16-720B Homework 4 Write-up

Gu, Qiao

October 30, 2019

Q1.1

Consider the point $\mathbf{w}$ where the principle axes of the two cameras intersect, and we can see that $\tilde{\mathbf{x}}_1 = [0, 0, 1]^T$ and $\tilde{\mathbf{x}}_2 = [0, 0, 1]^T$ corresponding one point in 3D. Therefore

$$\tilde{\mathbf{x}}_2^T \mathbf{E} \tilde{\mathbf{x}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{21} & \mathbf{E}_{22} & \mathbf{E}_{23} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{E}_{33} = 0 \tag{1}$$

Suppose the intrinsic matrix for two cameras are

$$\mathbf{K}_1 = \begin{bmatrix} f_{1x} & \gamma_1 & 0 \\ 0 & f_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{K}_2 = \begin{bmatrix} f_{2x} & \gamma_2 & 0 \\ 0 & f_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Then

$$\mathbf{E} = \mathbf{K}_1^T \mathbf{F} \mathbf{K}_2 = \begin{bmatrix} f_{1x} & 0 & 0 \\ \gamma_1 & f_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} \\ \mathbf{F}_{21} & \mathbf{F}_{22} & \mathbf{F}_{23} \\ \mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{F}_{33} \end{bmatrix} \begin{bmatrix} f_{2x} & \gamma_2 & 0 \\ 0 & f_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} f_{1x} & 0 & 0 \\ \gamma_1 & f_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dots & \dots & \mathbf{F}_{13} \\ \dots & \dots & \mathbf{F}_{23} \\ \dots & \dots & \mathbf{F}_{33} \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \mathbf{F}_{33} \end{bmatrix} \tag{4}$$

Therefore, $\mathbf{E}_{33} = \mathbf{F}_{33} = 0$.

Q1.2

Suppose the cameras are normalized in the sense that their intrinsic matrices are both identity. Now that the translation and rotation from camera 1 to camera 2 are

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

According to the dfinition of essential matrix

$$\tilde{\mathbf{x}}_2^T \mathbf{t} \times \mathbf{R} \tilde{\mathbf{x}}_1 = \tilde{\mathbf{x}}_2^T \mathbf{t}_\times \mathbf{R} \tilde{\mathbf{x}}_1 = \tilde{\mathbf{x}}_2^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}_1 \tag{6}$$

$$= \tilde{\mathbf{x}}_2^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \tilde{\mathbf{x}}_1 = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -t_x \\ t_x y_1 \end{bmatrix} \tag{7}$$

$$= -t_x y_2 + t_x y_1 = 0 \tag{8}$$

$$\Rightarrow y_1 = y_2 \tag{9}$$

For a certain epipolar line in camera 2, it has a fixed epipole with fixed y_1 in camera 1, and thus every 3D point on the corresponding incident ray to camera 1 has a projection with a fixed $y_2 = y_1$ in camera 2, which means the epipolar line has a fixed y coordinate, and thus parallel to x-axis.

The above deduction also holds if camera 1 and 2 are exchanged.

Q1.3

Assume $(\mathbf{R}_i, \mathbf{t}_i)$ and $(\mathbf{R}_i, \mathbf{t}_i)$ are the rotation and translation at time i and time j. Then for a point $3d$ in the 3D world

$$\begin{aligned}
&\lambda_i \tilde{\mathbf{x}}_i = \mathbf{R}_i \mathbf{w} + \mathbf{t}_i, \quad \lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{w} + \mathbf{t}_j \\
\Rightarrow &\mathbf{w} = \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{R}_i^T (\lambda_i \tilde{\mathbf{x}}_i - \mathbf{t}_i) + \mathbf{t}_j \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \mathbf{R}_j \mathbf{R}_i^T \lambda_i \tilde{\mathbf{x}}_i - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i + \mathbf{t}_j \\
\Rightarrow &\lambda_j \tilde{\mathbf{x}}_j = \lambda_i \mathbf{R}_{rel} \tilde{\mathbf{x}}_i + \mathbf{t}_{rel}
\end{aligned} \tag{10}$$

Therefore

$$\mathbf{R}_{rel} = \mathbf{R}_j \mathbf{R}_i^T, \quad \mathbf{t}_{rel} = \mathbf{t}_j - \mathbf{R}_j \mathbf{R}_i^T \mathbf{t}_i \tag{11}$$

Q1.4