

# 16-720B Homework 4 Write-up

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November 9, 2019

## Q1.1

Consider the point  $\mathbf{w}$  where the principle axes of the two cameras intersect, and we can see that  $\tilde{\mathbf{x}}_1 = [0, 0, 1]^T$  and  $\tilde{\mathbf{x}}_2 = [0, 0, 1]^T$  corresponding one point in 3D. Therefore

$$\tilde{\mathbf{x}}_2^T \mathbf{E} \tilde{\mathbf{x}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{21} & \mathbf{E}_{22} & \mathbf{E}_{23} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{E}_{33} = 0 \quad (1)$$

Since two cameras are normalized, the intrinsic matrices for them are identity:  $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{I}$ . Then  $\mathbf{E} = \mathbf{K}_1^T \mathbf{F} \mathbf{K}_2 = \mathbf{E}$ . Therefore,  $\mathbf{E}_{33} = \mathbf{F}_{33} = 0$ .

### Q1.2

Suppose the cameras are normalized in the sense that their intrinsic matrices are both identity:  $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{I}$ .

Now that the translation and rotation from camera 1 to camera 2 are

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

And thus the essential matrix are

$$\mathbf{E} = \mathbf{t}_{\times} \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \quad (3)$$

Therefore for an epipolar line in camera 1  $\mathbf{l}_1^T \tilde{\mathbf{x}}_1 = 0$  and  $\tilde{\mathbf{x}}_2^T \mathbf{E} \tilde{\mathbf{x}}_1 = 0$ , where  $\tilde{\mathbf{x}}_2$  is a fixed point on the image plane of camera 2 resulting from the ray corresponding to the epipolar line, then we can see that

$$\mathbf{l}_1^T = \tilde{\mathbf{x}}_2^T \mathbf{E} = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & t_x & -t_x y_2 \end{bmatrix} \quad (4)$$

Similarly we can see that any epipolar line in camera 1 has  $\mathbf{l}_2^T = [0 \ -t_x \ t_x y_1]$ . Since the first elements in both  $\mathbf{l}_1$  and  $\mathbf{l}_2$  are zero, the epipolar lines are parallel to  $x$  axis.

### Q1.3

Assume  $(\mathbf{R}_i, \mathbf{t}_i)$  and  $(\mathbf{R}_j, \mathbf{t}_j)$  are the rotation and translation from the world coordinate frame to the camera coordinate frame at time  $i$  and time  $j$ . And suppose  $\mathbf{R}_{rel}$  and  $\mathbf{t}_{rel}$  are the rotation and translation from camera at time  $i$  to the camera at time  $j$ . Then for a point  $\mathbf{w}$  in the 3D world

$$\begin{aligned}
\lambda_i \tilde{\mathbf{x}}_i &= \mathbf{R}_i \mathbf{w} + \mathbf{t}_i, & \lambda_j \tilde{\mathbf{x}}_j &= \mathbf{R}_j \mathbf{w} + \mathbf{t}_j \\
\Rightarrow \mathbf{w} &= \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) \\
\Rightarrow \lambda_j \tilde{\mathbf{x}}_j &= \mathbf{R}_j \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) + \mathbf{t}_j \\
\Rightarrow \lambda_j \tilde{\mathbf{x}}_j &= \mathbf{R}_j \mathbf{R}_i^T \lambda_i \tilde{\mathbf{x}}_i - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i + \mathbf{t}_j \\
\Rightarrow \lambda_j \tilde{\mathbf{x}}_j &= \lambda_i \mathbf{R}_{rel} \tilde{\mathbf{x}}_i + \mathbf{t}_{rel}
\end{aligned} \tag{5}$$

Therefore

$$\mathbf{R}_{rel} = \mathbf{R}_j \mathbf{R}_i^T, \quad \mathbf{t}_{rel} = \mathbf{t}_j - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i \tag{6}$$

Then the essential and fundamental matrix can be derived as

$$\mathbf{E} = (\mathbf{t}_{rel})_{\times} \mathbf{R}_{rel} \tag{7}$$

$$\mathbf{F} = (\mathbf{K}^{-1})^T \mathbf{E} \mathbf{K}^{-1} = (\mathbf{K}^{-1})^T (\mathbf{t}_{rel})_{\times} \mathbf{R}_{rel} \mathbf{K}^{-1} \tag{8}$$

#### Q1.4

Suppose the mirror is orthogonal to a unit vector  $\mathbf{v}$  pointing in to the mirror, then we can have the Householder transformation matrix  $\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T$ . Suppose the real world coordinate has its origin at a point on the mirror, then for any point  $\mathbf{w}$  in the world coordinate, the mirror produce its reflection  $\mathbf{w}_2 = \mathbf{H}\mathbf{w}_1 = \mathbf{w}_1 - 2\mathbf{v}\mathbf{v}^T\mathbf{w}_1 = \mathbf{w}_1 + 2\alpha\mathbf{v}$ , where  $\alpha = -\mathbf{v}^T\mathbf{w}_1$  is the dixed distance from  $\mathbf{w}_2$  to the mirror.

This two points in 3D produce two point on the image plane as follows

$$\lambda_1\tilde{\mathbf{x}}_1 = \mathbf{w}_1 \quad (9)$$

$$\lambda_2\tilde{\mathbf{x}}_2 = \mathbf{w}_2 = \mathbf{w}_1 + 2\alpha\mathbf{v} \quad (10)$$

This is equivalent to a two-camera system where  $\mathbf{R} = \mathbf{I}$  and  $\mathbf{t} = 2\alpha\mathbf{v}$ . Therefore

$$\mathbf{E} = \mathbf{t}_\times \mathbf{R} = 2\alpha\mathbf{v}_\times \mathbf{I} = 2\alpha\mathbf{v}_\times \quad (11)$$

is skew-symmetric. Since there are only one camera with only one intrinsic  $\mathbf{K}$ , for fundamental matrix

$$\mathbf{F} = (\mathbf{K}^{-1})^T \mathbf{E} \mathbf{E}^{-1} \mathbf{F}^T = (\mathbf{K}^{-1})^T \mathbf{E}^T \mathbf{E}^{-1} = -(\mathbf{K}^{-1})^T \mathbf{E} \mathbf{E}^{-1} = -\mathbf{F} \quad (12)$$

Therefore the fundamental matrix  $\mathbf{F}$  of this equivalent two-camera system is symmetric.

### Q2.1

The fundamental matrix  $\mathbf{F}$  given by the 8-point algorithm is

```
[[ 9.80213861e-10 -1.32271663e-07  1.12586847e-03]
 [-5.72416248e-08  2.97011941e-09 -1.17899320e-05]
 [-1.08270296e-03  3.05098538e-05 -4.46974798e-03]] .
```

And visualization result is shown in Figure. 1

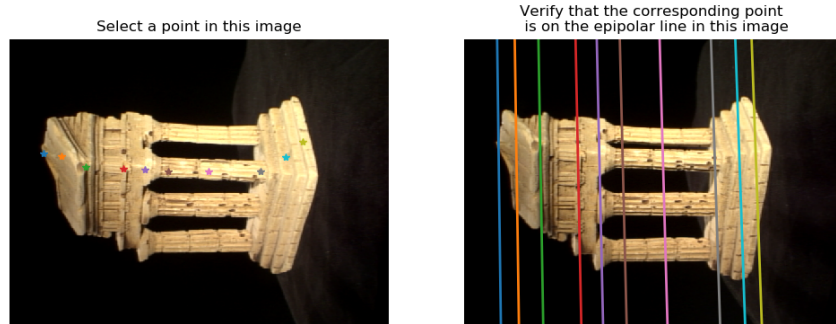


Figure 1: The visualization results showing the fundamental matrix given by 8-point algorithm.

## Q2.2

To expedite the process of searching for the best  $F$ , I iteratively choose 7 points randomly and keep  $F$  with the minimum 2-norm difference from  $F$  given by 8-point algorithm. And the best reasonable  $F$  is given when I use the points at indices [ 10 3 92 108 41 30 99], which is

```
[[-1.29290341e-08  1.81485505e-07  8.27390734e-04]
 [-3.52772006e-07  1.05936161e-09  4.14750371e-05]
 [-7.87641850e-04 -1.90120686e-05 -4.69857949e-03]]
```

And the visualization result is shown in Figure. 2

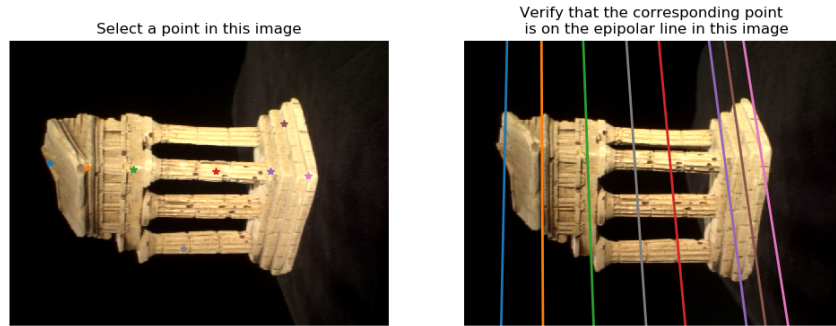


Figure 2: The visualization results showing the fundamental matrix given by 7-point algorithm.

### Q3.1

By applying the equation  $\mathbf{E} = \mathbf{K}_2^T \mathbf{F} \mathbf{K}_1$ , we can get the  $\mathbf{E} =$

```
[[ 2.26587820e-03 -3.06867395e-01  1.66257398e+00]
 [-1.32799331e-01  6.91553934e-03 -4.32775554e-02]
 [-1.66717617e+00 -1.33444257e-02 -6.72047195e-04]]
```

### Q3.2

Suppose  $\tilde{\mathbf{w}}$  is the homogenous coordinate of the 3D point  $\mathbf{w}$ , and it projects  $\tilde{\mathbf{x}}_{i1}$  and  $\tilde{\mathbf{x}}_{i2}$  on camera 1 and camera 2 respectively, which means

$$\mathbf{C}_1 \tilde{\mathbf{w}}_i = \lambda_1 \tilde{\mathbf{x}}_{i1}, \quad \mathbf{C}_2 \tilde{\mathbf{w}}_i = \lambda_2 \tilde{\mathbf{x}}_{i2} \quad (13)$$

Now only consider  $\mathbf{C}_1 \tilde{\mathbf{w}} = \lambda_1 \tilde{\mathbf{x}}_{i1}$ . Suppose  $\tilde{\mathbf{x}}_{i1} = [x_{i1}, y_{i1}, 1]^T$  and  $\mathbf{C}_{11}^T, \mathbf{C}_{12}^T, \mathbf{C}_{13}^T$  are the first, the second, and the third row of the camera matrix  $\mathbf{C}_1$ . We get

$$\begin{cases} \lambda_1 x_{i1} &= \mathbf{C}_{11}^T \tilde{\mathbf{w}}_i \\ \lambda_1 y_{i1} &= \mathbf{C}_{12}^T \tilde{\mathbf{w}}_i \\ \lambda_1 &= \mathbf{C}_{13}^T \tilde{\mathbf{w}}_i \end{cases} \Rightarrow \begin{cases} x_{i1} \mathbf{C}_{13}^T \tilde{\mathbf{w}}_i &= \mathbf{C}_{11}^T \tilde{\mathbf{w}}_i \\ y_{i1} \mathbf{C}_{13}^T \tilde{\mathbf{w}}_i &= \mathbf{C}_{12}^T \tilde{\mathbf{w}}_i \end{cases} \Rightarrow \begin{bmatrix} x_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{11}^T \\ y_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{12}^T \end{bmatrix} \tilde{\mathbf{w}}_i = \mathbf{0} \quad (14)$$

We can get Similar constraints from the projection on camera 2, and by concatenate the constraints together we can get  $\mathbf{A}_i \mathbf{w}_i = 0$  as follows:

$$\mathbf{A}_i \mathbf{w}_i = \begin{bmatrix} x_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{11}^T \\ y_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{12}^T \\ x_{i2} \mathbf{C}_{23}^T - \mathbf{C}_{21}^T \\ y_{i2} \mathbf{C}_{23}^T - \mathbf{C}_{22}^T \end{bmatrix} \tilde{\mathbf{w}}_i = \mathbf{0} \text{ and } \mathbf{A}_i = \begin{bmatrix} x_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{11}^T \\ y_{i1} \mathbf{C}_{13}^T - \mathbf{C}_{12}^T \\ x_{i2} \mathbf{C}_{23}^T - \mathbf{C}_{21}^T \\ y_{i2} \mathbf{C}_{23}^T - \mathbf{C}_{22}^T \end{bmatrix} \quad (15)$$

Then  $\tilde{\mathbf{w}}_i$  is in the null space of  $\mathbf{A}_i$ , which we can get by solving SVD decomposition of  $\mathbf{A}_i$  and getting the last column of  $V$ .



#### Q4.1

Some detected correspondences are shown in Figure. 4

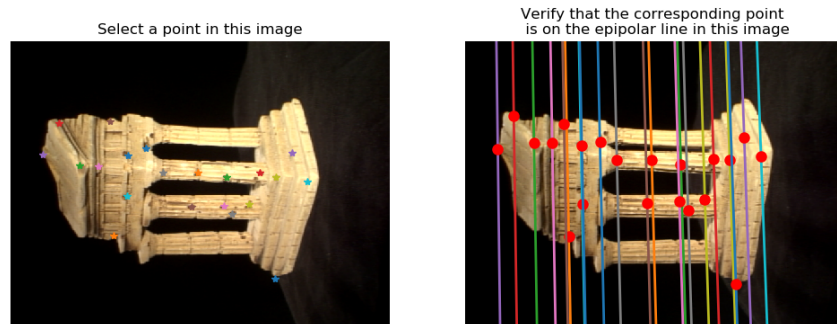
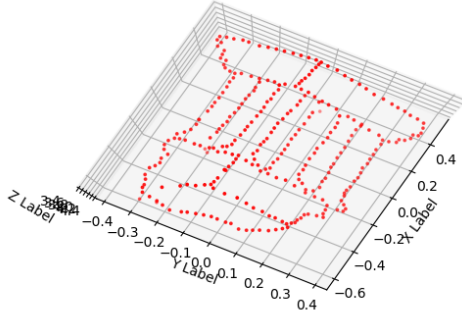


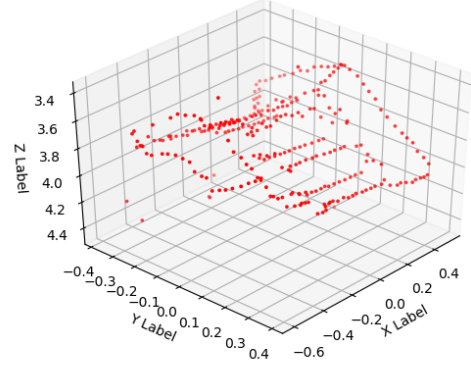
Figure 3: Visualization of some detected correspondences.

## Q4.2

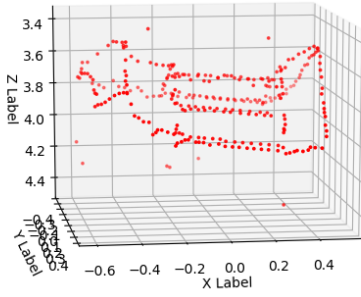
The recovered point cloud can be viewed in Figure. 4.



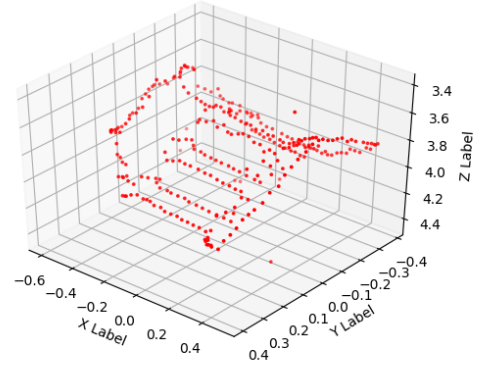
(a) view 1



(b) view 2



(c) view 3



(d) view 4

Figure 4: Different views of the point cloud recovered from `templeCoords`.

### Q5.1

As suggested by the lecture notes: When the fundamental matrix is correct, the epipolar line induced by a point in the first image should pass through the matching point in the second image and vice-versa. Therefore, we used the distance of points to corresponding epipolar lines as the criterion in deciding inliers during RANSAC process. More specifically

$$Err(\tilde{\mathbf{x}}_2, \mathbf{F}, \tilde{\mathbf{x}}_1) = dist^2(\tilde{\mathbf{x}}_2, \mathbf{F}\tilde{\mathbf{x}}_1) + dist^2(\tilde{\mathbf{x}}_1, \mathbf{F}^T\tilde{\mathbf{x}}_2), \quad (16)$$

$$\text{where } dist(\tilde{\mathbf{x}}_2, \mathbf{F}\tilde{\mathbf{x}}_1) = \frac{\tilde{\mathbf{x}}_2^T \mathbf{F} \tilde{\mathbf{x}}_1}{\sqrt{l_1^2 + l_2^2}}, \mathbf{F}\tilde{\mathbf{x}}_1 = [l_1, l_2, l_3]^T \quad (17)$$

And a correspondence is considered an inlier if its error is below a threshold  $\epsilon$ . To decide  $\epsilon$ , I tried different values and found that when  $\epsilon = 2$ , the RANSAC yield around  $140 * 0.75 = 105$  inliers, and thus decide the threshold to be 2.