Exercise 2

Chapter 8, Page 360

Introduction to Electrodynamics
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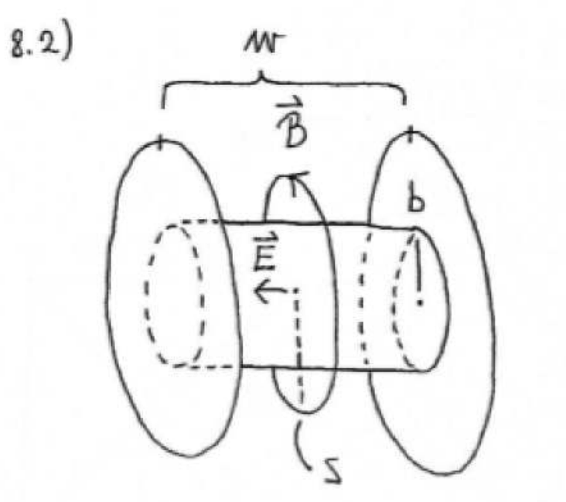
Solution

Verified

Step 1



Step 2

a)

From the solution to the problem 7.34), the magnetic and electric fields are:

$$\boxed{\vec{E} = \frac{It}{\epsilon_0 A}\hat{x}}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2}\frac{s}{A}\hat{\phi}}$$

, where the x-direction points along the electric field in the picture.

Step 3

b)

The energy density is:

$$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \boxed{\frac{1}{2\epsilon_0}\left(\frac{It}{A}\right)^2 + \frac{\mu_0}{4}\left(\frac{Is}{A}\right)^2}$$

, where A is the area of the plates. The Poynting vector is equal to:

$$\begin{aligned}\vec{S} =& \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{I^2 ts}{2\epsilon_0 A^2}\hat{z} \times \hat{\phi} = \frac{I^2 ts}{2\epsilon_0 A^2}(-\hat{s}) \\ =& \boxed{-\frac{I^2 ts}{2\epsilon_0 A^2}\hat{s}}\end{aligned}$$

Step 4

Now we check whether the energy conservation is satisfied:

$$\begin{aligned}\frac{\partial u}{\partial t} =& \frac{\partial}{\partial t}\left[\frac{1}{2\epsilon_0}\left(\frac{It}{A}\right)^2 + \frac{\mu_0}{4}\left(\frac{Is}{A}\right)^2\right] \\ =& \frac{I^2 t}{\epsilon_0 A^2}\end{aligned}$$

$$\nabla \cdot \vec{S} = \frac{1}{s}\frac{\partial}{\partial s}(sS) = -\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{I^2 ts}{2\epsilon_0 A^2}\right) = -\frac{I^2 t}{\epsilon_0 A^2}$$

Comparing the two expressions, the energy conservation condition is satisfied:

$$\boxed{\nabla \cdot \vec{S} = -\frac{\partial u}{\partial t}}$$

Step 5

c)

The total energy contained within a cyllinder of height w, but radius $b < a$ (to avoid fringing fields) is:

$$
\begin{aligned}
U = \int_V u dV = \int_0^w \int_0^{2\pi} \int_0^b u s ds d\phi dz = w 2\pi \int_0^b u s ds \\
&= 2\pi w \int_0^b \left[\frac{1}{2\epsilon_0} \left(\frac{It}{A} \right)^2 + \frac{\mu_0}{4} \left(\frac{Is}{A} \right)^2 \right] s ds \\
&= 2\pi w \left[\frac{1}{2\epsilon_0} \left(\frac{It}{A} \right)^2 \frac{1}{2} b^2 + \frac{\mu_0}{4} \left(\frac{I}{A} \right)^2 \int_0^b s^3 ds \right] \\
&= 2\pi w \left[\frac{1}{2\epsilon_0} \left(\frac{It}{A} \right)^2 \frac{1}{2} b^2 + \frac{\mu_0}{4} \left(\frac{I}{A} \right)^2 \frac{1}{4} b^4 \right] \\
&= \boxed{2\pi \mu_0 w \left(\frac{Ib}{2A} \right)^2 \left[(ct)^2 + \left(\frac{b}{2} \right)^2 \right]}
\end{aligned}
$$

Step 6

Now, for the same cyllinder choose the normal to the surface to be pointing inside, then the integral of the Poynting vector over this surface (i.e. the power) will be positive:

$$
P = \int_0^{2\pi} \int_0^w S b d\phi dz = 2b\pi w S = 2b\pi w \frac{I^2 t b}{2\epsilon_0 A^2} = \boxed{\frac{\pi w}{\epsilon_0} \left(\frac{Ib}{A} \right)^2 t}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial t} 2\pi \mu_0 w \left(\frac{Ib}{2A} \right)^2 \left[(ct)^2 + \left(\frac{b}{2} \right)^2 \right] \\
&= 2\pi \mu_0 w \left(\frac{Ib}{2A} \right)^2 c^2 2t = \frac{\pi w}{\epsilon_0} \left(\frac{Ib}{A} \right)^2 t
\end{aligned}
$$

Thus:

$$
\boxed{\frac{\partial U}{\partial t} = P}
$$

Result

a) $\boxed{\vec{E} = \frac{It}{\epsilon_0 A}\hat{x} \qquad \vec{B} = \frac{\mu_0 I}{2}\frac{s}{A}\hat{\phi}}$

b) $\boxed{u = \frac{1}{2\epsilon_0}\left(\frac{It}{A}\right)^2 + \frac{\mu_0}{4}\left(\frac{Is}{A}\right)^2 \qquad \vec{S} = -\frac{I^2 ts}{2\epsilon_0 A^2}\hat{s}}$

c) $\boxed{U = 2\pi\mu_0 w\left(\frac{Ib}{2A}\right)^2\left[(ct)^2 + \left(\frac{b}{2}\right)^2\right] \qquad P = \frac{\pi w}{\epsilon_0}\left(\frac{Ib}{A}\right)^2 t}$