Model Reference Adaptive Control

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Intelligent & Adaptive Control @ ECE AUTh, January 2022

1. Introduction

This project deals with the control of a given system that includes uncertainties on its plant. We assume the system

$$\begin{cases} \dot{x_p} = A_p x_p + B_p D\big(u + f(x_p)\big) \\ y_p = C_p^T x_p \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector, $u \in R$ is the control input and $y_p \in \mathbb{R}$ is the system output. The matrix $A_p \in \mathbb{R}^{n_p \times n_p}$ and the vectors $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{n_p \times 1}$ are known and constant. The unknown constant $D \in \mathbb{R}$, with $D > 0$, and the $f(x_p) = \theta^T \Phi(x_p) \in \mathbb{R}$ define the uncertainties of the system. The vector $\theta \in \mathbb{R}^{N \times 1}$ consists of unknown but constant parameters and $\Phi(x_p) \in \mathbb{R}^{N \times 1}$ consists of known, non-linear and locally *Lipschitz* continuous functions. The pair (A_p, B_p) is considered controllable. Our goal is to design a **linear state feedback controller** with proportional-integral characteristics, so that the output y_p of the **real system** (1) tracks the output of a **reference model**, which subsequently tracks (with bounded errors) any bounded *command signal*. As a result, the output of the *real system* tracks (with bounded errors) the *command signal*. Moreover, we need all the signals of the closed-loop to be bounded.

2. System without uncertainties

2.1. Real plant

Firstly, we are going to assume a system without uncertainties and we will reach our goal in the ideal case. We assume $D = 1$ and $f(x_p) = 0 \ \ \forall x_p \in \mathbb{R}^{n_p}$, so the system becomes

$$\begin{cases} \dot{x_p} = A_p x_p + B_p u \\ y_p = C_p^T x_p \end{cases} \tag{2}$$

2.2. Augmented real plant

We declare the *augmented state vector*

$$z_p = [e_{y_p I} \ \ x_p]^T \in \mathbb{R}^{(n_p+1) \times 1} \tag{3}$$

where

$$e_{y_p I} = \int_0^t e_{y_p}(\tau)\, d\tau \in \mathbb{R} \tag{4}$$

and

$$e_{y_p} = y_p - r \in \mathbb{R} \tag{5}$$

with $r \in \mathbb{R}$ being an *external command singal*, which we want our system output y_p to boundly track. The *augmented state vector* defines the *augmented real plant*

$$\begin{cases} \dot{z_p} = \tilde{A_p} z_p + \tilde{B_p} u + \tilde{B_m} r \\ y_p = \tilde{C_p}^T z_p \end{cases} \tag{6}$$

where

$$\tilde{A_p} = \begin{bmatrix} 0 & C_p^T \\ 0_{n_p \times 1} & A_p \end{bmatrix} \in \mathbb{R}^{(n_p+1) \times (n_p+1)},$$

$$\tilde{B}_p = \begin{bmatrix} 0 \\ B_p \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1},$$

$$\tilde{B_m} = \begin{bmatrix} -1 \\ 0_{n_p\times 1} \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1}$$

and

$$\tilde{C_p} = \begin{bmatrix} 0 \\ C_p \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1}$$

It is easy to prove that (6) is equivalent with (2) and (5) by simply replacing the *augmented* vectors and matrices in (6). We also suppose that the pair $(\tilde{A_p}, \tilde{B_p})$ is controllable.

2.3. Controller

We choose the controller

$$u = K^T z_p \tag{7}$$

with $K \in \mathbb{R}^{(n_p+1)\times 1}$ being the constant *gain vector*. Now, replacing (7) to (6), we get

$$\dot{z_p} = \tilde{A_p} z_p + \tilde{B_p}(K^T z_p) + \tilde{B_m} r \Rightarrow$$

$$\dot{z_p} = (\tilde{A_p} + \tilde{B_p} K^T) z_p + \tilde{B_m} r \tag{8}$$

2.4. Reference model

We assume the *reference model*

$$\begin{cases} \dot{x_m} = A_m x_m + B_m r + E_m e_{y_m I} \\ y_m = C_m^T x_m \end{cases} \tag{9}$$

so that

$$\tilde{B_m} = \begin{bmatrix} -1 \\ B_m \end{bmatrix} \Rightarrow B_m = \begin{bmatrix} 0_{n_p\times 1} \end{bmatrix} \in \mathbb{R}^{n_p\times 1},$$

$$C_m = C_p \in \mathbb{R}^{n_p\times 1}$$

and $A_m \in \mathbb{R}^{n_p\times n_p}$, $E_m \in \mathbb{R}^{n_p\times 1}$ can be chosen in order to achieve the control goal. Note that $e_{y_m I}$ is defined in the next subsection.

2.5. Augmented reference model

The *augmented state vector* for the *reference model* is

$$z_m = [e_{y_m I} \ \ x_m]^T \in \mathbb{R}^{(n_p+1)\times 1} \tag{10}$$

where

$$e_{y_m I} = \int_0^t e_{y_m}(\tau)\, d\tau \in \mathbb{R} \tag{11}$$

and

$$e_{y_m} = y_m - r \in \mathbb{R} \tag{12}$$

The *augmented reference model* is

$$\begin{cases} \dot{z_m} = \tilde{A_m} z_m + \tilde{B_m} r \\ y_m = \tilde{C_m}^T z_m \end{cases} \tag{13}$$

where

$$\tilde{A_m} = \begin{bmatrix} 0 & C_m^T \\ E_m^T & A_m \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times(n_p+1)},$$

$$\tilde{B_m} = \begin{bmatrix} -1 \\ B_m \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1},$$

and

$$\tilde{C_m} = \begin{bmatrix} 0 \\ C_m \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1}$$

Replacing the *augmented* vectors and matrices in (13), we get (9) and (12):

$$\begin{bmatrix} \dot{e}_{y_m I} \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} 0 & C_m^T \\ E_m & A_m \end{bmatrix} \begin{bmatrix} e_{y_m I} \\ x_m \end{bmatrix} + \begin{bmatrix} -1 \\ 0_{n_p \times 1} \end{bmatrix} r \Rightarrow$$

$$\begin{cases} e_{y_m} = C_m^T x_m - r \\ \dot{x}_m = A_m x_m + E_m e_{y_m I} \end{cases} \Rightarrow$$

$$\begin{cases} e_{y_m} = y_m - r \\ \dot{x}_m = A_m x_m + B_m r + E_m e_{y_m I} \end{cases}$$

and

$$y_m = \begin{bmatrix} 0 \\ C_m \end{bmatrix}^T \begin{bmatrix} e_{y_m I} \\ x_m \end{bmatrix} \Rightarrow$$

$$y_m = C_m^T x_m$$

2.6. Model matching

We write again the *augmented real plant* (6) and the *augmented reference model* (13) equations:

$$\begin{cases} \dot{z_p} = (\tilde{A_p} + \tilde{B_p} K^T) z_p + \tilde{B_m} r \\ y_p = \tilde{C_p}^T z_p \end{cases} , \quad \begin{cases} \dot{z_m} = \tilde{A_m} z_m + \tilde{B_m} r \\ y_m = \tilde{C_m}^T z_m \end{cases}$$

The *augmented system* is visualized below:

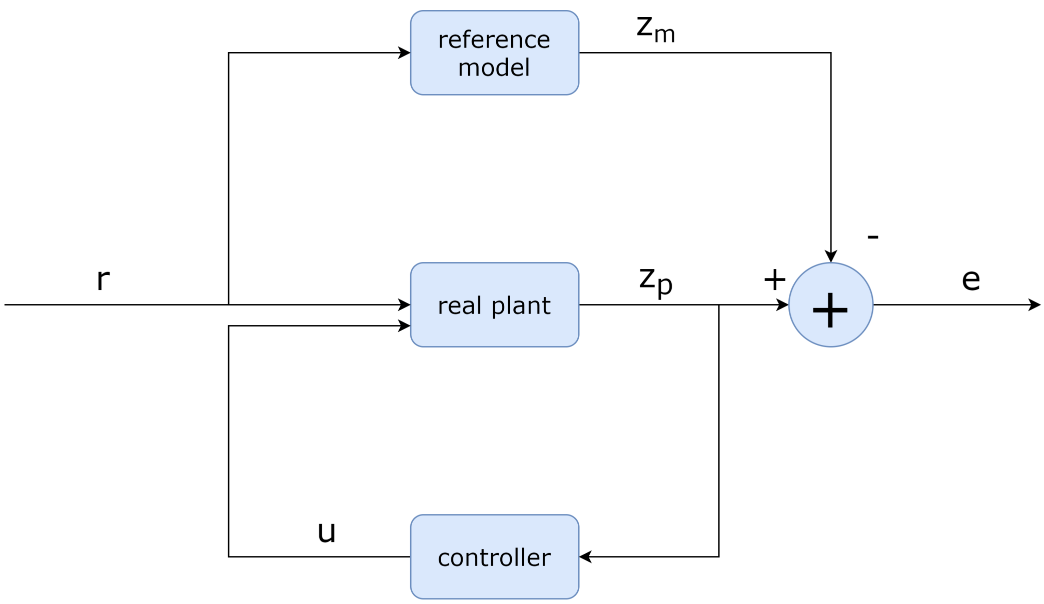

Figure 1: *Augmented MRAC system*

Since there are no uncertainties, we can apply the **model matching conditions** in order to find the value of the *gain vector* K, that leads the state tracking error $e = z_p - z_m$ between the *real plant state vector* and the *reference model state vector* to zero. We select K so that

$$\tilde{A_p} + \tilde{B_p} K^T = \tilde{A_m} \Rightarrow$$

$$\tilde{B_p}K^T = \tilde{A_m} - \tilde{A_p} \tag{14}$$

Note that (14) is a system with $n_p + 1$ unknown gains k_i and $(n_p + 1)^2$ equations. As a result, it is not guaranteed that there is a solution vector K, but we suppose that it does.

2.7. Results

Now, we take the state tracking error

$$e = z_p - z_m \Rightarrow$$
$$\dot{e} = \dot{z_p} - \dot{z_m} = (\tilde{A_p} + \tilde{B_p}K^T)z_p + \tilde{B_m}r - \tilde{A_m}z_m - \tilde{B_m}r \Rightarrow$$
$$\dot{e} = \tilde{A_m}z_p - \tilde{A_m}z_m = \tilde{A_m}(z_p - z_m) \Rightarrow$$

$$\dot{e} = \tilde{A_m}e \tag{15}$$

From (15) we get that $e \to 0$ for $\tilde{A_m} < 0$ and bounded input r. So, if we create the *augmented reference model* with $\tilde{A_m}$ being a *negative-definite* matrix, then we assure that the *augmented real system state vector* tends to the *augmented reference model state vector* ($z_p \to z_m$), which also means that the output of the *real plant* tends to the output of the *reference model* ($y_p \to y_m$). Moreover, $e \to 0$ means that $z_p = [e_{y_pI} \;\; x_p]^T$ is bounded, so $e_{y_pI} = \int_0^t e_{y_p}(\tau)\, d\tau$ is bounded, that means the error $e_{y_p} = y_p - r$ is bounded too. As a result, the *real system output* boundly tracks the *external command signal* r.

3. System with uncertainties

3.1. Real plant

In the non-ideal case, we assume $0 < D < 1$ and $f(x_p)$ non-zero function in (1). We are going to modify the controller of the previous section by adding a step of adaptive control to reach the same goal. So, we have the *real plant* of (1):

$$\begin{cases} \dot{x_p} = A_p x_p + B_p D\big(u + f(x_p)\big) \\ y_p = C_p^T x_p \end{cases}$$

with $0 < D < 1$ and $f(x_p) = \theta^T \Phi(x_p)$ being not a zero function. The analysis for this section is based on a chapter of *Lavretsky's* book [1].

3.2. Augmented real plant

We declare the same *augmented state vector* as previously:

$$z_p = [e_{y_pI} \;\; x_p]^T \in \mathbb{R}^{(n_p+1)\times 1} \tag{16}$$

where

$$e_{y_pI} = \int_0^t e_{y_p}(\tau)\, d\tau \in \mathbb{R} \tag{17}$$

and

$$e_{y_p} = y_p - r \in \mathbb{R} \tag{18}$$

The *augmented real plant* is

$$\begin{cases} \dot{z_p} = \tilde{A}_p z_p + \tilde{B}_p D\big(u + f(x_p)\big) + \tilde{B_m}r \\ y_p = \tilde{C_p}^T z_p \end{cases} \tag{19}$$

where

$$\tilde{A_p} = \begin{bmatrix} 0 & C_p^T \\ 0_{n_p\times 1} & A_p \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times(n_p+1)},$$

$$\tilde{B_p} = \begin{bmatrix} 0 \\ B_p \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1},$$

$$\tilde{B_m} = \begin{bmatrix} -1 \\ 0_{n_p \times 1} \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1}$$

and

$$\tilde{C_p} = \begin{bmatrix} 0 \\ C_p \end{bmatrix} \in \mathbb{R}^{(n_p+1)\times 1}$$

Note that the augmented open-loop system matrices and vectors ($\tilde{A_p}$, $\tilde{B_p}$, $\tilde{B_m}$ and $\tilde{C_p}$) are defined in the exact same way with those of the previous section. We also suppose that the pair $(\tilde{A_p}, \tilde{B_p})$ is controllable. According to [1], the augmented pair is controllable if and only if the original pair (A_p, B_p) is controllable and $\det \begin{pmatrix} A_p & B_p D \\ Cp & 0 \end{pmatrix} \neq 0$.

3.3. Controller

In this case, we choose the controller

$$u = \hat{K}^T z_p - \hat{\theta}^T \Phi(x_p) \tag{20}$$

with $\hat{K}(t) \in \mathbb{R}^{(n_p+1)\times 1}$ and $\hat{\theta}(t) \in \mathbb{R}^{N\times 1}$ being the adaptive *gain vectors*. Replacing (20) to (19), we get

$$\dot{z_p} = \tilde{A_p} z_p + \tilde{B_p} D\big(\hat{K}^T z_p - \hat{\theta}^T \Phi(x_p) + \theta^T \Phi(x_p)\big) + \tilde{B_m} r \Rightarrow$$

$$\dot{z_p} = (\tilde{A_p} + \tilde{B_p} D \hat{K}^T) z_p + \tilde{B_p} D(\theta^T - \hat{\theta}^T)\Phi(x_p) + \tilde{B_m} r \tag{21}$$

3.4. Reference model

We assume the *reference model* of (9) with the matrices as defined before:

$$\begin{cases} \dot{x_m} = A_m x_m + B_m r + E_m e_{y_m I} \\ y_m = C_m^T x_m \end{cases}$$

3.5. Augmented reference model

We also assume the *augmented reference model* of (13):

$$\begin{cases} \dot{z_m} = \tilde{A_m} z_m + \tilde{B_m} r \\ y_m = \tilde{C_m}^T z_m \end{cases}$$

3.6. Model matching

We assume a constant *gain vector* $K \in \mathbb{R}^{(n_p+1)\times 1}$, so that

$$\tilde{A_m} = \tilde{A_p} + \tilde{B_p} D K^T \Rightarrow \tilde{A_p} = \tilde{A_m} - \tilde{B_p} D K^T \tag{22}$$

Replacing (27) in (21) we get

$$\dot{z_p} = \tilde{A_m} z_p + \tilde{B_p} D(\hat{K}^T - K^T) z_p + \tilde{B_p} D(\theta^T - \hat{\theta}^T)\Phi(x_p) + \tilde{B_m} r \Rightarrow$$

$$\dot{z_p} = \tilde{A_m} z_p + \tilde{B_p} D\Bigg((\hat{K}^T - K^T) z_p + (\theta^T - \hat{\theta}^T)\Phi(x_p)\Bigg) + \tilde{B_m} r \Rightarrow$$

$$\dot{z_p} = \tilde{A_m} z_p + \tilde{B_p} D\Bigg((\hat{K}^T - K^T) z_p - (\hat{\theta}^T - \theta^T)\Phi(x_p)\Bigg) + \tilde{B_m} r \Rightarrow$$

$$\dot{z_p} = \tilde{A_m} z_p + \tilde{B_p} D\Bigg(\Delta K^T z_p - \Delta \theta^T \Phi(x_p)\Bigg) + \tilde{B_m} r \tag{23}$$

where $\Delta K = \hat{K} - K$ and $\Delta\theta = \hat{\theta} - \theta$.

3.7. Adaptive laws

We have to define the dymanics of the adaptive *gain vectors* $\hat{K}(t)$ and $\hat{\theta}(t)$, so that the *augmented real plant state vector* tracks the *augmented reference model state vector*, preserving the stability of the system. The state tracking error is

$$e = z_p - z_m \Rightarrow \dot{e} = \dot{z_p} - \dot{z_m} \Rightarrow$$

$$e = \tilde{A_m} z_p + \tilde{B_p} D\Big(\Delta K^T z_p - \Delta\theta^T \Phi(x_p)\Big) + \tilde{B_m} r - \tilde{A_m} z_m - \tilde{B_m} r \Rightarrow$$

$$e = \tilde{A_m} e + \tilde{B_p} D\Big(\Delta K^T z_p - \Delta\theta^T \Phi(x_p)\Big) \tag{24}$$

In order to reach our goal, we consider the *Lypaunov* function candidate

$$V(e, \Delta K, \Delta\theta) = e^T P e + tr(\Delta K^T \Gamma_K^{-1} \Delta K D) + tr(\Delta\theta^T \Gamma_\theta^{-1} \Delta\theta D) \tag{25}$$

where Γ_K and Γ_θ are *positive-definite* and *symmetric* matrices and define the rates of adaption. P is the unique *positive-definite* and *symmetric* solution of the *algebraic Lyapunov equation*

$$P\tilde{A_m} + \tilde{A_m}^T P = -Q$$

where Q is *symmetric positive-definite* matrix. The time derivative of the *Lyapunov* function candidate is

$$\dot{V}(e, \Delta K, \Delta\theta) = -e^T Q e + 2e^T P \tilde{B_p} D\big(\Delta K^T z_p - \Delta\theta^T \Phi(x_p)\big) + 2\,tr(\Delta K^T \Gamma_K^{-1} \dot{\hat{K}} D) + 2\,tr(\Delta\theta^T \Gamma_\theta^{-1} \dot{\hat{\theta}} D) \Rightarrow$$

$$\dot{V}(e, \Delta K, \Delta\theta) = -e^T Q e + 2\,tr\Big(\Delta K^T (\Gamma_K^{-1} \dot{\hat{K}} + z_p e^T P \tilde{B_p}) D\Big) + 2\,tr\Big(\Delta\theta^T \big(\Gamma_\theta^{-1} \dot{\hat{\theta}} - \Phi(x_p) e^T P \tilde{B_p}\big) D\Big) \tag{26}$$

Now, we select the following adaptive laws:

$$\begin{cases} \dot{\hat{K}} = -\Gamma_K z_p e^T P \tilde{B_p} \\ \dot{\hat{\theta}} = \Gamma_\theta \Phi(x_p) e^T P \tilde{B_p} \end{cases} \tag{27}$$

So, we have

$$\dot{V}(e, \Delta K, \Delta\theta) = -e^T Q e \ \leq 0 \tag{28}$$

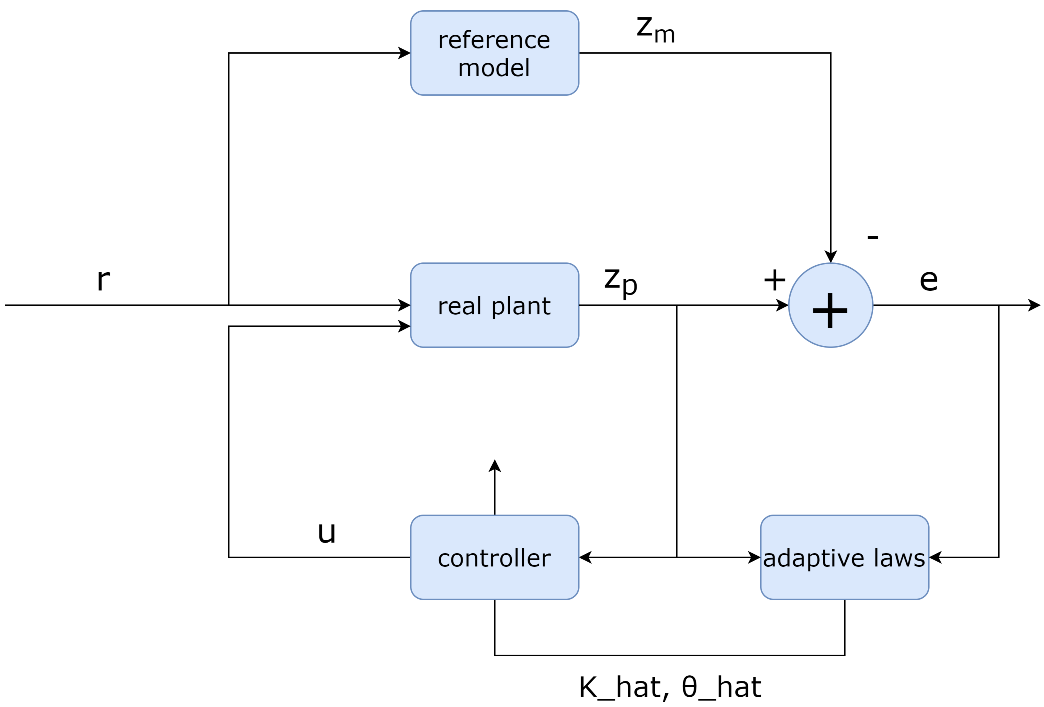

Figure 2: *Augmented MRAC system with uncertainties*

3.8. Results

From (33) and the *Lyapunov's second method for stability* we get that e, ΔK, $\Delta\theta$ are bounded, thus all the singals in the closed-loop are bounded too. In addition, from *Barbalat's lemma* we get that $e \to 0$, $\dot{\hat{K}} \to 0$, $\dot{\hat{\theta}} \to 0$ asymptotically while $t \to \infty$. Overall, we reach our goal as the output y_p tracks the *external command signal* r, with bounded errors, in the same way that we described in 2.7, while in this case, we achieved to cancel out the uncertainties of the *real plant* too.

4. Simulation

We will use the previous analysis in order to design a longitudinal motion controller for a conventional aircraft and simulate the system in *MATLAB*.

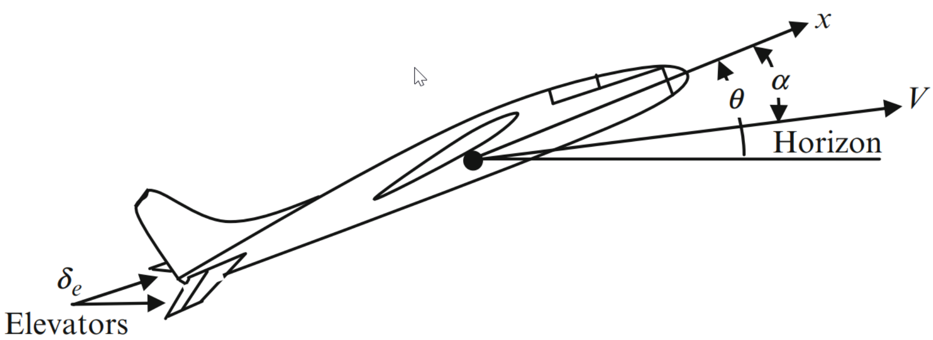

Figure 3: *Conventional aircraft - system representation*

4.1. System without uncertainties

As we can see in figure 3, we declare as x the vector that passes through the aircraft and V the velocity vector of the aircraft. Thus, the angle between x and V will be called *angle of attack* α and the angle between x and the horizon will be called *pitch angle* θ. The elevetors deflection by angle δ_e affects both the *angle of attack* and the *pitch angle*. We assume $x_p = [\alpha \quad q]^T$ in rad and rad/s respectively, where $q = \dot{\theta}$ is the *angular pitch velocity*. If the velocity of the aircraft is constant, the dynamics of the longitudinal motion of the aircraft for relatively small deflections of the elevators δ_e in rad, are

$$\dot{x}_p = \begin{bmatrix} -0.8060 & 1 \\ -9.1486 & -4.59 \end{bmatrix} x_p + \begin{bmatrix} -0.04 \\ -4.59 \end{bmatrix} \delta_e \tag{29}$$

and

$$y_p = \alpha \tag{30}$$

We assume

$$r(t) = \begin{cases} 0^\circ & 0 \le t < 1s \\ 0.5^\circ & 1s \le t < 10s \\ 0^\circ & 10s \le t < 22s \\ -0.5^\circ & 22s \le t < 32s \\ 0^\circ & 32s \le t < 45s \\ 1^\circ & 45s \le t < 55s \\ 0^\circ & 55s \le t < 65s \\ -1^\circ & 65s \le t < 75s \\ 0^\circ & 75s \le t < 85s \\ 0.5^\circ & 85s \le t < 95s \\ 0^\circ & 95s \le t \end{cases} \tag{31}$$

the reference input, which we want the system output y_p to track.

First of all, we write the dynamics of the system as

$$\begin{cases} \dot{x_p} = A_p x_p + B_p u \\ y_p = C_p^T x_p \end{cases} \tag{32}$$

where

$$A_p = \begin{bmatrix} -0.8060 & 1 \\ -9.1486 & -4.59 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.04 \\ -4.59 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Also, we define $u = \delta_e \in \mathbb{R}$ the control input. For the needs of simulation, we converted the control input in rad, while for the diagrams that will be presented later, we used deg for all the angles. Following the procedure of section 2, we design the controller of (7) and we reach

$$\begin{cases} \dot{z_p} = (\tilde{A_p} + \tilde{B_p}K^T)z_p + \tilde{B_m}r \\ y_p = \tilde{C_p}^T z_p \end{cases}, \quad \begin{cases} \dot{z_m} = \tilde{A_m}z_m + \tilde{B_m}r \\ y_m = \tilde{C_m}^T z_m \end{cases}$$

We select the *negative-definite*

$$\tilde{A_m} = \begin{bmatrix} 0 & 1 & 0 \\ -0.12 & -0.886 & 0.96 \\ -13.77 & -18.3286 & -9.18 \end{bmatrix}$$

and using *model matching condition* (14) we compute the *gain vector*

$$K = [3 \ \ 2 \ \ 1]^T$$

For the selected $\tilde{A_m}$ we can notice that the *model output* boundly tracks the *reference input*:

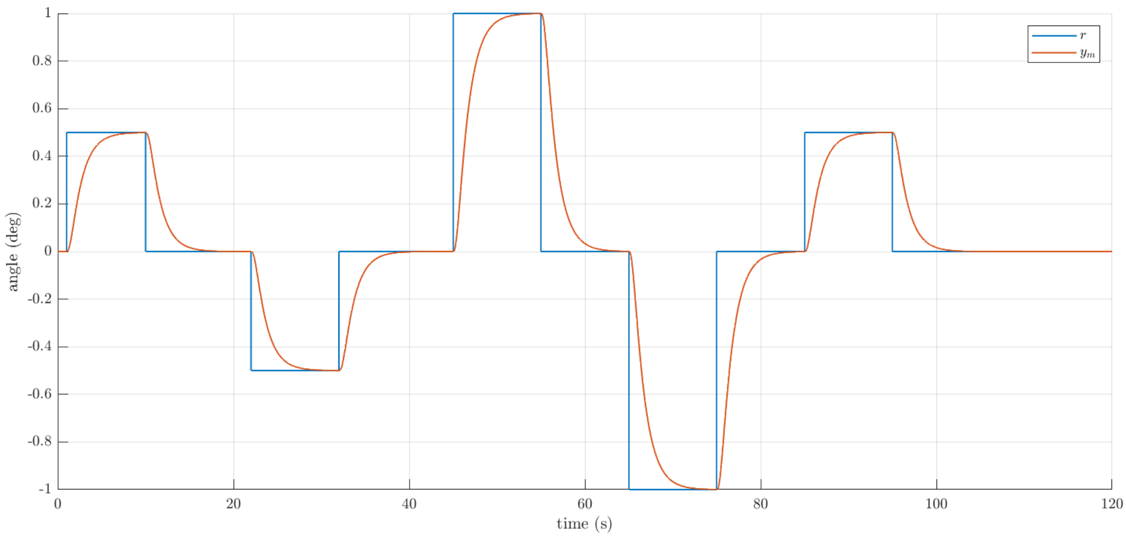

Figure 4: *Model output boundly tracks reference input*

As a result, if we achieve to make the output of the *real system* to track the *model output*, it will consequently boundly track the *reference input* too, which is our control goal.

For the computed *gain vector* K, we present the behavior of the *system output* under the effect of the controller, alongside with the *model output* and the *reference input* (zero initial conditions for both the plant and the model):

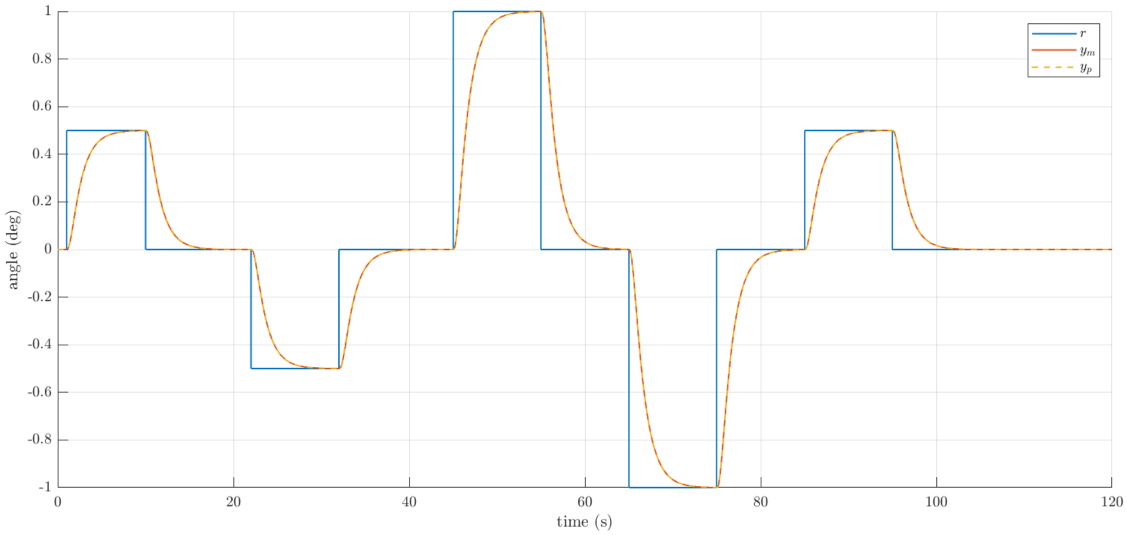

Figure 5: *Real system output tracks model output*

In figure 5 we can see that the control goal is achieved, as the *real system output* perfectly tracks the *model output*, thus, the *real system output* boundly tracks the *reference input*. The desired control signal $u(t) = K^T z_p(t)$ is presented in the next figure (transformed in *deg*) :

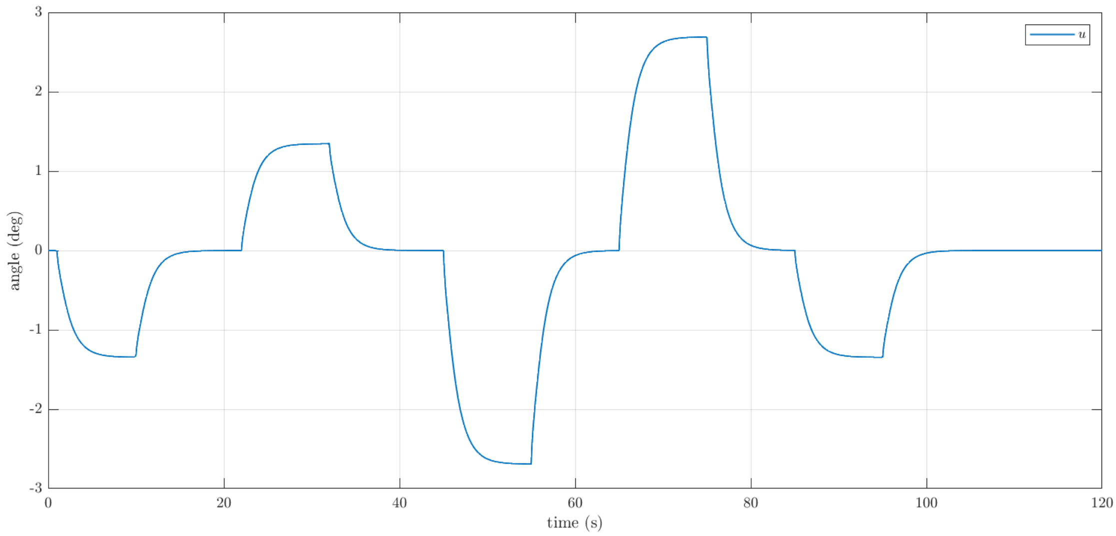

Figure 6: *Elevator deflection angle (control input) with respect to time*

4.2. System with uncertainties

Considering the data of the previous subsection, we add uncertainties in (32) and we have:

$$\begin{cases} \dot{x_p} = A_p x_p + B_p D\big(u + f(x_p)\big) \\ y_p = C_p^T x_p \end{cases} \tag{33}$$

with $0 < D < 1$ and $f(x_p) = k_\alpha \alpha + k_q q$. If we assume $\theta = [k_\alpha \;\; k_q]^T$ and $\Phi(x_p) = [x_{p_1} \;\; x_{p2}]^T$, we can write

$$f(x_p) = \theta^T \Phi(x_p)$$

We are going to set some values to the unknown parameters D, k_α and k_q in order to simulate the new system under the effect of the designed controller of 4.1 ($u = K^T z_p$). We set $D = 0.5$, $k_\alpha = -10$ and $k_q = -10$ and present the results of the simulation:

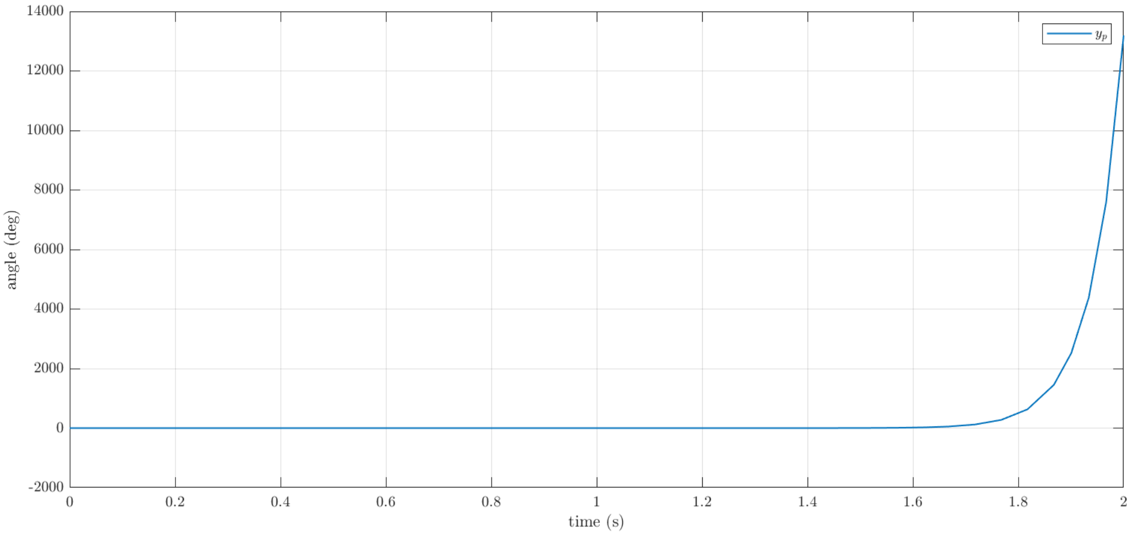

Figure 7: *Real system output with respect to time (0-2s)*

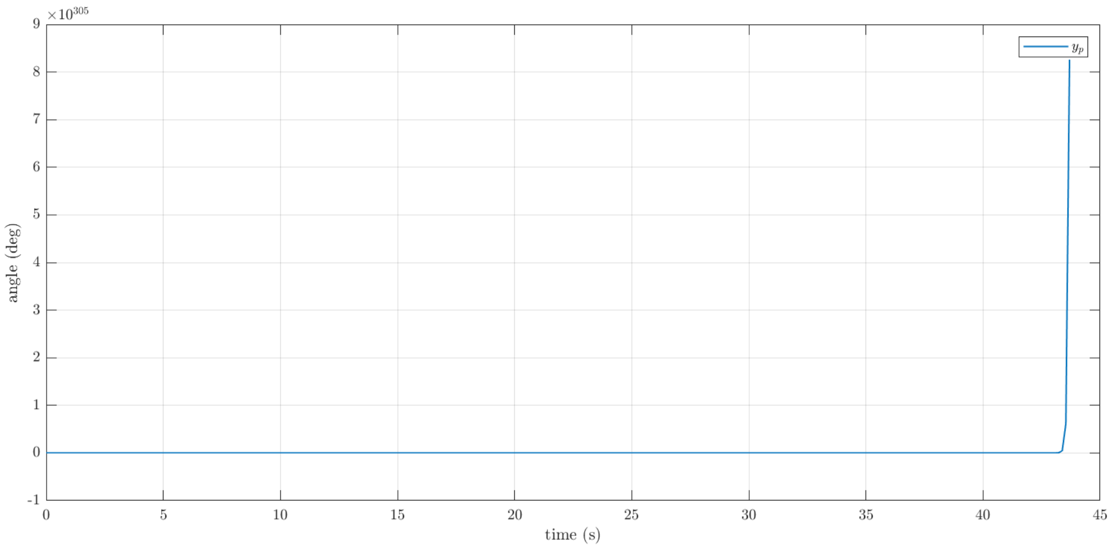

Figure 8: *Real system output with respect to time (0-45s)*

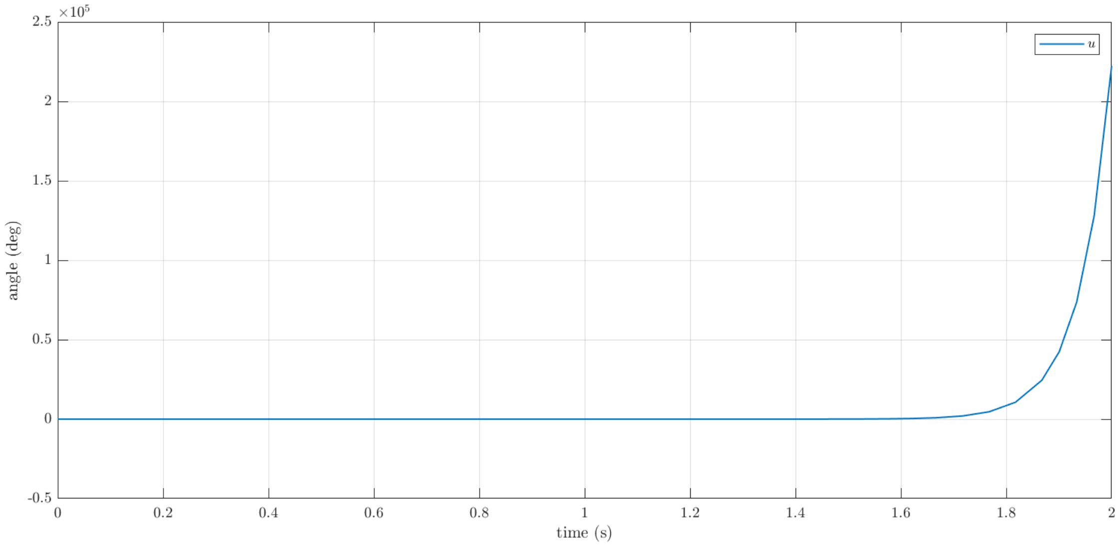

Figure 9: *Elevator deflection angle (control input) with respect to time (0-2s)*

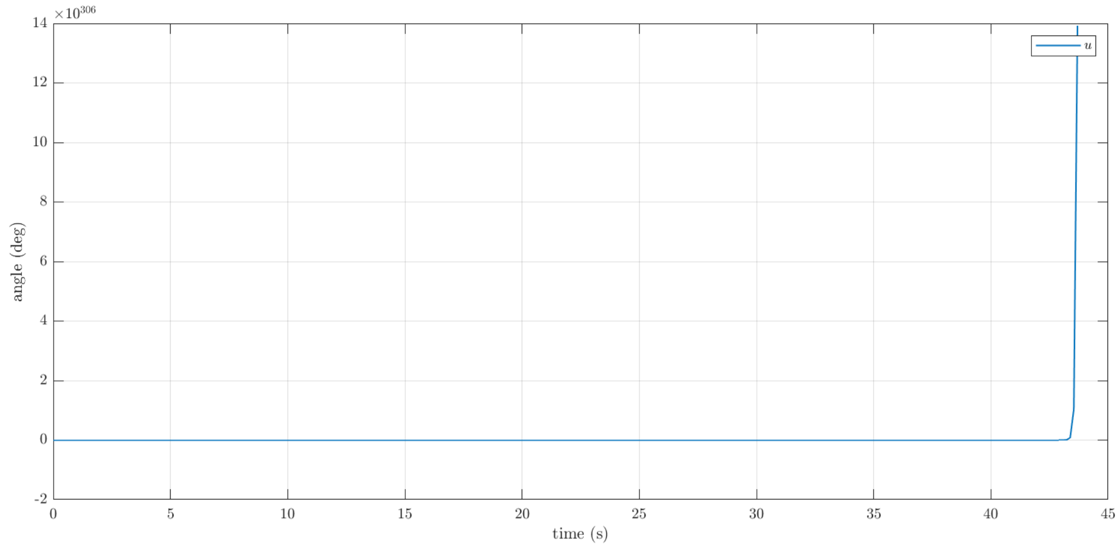

Figure 10: *Elevator deflection angle (control input) with respect to time (0-45s)*

We can see that in presence of uncertainties, the system becomes unstable as the designed controller is unable to handle uncertainties. As a result, both the *system output* and the *control input* increase exponentially.

In order to avoid the undesirable behavior of the system, we will use the adaptive control method of section 3. We design the same *augmented reference model* as previously, which makes y_m to track r and following the described procedure we reach the adaptive laws of (27), so we implement the controller of (20).

Considering as unknown the values of uncertainties that we have set, we have to select the adaption rate matrices Γ_K and Γ_θ. After tuning, we select

$$\Gamma_K = \begin{bmatrix} 5,000,000 & 0 & 0 \\ 0 & 5,000,000 & 0 \\ 0 & 0 & 5,000,000 \end{bmatrix},$$

$$\Gamma_\theta = \begin{bmatrix} 10,000,000 & 0 \\ 0 & 10,000,000 \end{bmatrix}$$

and the results are presented below:

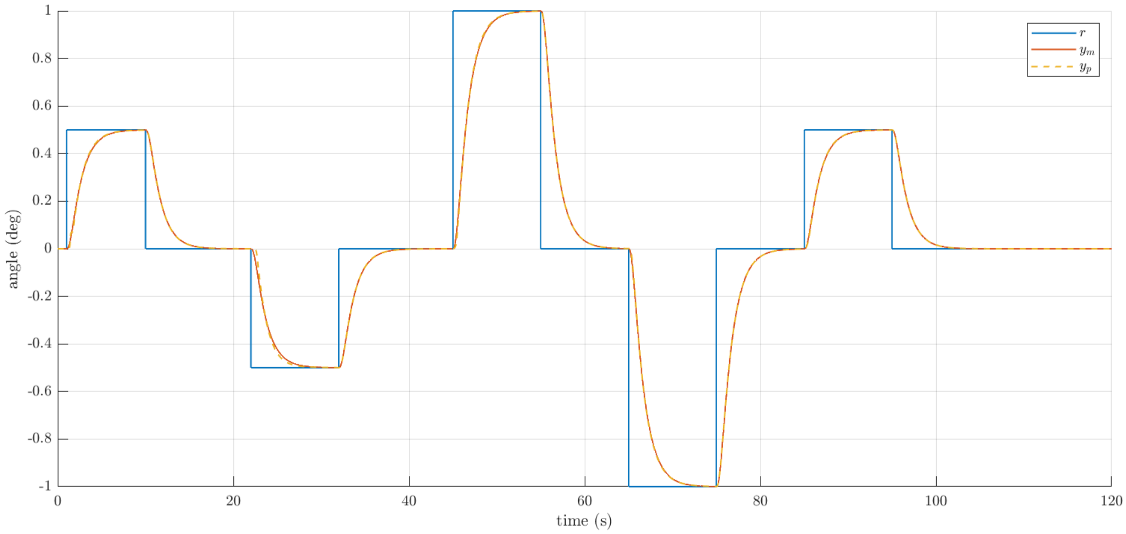

Figure 11: *Reference input, model output and real system output with respect to time*

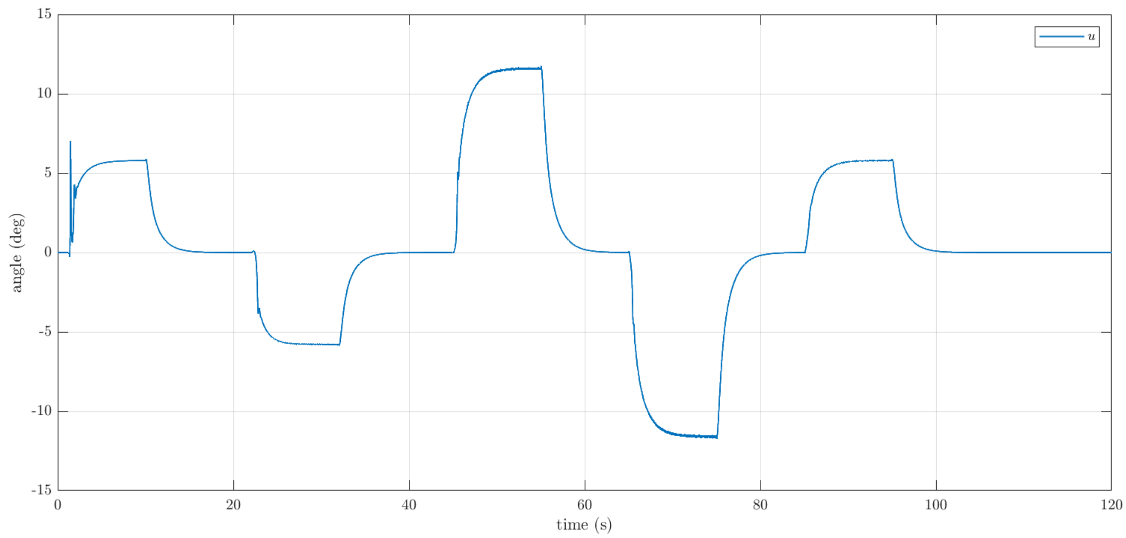

Figure 12: *Elevator deflection angle (control input) with respect to time*

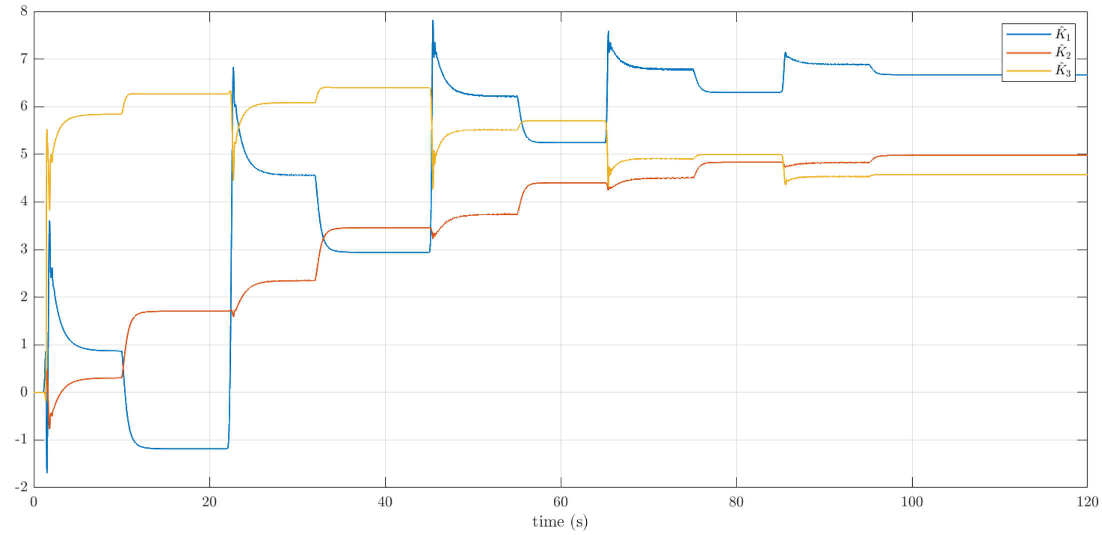

Figure 13: *Adaptive gain estimations $\hat{K}_i$ with respect to time*

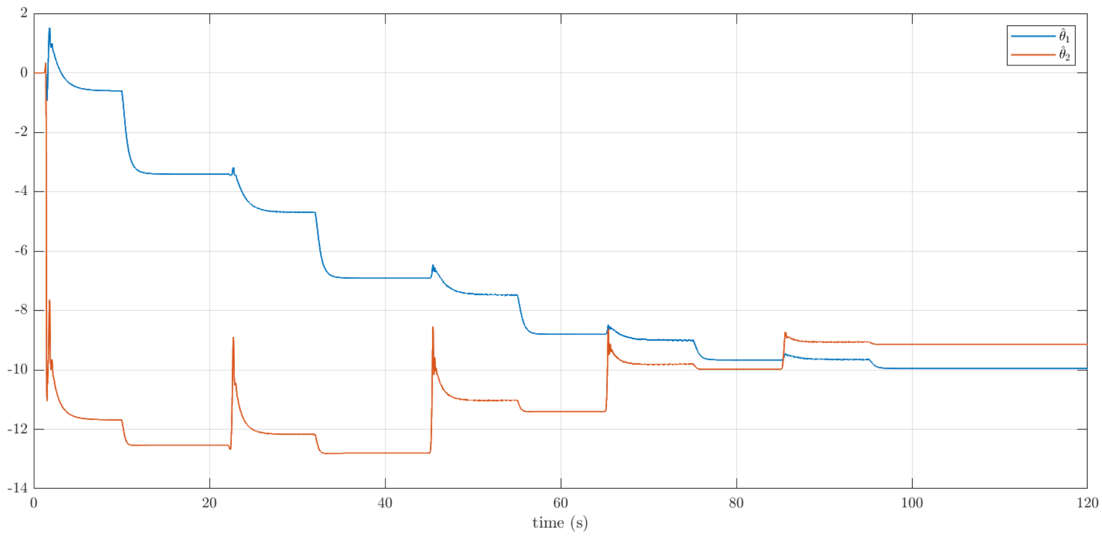

Figure 14: *Adaptive parameter estimations $\hat{\theta}_i$ with respect to time*

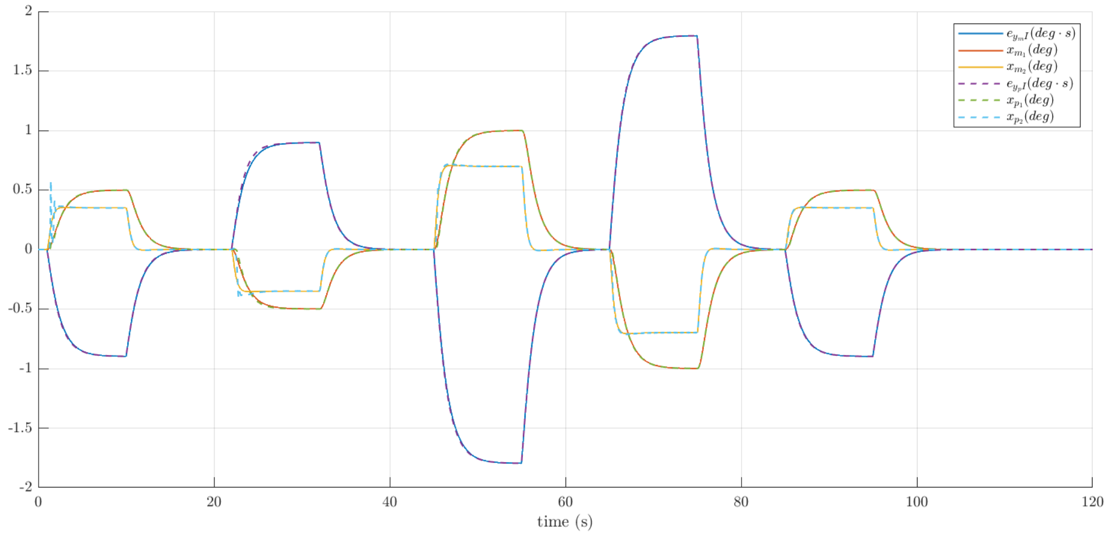

Figure 15: *Augmented state vectors of reference model (z_m) and real plant (z_p)*

- As we can see in figure 11, the adaptive controller can achieve the control goal, canceling out the uncertainties of the system plant.
- In figure 12, we can notice that the control input has much larger amplitude than the case without uncertainties (figure 6), but the two shapes are similar to each other.
- The adaptive signals (figure 13 and 14) look to converege to some values, but the discontinuities of the reference input cause some discontinuities to them too.
- From (22) we can compute the optimal $K = [6 \;\; 3 \;\; 2]^T$. The estimation $\hat{K}$ convereges to a vector near the optimal vector (figure 13).
- The estimation vector $\hat{\theta}$ (figure 14) looks to converege to values near to the real $\theta = [-10 \quad -10]^T$.
- The convergence of the adaptive estimations to their optimal values is not necessary, as it can be exported from the theoretical results too (3.8). Although it is guaranteed that the estimations will converege to constant values, the uncertainties can be cancelled out, thus the control goal can be achieved, without convergence to the optimal values of the parameters.
- In figure 15, we can see the *augmented state vector* of the *real system* tracking *augmented state vector* of the *reference model*.

5. References

[1] Eugene Lavretsky (2012): "*Robust and Adaptive Control: With Aerospace Applications*" - *Chapter 10*: "*Model Reference Adaptive Control with Integral Feedback Connections*"