b) (7 points) Now define $f : X \to \mathbb{R}$ by

$$f(x) \coloneqq \operatorname{dist}(x, E) \quad (x \in X).$$

i. (4 points) Assume that F is closed, and E is compact. Prove that

$$E \cap F = \varnothing \iff \operatorname{dist}(F, E) > 0.$$

ii. (3 points) Prove that the previous statement is not true if E and F are merely closed. That is, find an example of two closed sets E and F for which $\operatorname{dist}(F, E) = 0$ and $E \cap F = \varnothing$. (Hint: To produce an example in $\mathbb{R}$, by Heine-Borel both sets need to be closed and unbounded, e.g., consider $E \coloneqq \mathbb{N}$.)