# Scalars in a Tangent Category

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# Motivation

## Classical differential geometry

Everything is defined in terms of the ring $\mathcal{R}$.

1. Every manifold is locally $\mathcal{R}^n$
2. Tangent vectors: equivalence classes of curves $\mathcal{R} \to M$.

## Synthetic Differential Geometry (SDG)

Still the case.

1. $\mathcal{R}$ has infinitesimals $D = [d : R | d^2 = 0]$.
2. Must satsify *Kock-Lawvere* axiom:
   $\nu : R \times R \to [D, R] := (a, b) \mapsto \lambda d.ad + b$
   is an iso
3. Tangent vectors are $D \to M$.

# Tangent Categories

## Definition (Rosický, Cockett-Cruttwell)

Abstract setting for differential geometry: only the behaviour of the *tangent bundle* is axiomatized.

- No ring object.
- Tangent vector addition - but no tangent vector subtraction.

This captures examples from computer science:

- REL: differential structure, no subtraction or ring of scalars.
- The classifying category for the $\partial - \lambda$ calculus.

Modular: Add a structure and see how it fits:

- A class of "submersions": *display* tangent categories
- Solutions to ODEs: *curve* objects

## Goal

Sort out "what does a scalar rig look like in a tangent category"

- What universal properties should it satisfy?
- Does it make "calculus" better behaved?

Result: The *Kock-Lawvere* axiom without infinitesimals.

Tangent categories are *enriched* categories - extend those results.

- Embed a tangent category into one with a scalar unit.
- Show that differential objects are (enriched) sketchable.

# Tangent Categories

A *tangent category* is a category $\mathbb{X}$ equipped with an endofunctor $T$ on $\mathbb{X}$ and natural transformations

$$p : T \Rightarrow id, 0 : id \Rightarrow T, + : T{}_p\!\times_p T \Rightarrow T, \ell : T \Rightarrow T^2, c : T^2 \Rightarrow T^2$$

All pullback powers of $p$ exist and are preserved by $T$
$(p_M, +_M, 0_M)$ is an additive bundle
$(\ell, 0)$, $(c, id)$ are additive.

$\ell$ is *universal*: $T(M)_p\!\times_p T(M) \xrightarrow{\mu} T^2(M) \underset{pp0}{\overset{T(p)}{\rightrightarrows}} T(M)$

(where $\mu = \langle \pi_0 \ell, \pi_1 0 \rangle T(+)$.)
$c$ is a symmetry $cc = id$ and $T(c)cT(c) = cT(c)c$
Symmetric cosemigroup: $\ell c = \ell$, $\ell T(\ell) = \ell\ell$ and $cT(c)\ell = T(\ell)c$

# Differential Objects

## Definition

Easier to do differential bundle over 1 definition $(V, \sigma_V, \xi_V)$ is a commutative monoid object such that.

1. Biproduct in CMon

$$\begin{tikzcd}
V \arrow[rr, equal] \arrow[dr, "\lambda"] & & V \\
& T(V) \arrow[ur, "\hat{p}"] \arrow[dr, "p"] & \\
V \arrow[ur, "0"] \arrow[rr, equal] & & V
\end{tikzcd}$$

2. Compatible addition

$$\begin{tikzcd}
V \arrow[r, "!_V"] \arrow[d, "0_V"'] & 1 \arrow[d, "\xi"] \\
T(V) \arrow[r, "\hat{p}"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2V \arrow[r, "{(\hat{p}\pi_0, \hat{p}\pi_1)}"] \arrow[d, "+_V"] & V \times V \arrow[d, "\sigma_V"] \\
T(V) \arrow[r, "\hat{p}"'] & V
\end{tikzcd}$$

3. Compatible lift:

$$\begin{tikzcd}
TV \arrow[r, "l"] \arrow[d, "\hat{p}"'] & T^2V \arrow[d, "\hat{p}"] \\
V & TV \arrow[l, "\hat{p}"]
\end{tikzcd}$$

# Differential objects ii

## Theorem (Cockett and Cruttwell)

*The full subcategory of differential objects is cartesian differential category.*

Set $\nu = (\lambda \times 0)T(\sigma_V)$

The derivative:
$$\begin{tikzcd} V \times V \arrow[r, "D[f]"] \arrow[d, "\nu"'] & W \\ T(V) \arrow[r, "T(f)"] & T(W) \arrow[u, "\hat{p}"'] \end{tikzcd}$$

Linear in $V$:
$$\begin{tikzcd} T(V \times X) \arrow[r, "T(f)"] & T(W) \\ V \times X \arrow[r, "f"] \arrow[u, "\lambda_V \times 0_X"] & W \arrow[u, "\lambda_W"'] \end{tikzcd}$$

Note: $\lambda$ is universal
$$\begin{tikzcd} V \arrow[r, "\lambda"] & T(V) \arrow[r, shift left, "p"] \arrow[r, shift right, "!\xi"'] & V \end{tikzcd}$$

# Linear Classifier

Under a very mild assumption, we can add a universal ring object to a tangent category.

## Definition (Blute-Cockett-Seeley)

A *Scalar Unit* is a differential object with a point $1 \xrightarrow{u} R$ with the universal property that for all

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\langle 1,u \rangle"'] & W \\ V \times R \arrow[ur, dotted, "\exists! \hat{f} \text{linear in } R"'] & \end{tikzcd}$$

$f$ (multi)-linear in $V \Rightarrow \hat{f}$ is (multi)-linear in $V$

# Consequences of a Linear Point Classifier

- The unit object is a commutative rig $R$.

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d, "\langle 1, u \rangle"'] & R \\ R \times R \arrow[ur, "\cdot"'] & \end{tikzcd}$$

- Every differential object is an $R$-module.

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "\langle 1, u \rangle"'] & V \\ V \times R \arrow[ur, "\cdot"'] & \end{tikzcd}$$

- Every linear map preserves the $R$-module action (persistence).

# Rewriting the lift

Every $R$-module has the map $\lambda^R$:

$$V \xrightarrow{\langle 1,u\rangle} V \times R \xrightarrow{0\times\lambda} T(V \times R) \xrightarrow{T(\cdot)} T(V)$$

In SDG: $v \mapsto \lambda d.vd$.

## Lemma

*For a differential object,* $\lambda^R$ *satisfies the equalizer*

$$V \xrightarrow{\lambda^R_V} T(V) \underset{!\xi}{\overset{p}{\rightrightarrows}} V$$

*so* $\lambda_V = \lambda^R_V$.

## Corollary

*Homogenous morphisms of differential objects are linear.*

# KL-Modules

Set $\nu^T := \; V \times V \xrightarrow{\lambda^R \times 0} T(V) \times T(V) \xrightarrow{T(\sigma)} T(V)$

**Definition (*Kock-Lawvere R*-module)**

$V$ is a KL-module if there is an $R$-module map $\hat{p}$ making

$$((\lambda^R \times 0)T(\sigma))^{-1} = \langle \hat{p}, p_V \rangle$$

- The category of KL-modules is equivalent to the category of differential objects.
- In a locally presentable tangent category, KL-modules is a full reflective subcategory of $R$-modules.
- If $\mathbb{X}$ has negatives, then KL-modules are a completion of $R$-modules.

# New Questions

The notion of a scalar unit allows one to use the simpler definition of KL-modules.

If $R$ is a ring, KL-modules are a completion of $R$-modules - is there a sketch of KL-modules?

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Move to *enriched category theory*.

# Weil Algebras, Microlinear Weil Spaces

## Definition (The category Weil)

The full subcategory of $\pi : W \to R$ in unital $R\mathrm{Alg}/R$ so that:

- $\ker(\pi)$ is a nilpotent ideal
- $U(W) = R^n$.

e.g. the dual numbers $R[x]/x^2$

A *Microlinear Weil Space* is a presheaf Weil $\to$ Set preserving connected limits. Call the category of microlinear weil spaces $\mathcal{E}$.

## Theorem

*The category of microlinear weil spaces is*

- *Locally finitely presentable*
- *A coherently closed tangent category*
- *Has a scalar unit* $R = [y(R[x]/x^2), y(R[x]/x^2)]$

# Microlinear Weil Spaces

Recall that in a $\mathcal{E}$-category $\mathcal{C}$

- Power: $\mathcal{E}(w, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, w \pitchfork Y)$
- Copower: $\mathcal{E}(w \bullet X, Y) \cong \mathcal{E}(w, \mathcal{C}(X, Y))$

### Theorem (Garner, Leung)

*A tangent category is equivalently a category enriched in $\mathcal{E}$ with powers by representables.*

$$T(X) := y(R[x]/x^2) \pitchfork X$$

### Observation

The tangent bundle is now a weighted limit.

# Units in Presheaf Categories

**Theorem**

*The enriched Yoneda embedding preserves differential objects.*

**Theorem**

*The enriched presheaf category of a tangent category has a representable unit:*

$$1 \bullet R \cong [1 \bullet y(x^2), 1 \bullet y(x^2)]$$

**Observation**

Every differential objects is a KL-module in the presheaf category.

# KL-modules as sketches

Define the $\mathcal{E}$-sketch KLMod.

- Objects: $n \in \mathbb{N}$
- Hom-Objects: $[n, m] = R^{n \times m}$
- Composition: Matrix multiplication

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- Powers on objects: $y(R[x]/x^2) \pitchfork n := 2n$, and fix:
  - $0_n : 1 \to R^{n \times 2n}$ picks out the matrix $\begin{bmatrix} 0 \\ I \end{bmatrix}$
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  We derive:
  - $\lambda_n$ picks out $\begin{bmatrix} I \\ 0 \end{bmatrix}$
  - $\hat{p}$ picks out $\begin{bmatrix} I & 0 \end{bmatrix}$

  ($I = n \times n$ identity matrix)
- Powers on homs: $\Delta_{n,m} : R^{n \times m} \to R^{2n \times 2m}$

# Conclusions and Future Work

We used the notion of a scalar unit to simplify Differential Objects and find a $\mathcal{E}$-sketch.

Opens the door for sketch theory to be applied in tangent categories

- Gabriel-Ulmer duality: a free KL-module construction?
- Differential Bundles
- *Involution Algebroids* a sketch for Lie theory (Joint work with Matthew Burke)

Sketches can be interpreted as *Abstract Data Types*

- $\mathcal{E}$-sketches as $\partial$-ADTs in differential programming?

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