Real Analysis Midterm II
Fall 2021
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1. (12 points) Define:
   (a) (4 points) The upper Lebegue integral of a bounded measurable function.
   (b) (4 points) Absolute continutity for a real valued function on a closed, bounded interval $[a, b]$.
   (c) (4 points) Convergence in measure.
2. (6 points) Assume that f is integrable over E, and let
$$F := \{x \in E \mid f(x) \neq 0\}.$$
   Prove that F can be written as a countable union
$$F = \bigcup_{n=1}^{\infty} F_n,$$
   where $m(F_n) < \infty$ for every $n \in \mathbb{N}$.
3. (20 points) On $\mathbb{R}$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let f be a measurable function. We say that $\{f_n\}$ converges to f in probability if for every measurable subset of $\mathbb{R}$ of finite measure and for every $\eta > 0$,
$$m(\{x \in F \mid |f_n(x) - f(x)| > \eta\}) \to 0,$$
   as $n \to \infty$.
   (a) (4 points) Prove that convergence in measure implies convergence in probability.
   (b) (16 points) Consider now the sequence
$$f_n := \chi_{[n,n+1]}.$$
      i. (4 points) Prove that $f_n \to 0$ pointwise almost everywhere.
      ii. (8 points) Prove that $f_n \to 0$ in probability. (Hint: a set of finite measure is, up to small measure, always contained in a bounded interval.)
      iii. (4 points) Prove that no subsequence f_{n_k} converges to the function zero in measure.
4. (14 points) On the closed interval $[0, 1]$, define a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions by
$$f_n(x) := \begin{cases} 1 & x = \frac{i}{k}, \text{ if } 1 \leq k \leq n \text{ and } 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$
   For instance,
$$f_1(x) := \begin{cases} 1 & x = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$
$$f_2(x) := \begin{cases} 1 & x = 0, \frac{1}{2}, 1 \\ 0 & \text{otherwise.} \end{cases}$$
$$f_1(x) := \begin{cases} 1 & x = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) (4 points) Prove that $\{f_n\}$ is an increasing sequence.

(b) (6 points) Prove that $f_n \to \chi_{\mathbb{Q}}$ as $n \to \infty$.

(c) (4 points) Use this result to prove that the monotone convergence theorem does not hold for the Riemann integral.

5. (28 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Royden-Fitzpatrick."

   (a) (10 points) Let E be a measurable subset of $[0, 1]$. Consider the function $f : [0, \pi] \to \mathbb{R}$ given by

   $$f(x) := m(E \cap [0, x]).$$

      i. (6 points) Is f absolutely continuous on $[0, \pi]$?

      ii. (4 points) Is f differentiable at $x = 2$?

   (b) (6 points) True or false? Every bounded measurable function defined on $[0, 1]$ is the uniform limit of step functions.

   (c) (6 points) True or false? On $[0, \infty)$, the sequence $\{\chi_{[n,\infty)}\}_{n=1}^{\infty}$ converges to the function zero in measure.

   (d) (6 points) True or false? On $\mathbb{R}$, the sequence $\{\frac{1}{n}\chi_{[n,n+1]}\}_{n=1}^{\infty}$ converges to the function zero uniformly.

6. (6 points) On $[0, \infty)$, let $f(x) = x$, and for $n \in \mathbb{N}$ consider the sequence of functions

   $$f_n(x) := x + \frac{1}{n}.$$

   Prove that $f_n \to f$ in measure, but $f_n^2 \not\to f^2$ in measure.

7. (6 points) Prove that

   $$\lim_n \int_0^1 e^{-\sin^2(nx)x^n dx = 0.}$$

8. (8 points) Recall that Dirichlet's function $\chi_{\mathbb{Q}}$ is not Riemann integrable. Consider Thomae's function

   $$f(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   (a) (6 points) Prove that f is Riemann integrable. (Hint: first determine the set of points of discontinuity for f.)

   (b) (2 points) Find

   $$\int_a^b f,$$

   where $\int$ denotes the Riemann integral.