Real Analysis Midterm I
Fall 2021
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1. (12 points) First, recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz if there exists a constant $C \geq 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$.

   Now let E be a measurable subset of $\mathbb{R}$, and assume that $0 \leq m(E) < \infty$. Consider the function $f : [0, \infty) \to \mathbb{R}$ given by
$$f(x) := m(E \cap [0, x]).$$
   (a) (6 points) Prove that f is Lipschitz.
   (b) (6 points) Find the best possible constant C_0; that is, calculate
$$C_0 := \inf\{C \mid f \text{ is Lipschitz with constant } C\}.$$

2. (12 points) Define the following notions.
   (a) (4 points) A Borel set in $\mathbb{R}$.
   (b) (4 points) The outer measure m^* as a set function from the power set $2^{\mathbb{R}}$ to the extended reals. Concretely, what is the formal definition of $m^*(A)$, for $A \subseteq \mathbb{R}$?
   (c) (4 points) A measurable set. That is, given a set $E \subseteq \mathbb{R}$, write down the formal definition of "E is measurable."

3. (12 points) First, observe that the interval $[0, 1]$ can be written as the disjoint union of $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and a set $A \subseteq [0, 1]$. Consider now a half-open interval $[0, 1)$, which admits a similar disjoint union decomposition, using the same set A.
   (a) (6 points) Find a simple description of $B := [0, 1) \setminus A$, along the lines of what works for $[0, 1)$.
   (b) (6 points) Use the description in (a) to establish a one-to-one map from $[0, 1]$ onto $[0, 1)$; in other words, prove that $[0, 1]$ and $[0, 1)$ are equipotent by writing down a function $f : [0, 1] \to [0, 1)$ which is one-to-one and onto.

4. (12 points) Consider the following collection of sets in $\mathbb{R}$:
$$\mathcal{A} := \{B \subseteq \mathbb{R} \mid B \text{ or } B^C \text{ is countable.}\}$$
   (a) (6 points) Prove that $\mathcal{A}$ is a σ-algebra.
   (b) (6 points) Prove that for $a, b \in \mathbb{R}$ with $a < b$, the interval $[a, b]$ is not in $\mathcal{A}$. That is, $\mathcal{A}$ provides an example of a σ-algebra that does not contain any nondegenerate intervals.

5. (24 points) Consider a nonmeasurable set $A \subseteq [0, 1]$. Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by
$$f(x) := \chi_A(x) - \chi_{A^C}(x)$$
   and
$$g(x) := e^x \chi_A(x) - e^x \chi_{A^C}(x),$$
   where for a set $B \subseteq \mathbb{R}$, χ_B denotes the characteristic function of B.
   (a) (6 points) Prove that f is not measurable.
   (b) (6 points) Prove that $|f|$ is measurable.
   (c) (6 points) Prove that for any $c \in \mathbb{R}$, the set $\{x \in \mathbb{R} \mid g(x) = c\}$ is Borel.
   (d) (6 points) Prove that g is not measurable.

6. (16 points) Determine if each of the following statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, you are free to cite the textbook, and provide a rationale along the lines of "...by a proposition in Section a.b of Royden-Fitzpatrick."

   (a) (4 points) True or false? There exists a continuous function $f : [0, 1] \to \mathbb{R}$ mapping the Cantor set $\mathcal{C}$ onto a measurable set of measure 1.

   (b) (4 points) True or false? The set of discontinuities of an increasing function defined on the interval $[0, 1]$ is countable.

   (c) (4 points) True or false? The proof of Lusin's theorem uses Egoroff's theorem.

   (d) (4 points) True or false? For a set A of real numbers, let χ_A denote the characteristic function of A. Then, given two arbitrary sets $A, B \subseteq \mathbb{R}$, we always have

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B.$$

7. (12 points) Consider the set of natural numbers $\mathbb{N}$. On its power set $2^{\mathbb{N}}$ define the set function

$$\mu(A) := \begin{cases} \sum_{n\in A} 2^{-n} & A \text{ is finite} \\ \infty & A \text{ is infinite.} \end{cases}$$

   (a) (4 points) Prove that μ is monotone.

   (b) (4 points) Prove that μ is additive. (By induction, it suffices to prove that $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \subseteq \mathbb{N}$ with $A \cap B = \varnothing$.)

   (c) (4 points) Prove that μ is not countably additive.