

Homework 3: Convex Sets

Instructions

Please submit a .qmd file along with a rendered pdf to the Brightspace page for this assignment. You may use whatever language you like within your qmd file, I recommend python, julia, or R.

Problem 1 (cvx-book 2.12):

Which of the following sets are convex? For each case give the reason(s) why or why not

- A slab, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq \mathbf{a}^T \mathbf{x} \leq \beta\}$.
- A rectangle, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$. A rectangle is sometimes called a hyperrectangle when $n > 2$.
- A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_1^T \mathbf{x} \leq \mathbf{b}_1, \mathbf{a}_2^T \mathbf{x} \leq \mathbf{b}_2\}$
- The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2 \text{ for all } y \in S\}$ where $S \subset \mathbb{R}^n$.
- The set of points closer to one set than another, i.e., $\{\mathbf{x} \mid \text{dist}(\mathbf{x}, S) \leq \text{dist}(\mathbf{x}, T)\}$, where S, T are subsets of \mathbb{R}^N , and $\text{dist}(x, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|^2 \mid \mathbf{z} \in S\}$.

Problem 2 (cvx-book 2.15):

Some sets of probability distributions. Let x be a real-valued random variable with probability distribution

$$\mathbf{prob}(x = a_i) = p_i, i = 1, \dots, n$$

,

where: $a_1 < a_2 < \dots < a_n$. Of course $\mathbf{p} \in \mathbb{R}^n$ lies in the standard probability simplex $P\{\mathbf{p} \mid 1^T \mathbf{p} = 1, \mathbf{p} \succeq 0\}$. Which of the following conditions are convex in \mathbf{p} ? (That is, for which of the following conditions is the set of $\mathbf{p} \in P$ that satisfy the condition convex?) For each case give the reason(s) why or why not.

- The set of all \mathbf{p} where the expectation of the function $f(x)$ is between two limits: $\alpha \leq Ef(x) \leq \beta$, $Ef(x) = \sum_{i=1}^n p_i f(a_i)$. Here $f(x)$ is a function from \mathbb{R} to \mathbb{R} .
- The set of all \mathbf{p} such that the probability that $\mathbf{prob}(x \leq \alpha) \leq \beta$
- The set of all \mathbf{p} such that the expectation of $|x|^3$ is greater than a given constant α times the expectation of $|x|$: $E|x^3| \leq \alpha E|x|$
- The set of all \mathbf{p} such that the expectation of x^2 is less than a given constant α : $Ex^2 \leq \alpha$

Problem 3: Bounded Value Least Squares for Wine Mixing

We have seen several examples so far in the course where we would like to have inequality constraints on the decision variable for our least squares problem, for example to prevent non-sensical solutions like spending a negative amount of money on advertising, limiting the total investment in certain types of assets, or perhaps bounding the value of a statistical coefficient to a certain range. Non-negative least squares is a type of least squares problem where the decision variables $\mathbf{x} \geq 0$, and Bounded-Value Least Squares allows for more general constraints.

This type of least-squares problem needs to be solved algorithmically, and we will use it to get our first practice using the CVX software package. You should install CVXPY, CVXR, or a flavor of CVX compatible with whatever software you are using to solve the problem and use CVX to solve this problem.

The problem is one of finding a mixture of wines which achieves certain chemical characteristics. I have attached a dataset which contains data on the chemical composition of 6 different wines (the dataset originates from kaggle but is reduced for our purposes). Each wine is described according to 11 chemical characteristics, including `alcohol`, `residual sugar`, `chlorides`, etc. I have also provided data for the chemical composition for a target wine.

- [wine_data.csv](#)
- [target.csv](#)

The goal of this problem is to find the blend of wines which has chemical characteristics closest to the target wine.

Concretely, you are solving for weights \mathbf{p} . The concentration of chemical i in wine j is given by the matrix C_{ij} , and the concentration in the blended wine is:

$$c_{blend,i} = \sum_{j=1}^6 C_{ij} p_j,$$

so that the overall concentration vector in the blend satisfies:

$$\mathbf{c}_{blend} = C\mathbf{p}$$

The vector \mathbf{p} is a discrete probability distribution, meaning that all entries are non-negative and must sum to 1 (you can't add negative wine). The range of each chemical varies greatly, so our objective function should incorporate a penalty that is weighted according to the magnitude of the value in the target function:

$$\min_{\mathbf{p}} \sum_{i=1}^{11} \left(\frac{c_i - c_{blend,i}}{c_i} \right)^2$$

Implement this least squares optimization problem using CVX and determine the optimal blend of wines to match the target.