Financial Economics

7 Dynamic Finance Economy

LEC, SJTU

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk
aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

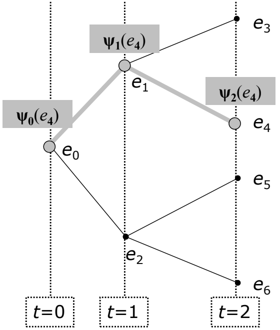
Theoretical resolutions

Introduction

- We will extend the model of the previous chapter so that it will accommodate multiple and even infinitely many periods. We need to think about several issues:
  - How to define assets in a multi-period model
  - How to model intertemporal preferences
  - What market completeness means in this environment
  - How the infinite horizon may affect the sensible definition of a budget constraint (Ponzi schemes)
  - And how the infinite horizon may affect pricing (bubbles)

Multiple Period Uncertainty

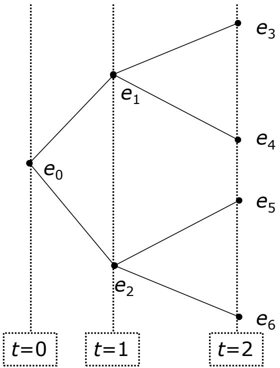
- Multiple period uncertainty: We will define assets to be dividend streams which are conditional on events (node of the uncertainty tree) and derive the fundamental pricing formula



- We recall the event tree that captures the gradual resolution of uncertainty
- This tree has 7 events (e_0 to e_6)
- They belong to 3 time periods (0 to 2)
- If e is some event, we denote the period it belongs to as $\tau(e)$
- So for instance, $\tau(e_2) = 1$, $\tau(e_4) = 2$
- We denote a path with ψ as the figure shows

Multiple Period Uncertainty

- The events of the last period are associated with probabilities, $\pi_3, \ldots, \pi_6$



- The earlier events also have probabilities
- To be consistent, the probability of an event is equal to the sum of the probabilities of its successor events
- So for instance, $\pi_1 = \pi_3 + \pi_4$
- $x^{\langle t \rangle}$ is the random variable that consists of the realizations of x in the events that belong to period t
- For instance, if w is aggregate event-contingent endowment, then $w^{\langle t \rangle}$ is the random aggregate endowment in period t

Multiple Period Assets

- A typical multiple period asset is a risk-free coupon bond:

$$r_e \coloneqq \begin{cases} \text{coupon} & \text{if } 0 < \tau(e) < t^* \\ 1 + \text{coupon} & \text{if } \tau(e) = t^* \\ 0 & \text{if } \tau(e) > t^* \end{cases}$$

- The coupon bond pays the coupon in each period before it expires, and pays the coupon plus the principal in the expiration period t^*.
- A consol is a coupon bond with $t^* = \infty$; it pays a coupon forever
- A discount bond (or zero-coupon bond) is a coupon bond with finite time to maturity but no coupon. It just pays 1 at expiration, and nothing otherwise

Multiple Period Assets (STRIPS)

- From an ordinary coupon bond, one can create so-called STRIPS by extracting only those payments that occur in a particular period
- Mathematically, STRIPS are the same as discount bonds
- This means that a coupon bond can be disaggregated into a collection of STRIPS, which is useful for empirical and practical work
- More generally, arbitrary assets (not just bonds) could be striped

Multiple Period Assets (Shares and Derivatives)

- Shares are another class of assets. These are claims to the future dividends of a particular firm
- Various derivative assets are routinely traded as well
  - A call option on a share is an asset that pays either zero or $q - x$, where q is the (random) price of the share on which the call is written, and x is the strike price
  - For instance, if the exercise is 100 and the share price is 97 at the time the option exires, then the cash flow of the option is zero
  - If the share price would have been 102 instead, the call would have delivered a payment of two
  - Options of this kind will be helpful when thinking about how to make a market system complete

Time Preferences with Many Periods

- Most of us are impatient in the sense that we would rather consume earlier than later if everything else remains the same
- In other words, most people do not like to wait
- A simple way to capture this idea is to assume that utility is additively separable through time and has the same form in all periods, but is weighted less the further in the future consumption takes place

Time Preference

- The utility can be expressed as

$$v(y_0) + \sum_{t=1}^{T} \delta(t) E\{v(y^{\langle t \rangle})\}$$

- or without uncertainty for simplicity:

$$v(y_0) + \sum_{t=1}^{T} \delta(t) v(y_t)$$

- $\delta(t)$ is a number between 0 and 1, indicating the weight given to utility in period t
- We assume $\delta(t) > \delta(t+1)$ for all t
- Suppose you are in period 0 and you make a plan of your present and future consumption: $y^0, y^1, \dots$
- The relation between consecutive consumption will depend on the interpersonal rate of substitution, which is $\delta(t)$

Time Consistency

- Suppose you lived through the first period and start the second...
- ... and suppose you have the possibility to revise your consumption plan. Will you do that, or will you stick to your original plan?
  - We say that your plan is time-consistent if you will stick to it in the future
- You will stick to your plan if and only if your intertemporal rates of substitution remain unchanged
- This is the case if we interpret $\delta(t)$ as being the weight of utility depending on calendar time

Time Consistency and Exponential Discounting

- But if $\delta(t)$ is meant to capture weights in relative time, i.e.
  - $\delta(0)$ is the weight of present utility,
  - $\delta(1)$ is the weight of tomorrow's utility,
  - etc...
- In that case, the marginal rate of intertemporal substitution between period 1 and 2 as of period 0 is $\delta(3)/\delta(2)$...
- ... but as of period 1, this rate is $\delta(2)/\delta(1)$
- Thus, time consistency requires $\delta(3)/\delta(2) = \delta(2)/\delta(1)$
- This implies that the δ-function is a power function, $\delta(t) = \delta^t$, with δ a constant

Pricing in a Static Dynamic Model

- We consider pricing in a model that contains many periods (possibly infinitely many)...
- ...and we assume that information is gradually revealed (this is the dynamic part)...
- ...but we also assume that all assets are only traded "at the beginning of time" (this is the static part)
- There is dynamics in the model because there is time, but the decision making is completely static
- Only after that will we move to a dynamic-dynamic model, one in which assets are repeatedly traded

Maximization over Many Periods

- Consider a representative agent who exponentially discounts a von Neumann-Morgenstern utility. The max-problem is:

$$\max \left\{ v(y^0) + \sum_{t=1}^{T} \delta^t E\{v(y^{\langle t \rangle})\} \middle| y - w \in \mathcal{M}(q) \right\}$$

- If all Arrow securities (conditional on each event) are traded, we can express the first-order conditions as follows:

$$v'(w^0) = \lambda, \quad \delta^{\tau(e)} \pi_e v'(w^e) = \lambda \alpha_e$$

Multi-Period SDF

- The equilibrium SDF is computed in the same fashion as in the static model we saw before

$$\frac{\alpha_e}{\pi_e} = \delta^{\tau(e)} \frac{v'(w^e)}{v'(w^0)}$$

$$= \left(\delta \frac{v'(w^{\psi_1(e)})}{v'(w^0)}\right) \left(\delta \frac{v'(w^{\psi_2(e)})}{v'(w^{\psi_1(e)})}\right) \cdots \left(\delta \frac{v'(w^e)}{v'(w^{\psi_{\tau(e)-1}(e)})}\right)$$

- We call $M_e = \delta \frac{v'(w^e)}{v'(w^{\psi_{\tau(e)-1}(e)})}$ the "one-period ahead" SDF and $\mathbf{M}_e$ the multi-period SDF

$$\mathbf{M}_e := M_{\psi_1(e)} M_{\psi_2(e)} \cdots M_{\psi_{\tau(e)}(e)} = \prod_{t'=1}^{\tau(e)} M_{\psi_{t'}(e)}$$

The Fundamental Pricing Formula

- To price an arbitrary asset r, we will consider it as a portfolio of STRIPed cash flows, so $r^j = r^j_{\langle 1 \rangle} + r^j_{\langle 2 \rangle} + \ldots + r^j_{\langle t \rangle}$, where $r^j_{\langle t \rangle}$ denotes the cash flows of r^j that take place in period t
- The price of asset r^j is simply the sum of the prices of its STRIPed payoffs, so:

$$q_j = \sum_{t=1}^{t} E\{\mathbf{M}_{\langle t \rangle} r^j_{\langle t \rangle}\}$$

- This is the fundamental pricing formula
- Note that $\mathbf{M}_{\langle t \rangle} = \delta^t$ if the representative agent is risk-neutral. The fundamental pricing formula then just reduces to the present value of expected dividends, $q_j = \sum_{t=1}^{T} \delta^t E\{r^j_{\langle t \rangle}\}$

Lucas's Tree Model

- Think of an economy in which the only endowment of agent i is, metaphorically speaking, a "tree"
- The tree produces a stochastic amount of fruit each period, $(r_i^0, r_i^{\langle 1\rangle}, r_i^{\langle 2\rangle}, \ldots)$, which is purely exogenous
- Assume that there is a (quasi-)complete market that allows them to mutually insure each other. The only risk is aggregate, $w := \sum_{i=1}^{I} w_i / I$
- This is the setup of Robert Lucas' s (1978) famous "tree model"
- For $r^j = w$, there is one special case if the representative agent has a Bernoulli' s specification, $v = \ln$:

$$q = \sum_{t=1}^{T} E\left\{\delta^t \frac{1/w^{\langle t\rangle}}{1/w^0} w^{\langle t\rangle}\right\} = \sum_{t=1}^{T} \delta^t w^0 = \frac{\delta(1-\delta^T)}{1-\delta} w^0$$

- With other utility functions, no concrete statements are possible unless we impose some assumptions on the endowment process

Dynamic Completion

- In the "static dynamic" model we assumed that there were many periods and information was gradually revealed (this is the dynamic part)...
- ... but all assets are traded "at the beginning of time" (this is the static part)
- Now we want to explore the consequences of re-opening financial markets. Assets can be traded at each instant
- This has deep implications. It opens up some nasty possibilities (Ponzi schemes and bubbles), but also allows us to reduce the number of assets available at each instant through dynamic completion. We explore the mechanics of this first

Completion with Short-Lived Assets

- If the horizon is infinite, the number of events is also infinite. Does that imply that we need an infinite number of assets to make the market complete?
- More generally, do we need assets with all possible times to maturity to have a complete market?
- The answer is no. This was shown by Guesnerie and Jaffray (1974) building on work by Arrow (1953)

Completion with Short-Lived Assets

- Call an asset 'short lived' if it pays out only in the period immediately after the asset is issued
- Suppose for each event e and each successor event e' there is an asset that pays in e' and nothing otherwise
- It is possible to achieve arbitrary transfers between all events in the event tree by trading only these short-lived assets
- This is straightforward if there is no uncertainty

Completion with Short-Lived Assets

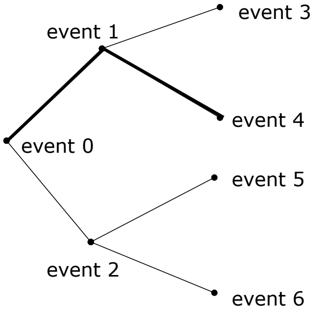
- Without uncertainty, and T periods (T can be infinite), there are T one period assets, from period 0 to period 1, from period 1 to 2, etc
- Let q_t be the price of the bond that begins in period $t-1$ and matures in period t
- For the market to be complete we need to be able to transfer wealth between any two periods, not just between consecutive periods
- This can be achieved with a trading strategy

Completion with Short-Lived Assets

- Example: Suppose we want to transfer wealth from period 1 to period 3
- In period 1 we cannot buy a bond that matures in period 3, because such a bond is not traded then
- Instead buy a bond that matures in period 2, for price q_2
- In period 2, use the payoff of the period-2 bond to buy period-3 bonds
- In period 3, collect the payoff
- The result is a transfer of wealth from period 1 to period 3. The price, as of period 1, for one unit of purchasing power in period 3, is $q_2 q_3$

Completion with Short-Lived Assets

- With uncertainty the process is only slightly more complicated. It is easily understood with a graph



- Let q_e be the price of the asset that pays one unit in event e. This asset is traded only in the event immediately preceeding e
- We want to transfer wealth from event 0 to event 4
- Go backwards: in event 1, buy one event 4 asset for a price q_4
- In event 0, buy q_4 event 1 assets
- The cost of this today is $q_1 q_4$. The payoff is one unit in event 4 and nothing otherwise

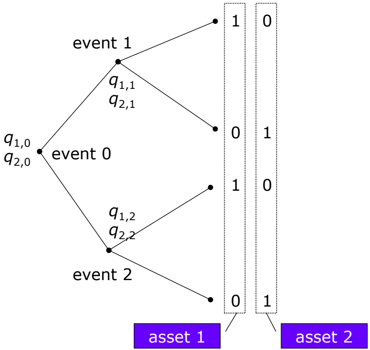
Completion with Long-Lived Assets

- David Kreps (1982) has shown that dynamic completion can also be achieved using only long-lived assets
- This is most easily seen without uncertainty
- Consider a T-period model without uncertainty (assume T is finite for now)
- A complete asset structure must allow agents to transfer purchasing power between any two periods
- But assume there is a single asset: a discount bond maturing in T
- This bond can be purchased and sold in each period, for price $q_t, t = 1, \ldots, T$

Completion with a Long Maturity Bond

- So there are T prices (not simultaneously, but sequentially, but this is good enough)
- Purchasing power can be transferred from period t to period $t' > t$ by purchasing the bond in period t and selling it in period t'
- If t' is earlier than t then the same can be achieved by first selling the bond short
- This is how a dynamic trading plan (when to buy and sell which assets) extends the market span, in this case, makes it complete
- The argument is more subtle with uncertainty

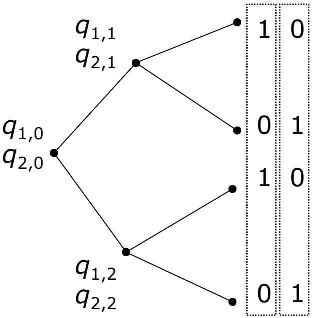
A Simple Information Tree



- This information tree has three non-trivial events plus four final states, so seven events altogether
- It seems as if we would need six Arrow securities (for events 1 and 2 and for the four final states) to have a complete market. Yet we have only two assets. So the market cannot be complete, right?
- Wrong! Dynamic trading provides a way to fully insure each event separately
- Note that there are six prices because each asset is traded in three events

One-Period Holding

- Call "asset $[j, e]$" the cash flow of asset j that is purchased in event e and is sold one period later
- How many such assets exist? What are their cash flows?



- There are six such assets: $[1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]$ (Note that this is potentially sufficient to span the complete space)
- "Asset $[1, 1]$" costs $q_{1,1}$ and pays out 1 in the first final state and zero in all other events
- "Asset $[1, 0]$" costs $q_{1,0}$ and pays out $q_{1,1}$ in event 1, $q_{1,2}$ in event 2, and zero in all the final states

The Extended Return Matrix

- The trading strategies $[1,0] \ldots [2,2]$ give rise to a new 6×6 return matrix
- This matrix is regular (and hence the market complete) if the grey submatrix is regular

asset	[1,0]	[2,0]	[1,1]	[2,1]	[1,2]	[2,2]
event 0	$-q_{1,0}$	$-q_{2,0}$	0	0	0	0
event 1	$q_{1,1}$	$q_{2,1}$	$-q_{1,1}$	$-q_{2,1}$	0	0
event 2	$q_{1,2}$	$q_{2,2}$	0	0	$-q_{1,2}$	$-q_{2,2}$
state 1	0	0	1	0	0	0
state 3	0	0	0	1	0	0
state 3	0	0	0	0	1	0
state 4	0	0	0	0	0	1

The Extended Return Matrix

- Is the grey submatrix regular?
- Components of submatrix are prices of the two assets, conditional on period 1 events
- There are cases in which (q_{11}, q_{21}) and (q_{12}, q_{22}) are collinear in equilibrium
- If per capita endowment is the same in event 1 and 2, in state 1 and 3, and in state 2 and 4, respectively, and if the probability of reaching state 1 after event 1 is the same as the probability of reaching state 3 after event 2, then the submatrix is singular
- But then events 1 and 2 are effectively identical, and we may collapse them into a single event

The Extended Return Matrix

- A random square matrix is regular. So outside of special cases, the grey submatrix is regular (it is regular "almost surely")
- The 2x2 submatrix may still be singular simply by accident
- In that case, it can be made regular again by applying a small perturbation of the returns of the long-lived assets, by perturbing aggregate endowment, the probabilities, or the utility function
- Generically, the market is dynamically complete

How Many Assets?

- In the example we just studied, two long-lived assets were sufficient to complete the market
- How many assets are necessary in general?
- The maximum number of branches fanning out from any event in the uncertainty tree is called the branching number. This is also the number of assets necessary to achieve dynamic completion
- Generalization by Duffie and Huang (1985): continuous time leads to continuity of events, but a small number of assets is sufficient
- The large power of the event space is matched by continuously trading few assets, thereby generating a continuity of trading strategies and of prices

Ponzi Schemes: Infinite Horizon Max. Problem

- Infinite horizon allows agents to borrow an arbitrarily large amount without effectively ever repaying, by rolling over the principal and the interest on this debt forever
- Such a scheme is known as a Ponzi scheme. It allows infinite consumption
- Consider an infinite horizon model, no uncertainty, and a complete set of short-lived bonds

$$\max \left\{ \sum_{t=0}^{\infty} \delta^t v(y^t) \;\middle|\; \begin{array}{ll} y^0 - w^0 \leqslant -\beta_1 z^1 & \\ y^t - w^t \leqslant z^t - \beta_{t+1} z^{t+1} & \text{for } t > 0 \end{array} \right\}$$

- z_t is the amount of bonds maturing in period t in the portfolio, β_t is the price of this bond as of period $t-1$

Ponzi Schemes: Rolling Over Debt Forever

- Note that the following consumption path is possible: $y_t = w^t + 1$ for all t
- It says that the agent consumes more than his endowment in each period, forever
- This can be financed with ever-increasing debt: $z^1 = -1/\beta_1$, $z^2 = (-1 + z^1)/\beta_2$, $z^3 = (-1 + z^2)/\beta_3$, etc
- Of course, Ponzi schemes can never be part of an equilibrium. In fact, such a scheme even destroys the existence of a utility maximum because the choice set of an agent is unbounded above. We need an additional constraint

Ponzi Schemes: Transversality

- The constraint that is typically imposed on top of the budget constraint is the transversality condition, $\lim_{t\to\infty} \beta_t z^t \geq 0$.
- This constraint implies that the value of debt cannot diverge to infinity
- More precisely, it requires that all debt must be redeemed eventually (i.e., in the limit)

Bubbles

- In an infinite dynamic model, in which assets are traded repeatedly, there are additional solutions besides the "fundamental pricing formula."
- These new solutions have an additional "bubble component."
- This phenomenon is easiest to study in a model without uncertainty
- Consider a consol, i.e. an asset delivering 1 in each period, forever

The Price of a Consol

- According to the static-dynamic model (the fundamental pricing formula), the price of the consol is $q = \sum_{t=1}^{\infty} \mathbf{M}_t$
- Consider re-opening markets now. Let q_t denote the price of the consol as traded at time t
- The price at time 0 is just the sum of all Arrow prices, so

$$q_0 = \sum_{t=1}^{\infty} \alpha_t = \sum_{t=1}^{\infty} \delta^t \frac{v'(w^t)}{v'(w^0)}$$

The Price of a Consol at $t = 1$

- At time 1, the price of the consol is defined analogously: it is the sum of all marginal rates of intertemporal substitution

$$q_1 = \sum_{t=2}^{\infty} \delta^{t-1} \frac{v'(w^t)}{v'(w^1)}$$

- This can be reformulated,

$$q_1 = \delta^{-1} \frac{v'(w^0)}{v'(w^1)} \sum_{t=2}^{\infty} \delta^t \frac{v'(w^t)}{v'(w^0)}$$

- The second part (the sum from 2 to ∞) is almost equal to q_0

$$\sum_{t=2}^{\infty} \delta^t \frac{v'(w^t)}{v'(w^0)} = \sum_{t=1}^{\infty} \delta^t \frac{v'(w^t)}{v'(w^0)} - \delta \frac{v'(w^1)}{v'(w^0)} = q_0 - \delta \frac{v'(w^1)}{v'(w^0)}$$

Solving Forward

- More generally, we can express the price at time $t+1$ as a function of the price at time t

$$q_{t+1} = \delta^{-1} \frac{v'(w^t)}{v'(w^{t+1})} \left(q_t - \delta \frac{v'(w^{t+1})}{v'(w^t)} \right)$$

$$q_t = M_{t+1} + M_{t+1} q_{t+1}$$

- We can solve this forward by substituting the $t+1$ version of this equation into the t version, ad infinitum

$$q_0 = M_1 + M_1 q_1 = M_1 + M_1[M_2 + M_2 q_2] = \cdots$$

$$q_0 = \underbrace{\sum\nolimits_{t=1}^{\infty} \mathbf{M}_t}_{\text{fundamental value}} + \underbrace{\lim\nolimits_{T\to\infty} \mathbf{M}_T \, q_T}_{\text{bubble}} .$$

Money as a Bubble

- The fundamental value is the price in the static-dynamic model
- Repeated trading gives rise to the possibility of a bubble component
- Fiat money can be understood as an asset with no dividends. In the static-dynamic model, such an asset would have no value (the present value of zero is zero)
- But if there is a bubble on the price of fiat money, then it can have positive value.
- In asset pricing theory, we often rule out bubbles simply by imposing $\lim_{T\to\infty} M_T q_T = 0$

Martingales

- Stochastic process:
  - Let X_1 be a random variable and let x_1 be the realization of this random variable
  - Let X_2 be another random variable and assume that the distribution of X_2 depends on x_1
  - Let X_3 be a third random variable and assume that the distribution of X_3 depends on x_1, x_2
  - Such a sequence of random variables, $(X_1, X_2, X_3, \ldots)$, is called a stochastic process
- A stochastic process is a martingale if

$$E\{X_{t+1}|x_t, \ldots, x_1\} = x_t$$

Prices are Martingales...Not Really!

- Samuelson (1965) has argued that prices have to be martingales in equilibrium
- This requires the representative agent to be risk-neutral and future prices and cash flows to be discounted.
- The statement that $E\{q_{t+1}|q_t\} = \delta q_t$ cannot be quite right, because q_t depends on the dividend of the asset in period $t+1$, but q_{t+1} does not.
- However, consider the value of a fund that starts with one unit of the asset, and then keeps reinvesting the dividends of this asset into the fund again
- LeRoy (1989) explains that it is the value of this fund that is a martingale

Value of a Fund

- Consider a fund owning nothing but one unit of asset j. The value of this fund at time 0 is $f_0 = q_{j,0} = E\{\sum_{t=1}^{T} \delta^t r^j_{\langle t \rangle}\} = \delta E\{r^j_{\langle 1 \rangle} + q_{j,\langle 1 \rangle}\}$
- After receiving dividends $r^j_{\langle 1 \rangle}$ (which are state contingent), it buys more of asset j at the then current price $q_{j,\langle 1 \rangle}$, so the fund then owns $1 + \frac{r^j_{\langle 1 \rangle}}{q_{j,\langle 1 \rangle}}$ units of the asset
- The discounted value of the fund is then $\delta E\{f_{\langle 1 \rangle}\} = \delta E\left\{q_{j,\langle 1 \rangle}\left(1 + \frac{r^j_{\langle 1 \rangle}}{q_{j,\langle 1 \rangle}}\right)\right\} = \delta E\{(q_{j,\langle 1 \rangle} + r^j_{\langle 1 \rangle})\} = q_{j,0} = f_0$, so the discounted value of the fund is indeed a martingale

And with Risk Aversion?

- A similar statement is true if the representative agent is not risk-neutral
- We must discount with the risk-free interest rate, not with the discount factor, and use the risk-free probabilities instead of the objective probabilities.
- We define the risk-neutral probabilities as

$$\tilde{\alpha}_{\langle t\rangle} = \frac{\alpha_{\langle t\rangle}}{\beta_t} = \pi_{\langle t\rangle}\frac{\mathbf{M}_{\langle t\rangle}}{\beta_t}$$

And with Risk Aversion?

- The initial value of the fund (according to the fundamental pricing formula) is
$$f_0 = q_{j,0} = \sum_{t=1}^{T} E\{\mathbf{M}_{\langle t\rangle} r^j_{\langle t\rangle}\}$$
- Let us elaborate on this a bit,
$f_0 = E\{\sum_{t=1}^{T} \mathbf{M}_{\langle t\rangle} r^j_{\langle t\rangle}\} = E\{M_{\langle 1\rangle} r^j_{\langle 1\rangle} + \sum_{t=2}^{T} \mathbf{M}_{\langle t\rangle} r^j_{\langle t\rangle}\} =$
$E\{M_{\langle 1\rangle} r^j_{\langle 1\rangle} + M_{\langle 1\rangle} \sum_{t=2}^{T} (\prod_{t'=2}^{t} M_{\langle t'\rangle}) r^j_{\langle t\rangle}\} = E\{M_{\langle 1\rangle}(r^j_{\langle 1\rangle} + q_{j,\langle 1\rangle})\}$
- Using $\tilde{E}$ to denote expectations under the risk-neutral distribution $\tilde{\alpha}$, this can be rewritten as $\beta_1 \tilde{E}\{r^j_{\langle 1\rangle} + q_{j,\langle 1\rangle}\}$
- The properly discounted (β instead of δ) and properly expected ($\tilde{\alpha}$ instead of π) value of the fund is indeed a martingale
  - The risk-neutral probability distribution is also called the equivalent martingale measure

Models of the Real Interest Rate

- Bond prices carry all the information on intertemporal rates of substitution, which are primarily affected by expectations, and only indirectly by risk considerations
- The collection of interest rates for different times to maturity is a meaningful predictor of future economic developments
- For instance, we know that higher expected growth implies a greater interest rate in the 2-period model
- In the multiple period model, this translates into the statement that more optimistic expectations produce an upward-sloping term structure of interest rates

The Term Structure of Real Interest Rates

- The price of a risk-free discount bond which matures in period t is, according to the fundamental pricing formula, $\beta_t = E\{M_{\langle t \rangle}\}$.
- The corresponding yield or interest rate is

$$\rho_t = (\beta_t)^{-1} = \delta^{-1} \left[\frac{E\{v'(w^{\langle t \rangle})\}}{v'(w^0)} \right]^{-1/t}$$

- The collection of all these interest rates is the term structure of interest rates, $(\rho_1, \rho_2, \rho_3, \ldots)$
- Note that these are real interest rates (net of inflation), as are all prices and return rates in this course

The Term Structure of Real Interest Rates

- Let $g_{\langle t \rangle}$ be the (state dependent) growth rate per period between period t and period 0, so $(1 + g_{\langle t \rangle})^t = w^{\langle t \rangle}/w^0$
- Assume further that the representative agent has CRRA utility and use the same first-order approximations we did in the two period model. This yields

$$\ln \rho_t \approx \gamma E\{g_{\langle t \rangle}\} - \ln \delta$$

- The yield curve measures expected growth rates over different horizons

The Term Structure of Real Interest Rates

- This formula ignores second-order effects of uncertainty (these effects dropped out in the approximation)
- but we know that more uncertainty depresses interest rates if the representative agent is prudent
- Thus, if long horizon uncertainy about the per capita growth rate is smaller than about short horizons (for instance if growth rates are mean reverting), then the term structure of interest rates will be upward sloping

The Expectations Hypothesis

- The term structure we have considered is made up of bond prices at a particular point in time. This is a cross section of prices
- A different view on interest rates is to consider their time series properties: how do interest rates evolve as time goes by?
- This view is the relevant view for an investor how tries to decide what kind of bonds to invest into, or what kind of loan to take
- We enhance our notation slightly and write $\rho_{t,t'}$ to denote the return rate of a bond that begins in period t and that ends in period $t + t'$

The Expectations Hypothesis

- Suppose you have some spare capital that you will not need for 2 years
- You could invest it into 2 year discount bonds, yielding a return rate of $\rho_{0,2}$
- Of course, since bonds are continuously traded, you could alternatively invest into 1-year discount bonds, and then roll over these bonds when they mature. The expected yield is $\rho_{0,1}E\{\rho_{1,1}\}$
- Or you could buy a 3-year bond and sell it after 2 years
- Which of these possibilities is the best?

The Expectations Hypothesis

- Note first that only the first strategy is truly free of risk
- The other two strategies are risky, because the price that the 3-year bond will have in period 2 is unknown today, and tomorrow's yield of a 1-year bond is also not known today
- The possible premium that these risky strategies have over the risk-free strategy are called term premia. They are a special form of risk premium

The Expectations Hypothesis

- Consider a t-period discount bond. The price of this bond is

$$\beta_{0,t} = E\{\mathbf{M}_{\langle t \rangle}\} = E\{M_{\langle 1 \rangle} \cdots M_{\langle t \rangle}\}$$

- That means, that one has to invest $\beta_{0,t}$ today in order to receive one consumption unit in period t
- Alternatively, one could buy 1-period discount bonds and roll them over t-times. The investment that is necessary today to get one consumption unit (in expectation) in period t with this strategy is $E\{M_{\langle 1 \rangle}\} \cdots E\{M_{\langle t \rangle}\}$

The Expectations Hypothesis

- The two strategies yield the same expected return rate if and only if

$$E\{M_{\langle 1\rangle} \cdots M_{\langle t\rangle}\} = E\{M_{\langle 1\rangle}\} \cdots E\{M_{\langle t\rangle}\}$$

  which holds if $M_{\langle t\rangle}$ is serially uncorrelated
- In that case, there are no term premia —an assumption known as the expectations hypothesis
- Whenever there is reason to believe that $M_{\langle t\rangle}$ is serially correlated (for instance because the growth process is serially correlated), then we should expect the expectations hypothesis to fail

Portfolio Selection

- Financial planners often recommend that customers decrease the share of wealth invested into risky assets, such as equities, with age
  - The usual, but questionable argument to support such an advice is that equities carry less risk over a longer horizon
- Holding the equity for two periods is the same as taking this bet twice (if the returns to equity are serially uncorrelated, which we assume here)
- It is optimal for a young investor to hold the equity, but for an old investor not to hold it, if and only if one such bet produces a negative expected utility (because it contains too little reward on average, given its risks) but two such bets produce a positive expected utility

Portfolio Selection

- Whether you accept the lottery $[w-100, 0.5; w+200, 0.5]$or not depends on your absolute risk aversion, since it is an additive risk
- If you reject this bet at all wealth levels w, your absolute risk aversion must be constant
- A person with non-CARA utility may very well accept a long enough sequence of (on average) beneficial bets, while rejecting a single such bet
- If an investor holds constant over several periods the share of wealth invested into risky assets, he faces a sequence of multiplicative risks, $[w(1-d), 0.5; w(1+u), 0.5]$

Portfolio Selection

- Mossin (1968) has concluded from this that myopic behavior is optimal for a CRRA agent: such an agent decides on the basis of current wealth and maximizes expected utility over one period, so that there is no intrinsic difference between young and old investors
- Empirically, the results on the age-contingent behavior of investors is mixed
  - Ameriks & Zeldes (2001) stress the identification problem that time, cohort, and age effects cannot be separated without imposing arbitrary assumptions, because age is by definition the difference between the observation period and the date of birth

Summary

- Multi-period Economy and multi-period asset
- Static-dynamic economy
- Dynamic trading
- Ponzi Scheme and bubbles
- Martingales
- Term structure
- Portfolio selection