

REAL ANALYSIS FINAL  
FALL 2021  
RAÚL CURTO

1. (12 points) Define:
  - (a) (2 points) Uniform convergence of a sequence of functions on a set  $E \subseteq \mathbb{R}$ .
  - (b) (3 points) Variation of a function  $f$  on a closed, bounded interval  $[a, b]$ , with respect to a partition  $P$  of  $[a, b]$ .
  - (c) (2 points) Point of closure of a subset  $E$  of a metric space  $X$ .
  - (d) (2 points) Equicontinuity for a collection  $\mathcal{F}$  of real-valued functions on a metric space  $X$ .
  - (e) (3 points) State the Cantor intersection theorem.
2. (8 points) Let  $f$  be a continuous function on a closed, bounded, nondegenerate interval  $[a, b]$  such that
  - (i)  $f$  is of bounded variation on  $[a, b]$ ; and
  - (ii)  $f$  maps sets of measure zero to sets of measure zero; that is, for  $E$  a measurable subset of  $[a, b]$ ,  $m(E) = 0 \implies m(f(E)) = 0$ .Prove that  $f$  is absolutely continuous on  $[a, b]$ .

3. (8 points) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel function, and define

$$\mu(E) := m(f^{-1}(E)) \quad (E \subseteq \mathbb{R}, E \text{ Borel}).$$

Prove:

- (a)  $\mu(E) \geq 0$  for all  $E$  Borel;
  - (b)  $\mu$  is monotone;
  - (c)  $\mu$  is countably additive;
  - (d)  $\mu$  is not translation invariant; e.g., find a counterexample of a Borel function  $f$  and a set  $E$  such that  $\mu(E + 1) \neq \mu(E)$ , where  $E + 1 := \{x + 1 \mid x \in E\}$ .
4. (20 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Royden-Fitzpatrick."
  - (a) (4 points) True or false? Let  $\mathcal{F}$  be a collection of measurable functions on  $\mathbb{R}$ , and let

$$g := \sup_{f \in \mathcal{F}} f.$$

Then  $g$  is measurable on  $\mathbb{R}$ .

- (b) (6 points) True or false? Consider the normed linear space  $X$  of Riemann integrable functions on  $[0, 1]$ , with the norm  $\|f\|_R := (R) \int_0^1 |f(x)| dx$ . Then  $X$  is a Banach space. (Hint: the sequence  $\{f_n\}_{n=1}^\infty$  of measurable functions given by

$$f_n(x) := \begin{cases} 1 & x = \frac{i}{k}, \text{ if } 1 \leq k \leq n \text{ and } 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

The sequence  $\{f_n\}$  is increasing and  $f_n \rightarrow \chi_{\mathbb{Q}}$  as  $n \rightarrow \infty$ .)

- (c) (4 points) True or false? Recall that  $\ell^\infty$  is the Banach space of real bounded sequences, equipped with the supremum norm. The space  $\ell^\infty$  is separable. (Hint:  $2^{\mathbb{N}}$  is not countable.)

- (d) (6 points) True or false? Let  $g$  be strictly increasing and absolutely continuous on a closed, bounded, nondegenerate interval  $[a, b]$ , and let  $\mathcal{O}$  be an open subset of  $(a, b)$ . Then

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'.$$

5. (16 points) (a) (6 points) A Hilbert space  $\mathcal{H}$  may be characterized as a Banach space in which the so-called parallelogram law holds; that is, for each pair of vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathcal{H}$ , we have

$$\|\mathbf{x} + \mathbf{y}\|_{\mathcal{H}}^2 + \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2 = 2 \cdot \|\mathbf{x}\|_{\mathcal{H}}^2 + 2 \cdot \|\mathbf{y}\|_{\mathcal{H}}^2.$$

Using only characteristic functions of measurable subsets of  $[0, 1]$ , prove that  $L^1([0, 1])$  is not a Hilbert space.

- (b) (10 points) (Embedding of  $\ell^p$  in  $L^p[1, \infty)$ .) Let  $1 \leq p < \infty$ . For  $a = (a_1, a_2, \dots) \in \ell^p$ , define  $f_a : [1, \infty) \rightarrow \mathbb{R}$  by

$$f_a(x) := \sum_{k=1}^{\infty} a_k \chi_{[k, k+1)}(x) \quad (x \in [1, \infty)).$$

- (2 points) Prove that  $f_a \in L^p[1, \infty)$ .
  - (2 points) Prove that  $\|f_a\|_p = \|a\|_p$ .
  - (3 points) Prove that the map  $T : \ell^p \rightarrow L^p[1, \infty)$  given by  $T(a) := f_a$  is linear and injective, but not surjective.
  - (3 points) True or false? The range of  $T$  is dense in  $L^p[1, \infty)$ .
6. (6 points) Let  $f$  be a Lipschitz function on  $[0, 1]$  and let  $g$  an absolutely continuous function on  $[0, 1]$ . Prove that  $f \circ g$  is absolutely continuous on  $[0, 1]$ .
7. (16 points) On a closed, bounded, nondegenerate interval  $[a, b]$ , consider a sequence  $\{f_n\}$  of increasing, absolutely continuous functions on  $[a, b]$ . Assume:
- $f_n(a) = 0$  for all  $n \in \mathbb{N}$ ;
  - $f'_n \leq f'_{n+1}$  a.e. on  $[a, b]$ , for all  $n \in \mathbb{N}$ ;
  - The sequence  $\{f_n(b)\}$  is bounded.
- Prove:
- (4 points) There exists  $f : [a, b] \rightarrow \mathbb{R}$  such that  $\{f_n\} \rightarrow f$  pointwise on  $[a, b]$ ;
  - (6 points)  $f$  is absolutely continuous on  $[a, b]$ ; and
  - (6 points)  $\{f_n\}$  converges to  $f$  uniformly on  $[a, b]$ .
8. (14 points) (a) (7 points) For a nonempty subset  $E$  of a metric space  $(X, \rho)$  and a point  $x \in X$ , define the distance from  $x$  to  $E$  by

$$\text{dist}(x, E) := \inf\{\rho(x, y) \mid y \in E\}$$

- (4 points) Show that  $f$  is continuous on  $X$ .
  - (3 points) Show that  $\overline{E} = f^{-1}(\{0\})$ .
- (b) (7 points) Now define  $f : X \rightarrow \mathbb{R}$  by

$$f(x) := \text{dist}(x, E) \quad (x \in X).$$

- i. (4 points) Assume that  $F$  is closed, and  $E$  is compact. Prove that

$$E \cap F = \emptyset \iff \text{dist}(F, E) > 0.$$

- ii. (3 points) Prove that the previous statement is not true if  $E$  and  $F$  are merely closed. That is, find an example of two closed sets  $E$  and  $F$  for which  $\text{dist}(F, E) = 0$  and  $E \cap F = \emptyset$ . (Hint: To produce an example in  $\mathbb{R}$ , by Heine-Borel both sets need to be closed and unbounded, e.g., consider  $E := \mathbb{N}$ .)