Financial Economics

2 Contingent Claim Economy

LEC, SJTU

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

Theoretical resolutions

Contingent Claim Economy

- Commodity Space
- Preferences and Utility Function
- Consumer Choices and Maximization
- General Equilibrium
- Social Welfare
- The Representative Agent

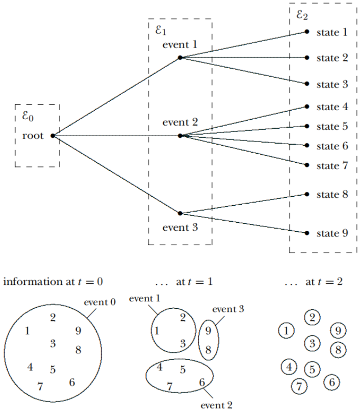
Commodity

- What is a *commodity* (or *good*) ?
  - Physical characteristics
  - Geographical place of availability
  - Time of availability
- Example: An umbrella in London in the summer
- Fourth property: Conditionality
  - A good may or may not be useful conditional on a random (exogenous) event
- Example: An umbrella when it rains v. umbrella when it does not rain –are two different commodities

Events and States

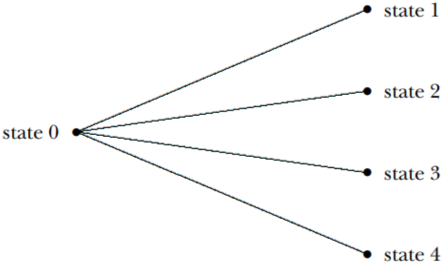
- $\mathcal{S}$ is a finite set with S elements
  - Each element in $\mathcal{S}$ represents a possible *state* of the world
- $\mathcal{E}_t$ is a partition of set $\mathcal{S}$
  - A partition is a collection of non-empty and pairwise disjoint subsets whose union makes up the whole set
  - The elements of $\mathcal{E}_t$ are the *events* that can happen at time t.

Resolution of Uncertainty



- Commodities are event contingent means that for each point in time, the commodity is available if and only if a specific event of this period of time is realized

Two Period Model



- The event tree simplifies considerably in a two-period model
  - Period 1: Complete uncertainty about state of world
  - Period 2: All uncertainty resolved and states of world revealed
- The commodity in two period model is simplified to be state contingent instead of event contingent

Commodities

Definition of a commodity

A complete description of a commodity requires a specification of the following components;

- physical specification,
- place of availability,
- event contingency (or state contingency in a two-period model).

Preferences

- Commodity space
  - let ℓ be the number of different commodities
  - a *consumption bundle* is a point in *commodity space* $\mathbb{R}^\ell$
- Preferences
  - If an agent prefers bundle 1 over bundle 2, we write bundle 1 $\succ$ bundle 2
  - If an agent thinks bundle 1 is at least as good as bundle 2, we write bundle 1 $\succsim$ bundle 2
- Utility function
  - Under some assumptions, preferences can be represented by a utility function, u: $\mathbb{R}^\ell \to \mathbb{R}$, such that

$$\text{bundle 1} \succ \text{bundle 2} \iff u(\text{bundle 1}) > u(\text{bundle 2})$$

Rationality

Definition of rational preference

The preference relation $\succsim$ is rational if it possesses the following two properties:

- Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$, or both;
- Transitivity: for all $x, y, z \in X$, if $x \succsim y, y \succsim z$, then $x \succsim z$.

Properties of Utility Function

- Continuous
  - no jumps
- Increasing
  - more is better than less
- Strictly quasi-concave
  - some of everything is better than lots of something and nothing of other things
  - indifference curve is convex
- Smooth
  - differentiable arbitrarily many times

Preferences and Ordinal Utility

- Utility function orders the points in commodity space
- Positive transformations of utility functions are equivalent
- Utility function that represents a preference ordering is called ordinal
  - Ordinal utility allows ranking of choices, not levels or differences
  - the utility functions $\sqrt{x_1 x_2}$ and $\ln x_1 + \ln x_2$ are equivalent

More on Rationality

- We have introduced the definition of rational preference. With rational preference (together with other assumption), we can introduce utility function
- What is rational decision?
- In neo-classical framework, rationality means that one chooses the consumption bundle he deems the best among the set of consumption bundles he can afford

Endowment

- Agent's endowment: List of quantities of all commodities you own before any trade takes place
  - Suppose there are ℓ commodities and you own amounts: $\omega_1, \omega_2, \cdots, \omega_\ell$
- Wealth: monetary value of all commodities you own
  - In a perfectly competitive economy, the prices of commodities are $p_1, p_2, \cdots, p_\ell$, then the wealth$= \sum_{c=1}^{\ell} p_c \omega_c$
- Budget constraint: you can consume any combination of goods $x_1, x_2, \cdots, x_\ell$, whose monetary value is not more than your wealth

$$p \cdot x \leq p \cdot \omega, \text{or } p \cdot (x - \omega) \leq 0$$

  - Here $(x - \omega)$ is the excess demand

Maximizing Preference Subject to Budget Constraint

- Revisit *rationality*: choose the bundle one likes best given the constraints imposed. Formally, the problem of an agent becomes

$$\max\{u(x)|p \cdot (x - \omega) \leq 0\}$$

- Additional assumptions on the utility function: (i) strictly convex preference ; (ii) differentiable utility function. (convex and smooth indifference curves)

Kuhn-Tucker Theorem

- Maximization of the agent's consumption problem implies the first order condition (FOC):

$$\partial_c u(x) = \lambda p_c, \text{for } c = 1, \cdots, \ell$$

  where λ is a positive number which is called the Lagrangian multiplier

- Or in vector form:

$$\nabla u(x) = \lambda p$$

  where $\nabla u(x) \coloneqq (\partial_1 u(x), \cdots, \partial_\ell u(x))$ is the gradient of u at x

Geometry of Maximization

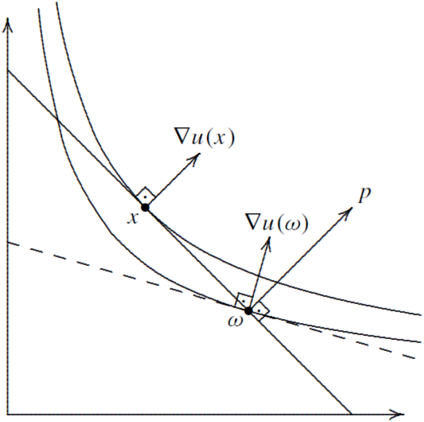


Figure 2.3. *Maximization of a standard preference subject to a budget requires that the gradient of the utility at the maximum be collinear to the price vector, i.e.* $\nabla u(x) = \lambda p$ *for some* $\lambda > 0$.

Geometry of Maximization

- Lagrange multiplier λ measures marginal utility of wealth
- The first order condition means that the gradient of the utility function at the optimal consumption bundle points in the same direction as the price vector
- For any pair of commodities (i, j) we have

$$\frac{\partial_i u(x)}{\partial_j u(x)} = \frac{p_i}{p_j}$$

  Marginal Rate of Substitution (MRS) equals relative price
  - Only the relative prices affect behavior. The price level is irrelevant.

Interest Rates as Relative Prices

- Decision problem of saving for future consumption
  - Suppose your current endowment of wealth$= w$
  - If you save s, you will be able to consume $w - s$ today
  - Let's assume the gross interest rate is ρ, then you will be able to consume ρs tomorrow
  - Your problem becomes:

$$\max_s u(w - s, \rho s)$$

- The first-order condition of this problem is

$$-\partial_1 u + \rho \partial_2 u = 0 \implies \frac{\partial_1 u}{\partial_2 u} = \rho = \frac{p_1}{p_2}$$

  - p_1 is the price of asset that delivers \$1 today; p_2 is the price of asset that delivers \$1 tomorrow
- First Order condition: Real interest rate as MRS between today's and tomorrow's purchasing power

Insurance Premia as Relative Prices

- Suppose you have wealth w
  - In state 1, you are lucky and will keep wealth w
  - In state 2, you are unlucky and will suffer a damage d
- There is an insurance company that offers to cover the loss in exchange for a premium
  - You can choose the coverage rate c (meaning that you get paid cd in state 2) at the cost of $c\mu$ (where μ is the premium of full coverage)
- Your decision problem becomes:

$$\max_c u(w - c\mu, w - c\mu - d + cd)$$

- The first-order condition yields

$$\frac{\partial_1 u}{\partial_2 u} = \frac{d - \mu}{\mu}$$

- This can be rearranged into :$\frac{\mu}{d} = \frac{p_2}{p_1 + p_2}$
  where p_1 and p_2 is the price for state-1-contigent and state-2-contigent commodities, respectively

General Equilibrium

- GE theory –concerns interaction of optimizing agents through markets
- Questions: Does an equilibrium exist? Is it unique? Are the equilibrium allocations efficient?
- Answers: Yes. Usually not. Yes.
- Finance (or macrofinance - asset pricing for example) is not concerned with existence or equilibrium allocation
- Finance focuses on equilibrium prices and how they relate to utilities (average tastes) and (average) endowments
- What does "average" tastes mean? Aggregation Problem: to find the representative agent
- Finance - need agent who is - "local representative agent" or behaves like one only at equilibrium - We now see how we can construct this local representative

Contingent Claim Economy

- Two-period model - M spot assets today and M spot assets in each state of S states tomorrow
- Agent i's utility function: $u_i : \mathbb{R}^{(S+1)M} \to \mathbb{R}$
- Agent i's endowment: $\omega(i) \in \mathbb{R}^{(S+1)M}$
- Collection of all agents in the contingent claim economy is:

$$\{(u_i, \omega(i)) : i = 1, \cdots, I\}$$

- Decision Problem: Agent must choose bundle today $x^0(i)$ and state contingent bundle tomorrow $x^1(i), \ldots, x^S(i)$ s.t.

$$\max \left\{ u_i(x(i)) \,\middle|\, \sum_{s=0}^{S} p_s \cdot (x^s(i) - \omega^s(i)) \leq 0 \right\}$$

Contingent Claim Economy and Equilibrium

- We have many agents each one using this optimization rule.
- What can happen? Suppose a good is very cheap - many people would like to buy it and few will want to sell it $\Rightarrow$ Demand exceeds supply.
- A competitive equilibrium is a pair (p, x); matrix of prices and collection of consumption bundles; one for each agent, such that for each i, $x(i)$ maximizes i's utility s.t. the budget constraint, given p, and all markets clear (aggregate demand equals aggregate supply for each commodity simultaneously).

$$\sum_{i=1}^{I} x_m^s(i) = \sum_{i=1}^{I} \omega_m^s(i), s = 0, \ldots, S; m = 1, \ldots, M$$

Some General Points

- Step #1: How to construct a representative agent
- Step #2: How to go from many commodities into one aggregate commodity namely wealth
- Objective: **one-good and one-agent economy**
- We want to use this to determine the macroeconomic determinants of asset prices
- Existence of equilibrium is an important issue in GE theory - but not in finance - we will not go into this in this course
- However here's why this is important: A model should at least guarantee an equilibrium - otherwise it is incomplete
- GE theory uses things like fixed point theorems to prove these things (see any text like MWG for details)

Pareto Efficiency

- Consider an economy with I agents - aggregate endowment Ω - no markets, prices or budgets
- People vote how best to distribute endowment - start by randomly assigning an endowment to each agent: $(\omega(1), \ldots, \omega(I)), \text{s.t. } \sum_{i=1}^{I} \omega(i) = \Omega$
- Every allocation x that is feasible is proposed: $x = (x(1), \ldots, x(I)), \text{s.t. } \sum_{i=1}^{I} x(i) \leq \Omega$
- Voting must be unanimous - as any agent disagrees with the proposed allocation it will not be implemented
- An allocation x is **Pareto efficient** if there is no alternate allocation y that could be unanimously accepted given any initial distribution w - Not possible to redistribute consumption among agents so that no one is worse off and at least some one is better off by the redistribution.

First Welfare Theorem

- Equilibrium allocations are Pareto efficient. Why?
- Everyone's maximum indifference curve is tangent to the budget hyperplane in equilibrium $\Rightarrow$ no unexploited gains by trade

First Welfare Theorem

Everyone is marginally identical in equilibrium - hence there are no further gains from trade and the equilibrium allocation is Pareto efficient

- In other words, given a competitive equilibrium allocation there is no redistribution that would be accepted unanimously

Social Welfare Function

- Given the utility functions of agents and an aggregate endowment we can generate all Pareto-efficient allocations using a **Social Welfare Function** (SWF)
- SWF is a weighted sum of individual utilities maximized subject to feasibility constraints:

$$U(z) = \max \left\{ \frac{1}{I} \sum_{i=1}^{I} \sigma_i u_i(y(i)) \,\middle|\, \sum_{i=1}^{I} (y(i) - z) \leq 0 \right\}$$

- $\sigma_1, \ldots, \sigma_I, > 0$ are the weights assigned to the respective agents' utility; $z = \Omega / I$ is the mean endowment of each individual
- Setting z equal to the mean endowment of the original economy we can generate every Pareto-efficient allocation by an appropriate choice of weights

Choosing the Competitive SWF-I

- How can we choose $\sigma_1, \ldots, \sigma_I, > 0$, the weights to construct this competitive SWF?
- We use the FOC's of the individual's maximization problem. We know that $\exists \lambda_i > 0$, for each agent s.t.

$$p = \lambda_1^{-1} \nabla u_1(x(1)) = \cdots = \lambda_I^{-1} \nabla u_I(x(I))$$

- λ_i measures the agent's marginal utility of wealth
- The FOC's for the SWF are:

$$\frac{1}{I} \sigma_i \nabla u_i(y(i)) = \mu, i = 1, \ldots, I$$

$$\mu_c \sum_{i=1}^{I} (y_c(i) - z_c) = 0, c = 1, \ldots, (S+1)M$$

Choosing the Competitive SWF-II

- If μ is the vector of Lagrange multipliers, then we search for weights $\sigma_1, \ldots, \sigma_I, > 0$, such that $y = x$ is a solution if $z = \Omega/I$
- Consider the weight $\sigma_i = \lambda_i^{-1}$, substituting x for y and Ω/I for z and bear in mind that the $\mu \gg 0$ (strictly positive)

$$\frac{1}{I}\lambda_i^{-1}\nabla u_i(x(i)) = \mu, i = 1, \ldots, I$$

$$\sum_{i=1}^{I}(x(i) - \Omega/I) = 0$$

- This is a market clearing condition and is satisfied in equilibrium. We know that the equilibrium allocation x satisfies this condition because it is an efficient allocation (by the First Welfare Theorem).
- Thus $\mu = p/I$ is a solution

The Competitive SWF

- We conclude that the equilibrium allocation maximizes a Social Welfare Function that weights agents according to the reciprocal of their marginal utility of wealth.
- Competitive SWF is:

$$U(z) = \max \left\{ \frac{1}{I} \sum_{i=1}^{I} \lambda_i^{-1} u_i(y(i)) \,\middle|\, \sum_{i=1}^{I} (y(i) - z) \leq 0 \right\}$$

- Shadow price is the marginal increase that can be achieved in the objective function if the constraint is eased marginally - the Lagrange multiplier is equal to the shadow price. Enlarging z by dz in the constraint $\sum_{i=1}^{I} y(i) \leq Iz$ eases the constraint by I times dz, hence we get

$$\nabla U(z) = I\mu = p$$

- Equilibrium price $\equiv$ marginal social value of goods.

Some Summing up

- We studied an *abstract contingent claim economy*
- We defined a Pareto efficient allocation and a competitive equilibrium in a contingent claim economy
- A *SWF* is the value of a problem that maximizes a weighted sum of individual utilities subject to the material limitations of the economy
- An allocation is Pareto efficient if and only if it is the solution to some SWF
- A competitive equilibrium is a price-allocation pair in which all markets clear and every agent maximizes utility subject to a budget constraint
- Key result: **First Welfare Theorem - A competitive equilibrium allocation is Pareto efficient**

Features of Representative Agents

- Economists are interested in aggregate data that we can observe and society's utility function - not necessarily in an individual's endowment and decisions
- If we have an economy (u, ω) with I agents - to solve for a competitive equilibrium (p, x) would be cumbersome - we need to do this for every agent
- If there is only one agent - we know the equilibrium allocation because there is nobody else to trade with - equilibrium prices are just the gradient of this agent's utility function at his endowment point
- Given a multi-agent economy (u, ω) and a competitive equilibrium (p, x) we define a locally representative agent as an artificial agent (u_0, ω_0) such that (p, ω_0) is a competitive equilibrium of this one one-agent economy (u_0, ω_0)
- The equilibrium allocation is ω_0 or it is a no-trade situation in this one-agent economy

Representative Agents - some discussion

- If we work with a representative agent we lose all information on the *inter-personal equilibrium distribution* - we don't know now who consumes what - however in finance we are not interested in this micro-information.
- An arbitrary representative agent: Take any utility function v and a point x where the FOC is achieved: $\nabla v(x) = \lambda p$, then letting $\lambda = 1$, (v, x) is a representative agent.
- Why? Faced with prices p he will not wish to trade forming a one-person general equilibrium.
- However this arbitrary R.A. is not very useful - he has no relation to the original data of the multi-person economy.
- How can we construct a representative from the data of the original economy?
- Use result of the Pareto efficiency of an equilibrium - everyone is marginally identical in equilibrium $\Rightarrow$ everyone is a representative agent in equilibrium.

Competitive SWF as Representative Agent-I

- We want to construct a representative agent using only the aggregate data of the original economy. How can we do that?
- Using the SWF, the gradient (derivative) of the competitive SWF at the point $z = \Omega/I$ just equals the equilibrium prices:

$$\nabla U \left(\frac{\Omega}{I} \right) = p$$

- Thus $(U, \Omega/I)$ is a representative agent
- This is good news: the R.A.'s endowment is just the per capita endowment in the original economy and we have this data
- What about the construction of U?
- This still requires micro-level data: we need information on the inter-personal distribution of preferences and endowments

Competitive SWF as Representative Agent-II

- What about the construction of U?
- We can estimate U or
- We can assume that everyone has the same utility function u but different endowments $\omega(i)$ and then use data on endowment distributions to compute U
- However we do not need to do this
- We next investigate a class of utility functions that can be aggregated without the knowledge of the distribution of endowments
- These arise from the von Neumann-Morgenstern utility theory - we look at this topic next

Contingent Claim Economy

- Commodity Space
- Preferences and Utility Function
- Consumer Choices and Maximization
- General Equilibrium
- Social Welfare
- The Representative Agent