Complex Analysis Midterm I
Spring 2022
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1. (20 points) (a) (4 points) Evaluate the cross ratio

$$(1+i, 1, 0, \infty).$$

   (b) (5 points) Let γ be the right half of the unit circle from i to $-i$. Calculate

$$\int_\gamma z^{-\frac{3}{2}} dz.$$

   (c) (5 points) If

$$T(z) = \frac{az+b}{cz+d},$$

   find complex numbers z_2, z_3, z_4 in terms of a, b, c, d, and such that

$$T(z) = (z_1, z_2, z_3, z_4).$$

   (d) (6 points) Evaluate the line integral

$$\int_\gamma \frac{\log z}{z^n} dz,$$

   where $\gamma(t) = 1 + \frac{1}{2}e^{it}$, for $0 \leq t \leq 2\pi$ and $n \geq 0$.

2. (10 points) Show that the series

$$\sum_{n=1}^{\infty} \left(\frac{z+i}{z-i} \right)^n \tag{1}$$

defines an analytic function on the disc of radius 1 centered at $-i$.

Hint: for $0 < s < 1$, first prove that

$$\left| \frac{z+i}{z-i} \right| \leq \frac{2}{2-s}. \tag{2}$$

Next, prove that the series

$$\sum_{n \geq 1} \left(\frac{s}{2-s} \right)^n \tag{3}$$

converges. With these two results in hand, prove (1).

3. (16 points) (a) (4 points) State the Cantor intersection theorem.

   (b) (6 points) Recall the stereographic projection $\Pi : S \to \mathbb{C}_\infty$, and let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$. In $\mathbb{R}^3$, consider the vertical plane P_α given by $x_2 = \alpha$, and let C_α denote the intersection of P_α with the unit sphere S.

      i. (3 points) In terms of z and $\bar{z}$, describe $\Pi(C_\alpha)$.

      ii. (3 points) What kind of geometric figure is $\Pi(C_\alpha)$: a straight line, a circle, a parabola, an ellipse with major and minor axes of different length, or none of the above? Justify your answer.

   (c) (6 points) Find the domain of analyticity of

$$f(z) := \log \left(\frac{z-1}{z+1} \right),$$

   where log denotes the principal branch of the logarithm.

4. (14 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Conway."

   (a) (5 points) True or false? Let T be a Möbius transformation that has ∞ as its only fixed point. Then T is a translation, but not the identity map.

   (b) (4 points) True or false? For $z, w \in \mathbb{C}$ the following identity holds:
   $$|z + \bar{w}|^2 - |z - \bar{w}|^2 = 4\Re(zw).$$

   (c) (5 points) True or false? Let D be the open unit disk and let $f : D \to \mathbb{C}$ be an analytic function. Assume that the set of zeros of f include the sequence
   $$\left\{\frac{1}{2}e^{in} \mid n \in \mathbb{N}\right\}.$$
   Then f is identically equal to zero.

5. (8 points) Let f be an entire function such that $f(x) = e^x$ for all x real and positive. Prove that $f(z) = e^z$ for all $z \in \mathbb{C}$.

6. (10 points) Calculate the radius of convergence R for the power series
   $$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$$
   Hint: Stirling's formula may be helpful: for every $n \in \mathbb{N}$,
   $$n! = n^n e^{-n} u_n,$$
   where the sequence $\{u_n\}$ satisfies the condition
   $$\lim_n u_n^{1/n} = 1.$$

7. (10 points) Let $D = \{z \mid |z| < 1\}$ be the open unit disk. Find all Möbius transformations T such that $T(D) = D$.

8. (12 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Conway."

   (a) (4 points) True or false? Every connected component of a nonempty open set in $\mathbb{C}$ is open and closed.

   (b) (4 points) True or false? If F_1 and F_2 are primitives for $f : G \to \mathbb{C}$ and G is open and connected, then there is a constant c such that $F_1(z) = c + F_2(z)$ for each z in G.

   (c) (4 points) True or false? If $z \in \mathbb{C}$ and $\Re(z^n) \leq 0$ for all $n \in \mathbb{N}$, then $z = 0$.