Topology PhD Qualifying Exam
August 19, 2022
Mohammad Farajzadeh, Keiko Kawamuro

Part A

1. Prove that for every group $G = \langle g_1, \dots, g_k \mid r_1, \dots, r_k \rangle$ there is a 2-dimensional cell complex X_G with $\pi_1(G) \simeq G$.

2. (a) Let α be an oriented closed loop in a topological space X. Let $x_0 \in X$ and $p, q \in \alpha$ be points. (They are possibly the same points.) Let γ (resp. δ) be an oriented path in X that starts at x_0 and ends at p (resp. q).

   Show that homotopy classes $[\gamma * \alpha * \bar{\gamma}]$ and $[\delta * \alpha * \bar{\delta}]$ in $\pi_1(X, x_0)$ are conjugate to each other.

   (b) Let x_0 and x_0' be points in X. Show that $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ are group isomorphic.

3. Let D be the unit disk in $\mathbb{R}^2$. Show that every continuous map $h : D \to D$ has a fixed point.

4. Let $i = 1, 2$. Let $h_i : D^2 \times S^1 \to A_i$ be a homeomorphism. The boundary of A_i is a torus $\partial A_i = \partial D \times S^1 = S^1 \times S^1$. We define two curves on ∂A_i, a meridian m_i and a longitude l_i, as follows: Fix points $x \in \partial D^2$ and $y \in S^1$. The meridian is defined by $m_i = h_i(\partial D^2 \times \{y\})$ and the longitude is defined by $l_i = h_i(\{x\} \times S^1)$.

   Let (p, q) be coprime integers. Let $\phi : \partial A_1 \to \partial A_2$ be a homeomorphism that takes the meridian m_1 to a simple closed curve whose homotopy type is equal to $p[m_1] + q[l_2]$. Construct a space X gluing the solid tori A_1 and A_2 along their boundary using the map ϕ.

   Using the van Kampen theorem, compute the fundamental group of the space X.

5. Let $X = S^1 \vee S^1$.

   (a) Find a $3 : 1$ (non-trivial) normal covering of X and find the corresponding normal subgroup of $\pi_1(X)$.

   (b) Find a $3 : 1$ non-normal covering of X.

   (c) Describe the universal covering of X.

6. (a) Describe a cell-complex structure on the n-dimensional real projective space $\mathbb{R}P^n$ where $n \geq 2$.

   (b) Find a non-trivial covering of $\mathbb{R}P^n$ and compute its deck transformation group.

Part B

1. Let S be the unit sphere in $\mathbb{R}^3$. Find a C^∞ atlas on S that consists of two charts.

2. State the regular level set theorem, then show that the unit sphere S is a 2-dimensional manifold.

3. Let U be an open subset of $\mathbb{R}^n$. Show that the exterior derivative $d : \Omega^*(U) \to \Omega^*(U)$ satisfies the *cocycle condition*; that is, $d^2 = 0$.

4. (a) Show that the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\phi(x, y, z) = (2y, -x, -xy + z)$$

is a diffeomorphism.

(b) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^3$. Compute $\phi_*(X)$ at $p = (x, y, z)$.

(c) Let $\alpha = dz - ydx$ be a 1-form on $\mathbb{R}^3$. Compute the pullback $\phi^*(\alpha)$ at $p = (x, y, z)$.

5. Let ω be a 2-form on the unit sphere S in $\mathbb{R}^3$ defined by:

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz\wedge dx}{y} & \text{for } y \neq 0, \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

Show that this ω is well-defined. (In other words, if $x, y, z \neq 0$ then the three expressions give the same 2-form.) Then compute the integral $\int_S \omega$.

6. Let $O(n)$ be the orthogonal group, the group of linear transformations of $\mathbb{R}^n$ that preserve distance. In other words, the group of $n \times n$ matrices such that $AA^T = I$ where A^T is the transpose of A and I is the identity matrix. Show that $O(n)$ is a manifold of dimension $n(n-1)/2$.