Scalars in a Tangent Category

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Motivation

Classical differential geometry

Everything is defined in terms of the ring \mathcal{R} .

- 1. Every manifold is locally \mathcal{R}^n
- 2. Tangent vectors: equivalence classes of curves $\mathcal{R} \to M$.

Synthetic Differential Geometry (SDG)

Still the case.

- 1. \mathcal{R} has infinitesimals $D = [d : R|d^2 = 0]$.
- 2. Must satsify *Kock-Lawvere* axiom: $\nu: R \times R \rightarrow [D, R] := (a, b) \mapsto \lambda d.ad + b$ is an iso
- 3. Tangent vectors are $D \rightarrow M$.

Tangent Categories

Definition (Rosický, Cockett-Cruttwell)

Abstract setting for differential geometry: only the behaviour of the *tangent bundle* is axiomatized.

- No ring object.
- Tangent vector addition but no tangent vector subtraction.

This captures examples from computer science:

- ▶ REL: differential structure, no subtraction or ring of scalars.
- ▶ The classifying category for the $\partial \lambda$ calculus.

Modular: Add a structure and see how it fits:

- A class of "submersions": display tangent categories
- Solutions to ODEs: curve objects

Goal

Sort out "what does a scalar rig look like in a tangent category"

- What universal properties should it satisfy?
- ▶ Does it make "calculus" better behaved?

Result: The Kock-Lawvere axiom without infinitesimals.

Tangent categories are enriched categories - extend those results.

- Embed a tangent category into one with a scalar unit.
- Show that differential objects are (enriched) sketchable.

Tangent Categories

A tangent category is a category $\mathbb X$ equipped with an endofunctor $\mathcal T$ on $\mathbb X$ and natural transformations

$$p: T \Rightarrow id, 0: id \Rightarrow T, +: T_{p \times_{p}} T \Rightarrow T, \ell: T \Rightarrow T^{2}, c: T^{2} \Rightarrow T^{2}$$

All pullback powers of p exist and are preserved by T $(p_M, +_M, 0_M)$ is an additive bundle $(\ell, 0), (c, id)$ are additive.

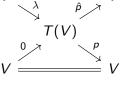
$$\ell$$
 is universal: $T(M)_p \times_p T(M) \xrightarrow{\mu} T^2(M) \xrightarrow{T(p)} T(M)$
(where $\mu = \langle \pi_0 \ell, \pi_1 0 \rangle T(+)$.)
 c is a symmetry $cc = id$ and $T(c)cT(c) = cT(c)c$
Symmetric cosemigroup: $\ell c = \ell$, $\ell T(\ell) = \ell \ell$ and $cT(c)\ell = T(\ell)c$

Differential Objects

Definition

Easier to do differential bundle over 1 definition (V, σ_V, ξ_V) is a commutative monoid object such that.

1. Biproduct in CMon



3. Compatible lift: $\begin{array}{ccc} TV & \stackrel{/}{\longrightarrow} & T^2V \\ \hat{\rho} & & & \downarrow \hat{\rho} \\ V & \longleftarrow & TV \end{array}$

Differential objects ii

Theorem (Cockett and Cruttwell)

The full subcategory of differential objects is cartesian differential category.

Set
$$\nu = (\lambda \times 0) T(\sigma_V)$$

The derivative:
$$V \times V \xrightarrow{D[f]} W$$

$$T(V) \xrightarrow{\hat{p}} T(W)$$

$$T(V \times X) \xrightarrow{T(f)} T(W)$$
Linear in $V: \lambda_{V} \times 0_{X} \qquad \lambda_{W} \qquad \lambda_{W} \qquad V \times X \xrightarrow{f} W$
Note: λ is universal $V \xrightarrow{\lambda} T(V) \xrightarrow{P} V$

Linear Classifier

Under a very mild assumption, we can add a universal ring object to a tangent category.

Definition (Blute-Cockett-Seeley)

A *Scalar Unit* is a differential object with a point $1 \xrightarrow{u} R$ with the universal property that for all

$$V \xrightarrow{f} W$$

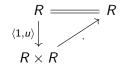
$$\langle 1, u \rangle \downarrow \qquad \exists ! \hat{f} \text{ linear in } R$$

$$V \times R$$

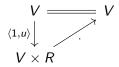
f (multi)-linear in $V \Rightarrow \hat{f}$ is (multi)-linear in V

Consequences of a Linear Point Classifier

▶ The unit object is a commutative rig *R*.



Every differential object is an R-module.



Every linear map preserves the R-module action (persistence).

Rewriting the lift

Every R-module has the map λ^R :

$$V \xrightarrow{\langle 1, u \rangle} V \times R \xrightarrow{0 \times \lambda} T(V \times R) \xrightarrow{T(\cdot)} T(V)$$

In SDG: $v \mapsto \lambda d.vd$.

Lemma

For a differential object, λ^R satisfies the equalizer

$$V \xrightarrow{\lambda_V^R} T(V) \xrightarrow{p \atop ! \varepsilon} V$$

so
$$\lambda_V = \lambda_V^R$$
.

Corollary

Homogenous morphisms of differential objects are linear.

KL-Modules

Set
$$\nu^T := V \times V \xrightarrow{\lambda^R \times 0} T(V) \times T(V) \xrightarrow{T(\sigma)} T(V)$$

Definition (Kock-Lawvere R-module)

V is a KL-module if there is an R-module map \hat{p} making

$$((\lambda^R \times 0)T(\sigma))^{-1} = \langle \hat{p}, p_V \rangle$$

- The category of KL-modules is equivalent to the category of differential objects.
- ▶ In a locally presentable tangent category, KL-modules is a full reflective subcategory of *R*-modules.
- ▶ If X has negatives, then KL-modules are a completion of R-modules.

New Questions

The notion of a scalar unit allows one to use the simpler definition of KL-modules.

If R is a ring, KL-modules are a completion of R-modules - is there a sketch of KL-modules?

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Move to *enriched category theory*.

Weil Algebras, Microlinear Weil Spaces

Definition (The category Weil)

The full subcategory of $\pi:W\to R$ in unital RAlg/R so that:

- \blacktriangleright ker (π) is a nilpotent ideal
- $ightharpoonup U(W) = R^n.$

e.g. the dual numbers $R[x]/x^2$

A *Microlinear Weil Space* is a presheaf Weil \rightarrow Set preserving connected limits. Call the category of microlinear weil spaces \mathcal{E} .

Theorem

The category of microlinear weil spaces is

- Locally finitely presentable
- A coherently closed tangent category
- ► Has a scalar unit $R = [y(R[x]/x^2), y(R[x]/x^2)]$

Microlinear Weil Spaces

Recall that in a \mathcal{E} -category \mathcal{C}

- ▶ Power: $\mathcal{E}(w, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, w \pitchfork Y)$
- ▶ Copower: $\mathcal{E}(w \bullet X, Y) \cong \mathcal{E}(w, \mathcal{C}(X, Y))$

Theorem (Garner, Leung)

A tangent category is equivalently a category enriched in $\mathcal E$ with powers by representables.

$$T(X) := y(R[x]/x^2) \pitchfork X$$

Observation

The tangent bundle is now a weighted limit.

Units in Presheaf Categories

Theorem

The enriched Yoneda embedding preserves differential objects.

Theorem

The enriched presheaf category of a tangent category has a representable unit:

$$1 \bullet R \cong [1 \bullet y(x^2), 1 \bullet y(x^2)]$$

Observation

Every differential objects is a KL-module in the presheaf category.



KL-modules as sketches

Define the \mathcal{E} -sketch KLMod.

▶ Objects: $n \in \mathbb{N}$

▶ Hom-Objects: $[n, m] = R^{n \times m}$

Composition: Matrix multiplication

KL-modules as sketches

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- ▶ Objects: $n \in \mathbb{N}$
- ► Hom-Objects: $[n, m] = R^{n \times m}$
- Composition: Matrix multiplication
- ▶ Powers on objects: $y(R[x]/x^2) \pitchfork n := 2n$, and fix:
 - ▶ $0_n: 1 \to R^{n \times 2n}$ picks out the matrix $\begin{bmatrix} 0 \\ I \end{bmatrix}$ ▶ $p_n: 1 \to R^{2n \times n}$ picks out $\begin{bmatrix} 0 & I \end{bmatrix}$

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We derive:

- λ_n picks out $\begin{bmatrix} I \\ 0 \end{bmatrix}$
- \hat{p} picks out $\begin{bmatrix} I & 0 \end{bmatrix}$

 $(I = n \times n \text{ identity matrix})$

▶ Powers on homs: $\Delta_{n,m}: R^{n\times m} \to R^{2n\times 2m}$

Conclusions and Future Work

We used the notion of a scalar unit to simplify Differential Objects and find a \mathcal{E} -sketch.

Opens the door for sketch theory to be applied in tangent categories

- Gabriel-Ulmer duality: a free KL-module construction?
- Differential Bundles
- Involution Algebroids a sketch for Lie theory (Joint work with Matthew Burke)

Sketches can be interpreted as Abstract Data Types

 \triangleright \mathcal{E} -sketches as ∂ -ADTs in differential programming?

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