

## Part A

1. Prove that for every group  $G = \langle g_1, \dots, g_k \mid r_1, \dots, r_k \rangle$  there is a 2-dimensional cell complex  $X_G$  with  $\pi_1(G) \simeq G$ .
2. (a) Let  $\alpha$  be an oriented closed loop in a topological space  $X$ . Let  $x_0 \in X$  and  $p, q \in \alpha$  be points. (They are possibly the same points.) Let  $\gamma$  (resp.  $\delta$ ) be an oriented path in  $X$  that starts at  $x_0$  and ends at  $p$  (resp.  $q$ ).  
Show that homotopy classes  $[\gamma * \alpha * \bar{\gamma}]$  and  $[\delta * \alpha * \bar{\delta}]$  in  $\pi_1(X, x_0)$  are conjugate to each other.  
(b) Let  $x_0$  and  $x'_0$  be points in  $X$ . Show that  $\pi_1(X, x_0)$  and  $\pi_1(X, x'_0)$  are group isomorphic.
3. Let  $D$  be the unit disk in  $\mathbb{R}^2$ . Show that every continuous map  $h : D \rightarrow D$  has a fixed point.
4. Let  $i = 1, 2$ . Let  $h_i : D^2 \times S^1 \rightarrow A_i$  be a homeomorphism. The boundary of  $A_i$  is a torus  $\partial A_i = \partial D \times S^1 = S^1 \times S^1$ . We define two curves on  $\partial A_i$ , a meridian  $m_i$  and a longitude  $l_i$ , as follows: Fix points  $x \in \partial D^2$  and  $y \in S^1$ . The meridian is defined by  $m_i = h_i(\partial D^2 \times \{y\})$  and the longitude is defined by  $l_i = h_i(\{x\} \times S^1)$ .  
Let  $(p, q)$  be coprime integers. Let  $\phi : \partial A_1 \rightarrow \partial A_2$  be a homeomorphism that takes the meridian  $m_1$  to a simple closed curve whose homotopy type is equal to  $p[m_2] + q[l_2]$ . Construct a space  $X$  gluing the solid tori  $A_1$  and  $A_2$  along their boundary using the map  $\phi$ .  
Using the van Kampen theorem, compute the fundamental group of the space  $X$ .
5. Let  $X = S^1 \vee S^1$ .  
(a) Find a  $3 : 1$  (non-trivial) normal covering of  $X$  and find the corresponding normal subgroup of  $\pi_1(X)$ .  
(b) Find a  $3 : 1$  non-normal covering of  $X$ .  
(c) Describe the universal covering of  $X$ .
6. (a) Describe a cell-complex structure on the  $n$ -dimensional real projective space  $\mathbb{R}P^n$  where  $n \geq 2$ .  
(b) Find a non-trivial covering of  $\mathbb{R}P^n$  and compute its deck transformation group.

## Part B

1. Let  $S$  be the unit sphere in  $\mathbb{R}^3$ . Find a  $C^\infty$  atlas on  $S$  that consists of two charts.
2. State the regular level set theorem, then show that the unit sphere  $S$  is a 2-dimensional manifold.
3. Let  $U$  be an open subset of  $\mathbb{R}^n$ . Show that the exterior derivative  $d : \Omega^*(U) \rightarrow \Omega^*(U)$  satisfies the *cocycle condition*; that is,  $d^2 = 0$ .
4. (a) Show that the map  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$\phi(x, y, z) = (2y, -x, -xy + z)$$

is a diffeomorphism.

- (b) Let  $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$  be a vector field on  $\mathbb{R}^3$ . Compute  $\phi_*(X)$  at  $p = (x, y, z)$ .
  - (c) Let  $\alpha = dz - ydx$  be a 1-form on  $\mathbb{R}^3$ . Compute the pullback  $\phi^*(\alpha)$  at  $p = (x, y, z)$ .
5. Let  $\omega$  be a 2-form on the unit sphere  $S$  in  $\mathbb{R}^3$  defined by:

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz \wedge dx}{y} & \text{for } y \neq 0, \\ \frac{dx \wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

Show that this  $\omega$  is well-defined. (In other words, if  $x, y, z \neq 0$  then the three expressions give the same 2-form.) Then compute the integral  $\int_S \omega$ .

6. Let  $O(n)$  be the orthogonal group, the group of linear transformations of  $\mathbb{R}^n$  that preserve distance. In other words, the group of  $n \times n$  matrices such that  $AA^T = I$  where  $A^T$  is the transpose of  $A$  and  $I$  is the identity matrix. Show that  $O(n)$  is a manifold of dimension  $n(n-1)/2$ .