16-720B Homework 4 Write-up

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Q1.1

Consider the point $\mathbf{w}$ where the principle axes of the two cameras intersect, and we can see that $\tilde{\mathbf{x}}_1 = [0, 0, 1]^T$ and $\tilde{\mathbf{x}}_2 = [0, 0, 1]^T$ corresponding one point in 3D. Therefore

$$\tilde{\mathbf{x}}_2^T \mathbf{E} \tilde{\mathbf{x}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{21} & \mathbf{E}_{22} & \mathbf{E}_{23} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{E}_{33} = 0 \tag{1}$$

Since two cameras are normalized, the intrinsic matrices for them are identity: $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{I}$. Then $\mathbf{E} = \mathbf{K}_1^T \mathbf{F} \mathbf{K}_2 = \mathbf{E}$. Therefore, $\mathbf{E}_{33} = \mathbf{F}_{33} = 0$.

Q1.2

Suppose the cameras are normalized in the sense that their intrinsic matrices are both identity: $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{I}$.

Now that the translation and rotation from camera 1 to camera 2 are

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

And thus the essential matrix are

$$\mathbf{E} = \mathbf{t}_\times \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \tag{3}$$

Therefore for an epipolar line in camera 1 $\mathbf{l}_1^T \tilde{\mathbf{x}}_1 = 0$ and $\tilde{\mathbf{x}}_2^T \mathbf{E} \tilde{\mathbf{x}}_1 = 0$, where $\tilde{\mathbf{x}}_2$ is a fixed point on the image plane of camera 2 resulting from the ray corresponding to the epipolar line, then we can see that

$$\mathbf{l}_1^T = \tilde{\mathbf{x}}_2^T \mathbf{E} = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & t_x & -t_x y_2 \end{bmatrix} \tag{4}$$

Similarly we can see that any epipolar line in camera 1 has $\mathbf{l}_2^T = [0 \ -t_x \ t_x y_1]$. Since the first elements in both $\mathbf{l}_1$ and $\mathbf{l}_2$ are zero, the epipolar lines are parallel to x axis.

Q1.3

Assume $(\mathbf{R}_i, \mathbf{t}_i)$ and $(\mathbf{R}_i, \mathbf{t}_i)$ are the rotation and translation from the world coordinate frame to the camera coordinate frame at time i and time j. And suppose $\mathbf{R}_{rel}$ and $\mathbf{t}_{rel}$ are the rotation and translation from camera at time i to the camera at time j. Then for a point $\mathbf{w}$ in the 3D world

$$\begin{aligned}
&\lambda_i \tilde{\mathbf{x}}_i = \mathbf{R}_i \mathbf{w} + \mathbf{t}_i, \quad \lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{w} + \mathbf{t}_j \\
\Rightarrow &\mathbf{w} = \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) + \mathbf{t}_j \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{R}_i^T \lambda_i \tilde{\mathbf{x}}_i - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i + \mathbf{t}_j \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \lambda_i \mathbf{R}_{rel} \tilde{\mathbf{x}}_i + \mathbf{t}_{rel}
\end{aligned} \tag{5}$$

Therefore

$$\mathbf{R}_{rel} = \mathbf{R}_j \mathbf{R}_i^T, \quad \mathbf{t}_{rel} = \mathbf{t}_j - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i \tag{6}$$

Then the essential and fundamental matrix can be derived as

$$\mathbf{E} = (\mathbf{t}_{rel})_\times \mathbf{R}_{rel} \tag{7}$$

$$\mathbf{F} = (\mathbf{K}^{-1})^T \mathbf{F} \mathbf{K}^{-1} = (\mathbf{K}^{-1})^T (\mathbf{t}_{rel})_\times \mathbf{R}_{rel} \mathbf{K}^{-1} \tag{8}$$

Q1.4

Suppose the mirror is orthogonal to a unit vector $\mathbf{v}$ pointing in to the mirror, then we can have the Householder transformation matrix $\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T$. Suppose the real world coordinate has its origin at a point on the mirror, then for any point $\mathbf{w}$ in the world coordinate, the mirror produce its reflection $\mathbf{w}_2 = \mathbf{H}\mathbf{w}_1 = \mathbf{w}_1 - 2\mathbf{v}\mathbf{v}^T\mathbf{w}_1 = \mathbf{w}_1 + 2\alpha\mathbf{v}$, where $\alpha = -\mathbf{v}^T\mathbf{w}_1$ is the dixed distance from $\mathbf{w}_2$ to the mirror.

This two points in 3D produce two point on the image plane as follows

$$\lambda_1\tilde{\mathbf{x}}_1 = \mathbf{w}_1 \tag{9}$$

$$\lambda_2\tilde{\mathbf{x}}_2 = \mathbf{w}_2 = \mathbf{w}_1 + 2\alpha\mathbf{v} \tag{10}$$

This is equivalent to a two-camera system where $\mathbf{R} = \mathbf{I}$ and $\mathbf{t} = 2\alpha\mathbf{v}$. Therefore

$$\mathbf{E} = \mathbf{t}_\times\mathbf{R} = 2\alpha\mathbf{v}_\times\mathbf{I} = 2\alpha\mathbf{v}_\times \tag{11}$$

is skew-symmetric. Since there are only one camera with only one intrinsic $\mathbf{K}$, for fundamental matrix

$$\mathbf{F} = (\mathbf{K}^{-1})^T\mathbf{E}\mathbf{E}^{-1}\mathbf{F}^T = (\mathbf{K}^{-1})^T\mathbf{E}^T\mathbf{E}^{-1} = -(\mathbf{K}^{-1})^T\mathbf{E}\mathbf{E}^{-1} = -\mathbf{F} \tag{12}$$

Therefore the fundamental matrix $\mathbf{F}$ of this equivalent two-camera system is symmetric.