Financial Economics

3 Asset Economy

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk
aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

Theoretical resolutions

Asset Economy

- Financial Assets
- Arrow Securities and Risk-neutal Pricing
- Radner Economies and Equilibrium
- Complete and Incomplete Markets

Financial Assets

- Many contingent claim markets do not exist in reality, but we do have **spot markets** and **financial assets**
- Spot Market: a market for a commodity today ($t = 0$)
- Spot commodity is not contingent on any event and is at the root of the event tree
- Financial assets are contracts that deliver some state-contingent amount of money in the future.
- Example: Bonds give you Cash Flows + Face Value if the firm is solvent or nothing if the firm is bankrupt.

Real and Nominal Assets

- Financial asset in a 2-period economy with J assets and S states

$$r^j = \begin{bmatrix} r_1^j \\ \vdots \\ r_S^j \end{bmatrix}, r = \begin{pmatrix} r_1^1 & \cdots & r_1^J \\ \vdots & \ddots & \vdots \\ r_S^1 & \cdots & r_S^J \end{pmatrix}$$

- r (confusingly) denotes cash-flow or the payoff in this text
- **Real asset**: its return (payoff) is in physical goods, e.g., a durable piece of machinery or a futures contract for the delivery of one ton of Copper metal.
- **Nominal asset**: its return is in the form of paper money.

Real and Nominal Assets (contd.)

- Let x be some bundle of spot commodities.
- Real asset: Cash flow is a linear function of spot prices, delivers the purchasing power necessary to buy some specific commodity bundle x on tomorrow' s spot markets

$$r_s^j = p_s \cdot x$$

- Cash flows of some assets are independent of spot prices, an example is a nominal bond.
- Bond delivers some specified (state-contingent) amount of money.
- Nominal asset: delivers some specified amount of state-contingent money, you cannot consume this money but can spend it on buying some commodity but the purchasing power is uncertain.

Arrow Securities

- **Risk-free asset** is one that delivers a fixed amount of money in all states. For a bond, let' s fix this amount of money to be 1.
- **Arrow security** - delivers one unit of purchasing power conditional on an event s or zero otherwise.
- Vector of state-contingent cash flows of a state-s Arrow security and payoff matrix of the collection of all S Arrow securities:

$$e^s = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Any financial asset can be represented by a portfolio of Arrow securities.

Law of One Price

- Suppose there are no frictions –no transaction costs and no bid-ask spreads
- LOP says that if two portfolios have the same payoffs, they must cost the same or have the same price
- Suppose the price of security j in period 0 is q_j and if two portfolios have the same cash flows, they have the same price:

$$r \cdot z = r \cdot z' \Rightarrow q \cdot z = q \cdot z'$$

- Suppose the prices of the Arrow securities are $\alpha = [\alpha_1, \cdots, \alpha_S]$, we can write the price of a security as:

$$q_j = \alpha \cdot r^j$$

Risk-free Asset

- The cash flow of a risk-free (safe) asset is given by

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

- We denote the price of a risk-free bond with β, which is the reciprocal of the gross risk-free interest rate: $\beta = \rho^{-1}$
- The price of the risk-free bond must be the same as the sum of the prices of all Arrow securities:

$$\beta = \rho^{-1} = \sum_{s=1}^{S} \alpha_s$$

Risk-Neutral Probabilities

Risk-neutral Probabilities

Let ρ be the risk-free interest rate and let α be the vector of Arrow security prices. The numbers

$$\tilde{\alpha}_s := \rho \alpha_s$$

are called the risk-neutral probabilities

Risk-neutral Pricing

The price of a security with cash flow r^j equals the expected cash flow of the security, using the risk-neutral probabilities, discounted with the risk-free interest rate. Formally,

$$q_j = \beta \tilde{E}\{r^j\}$$

If we define the gross return: $R^j_s := \frac{r^j_s}{q_j}$, we have the risk-neutral returns:

$$\tilde{E}\{R^j\} = \rho$$

Radner Economies

Asset Economy

An *asset economy* consists of a contingent claim economy and a cash flow matrix, (u, ω, r). The matrix r has S rows and J columns, with J denoting the number of financial assets. The cash flows defined in r are deflated by price level.

The Markets Span

- Consider a return matrix r and a vector of financial asset prices q
- Cost of a portfolio $z : q \cdot z$ yields a cash flow $= r_s \cdot z$ in state s tomorrow
- Collecting all portfolios and tomorrow's cash flows that can be created in this way, we get the market span:

$$\mathcal{M}(q) := \text{span} \begin{bmatrix} -q \\ r \end{bmatrix} := \left\{ \begin{bmatrix} -q \\ r \end{bmatrix} \cdot z \middle| z \in \mathbb{R}^J \right\}$$

- $\mathcal{M}(q)$ is a linear space of at most J dimensions, captures the choice set of agents. If two different return matrices and security price vectors give rise to the same market span, they are equivalent, it' s only a change of basis.
- Define $\alpha_+ := [1, \alpha_1, \ldots, \alpha_S]$. α_+ is orthogonal to $\mathcal{M}(q)$

Decision Problems and Beliefs-I

- Decision problem: Maximize utility by choosing the consumption bundle today (x^0) and the "planned" bundles tomorrow ($x^1, \ldots, x^S$) and a portfolio of securities z to fulfill the budget constraint at every time and in every state
- This is an *integrated consumption-portfolio problem*
- Assume agent does not know the spot prices in the future, he may have a belief about them. Let' s call this $B(p_1), \ldots, B(p_S)$
- Formal problem can be written at $t = 0$ before uncertainty is resolved as:

$$\max\left\{u(x) \;\middle|\; \begin{array}{l} \overbrace{p_0 \cdot (x^0 - \omega^0)}^{-\text{saving}} + \overbrace{q \cdot z}^{\text{investment}} \leqslant 0 \\ \underbrace{B(p_s) \cdot (x^s - \omega^s)}_{\substack{\text{value of excess}\\ \text{consumption}}} - \underbrace{r_s \cdot z}_{\text{return}} \leqslant 0 \quad \text{for } s = 1, \ldots, S \end{array}\right\}$$

Decision Problems and Beliefs-II

- Combining the constraints in each period (and using the fact that since the utility function is monotonic, the constraints bind with equality, we can write the formal problem compactly at $t = 0$ before uncertainty is resolved as:

$$\max\{u(x)|B(p)\cdot(x-\omega)\in\mathcal{M}(q)\}$$

- Note that we ignore issues about how people form beliefs, we do above given some set of beliefs
- Later in the definition of a Radner equilibrium, we will make an assumption about the mutual consistency of beliefs

No-arbitrage Condition

- How to ensure the maximization problem in the previous slide has a solution?
  - The objective function is continuous and the constraint set is closed, yet could be unbounded
  - If there are arbitrage opportunities, the consumption-portfolio problem does not have a solution
- (q, r) contains arbitrage opportunities if there exists a portfolio z such that

$$\begin{bmatrix} -q \\ r \end{bmatrix} \cdot z \geq 0, \text{ or } \mathcal{M}(q) \cap \mathbb{R}_{+}^{S+1} = 0$$

- It is also equivalent to say that the Arrow prices are strictly positive
  - (q, r) is arbitrage-free if and only if there exists an $\alpha \gg 0$ such that $\alpha \cdot r = q$

Towards Radner Equilibrium

- In a contingent claim economy, demand equals supply for each commodity in each state of equilibrium.
- What about an economy with financial assets? What does market clearing mean for financial assets?
- Every security bought by an investor must first be issued.
- If someone issues an asset, he is short in this asset.
- Aggregating over all individuals, the holdings must sum to zero, each security bought by an individual must be sold by someone.
- Market clearing condition: Financial assets are in zero net supply.

Radner Equilibrium: Plans, Prices, and Price Expectations

- Plans = consumption bundles today (x^0) and planned consumption bundles in all states that will materialize tomorrow ($x^1, \ldots, x^S$)
- Prices = spot prices that can be observed today (p^0) and the prices of the financial assets (q)
- Price Expectations = tomorrow's prices where each agent has some beliefs about these prices
- Here in addition to market clearing, an equilibrium requires that everyone has the same beliefs and that these beliefs are correct or $p_s = B_i(p_s)$, or *rational expectations*.

Radner Equilibrium: Plans, Prices, and Price Expectations

- A Radner equilibrium is a four-tuple: $p =$ spot prices, $q =$ security prices, $x(i)$ and $z(i) =$ collections of consumption matrices and security portfolios for each i where:

$$x(i) \in \arg\max\{u(y) | B^i(p) \cdot (y - \omega) \in \mathcal{M}(q)\}, \quad i = 1, \ldots, I$$

- Aggregate consumption equal to aggregate endowment today and in each state tomorrow

$$\sum_{i=1}^{I} x_m^s(i) = \sum_{i=1}^{I} \omega_m^s(i), \quad s = 0, 1, \ldots, S; \quad m = 1, \ldots, M$$

- Each security is in zero net supply

$$\sum_{i=1}^{I} z_j(i) = 0, \quad j = 1, \ldots, J$$

- Everyone has perfect conditional foresight

$$B^i(p_s^m) = p_s^m, \quad i = 1, \ldots, I; s = 1, \ldots, S; m = 1, \ldots, M$$

Agent's Problem in Radner Economies-I

- In a Radner economy, we can divide an agent's decision problem into two: the consumption-composition problem and the financial problem

$$\max\{u(x)|B(p)\cdot(x-\omega)\in\mathcal{M}(q)\}$$

  - Here if we replace $B^i\{p_s\}$ with p_s and denote w as the state-contingent value of the agent's endowment, evaluated at spot prices:

$$w^s := p_s \cdot \omega^s \quad \text{for } s = 0, \dots, S$$

  - w^0 is the agent's income today and $w^1, \dots, w^S$ is his state-contingent future income

Agent's Problem in Radner Economies-II

- Define the indirect utility function v as:

$$v(y) := \max\{u(x) | p_s \cdot x^s \leq y^s \quad \text{for } s = 0, \ldots, S\}$$

- $v(y)$ is the maximized utility if at most y^s can be spent in state s. The choice of x is the choice about the composition of consumption
- $y = (y^0, y^1, \ldots, y^S)$ is the distribution of incomes spent today and tomorrow in each state: summarizes the allocation of the financial means of the agent over time and across states. The choice of y is about savings and risk, the financial decision
- The financial problem alone is:

$$\max\{v(y) | y - w \in \mathcal{M}(q)\}$$

Why Are We Doing All This Work?

- Separation of the integrated consumption-portfolio problem into a financial part and a consumption composition part can be used to simplify the original economy (u, ω, r).
- Let (p, q, x, z) be an equilibrium of this economy.
- Consider a new economy (v, w) where:

$$w^s := p_s \cdot \omega^s \quad \text{for } s = 0, \ldots, S$$

$$v(y) := \max\{u(x) | p_s \cdot x^s \leq y^s \quad \text{for } s = 0, \ldots, S\}$$

- This is a contingent claim economy with I agents but with only one commodity: income or consumption today and in each of the future states
- By construction, (α_+, y), with $y^s(i) := p_s \cdot x^s(i)$ is a competitive equilibrium

Complete Markets

Definition of Complete Markets

We say that markets are complete if agents can insure each state separately, i.e., if they can trade assets in such a way as to affect the payoff in one specific state without affecting the payoff in other states.

- If markets are complete, there is a portfolio—for each state s a different one—that generates the state-contingent cash flows of the state-s Arrow security

$$r \cdot [z^1, \ldots, z^S] = e$$

- Markets are complete if and only if r is invertible. In this case, the Arrow prices can be computed as $\alpha = q \cdot r^{-1}$ which is unique

Equivalence to Contingent Claim Economy

- When markets are complete, the individual' s decision problem in an asset economy is the same as in a contingent claim economy

$$\max \left\{ u(x) \,\middle|\, \sum_{s=0}^{S} \tilde{p}_s \cdot (x^s - \omega^s) \leq 0 \right\}$$

$$\max \left\{ u(x) \,\middle|\, \begin{array}{ll} p_0 \cdot (x^0 - \omega^0) + \alpha \cdot z \leqslant 0 & \\ p_s \cdot (x^s - \omega^s) \leqslant z^s & \text{for } s = 1, \ldots, S \end{array} \right\}$$

One-good One-agent Economy

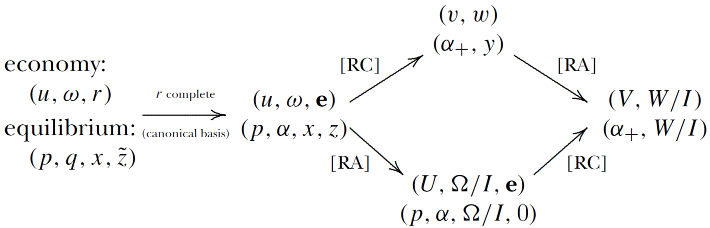

Figure 3.2. *Two routes from a multiple-agents multiple-goods asset economy to a one-agent one-good economy ([RC] means "make a representative commodity"; [RA] means "make a representative agent").*

Effects of Incomplete Markets

- Complete markets imply financial markets are such that individual states can be insured
- When markets are complete, the individual's decision problem in an asset economy is the same as in a contingent claim economy
- Hence for every competitive equilibrium of an abstract exchange economy, there is a corresponding economy with a Radner equilibrium.
- Equilibrium allocation is the same in contingent claim equilibrium and Radner equilibrium.
- Thus, the welfare theorem holds also in an asset economy provided markets are complete.
- We can therefore construct the competitive SWF and the representative agent in the same way
- But can we construct the representative if markets are incomplete?

Consequences of Incompleteness

- Arrow prices associated with equilibrium are not unique (pricing of new assets that are not in the span of existing assets is not well defined).
- The equilibrium allocation is not Pareto efficient.
- No locally representative agent based on a SWF (aggregate models do not exist).

Representative in an Incomplete Market?

- We cannot construct in general a representative if the market is incomplete. Why?
- Incomplete market implies return matrix is singular, and the market space has less than S dimensions
- Consequently, some income transfers from one state to another or one time period to another cannot be achieved independently of each other
- This has profound effects on equilibrium. Why?
- FOCs imply everyone's marginal rates of state-contingent intertemporal substitution of wealth are given by Arrow prices
- But Arrow prices are not now defined uniquely in a Radner equilibrium because there is an infinite combination of Arrow prices that are orthogonal to $\mathcal{M}(q)$
- Different agents have different MRS's and they would like to trade with each other because there are benefits from such trade

Representative in an Incomplete Market?-II

- However, they cannot perform this trade since the financial markets do not have the infrastructure to do so
- Now there is no SWF that is maximized, hence the equilibrium allocation is not Pareto efficient
- Lack of efficiency has grave implications for the equilibrium
- Now no representative agent can be computed on the basis of a SWF
- Does not affect ideas like no arbitrage pricing but affects models like the SDF of the CCAPM, etc., that we will develop later in this course

Quasi-complete Markets

- An incomplete market economy could be accidentally efficient.
- If the span of the incomplete market contains a Pareto-efficient point, then this allocation is an equilibrium of this economy which also happens to be efficient.
- Now all aggregations can be performed despite the incompleteness, termed a quasi-complete market.
- Suppose x is a Pareto-efficient allocation and $\exists$ q such that for each agent:

$$p \cdot (x(i) - \omega(i)) \in \mathcal{M}(q)$$

- Then we can say that the asset market r is quasi-complete and then $\exists$ z such that (p, q, x, z) is a Radner equilibrium.
- We can show now that goods and asset markets clear and that everyone behaves optimally.

Asset Economy

- Financial Assets
- Arrow Securities and Risk-neutal Pricing
- Radner Economies and Equilibrium
- Complete and Incomplete Markets