

REAL ANALYSIS FINAL  
FALL 2021  
RAÚL CURTO

1. (12 points) Define:
  - (a) (2 points) Uniform convergence of a sequence of functions on a set  $E \subseteq \mathbb{R}$ .
  - (b) (3 points) Variation of a function  $f$  on a closed, bounded interval  $[a, b]$ , with respect to a partition  $P$  of  $[a, b]$ .
  - (c) (2 points) Point of closure of a subset  $E$  of a metric space  $X$ .
  - (d) (2 points) Equicontinuity for a collection  $\mathcal{F}$  of real-valued functions on a metric space  $X$ .
  - (e) (3 points) State the Cantor intersection theorem.
2. (8 points) Let  $f$  be a continuous function on a closed, bounded, nondegenerate interval  $[a, b]$  such that
  - (i)  $f$  is of bounded variation on  $[a, b]$ ; and
  - (ii)  $f$  maps sets of measure zero to sets of measure zero; that is, for  $E$  a measurable subset of  $[a, b]$ ,  $m(E) = 0 \implies m(f(E)) = 0$ .Prove that  $f$  is absolutely continuous on  $[a, b]$ .
3. (8 points) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel function, and define
$$\mu(E) := m(f^{-1}(E)) \quad (E \subseteq \mathbb{R}, E \text{ Borel}).$$
Prove:
  - (a)  $\mu(E) \geq 0$  for all  $E$  Borel;
  - (b)  $\mu$  is monotone;
  - (c)  $\mu$  is countably additive;
  - (d)  $\mu$  is not translation invariant; e.g., find a counterexample of a Borel function  $f$  and a set  $E$  such that  $\mu(E + 1) \neq \mu(E)$ , where  $E + 1 := \{x + 1 \mid x \in E\}$ .
4. (20 points)
  - (a) (4 points)
  - (b) (6 points)
  - (c) (4 points)
  - (d) (6 points)
5. (16 points)
  - (a) (6 points)
  - (b) (10 points)
    - i. (2 points)
    - ii. (2 points)
    - iii. (3 points)
    - iv. (3 points)
6. (6 points)
7. (16 points)
  - (a) (4 points)
  - (b) (6 points)
  - (c) (6 points)
8. (14 points)
  - (a) (4 points)
  - (b) (3 points)
  - (c) (4 points)
  - (d) (3 points)