# Exercise 49

Chapter 2, Page 108

Introduction to Electrodynamics

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## Solution

Verified

### Step 1

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We have a sphere of radius $R$ carries a charge density $\rho(r) = kr$, where $k$ is constant. We need to find the energy of the configuration. First we need to find the electric field inside and outside the sphere using Gauss's law, as (consider a sphere with radius of $r$, first with $r < R$ and then for $r > R$),

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}$$

$$\epsilon_0 4\pi r^2 E = \int \rho d\tau$$

$$\epsilon_0 4\pi r^2 E = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} (k\bar{r})\bar{r}^2 \sin\theta d\bar{r} d\theta d\phi$$

$$\epsilon_0 4\pi r^2 E = 4\pi k \int_0^{r} \bar{r}^3 d\bar{r}$$

note that the integration is from 0 to $r$ for $r < R$ and from 0 to $R$ for $r < R$, so we get two field one for $r < R$ and the other for $r > R$, as,

$$\epsilon_0 4\pi r^2 E = \begin{cases} \pi k r^4 & (r < R) \\ \pi k R^4 & (r > R) \end{cases}$$

$$\mathbf{E} = \begin{cases} \dfrac{k}{4\epsilon_0} r^2 \hat{\mathbf{r}} & (r < R) \\ \\ \dfrac{kR^4}{4\epsilon_0 r^2} \hat{\mathbf{r}} & (r > R) \end{cases} \tag{1}$$

Now we need to find the the energy using another method. We can use equation 2.43,

$$W = \frac{1}{2}\int \rho V \, d\tau$$

but for this method we need to find the potential, as,

$$\begin{aligned}
V(r) &= -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_{\infty}^{R} \left(\frac{kR^4}{4\epsilon_0 r^2}\right) dr - \int_{R}^{r} \left(\frac{kr^2}{4\epsilon_0}\right) dr \\
&= -\frac{k}{4\epsilon_0}\left(R^4\left(-\frac{1}{r}\right)\Bigg|_{\infty}^{R} + \frac{r^3}{3}\Bigg|_{R}^{r}\right) \\
&= -\frac{k}{4\epsilon_0}\left(-R^3 + \frac{r^3}{3} - \frac{R^3}{3}\right) \\
&= \frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right)
\end{aligned}$$

substitute into the above equation,

$$\begin{aligned}
W &= \frac{1}{2}\int_{0}^{R} (kr)\left[\frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right)\right] 4\pi r^2 dr \\
&= \frac{2\pi k^2}{3\epsilon_0}\int_{0}^{R}\left(R^3 r^3 - \frac{1}{4}r^6\right) dr \\
&= \frac{2\pi k^2}{3\epsilon_0}\left(R^3\frac{R^4}{4} - \frac{1}{4}\frac{R^7}{7}\right) \\
&= \frac{2\pi k^2}{3\epsilon_0}\left(R^3\frac{R^4}{4} - \frac{1}{4}\frac{R^7}{7}\right) \\
&= \frac{\pi k^2 R^7}{2 \cdot 3\epsilon_0}\left(\frac{6}{7}\right) = \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}$$

$$\boxed{W = \frac{\pi k^2 R^7}{7\epsilon_0}}$$

## Result



$$W = \frac{\pi k^2 R^7}{7\epsilon_0}$$

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