Props in Network Theory

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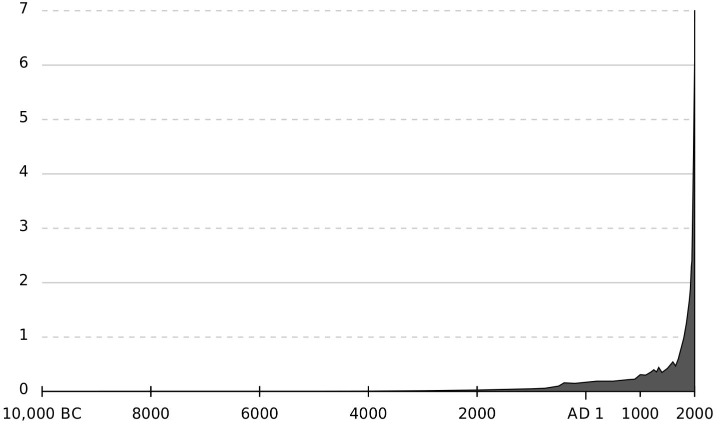
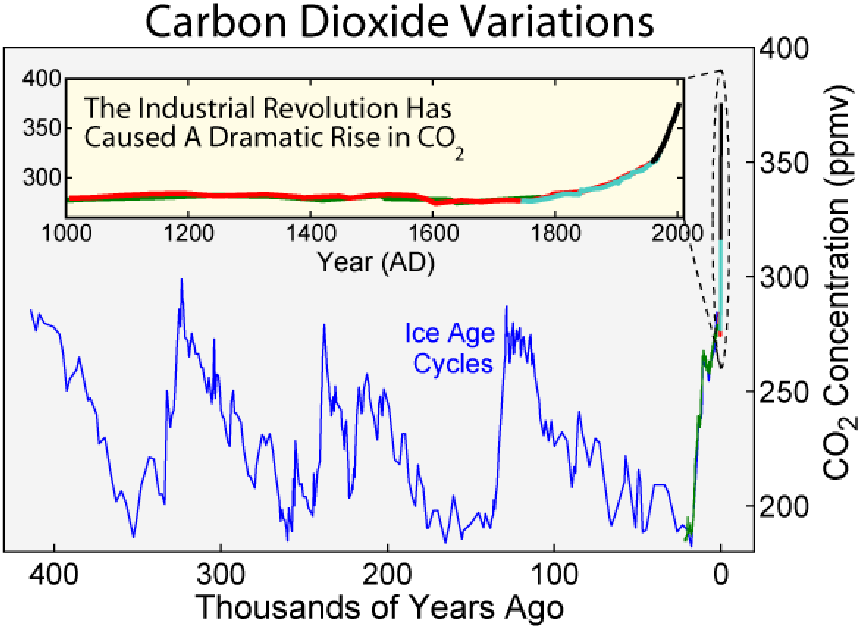
We have left the Holocene and entered a new epoch, the Anthropocene, in which the biosphere is rapidly changing due to human activities.



THE EARTH'S POPULATION, IN BILLIONS



Carbon Dioxide Variations
The Industrial Revolution Has Caused A Dramatic Rise in CO_2
400
350
300
1000
1200
1400
1600
1800
2000
Year (AD)
Ice Age Cycles
400
350
300
250
200
CO_2 Concentration (ppmv)
400
300
200
100
0
Thousands of Years Ago

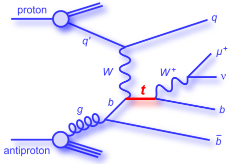
Climate change is not an isolated ‘problem’ of the sort routinely ‘solved’ by existing human institutions. It is part of a shift from the exponential growth phase of human impact on the biosphere to a new, uncharted phase.

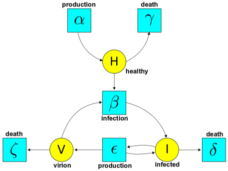
- About 1/4 of all chemical energy produced by plants is now used by humans.
- Humans now take more nitrogen from the atmosphere and convert it into nitrates than all other processes combined.
- 8-9 times as much phosphorus is flowing into oceans than the natural background rate.
- The rate of species going extinct is 100-1000 times the usual background rate.
- Populations of large ocean fish have declined 90% since 1950.

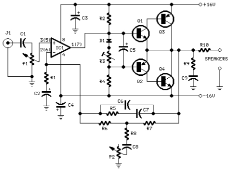
So, we can expect that in this century, scientists, engineers and mathematicians will be increasingly focused on *biology*, *ecology* and *complex networked systems* — just as the last century was dominated by physics.

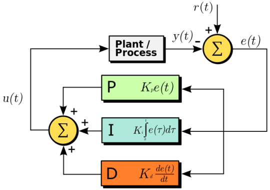
What can category theorists contribute?

One thing category theorists can do: *understand networks*.







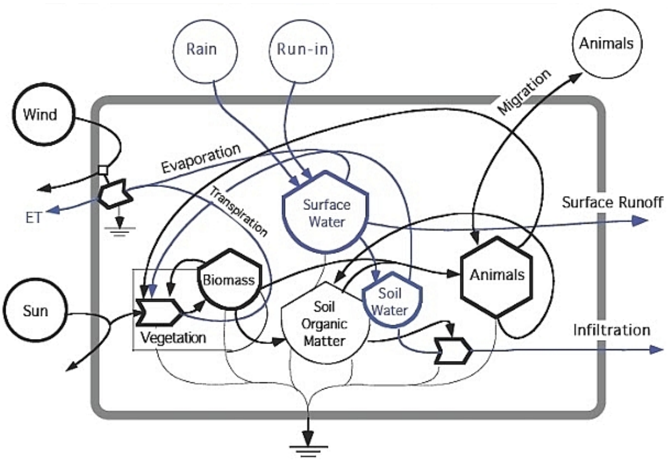


We need a good general theory of these. It will require category theory.

To understand ecosystems, ultimately will be to understand networks. — B. C. Patten and M. Witkamp

I believe biology proceeds at a *higher level of abstraction* than physics, so it calls for new mathematics.

Back in the 1950's, Howard Odum introduced an Energy Systems Language for ecology:

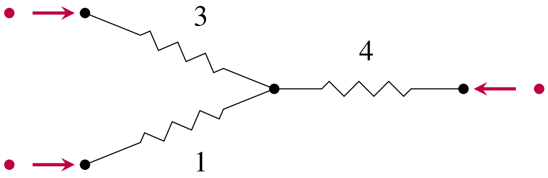


Maybe we are finally ready to develop these ideas.

The dream: each different kind of network or open system should be a morphism in a different symmetric monoidal category.

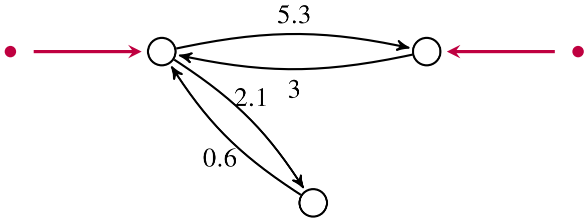
Some examples:

- ResCirc, where morphisms are circuits of resistors with inputs and outputs:



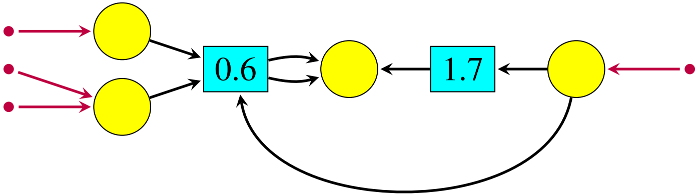
These, and many variants, are important in electrical engineering.

- Markov, where morphisms are **open Markov processes**:



These help us model stochastic processes: technically, they describe continuous-time finite-state Markov chains with inflows and outflows.

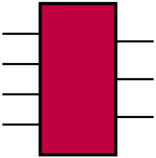
- RxNet, where morphisms are **open reaction networks with rates**:



Also known as **open Petri nets with rates**, these are used in chemistry, population biology, epidemiology etc. to describe changing populations of interacting entities.

All these examples can be seen as **props**: strict symmetric monoidal categories whose objects are natural numbers, with addition as tensor product.

A morphism $f\colon 4 \to 3$ in a prop can be drawn this way:



FinCospan

Steve Lack,

Composing PROPs

$$\mathrm{FinCospan} \longrightarrow \mathrm{FinCorel}$$

Brandon Coya & Brendan Fong, *Corelations are the prop for extraspecial commutative Frobenius monoids*

$$\mathrm{Circ} \longrightarrow \mathrm{FinCospan} \longrightarrow \mathrm{FinCorel}$$

R. Rosebrugh, N. Sabadini & R. F. C. Walters

Generic commutative separable algebras and cospans of graphs

$$\mathrm{Circ} \longrightarrow \mathrm{FinCospan} \longrightarrow \mathrm{FinCorel} \longrightarrow \mathrm{LagRel}$$

JB & Brendan Fong,
A compositional framework for passive linear circuits

JB, Brandon Coya & Franciscus Rebro,
Props in network theory

$$\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
 & & & \mathrm{LinRel}
\end{tikzcd}$$

Filippo Bonchi, Pawel Sobocinski & Fabio Zanasi,

Interacting Hopf algebras

JB & Jason Erbele,

Categories in control

$$
\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] \arrow[dl] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
\mathrm{ResCirc} \arrow[urrr] & & & \mathrm{LinRel}
\end{tikzcd}
$$

JB, Brendan Fong & Blake Pollard, *A compositional framework for Markov processes*

$$
\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] \arrow[dl] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
\mathrm{ResCirc} \arrow[urrr] \arrow[d] & & & \mathrm{LinRel} \\
\mathrm{Markov} & & &
\end{tikzcd}
$$

JB & Blake Pollard,
A compositional framework for reaction networks

$$
\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] \arrow[dl] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
\mathrm{ResCirc} \arrow[urrr] \arrow[d] & & & \mathrm{LinRel} \\
\mathrm{Markov} \arrow[r] & \mathrm{RxNet} & &
\end{tikzcd}
$$

JB & Blake Pollard,
A compositional framework for reaction networks

$$
\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] \arrow[dl] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
\mathrm{ResCirc} \arrow[urrr] \arrow[d] & & & \mathrm{LinRel} \\
\mathrm{Markov} \arrow[r] & \mathrm{RxNet} \arrow[r] & \mathrm{Dynam} &
\end{tikzcd}
$$

JB & Blake Pollard,
A compositional framework for reaction networks

$$
\begin{tikzcd}
\mathrm{Circ} \arrow[r] & \mathrm{FinCospan} \arrow[r] \arrow[dl] & \mathrm{FinCorel} \arrow[r] & \mathrm{LagRel} \arrow[d] \\
\mathrm{ResCirc} \arrow[urrr] \arrow[d] & & & \mathrm{LinRel} \arrow[d] \\
\mathrm{Markov} \arrow[r] \arrow[urrr] & \mathrm{RxNet} \arrow[r] & \mathrm{Dynam} \arrow[r] & \mathrm{SemiAlgRel}
\end{tikzcd}
$$

JB & Blake Pollard,
A compositional framework for reaction networks

Let's look at a little piece of this picture:

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}$$

The composite sends any circuit made just of purely conductive wires

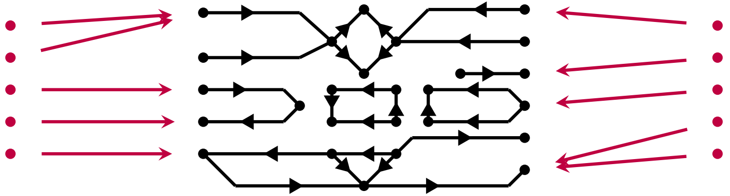
$$f \colon m \to n$$

to the linear relation

$$KHG(f) \subseteq \mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$$

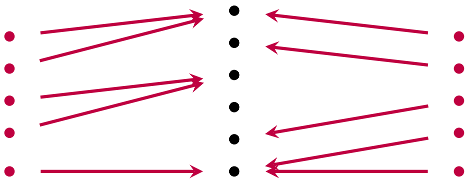
that this circuit establishes between the potentials and currents at its inputs and outputs.

In the prop Circ, a morphism looks like this:



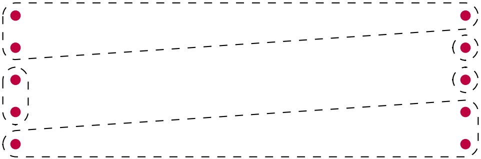
We can use such a morphism to describe an electrical circuit made of purely conductive wires.

In the prop FinCospan, a morphism looks like this:



We can use such a morphism to say which inputs and outputs lie in which connected component of our circuit.

In the prop FinCorel, a morphism looks like this:



Here a morphism $f: m \to n$ is a **corelation**: a partition of the set $m + n$. We can use such a morphism to say which inputs and outputs are connected to which others by wires.

In the prop LagRel, a morphism $L\colon m \to n$ is a **Lagrangian linear relation**

$$L \subseteq \mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$$

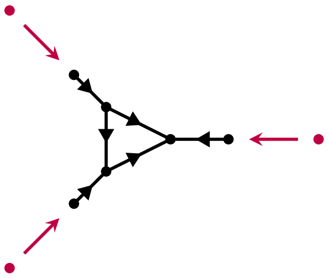
that is, a linear subspace of dimension $m + n$ such that

$$\omega(v, w) = 0 \text{ for all } v, w \in L.$$

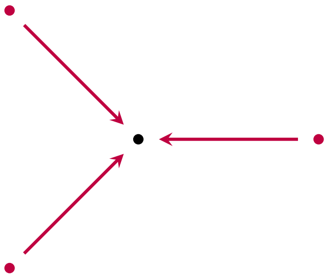
Here ω is a well-known bilinear form on $\mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$, called a "symplectic structure".

Remarkably, any circuit made of purely conductive wires establishes a linear relation between the potentials and currents at its inputs and its outputs that is *Lagrangian!*

A morphism $f\colon 2 \to 1$ in Circ:

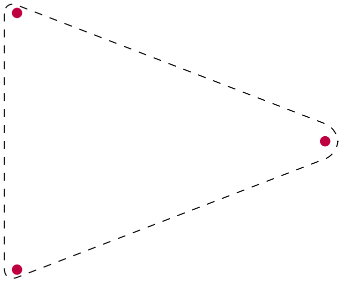


The morphism $G(f)\colon 2 \to 1$ in FinCospan:



$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan}$$

The morphism $HG(f)\colon 2 \to 1$ in FinCorel:



$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel}$$

The morphism $L = KHG(f)\colon 2 \to 1$ in LagRel:

$$L = \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) : \phi_1 = \phi_2 = \phi_3, I_1 + I_2 = I_3\}$$

(ϕ_1, I_1) •

• (ϕ_3, I_3)

(ϕ_2, I_2) •

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}$$

In working on these issues, three questions come up:

- When is a symmetric monoidal category equivalent to a prop?
- What exactly is a map between props?
- How can you present a prop using generators and relations?

Answers can be found here:

- John Baez, Brendan Coya and Franciscus Rebro, Props in network theory, arXiv:1707.08321.

We start with the 2-category SymMonCat, where:

- objects are symmetric monoidal categories,
- morphisms are symmetric monoidal functors,
- 2-morphisms are monoidal natural transformations.

We often prefer to think about the category PROP, where:

- object are props: strict symmetric monoidal categories with natural numbers as objects and addition as tensor product,
- morphisms are strict symmetric monoidal functors sending 1 to 1.

This is evil, but convenient. When can we get away with it?

Theorem. $C \in \mathrm{SymMonCat}$ is equivalent to a prop iff there is an object $x \in C$ such that every object of C is isomorphic to

$$x^{\otimes n} = x \otimes (x \otimes (x \otimes \cdots))$$

for some $n \in \mathbb{N}$.

Theorem. Suppose $F\colon C \to D$ is a symmetric monoidal functor between props. Then F is isomorphic, in SymMonCat, to a *strict* symmetric monoidal functor $G\colon C \to D$.

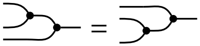
If $F(1) = 1$, G is a morphism of props.

We all “know” how to describe props using generators and relations. For example, the prop for commutative monoids can be presented with two generators:

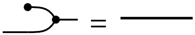
$\mu\colon 2 \to 1$

$\iota\colon 0 \to 1$

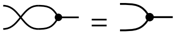
and three relations:



(associativity)



(unitality)



(commutativity)

But what are we really doing here?

There is a forgetful functor from props to signatures:

$$U\colon \mathrm{PROP} \to \mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$$

A **signature** just gives a set $\hom(m,n)$ for each $(m,n) \in \mathbb{N} \times \mathbb{N}$.

Theorem. The forgetful functor U is **monadic**, meaning that it has a left adjoint

$$F\colon \mathrm{Set}^{\mathbb{N}\times\mathbb{N}} \to \mathrm{PROP}$$

and PROP is equivalent to the category of algebras of the resulting monad $UF\colon \mathrm{Set}^{\mathbb{N}\times\mathbb{N}} \to \mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$.

Everything one wants to do with generators and relations follows from $U\colon \mathrm{PROP} \to \mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$ being monadic.

For example:

Corollary. Any prop T is a coequalizer

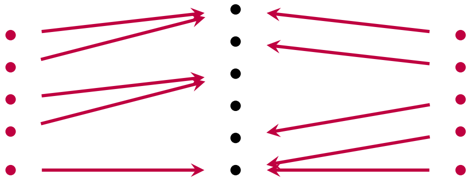
$$\begin{tikzcd} F(R) \arrow[r, shift left] \arrow[r, shift right] & F(G) \arrow[r] & T \end{tikzcd}$$

for some signatures G, R.

We call elements of G **generators** and elements of R **relations**.

Example. The symmetric monoidal category where

- objects are finite sets
- morphisms are isomorphism classes of cospans of finite sets:

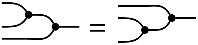


- the tensor product is disjoint union

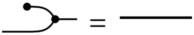
is equivalent to a prop, FinCospan.

Theorem (Lack). The prop FinCospan has generators

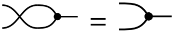
and relations:



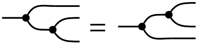
associativity



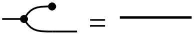
unitality



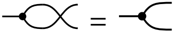
commutativity



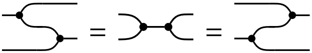
coassociativity



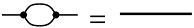
counitality



cocommutativity



Frobenius law



special law

Thus, for any strict symmetric monoidal category C, there's a 1-1 correspondence between:

- strict symmetric monoidal functors $F\colon \mathrm{FinCospan} \to C$

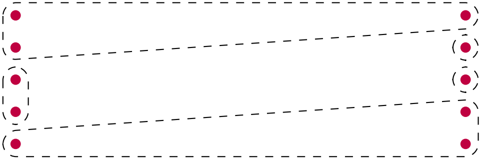
and

- special commutative Frobenius monoids in C.

We summarize this by saying FinCospan is "the prop for special commutative Frobenius monoids".

Example. The symmetric monoidal category where:

- objects are finite sets,
- morphisms are corelations:



- the tensor product is disjoint union

is equivalent to a prop, FinCorel.

Theorem (Coya, Fong). The prop FinCorel has the same generators as FinCospan:

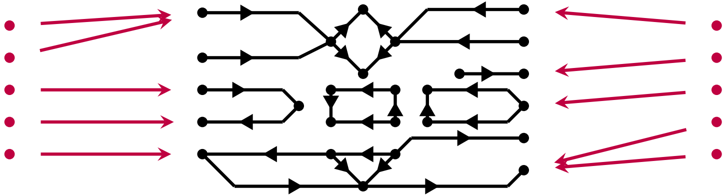
and all the same relations, together with one more:

extra law

Thus, FinCorel is the prop for extraspecial commutative Frobenius monoids.

Example. The symmetric monoidal category where:

- objects are finite sets,
- morphisms are circuits made solely of wires:



- the tensor product is disjoint union

is equivalent to a prop, Circ.

Theorem (Rosebrugh, Sabadani, Walters). The prop Circ has all the same generators and relations as Cospan, together with one additional generator $f\colon 1 \to 1$.

Thus, Circ is the prop for special commutative Frobenius monoids X equipped with a morphism $f\colon X \to X$.

In applications to electrical circuits, this morphism describes a *purely conductive wire*:

We can now understand these maps of props:

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}$$

using generators and relations:

- Circ is the prop for special commutative Frobenius monoids with endomorphism f.
- FinCospan is the prop for special commutative Frobenius monoids. G sends f to the identity.
- FinCorel is the prop for extraspecial commutative Frobenius monoids. H does the obvious thing.
- K sends the extraspecial commutative Frobenius monoid $1 \in \mathrm{FinCorel}$ to $\mathbb{R}^2 \in \mathrm{LagRel}$, which becomes an extraspecial commutative Frobenius monoid by 'duplicating potentials and adding currents'. For example

  gets sent to the Lagrangian relation

  $$L = \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) : \phi_1 = \phi_2 = \phi_3, I_1 + I_2 = I_3\} \subseteq \mathbb{R}^4 \oplus \mathbb{R}^2.$$

This is just the tip of the iceberg. Many fields of science and engineering use networks. A unified theory of networks will:

- reveal and clarify the mathematics underlying these fields,
- help integrate these fields,
- enhance interoperability of human-designed systems,
- focus attention on *open* systems: systems with inflows and outflows.

References

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