Real Analysis Final
Fall 2021
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1. (12 points) Define:
   (a) (2 points) Uniform convergence of a sequence of functions on a set $E \subseteq \mathbb{R}$.
   (b) (3 points) Variation of a function f on a closed, bounded interval $[a, b]$, with respect to a partition P of $[a, b]$.
   (c) (2 points) Point of closure of a subset E of a metric space X.
   (d) (2 points) Equicontinuity for a colleciton $\mathcal{F}$ of real-valued functions on a metric space X.
   (e) (3 points) State the Cantor intersection theorem.

2. (8 points) Let f be a continuous function on a closed, bounded, nondegenerate interval $[a, b]$ such that

   (i) f is of bounded variation on $[a, b]$; and

   (ii) f maps sets of measure zero to sets of measure zero; that is, for E a measurable subset of $[a, b]$, $m(E) = 0 \implies m(f(E)) = 0$.

   Prove that f is absolutely continuous on $[a, b]$.

3. (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, and define

$$\mu(E) := m(f^{-1}(E)) \quad (E \subseteq \mathbb{R}, E \text{ Borel}).$$

   Prove:
   (a) $\mu(E) \geq 0$ for all E Borel;
   (b) μ is monotone;
   (c) μ is countably additive;
   (d) μ is not translation invariant; e.g., find a counterexample of a Borel function f and a set E such that $\mu(E + 1) \neq \mu(E)$, where $E + 1 := \{x + 1 \mid x \in E\}$.

4. (20 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Royden-Fitzpatrick."
   (a) (4 points) True or false? Let $\mathcal{F}$ be a collection of measurable functions on $\mathbb{R}$, and let

$$g := \sup_{f \in \mathcal{F}} f.$$

   Then g is measurable on $\mathbb{R}$.
   (b) (6 points) True or false? Consider the normed linear space X of Riemann integrable functions on $[0, 1]$, with the norm $\|f\|_R := (R) \int_0^1 |f(x)|dx$. Then X is a Banach space. (Hint: the sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions given by

$$f_n(x) := \begin{cases} 1 & x = \frac{i}{k}, \text{ if } 1 \leq k \leq n \text{ and } 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

   The sequence $\{f_n\}$ is increasing and $f_n \to \chi_{\mathbb{Q}}$ as $n \to \infty$.)
   (c) (4 points) True or false? Recall that ℓ^∞ is the Banach space of real bounded sequences, equipped with the supremum norm. The space ℓ^∞ is separable. (Hint: $2^{\mathbb{N}}$ is not countable.)

(d) (6 points) True or false? Let g be strictly increasing and absolutely continuous on a closed, bounded, nondegenerate interval $[a, b]$, and let $\mathcal{O}$ be an open subset of (a, b). Then

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'.$$

5. (16 points) (a) (6 points)
   (b) (10 points) i. (2 points)
   ii. (2 points)
   iii. (3 points)
   iv. (3 points)

6. (6 points)

7. (16 points) (a) (4 points)
   (b) (6 points)
   (c) (6 points)

8. (14 points) (a) (4 points)
   (b) (3 points)
   (c) (4 points)
   (d) (3 points)