Exercise 34

Chapter 2, Page 95

Introduction to Electrodynamics
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Solution

Verified

Step 1

a)

In the problem 2.21) we found that the potential inside of a sphere of uniform charge is:

$$V = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

So:

$$\begin{aligned}
U = \frac{1}{2}\int_\tau \rho V d\tau =& \frac{\rho}{2}\int_\tau V d\tau \\
=& \frac{\rho}{2}\int_0^R \int_0^{2\pi}\int_0^{2\pi} \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) r^2 \sin\theta d\theta d\phi \\
=& \frac{3q^2}{64\pi^2\epsilon_0 R^4} 4\pi \int_0^R \left(3 - \frac{r^2}{R^2}\right) r^2 dr \\
=& \frac{3q^2}{16\pi\epsilon_0 R^4}\left(R^3 - \frac{1}{5}\frac{R^5}{R^2}\right) = \boxed{\frac{3q^2}{20\pi\epsilon_0 R}}
\end{aligned}$$

Step 2

b)

The electric fields inside and outside of an electric ball of charge are:

$$\vec{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} \qquad \vec{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Total energy is then:

$$\begin{aligned} U = \frac{\epsilon_0}{2}\int E^2 d\tau =& \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left[4\pi\int_0^R \frac{r^2}{R^6}r^2 dr + 4\pi\int_R^\infty \frac{1}{r^4}r^2 dr\right] \\ =& \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} - \frac{1}{r}\bigg|_R^\infty\right] = \boxed{\frac{3q^2}{20\pi\epsilon_0 R}} \end{aligned}$$

Step 3

c)

We will integrate over the sphere of radius a, $a > R$, so it encompasses all charge. The energy is then:

$$\begin{aligned} U =& \frac{\epsilon_0}{2}\left(\int E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right) \\ =& \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} - \frac{1}{r}\bigg|_R^a\right] + \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\int_0^{2\pi}\int_0^\pi \frac{1}{a}\frac{1}{a^2}a^2\sin\theta d\theta d\phi \\ =& \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} - \frac{1}{r}\bigg|_R^a\right] + \frac{q^2}{8\pi\epsilon_0}\frac{1}{a} \\ =& \boxed{\frac{3q^2}{20\pi\epsilon_0 R}} \end{aligned}$$

As the sphere of integration expands ($a \to \infty$), the surface integral contribution goes to zero.

Result

$$\boxed{U = \frac{3q^2}{20\pi\epsilon_0 R}}$$

, in all three cases.

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