Lecture Notes
Linear Programming

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Chapter 1

The Conventional Linear Programming Model

1.1 Models and Model Types

Some of the Best Known Models:

- Scale model

  A typical example is an aerodynamic model of an airplane which looks exactly like the actual aircraft but smaller in size. Such a model serves as an evaluative tool rather than a means for system optimization.

- Pictorial model

  Usually, this is a two-dimensional photograph or sketch of a system. Such a model aids the decision process but does not actually lend itself to an optimizing procedure.

- Flow chart (or network)

  This is a special type of a pictorial model which illustrates the interrelationships among the components. In some instances, the flow chart is used only as a decision aid, but one can actually perform an optimization of the model of the system through network analysis.

- Matrix

  In many instances, the flow chart model and the matrix model may be interchangeable since a flow chart can be represented by a matrix.

- Mathematical Model

  This is a set of mathematical functions that represent the problem under consideration.

1.2 General Guidelines in Model Building

1. What is the primary purpose of the model?

2. What are the results going to be used for?

3. How much accuracy is required?

4. Over what time scale is the model to be used?
5. Over what range of inputs must it respond?
6. What are the budget and time restrictions on model development?
7. Who will use the final model and/or its results?

The model that seems to most naturally fit the problem should be employed.

1.3 Basic Steps in Linear Programming Model Formulation

1. **Determine the decision variables.**

   The decision variables within a problem are those over which one actually has control. Consider for example, the problem involved in insulating your house so as to reduce utility costs. Numerous variables exist in such a problem, including the following:

   (a) Amount of attic insulation to be installed.
   (b) Amount of side-wall insulation to be installed.
   (c) Amount of caulking to be done.
   (d) Number (and perhaps types) of storm windows.
   (e) Number (and perhaps types) of insulating draperies or curtains used.
   (f) Amount of insulation to be installed around the hot-water tank.
   (g) Temperatures experienced.
   (h) Wind velocity and direction.
   (i) Amount of sunshine incident on the house.
   (j) Number of individuals within the house.
   (k) Number of times per day that a door or garage door is opened.
   (l) Cost of utilities furnished.

   Only the first six are directly controllable. Hence, the only first six variables should appear in our model as decision variables.

2. **Formulation of the Objective**

   The objective is generally the result of the desire of the decision maker and may typically be one of the following:

   - Maximize profit
   - Minimize costs
   - Minimize overtime
   - Maximize resource utilization (personnel, machinery or processes)
   - Minimize labor turnover
   - Minimize machine downtime
   - Maximize the probability that a given process remains within certain control limits

- Minimize the deviation from the standard

3. **Formulation of the Constraints**

   Generally, a constraint (or restriction) is the result of a resource of technological limitation such as

   - Limited raw material
   - Limited budget
   - Limited time
   - Limited personnel
   - Limited ability or skills

1.4 The General Form of the Linear Programming Model

Find $\mathbf{X} = (x_1, x_2, \ldots x_n)^T$ so as to optimize the objective function subject to the specified constraints.

$$\text{optimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{subject to}$$

$$\begin{array}{rcl}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n & \{\leq, =, \geq\} & b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n & \{\leq, =, \geq\} & b_2 \\
 & \vdots & \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n & \{\leq, =, \geq\} & b_m
\end{array}$$

$$x_1, x_2, \ldots x_n \geq 0$$

1.5 Assumptions of the LP Model

1. **Certainty**. The problem data (c_j, b_i, and $a_{i,j}; i = 1, \ldots m; j = 1, \ldots n$) are assumed to be known with certainty.

2. **Proportionality**. The contribution of the decision variable, x_j, to the objective function is c_jx_j and its contribution to the ith constraint is $a_{i,j}x_j$. That is, the contribution of x_j is always directly proportional to the level of the variable x_j.

3. **Additivity**. This assumption ensures, for example, that the total cost is the sum of the cost contributions of each individual variable. That is, the contributions from individual variables combine linearly in both the objective function and the constraints.

4. **Divisibility**. Divisibility implies that the decision variables are continuous variables, that is, they can be divided into fractional parts.

1.6 Examples of Linear Programming Model Formulation

Example 1.1. A Product-Mix Problem

The product-mix problem is typical of linear programming problems in which there is competition for limited resources among several products.

HiTech Inc., a small manufacturing firm produces two microwave switches, Switch A and Switch B. The return per unit of Switch A is $20, whereas the return per unit of Switch B is $30. Because of contractual commitments, HiTech must manufacture at least 25 units of Switch A per week and based on the present demand for its products, it can sell all that it can manufacture. However, it wishes to maximize profit while determining the production sizes to satisfy various limits resulting from a small production crew. These include

Assembly hours: 240 hours available per week

Testing hours: 140 hours available per week

Switch A requires 4 hours of assembly and 1 hour of testing, and switch B requires 3 hours and 2 hours, respectively.

Determination of the decision variables. The variables directly under HiTech's control are

$$\begin{aligned} x_1 &= \text{ amount of Switch A manufactured per week} \\ x_2 &= \text{ amount of Switch B manufactured per week} \end{aligned}$$

Formulation of the objective.

$$\text{maximize } z = 20x_1 + 30x_2 \text{ (profit per week)}$$

Formulation of the constraints.

$$\begin{aligned} 4x_1 + 3x_2 &\le 240 \quad \text{(assembly hours per week)} \\ x_1 + 2x_2 &\le 140 \quad \text{(testing hours per week)} \\ x_1 &\ge 25 \quad \text{(Switch A demand per week)} \\ x_1, x_2 &\ge 0 \quad \text{(Nonnegativity constraint)} \end{aligned}$$

As a result, the mathematical model for the problem may be summarized as follows:

$$\text{maximize } z = 20x_1 + 30x_2$$

subject to

$$\begin{aligned} 4x_1 + 3x_2 &\le 240 \\ x_1 + 2x_2 &\le 140 \\ x_1 &\ge 25 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Example 1.2. An Investment Planning Problem

An investment-planning problem is another example of the optimal allocation of limited resources, in this case, investment capital.

An investor has decided to invest a total of \$50,000 among three investment opportunities: savings certificates, municipal bonds and stocks. The annual return on each investment is estimated to be 7%, 9% and 14% respectively. The investor does not intend to invest his annual interest returns, (that is, he plans to use the interest to finance his desire to travel). He would like to maximize his yearly return while investing a minimum of \$10,000 in bonds. Also, the investment in stocks should not exceed the combined total investment in bonds and savings certificates. And finally, he should invest between \$5,000 and \$15,000 in savings certificates.

Determination of the decision variables.

$$\begin{aligned} x_1 &= \text{dollars invested in savings certificates} \\ x_2 &= \text{dollars invested in municipal bonds} \\ x_3 &= \text{dollars invested in stocks} \end{aligned}$$

Formulation of the objective.

$$\text{maximize } z = 0.07x_1 + 0.09x_2 + 0.14x_3 \text{ (yearly return)}$$

Formulation of the constraints.

$$\begin{aligned} x_2 &\ge 10000 \quad \text{(investment in bonds)} \\ x_3 &\le x_1 + x_2 \quad \text{(stock restriction)} \\ 5000 &\le x_1 \le 15000 \quad \text{(savings certificates)} \\ x_1 + x_2 + x_3 &\le 50000 \quad \text{(total investment)} \\ x_1, x_2, x_3 &\ge 0 \quad \text{(Nonnegativity constraint)} \end{aligned}$$

As a result, the mathematical model for the problem may be summarized as follows:

$$\text{maximize } z = 0.07x_1 + 0.09x_2 + 0.14x_3$$

subject to

$$\begin{aligned} x_2 &\geq 10000 \\ -x_1 - x_2 + x_3 &\leq 0 \\ x_1 &\geq 5000 \\ x_1 &\leq 15000 \\ x_1 + x_2 + x_3 &\leq 50000 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Example 1.3. A Product-Blending Problem

There is a wide variety of problems in which certain basic components of raw materials are combined, or blended, to produce a product that satisfies certain specifications.

The Sierra Refining Company produces two grades of unleaded gasoline, Grade 1 and Grade 2, which it supplies to its chain of service stations for \$48 and \$53 per barrel, respectively. Both grades of gasoline are blended from Sierra's inventory of gasoline components and must meet the specification in Table 1. The characteristics of the components in inventory are found in Table 2.

TABLE 1 GASOLINE SPECIFICATIONS

Gasoline	Minimum octane rating	Maximum demand (barrels/wk)	Minimum deliveries (barrels/wk)
Grade 1	87	80,000	60,000
Grade 2	93	40,000	15,000

TABLE 2 COMPONENTS CHARACTERISTICS

Gasoline component	Octane rating	Inventory (barrels)	Cost (\$ /barrel)
1	86	70,000	33
2	96	60,000	37

What quantities of the two components should be blended into the two gasolines in order to maximize weekly profit?

Determination of the decision variables.

$$\begin{aligned} x_{i,j} &= \text{ barrels of Component } i \text{ blended into Grade } j \text{ gasoline per week;} \\ i &= 1, 2; j = 1, 2 \end{aligned}$$

Formulation of the objective. Assuming that gasoline components combine linearly, the total amount of Grade 1 gasoline is given by $(x_{1,1} + x_{2,1})$ and the total amount of Grade 2 gasoline is given by $(x_{1,2} + x_{2,2})$. Similarly the total amounts of Component 1 and Component 2 used are given by $(x_{1,1} + x_{1,2})$ and $(x_{2,1} + x_{2,2})$, respectively. The objective function can now be formulated by noting that profit = sales - cost.

$$\text{maximize } z = 48(x_{1,1} + x_{2,1}) + 53(x_{1,2} + x_{2,2}) - 33(x_{1,1} + x_{1,2}) - 37(x_{2,1} + x_{2,2})$$
(profit per week)

which simplifies to

$$\text{maximize } z = 15x_{1,1} + 20x_{1,2} + 11x_{2,1} + 16x_{2,2}$$

Formulation of the constraints. In order to model the minimum octane-rating requirement, one has to be able to determine the octane level when varying quantities of the two refined components are combined. Again, it is necessary to assume that octanes blend linearly. The octane level in a mixture of components is obtained by computing the weighted average of the octane in the mixture. This is done by dividing the total octane in the mixture by the number of barrels in the mixture. Thus, the minimum octane constraint for Grade 1 gasoline can be written mathematically as

$$\frac{86x_{1,1} + 96x_{2,1}}{x_{1,1} + x_{2,1}} \geq 87 \qquad \text{(minimum octane rating for Grade1)}$$

Note that this constraint is nonlinear; however, multiplying both sides by $(x_{1,1} + x_{2,1})$ and simplifying yield the equivalent linear constraint:

$$-x_{1,1} + 9x_{2,1} \geq 0$$

Similarly, the minimum octane restriction for Grade 2 gasoline is given by

$$\frac{86x_{1,2} + 96x_{2,2}}{x_{1,2} + x_{2,2}} \geq 93 \qquad \text{(minimum octane rating for Grade2)}$$

or

$$-7x_{1,2} + 3x_{2,2} \geq 0$$

The minimum and maximum distribution requirements for the two grades of gasoline may be formulated as follows:

$$\begin{array}{rcll}
x_{1,1} + x_{2,1} & \geq & 60000 & \text{(minimum deliveries of Grade 1)} \\
x_{1,2} + x_{2,2} & \geq & 15000 & \text{(minimum deliveries of Grade 2)} \\
x_{1,1} + x_{2,1} & \leq & 80000 & \text{(maximum demand for Grade 1)} \\
x_{1,2} + x_{2,2} & \leq & 40000 & \text{(maximum demand for Grade 2)}
\end{array}$$

Similarly, the supply constraints are given by

$$\begin{array}{rcll}
x_{1,1} + x_{1,2} & \leq & 70000 & \text{(inventory of Component 1)} \\
x_{2,1} + x_{2,2} & \leq & 60000 & \text{(inventory of Component 2)}
\end{array}$$

and the nonnegativity constraints are

$$x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \geq 0$$

As a result, the mathematical model for the problem may be summarized as follows:

$$\text{maximize } z = 15x_{1,1} + 20x_{1,2} + 11x_{2,1} + 16x_{2,2}$$

subject to

$$
\begin{aligned}
-x_{1,1} + 9x_{2,1} &\geq 0 \\
-7x_{1,2} + 3x_{2,2} &\geq 0 \\
x_{1,1} + x_{2,1} &\geq 60000 \\
x_{1,2} + x_{2,2} &\geq 15000 \\
x_{1,1} + x_{2,1} &\leq 80000 \\
x_{1,2} + x_{2,2} &\leq 40000 \\
x_{1,1} + x_{1,2} &\leq 70000 \\
x_{2,1} + x_{2,2} &\leq 60000 \\
x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} &\geq 0
\end{aligned}
$$

Example 1.4. A Transportation Problem

In many applications, it is necessary to determine a shipping schedule for distributing goods from several warehouses to several retail outlets. Due to proximity and mode of transportation, the cost of shipping a unit between each warehouse and retail outlet may vary from location to location. In addition, the supplies available for shipping from the warehouses and units demanded at the retail outlets may also vary. The task then is to determine the number of units to ship from each warehouse to each retail outlet while minimizing shipping costs.

A manufacturer has three warehouses that supply finished product to four retail outlets. The warehouses have 6000, 9000, and 4000 units available, and the demands at the retail outlets are projected to be 3900, 5200, 2700, and 6400 units. The per unit costs (in dollars) of shipping from each warehouse to each retail outlet are given in Table 3. The manufacturer need to determine the minimum-cost shipping schedule that satisfies all demands.

TABLE 3 UNIT SHIPPING COSTS

	Retail Outlets			
Warehouse	1	2	3	4
1	7	3	8	4
2	9	5	6	3
3	4	6	9	6

Determination of the decision variables.

$$
\begin{aligned}
x_{i,j} &= \text{quantity shipped from Warehouse } i \text{ to Retail Outlet } j\ ; \\
i &= 1, 2, 3; j = 1, 2, 3, 4
\end{aligned}
$$

Formulation of the objective.

minimize $z = 7x_{1,1} + 3x_{1,2} + 8x_{1,3} + 4x_{1,4} + 9x_{2,1} + 5x_{2,2} + 6x_{2,3} + 3x_{2,4} + 4x_{3,1}$
$+6x_{3,2} + 9x_{3,3} + 6x_{3,4}$ (total shipping cost)

Formulation of the constraints.

$$\begin{aligned}
x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4} &\leq 6000 \quad \text{(supply at Warehouse 1)}\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4} &\leq 9000 \quad \text{(supply at Warehouse 2)}\\
x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4} &\leq 4000 \quad \text{(supply at Warehouse 3)}\\
x_{1,1}+x_{2,1}+x_{3,1} &= 3900 \quad \text{(demand at Retail Outlet 1)}\\
x_{1,2}+x_{2,2}+x_{3,2} &= 5200 \quad \text{(demand at Retail Outlet 2)}\\
x_{1,3}+x_{2,3}+x_{3,3} &= 2700 \quad \text{(demand at Retail Outlet 3)}\\
x_{1,4}+x_{2,4}+x_{3,4} &= 6400 \quad \text{(demand at Retail Outlet 4)}\\
x_{1,1},x_{1,2},x_{1,3},x_{1,4},x_{2,1},x_{2,2},x_{2,3},x_{2,4},x_{3,1},x_{3,2},x_{3,3},x_{3,4} &\geq 0 \quad \text{(Nonnegativity constraint)}
\end{aligned}$$

The complete model may be written as follows

$$\text{minimize } z = 7x_{1,1}+3x_{1,2}+8x_{1,3}+4x_{1,4}+9x_{2,1}+5x_{2,2}+6x_{2,3}+3x_{2,4}+4x_{3,1} \\ +6x_{3,2}+9x_{3,3}+6x_{3,4}$$

subject to

$$\begin{aligned}
x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4} &\leq 6000\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4} &\leq 9000\\
x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4} &\leq 4000\\
x_{1,1}+x_{2,1}+x_{3,1} &= 3900\\
x_{1,2}+x_{2,2}+x_{3,2} &= 5200\\
x_{1,3}+x_{2,3}+x_{3,3} &= 2700\\
x_{1,4}+x_{2,4}+x_{3,4} &= 6400\\
x_{1,1},x_{1,2},x_{1,3},x_{1,4},x_{2,1},x_{2,2},x_{2,3},x_{2,4},x_{3,1},x_{3,2},x_{3,3},x_{3,4} &\geq 0
\end{aligned}$$

1.7 Model Validity

On what basis may the validity of models be compared?

1. **An Evaluation of Model Structure**

   There are two basic approaches commonly used in model development. The first is to start with a model that contains all aspects and variables of the system. One then attempts to simplify this model step by step to the point at which any further simplification would so distort the model as to be unacceptable. The second method, and by far the more preferable, is to begin with a preliminary, simplified model of the system. Such a model represents only the most basic factors and operations of the system. Detail is added to this model until one is satisfied that the responses of interest are accurately represented.

2. **An Evaluation of Model Logic**

   The first evaluation examined the representation of those elements within the system that are believed to have a significant impact on the system's responses. The accuracy of the representation of the interrelationships and interactions between those elements is now determined. If the model logic truly reflects the system logic, the model will react to a stimulus in the same manner as would the actual system. A common procedure for the evaluation of model logic is to stimulate the model with a representative range of inputs and observe the resultant model output.

Generally, it is not essential that the output response values of the model be of the same value as the actual system. Rather, it is the relative difference in the outputs that is of importance. For example, for a given policy, do profits rise or fall?

3. **An Evaluation of the Design and/or Input Data**

   The data used in model development may be roughly classified into two types: (1) the design data, or that information used to actually construct the model, and (2) the input data, or data used to stimulate the system. Inattention to either may seriously degrade the validity of a model.

   Data collection and verification may well be the most overlooked portion of model construction. Textbook presentations of data-collection process are simplified. One common simplification is that the data needed for that model development and/or solution are usually presented to the reader as was done in the foregoing examples. Thus, the reader seldom, if ever, faces the problems involved in the actual collection of data. Unfortunately, not only is the validity of the model dependent on these data, but, also, the process of data collection often consumes the major amount of time and resources when dealing with actual problems.

   All too often data are collected before one has decided on the basic form of the model to be used. As a result, many of the data are useless in that they do not fit the requirements of the model finally developed. What one should do first is decide on the basic form of the model, construct a preliminary model and then identify its specific data needs.

4. **An Evaluation of Model Response**

   The true validation of a model is often said to be reflected solely in its ability to predict the behavior of the system that has been modeled. Such a premise can be absurd.

   Even if one accepts the premise of future verification, it is not always possible to wait for such results. Validation of model response must then be accompanied through the input of estimated, historical and ongoing data. However, the fact that the model performs "reasonably" with such inputs is not, by any means, an absolute guarantee of its validity or reaction to future data.

Although consideration of the four evaluation areas is by no means perfect nor completely objective, it does provide a practical means to consider validation on a relative basis and to compare the validity of models.

EXERCISES.

1. A furniture maker has 6 units of wood and 28 hours of free time, in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model I requires 2 units of wood and 7 hours time, while model II requires 1 unit of wood and 8 hours of time. The prices of the models are $120 and $80, respectively. Develop a linear programming model that maximizes his sales revenue.

2. Two recent graduates have decide to enter the field of microcomputers. They intend to manufacture two types of microcomputers, CompA and CompB. Because of the interest in microcomputers, they can sell all that they could possibly produce. However, they wish to size the production rate so as to satisfy various estimated limits with a small production crew. These include

   Assembly hours: 150 hours per week

   Test hours: 70 hours per week

   CompA requires 4 hours of assembly and 3 hours of testing, and CompB consumes 6 hours and 3.5 hours, respectively. Profit to the CompA is estimated at $300 per unit; that of CompB is $450 per unit. Develop a linear programming model that maximizes weekly profit.

3. Wild West produces two types of cowboy hats. Type 1 hat requires twice as much labor time as does each to the type 2. If all produced hats are of type 2 only, the company can produce a total of 400 hats a day. The market daily limits are 150 and 200 hats for types 1 and 2, respectively. The profit per type 1 hat is \$8 and that of type 2 hat is \$5. Develop a linear programming model that maximizes profit.

4. A container manufacturer is considering the purchase of two different types of cardboard folding machines: model A and model B. Model A can fold 30 boxes per minute and and requires 1 attendant, whereas model B can fold 50 boxes per minute and requires 2 attendants. Suppose the manufacturer must fold at least 320 boxes per minute and cannot afford more than 12 employees for the folding operation. If a model A machine costs \$15,000 and a model B machine costs \$20,000, how many machines of each type should be bought to minimize the cost?

5. A lumber mill saws both finish-grade and construction-grade boards from the logs that it receives. Suppose that it takes 2 hr to rough-saw each 1000 board feet of the finish-grade boards and 5 hr to plane each 1000 board feet of these boards. Suppose also that it takes 2 hr to to rough-saw each 1000 board feet of the construction-grade boards, but it takes only 3 hr to plane each 1000 board feet of these boards. The saw is available 8 hr per day, and the plane is available 15 hr per day. If the profit on each 1000 board feet of finish-grade boards is \$120 and the profit on each 1000 board feet of construction-grade boards is \$100, how many board feet of each type of lumber should be sawed to maximize the profit?

6. A nutritionist is planning a menu consisting of two main foods A and B. Each ounce of A contains 2 units of fat, 1 unit of carbohydrates, and 4 units of protein. Each ounce of B contains 3 units of fat, 3 units of carbohydrates, and 3 units of protein. The nutritionist wants the meal to provide at least 18 units of fat, at least 12 units of carbohydrates, and at least 24 units of protein. If an ounce of A costs 20 cents and an ounce of B costs 25 cents, how many ounces of each food should be served to minimize the cost of the meal yet satisfy the nutritionist's requirements?

7. Three available investment options are available at the beginning of each year during the next 6-year period. The durations of the investments are 1 year, 3 years, and 5 years. The 1-year investment yields a total return of 5.1%, the 3-year investment yields a total return of of 16.2%, and the 5-year investment yields a total return of 28.5%. If an initial investment of \$10,000 dollars is made and all available funds are invested at the beginning of each year, formulate a linear programming model to determine the investment pattern that results in the maximum available cash at the end of the sixth year.

8. Acme Fuel, Inc., has two refineries where fuel oil is produced. Refinery A has the capacity to produce a maximum of 275,000 gallons per week, and the corresponding figure for Refinery B is 350,000 gallons. Acme has four regional distribution centers that receive fuel oil directly from the refineries. The shipping cost per gallon are summarized in the table below.

Refinery	Center 1	Center 2	Center 3	Center 4
A	\$0.12	\$0.07	\$0.09	\$0.11
B	\$0.08	\$0.10	\$0..09	\$0.10

The projected demands at Distribution Centers 1, 2, 3, and 4 are 120,000, 70,0000, 185,000, and 200,000 gallons, respectively. To help maintain a uniform work load, it is management policy that the ratio of scheduled production to a refinery's capacity must be the same for the two refineries. Formulate a linear programming problem to find the minimum-cost shipping pattern.

9. A firm manufactures chicken feed by mixing three different ingredients. Each ingredient contains four key nutrients: protein, fat, vitamin s and mineral t. The amount of each nutrient contained in 1 kilogram of the three basis ingredients is summarized in the table below.

Ingredient	Protein (grams)	Fat (grams)	Vitamin s (units)	Mineral t (grams)
1	25	11	235	12
2	45	10	160	6
3	32	7	190	10

The cost per kilogram of Ingredients 1,2, and 3 are \$0.55, \$0.42, and \$0.38, respectively. Each kilogram of the feed must contain at least 35 grams of protein, a minimum of 8 grams of fat and a maximum of 10 grams of fat, at least 200 units of vitamin s, and at least 10 units of mineral t. Formulate a linear programming problem for finding the feed mix that has the minimum cost per kilogram.

Chapter 2

Foundations of the Simplex Method

2.1 Converting a Linear Program into Standard Form

Recall that a linear program may be written in the general form:

$$\text{optimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{subject to}$$

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n & \{\leq, =, \geq\} & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n & \{\leq, =, \geq\} & b_2 \\ & \vdots & \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n & \{\leq, =, \geq\} & b_m \end{array}$$

$$x_1, x_2, \ldots x_n \geq 0$$

or written more compactly, in summation notation,

$$\text{optimize } z = \sum_{j=1}^{n} c_jx_j$$

$$\text{subject to}$$

$$\begin{array}{rcl} \sum_{j=1}^{n} a_{i,j}x_j & \{\leq, =, \geq\} & b_i;\ i = 1, \ldots m \\ x_j & \geq & 0;\ j = 1, \ldots n \end{array}$$

Constraint Conversion

First, consider an inequality, say, constraint r, of the following form:

$$\sum_{j=1}^{n} a_{r,j}x_j \leq b_r$$

We introduce a new variable, $s_r \geq 0$, called the *slack* variable, so that

$$\sum_{j=1}^{n} a_{r,j} x_j + s_r = b_r$$

Next, consider an inequality, say, constraint t, of the form:

$$\sum_{j=1}^{n} a_{t,j} x_j \geq b_t$$

In this case, we introduce a new variable, $s_t \geq 0$, called the *surplus* variable, so that

$$\sum_{j=1}^{n} a_{t,j} x_j - s_t = b_t$$

The Objective Function

We shall consider the standard form of the objective function to be *maximization*. This in no way eliminates the consideration of minimization-type objectives because if a function z is to be minimized, we can use the simple equivalence:

$$\text{minimize } z = \text{- maximize } (-z)$$

Thus, given a maximization objective z and p surplus and slack variables, the modified objective is

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} c_k s_k$$

Example 2.1. Converting a Linear Program into Standard Form Given the following linear programming model, convert both the constraints and objective function into standard form.

$$\text{minimize } z = 7x_1 - 3x_2 + 5x_3 \qquad \text{(cost in dollars)}$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 9 \\ 3x_1 + 2x_2 + x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Let us assume that the cost of a surplus unit in the first constraint is zero, and the cost of the slack in the second is \$1.50 per unit. The resultant model is

$$\text{maximize } z' = -7x_1 + 3x_2 - 5x_3 + 0s_1 - 1.5s_2 \qquad \text{(cost in dollars)}$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + x_2 + x_3 - s_1 &= 9 \\ 3x_1 + 2x_2 + x_3 + s_2 &= 12 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

Quite often, textbooks do not distinguish between a decision variable (x_j) or a slack or surplus variable and the foregoing model would be written:

$$\text{maximize } z' = -7x_1 + 3x_2 - 5x_3 + 0x_4 - 1.5x_5$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 9 \\ 3x_1 + 2x_2 + x_3 + x_5 &= 12 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The latter equivalent formulation will be easier to deal with in subsequent lessons. In general, this formulation will be referred to as the *standard form* of a linear program and can be summarized as follows:

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to}$$

$$\begin{aligned} \sum_{j=1}^{n} a_{i,j} x_j &= b_i;\ i = 1, \ldots m \\ x_j &\geq 0;\ j = 1, \ldots n \end{aligned}$$

Notation and Definitions

We will introduce some vector and matrix notation. First, let $\mathbf{X}$ represent all the variables in the standard form of an LP model.

We may then write the standard linear programming model as

$$\text{maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq 0 \end{aligned}$$

where the data are given by

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \\ \mathbf{b} &= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\ \mathbf{c} &= (c_1, c_2, \ldots, c_n) \\ \mathbf{X} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

Example 2.2. Using Matrix Notation

Consider the following linear programming model, in which x_4 and x_5 are slack variables.

$$\text{maximize } z = 5x_1 + 7x_2 + x_3 + 0x_4 + 0x_5$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 &= 12 \\ 5x_1 + 6x_2 + x_5 &= 24 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Then this problem may be written as

$$\text{maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq 0 \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 3 & -1 & 1 & 0 \\ 5 & 6 & 0 & 0 & 1 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 12 \\ 24 \end{bmatrix} \\ \mathbf{c} &= (5\ 7\ 1\ 0\ 0) \\ \mathbf{X} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} \end{aligned}$$

We now introduce some terminology concerning the solution space of a linear programming problem.

1. **FEASIBLE SOLUTION** A solution is feasible if it satisfies all the constraints of the linear program, for example, $\mathbf{X}$ is a feasible solution of a problem if $\mathbf{AX} = \mathbf{b}$ and $\mathbf{X} \geq \mathbf{0}$. The set of all feasible solutions is called the *feasible region*.

2. **INFEASIBLE SOLUTION** Any point that does not satisfy all the constraints the nonnegativity conditions is infeasible.

3. **OPTIMAL SOLUTION** A point $\mathbf{X}^*$ is an optimal solution to a maximization linear problem if $\mathbf{X}^*$ is a feasible solution and $\mathbf{cX}^* \geq \mathbf{cX}$ for all feasible solutions $\mathbf{X}$.

2.2 Graphical Solution of Two-Dimensional Linear Programs

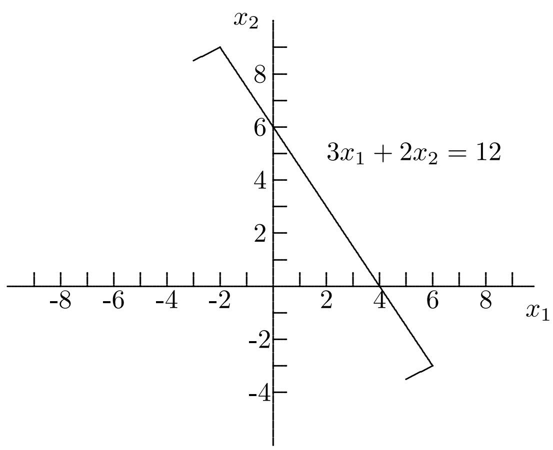
The mechanics of the graphical procedure are brief and straightforward. We first plot all constraints on a rectangular coordinate system in which each axis represents one decision variable. If a constraint is *equality*, it will plot as a *straight line* (in general, this is called a *hyperplane*). However, if the constraint is an *inequality*, it defines a *region* (this region is formally called a *halfplane* or *halfspace*) that is bounded by the straight line obtained when the constraint is considered as an equality. Suppose we would like to graph the set of points (x_1, x_2) satisfying the inequality constraint

$$3x_1 + 2x_2 \leq 12$$

To identify the region defined by this constraint, we begin by graphing the corresponding linear equation,

$$3x_1 + 2x_2 = 12$$

This straight line divides the Cartesian plane into two regions(halfplanes). It is quite easy to determine which side of the line corresponds to the region defined by the inequality. Simple choose any point P, that is not on the line and check if P satisfies the inequality. If so, then all points that lie on the same side as P satisfy the inequality. Otherwise, all points on the opposite side as P satisfy the inequality. For example, choosing $P = (0, 0)$, we see that $3(0)+2(0) = 0 < 6$; therefore all points on the same side as $(0, 0)$ satisfy the inequality. The region corresponding to $3x_1+2x_2 \leq 12$ is plotted below.



We now use the following example to illustrate how to graphically solve a linear program with two decision variables.

Example 2.3. A Maximization Problem

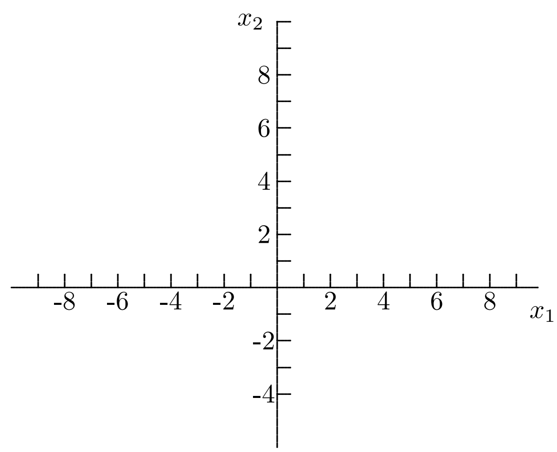
Find x_1 and x_2 so as to

maximize $z = 2x_1 + 3x_2$

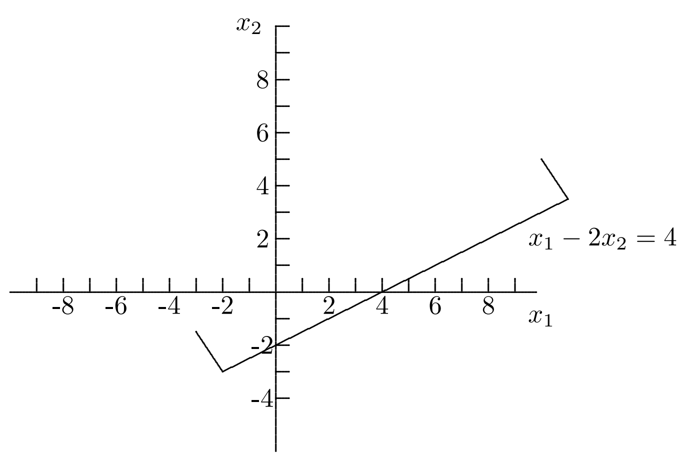
subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 4 \\ 2x_1 + x_2 &\leq 18 \\ x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

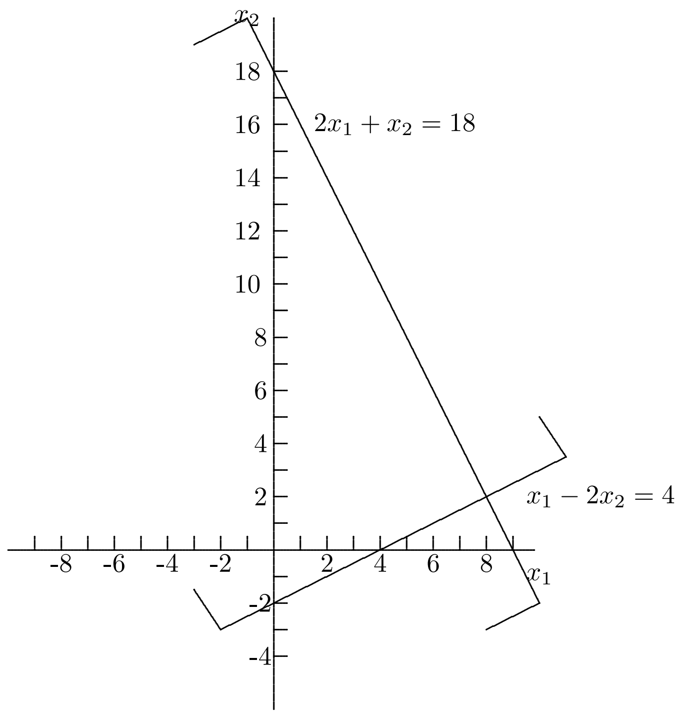
First, we must identify the feasible region of the model. Labeling one axis x_1 and the other x_2, we establish our coordinate system as shown below. Note that the nonnegativity constraint require that we only consider points (x_1, x_2), in the first quadrant.

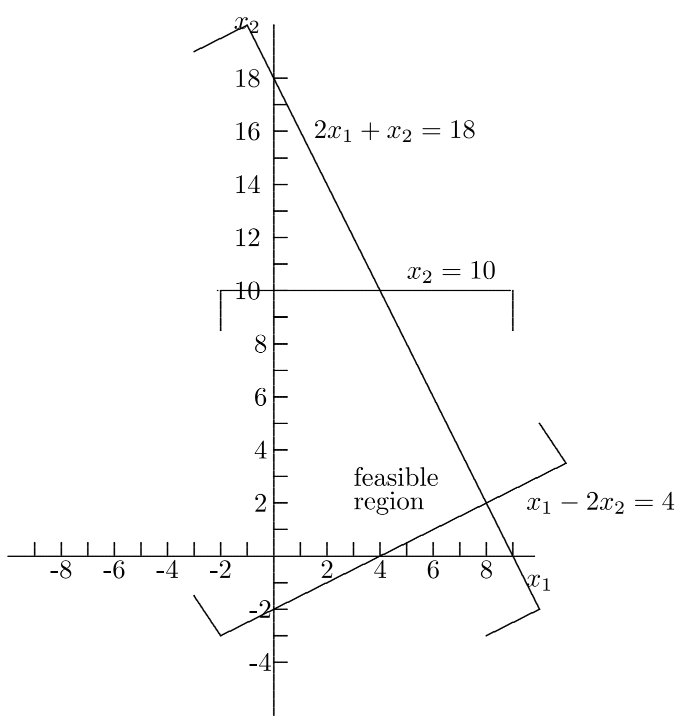


Next, the region identified by each constraint is plotted. Considering the first constraint ($x_1 - 2x_2 \leq 4$). The region is depicted below.



We repeat this process with the second and third constraints.





The final graph above represents the feasible region of the problem.

The final step is to determine the point(s) in the feasible region that yield the maximum value of the objective function:

$$z = 2x_1 + 3x_2$$

Let us begin by examining the level curves(isoprofit lines) of the objective function. For example, $z = 12$ defines the line

$$2x_1 + 3x_2 = 12$$

That is, any point on this line gives an objective function value of $z = 12$. Similarly, $z = 24$ defines the line

$$2x_1 + 3x_2 = 24$$

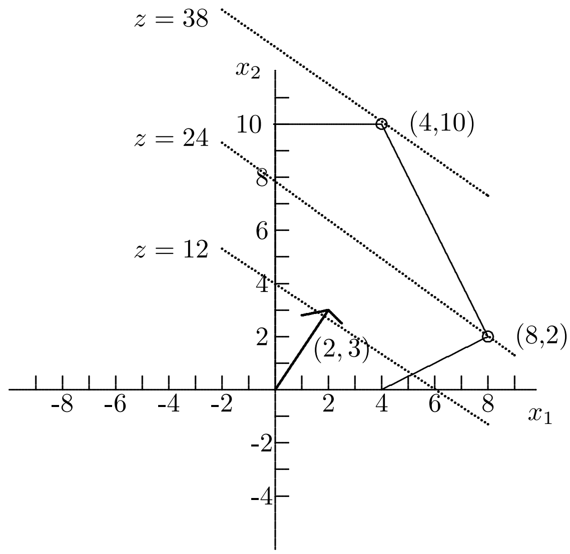
Clearly, these represent parallel lines because they have the same slope. Thus, the level curves of the objective function are a family of parallel lines. We simply need to identify the level curve that contacts the feasible region and corresponds to the greatest objective value. Thus, once we have defined the slope of the parallel lines, we only need to slide this line of fixed slope through the set of feasible points in the direction of improving z. The direction of improving z can be quite easily identified by examining the *gradient* of the objective function. The gradient of the function $z = f(x_1) + f(x_2) = c_1x_1 + c_2x_2$ is given by

$$\nabla f(x_1, x_2) = \begin{bmatrix} \dfrac{\partial z}{\partial x_1} \\ \\ \dfrac{\partial z}{\partial x_2} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and for our example,

$$\nabla f(x_1, x_2) = \begin{bmatrix} \dfrac{\partial z}{\partial x_1} \\ \\ \dfrac{\partial z}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The gradient of a function is *normal* to the level curve of the function and always points in the *direction of steepest ascent*, that is the direction of greatest increase of the objective function. Thus, to find the optimal solution of a two-variable linear program, we only need to sketch the vector corresponding to the gradient of the objective function. This is illustrated below.



For a maximization problem, we would slide the level curves in the direction of the gradient until they reach the boundary of the solution space. Similarly, for a minimization problem, we would slide the level curves in the direction opposite the gradient until they reach the boundary of the solution space.

By using the foregoing technique, the optimal solution to the previous example is determined to be $(x_1*, x_2*) = (4, 10)$ as illustrated above. The corresponding optimal objective value as $z* = 2(4) + 3(10) = 38$.

Note that the optimal point $\mathbf{X}*$ lies directly on the constraints $x_2 \leq 10$ and $2x_1 + x_2 \leq 18$; that is, these constraints are satisfied as equalities by $\mathbf{X}*$. In general, those constraints that are satisfied as equalities by a given point say $\overline{\mathbf{X}}$, are said to be *binding* at $\overline{\mathbf{X}}$. Those constraints that are not satisfied as equalities at $\overline{\mathbf{X}}$ are said to be *nonbinding*. For example, in the previous problem, the constraint $x_1 - 2x_2 \leq 24$ is nonbinding at $\mathbf{X}*$.

We now state the steps involved in the graphical solution of a two-dimensional linear programming problem.

Graphical solution procedure.

1. *Define the coordinate system.* Sketch the axes of the coordinate system and associate, with each axis, a specific decision variable.

2. *Plot the constraints.* Establish the line (in case of an equality) or region(in the case of an inequality) associated with each constraint.

3. *Identify the resultant solution space.* The *intersection* of all the regions in step 2 determines the feasible region, the set of all points that all simultaneously satisfy all problem constraints. In the event that the intersection is empty, no solution exists that satisfy *all* constraints. In this case, the problem is termed *infeasible.* If there is a nonempty feasible region, go to Step 4.

4. *Identify the gradient of the objective function.* The level curves of the objective function are normal to the gradient of the objective and the vector corresponding to the gradient points in the direction of increasing z.

5. *Identify the optimal solution(s).* For a maximization problem, slide the level curves in the direction of the gradient until they reach the boundary of the feasible region. Similarly, for a minimization problem, slide the level curves in the direction opposite the gradient until they reach the boundary of the feasible region.

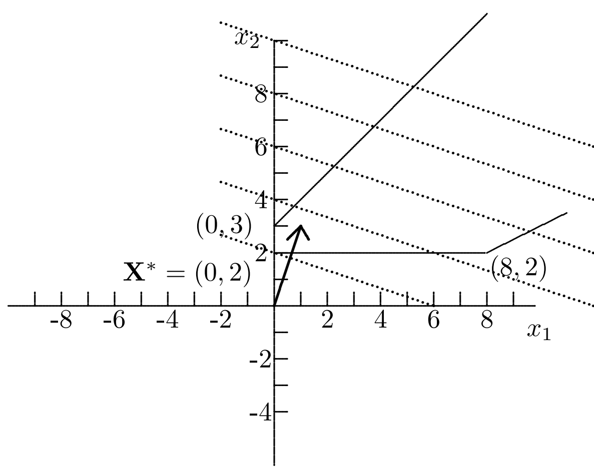
Example 2.4. A Minimization Problem

Find x_1 and x_2 so as to

$$\text{minimize } z = x_1 + 3x_2$$
subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 4 \\ -x_1 + x_2 &\leq 3 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The feasible region defined by the constraints is plotted below. As before, consider the level curves associated with the objective function. The set of parallel lines corresponding to these level curves are normal to the gradient of the objective function. Because the objective is to minimize z, we slide the level curves in the direction opposite the gradient. The optimal solution is $(x_1^*, x_2^*) = (0, 2)$ and $z^* = 6$.



Notice that this feasible region is *unbounded*, whereas the feasible region associated with the previous example is *bounded*.

Now, suppose that the objective is to maximize z instead of minimize z. Observe from the above figure that we can slide the level curves in the direction of the gradient and never reach the boundary of the feasible region. That is, the value of z can be made arbitrarily large. When this occurs we say that the linear programming problem has an *unbounded objective value* or that the linear programming problem has no *finite optimum*.

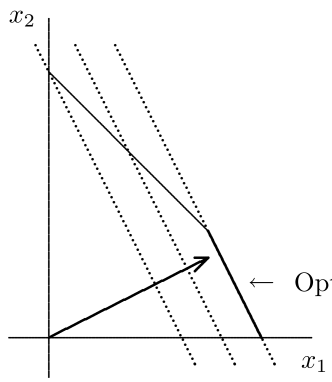
Example 2.5. Alternative Optimal Solutions

Find x_1 and x_2 so as to

$$\text{maximize } z = 6x_1 + 3x_2$$
subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 16 \\ -x_1 + x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The feasible region for this example is plotted below.



In this case, we see that the feasible region is nonempty and bounded. Sliding the level curves in the direction of the gradient does not satisfy a single extreme point as the optimal solution. Instead, the entire line segment connecting the extreme points $(6, 4)$ and $(8, 0)$ coincides with the level curve, $z^* = 48$, which corresponds to the optimal value of z. When this occurs, we say that the we have *alternative optimal solutions* and any point on the line segment joining $(6, 4)$ and $(8, 0)$ is an optimal solution.

2.3 Convex Sets and Polyhedral Sets

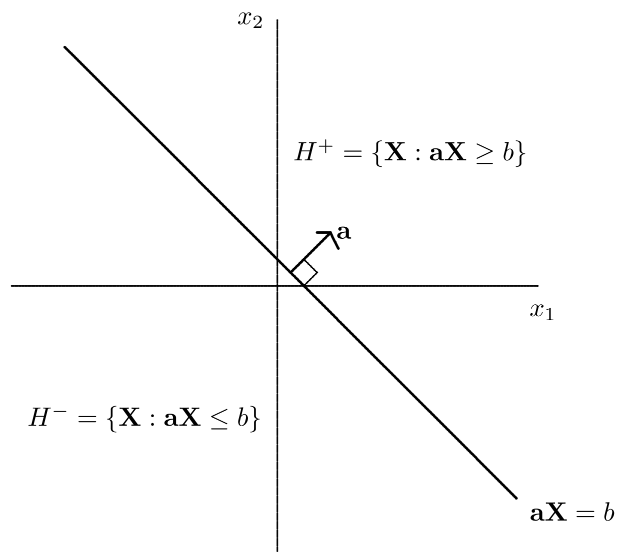
The graphical procedure was introduced to motivate the geometrical aspects of the simplex method. We present several definitions that form the foundation of the development that is to follow.

Definition 2.1. *A* **hyperplane** *(line in two dimensions, plane in three dimensions) is the set of points* $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T \in E^n$ *(Euclidean* n *space) that satisfy* $\mathbf{aX} = b$*, where* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in E^n, \mathbf{a} \neq \mathbf{0}$*, and* $b \in E^1$ *(i.e.,* b *is a scalar).*

Utilizing this definition, note that the linear system, defined by $\mathbf{AX} = \mathbf{b}$ is simply a collection of m hyperplanes.

Definition 2.2. *A* **closed halfspace** *corresponding to the hyperplane* $\mathbf{aX} = b$ *is either of the sets* $H^+ = \{\mathbf{X} : \mathbf{aX} \geq b\}$ *or* $H^- = \{\mathbf{X} : \mathbf{aX} \leq b\}$. *When these halfspaces are defined as* $\{\mathbf{X} : \mathbf{aX} > b\}$ *or* $\{\mathbf{X} : \mathbf{aX} < b\}$*, they are called* **open halfspaces**.

It is easily seen that vector $\mathbf{a}$ is the gradient of the linear function $f(x) = \mathbf{aX}$, and thus is normal to the hyperplane and points in the direction of increasing $\mathbf{aX}$, as shown in the figure below.



Definition 2.3. *A* **polyhedral set** *is the intersection of a finite number of halfspaces. Thus, the constraint set* $S = \{\mathbf{X} : \mathbf{AX} \leq b, \mathbf{X} \geq \mathbf{0}\}$ *is a polyhedral set because it is the intersection of* m *halfspaces corresponding to* $\mathbf{AX} \leq b$ *and* n *halfspaces corresponding to* $\mathbf{X} \geq \mathbf{0}$.

Definition 2.4. *A set* S *is* **convex** *if, for any two points, say,* $\mathbf{X}_1, \mathbf{X}_2 \in S$*, then the line segment joining these two points lies entirely within* S*. Mathematically this means that if* $\mathbf{X}_1, \mathbf{X}_2 \in S$*, then* $\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 \in S$ *for all* $\alpha \in [0, 1]$.

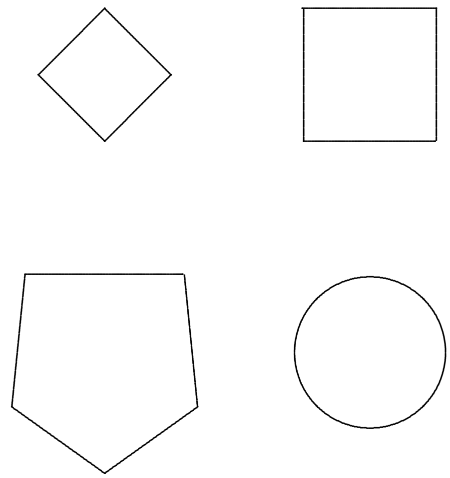
The expression $\mathbf{X} = \alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 \in S, \alpha \in [0, 1]$ parametrically defines the line segment joining $\mathbf{X}_1$ and $\mathbf{X}_2$ and is called the *convex combination* of $\mathbf{X}_1$ and $\mathbf{X}_2$. If $\alpha \in (0, 1)$ then it is called a *strict convex combination*. The concept of convex combination can be generalized to any finite number of points as follows:

$$\mathbf{X} = \sum_{i=1}^{n} \alpha_i \mathbf{X}_i$$

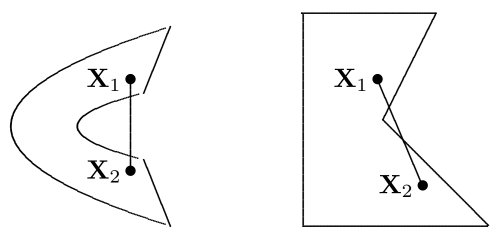
where

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i &= 1 \\ \alpha_i &\geq 0; i = 1, \ldots, p \end{aligned}$$

By convention, a point itself is a convex set. Below are some examples of convex and nonconvex sets.



Convex sets



Nonconvex sets

Theorem 2.1. *The set $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$ is a convex set.*

Proof. Let $\mathbf{X}_1, \mathbf{X}_2 \in S$ and let $\alpha \in [0, 1]$. To complete the proof, it is sufficient to show that $\overline{\mathbf{X}} = \alpha\mathbf{X}_1 + (1 - \alpha)\mathbf{X}_2 \in S$.
Because $\mathbf{X}_1 \in S$, it follows from the definition of S that $\mathbf{AX}_1 = \mathbf{b}$ and $\mathbf{X}_1 \geq \mathbf{0}$. Similarly, $\mathbf{AX}_2 = \mathbf{b}$ and $\mathbf{X}_2 \geq \mathbf{0}$. Also $\alpha \in [0, 1]$ implies that $\alpha \geq 0$ and $(1 - \alpha) \geq 0$. Now combining these results yields

$$\begin{aligned}
\alpha\mathbf{AX}_1 &= \alpha\mathbf{b} \\
\alpha\mathbf{X}_1 &\geq \mathbf{0} \\
(1-\alpha)\mathbf{AX}_2 &= (1-\alpha)\mathbf{b} \\
(1-\alpha)\mathbf{X}_2 &\geq \mathbf{0}
\end{aligned}$$

Summing these expressions yields

$$\begin{aligned}
\alpha\mathbf{AX}_1 + (1-\alpha)\mathbf{AX}_2 &= \alpha\mathbf{b} + (1-\alpha)\mathbf{b} \\
\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 &\geq \mathbf{0}
\end{aligned}$$

Now, rearranging,

$$\mathbf{A}[\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 = [\alpha + (1-\alpha)]\mathbf{b} = \mathbf{b}$$

and

$$\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 \quad \geq \quad \mathbf{0}$$

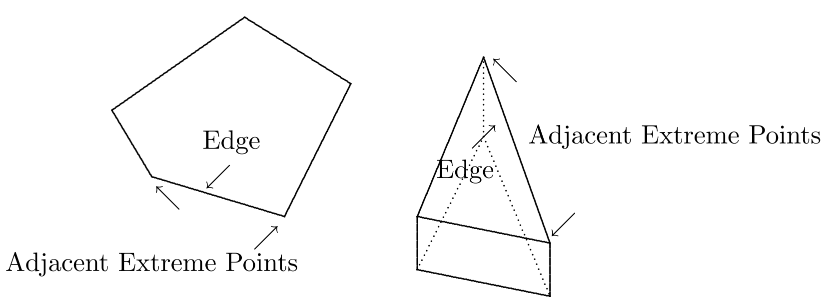
Hence, $\mathbf{A}\overline{\mathbf{X}} = \mathbf{b}$ and $\overline{\mathbf{X}} \geq \mathbf{0}$, and therefore, $\overline{\mathbf{X}} \in S$. □

Definition 2.5. *A point* $\mathbf{X}$ *is an* **extreme point** *of a given convex set* S *if it cannot be written as a strict convex combination of two other distinct points of* S*. Geometrically, this means that* $\mathbf{X}$ *is an extreme point of* S *if it does not lie on the interior of the line segment joining two other distinct points of* S*. Mathematically, there does not exist,* $\mathbf{X}_1, \mathbf{X}_2 \in S, \mathbf{X}_1 \neq \mathbf{X}_2$*, and* $\alpha \in (0,1)$ *such that* $\mathbf{X} = \alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2$*. Or equivalently, if* $\mathbf{X}_1, \mathbf{X}_2 \in S$*, and* $\alpha \in (0,1)$ *and* $\mathbf{X} = \alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2$*, then* $\mathbf{X} = \mathbf{X}_1 = \mathbf{X}_2$*.*

In polyhedral sets, these extreme points occur only at the intersection of the hyperplanes that form the boundaries of the polyhedral set. In contrast, all points that lie on the boundary of a closed circle are extreme points.

Definition 2.6. *Two distinct extreme points, say* $\mathbf{X}_1$ *and* $\mathbf{X}_2$*, are adjacent if the line segment joining them is an edge of the convex set.*

As we noticed when graphically solving two-dimensional linear programming problems, if an optimal solution exists, then at least one extreme-point (corner-point) optimal solution exists. As such, the concept of the extreme point is very important and, as we will see, the algebraic technique for solving linear programming problems involving involves determining these extreme points and systematically moving between adjacent extreme points. Adjacent extreme points and edges in E^2 and E^3 are illustrated below.

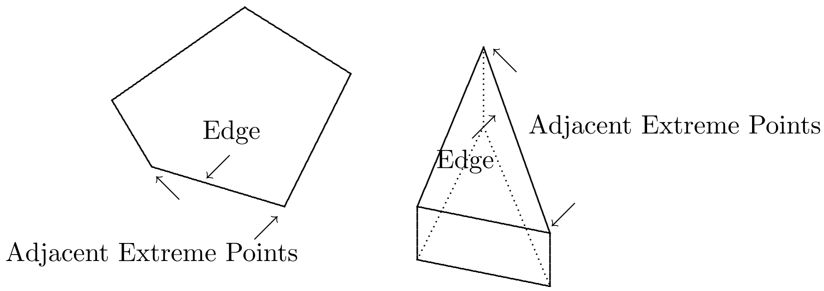


2.4 Extreme Points, Extreme Directions and Optimality

Theorems in this section will be cited without proofs.

Theorem 2.2. *Let* $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$*, where* $\mathbf{A}$ *is* $m \times n$*, and rank*$(\mathbf{A}) = m < n$*.* $\overline{\mathbf{X}}$ *is an extreme point of* S *if and only if* $\overline{\mathbf{X}}$ *is the intersection of* n *linearly independent hyperplanes.*

A simple example of this can be observed in the previous figure:



Note that in E^2, each extreme point is the intersection of two linearly independent hyperplanes(lines), and in E^3, each extreme point is the intersection of three linearly independent hyperplanes(planes).

Definition 2.7. *A nonzero vector* $\mathbf{d} = (d_1, d_2, \ldots, d_n)^T \in E^n$ *is a* **direction** *of a convex set* S *if* $\mathbf{X} + \lambda\mathbf{d} \in S$ *for all* $\mathbf{X} \in S$ *and* $\lambda \geq 0$.

Note that the set $\mathbf{X} + \lambda\mathbf{d}, \lambda \geq 0$, defines a *ray* with vertex $\mathbf{X}$ in the direction $\mathbf{d}$.

Definition 2.8. *A direction* $\mathbf{d}$ *of a set* S *is an* **extreme direction** *it cannot be written as a positive combination of two distinct directions of* S*. That is, there does not exist directions* $\mathbf{d}_1, \mathbf{d}_2$ *of* S*,* $\mathbf{d}_1 \neq \mathbf{d}_2$ *and* $\alpha_1, \alpha_2 > 0$ *such that* $\mathbf{d} = \alpha_1\mathbf{d}_1 + \alpha_2\mathbf{d}_2$.

It should be clear from the definition of direction that a convex set must be unbounded to have a nonempty direction set.

Theorem 2.3. *Let* $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$*. Then* $\mathbf{d}$ *is a direction of* S *if and only if* $\mathbf{d} \in D = \{\mathbf{d} : \mathbf{Ad} = 0, \mathbf{d} \neq \mathbf{0}\}$. *(Note that* D *is simply the set of nonnegative, nonzero solutions satisfying the homogenous system corresponding to* $\mathbf{AX} = b$*.)*

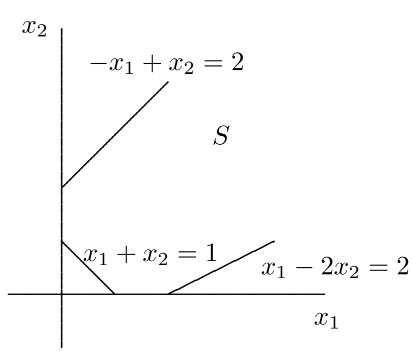
Example 2.6. Directions and Extreme Directions
Consider the set

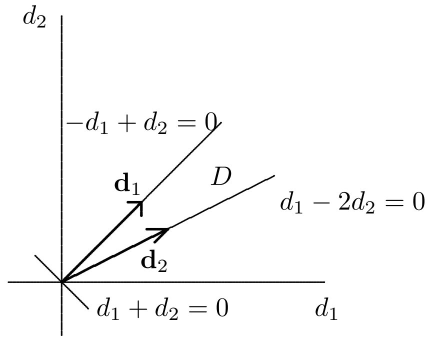
$$S = \{\mathbf{X} = (x_1, x_2)^T : -x_1 + x_2 \leq 2, x_1 + x_2 \geq 1, x_1 - 2x_2 \leq 2, x_1, x_2 \geq 0\}$$

Then the direction set D of S is given by

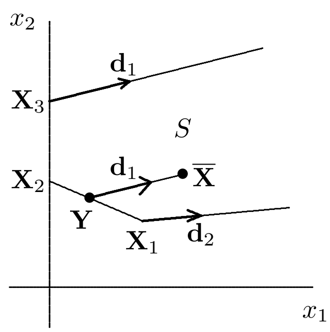
$$D = \{\mathbf{d} = (d_1, d_2)^T : -d_1 + d_2 \leq 0, d_1 + d_2 \geq 0, d_1 - 2d_2 \leq 0, d_1, d_2 \geq 0, (d_1, d_2) \neq (0, 0)\}$$

These two sets are illustrated graphically below.





Consider the unbounded polyhedral set S below. Clearly, set S has three extreme points, $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$, and two extreme directions $\mathbf{d}_1$ and $\mathbf{d}_2$. Consider the point $\overline{\mathbf{X}}$.



Observe that $\overline{\mathbf{X}}$ can be written in the form

$$\overline{\mathbf{X}} = \mathbf{Y} + \lambda \mathbf{d}_1$$

for some $\lambda > 0$. That is, $\overline{\mathbf{X}}$ lies on the ray with vertex $\mathbf{Y}$ and direction $\mathbf{d}_1$. Also, because $\mathbf{Y}$ lies on the edge connecting $\mathbf{X}_1$ and $\mathbf{X}_2$, then $\mathbf{Y}$ can be written as a convex combination of $\mathbf{X}_1$ and $\mathbf{X}_2$. That is,

$$\overline{\mathbf{Y}} = \alpha \mathbf{X}_1 + (1-\alpha)\mathbf{X}_2$$

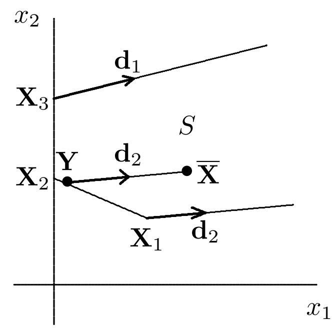
for some $\alpha \in (0,1)$. By substitution we have

$$\overline{\mathbf{X}} = \alpha \mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 + \lambda \mathbf{d}_1, \qquad \alpha \in (0,1) \qquad \lambda > 0$$

or written more completely

$$\overline{\mathbf{X}} = \alpha \mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 + 0\mathbf{X}_3 + \lambda \mathbf{d}_1 + 0d_2, \qquad \alpha \in (0,1) \qquad \lambda > 0$$

Thus, the arbitrary point $\overline{\mathbf{X}}$ can be written as a convex combination of the extreme points $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$, and a nonnegative combination of the extreme directions $\mathbf{d}_1$ and $\mathbf{d}_2$. Observe that this representation is not unique as indicated in the following figure.



Theorem 2.4. *Let $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$ and let E be the set of extreme points of S and let D be the extreme directions of S. Then:*

1. *S has at least one extreme point and at most a finite number of extreme points, that is the set E is nonempty and finite, $E = \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p\} \neq \emptyset$*
2. *S is unbounded if and only if S has at least one extreme direction, that is, S is unbounded if and only if D is nonempty*
3. *if S is unbounded, then S has a finite number of extreme directions, that is $D = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_q\} \neq \emptyset$*
4. *if $\mathbf{X} \in S$, then $\mathbf{X}$ can be written as a convex combination of the extreme points plus a nonnegative combination of the extreme directions, that is,*
$$\mathbf{X} = \sum_{i=1}^{p} \alpha_i \mathbf{X}_i + \sum_{j=1}^{q} \lambda_j \mathbf{d}_j$$
where
$$\begin{array}{rrcl} & \displaystyle\sum_{i=1}^{p} \alpha_i & = & 1 \\ \alpha_i \geq 0, & \text{for all } i & = & 1, \ldots, p \\ \lambda_j \geq 0, & \text{for all } j & = & 1, \ldots, q \end{array}$$

The following theorem specifies that if an optimal solution exists, we only need to examine the extreme points of a feasible region in order to find an optimal solution.

Theorem 2.5. *Let $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$ and consider the following linear program:*

$$\text{(LP) maximize } z = \mathbf{cX}$$

$$\text{subject to } \mathbf{X} \in S$$

Suppose S is an unbounded set with extreme points $E = \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_3 \neq \emptyset$ and extreme directions $D = \mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_3 \neq \emptyset$. Let z^* represent the optimal objective value of (LP). Then z^* is finite, (that is, $z^* < \infty$), if and only if $\mathbf{cd}_j \leq 0$ for all $\mathbf{d}_j \in D$. And, furthermore, if a finite optimal solution exists, then an extreme-point optimal solution exists.

Corollary 2.5.1. *Suppose S is a nonempty bounded set with extreme points $E = \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_3 \neq \emptyset$. Let z^* represent the optimal objective value of (LP). Then z^* is finite, (that is, $z^* < \infty$)and, furthermore, an extreme-point optimal solution exists.*

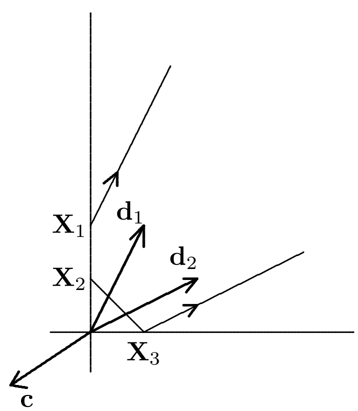
Example 2.7. Using Theorem 2.5 to Characterize the Optimum of a Linear Program

Consider the following linear program:

$$
\begin{aligned}
&\text{maximize } z = -3x_1 - 2x_2\\
&\text{subject to}
\end{aligned}
$$

$$
\begin{aligned}
-2x_1 + x_2 &\leq 2\\
x_1 + x_2 &\geq 1\\
x_1 - 2x_2 &\leq 1\\
x_1, x_2 &\geq 0
\end{aligned}
$$

The feasible region of this linear program is shown below.



Note that the feasible region has three extreme points, $\mathbf{X}_1 = (0,2)^T, \mathbf{X}_2 = (0,1)^T$, and $\mathbf{X}_3 = (1,0)^T$, and two extreme directions, $\mathbf{d}_1 = (1,2)^T$ and $\mathbf{d}_2 = (2,1)^T$. (These representations of $\mathbf{d}_1$ and $\mathbf{d}_2$ are not unique). The objective gradient is $\mathbf{c} = (-3, -2)$, which yields

$$\mathbf{c}\mathbf{X}_1 = (-3 - 2)\begin{pmatrix} 0 \\ 2 \end{pmatrix} = -4$$

Similarly,

$$
\begin{aligned}
\mathbf{c}\mathbf{X}_2 &= -2\\
\mathbf{c}\mathbf{X}_3 &= -3\\
\mathbf{c}\mathbf{d}_1 &= -7\\
\mathbf{c}\mathbf{d}_2 &= -8
\end{aligned}
$$

Because $\mathbf{c}\mathbf{d}_1 < 0$ and $\mathbf{c}\mathbf{d}_2 < 0$, the linear program has a finite optimal solution. And because it has a finite optimal solution, it has an extreme point solution. Finally, because $\mathbf{c}\mathbf{X}_2 > \mathbf{c}\mathbf{X}_1$ and $\mathbf{c}\mathbf{X}_2 > \mathbf{c}\mathbf{X}_3$, then clearly the optimal extreme point solution is $\mathbf{X}_2 = (0,1)^T$ with $z^* = \mathbf{c}\mathbf{X}_2 = -2$.

Alternatively, one may look at the reformulation provided by Theorem 2.4

$$\text{maximize } z = -4\alpha_1 - 2\alpha_2 - 3\alpha_3 - 7\lambda_1 - 8\lambda_2$$

subject to

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \alpha_1, \alpha_2, \alpha_3 &\geq 0 \\ \lambda_1, \lambda_2 &\geq 0 \end{aligned}$$

Obviously, the optimal solution of this problem is $\alpha_2 = 1, \alpha_1 = \alpha_3 = \lambda_1 = \lambda_2 = 0$, which corresponds precisely to extreme point $\mathbf{X}_2$.

2.5 Basic Feasible Solutions and Extreme Points

In this section, we present a method for characterizing extreme points algebraically. This will enable us to develop a systematic procedure for determining an optimal extreme point, if one exists. The resulting systematic algebraic approach is in fact the simplex method.

Consider a linear system of equations given by

$$\mathbf{AX} = \mathbf{b}$$

where

$$\mathbf{A} \text{ is a given } m \times n \text{ matrix,}$$

$$\mathbf{b} \text{ is a given } m\text{-vector, i.e, } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in E^n$$

Assume that the rank$(\mathbf{A}) = m \leq n$. That is, assume that $\mathbf{A}$ has full row rank, or, equivalently, the rows of $\mathbf{A}$ are linearly independent. Also assume that the columns of $\mathbf{A}$ can be reordered so that $\mathbf{A}$ can be written in partitioned form as

$$\mathbf{A} = (\mathbf{B} : \mathbf{N})$$

where

$$\begin{aligned} \mathbf{B} = &\quad m \times m \text{ nonsingular matrix, designated the } \textit{basis matrix} \\ \mathbf{N} = &\quad m \times (n-m) \text{ matrix (the matrix of nonbasic columns)} \end{aligned}$$

Based on this partitioning of matrix $\mathbf{A}$, the linear system $\mathbf{AX} = \mathbf{b}$ can be rewritten in the form

$$\mathbf{BX}_B + \mathbf{NX}_N = \mathbf{b}$$

where vector $\mathbf{X}$ has been partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{bmatrix}$$

to correspond precisely to the partitioning of matrix $\mathbf{A}$. Now because $\mathbf{B}$ is nonsingular, the inverse of $\mathbf{B}$ exists, and we may premultiply both sides by $\mathbf{B}^{-1}$ to obtain

$$\mathbf{B}^{-1}\mathbf{BX}_B + \mathbf{B}^{-1}\mathbf{NX}_N = \mathbf{B}^{-1}\mathbf{b}$$

This simplifies to

$$\mathbf{X}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N = \mathbf{B}^{-1}\mathbf{b}$$

and solving for $\mathbf{X}_B$ yields

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N$$

Setting $\mathbf{X}_N = \mathbf{0}$, we have $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b}$. The solution

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

is called a *basic solution*, with vector $\mathbf{X}_B$ called the vector of *basic variables*, and $\mathbf{X}_N$ is called the vector of *nonbasic variables*.

If, in addition, $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then

$$\mathbf{X} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

is called a *basic feasible solution* of the system defined by

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

Finally, if $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} > \mathbf{0}$, then

$$\mathbf{X} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

is called a *nondegenerate* basic feasible solution. Otherwise, if at least one element of $\mathbf{X}_B$ is zero, then $\mathbf{X}$ is called a *degenerate* basic feasible solution.

Example 2.8. Basic Solutions

Consider the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 15 \\ x_1 + 3x_2 + x_3 - x_5 &= 12 \end{aligned}$$

This linear system can be written in the form

$$\mathbf{AX} = \mathbf{b}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & -1 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 15 \\ 12 \end{bmatrix} \\ \mathbf{X} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

Because $\mathbf{A}$ is a 2×5 matrix, each basic solution will have $m = 2$ basic variables and $n - m = 3$ nonbasic variables. Now consider the basis matrix $\mathbf{B}$ formed from the first and third columns of $\mathbf{A}$, that is,

$$\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_3) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Because $\mathbf{B}$ is nonsingular, it is a suitable basis matrix and

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Thus, the basic solution corresponding to the basis matrix $\mathbf{B}$ is

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{X}_N &= \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{X}_B &= \begin{bmatrix} x_{B,1} \\ x_{B,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \end{aligned}$$

This solution is a nondegenerate basic feasible solution because $\mathbf{X}_B > 0$.

Theorem 2.6. *Let $S = \{\mathbf{X} : \mathbf{AX} = b, \mathbf{X} \geq \mathbf{0}\}$, where $\mathbf{A}$ is $m \times n$, and rank$(\mathbf{A}) = m < n$. $\mathbf{X}$ is an extreme point of S if and only if $\mathbf{X}$ is a basic feasible solution.*

Proof. ($\Longrightarrow$) Let $\mathbf{X}$ be an extreme point of S. We need to show that $\mathbf{X}$ is a basic feasible solution.

Because $\mathbf{X}$ is an extreme point, $\mathbf{X}$ is the intersection of n linearly independent hyperplanes. Note from the definition of S, that $\mathbf{AX} = \mathbf{b}$ provides m of these hyperplanes. Thus the remaining $n - m$ hyperplanes must come from the nonnegativity constraints $\mathbf{X} \geq \mathbf{0}$. That is at least $n - m$ of the constraints $\mathbf{X} \geq \mathbf{0}$ are satisfied as equalities by the extreme point $\mathbf{X}$. Let us denote $n - m$ of these binding constraints by $\mathbf{X}_N = \mathbf{0}$. Then the extreme point $\mathbf{X}$ is the unique solution of the n linearly independent hyperplanes, $\mathbf{AX} = \mathbf{b}, \mathbf{X}_N = \mathbf{0}$. Let $\mathbf{X}_B$ represent the remaining m components of $\mathbf{X}$ and partition the matrix $\mathbf{A}$ to correspond to the vectors $\mathbf{X}_B$ and $\mathbf{X}_N$; that is, $\mathbf{A} = (\mathbf{B} : \mathbf{N})$. Then the extreme point $\mathbf{X}$ is the unique solution of the system, $\mathbf{BX}_B + \mathbf{NX}_N = \mathbf{b}, \mathbf{X}_N = \mathbf{0}$. It then follows that $\mathbf{X}_B$ is the unique solution of $\mathbf{BX}_B = \mathbf{b}$ and thus, $\mathbf{B}$ is invertible and a basis matrix. Therefore,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{pmatrix}$$

is a basic solution and, clearly $\mathbf{X}$ is also feasible because $\mathbf{X}$ is an extreme point.

($\Longleftarrow$) Let $\mathbf{X}$ be a basic feasible solution. Show that $\mathbf{X}$ is an extreme point of S.

Because $\mathbf{X}$ is a basic feasible solution, there exists a basis matrix $\mathbf{B}$ such that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

But this implies that $\mathbf{X}$ is the unique solution to the system, $\mathbf{BX}_B + \mathbf{NX}_N = \mathbf{b}, \mathbf{X}_N = \mathbf{0}$, or, equivalently, $\mathbf{AX} = \mathbf{b}, \mathbf{X}_N = \mathbf{0}$. Because $\mathbf{AX} = \mathbf{b}$ represents m hyperplanes and $\mathbf{X}_N = \mathbf{0}$ is an additional $n - m$ hyperplanes, $\mathbf{X}$ is the intersection of n linearly independent hyperplanes and, hence, is an extreme point.

$\square$

Example 2.9. Basic Feasible Solutions and Extreme Points Consider again, the two variable problem of Example 2.3. After the addition of the slack variables we obtain

maximize $z = 2x_1 + 3x_2$

subject to

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 4 \\
2x_1 + x_2 + x_4 &= 18 \\
x_2 + x_5 &= 10 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

Thus, the problem can be rewritten as

maximize $z = \mathbf{cX}$

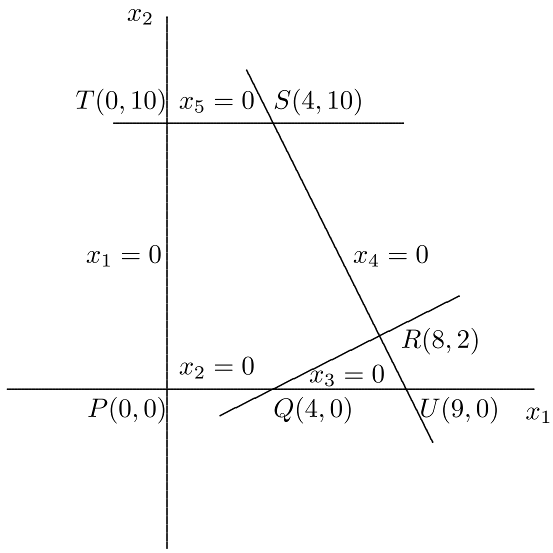
subject to

$$
\begin{aligned}
\mathbf{AX} &= \mathbf{b} \\
\mathbf{X} &\geq 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{b} &= \begin{bmatrix} 4 \\ 18 \\ 10 \end{bmatrix} \\
\mathbf{c} &= (2\ 3\ 0\ 0\ 0) \\
\mathbf{X} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix}
\end{aligned}
$$

The feasible region is graphed below.



The boundary of the feasible region consists of the hyperplanes:

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ 2x_1 + x_2 &= 18 \\ x_2 &= 10 \\ x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

Observe also that each of these hyperplanes is associated with a particular variable which has the value zero along the hyperplane. For example, from $x_1 - 2x_2 + x_3 = 4$, x_3 has the value zero along the hyperplane defined in $x_1 - 2x_2 = 4$. Thus, x_3 can be thought of as being nonbasic to this hyperplane. The nonbasic variable associated with each bounding hyperplane is noted in the above figure.

The coefficient matrix $\mathbf{A}$ is 3×5, therefore, each basic solution will have $m = 3$ basic variables and $n - m = 2$ nonbasic variables. Consider the basis matrix

$$\mathbf{B} = (\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Because $\mathbf{B}$ is nonsingular, it is invertible and forms a suitable basis. The basic solution corresponding to $\mathbf{B}$ can now be computed as

$$\begin{aligned} \mathbf{X}_N &= \begin{bmatrix} x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{X}_B &= \begin{bmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 10 \\ 24 \\ 8 \end{bmatrix} \end{aligned}$$

Notice that because x_1 and x_5 are nonbasic variables, this basic solution corresponds to extreme point T in the above figure. This is true because point T is the intersection of the hyperplanes associated with nonbasic variables x_1 and x_5. This also checks out mathematically because the coordinates of this extreme point are $(x_1, x_2) = (0, 10)$.

Now consider the basic solution corresponding to the basis matrix

$$\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Following the procedure defined earlier, we obtain

$$\begin{aligned} \mathbf{X}_N &= \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{X}_B &= \begin{bmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 9 \\ -5 \\ 10 \end{bmatrix} \end{aligned}$$

Because $x_3 < 0$, this is a basic infeasible solution. Also, because x_2 and x_4 are nonbasic, observe that this nonbasic solution corresponds to point U in the above figure. Obviously, this is an infeasible point. Finally, notice that the constraint that is violated corresponds precisely to the variable x_3, which has a negative value in the basic solution. The basic feasible solutions for this linear programming problem is summarized in the table below.

BASIC FEASIBLE SOLUTIONS

Extreme point	Basic variables	Nonbasic variables
P	$x_3 = 4, x_4 = 18, x_5 = 10$	$x_1 = 0, x_2 = 0$
Q	$x_1 = 4, x_4 = 10, x_5 = 10$	$x_2 = 0, x_3 = 0$
R	$x_1 = 8, x_2 = 2, x_5 = 8$	$x_3 = 0, x_4 = 0$
S	$x_1 = 4, x_2 = 10, x_3 = 20$	$x_4 = 0, x_5 = 0$
T	$x_2 = 10, x_3 = 24, x_4 = 8$	$x_1 = 0, x_5 = 0$

The preceding example illustrates the relationship between the geometry of a linear program and the algebra of the problem. It should also be observed that adjacent extreme points differ by exactly one basic and nonbasic variable. Thus, we can move from one extreme point to an adjacent extreme point by simply interchanging one basic and one nonbasic variable. This idea is central to the operation of the simplex method.

EXERCISES.

1. Transform the following linear programming program into the standard form.

minimize $z = 2x_1 - 3x_2 + 5x_3 + x_4$

subject to

$$\begin{aligned} -x_1 + 3x_2 - x_3 + 2x_4 &\leq -12 \\ 5x_1 + x_2 + 4x_3 - x_4 &\geq 10 \\ 3x_1 - 2x_2 + x_3 - x_4 &= -8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

2. Solve the following linear programming program graphically.

minimize $z = 4x_1 + 5x_2$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\leq 24 \\ x_1 &\geq 5 \\ 3x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Solve the following linear programming program graphically.

minimize $z = x_1 - 4x_2$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 12 \\ -2x_1 + x_2 &\leq 4 \\ x_2 &\leq 8 \\ x_1 - 3x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. Solve the following linear programming program graphically.

maximize $z = 6x_1 + 8x_2$

subject to

$$\begin{aligned} x_1 + 4x_2 &\leq 16 \\ 3x_1 + 4x_2 &\leq 24 \\ 3x_1 - 4x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. Solve the following linear programming program graphically.

maximize $z = x_1 + 2x_2$

subject to

$$\begin{aligned} -2x_1 + x_2 &\leq 2 \\ 2x_1 + 5x_2 &\geq 10 \\ x_1 - 4x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6. Show that the halfspace $H^- = \{\mathbf{X} : \mathbf{aX} \leq \alpha\}$ is a convex set.

7. Let

$$\mathbf{a}_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Illustrate graphically the following:

(a) The set of all linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

(b) The set of all nonnegative linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

(c) The set of all convex combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

8. Given the polyhedral set $S = \{(x_1, x_2) : x_1 + x_2 \leq 10, -x_1 + x_2 \leq 6, x_1 - 4x_2 \leq 0\}$.

(a) Find all the extreme points of S.

(b) Represent the point $\mathbf{X} = (2, 4)$ as a convex combination of the extreme points.

9. Let S_1 and S_2 be convex sets. Show that the set $S_1 \cap S_2$ is a convex set. Is this also true of $S_1 \cup S_2$?

10. Given the following system of linear equations.

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 &= 30 \\ 2x_1 + x_2 + 2x_3 + x_4 &= 15 \end{aligned}$$

    (a) Find all basic solutions.
    (b) For each basic solution, specify the basic and nonbasic variables and the basis matrix $\mathbf{B}$

11. Consider the following linear programming problem.

maximize $z = 2x_1 + x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 20 \\ -3x_1 + 4x_2 &\leq 10 \\ 3x_1 + 2x_2 &\leq 36 \\ x_1, x_2 &\geq 0 \end{aligned}$$

    (a) Sketch the feasible region.
    (b) Write the problem in standard form.
    (c) Identify the defining variable for each hyperplane bounding the feasible region, and specify the basic and nonbasic variables for each extreme point.
    (d) Graphically determine the optimal extreme point and specify the optimal basis matrix.

12. Given the polyhedral set $S = \{(x_1, x_2) : x_1 + x_2 \geq 6, x_1 \geq 2, -2x_1 + x_2 \leq 4, x_1 - x_2 \leq 4, x_1, x_2 \geq 0\}$.

    (a) Find all extreme points and extreme directions of S.
    (b) Represent the point $\mathbf{X} = (5, 8)$ as a convex combination of the extreme points and a nonnegative combination of the extreme directions.

13. Consider the following linear programming problem.

maximize $z = -7x_1 + 2x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ -3x_1 + x_2 &\leq 9 \\ x_2 &\geq 4 \\ 2x_1 - 4x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

    (a) Sketch the feasible region and identify the optimal solution.
    (b) Identify all extreme points and extreme directions.
    (c) Reformulate the problem in terms of convex combination of the extreme points and nonnegative combinations of the extreme directions. Solve the resulting problem and interpret the solution.
    (d) Change the objective function to maximize $z = 4x_1 - x_2$ and repeat part (c).
    (e) Discuss the practicality of using the procedure in parts (c) and (d) for large problems.

Chapter 3

The Simplex Algorithm: Tableau and Computation

3.1 Algebra of the Simplex Method

Consider the standard linear programming problem:

$$\text{maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq 0 \end{aligned}$$

Recall that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

is called a *basic solution*, with vector $\mathbf{X}_B$ called the vector of basic variables, and $\mathbf{X}_N$ is called the vector of *nonbasic variables*. If in addition, $\mathbf{X}_B \geq 0$ then

$$\mathbf{X} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

is called a *basic feasible solution*.

Now consider the objective function $z = \mathbf{cX}$. Partitioning the cost vector $\mathbf{c}$ into basic and nonbasic components, the objective function can be written as

$$z = \mathbf{c}_B\mathbf{X}_B + \mathbf{c}_N\mathbf{X}_N$$

Substituting $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N$, we have

$$z = \mathbf{c}_B(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N) + \mathbf{c}_N\mathbf{X}_N$$

which can be rewritten as

$$z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{X}_N$$

Setting $\mathbf{X}_N = \mathbf{0}$, we see that $z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$, which is the objective value corresponding to the current basic feasible solution. Therefore, the current extreme-point solution can be represented in *canonical form*:

$$\begin{aligned} z &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{X}_N \\ \mathbf{X}_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N \end{aligned}$$

with the current basic feasible solution given as

$$\begin{aligned} z &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{X} &= \begin{pmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} \geq \mathbf{0} \end{aligned}$$

The canonical representation forms the foundation upon which the simplex method is built. Letting J denote the index set of the nonbasic variables, observe that this canonical representation can be rewritten as follows.

$$\begin{aligned} z &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j)x_j \\ \mathbf{X}_B &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(\mathbf{B}^{-1}\mathbf{a}_j)x_j \end{aligned}$$

The central idea behind the simplex method is to move from an extreme point to an improving adjacent extreme point by interchanging a column of $\mathbf{B}$ and $\mathbf{N}$.

3.1.1 Checking for Optimality

When will such an exchange improve the objective function? In the current basic feasible solution, $\mathbf{X}_B = \mathbf{0}$, that is, the nonbasic variables are at *lower bound* and can only be increased from their current value of zero. Observe that the coefficient $-(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j)$ of x_j represents the rate of change of z with respect to the nonbasic variable x_j. That is,

$$\frac{\partial z}{\partial x_j} = -(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j)$$

Thus, if $\partial z/\partial x_j > 0$, then increasing the nonbasic variable x_j will increase z. The quantity $(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j)$ is sometimes referred to as the *reduced cost* and for convenience is usually denoted by $(z_j - c_j)$.

Optimality conditions (maximization problem). The basic feasible solution will be optimal to (LP) if

$$\frac{\partial z}{\partial x_j} = -(z_j - c_j) = -(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j) \leq 0 \quad \text{for all} j \in J$$

or equivalently, if

$$(z_j - c_j) = (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j) \geq 0 \quad \text{for all} j \in J$$

Note that because $(z_j - c_j) = 0$ for all basic variables, then the optimality conditions could also be stated simply as $(z_j - c_j) \geq 0$, for all $j = 1, \ldots, n$.

If $(z_j - c_j) > 0$, for all $j \in J$, then the current basic solution will be the unique optimal solution because increasing any nonbasic variable results in a strict decrease in the objective function. However, if some nonbasic variable x_k has $(z_k - c_k) = 0$ then increasing x_k does not change the objective value, and in the absence of degeneracy, entering x_k will lead to an alternative extreme point with the same objective value. When this occurs, we say that there are *alternative optimal solutions*.

3.1.2 Determining the Entering Variable

Suppose there exists some nonbasic variable x_k with a reduced cost $z_k - c_k < 0$. Then $\partial z/\partial x_k > 0$ and the objective function can be improved by increasing x_k from its current value of zero. Typically, we choose to increase that nonbasic variable that forces the greatest rate of change of the objective, that is, the nonbasic variable with the most negative $z_j - c_j$. The selected variable x_k is called the *entering variable*. That is, x_k is going to enter the basic vector. To maintain a basic vector with m components, we must exchange it for some variable that is currently basic; this variable is called the *departing variable*. Mathematically, we need to form a new basis by exchanging $\mathbf{a}_k$ with some column of the current basis matrix $\mathbf{B}$.

Theorem 3.1. *Let $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ be a basis for E^m, and let $\mathbf{a} \in E^m, \mathbf{a} \neq 0$. Then $\mathbf{a}$ can be written uniquely as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$.*

Proof. Since $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ is a basis for E^m, then clearly it is possible to write $\mathbf{a}$ as linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$. We need to show that this representation is unique. Suppose it is not unique. That is, suppose that $\mathbf{a}$ can be written as

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j, \text{ where } \lambda_j \in E^1, \text{ for all } j = 1, \ldots, m$$

and

$$\mathbf{a} = \sum_{j=1}^{m} \mu_j \mathbf{b}_j, \text{ where } \mu_j \in E^1, \text{ for all } j = 1, \ldots, m$$

Subtracting these two equations will give us

$$\mathbf{0} = \sum_{j=1}^{m} (\lambda_j - \mu_j)\mathbf{b}_j$$

But because $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ forms a basis, they are linearly independent. Thus,

$$\lambda_j - \mu_j, \qquad \text{for all } j = 1, \ldots, m$$

It then follows that $\lambda_j = \mu_j$, for all $j = 1, \ldots, m$. □

Theorem 3.2. *Let $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ be a basis for E^m, and let $\mathbf{a} \in E^m, \mathbf{a} \neq 0$ be represented by $\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j$. Without loss of generality, suppose $\lambda_m \neq 0$. Then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m-1}, \mathbf{a}$ form a basis for E^m.*

Proof. We simply need to show that $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m-1}, \mathbf{a}$ are linearly independent. By contradiction, suppose they are not. Then there exists $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \delta \in E^1$, which are not all zero such that

$$\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j + \delta \mathbf{a} = \mathbf{0}$$

Clearly, $\delta \neq 0$, otherwise, $\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j = \mathbf{0}$, which contradicts the fact that $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m-1}$ are linearly independent. But

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j$$

By substitution,

$$\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j + \delta \sum_{j=1}^{m} \lambda_j \mathbf{b}_j = \sum_{j=1}^{m-1} (\gamma_j + \delta\lambda_j)\mathbf{b}_j + \delta\lambda_m \mathbf{b}_m = \mathbf{0}$$

Because $\delta \neq 0$ and $\lambda_m \neq 0$, it follows that $\delta\lambda_m \neq 0$. We now have a contradiction since $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ are linearly independent. Thus, $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{m-1}, \mathbf{a}$ are linearly independent and form a basis for E^m. □

3.1.3 Determining the Departing Variable

Consider the vector of coefficients of the nonbasic variable x_k in $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in J} (\mathbf{B}^{-1}\mathbf{a}_j)x_j$ and let

$$\alpha_k = \mathbf{B}^{-1}\mathbf{a}_k$$

The elements of vector α_k can be interpreted in two different ways.

First, multiplying both sides by $\mathbf{B}$ yields

$$\mathbf{a}_k = \mathbf{B}\alpha_k = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m) \begin{pmatrix} \alpha_{1,k} \\ \alpha_{2,k} \\ \vdots \\ \alpha_{m,k} \end{pmatrix} = \sum_{j=1}^{m} \alpha_{j,k} \mathbf{b}_j$$

Recall that $\mathbf{a}_k$ can be written uniquely as a linear combination of the columns of the basis matrix $\mathbf{B}$. Thus, the above equation identifies that, in fact, the elements of α_k specify how to write $\mathbf{a}_k$ as a linear combination of the columns of $\mathbf{B}$. Therefore, $\mathbf{a}_k$ may be exchanged with any column of $\mathbf{b}_j$ of $\mathbf{B}$ for which $\alpha_{j,k} \neq 0$.

Second, note that the rate of change of the basic variables with respect to the nonbasic variable x_k is given by

$$\frac{\partial \mathbf{X}_B}{\partial x_k} = -\mathbf{B}^{-1}\mathbf{a}_k = -\alpha_k$$

That is, if the nonbasic variable x_k is increased from its current value of zero while holding all other nonbasic variables at zero, then the basic variables will change according to the relationship

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} + x_k(-\mathbf{B}^{-1}\mathbf{a}_k) = \mathbf{B}^{-1}\mathbf{b} - x_k\alpha_k$$

And because all variables must remain nonnegative, it follows that

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} - x_k\alpha_k \geq \mathbf{0}$$

Now let

$$\mathbf{B}^{-1}\mathbf{b} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

Then

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} - x_k \begin{pmatrix} \alpha_{1,k} \\ \alpha_{2,k} \\ \vdots \\ \alpha_{m,k} \end{pmatrix} \geq \mathbf{0}$$

and an upper bound on x_k can be found quite easily as

$$x_k \leq \text{minimum} \left\{ \frac{\beta_i}{\alpha_{i,k}} : \alpha_{i,k} > 0 \right\}$$

This process is termed the *minimum ratio test* and provides a very simple method for determining the maximum value of the entering variable. Essentially, we are finding the smallest value of the entering variable x_k, which results in a basic variable assuming the value zero. The basic variable that is forced to zero as a result of this increase in x_k is called the *departing variable.*

3.1.4 Checking for Unbounded Objective

Suppose that x_k is chosen as the entering variable. However, when examining vector α_k for the minimum ratio test, we find that $\alpha_{i,k} \leq 0$, for all i. Then from

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} - x_k \begin{pmatrix} \alpha_{1,k} \\ \alpha_{2,k} \\ \vdots \\ \alpha_{m,k} \end{pmatrix} \geq \mathbf{0}$$

x_k can be increased without bound, that is, as x_k is increased from its current value of zero, no basic variable decreases in value. Thus, if $z_k - c_k < 0$ and $\alpha_k \leq \mathbf{0}$, then the objective function can be increased indefinitely and no finite optimal solution exists. In fact, the objective function can be increased indefinitely by moving along the ray defined by

$$\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} + x_k \begin{pmatrix} -\alpha_k \\ \mathbf{e}_k \end{pmatrix}$$

where

$$\mathbf{e}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and the 1 appears in the kth position. Note that the vertex of this ray is the current basic feasible solution that is given by

$$\begin{pmatrix} \mathbf{X}_B \\ \mathbf{X}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

and the direction of the ray is given by

$$\mathbf{d} = \begin{pmatrix} -\alpha_k \\ \mathbf{e}_k \end{pmatrix} = \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{a}_k \\ \mathbf{e}_k \end{pmatrix}$$

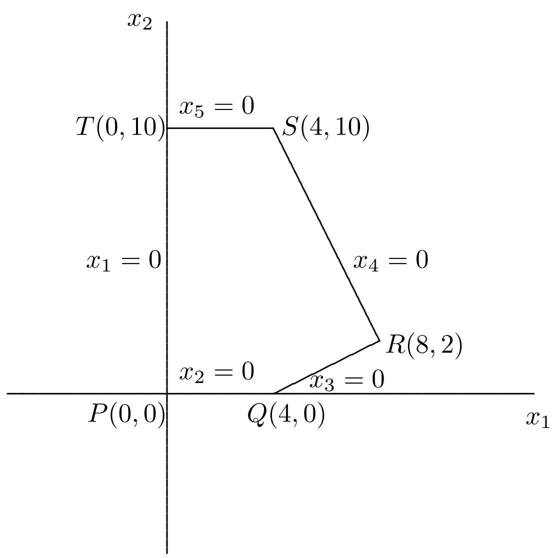
Example 3.1. Algebra of the Simplex Method

Let us again consider Example 2.9.

maximize $z = 2x_1 + 3x_2$
subject to

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 4 \\
2x_1 + x_2 + x_4 &= 18 \\
x_2 + x_5 &= 10 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

Also the feasible region is depicted below.



We begin the solution process by choosing a convenient starting basis matrix $\mathbf{B}$. We do not want to choose an arbitrary matrix $\mathbf{B}$, but instead, because that solution will be determined by $\mathbf{B}^{-1}$, we will always choose the starting basis matrix $\mathbf{B} = \mathbf{I}$. Observe that this results in

$$
\begin{aligned}
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{X}_B &= \begin{pmatrix} X_{B,1} \\ X_{B,2} \\ X_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}
\end{aligned}
$$

Now form the canonical representation by solving for z and $\mathbf{X}_B$ in terms of $\mathbf{X}_N$. Because $\mathbf{B} = \mathbf{I}$, this is a trivial process and yields

$$
\begin{aligned}
z &= 2x_1 + 3x_2 \\
x_3 &= 4 - x_1 + 2x_2 \\
x_4 &= 18 - 2x_1 - x_2 \\
x_5 &= 10 - x_2
\end{aligned}
$$

The starting solution, which is obtained by setting the nonbasic variables equal to zero, can be summarized as follows:

$$
\begin{aligned}
z &= 0 \\
\mathbf{X}_B &= \begin{pmatrix} X_{B,1} \\ X_{B,2} \\ X_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 14 \\ 18 \\ 10 \end{pmatrix} \\
\mathbf{X}_N &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Observe that this starting solution corresponds to extreme point P, the origin. This can be seen quite readily by noting that x_1 and x_2 are the nonbasic variables.

$$
\begin{aligned}
\frac{\partial z}{\partial x_1} &= -(z_1 - c_1) = 2 > 0 \quad \Leftrightarrow \quad (z_1 - c_1) = -2 < 0 \\
\frac{\partial z}{\partial x_2} &= -(z_2 - c_2) = 3 > 0 \quad \Leftrightarrow \quad (z_2 - c_2) = -3 < 0
\end{aligned}
$$

Thus, increasing either of the nonbasic variable x_1 or x_2 will increase the values of z and the current solution is not optimal. Because $\partial z/\partial x_2 > \partial z/\partial x_1$, we will choose to increase x_2 (that is, x_2 is the entering variable.) The next step is to find the departing variable using the minimum ratio test. Note that the nonnegativity restrictions are not included in the canonical representation and must be enforced implicitly. As x_2 is increased, we must ensure that x_3, x_4 and x_5 remain nonnegative. Because nonbasic variable x_1 is being held at zero, we see that the values of the basic variables are given by

$$
\mathbf{X}_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \beta - x_2\alpha_2 = \begin{pmatrix} 14 \\ 18 \\ 10 \end{pmatrix} - x_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \geq \mathbf{0}
$$

From this equation, $\partial x_3/\partial x_2 = 2$, and thus x_3 increases 2 units for each unit increase in x_2. Therefore, x_3 will always remain positive as x_2 increased. However, this is not true of x_4 and x_5 because $\partial x_4/\partial x_2 = -1$ and $\partial x_5/\partial x_2 = -1$. Notice from the above equation that x_4 will remain nonnegative as long as $x_2 \leq 18/1$, and similarly, x_5 will remain nonnegative as long as $x_2 \leq 10/1$. Thus, by the minimum ratio test, the maximum value of x_2 is equal to minimum $\{18, 10\} = 10$. Equation $x_5 = 10 - x_2$ is called the *blocking equation* and x_5 is called the blocking variable, or *departing variable*. A new canonical representation is now derived by solving for z and the new set of basic variables in terms of the new set of nonbasic variables. This can be done by solving for $x_2 = 10 - x_5$ in the blocking equation and using this representation of x_2 to eliminate x_2 from the remaining equations. This process is called a *pivot* and results in the new canonical representation:

$$
\begin{aligned}
z &= 2x_1 + 3(10 - x_5) = 30 + 2x_1 - 3x_5 \\
x_3 &= 4 - x_1 + 2(10 - x_5) = 24 - x_1 - 2x_5 \\
x_4 &= 18 - 2x_1 - (10 - x_5) = 8 - 2x_1 + x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

The current solution and basis matrix can be summarized as follows:

$$
\begin{aligned}
z &= 30 \\
\mathbf{X}_B &= \begin{pmatrix} X_{B,1} \\ X_{B,2} \\ X_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{X}_N &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Note that $\mathbf{a}_2$ has replaced $\mathbf{a}_5$ in the basis matrix $\mathbf{B}$. Also observe, that graphically, we have moved from extreme point P to extreme point T.

This solution is not yet optimal because $(z_1 - c_1) = -2 < 0$. Thus, x_1 is chosen as the entering variable. As before, the basic variables can be written as

$$
\mathbf{X}_B = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \beta - x_1\alpha_1 = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} - x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \geq \mathbf{0}
$$

From this equation, x_3 will remain nonnegative as long as $x_1 \leq 24/1 = 24$, and similarly, x_4 and x_2 will remain nonnegative as long as $x_1 \leq 8/2 = 4$ and $x_1 \leq \infty$, respectively. Therefore, the minimum ratio test yields the minimum $\{24, 4\} = 4$ and x_4 is the departing variable. The pivot operation results in

$$
\begin{aligned}
z^* &= 38 \\
\mathbf{X}_B^* &= \begin{pmatrix} X_{B,1}^* \\ X_{B,2}^* \\ X_{B,3}^* \end{pmatrix} = \begin{pmatrix} x_3^* \\ x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 20 \\ 4 \\ 10 \end{pmatrix} \\
\mathbf{X}_N^* &= \begin{pmatrix} x_4^* \\ x_5^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_2) = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Note again that the basis matrix has changed by one column.

The algebraic process just described is usually summarized in tabular form. These tabular formats are referred to as the *simplex tableau.*

3.2 The Simplex Method in Tableau Form

Consider again the canonical form

$$
\begin{aligned}
z &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{X}_N \\
\mathbf{X}_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N
\end{aligned}
$$

Now, rearranging terms so that all variables are on the left-hand side of the equation, with the constants on the right-hand side, we have

$$\begin{aligned} z + (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{X}_N &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{X}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N &= \mathbf{B}^{-1}\mathbf{b} \end{aligned}$$

The simplex tableau is simply a table used to store the coefficients of the algebraic representation in the above equations. The top row of the tableau consists of the coefficients in the objective function, and the body of the tableau records of the coefficient of the constraint equations. The general form is shown in the following table.

	z	$\mathbf{X}_B$	$\mathbf{X}_N$	RHS
z	1	$\mathbf{0}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}\mathbf{X}_N$	$\mathbf{B}^{-1}\mathbf{b}$

3.2.1 Identifying $\mathbf{B}^{-1}$ from the Simplex Tableau

Observe that each x_j column in the above tableau is of the form

$$\begin{array}{c} x_j \\ \begin{pmatrix} z_j - c_j \\ \hline \alpha_j \end{pmatrix} \end{array} = \begin{array}{c} x_j \\ \begin{pmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j \\ \hline \mathbf{B}^{-1}\mathbf{a}_j \end{pmatrix} \end{array}$$

That is, each original $\mathbf{a}_j$ column is updated by multiplying $\mathbf{B}^{-1}$ to get $\mathbf{B}^{-1}\mathbf{a}_j$, and $z_j - c_j$ is computed by multiplying the updated column $\mathbf{B}^{-1}\mathbf{a}_j$ by $\mathbf{c}_B$ and then subtracting c_j. This is an important observations for two reasons. First, it identifies the key elements in constructing any simplex tableau as $\mathbf{B}^{-1}$ and $\mathbf{c}_B$ along with the original data columns. Second, because all the columns are updated by multiplying by $\mathbf{B}^{-1}$, if the original matrix $\mathbf{A}$ contains the identity matrix $\mathbf{I}$, then $\mathbf{B}^{-1}$ will occupy the position in the updated tableau that was occupied by $\mathbf{I}$ in the original tableau. For example, suppose the original identity $\mathbf{I} = (\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k)$. Then $\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k) = (\mathbf{B}^{-1}\mathbf{a}_i, \mathbf{B}^{-1}\mathbf{a}_j, \mathbf{B}^{-1}\mathbf{a}_k) = \alpha_i, \alpha_j, \alpha_k$. Thus, $\mathbf{B}^{-1}$ can be found by a suitable rearrangement of a subset of the α_j columns. This provides a method for conveniently identifying $\mathbf{B}^{-1}$ for a given tableau.

3.2.2 The Simplex Algorithm (Maximization Problem)

1. *Check for possible improvement.* Examine the $z_j - c_j$ values in the top row of the simplex tableau. If these are all nonnegative, then the current basic feasible solution is optimal; stop. If, however, any $z_j - c_j$ is negative, go to Step 2.

2. *Check for unboundedness.* If for any $z_j - c_j < 0$, there is no positive element in the associated α_j vector (i.e. $\alpha_j \leq \mathbf{0}$), then the problem has an unbounded objective value. Otherwise, finite improvement in the objective is possible and we go to Step 3.

3. *Determine the entering variable.* Select as the entering variable, the nonbasic variable with the most negative $z_j - c_j$. Designate this variable as x_k. Ties in the selection of x_k may be broken arbitrarily. The column associated with x_k is called the pivot column. Go to Step 4.

4. *Determine the entering variable.* Use the minimum ratio test to determine the departing basic variable. That is, let

$$\frac{\beta_r}{\alpha_{r,k}} = \text{minimum} \left\{ \frac{\beta_i}{\alpha_{i,k}} : \alpha_{i,k} > 0 \right\}$$

Row r is called the *pivot row*, $\alpha_{r,k}$ is called the *pivot element*, and the basic variable, $x_{B,r}$ associated with row r is the departing variable. Go to Step 5.

5. *Pivot and establish a new tableau.*
   (a) The entering variable x_k is the new basic variable in row r.
   (b) Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consists of all zero except for a 1 at the pivot position $\alpha_{r,k}$.
   (c) Return to Step 1.

Example 3.2. The initial tableau for the problem of Example 3.1 is given below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-3	0	0	0	0
x_3	0	1	-2	1	0	0	4
x_4	0	2	1	0	1	0	18
x_5	0	0	1	0	0	1	10

Because there are negative $z_j - c_j$, the tableau is not yet optimal. And since no $\alpha_j \leq \mathbf{0}$ associated with a $z_j - c_j < 0$, finite improvement in the objective is possible. The entering variable is x_2 and to determine the departing variable, we examine the ratios $\beta_i/\alpha i, 2$ where $\alpha i, 2 > 0$. We add another column to the initial tableau for this process.

	z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio
z	1	-2	-3	0	0	0	0	-
x_3	0	1	-2	1	0	0	4	-
x_4	0	2	1	0	1	0	18	18
x_5	0	0	1	0	0	1	10	10

Thus, the departing variable is in row 3 which is x_5 and 1 is the pivot element. We perform pivotal elimination and obtain the tableau below.

	z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio
z	1	-2	0	0	0	3	30	-
x_3	0	1	0	1	0	2	24	24
x_4	0	2	0	0	1	-1	8	4
x_2	0	0	1	0	0	1	10	-

Recall from the previous section that $\mathbf{B}^{-1}$ occupies that portion of the tableau associated with the *original identity*. Note that $\mathbf{I} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$. Thus, $\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}(\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = (\mathbf{B}^{-1}\mathbf{a}_3, \mathbf{B}^{-1}\mathbf{a}_4, \mathbf{B}^{-1}\mathbf{a}_5) = (\alpha_3, \alpha_4, \alpha_5)$. That is $\mathbf{B}^{-1}$ is located beneath the slack variables x_3, x_4, x_5. For the tableau above, this corresponds to

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Going back to the simplex algorithm, we repeat the process since $z_1 - c_1 < 0$ and finite improvement in the objective is possible. This time, the entering variable is x_1 and the departing variable is x_4. After the pivotal elimination procedure, we obtain the tableau below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0	1	2	38
x_3	0	0	0	1	$-\frac{1}{2}$	$\frac{5}{2}$	20
x_1	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	4
x_2	0	0	1	0	0	1	10

As before, the basis inverse can be identified as

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

The solution given in the last table is now optimal. In fact, because $z_j - c_j > 0$ for the nonbasic variables x_4 and x_5, then this tableau represents the unique optimal solution. Thus, the optimal solution is

$$\begin{aligned} z^* &= 38 \\ x_1^* &= 4 \\ x_2^* &= 10 \\ x_3^* &= 20 \\ x_4^* &= 0 \\ x_5^* &= 0 \end{aligned}$$

Example 3.3. Unbounded Objective

maximize $z = 5x_1 + 3x_2$
subject to

$$\begin{aligned} -x_1 + x_2 &\leq 4 \\ x_1 - 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Adding slack variables results in

maximize $z = 5x_1 + 3x_2$
subject to

$$\begin{aligned} -x_1 + x_2 + x_3 &= 4 \\ x_1 - 2x_2 + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The following tables present the first two tableaux when applying the simplex algorithm to the preprocessed algorithm.

	z	x_1	x_2	x_3	x_4	RHS	Ratio
z	1	-5	-3	0	0	0	-
x_3	0	-1	1	1	0	4	-
x_4	0	1	-2	0	1	6	6
z	1	0	-13	0	5	30	
x_3	0	0	-1	1	1	10	
x_1	0	1	-2	0	1	6	

The objective can be designated as unbounded because $z_2 - c_2 < 0$ and $\alpha_2 \leq \mathbf{0}$

For a real-life problem, the next step would be to review the model. Has an error been made in the constraint formulation, or more likely, have we overlooked a limited resource or other restriction that when included in the formulation will bound the objective value?

Example 3.4. Alternative Optimal Solutions maximize $z = 2x_1 + x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 8 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Adding slack variables results in

maximize $z = 2x_1 + x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

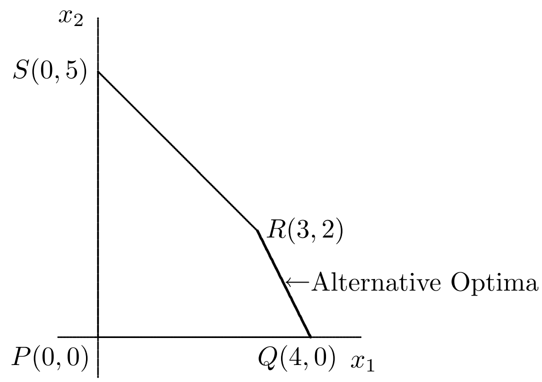
The tableaux for this problem are given below.

	z	x_1	x_2	x_3	x_4	RHS	Ratio
z	1	-2	-1	0	0	0	-
x_3	0	2	1	1	0	8	4
x_4	0	1	1	0	1	5	5
z	1	0	0	1	0	8	
x_1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	4	
x_4	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	1	

Note that the basic feasible solution is an optimal solution because $z_j - c_j \geq 0$ for all j. However, note also that $z_j - c_j = 0$ for the nonbasic variable x_2. Thus increasing x_2 from its current value of zero will not change the objective value. Entering x_2 into the basis results in the following tableau.

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	1	0	8
x_1	1	0	1	-1	0	3
x_2	0	1	-1	2	1	2

Observe that we have moved from the optimal extreme point Q to the optimal extreme point R in the following figure.



3.3 Finding an Initial Basic Feasible Solution

Example 3.5.

maximize $z = 8x_1 + 10x_2$
subject to

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + x_2 &\le 9 \\ x_1 + \frac{1}{2}x_2 &\ge 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Converting the problem to standard form yields

maximize $z = 8x_1 + 10x_2$
subject to

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + x_2 + x_3 &= 9 \\ x_1 + \frac{1}{2}x_2 - x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Therefore, the coefficient matrix is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & -1 \end{pmatrix}$$

Observe that the matrix $\mathbf{A}$ does not contain the identity as submatrix. We cannot use $\mathbf{B} = \mathbf{I}$ as a convenient starting basis. Artificial-variable techniques were developed to find a starting basis feasible solution in this all-too-common situation when a nice starting basis is not available. In the following section, we present one of the most common artificial-variable techniques, the two-phase method.

3.3.1 The Two-Phase Method

First, we create an identity submatrix by adding the necessary artificial variables to the original constraints. For Example 3.5, it would be necessary to add two artificial variables, say, x_5 and x_6, to constraints 1 and 3, respectively.

This would result in the following system of constraints.

$$\begin{aligned}
x_1 - x_2 + x_5 &= 1 \\
x_1 + x_2 + x_3 &= 9 \\
x_1 + \frac{1}{2}x_2 - x_4 + x_6 &= 4 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

Thus the coefficient matrix becomes

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & -1 & 0 & 1 \end{pmatrix}$$

Clearly, the identity submatrix is now available with $\mathbf{I} = (\mathbf{a}_5, \mathbf{a}_3, \mathbf{a}_6)$. Note that it was not necessary to add an artificial variable to the second constraint because x_3 appears only in the second constraint with a coefficient of 1. However, by adding these variables, we have changed the problem, and in order to have a solution to the original problem, the artificial variables must be zero. Thus, in phase I, an artificial objective function is used and an attempt is made to drive all artificial variables to zero. This artificial objective is to minimize the sum of the artificial variables. If all the artificial variables cannot be driven to zero, then at least one constraint of the original constraint is violated, and, consequently, the original problem is infeasible.

Phase II consists of replacing the artificial objective function by the original objective function and using the basic feasible solution in phase I as a starting point. If no artificial variables were left in the basis at the end of phase I, we simply perform the simplex algorithm until an optimum is reached. If, however, an artificial variable was in the basis (at a zero value) at the conclusion of phase I, we slightly modify the departing-variable rule. The specific steps of the two-phase algorithm follow and are illustrated via some examples.

The two-phase algorithm

1. Establish the problem formulation in a suitable form for the implementation of the simplex algorithm.

2. The artificial objective function of phase I is to maximize the negative sum of the artificial variables.

3. *Phase I*: Employ the simplex algorithm on the problem constructed in Steps 1 and 2. If, at optimality, there are no artificial variables in the basis at a positive value, go to Step 4. Otherwise, the problem is infeasible and we stop.

4. *Phase II*: Assign the actual objective function coefficient to each variable except for the artificial variables. Any artificial variables in the basis at a zero level are given a c_j value of zero in phase II. Any artificial variables that are not in the basis may be dropped from consideration by striking out their entire associated column in the tableau.

5. The first tableau of phase II is the final tableau of phase I except for the objective row. Update the objective row using the relationships

$$z_j - c_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{c}_B \alpha_j - c_j$$

$$z = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}_B \beta$$

6. If no artificial variables were in the basis at the end of phase I, we now simply use the simplex algorithm and proceed as usual. If, however, there are artificial variables in the basis, go to Step 7.

7. We must make sure that the artificial variables in the basis do not ever become positive in phase II. This is accomplished by modifying the departing variable rule of the simplex algorithm as follows:

(a) Determine the entering variable x_k in the usual manner.

(b) Examine the entering variable column $\mathbf{a}_k = \mathbf{B}^{-1}\mathbf{a}_k$. If the $\alpha_{i,k}$ values for any of the artificial variables left in the basis are negative, then choose an artificial variable with a negative $\alpha_{i,k}$ as the departing variable. Otherwise, employ the usual departing variable rule.

Example 3.6. The Two-Phase Algorithm

Returning to Example 3.5, we see that the phase I problem is as follows:

maximize $Z = -x_5 - x_6 \equiv$ minimize $x_5 + x_6$

subject to

$$\begin{aligned} x_1 - x_2 + x_5 &= 1 \\ x_1 + x_2 + x_3 &= 9 \\ x_1 + \frac{1}{2}x_2 - x_4 + x_6 &= 4 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Note that x_5 and x_6 are artificial variables, whereas x_3 and x_4, are respectively, slack and surplus variables in the original formulation. Notice also that the objective function of the phase I problem includes only the artificial variables.

$$\begin{aligned}
z_1 - c_1 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_1 - c_1 &= (-1\ 0\ -1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 0 = -2 \\
z_2 - c_2 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_2 - c_2 &= (-1\ 0\ -1)\begin{pmatrix} -1 \\ 1 \\ \frac{1}{2} \end{pmatrix} - 0 = \frac{1}{2} \\
z_3 - c_3 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_3 - c_3 &= (-1\ 0\ -1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 0 \\
z_4 - c_4 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_4 - c_4 &= (-1\ 0\ -1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - 0 = 1 \\
z_5 - c_5 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_5 - c_5 &= (-1\ 0\ -1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (-1) = 0 \\
z_6 - c_6 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_6 - c_6 &= (-1\ 0\ -1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - (-1) = 0 \\
Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} &= (-1\ 0\ -1)\begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix} = -5
\end{aligned}$$

The result of phase I is given below.

	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Ratio
Z	1	-2	$\frac{1}{2}$	0	1	0	0	-5	
x_5	0	1	-1	0	0	1	0	1	1
x_3	0	1	1	1	0	0	0	9	9
x_6	0	1	$\frac{1}{2}$	0	- 1	0	1	4	4
Z	1	0	$-\frac{3}{2}$	0	1	2	0	-3	
x_1	0	1	-1	0	0	1	0	1	
x_3	0	0	2	1	0	-1	0	8	4
x_6	0	0	$\frac{3}{2}$	0	-1	-1	1	3	2
Z	1	0	0	0	0	1	0	0	
x_1	0	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	3	
x_3	0	0	0	1	$\frac{4}{3}$	$\frac{1}{3}$	$-\frac{4}{3}$	4	
x_2	0	0	1	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	2	

Since $z_j - c_j \geq 0$ for all j, the last table is optimal. Because $Z = 0$, then all the artificial variables have been driven to zero. Also, because the artificial variables, x_5 and x_6, have been driven from the basis, their columns may be dropped from the phase II tableau. Updating the objective row using the original objective coefficients, we begin phase II with the last table of phase I. Note that we only need to compute $z_4 - c_4$ and z because x_1, x_2 and x_3 are basic variables.

$$\begin{aligned} z_4 - c_4 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_4 - c_4 &= (8\ 0\ 10)\begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix} - 0 = -12 \\ z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} &= (8\ 0\ 10)\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = 44 \end{aligned}$$

Phase II is given below.

	z	x_1	x_2	x_3	x_4	RHS	Ratio
z	1	0	0	0	-12	44	
x_1	0	1	0	0	$-\frac{2}{3}$	3	
x_3	0	0	0	1	$\frac{4}{3}$	4	3
x_2	0	0	1	0	$-\frac{2}{3}$	2	
z	1	0	0	9	0	80	
x_1	0	1	0	$\frac{1}{2}$	0	5	
x_4	0	0	0	$\frac{3}{4}$	1	3	
x_2	0	0	1	$\frac{1}{2}$	0	4	

Since $z_j - c_j \geq 0$ for all j, we have found the optimal solution:

$$\begin{aligned} z^* &= 80 \\ x_1^* &= 5 \\ x_2^* &= 4 \\ x_3^* &= 0 \\ x_4^* &= 3 \end{aligned}$$

Example 3.7. An Infeasible Problem

maximize $z = -5x_1 + x_2$
subject to

$$
\begin{aligned}
2x_1 + x_2 &\geq 5 \\
x_2 &\geq 1 \\
2x_1 + 3x_2 &\leq 6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The resulting phase I problem is given as

maximize $Z = -x_6 - x_7$

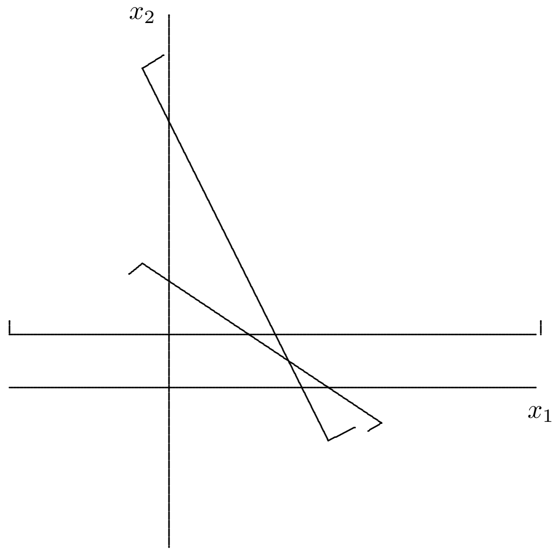
subject to

$$
\begin{aligned}
2x_1 + x_2 - x_3 + x_6 &= 5 \\
x_2 - x_4 + x_7 &= 1 \\
2x_1 + 3x_2 + x_5 &= 6 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0
\end{aligned}
$$

Phase I:

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Ratio
Z	1	-2	-2	1	1	0	0	0	-6	
x_6	0	2	1	-1	0	0	1	0	5	$\frac{5}{2}$
x_7	0	0	1	0	-1	0	0	1	1	
x_5	0	2	3	0	0	1	0	0	6	3
Z	1	0	-1	0	1	0	1	0	-1	
x_1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{5}{2}$	5
x_7	0	0	1	0	-1	0	0	1	1	1
x_5	0	0	2	1	0	1	-1	0	1	$\frac{1}{2}$
Z	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	
x_1	0	1	0	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	$\frac{9}{4}$	
x_7	0	0	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	
x_2	0	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	

The final tableau indicates optimality, but Z is not zero (because artificial variable x_6 is in the basis at a positive value). Consequently, this problem, as modeled, is mathematically infeasible and we may terminate the solution procedure. The feasible region of the original problem is graphed below, and clearly the feasible region is empty.



Example 3.8. Artificial Variables Left in the Basis

Our final illustration of the two-phase method involves a problem in which an artificial variable remains in the basis, at zero value, at the end of phase I. This is often caused by redundant constraints in the model formulation.

The table below is the first tableau, for a particular problem of phase II. Variables x_6 and x_7 are artificial variables and are both in the basis at zero values. Under normal simplex rules, variable x_2 would enter the basis and x_4 would leave, resulting in both x_6 and x_7 going positive. This would result in an infeasible solution.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	Ratio
z	1	0	-3	3	0	2	0	0	6	
x_1	0	1	-1	3	0	1	0	0	3	
x_6	0	0	-2	1	0	3	1	0	0	
x_4	0	0	3	-1	1	- 1	0	0	9	3
x_7	0	0	-1	0	0	2	0	1	0	
z	1	0	0	2	1	1	0	0	15	
x_1	0	1	0	$\frac{8}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	6	
x_6	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{7}{3}$	1	0	6	
x_2	0	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	3	
x_7	0	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	0	1	3	

However, using the modified departing variable rule of the two-phase method, we would select either x_6 or x_7 as the departing variable. You are asked to complete the operations of phase II.

3.3.2 The Big-M Method

The Big-M method is a technique that essentially combines the phase I and phase II problems of the two-phase method into a single problem. This is done by including the artificial variables in the original objective function with cost coefficients that implicitly try to drive the artificial variables to zero. To illustrate, consider the following example.

Example 3.9. The Big-M Method

$$\text{minimize } z = 3x_1 + x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 12 \\ 4x_1 - x_2 + x_3 &\geq 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Converting the problem to a maximization problem and preprocessing the constraints yields

maximize $z' = -3x_1 - x_2 - 4x_3$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 12 \\ 4x_1 - x_2 + x_3 - x_5 &= 6 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Because neither column of the identity is available, we supplement the problem with two artificial variables, x_6 and x_7, and form the *Big-M problem*:

maximize $z' = -3x_1 - x_2 - 4x_3 - Mx_6 - Mx_7$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 + x_6 &= 12 \\ 4x_1 - x_2 + x_3 - x_5 + x_7 &= 6 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

We now solve this problem by the standard simple procedure, while assuming that M is a large positive number.

$$\begin{aligned}
z_1 - c_1 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_1 - c_1 &= (-M \ \ -M)\begin{pmatrix} 1 \\ 4 \end{pmatrix} + 3 = -5M + 3 \\
z_2 - c_2 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_2 - c_2 &= (-M \ \ -M)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 = 1 \\
z_3 - c_3 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_3 - c_3 &= (-M \ \ -M)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 = -2M + 4 \\
z_4 - c_4 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_4 - c_4 &= (-M \ \ -M)\begin{pmatrix} -1 \\ 0 \end{pmatrix} - 0 = M \\
z_5 - c_5 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_5 - c_5 &= (-M \ \ -M)\begin{pmatrix} 0 \\ -1 \end{pmatrix} - 0 = M \\
z_6 - c_6 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_6 - c_6 &= (-M \ \ -M)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + M = 0 \\
z_7 - c_7 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_7 - c_7 &= (-M \ \ -M)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + M = 0 \\
z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} &= (-M \ \ -M)\begin{pmatrix} 12 \\ 6 \end{pmatrix} = -18M
\end{aligned}$$

The following table present the initial tableau.

	z'	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z'	1	$-5M+3$	1	$-2M+4$	M	M	0	0	$-18M$
x_6	0	1	1	1	-1	0	1	0	12
x_7	0	4	-1	1	0	-1	0	1	6

The details to arrive at the final tableau below is left as an exercise.

	z'	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z'	1	0	0	$\frac{11}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$M-\frac{7}{5}$	$M-\frac{2}{5}$	$-\frac{106}{5}$
x_2	0	0	1	$\frac{3}{5}$	$-\frac{4}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$-\frac{1}{5}$	$\frac{42}{5}$
x_1	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{18}{5}$

Because all $z_j - c_j \geq 0$ and the artificial variables have value zero, then the above table represents the optimal solution to the original problem. Thus,

$$
\begin{aligned}
z^* &= -z' = -(-\frac{106}{5}) = -\frac{106}{5} \\
x_1^* &= \frac{18}{5} \\
x_2^* &= \frac{42}{5} \\
x_3^* &= 0 \\
x_4^* &= 0 \\
x_5^* &= 0
\end{aligned}
$$

The use of $-M$ as the objective coefficient for artificial variables serves its purpose, but at the cost of tedious hand calculations. It also presents problems when using the computer. On the computer, M must be assigned some numerical value that must be considerably larger than any of the other objective coefficients. If M is too small, we might obtain a solution with an artificial variable in the basis at a positive value (signifying an infeasible problem) when actually the problem is feasible. However, if M is too large, it may tend to dominate the $z_j - c_j$ values. Roundoff errors (inherent in any digital computer) could well result and impact the final solution. For these reasons, the Big-M method is seldom used in practice. The two-phase method was developed to avoid, or at least alleviate these difficulties.

3.4 Unrestricted Variables and Variables With Negative Lower Bounds

All of the material presented so far has stressed that all variables must be restricted to nonnegative values when employing simplex algorithm. This restriction is one that may easily be circumvented by simple substitutions. For example, if x_k is an unrestricted variable, we may simply let

$$x_k = x_k^+ - x_k^-$$

where

$$-\infty \leq x_k \leq +\infty$$
$$x_k^+, x_k^{\geq} 0$$

Wherever x_k appears in the problem, we substitute $x_k^+ - x_k^-$. Because the columns associated with x_k^+ and x_k^- are linearly dependent (one column is simply the negative of the other), at most one can appear in the basis at a positive value. If x_k^+ is in the basis, then $x_k \geq 0$, whereas if x_k^- is in the basis, then $x_k \leq 0$.

In a similar way, we can also handle variables with negative lower bounds. For example, suppose $x_l \geq -5$. Then we simply let

$$x_l' = x_l + 5 \geq 0$$

Whenever x_l appears in the problem, we substitute $x'_l - 5$. Both of these techniques are illustrated in the following example.

Example 3.10. Unrestricted Variables and Variables With Negative Lower Bounds

$$
\begin{array}{l}
\text{maximize } z = -3x_1 + x_2 \\
\text{subject to} \\
\qquad \begin{array}{rcl}
-2x_1 + 3x_2 & \leq & 4 \\
2x_1 + x_2 & \leq & 8 \\
4x_1 - x_2 & \leq & 16 \\
x_1 & & \text{unrestricted} \\
x_2 & \geq & -4
\end{array}
\end{array}
$$

Let $x'_2 = x_2 + 4 \geq 0$. Also let $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geq 0$. Now transform the original problem by substituting $x_2 = x'_2 - 4$ and $x_1 = x_1^+ - x_1^-$. This results in

$$
\begin{array}{l}
\text{maximize } z = -3(x_1^+ - x_1^-) + (x'_2 - 4) = -3x_1^+ + 3x_1^- + x'_2 - 4 \\
\text{subject to} \\
\qquad \begin{array}{rcl}
-2(x_1^+ - x_1^-) + 3(x'_2 - 4) & \leq & 4 \\
2(x_1^+ - x_1^-) + (x'_2 - 4) & \leq & 8 \\
4(x_1^+ - x_1^-) - (x'_2 - 4) & \leq & 16 \\
x_1^+, x_1^-, x'_2 & \geq & 0
\end{array}
\end{array}
$$

The constant (-4) in the objective has no effect on the optimization process; therefore, let $z' = z + 4$. Simplification and the addition of slack variables yields

$$
\begin{array}{l}
\text{maximize } z' = -3x_1^+ + 3x_1^- + x'_2 \\
\text{subject to} \\
\qquad \begin{array}{rcl}
-2x_1^+ + 2x_1^- + 3x'_2 + x_3 & = & 16 \\
2x_1^+ - 2x_1^- + x'_2 + x_4 & = & 12 \\
4x_1^+ - 4x_1^- - x'_2 + x_5 & = & 12 \\
x_1^+, x_1^-, x'_2, x_3, x_4, x_5 & \geq & 0
\end{array}
\end{array}
$$

The initial tableau is shown below.

	z'	x_1^+	x_1^-	x'_2	x_3	x_4	x_5	RHS
z'	1	3	-3	-1	0	0	0	0
x_3	0	-2	2	3	1	0	0	16
x_4	0	2	-2	1	0	1	0	12
x_5	0	4	-4	-1	0	0	1	12

After using the simplex algorithm, the final tableau is obtained. This is shown below.

	z'	x_1^+	x_1^-	x'_2	x_3	x_4	x_5	RHS
z'	1	0	0	$\frac{7}{2}$	$\frac{3}{2}$	0	0	24
x_1^-	0	-1	1	$\frac{3}{2}$	$\frac{1}{2}$	0	0	8
x_4	0	0	0	4	1	1	0	28
x_5	0	0	0	5	2	0	1	44

Thus, the optimal solution to the original problem can be recovered as

$$\begin{aligned}
z^* &= z' - 4 = 24 - 4 = 20 \\
x_1^* &= x_1^+ - x_1^- = 0 - 8 = -8 \\
x_2^* &= x_2' - 4 = 0 - 4 = -4 \\
x_3^* &= 0 \\
x_4^* &= 28 \\
x_5^* &= 44
\end{aligned}$$

3.5 Degeneracy and Cycling

Recall that a degenerate basic feasible solution is one in which at least one basic variable has a value of zero. This does not mean that anything is wrong with the solution. However, degeneracy could possibly create two related problems:

1. The objective function z may not improve when we move from one basis to another.

2. We might, in fact, cycle forever (repeating a sequence of bases) and not ever reach the optimal solution.

We illustrate the concept of a degenerate extreme point and a degenerate pivot via the following example.

Example 3.11. Degeneracy

maximize $z = 4x_1 + 3x_2$
subject to

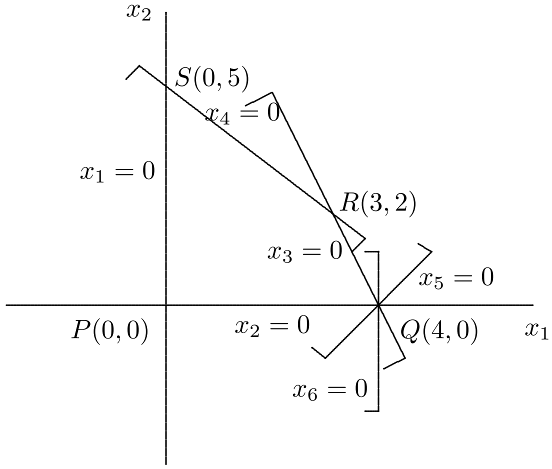
$$\begin{aligned}
2x_1 + x_2 &\leq 8 \\
x_1 + x_2 &\leq 5 \\
x_1 - x_2 &\leq 4 \\
x_1 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Preprocessing, we have

maximize $z = 4x_1 + 3x_2$
subject to

$$\begin{aligned}
2x_1 + x_2 + x_3 &= 8 \\
x_1 + x_2 + x_4 &= 5 \\
x_1 - x_2 + x_5 &= 4 \\
x_1 + x_6 &= 4 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

The feasible region for this problem is graphed below.



Obviously, in this case, constraints 3 and 4 are redundant and have no effect on the feasible region. However, detecting such redundant constraints is not an easy task for problems of nontrivial size. Let us ignore this redundancy so that we can illustrate the concept of a degenerate extreme point. Because $m = 4$ and $n = 6$, each basic feasible solution will be characterized by $n - m = 2$ nonbasic variables. Observe, however, that extreme point Q has $x_2 = x_3 = x_5 = x_6 = 0$. Thus, there are $C_2^4 = 6$ basic feasible solutions representing Q. The following table gives the simplex tableaux.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Ratio
z	1	-3	-2	0	0	0	0	0	
x_3	0	2	1	1	0	0	0	8	4
x_4	0	1	1	0	1	0	0	5	5
x_5	0	1	-1	0	0	1	0	4	4
x_6	0	1	0	0	0	0	1	4	4
z	1	0	-5	0	0	3	0	12	
x_3	0	0	3	1	0	-2	0	0	0
x_4	0	0	2	0	1	-1	0	1	$\frac{1}{2}$
x_1	0	1	-1	0	0	1	0	4	
x_6	0	0	1	0	0	-1	1	0	0
z	1	0	0	0	0	-2	5	12	
x_3	0	0	0	1	0	1	-3	0	0
x_4	0	0	0	0	1	1	-2	1	1
x_1	0	1	0	0	0	0	1	4	
x_2	0	0	1	0	0	-1	1	0	
z	1	0	0	2	0	0	-1	12	
x_5	0	0	0	1	0	1	-3	0	
x_4	0	0	0	-1	1	0	1	1	1
x_1	0	1	0	0	0	0	1	4	4
x_2	0	0	1	1	0	0	-2	0	
z	1	0	0	1	1	0	0	13	
x_5	0	0	0	-2	3	1	0	3	
x_6	0	0	0	-1	1	0	1	1	
x_1	0	1	0	0	-1	0	0	3	
x_2	0	0	1	-1	2	0	0	2	

Note that in the initial tableau, x_1 chosen as the entering variable, and the minimum ratio test results in a three-way tie. That is, each of x_3, x_5 or x_6 could be chosen as the departing variable. This should be the signal that the next basic feasible solution will be degenerate. Choosing x_5 as the departing variable results in a degenerate solution which represents extreme point Q. Note that as expected, the basic variables x_3 and x_6 are equal to zero. After the first iteration, x_2 is chosen as the entering variable. This results in a minimum ratio of zero with either x_3 or x_6 as the departing variable. Choosing x_6 as the departing variable results to the next tableau. Note that although the basis has changed, the values of all the variables are the same, and this new solution is also a basic feasible solution representing extreme point Q. Continuing the simplex procedure, we see that after an additional degenerate pivot, we eventually end up with the optimal solution in the final tableau corresponding to the extreme point R.

In this example, we saw that in the presence of degeneracy, it is possible to perform pivot operations and remain at the same extreme point. Although, this example did not exhibit the phenomenon of cycling it is theoretically possible for degeneracy to lead a computational loop or cycle that is repeated infinitely. In practice, however, this has not occurred in actual problems and is highly unlikely to present a problem.

In the absence of degeneracy, the simplex method is guaranteed to stop a finite number of iterations. This follows directly because there are a finite number of extreme points and each nondegenerate pivot forces the objective function to strictly increase. Thus, it is not possible to visit the same extreme point twice.

However, in the presence of degeneracy, finite convergence is only guaranteed if a cycling-prevention is employed. The basic idea behind cycling-prevention rules is to provide a method for breaking ties in the minimum ratio test in such a way that it is not possible to repeat a basis matrix during a sequence of degenerate pivots.

EXERCISES.

1. Consider the following linear programming problem.

$$\text{maximize } z = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 20 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) Solve this problem graphically.

   (b) Solve this problem by the algebraic simplex method and at each iteration, identify the corresponding extreme-point solution on the graph.

   (c) Identify the basic variables, the nonbasic variables, and the basis matrix **B** at each iteration.

2. Consider the following constraint set.

$$\begin{aligned} x_2 &\leq 9 \\ x_1 - x_2 &\leq 5 \\ 2x_1 + x_2 &\leq 22 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) Sketch the feasible region.

   (b) Identify the basic variables, the nonbasic variables, and the basis matrix associated with each extreme point of the feasible region.

(c) Suppose that x_2 and the slack variable in the second constraint are the nonbasic variables defining the current extreme-point solution. If x_2 is chosen as the entering variable, what would be the departing variable and which extreme point would correspond to the next basic feasible solution?

3. Consider the following linear programming problem.

$$\text{maximize } z = x_1 + 2x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_1 + x_2 &\le 16 \\ -x_1 + x_2 &\le 5 \\ x_1 &\le 12 \\ -x_1 + 3x_2 &\le 16 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(a) Solve this problem graphically.

(b) Solve this problem by the algebraic simplex method.

(c) Solve this problem by the simplex algorithm using the simplex tableau.

4. Solve the following linear programming problem by the simplex algorithm.

$$\text{minimize } z = 3x_1 - 4x_2 - x_3 - 2x_4 - 3x_5$$
$$\text{subject to}$$
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 + 2x_5 &\le 12 \\ x_1 - 2x_2 - x_3 - x_4 - x_5 &\ge -30 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

5. Solve the following linear programming problem by the simplex algorithm, identifying the basis matrix $\mathbf{B}$ and the basis inverse $\mathbf{B}^{-1}$ at each iteration. Is the optimal solution unique? Explain.

$$\text{maximize } z = x_1 + 2x_2 + 5x_3 + x_4$$
$$\text{subject to}$$
$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &\le 20 \\ -x_1 + x_2 + x_3 + x_4 &\le 12 \\ 2x_1 + x_2 + x_3 - x_4 &\le 30 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

6. Consider the following linear programming problem.

$$\text{maximize } z = 2x_1 + 2x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_2 &\le 10 \\ x_1 - 3x_2 &\le 2 \\ x_1 + x_2 &\le 16 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(a) Verify graphically that this problem has alternative optimal solutions.

(b) Use the simplex algorithm to find all alternative optimal solutions.

7. Three products A, B, and C, are made using two manufacturing processes. The unit production times in hours are given in the following table. The time available for Process 1 is 36 hours, and for Process 2 is 40 hours. Products A, B, and C sell for \$9, \$6, and \$8, respectively. In addition, it is estimated that no more than 6 units of Product C can be sold. Formulate a linear programming problem for determining the optimal product mix, and solve by the simplex method.

	Unit times	production (hours)
Product	Process 1	Process 2
A	2	2
B	1	3
C	2	1

8. Consider the basic solution defined by the following tableau. Suppose that the objective function of this problem is given by maximize $z = c_1x_1 + c_2x_2$. Give mathematical condition(s) in terms of c_1 and c_2 such that the given basic feasible solution is the unique optimal solution.

	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	$-\frac{1}{2}$	$\frac{5}{2}$	20
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	4
x_2	0	1	0	0	1	10

9. Consider the following linear programming problem.

$$\text{maximize } z = -5x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 &\leq 12 \\ -x_1 + 2x_2 &\geq -8 \\ x_2 &\leq 8 \\ 2x_1 + 3x_2 &\leq 36 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Solve this problem graphically.

(b) Solve this problem by the simplex algorithm.

10. Consider the optimal tableau below. Characterize mathematically the set of all optimal solutions to this problem.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	3	0	0	4	8
x_1	0	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	4
x_4	0	0	0	3	1	1	0	4
x_2	0	0	1	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	2

11. Consider the following problem.

$$\text{maximize } z = 4x_1 + x_2$$
subject to
$$\begin{aligned} 2x_1 - 3x_2 &\le 12 \\ -4x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$

    (a) Verify graphically that this problem does not have a finite optimal solution.
    (b) Use the simplex method to show that this problem does not have a finite optimal solution.
    (c) Use your final tableau to construct an extreme direction of the feasible region, $\mathbf{AX} = \mathbf{b}, \mathbf{X} \ge \mathbf{0}$.

12. Once a variable departs from a basis (during the simplex procedure), can it ever return (i.e., become, once again, basic)? Explain.

13. Consider the basic solution defined by the following tableau. Suppose that the objective function of this problem is given by maximize $z = c_1x_1 + c_2x_2$. Give a mathematical condition(s) in terms of c_1 and c_2 such that the given basic feasible solution indicates that no finite optimal solution exists.

14. Consider the following problem.

$$\text{maximize } z = 2x_1 + x_2$$
subject to
$$\begin{aligned} -x_1 + 2x_2 &\le 10 \\ x_1 &\le 6 \\ x_1 + x_2 &\le 14 \\ x_1 - x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

    (a) Sketch the feasible region and identify any degenerate extreme points.
    (b) For each extreme point, specify all possible basis matrices.
    (c) Solve the problem by the simplex algorithm.

15. Consider the following problem.

$$\text{maximize } z = 7x_1 + 3x_2 + 2x_3$$
subject to
$$\begin{aligned} 4x_1 + x_2 + x_3 &\le 18 \\ 3x_1 + 2x_2 + x_3 &\le 14 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Let the slack variables for the respective constraints be denoted by x_4 and x_5. The tableau below represents the current basic solution.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1				1	1	
x_1	0				1	-1	
x_3	0				-3	4	

(a) Identify the basis inverse corresponding to the given tableau.
(b) Determine the values of the missing entries in the tableau.
(c) Is the tableau optimal? If so, is the optimal solution unique?

16. Consider the following problem.

$$\begin{array}{l} \text{maximize } z = 2x_1 - x_4 - 5x_5 + 2x_7 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 + x_3 - 2x_5 - x_7 &=& 12 \\ -x_1 + x_2 + x_4 + 3x_5 - x_7 &=& 6 \\ 2x_1 - 2x_4 + +6x_5 + x_6 + 4x_7 &=& 18 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq& 0 \end{array}$$

The tableau below represents the current basic solution.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	0	1	1	0	a	1	0	e
0	0	3	1	1	b	1	0	f
0	0	1	0	0	c	$\frac{1}{2}$	1	g
0	1	1	1	0	d	$\frac{1}{2}$	0	h

(a) Identify the basic variables and the basis inverse corresponding to the given tableau.
(b) Determine the values of the unknowns in the tableau.
(c) What is the range of change of z with respect to x_2 (i.e. $\partial z/\partial x_2$)?
(d) What is the range of change of x_7 with respect to x_6 (i.e. $\partial x_7/\partial x_2$)?

17. Consider the following problem.

$$\begin{array}{l} \text{maximize } z = 2x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 + x_2 &\geq& 6 \\ x_1 + 3x_2 &\leq& 24 \\ x_2 &\geq& 2 \\ x_1 - x_2 &\leq& 8 \\ x_1, x_2 &\geq& 0 \end{array}$$

(a) Sketch the feasible region and find the optimal solution graphically.
(b) Solve the problem by the two-phase method and track the sequence of solutions on the graph. Note that each solution corresponds to a basic (but not necessarily feasible) solution of the problem.

18. Consider the following problem.

$$\begin{array}{l} \text{maximize } z = 2x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 - x_2 &\geq& 10 \\ 2x_1 + 3x_2 &\leq& 24 \\ 2x_1 + x_2 &\leq& 12 \\ x_1, x_2 &\geq& 0 \end{array}$$

(a) Verify graphically that this problem is infeasible.

(b) Use phase I of the two-phase method to show that the problem is infeasible.

19. Solve the following problem by the two-phase method.

$$\text{minimize } z = 2x_1 - 5x_2 + x_3$$
$$\text{subject to}$$
$$\begin{aligned} -x_1 + x_2 + x_3 &\geq 5 \\ -x_1 - x_2 + x_3 &= -1 \\ 5x_1 + 3x_2 - x_3 &\leq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. Consider the following problem.

$$\text{minimize } z = 7x_1 + 3x_2 + 4x_3$$
$$\text{subject to}$$
$$\begin{aligned} x_1 + x_2 &\geq 10 \\ 2x_1 + 2x_2 + x_3 &\leq 40 \\ -2x_1 + x_2 - x_3 &= 22 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Solve this problem by the two-phase method.

(b) Solve this problem by the big-M method.

21. The tableau below represents a basic feasible solution of a linear programming problem in which the objective is maximize $z = 10x_1 - 12x_2 + 8x_3$ and x_4, x_5 and x_6 are the slack variables for the respective constraints.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	a	d	2	6	0	88
x_1	0	1	b	1	e	3	0	10
x_2	0	0	c	-2	-1	f	0	g
x_6	0	0	0	3	-1	1	1	8

(a) Determine the values of the unknowns in the tableau.

(b) Identify $\mathbf{B}^{-1}$

(c) What is the rate of change of z with respect to x_5 (i.e. $\partial z/\partial x_5$)?

(d) What is the rate of change of x_2 with respect to x_3 (i.e. $\partial x_2/\partial x_3$)?

(e) Without explicitly finding vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_6$, write vector $\mathbf{a}_3$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_6$.

22. Consider the following problem.

$$\text{maximize } z = 5x_1 + x_2$$
$$\text{subject to}$$
$$\begin{aligned} x_1 - x_2 &\leq 9 \\ -x_1 - x_2 &\leq 2 \\ 2x_1 + x_2 &\leq 12 \\ x_1 &\geq 0 \\ x_2 &\quad \textit{unrestricted} \end{aligned}$$

(a) Solve this problem graphically

(b) Transform the problem so that all variables are nonnegative and solve the resulting problem by the simplex algorithm. Derive the solution to the original problem.

23. Consider the following problem.

$$\begin{aligned}
&\text{maximize } z = x_1 - 2x_2 + 3x_3\\
&\text{subject to}\\
&\qquad \begin{array}{rcl}
x_1 - x_2 + x_3 & \leq & 8\\
x_1 - x_2 - x_3 & \leq & 12\\
x_1 & \geq & -2\\
x_2 & \geq & 0\\
x_3 & & \textit{unrestricted}
\end{array}
\end{aligned}$$

(a) Transform the problem so that all variables are nonnegative.

(b) Solve the transformed problem by the simplex algorithm.

(c) Use the solution found in part (b) to derive the optimal solution of the original problem.

24. Consider the following problem.

$$\begin{aligned}
&\text{maximize } z = 2x_1 - 4x_2 + 7x_3\\
&\text{subject to}\\
&\qquad \begin{array}{rcl}
x_1 - x_2 + x_3 & = & 2\\
x_1 + x_2 + 2x_3 & \leq & 10\\
x_1 + x_2 - x_3 & \geq & 6\\
x_1, x_2, x_3 & \geq & 0
\end{array}
\end{aligned}$$

(a) Solve this problem by the two-phase method.

(b) Solve this problem by the big-M method.

25. Solve the following problem by the simplex method.

$$\begin{aligned}
&\text{maximize } z = x_1 + 5x_2 + x_3 + 2x_4\\
&\text{subject to}\\
&\qquad \begin{array}{rcl}
2x_2 + x_3 + 3x_4 & \geq & 100\\
3x_1 + x_2 + 2x_3 + x_4 & \leq & 900\\
-x_1 - x_2 + x_3 - 4x_4 & = & 300\\
x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
\end{aligned}$$

26. Consider the following linear programming problem and the associated optimal tableau shown below.

$$\begin{aligned}
&\text{maximize } z = c_1x_1 + c_2x_2 + c_3x_3\\
&\text{subject to}\\
&\qquad \begin{array}{rcl}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 & \leq & b_1\\
a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 & \leq & b_2\\
x_1, x_2, x_3 & \geq & 0
\end{array}
\end{aligned}$$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	2	0	1	$\frac{1}{2}$	11
x_3	0	0	-5	1	-2	$\frac{3}{2}$	3
x_1	0	1	3	0	1	$-\frac{1}{2}$	1

Find the values of the $a_{i,j}, b_i$ and c_j.

27. Consider the following system of linear equations in the form $\mathbf{AX} = \mathbf{b}$, where $\mathbf{A}$ is $m \times m$.

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 2 \\ -2x_1 + x_2 - 2x_3 &= 3 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

(a) Use phase I of the two-phase method to find the unique optimal solution to this system.

(b) Specify the inverse of the coefficient matrix $\mathbf{A}^{-1}$.

28. A company must distribute its product from two warehouse locations to two retail outlets. Warehouse A has a total of 48 units, and Warehouse B has a total of 60 units. Forecasting estimates a demand of at most 36 units for Retail Outlet 1 and 72 units for Retail Outlet. The unit shipping costs between each warehouse and retail outlet are given in the table below. The problem is to determine the minimum-cost shipping schedule. Formulate a linear programming model and solve by the simplex algorithm.

Warehouse	Retail Outlet 1	Retail Outlet 2
A	\$6	\$8
B	\$4	\$3

Chapter 4

Duality and Sensitivity Analysis

4.1 Formulation of the Linear Programming Dual

Associated with each (*primal*) linear programming problem is a companion problem called the *dual.* The formulation of the dual problem is actually a mechanical straightforward process.

4.1.1 The Canonical Form of the Dual

The canonical form of a linear programming problem is one in which the objective is to be maximized, all constraints are of the ($\leq$) form, and ll variables are restricted to nonnegative values. Thus, the *primal*, in canonical form is

$$\text{(P) maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &\leq \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

Notice in particular that, in the canonical form, elements of the right-hand-side vector $\mathbf{b}$ may be *negative.* Consequently, any linear programming problem may be placed into the above form.

If the above LP is specified to be the primal (P), its *dual* (D) is given by:

$$\text{(D) minimize } Z = \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{Y} = (y_1, y_2, \ldots, y_m)$ is the vector of dual variables.

Note that in specifying the dual from the primal, several simple rules have been followed.

1. The objective of the primal is to be maximized; the objective of the dual is to be minimized.
2. The maximization problem must have all ($\leq$) constraints and the minimization problem has all ($\geq$) constraints.
3. All primal and dual variables must be nonnegative.

4. Each *constraint* in one problem corresponds to a *variable* (and vice-versa) in the other. For example, given m primal constraints, there are n dual constraints. Consequently, if one problem is of order $m \times n$, the other is of order $n \times m$.
5. The elements of the right-hand side of the constraints in one problem are the respective coefficients of the objective function in the other problem.
6. The matrix of constant coefficients for one problem is the transpose of the matrix of constant coefficients for the other problem.

Maximization problem		Minimization problem
m Constraints		m Variables
$\leq$	$\leftrightarrow$	≥ 0
n Variables		n Constraints
≥ 0	$\leftrightarrow$	$\leq$

Example 4.1. Formulation of the Canonical Dual
Find the dual of the following problem:

$$\text{maximize } z = c_1x_1 + c_2x_2 + c_3x_3$$

$$\text{subject to}$$

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &\leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &\leq b_2 \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

The dual form is as follows:

$$\text{minimize } Z = b_1y_1 + b_2y_2$$

$$\text{subject to}$$

$$\begin{aligned} a_{1,1}y_1 + a_{2,1}y_2 &\geq c_1 \\ a_{1,2}y_1 + a_{2,2}y_2 &\geq c_2 \\ a_{1,3}y_1 + a_{2,3}y_2 &\geq c_3 \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

Example 4.2. Formulation of the Canonical Dual
Find the dual of the following problem:

$$\text{maximize } z = 4x_1 + 2x_2$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ x_1 + 2x_2 &\leq 15 \\ 2x_1 - x_2 &\leq 12 \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

The proper canonical form for the example is

$$\text{maximize } z = 4x_1 + 2x_2$$

$$\text{subject to}$$

$$\begin{aligned} -x_1 - x_2 &\leq -2 \\ x_1 + 2x_2 &\leq 15 \\ 2x_1 - x_2 &\leq 12 \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

The dual form is as follows:

$$\text{minimize } z = -2y_1 + 15y_2 + 12y_3$$

$$\text{subject to}$$

$$\begin{aligned} -y_1 + y_2 + 2y_3 &\geq 4 \\ -y_1 + 2y_2 - y_3 &\geq 2 \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

Theorem 4.1. *The dual of the dual is the primal.*

Proof. Consider again the problem given by

$$\text{minimize } \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

Now, the first step in writing the ducal of this problem is to transform it into the form of the canonical primal. This results in

$$\text{maximize } -\mathbf{b}'\mathbf{Y}'$$

$$\text{subject to}$$

$$\begin{aligned} -\mathbf{Y}'\mathbf{A}' &\leq -\mathbf{c}' \\ \mathbf{Y}' &\geq \mathbf{0} \end{aligned}$$

Now letting $\mathbf{W} = (w_1, w_2, \ldots, w_n)$ represent the dual variables of this transformed problem, writing the canonical dual yields

$$\text{minimize } \mathbf{W}(-\mathbf{c}')$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{W} - \mathbf{A}' &\geq -\mathbf{b}' \\ \mathbf{W} &\geq \mathbf{0} \end{aligned}$$

But this can be rewritten as

$$\text{maximize } \mathbf{cW'}$$

$$\text{subject to}$$

$$\begin{aligned}\mathbf{Aw'} &\leq \mathbf{b}\\ \mathbf{W'} &\geq \mathbf{0}\end{aligned}$$

Finally, letting $\mathbf{X} = \mathbf{W'}$ yields precisely the primal problem. □

4.1.2 General Duality

Consider the following linear program.

$$\text{(P) maximize } z = c_1x_1 + c_2x_2 + c_3x_3$$

$$\text{subject to}$$

$$\begin{aligned}a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &\leq b_1\\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &\geq b_2\\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 &= b_3\\ x_1 &\quad \textit{unrestricted}\\ x_2 &\geq 0\\ x_3 &\leq 0\end{aligned}$$

In order to transform this into canonical form, constraint 2 must be multiplied by -1 to reverse the direction of the inequality. It is also necessary to write the equality constraint as two inequality constraints. Unrestricted variable x_1 must be replaced by the difference of two nonnegative variables, that is, $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geq 0$. Finally, the nonpositive variable x_3 must be replaced by the nonnegative variable $x_3' = -x_3$. These changes can be summarized as follows:

$$\text{(P) maximize } z = c_1x_1^+ - c_1x_1^- + c_2x_2 + c_3x_3'$$

$$\text{subject to}$$

$$\begin{aligned}a_{1,1}x_1^+ - a_{1,1}x_1^- + a_{1,2}x_2 + a_{1,3}x_3' &\leq b_1\\ -a_{2,1}x_1^+ + a_{2,1}x_1^- - a_{2,2}x_2 + a_{2,3}x_3' &\leq -b_2\\ a_{3,1}x_1^+ - a_{3,1}x_1^- + a_{3,2}x_2 + a_{3,3}x_3' &\leq b_3\\ -a_{3,1}x_1^+ + a_{3,1}x_1^- - a_{3,2}x_2 - a_{3,3}x_3' &\leq -b_3\\ x_1^+, x_1^-, x_2, x_3' &\geq 0\end{aligned}$$

Now, letting $\omega_1, \omega_2, \omega_3$ and ω_4 denote the dual variables corresponding to the above constraints. The canonical dual problem can be written as

$$\text{minimize } b_1\omega_1 - b_2\omega_2 + b_3\omega_3 - b_3\omega_4$$

subject to

$$\begin{aligned}
a_{1,1}\omega_1 - a_{2,1}\omega_2 + a_{3,1}\omega_3 - a_{3,1}\omega_4 &\geq c_1 \\
-a_{1,1}\omega_1 + a_{2,1}\omega_2 - a_{3,1}\omega_3 + a_{3,1}\omega_4 &\geq -c_1 \\
a_{1,2}\omega_1 - a_{2,2}\omega_2 + a_{3,2}\omega_3 - a_{3,2}\omega_4 &\geq c_2 \\
-a_{1,3}\omega_1 + a_{2,3}\omega_2 - a_{3,3}\omega_3 + a_{3,3}\omega_4 &\geq -c_3 \\
\omega_1, \omega_2, \omega_3, \omega_4 &\geq 0
\end{aligned}$$

Rearranging we have

$$\text{minimize } b_1\omega_1 - b_2\omega_2 + b_3(\omega_3 - \omega_4)$$

subject to

$$\begin{aligned}
a_{1,1}\omega_1 - a_{2,1}\omega_2 + a_{3,1}(\omega_3 - \omega_4) &\geq c_1 \\
a_{1,1}\omega_1 - a_{2,1}\omega_2 + a_{3,1}(\omega_3 - \omega_4) &\leq c_1 \\
a_{1,2}\omega_1 - a_{2,2}\omega_2 + a_{3,2}(\omega_3 - \omega_4) &\geq c_2 \\
a_{1,3}\omega_1 - a_{2,3}\omega_2 + a_{3,3}(\omega_3 - \omega_4) &\leq c_3 \\
\omega_1, \omega_2, \omega_3, \omega_4 &\geq 0
\end{aligned}$$

Finally we can write the general dual of the original problem (P) by letting $y_1 = \omega_1, y_2 = \omega_2$, and $y_3 = \omega_3 - \omega_4$, and converting the first two constraints into a single equality constraint. Note that $y_2 \leq 0$ because $\omega_2 \geq 0$ and y_3 is an unrestricted variable because it is the difference of two nonnegative variables. The resulting dual problem is

$$\text{(D) minimize } b_1y_1 + b_2y_2 + b_3y_3$$

subject to

$$\begin{aligned}
a_{1,1}y_1 + a_{2,1}y_2 + a_{3,1}y_3 &= c_1\\
a_{1,2}y_1 + a_{2,2}y_2 + a_{3,2}y_3 &\geq c_2\\
a_{1,3}y_1 + a_{2,3}y_2 + a_{3,3}y_3 &\leq c_3\\
y_1 &\geq 0\\
y_2 &\leq 0\\
y_3 & \quad \textit{unrestricted}
\end{aligned}$$

PRIMAL-DUAL RELATIONSHIPS

Maximization problem		Minimization problem
Constraints		*Variables*
$\leq$	$\leftrightarrow$	≥ 0
$\geq$	$\leftrightarrow$	≤ 0
$=$	$\leftrightarrow$	unrestricted
Constraints		*Variables*
≥ 0	$\leftrightarrow$	$\geq$
≤ 0	$\leftrightarrow$	$\leq$
unrestricted	$\leftrightarrow$	$=$

Example 4.3. Formulation of the General Dual

Find the dual of

$$\text{maximize } z = 4x_1 + 2x_2 - x_3$$

subject to

$$\begin{aligned}
x_1 + x_2 + x_3 &= 20\\
2x_1 - x_2 &\geq 6\\
3x_1 + 2x_2 + x_3 &\leq 40\\
x_1, x_2 &\geq 0\\
x_3 & \quad \textit{unrestricted}
\end{aligned}$$

From the table above, the dual problem is

$$\text{minimize } z = 20y_1 + 6y_2 + 40y_3$$

subject to

$$\begin{aligned}
y_1 + 2y_2 + 3y_3 &\geq 4\\
y_1 - y_2 + 2y_3 &\geq 2\\
y_1 + 2y_2 + 3y_3 &= -1\\
y_1 & \quad \textit{unrestricted}\\
y_2 &\leq 0\\
y_3 &\geq 0
\end{aligned}$$

Example 4.4. Formulation of the General Dual
Find the dual of

$$\text{minimize } 5x_1 + 3x_2$$

$$\text{subject to}$$

$$\begin{aligned} x_1 - 6x_2 &\geq 2 \\ 5x_1 + 7x_2 &= -4 \\ x_1 + 2x_2 &\leq 10 \\ x_1 & \quad \textit{unrestricted} \\ x_2 &\geq 0 \end{aligned}$$

From the table above, the dual problem is

$$\text{maximize } 2y_1 - 4y_2 + 10y_3$$

$$\text{subject to}$$

$$\begin{aligned} y_1 + 5y_2 + y_3 &= 5 \\ -6y_1 + 7y_2 + 2y_3 &\leq 2 \\ y_1 &\geq 0 \\ y_2 & \quad \textit{unrestricted} \\ y_3 &\leq 0 \end{aligned}$$

4.1.3 The Standard Form

There is one other form of the primal that is commonly used. This is the form in which the constraints of the primal are all equalities. A problem in this form is said to be in *standard form*. The rules for the general form may be applied to the standard form and thus it simply represents a special case of the general form. Because all constraints in the primal are equalities, all dual variables will be unrestricted. By utilizing the previous table, the standard form can be written as follows:

$$\text{(P) maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

$$\text{(D) minimize } Z = \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} & \quad \text{unrestricted} \end{aligned}$$

4.2 Relationships in Duality

The concept of *weak duality* essentially states that the objective value associated with any feasible solution to the maximization problem is less than or equal to the objective value of any feasible solution to the minimization problem.

Theorem 4.2. Weak Duality
Consider the following primal-dual pair.

$$\text{(P) maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &\leq \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

$$\text{(D) minimize } Z = \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

Let $\overline{\mathbf{X}}$ be a feasible solution to the maximization problem (P) and let $\overline{\mathbf{Y}}$ be a feasible solution to the minimization problem (D). Then, $z = \mathbf{c}\overline{\mathbf{X}} \leq \overline{\mathbf{Y}}\mathbf{b} = Z$.

Proof. Because $\overline{\mathbf{X}}$ is feasible to (P), it follows that

$$\begin{aligned} \mathbf{A}\overline{\mathbf{X}} &\leq \mathbf{b} \\ \overline{\mathbf{X}} &\geq \mathbf{0} \end{aligned}$$

Similarly because $\overline{\mathbf{Y}}$ is feasible to (D) we have

$$\begin{aligned} \overline{\mathbf{Y}}\mathbf{A} &\geq \mathbf{c} \\ \overline{\mathbf{Y}} &\geq \mathbf{0} \end{aligned}$$

Now because $\overline{\mathbf{X}}$ is nonnegative,

$$\overline{\mathbf{Y}}\mathbf{A}\overline{\mathbf{X}} \geq \mathbf{c}\overline{\mathbf{X}}$$

Similarly,

$$\overline{\mathbf{Y}}\mathbf{A}\overline{\mathbf{X}} \leq \overline{\mathbf{Y}}\mathbf{b}$$

Combining these results, we get

$$\mathbf{c}\overline{\mathbf{X}} \leq \overline{\mathbf{Y}}\mathbf{A}\overline{\mathbf{X}} \leq \overline{\mathbf{Y}}\mathbf{b}$$

□

This is a very important result that forms the foundation for several other duality relationships. First of all, notice that each feasible solution to the maximization problem provides a lower bound for the objective of the minimization problem, and, likewise, each feasible solution to the minimization problem provides an upper bound for the objective of the maximization problem.

Corollary 4.2.1. *If the primal objective is unbounded, then the dual problem is infeasible.*

Corollary 4.2.2. *If the dual objective is unbounded, then the primal problem is infeasible.*

The converse of each corollary is not true. If one problem is infeasible, it is also possible for the other to be infeasible.

Example 4.5. Infeasible Primal and Dual

Consider the following canonical primal-dual pair:

$$\text{(P) maximize } x_1 + 2x_2$$

$$\text{subject to}$$

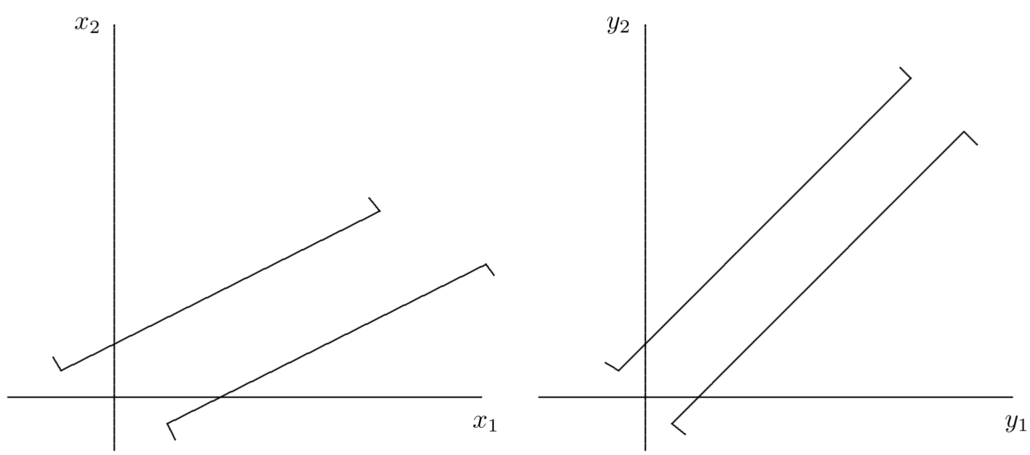
$$\begin{aligned} -x_1 + 2x_2 &\leq -2 \\ x_1 - 2x_2 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\text{(D) minimize } -2y_1 - 2y_2$$

$$\text{subject to}$$

$$\begin{aligned} -y_1 + y_2 &\geq 1 \\ 2y_1 - 2y_2 &\geq 2 \\ y_1 + 2y_2 + 3y_3 &= -1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Upon graphing, it is clear from the figure below that neither the primal nor the dual possesses a feasible solution.

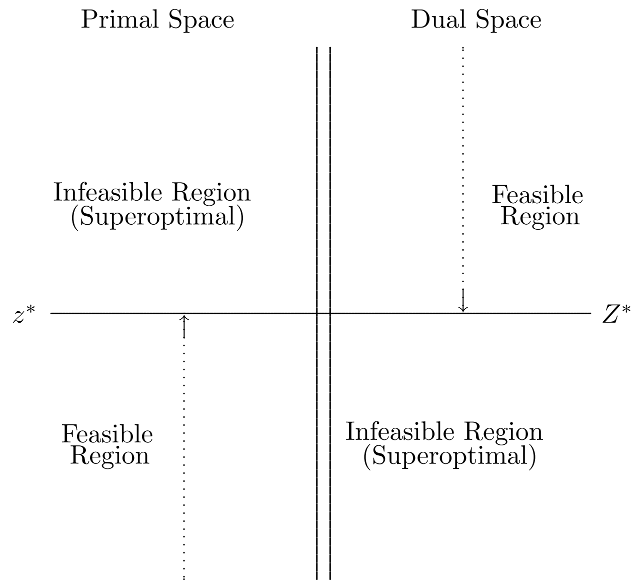


Based on the observations in the previous example and theorem 4.2, we can state the following:

Corollary 4.2.3. *If the primal is infeasible, then the dual is either infeasible or has an unbounded objective.*

Corollary 4.2.4. *If the dual is infeasible, then the primal is either infeasible or has an unbounded objective.*

The following figure depicts the primal-dual space for the canonical form.



Notice that the infeasible regions for the primal and dual. A value of z in the infeasible region of the primal is "superoptimal" because it is greater than z^*. However the constraints of the primal are violated. The same thing holds true for the dual except that we refer to minimization rather than maximization. Finally, note that the only time both problems are simultaneously feasible is when both are optimal.

Corollary 4.2.5. *If $\overline{\mathbf{X}}$ is feasible to (P), and $\overline{\mathbf{Y}}$ is feasible to (D), and $\mathbf{c}\overline{\mathbf{X}} = \overline{\mathbf{Y}}\mathbf{b}$, then $\overline{\mathbf{X}}$ is an optimal solution to (P) and $\overline{\mathbf{Y}}$ is an optimal solution to (D).*

Theorem 4.3. Strong Duality
Consider the following canonical primal-dual pair:

$$\textit{(P) maximize } z = \mathbf{cX}$$

$$\textit{subject to}$$

$$\begin{aligned} \mathbf{AX} &\leq \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

$$\textit{(D) minimize } Z = \mathbf{Yb}$$

$$\textit{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

If (P) has a feasible solution and (D) has a feasible solution, then both problems have finite optimal solutions with equal objectives.

Proof. Because (P) is feasible, it follows from corollary 4.2.1 that (D) has a finite optimal solution. Likewise, because (D) is feasible, corollary 4.2.2 implies that (P) has a finite optimal solution.

Now because (P) has a finite optimal solution, (P) has an extreme-point optimal solution corresponding to at least one basic feasible solution. Let $\mathbf{B}$ be an optimal basis matrix for problem (P) in standard form:

$$\text{maximize } \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} + \mathbf{X}_s &= \mathbf{b} \\ \mathbf{X}, \mathbf{X}_s &\geq \mathbf{0} \end{aligned}$$

Then, $z = \mathbf{cX}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$. By corollary 4.2.5, the proof will be complete if we can produce a feasible solution to (D), which has the same objective value. Consider $\overline{\mathbf{Y}} = \mathbf{c}_B\mathbf{B}^{-1}$. Clearly, $\mathbf{cX}^* = \overline{\mathbf{Y}}\mathbf{b} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$. Thus it only remains to show that $\overline{\mathbf{Y}}$ is a feasible solution to (D).

Consider the initial simples tableau corresponding to the problem above in standard form shown below.

	z	$\mathbf{X}$	$\mathbf{X}_s$	RHS
z	1	$-\mathbf{c}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{X}_B$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$

Now, because $\mathbf{B}$ is an optimal basis matrix, it follows that the optimal tableau will be as in the table below.

	z	$\mathbf{X}$	$\mathbf{X}_s$	RHS
z	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{I} - \mathbf{0}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

Because the above table is the optimal tableau, it follows that the optimality conditions are satisfied, that is,

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - c \geq 0$$

and

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{I} - \mathbf{0} = \mathbf{c}_B\mathbf{B}^{-1} \geq 0$$

But recall that $\overline{\mathbf{Y}} = \mathbf{c}_B\mathbf{B}^{-1}$. Now, by substitution,

$$\overline{\mathbf{Y}}\mathbf{A} - c \geq 0$$

and

$$\overline{\mathbf{Y}} \geq 0$$

which are precisely the dual feasibility conditions. Thus, $\overline{\mathbf{Y}}$ is dual feasible and $\overline{\mathbf{Y}}\mathbf{b} = \mathbf{cX}^*$

□

Observe that the above theorem provides a method for computing the values of the dual variables. That is, whereas the optimal solution can be written as

$$\mathbf{X}_N = \mathbf{0} \tag{4.1}$$

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} = \beta \tag{4.2}$$

the dual solution is given by

$$\mathbf{Y} = \mathbf{c}_B\mathbf{B}^{-1} \tag{4.3}$$
$$\lambda = \mathbf{YA} - \mathbf{c} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \tag{4.4}$$

where λ is the vector of dual surplus variables. Finally, the objective value of both problem is

$$z = \mathbf{cX} = \mathbf{Yb} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

Thus, given a basis matrix $\mathbf{B}$, the solutions to both problems can be determined directly from $\mathbf{B}^{-1}, \mathbf{b},$ and $\mathbf{c}$

4.2.1 Primal-Dual Tableau Relationships

Note that the tableaux depicted in the above proof also establish some relationships between the primal and dual variables. Let us rewrite the last table utilizing the fact that

$$\mathbf{Y} = \mathbf{c}_B\mathbf{B}^{-1} \tag{4.5}$$
$$\lambda = \mathbf{YA} - \mathbf{c} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \tag{4.6}$$

This results in the following table.

	z	$\mathbf{X}$	$\mathbf{X}_s$	RHS
z	1	λ	$\mathbf{Y}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

First note that the $z_j - c_j$ values for the primal decision variables $\mathbf{X}$ are given by the dual surplus variables λ. Just as $\mathbf{B}^{-1}$ resides in the portion of the tableau that was occupied by the original identity, $\mathbf{Y} = \mathbf{c}_B\mathbf{B}^{-1}$ is located in the top row immediately above $\mathbf{B}^{-1}$. However, this is only true if the original objective coefficients of the corresponding slack variables are zero. Thus, the $z_j - c_j$ values for the zero-cost primal slack variables $\mathbf{X}_s$ are given by the dual decision variables $\mathbf{Y}$.

Example 4.6. Tableau Relationships

Consider the following problem, which we shall arbitrarily designate as the primal.

maximize $z = 6x_1 + 3x_2 + 4x_3$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 8 \\ 5x_1 + 4x_2 + 3x_3 &\leq 25 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Its final tableau is given below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	0	1	1	33
x_1	0	1	-1	0	-$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
x_3	0	0	3	1	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{15}{2}$

This indicates that the optimal primal solution is given by

$$\begin{aligned} z^* &= 33 \\ x_1^* &= \frac{1}{2} \\ x_2^* &= 0 \\ x_3^* &= \frac{15}{2} \\ x_4^* &= 0 \\ x_5^* &= 0 \end{aligned}$$

Now denote the dual decision variables by y_1 and y_2 corresponding to constraints 1 and 2, respectively. Also let y_3, y_4, and y_5 represent the respective surplus variables for the three dual constraints. Then, by using the tableau relationships established in the above table, the top row of the tableau will in the following form:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	y_3	y_4	y_5	y_1	y_2	33

By comparing this with the above table, it immediately follows that the dual solution is given by

$$\begin{aligned} Z^* &= 33 \\ y_1^* &= z_4 - c_4 = 1 \\ y_2^* &= z_5 - c_5 = 1 \\ y_3^* &= z_1 - c_1 = 0 \\ y_4^* &= z_2 - c_2 = 3 \\ y_5^* &= z_3 - c_3 = 0 \end{aligned}$$

Note that x_1 was a basic variable in the optimal tableau and the associated dual variable $y_3 = z_1 - c_1$ was equal to zero. Similarly, x_3 was basic and $y_5 = z_3 - c_3$ was also zero. Finally, note that y_1, y_2 and y_4 are all positive in the optimal tableau, and consequently, the associated variables, x_4, x_5 and x_2, respectively, are all nonbasic with value zero. That is, when a particular variable has a positive value, then its associated dual variable has the value zero. This phenomenon is an important duality relationship called *complementary slackness.*

4.2.2 Complementary Slackness

Consider again the following canonical primal-dual pair.

$$\text{(P) maximize } \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &\leq \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

$$\text{(D) minimize } \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} &\geq \mathbf{c} \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

Recall that $\mathbf{Y} = (y_1, \ldots, y_m)$, where y_i is the dual variable corresponding to the constraint $\mathbf{a}^i\mathbf{X} \leq b_i$ where the notation $\mathbf{a}^i$ represents the ith row of matrix $\mathbf{A}$. Similarly, x_j is the dual variable for the constraint $\mathbf{Ya}_j \geq c_j$. The *complementary slackness conditions* can then be stated as follows:

$$\begin{aligned} y_i(b_i - \mathbf{a}^i\mathbf{X}) = 0, &\qquad \text{for all } i = 1, \ldots, m \\ (\mathbf{Ya}_j - c_j)x_j = 0, &\qquad \text{for all } j = 1, \ldots, n \end{aligned}$$

These conditions state that if a constraint is nonbinding, then the associated dual variable is zero. Or equivalently, if a dual variable is positive, then the corresponding constraint is binding.

Now, let $x_{s,i}$ be the slack variable in primal constraint i and let λ_j be the surplus variable in dual constraint j. That is,

$$\begin{aligned} \mathbf{a}^i\mathbf{X} + x_{s,i} &= b_i \\ \mathbf{Ya}_j - \lambda_j &= c_j \end{aligned}$$

or, in matrix form

$$\begin{aligned} \mathbf{AX} + \mathbf{X}_s &= \mathbf{b} \\ \mathbf{YA} - \lambda &= \mathbf{c} \end{aligned}$$

Then, the complementary slackness conditions can be written as

$$\begin{aligned} y_i x_{s,i} = 0, &\qquad \text{for all } i = 1, \ldots, m \\ \lambda_j x_j = 0, &\qquad \text{for all } j = 1, \ldots, n \end{aligned}$$

And because $\mathbf{X}, \mathbf{X}_s, \mathbf{Y}$ and λ are all nonnegative, the complementary slackness condition can also be written in the compact form:

$$\begin{aligned} \mathbf{YX}_s &= \sum_{i=1}^{m} y_i x_{s,i} = 0 \\ \lambda\mathbf{X} &= \sum_{j=1}^{n} \lambda_j x_j = 0 \end{aligned}$$

or

$$\mathbf{YX}_s + \lambda\mathbf{X} = 0$$

Theorem 4.4. ***(Complementary Slackness)***

Consider the following primal-dual pair that has been converted to standard form by adding the appropriate slack/surplus variables.

$$\text{(P) maximize } \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} + \mathbf{X}_s &= \mathbf{b} \\ \mathbf{X}, \mathbf{X}_s &\geq \mathbf{0} \end{aligned}$$

$$\text{(D) minimize } \mathbf{Yb}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{YA} - \lambda &= \mathbf{c} \\ \mathbf{Y}, \lambda &\geq \mathbf{0} \end{aligned}$$

Let $(\overline{\mathbf{X}}, \overline{\mathbf{X}}_s)$ be feasible to (P) and let $(\overline{\mathbf{Y}}, \overline{\lambda})$ be feasible to (D). Then $(\overline{\mathbf{X}}, \overline{\mathbf{X}}_s)$ is optimal to (P) and $(\overline{\mathbf{Y}}, \overline{\lambda})$ is optimal to (D) if and only if complementary slackness holds.

Proof. Because $(\overline{\mathbf{X}}, \overline{\mathbf{X}}_s)$ is feasible to (P), it follows that

$$\begin{aligned} \mathbf{A}\overline{\mathbf{X}} + \overline{\mathbf{X}}_s &= \mathbf{b} \\ \overline{\mathbf{X}}, \overline{\mathbf{X}}_s &\geq \mathbf{0} \end{aligned}$$

Similarly, $(\overline{\mathbf{Y}}, \overline{\lambda})$ is feasible to (D) implies that

$$\begin{aligned} \overline{\mathbf{Y}}\mathbf{A} - \overline{\lambda} &= \mathbf{c} \\ \overline{\mathbf{Y}}, \overline{\lambda} &\geq \mathbf{0} \end{aligned}$$

Now,

$$\begin{aligned} \overline{\mathbf{Y}}\mathbf{A}\overline{\mathbf{X}} + \overline{\mathbf{Y}\mathbf{X}}_s &= \overline{\mathbf{Y}}\mathbf{b} \\ \overline{\mathbf{Y}}\mathbf{A}\overline{\mathbf{X}} - \overline{\lambda\mathbf{X}} &= \mathbf{c}\overline{\mathbf{X}} \end{aligned}$$

Subtracting, we get

$$\overline{\mathbf{Y}\mathbf{X}}_s + \overline{\lambda\mathbf{X}} = \overline{\mathbf{Y}}\mathbf{b} - \mathbf{c}\overline{\mathbf{X}}$$

Note that because all variables are nonnegative, it follows that the left side of the above equation is zero if and only if complementary slackness holds. That is, the primal and dual solutions have the same objective value ($\overline{\mathbf{Y}}\mathbf{b} = \mathbf{c}\overline{\mathbf{X}}$) if and only if complementary slackness holds. The theorem then follows directly from Corollary 4.2.5 . □

The previous theorem establishes that in linear programming, primal and dual solutions satisfy the complementary slackness conditions if and only if they have the same objective value. Thus the optimality conditions for a linear programming problem consists of three important components: *primal feasibility, dual feasibility,* and *complementary slackness.*

Example 4.7. Using the Dual to Solve the Primal

Consider the following linear programming problem:

maximize $10x_1 + 6x_2 - 4x_3 + x_4 + 12x_5$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_5 &\leq 18 \\ x_1 + x_2 - x_3 + x_4 + 2x_5 &\leq 6 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

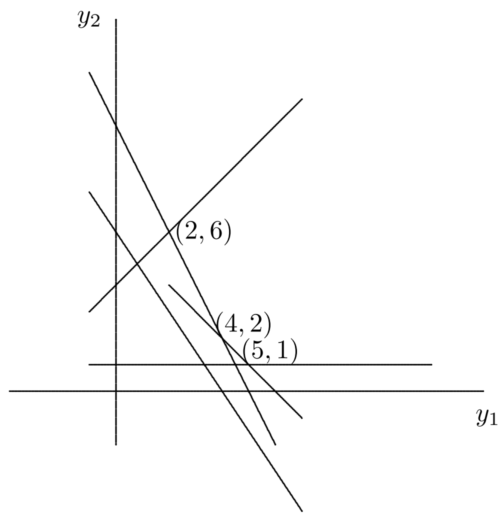
By denoting the dual variables by $\mathbf{Y} = (y_1, y_2)$, the dual problem is

minimize $18y_1 + 6y_2$

subject to

$$\begin{aligned} 2y_1 + y_2 &\geq 10 \\ y_1 + y_2 &\geq 6 \\ y_1 - y_2 &\geq -4 \\ y_2 &\geq 1 \\ 3y_1 + 2y_2 &\geq 12 \\ y_1, y_2, &\geq 0 \end{aligned}$$

Note that the dual problem involves only two decision variables, and thus, unlike the primal, the dual problem can be solved graphically. The feasible region is shown below.



We see that the optimal solution of the dual is $y_1^* = 2, y_2^* = 6$, with $Z^* = 72$.

Note from the graph that constraints 1 and 3 are binding at the optimal solution, whereas all other dual constraints are nonbinding at the optimal solution. Thus, by complementary slackness, the primal variables (x_2, x_4, x_5) corresponding to the nonbinding dual constraints are zero. Also, because $y_1 > 0$ and $y_2 > 0$, the corresponding primal constraints must be binding. Therefore, using these observations regarding complementary slackness, the primal constraint set reduces to

$$\begin{aligned} 2x_1 + x_3 &= 18 \\ x_1 - x_3 &= 6 \end{aligned}$$

Solving this simple system of linear equations yields the unique solution, $x_1 = 8$ and $x_3 = 2$, which is precisely the optimal primal solution.Also note that $z = 10(8) + 6(0) - 4(2) + 1(0) + 12(0) = 72$, which is exactly the same objective value as generated by the dual. Thus, we have solved the dual problem graphically and used the optimal dual solution along with the complementary slackness conditions to derive the optimal solution to the primal.

4.3 Economic Interpretation of the Dual

Consider again Example 4.1 in the context of a simple product-mix problem. That is, suppose that we are computing the optimal mix of three products (x_1, x_2, x_3) subject to two resource constraints.

$$\text{maximize } c_1x_1 + c_2x_2 + c_3x_3 \qquad \text{(Profit in \$)}$$

$$\text{subject to}$$

$$\begin{array}{rcll} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 & \leq & b_1 & \text{(Units of Resource 1)} \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 & \leq & b_2 & \text{(Units of Resource 2)} \\ \mathbf{X} & \geq & \mathbf{0} & \end{array}$$

Notice that we presently have b_1 units of resource 1 and b_2 units of resource 2. Suppose that a buyer offers to purchase our resources for \$$y_1$/unit of resource 1 and \$$y_2$/unit of resource 2. What restrictions would we place on the prices y_1, y_2?

First of all, we would place the restriction that y_1, y_2 be nonnegative. Next, we would require that the revenue gained from the sale of the resources offsets the loss in profit due to the resulting decrease in production. Suppose we were to forego the production of one unit of product $1(x_1)$. Although we lose $\$c_1$ in profit, the decrease in production releases $a_{1,1}$ units of resource 1 and $a_{2,1}$ units of resource 2, which may be sold to the buyer for $\$y_1$ and $\$y_2$ per unit, respectively, resulting in additional income of $a_{1,1}y_1 + a_{2,1}y_2$. Thus, in order to break even, we would require that the prices $\$y_1, \$y_2$ satisfy the following constraint

$$a_{1,1}y_1 + a_{2,1}y_2 \geq c_1$$

In an analogous manner, decreasing the production of products 2 and 3 would result in the following constraints:

$$\begin{aligned} a_{1,2}y_1 + a_{2,2}y_2 &\geq c_2 \\ a_{1,3}y_1 + a_{2,3}y_2 &\geq c_3 \end{aligned}$$

Observe that these restrictions are precisely the constraints of the dual problem. Thus if the prices y_1, y_2 satisfy these constraints, we will lose no money by accepting buyer's offer.

Now consider the buyer's point of view. Based on the offer to purchase our resources, the buyer will pay us an amount given by

$$\text{minimize } Z = b_1y_1 + b_2y_2$$

$$\text{subject to}$$

$$\begin{aligned} a_{1,1}y_1 + a_{2,1}y_2 &\geq c_1 \\ a_{1,2}y_1 + a_{2,2}y_2 &\geq c_2 \\ a_{1,3}y_1 + a_{2,3}y_2 &\geq c_3 \\ \mathbf{Y} &\geq \mathbf{0} \end{aligned}$$

But this is precisely the dual problem. Therefore, the dual problem can be thought of as the problem of determining fair market prices of the resources. For this reason, the dual variables are often referred to as *shadow prices* or *dual prices*.

4.3.1 The Dual Variables as Rates of Change

Consider the canonical form of the primal and dual in summation form:

$$\text{(P) maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to}$$

$$\begin{aligned} \sum_{j=1}^{n} a_{i,j}x_j &\leq b_i, \quad \text{for } i = 1, \ldots m \\ x_j &\geq 0, \quad \text{for } j = 1, \ldots n \end{aligned}$$

$$\text{(D) minimize } Z = \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to}$$

$$\begin{aligned} \sum_{i=1}^{m} a_{i,j}y_j &\geq c_j, \quad \text{for } j = 1, \ldots n \\ y_i &\geq 0, \quad \text{for } i = 1, \ldots m \end{aligned}$$

Notice that in the primal, we may define the dimensions of each parameter as follows:

$$\begin{aligned} z &= \text{return} \\ x_j &= \text{units of variable } j \\ c_j &= \text{return/(units of variable } j) \\ b_i &= \text{units of variable } i \\ c_{i,j} &= \text{(units of resource } i\text{)/(unit of variable } j) \end{aligned}$$

The only new parameters introduced in the dual formulation are Z and $\mathbf{Y} = (y_1, \ldots, y_m)$. It is obvious that because $z^* = Z^*$, the dimension of Z is in terms of "return." The question remaining is: What are the dimensions associated with the dual variables, y_i?

Noting that the dual objective is in the form

$$Z = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

we see that

$$\frac{\partial Z}{\partial b_i} = y_i$$

That is, y_i is the rate of change of the objective with respect to b_i. Thus, the dual variable is expressed in terms of the *return per unit of resource i.*

Given an optimal solution to a linear programming problem, the dual variable then indicates the per unit contribution of the ith resource toward the increase in the (presently) optimal value of the objective. For example, if $y_3 = 9$, we interpret this to mean that for every unit of resource 3(the resource associated with constraint 3 in the primal), the objective (z) will increase by 9 units.

Not only is this a useful interpretation, it is often of greater importance and interest than the optimal solution. The reason is that most companies wish to improve on the status quo. The optimal solution ($\mathbf{z}^*$ and $\mathbf{X}^*$) to a linear programming problem tells them only how to best allocate their resources for their present state. The dual variables, on the other hand, provide the company with the information needed to expand and to increase profit, if one knows how to interpret them.

Now consider again the complementary slackness conditions. Previously, we showed that the dual variables give the rate of change of the objective function with respect to the right-hand side of the associated primal constraint. Now suppose that primal constraint i is nonbinding at the optimal solution, that is, $\mathbf{a}^i\mathbf{X} < b_i$. Then we have an excess of resource i, and acquiring more of this resource will have no effect on the objective because we were not able to use all our current supply. Therefore, the rate of change of the objective with respect to resource i is zero, that is, $y_i = 0$. On the other hand, suppose $\partial z/\partial b_i = y_i > 0$. Then acquiring more of this resource will improve the objective function. However, because the current solution is optimal, the current supply of the resource must be depleted, and this the associated constraint is binding, that is $\mathbf{a}^i\mathbf{X} = b_i$.

Example 4.8. Economic Interpretation of the Dual

A company manufactures two products. Forecasting indicates that the maximum weekly demand for Product A is 900 units and for Product B is 600 units. The manufacture of each product requires raw material and two basic operations. The per unit consumption of the raw material and resources is summarized in the table below. Finally, the profit per unit of Product A is \$7 and the profit per unit of Product B is \$12.

Product	Forging (hr)	Machining (hr)	Raw material
A	0.15	0.10	2.5
B	0.20	0.20	4.0
Available per Week	200 hours	140 hours	3200 pounds

Now, let

$$
\begin{aligned}
x_1 &= \text{number of units of Product A manufactured per week} \\
x_2 &= \text{number of units of Product B manufactured per week}
\end{aligned}
$$

Then the linear programming model for this problem is as follows, with the optimal tableau shown below.

maximize $z = 7x_1 + 12x_2$ (Weekly profit in \$)
subject to

$$
\begin{array}{rcll}
0.15x_1 + 0.2x_2 & \leq & 200 & \text{Forging-capacity limit} \\
0.1x_1 + 0.2x_2 & \leq & 140 & \text{Machining-capacity limit} \\
2.5x_1 + 4.0x_2 & \leq & 3200 & \text{Raw-material limit} \\
x_1 & \leq & 900 & \text{Demand limit for product A} \\
x_2 & \leq & 600 & \text{Demand limit for product B} \\
x_1, x_2 & \geq & 0 &
\end{array}
$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	0	0	20	2	0	0	9200
x_3	0	0	0	1	1	-0.1	0	0	20
x_1	0	1	0	0	-40	2	0	0	800
x_2	0	0	1	0	25	-1	0	0	300
x_6	0	0	0	0	40	-2	1	0	100
x_7	0	0	0	0	-25	1	0	1	300

Now by letting y_1, y_2, y_3, y_4 and y_5 denote the dual variables corresponding to the constraints, the optimal primal and dual solutions can be read from the above tableau as follows:

$$
\begin{array}{rcl}
x_1^* & = & \text{800 units of Product A} \\
x_2^* & = & \text{300 units of Product B} \\
z^* & = & Z^* = \$92{,}000 \text{ (weekly profit)} \\
y_1^* & = & \text{0 per unit of resource 1 (\$/hr of forging)} \\
y_2^* & = & \text{\$20 per unit of resource 2 (\$/hr of machining)} \\
y_3^* & = & \text{\$2 per unit of resource 3 (\$/lb of raw material)} \\
y_4^* & = & \text{\$0 per unit of resource 4 (\$/unit of demand of Product A)} \\
y_5^* & = & \text{\$0 per unit of resource 5 (\$/unit of demand of Product B)}
\end{array}
$$

Notice that because $y_2 = 20$, the company may increase weekly profit by \$20 for each additional hour of machining time. Thus, if it is possible to purchase additional machining time for less that \$20 per hour, then profit will increase as a result. That is, if an additional hour of machining time actually costs $\$\alpha$, then the company will receive $\$(20 - \alpha)$ for each additional hour purchased. Obviously, there has to be an upper limit to this result, otherwise, the amount of additional profit is unlimited.

In addition, we see that the value of raw materials, over the maximum of 3200 pounds now available, is \$2 per pound per week. That is, the shadow price of raw materials is \$2/pound. Thus, if the company can purchase additional raw material (up to a certain limit) for less than \$2 per pound, it will increase profits.

4.4 The Dual Simplex Algorithm

From this point on, we shall refer to the original simplex algorithm as the *primal simplex* method. The dual simplex method also addresses the primal and its tableau. However, although it is the primal problem that we see in the tableau before us, it is its dual that is actually being operated on. Such an approach is possible only if the current solution is *dual feasible*. That is, primal optimality conditions are satisfied.

The basic thrust of the dual simplex algorithm is quite simple. One always attempts to retain dual feasibility (i.e., primal optimality, $z_j - c_j \geq 0$, for all j) while bringing the primal back to feasibility (i.e., $x_{B,i} \geq 0$, for all i). This may be accomplished by bringing the equivalent feasible dual solution to optimality, but all computations are performed with the primal tableau. Thus, the dual simplex algorithm maintains dual feasibility and complementary slackness throughout its operation, while trying to achieve primal feasibility. It is actually nothing more than the primal simplex method applied to the dual problem; however, we do so while utilizing the primal tableau.

The dual simplex algorithm

1. To employ this algorithm, the problem must be dual feasible, that is, all $z_j - c_j \geq 0$. If this condition is met, go to step 2.
2. *Determining the departing variable.* If $\beta_i \geq 0$ for all i, then the current solution is optimal; stop. Otherwise, select the row associated with the most negative β_i. Denote this row as row r. The basic variable $x_{B,r}$ associated with its row is the departing variable.
3. *Check for primal feasibility.* If $\alpha_{r,j} \geq 0$, for all j, then the primal problem is infeasible and the dual problem has an unbounded objective; stop. Otherwise, go to step 4.
4. *Determine the entering variable.* Use the following minimum ratio test to determine the entering basic variable. That is, let
$$\frac{z_k - c_k}{-\alpha_{r,k}} = \text{minimum}\left\{\frac{z_j - c_j}{-\alpha_{r,j}} : \alpha_{r,j} < 0\right\}$$
Column k is the pivot column, $\alpha_{r,k}$ is the pivot element, and the basic variable x_k associated with column k is the entering variable. Go to step 5.
5. *Pivot and establish a new tableau.*
   (a) The entering variable x_k is the new basic variable in row r.
   (b) Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consists of all zero elements except for a 1 at the pivot position $\alpha_{r,k}$.
   (c) Return to step 2.

Example 4.9. The Dual Simplex Method

minimize $z = 8x_1 + 15x_2$
subject to

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ 2x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Although we do not need to write down the dual problem to execute the dual simple algorithm, let us do so in this case so that we may track the solutions to both problems graphically. By letting y_1, y_2 designate the dual variables, the dual problem can be written as follows:

maximize $3y_1 + 4y_2$
subject to

$$\begin{aligned} y_1 + 2y_2 &\leq 8 \\ y_1 + y_2 &\leq 5 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Rather than preprocessing the primal problem as usual, instead we shall change the objective to maximization form and multiply both constraints through by -1. Adding slack variables x_3 and x_4,

maximize $z' = -8x_1 - 15x_2$
subject to

$$\begin{aligned} -x_1 - x_2 + x_3 &= -3 \\ -2x_1 - x_2 + x_4 &= -4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The initial tableau for the resulting problem is then given in the table below. Notice that the initial basis is primal infeasible ($x_3 = -3$ and $x_4 = -4$) and dual feasible (all $z_j - c_j \geq 0$). Thus, the dual simplex algorithm can be employed.

	z'	x_1	x_2	x_3	x_4	RHS
z'	1	8	5	0	0	0
x_3	0	-1	-1	1	0	-3
x_4	0	-2	-1	0	1	-4

We will add another row for the ratios $(z_k - c_k)/(-\alpha_{r,k})$ and apply the dual simplex algorithm.

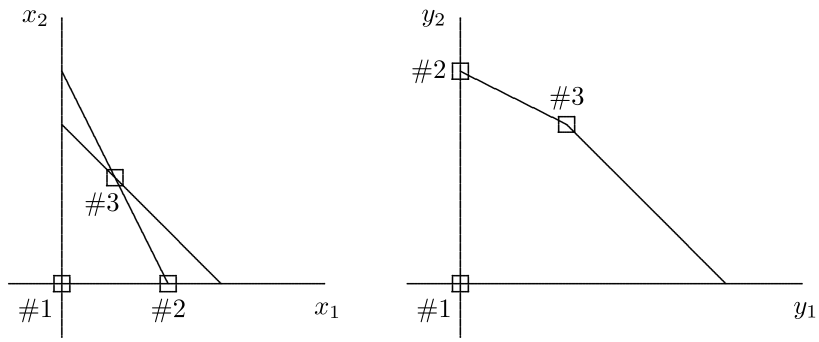
	z'	x_1	x_2	x_3	x_4	RHS
z'	1	8	5	0	0	0
x_3	0	-1	-1	1	0	-3
x_4	0	-2	-1	0	1	-4
Ratio		4	5			
z'	1	0	1	0	4	-16
x_3	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	2
Ratio			2		8	
z'	1	0	0	2	3	-18
x_2	0	0	1	-2	1	2
x_1	0	1	0	1	-1	1

We stop since both primal and dual feasibility are satisfied. Note that both the complementary primal and dual solutions can be read from the optimal tableau. These solutions are listed in the table below. Observe that complementary slackness is satisfied and both solutions correspond to an objective value $z^* = -z' = 18$.

COMPLEMENTARY
PRIMAL AND DUAL SOLUTIONS

Primal solution	Complementary dual solution
$x_1 = 1$	$y_3 = 0$
$x_2 = 2$	$y_4 = 0$
$x_3 = 0$	$y_1 = 2$
$x_4 = 0$	$y_2 = 3$

The following figure depicts the sequence of iterations followed by the primal and dual problems in their respective feasible regions. Because dual feasibility was maintained throughout the algorithm, note that the optimal solution was obtained as soon as the primal feasible region was reached.



4.5 Sensitivity Analysis in Linear Programming

Data used for linear programming problems may be subject to error, cost and resource availabilities can change with time, and the system itself may be modified. Management must deal with the future as well as the present, and deal with various price and commodity fluctuations, so it should be apparent that an approach is needed to include such considerations within the linear programming technique. This approach is generally referred to as *sensitivity analysis.*

In this section, we present an approach for determining the impact of discrete changes in linear programming model structure on the resulting problem solution. In the following section, we consider continuous systematic changes via a technique called *parametric programming.*

To clarify the difference between discrete and continuous changes, consider analyzing the impact of one factor, say c_3 (i.e., the ”‘cost”’ coefficient associated with decision variable x_3. Now, if we wish to determine the impact of a change in the value of c_3 from its present value (c_3) to a specific new value (c_3'), we are dealing with a discrete change. However, if what we want to examine is the impact over the entire range of values from, say, ($\overline{c_3}$) to ($\widetilde{c_3}$), then the problem deals with a parametric change.

Example 4.10. We shall refer to the following example frequently throughout this section.

maximize $z = 10x_1 + 7x_2 + 6x_3$ (Return in \$)

subject to

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &\leq 36 \qquad \text{(Units of Resource 1)}\\
x_1 + x_2 + 2x_3 &\leq 32 \qquad \text{(Units of Resource 2)}\\
2x_1 + x_2 + x_3 &\leq 22 \qquad \text{(Units of Resource 3)}\\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

By letting x_4, x_5 and x_6 denote the slack variables for the constraints above, the data for the problem in standard form can be summarized as follows:

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}\\
\mathbf{b} &= \begin{pmatrix} 36 \\ 32 \\ 22 \end{pmatrix}\\
\mathbf{c} &= (10\ 7\ 6\ 0\ 0\ 0)
\end{aligned}
$$

The optimal tableau is shown below.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	0	0	1	0	5	146
x_2	0	1	1	0	1	0	-1	14
x_5	0	-2	0	0	1	1	-3	2
x_3	0	1	0	1	-1	0	2	8

First note that the optimal solution is given by

$$\begin{aligned} z^* &= 146 \\ \mathbf{X}_B^* &= \begin{pmatrix} x_2 \\ x_5 \\ x_3 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \beta = \begin{pmatrix} 14 \\ 2 \\ 8 \end{pmatrix} \\ \mathbf{X}_N^* &= \begin{pmatrix} x_1 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Next, observe that the original identity was associated with slack variables x_4, x_5, and x_6, that is, $\mathbf{I} = (\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6$. Therefore the basis inverse $\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{I} = (\alpha_4, \alpha_5, \alpha_6)$ and may be read directly from the optimal tableau as

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}$$

Now, let $\mathbf{Y} = (y_1, y_2, y_3)$ represent the dual variables corresponding to the constraints. Then, again using the information in the above tableau, we see that

$$\mathbf{Y} = (y_1, y_2, y_3) = \mathbf{c}_B\mathbf{B}^{-1} = (1\ 0\ 5)$$

Which resources in this problem are fully utilized (or scarce)? This question is answered quite readily by noting the values of the slack variables. That is, because $x_4 = x_6 = 0$, we see that resources 1 and 3 are scarce resources. However, $x_5 = 2$ and thus we have 2 units or resource 2 remaining. Note also that y_1, y_2 and y_3 are the respective shadow prices of resources 1, 2 and 3. As expected, $y_2 = 0$ because resource 2 is not fully utilized. That is, acquiring more of resource 2 will not affect the value of the objective function. However, the rate of change of the objective with respect to additional units of resources 1 and 3 are given by the shadow prices, $y_1 = 1$ and $y_3 = 5$. That is, for each additional unit of resource 1, the objective value (return) will increase by \$1. It is important to realize that this rate of change is only valid for relatively small increases in the quantity of resource 1. More specifically, it is only valid as long as the current basis remain optimal. Likewise, for each additional unit of resource 3 up to a certain limit, the objective value will increase by \$5.

4.5.1 Changes in Objective Coefficients

1. **A change in the c_j of a nonbasic variable**

   Let us first consider a change in the cost coefficient c_k of a nonbasic variable x_k. Note that $\mathbf{c}_B$ is not affected; thus, the only impact of such a change is on the single tableau element, $z_k - c_k$. By letting c'_k be the new value of c_k, then $z_k - c'_k$ will replace $z_k - c_k$ in the optimal tableau. Of course, if $z_k - c'_k$ remains nonnegative, then the current basis remains optimal. However, if $z_k - c'_k < 0$, then dual feasibility (primal optimality) has been lost and must be restored by using the primal simplex method. The value of $z_k - c'_k$ can be computed quite easily using the following relationship.

$$\begin{aligned} z_k - c'_k &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c'_j \\ &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_k - c_k + c_k - c'_k = (z_k - c_k) + (c_k - c'_k) \end{aligned}$$

2. **A change in the c_k of a basic variable**

   Now consider a change in the cost coefficient c_k associated with a basic variable x_k. Because x_k is a basic variable, a change in c_k results in a change in the $\mathbf{c}_B$ vector. Thus, such a change can affect any or all of the $z_j - c_j$ elements and the value of z. Let c'_k be the new value of c_k and let $\mathbf{c}'_B$ denote the revised $\mathbf{c}_B$. The $z_j - c_j$ elements associated with the basic variables will remain zero, so we only need to update the $z_j - c_j$ for the nonbasic variables as follows:

$$z'_j - c_j = \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{c}'_B \alpha_j - c_j, \text{ for all nonbasic variables } x_j$$

   In addition, the updated value of the objective function is given by

$$z'_j = \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}'_B \beta$$

   If some $z'_j - c_j$ is negative, then dual feasibility (primal optimality) must be restored using the primal simplex method.

Example 4.11. A change in a nonbasic $\mathbf{c}_j$

Consider again the problem of Example 4.10.

maximize $z = 10x_1 + 7x_2 + 6x_3$ (Return in $)
subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\leq 36 \quad \text{(Units of Resource 1)} \\ x_1 + x_2 + 2x_3 &\leq 32 \quad \text{(Units of Resource 2)} \\ 2x_1 + x_2 + x_3 &\leq 22 \quad \text{(Units of Resource 3)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	0	0	1	0	5	146
x_2	0	1	1	0	1	0	-1	14
x_5	0	-2	0	0	1	1	-3	2
x_3	0	1	0	1	-1	0	2	8

Notice that x_1 is nonbasic in this final tableau. Let us assume that the unit return of x_1 changes from its present value of $c_1 = 10$ to $c'_1 = 14$. Then,

$$z_1 - c'_1 = (z_1 - c_1) + (c_1 - c'_1) = 3 + (10 - 14) = -1$$

or equivalently, one could compute

$$z_1 - c'_1 = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_1 - c'_1 = \mathbf{c}_B \alpha_1 - c'_1 = (7\ 0\ 6) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - 14 = -6$$

Thus, primal optimality (dual feasibility) has been lost. Therefore, we must perform at least one primal simplex pivot to restore optimality.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	1	0	0	7	154
x_2	0	0	1	-1	2	0	-3	6
x_5	0	0	0	2	-1	1	1	18
x_1	0	1	0	1	-1	0	2	8

Example 4.12. A change in a basic $\mathbf{c}_j$

Given the original linear programming model of Example 4.10, let us assume that the value of c_3 should be 5 rather that 6. Then,

$$\begin{aligned}
z_1' - c_1 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_1 - c_1' = \mathbf{c}_B\alpha_1 - c_1' = (7\ 0\ 5)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - 10 = 2 \\
z_4' - c_4 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_4 - c_4' = \mathbf{c}_B\alpha_4 - c_4' = (7\ 0\ 5)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 0 = 2 \\
z_6' - c_6 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_6 - c_6' = \mathbf{c}_B\alpha_6 - c_6' = (7\ 0\ 5)\begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} - 0 = 3 \\
z' &= \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B'\beta = (7\ 0\ 5)\begin{pmatrix} 14 \\ 2 \\ 8 \end{pmatrix} = 138
\end{aligned}$$

In this example, although the values of the entire top rows changed, the optimal basis remains the same, and thus no further processing is required. Note, however, that the optimal value of the objective has changed.

4.5.2 Changes in the Right-Hand Side

Recalling that $\mathbf{X}_B$ is given simply by $\mathbf{B}^{-1}\mathbf{b}$ and recalling that $\mathbf{B}^{-1}$ can be found from the tableau by a proper arrangement of the α_j column vectors, we have

$$\mathbf{X}_B' = \mathbf{B}^{-1}\mathbf{b}'$$

where

$$\begin{aligned}
\mathbf{X}_B' &= \text{new values of the basic variables in the tableau of interest} \\
\mathbf{B}^{-1} &= \text{inverse of the present basis matrix} \\
\mathbf{b}' &= \text{new set of right-hand side constraints}
\end{aligned}$$

Also,

$$z' = \mathbf{c}_B'\mathbf{X}_B' = \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{b}'$$

The basis inverse $\mathbf{B}^{-1}$ may contain negative elements, and thus there is always a possibility that $\mathbf{X}_B'$ may include some negative elements. That is, there is always a possibility that we may lose primal feasibility. However, because the dual simplex algorithm may be used to regain primal feasibility.

Example 4.13. A change in a b_i

Consider again the problem of Example 4.10 when b_3 is changed from 22 to 24.

As we observed in Example 4.10, the inverse basis matrix is given by

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}$$

Thus, we have

$$\begin{aligned}
\mathbf{X}'_B &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 36 \\ 32 \\ 24 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 12 \end{pmatrix} \\
z' &= (7\ 0\ 6) \begin{pmatrix} 12 \\ -4 \\ 12 \end{pmatrix} = 156
\end{aligned}$$

The new updated tableau is given below. Note that we have lost primal feasibility, and thus the dual simplex is used to obtain the new final tableau.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	0	0	1	0	5	156
x_2	0	1	1	0	1	0	-1	12
x_5	0	-2	0	0	1	1	-3	-4
x_3	0	1	0	1	-1	0	2	12
z	1	0	0	0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	140
x_2	0	0	1	0	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	10
x_1	0	1	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	2
x_3	0	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	10

4.5.3 Changes in the Technological Coefficients $a_{i,j}$

Discrete changes in the technological coefficients are relatively easy to handle if the $a_{i,j}$ to be changed are associated with a nonbasic variable. However, a change in an $a_{i,j}$ associated with a basic variable is considerably more involved, and thus, for such a case, we shall resort to simply resolving the problem from the beginning.

Restricting our attention then to changes in technological coefficients of nonbasic variables, we note that any change in the $\mathbf{a}_k$ column for a nonbasic variable x_k will directly affect the associated α_k vector (and, indirectly, the value of $z_k - c_k$. At any iteration, the α_k column vector is given by $\mathbf{B}^{-1}\mathbf{a}_k$, so we have

$$\alpha'_k = \mathbf{B}^{-1}\mathbf{a}'_k$$

where

$$\begin{aligned}
\mathbf{B}^{-1} &= \text{inverse of the present basis matrix} \\
\mathbf{a}'_k &= \text{new vector of technological coefficients} \\
&= \text{with nonbasic variable } x_k \\
\alpha'_k &= \text{new updated vector corresponding to} \\
&= x_k \text{ in the final simplex tableau}
\end{aligned}$$

the updated $z_k - c_k$ is given by

$$z'_k - c_k = \mathbf{c}_B\alpha'_k - c_k = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}'_k - c_k$$

and if $z'_k - c_k$ should go negative, primal optimality (dual feasibility) has been lost and the primal simplex method must be applied.

Example 4.14. A Change in a nonbasic coefficients $a_{i,j}$

Let us assume for the original problem given in Example 4.10, that we are informed that an error was made in data collection and that $a_{1,3}$ is really 1 rather than 2. Thus,

$$\mathbf{a}'_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$\begin{aligned} \mathbf{a}'_1 &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ z'_1 - c_1 &= (7\ 0\ 6) \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - 10 = -2 \end{aligned}$$

The resultant tableau, after the change in $a_{1,3}$, is shown below. The solution is no longer optimal and we apply the primal simplex method.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-2	0	0	1	0	5	146
x_2	0	2	1	0	1	0	-1	14
x_5	0	1	0	0	1	1	-3	2
x_3	0	-1	0	1	-1	0	2	8
z	1	0	0	0	3	2	-1	150
x_2	0	0	1	0	-1	-2	5	10
x_1	0	1	0	0	1	1	-3	2
x_3	0	0	0	1	0	1	-1	10
z	1	0	$\frac{1}{5}$	0	$\frac{14}{5}$	$\frac{8}{5}$	0	152
x_6	0	0	$\frac{1}{5}$	0	$-\frac{1}{5}$	$-\frac{2}{5}$	1	2
x_1	0	1	$\frac{3}{5}$	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	8
x_3	0	0	$\frac{1}{5}$	1	$-\frac{1}{5}$	$\frac{3}{5}$	0	12

4.5.4 Addition of a New Variable

Let us assume that once we have solved a linear programming problem for a company, they ask us to consider the impact of the introduction of a new product. This new product must be represented by a new decision variable.

The introduction of a new decision variable will wither affect the optimality (dual feasibility) of the present solution or else have no effect at all. In the first case, we must introduce the new x_j into the basis. In the second, the new x_j stays nonbasic (i.e., we would not introduce the new product into the market).

Let us assume that x_k is the new decision variable and the data associated with x_k is given by c_k and $\mathbf{a}_k$. That is, c_k is the objective coefficient of x_k and the vector $\mathbf{a}_k$ specifies the coefficients of x_k in the existing constraints. Because $\mathbf{a}_k$ specifies the rates at which x_k will be consuming the various resources, $\mathbf{a}_k$ is often referred to as a consumption vector. The process of checking the optimality condition of x_k is actually straightforward. We simply compute $z_k - c_k$ in the usual manner, that is,

$$z_k - c_k = \mathbf{c}_B \alpha_k - c_k = \mathbf{Y}\mathbf{a}'_k - c_k$$

If $z_k - c_k \geq 0$, then the present solution remains optimal. However, if $z_k - c_k < 0$, then x_k should be introduced into the basis by performing a primal simplex pivot in the updated x_k column given by

$$\alpha_k = \mathbf{B}^{-1}\mathbf{a}_k$$

Example 4.15. Addition of a new variable

Recall the original model of Example 4.10,

maximize $z = 10x_1 + 7x_2 + 6x_3$
subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + x_4 &= 36 \\ x_1 + x_2 + 2x_3 + x_5 &= 32 \\ 2x_1 + x_2 + x_3 + x_6 &= 22 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Recall that the optimal dual solution from the optimal tableau is

$$\mathbf{Y} = (y_1, y_2, y_3) = \mathbf{c}_B \mathbf{B}^{-1} = (1\ 0\ 5)$$

Let us now assume that a new decision variable, say, x_7 is to be evaluated with the following data

$$\begin{aligned} c_7 &= 8 \\ \mathbf{a}_7 &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \end{aligned}$$

Checking primal optimality (dual feasibility), we see that,

$$z_7 - c_7 = \mathbf{Y}\mathbf{a}_7 - c_7 = (1\ 0\ 5)\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - 8 = -1$$

Because $z_7 - c_7 < 0$, primal optimality has been lost. Therefore, we need to compute the updated x_7 column.

$$\alpha_7 = \mathbf{B}^{-1}\mathbf{a}_7 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

This results in the following tableau. You are invited to complete this example by using the primal simplex to restore optimality.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	3	0	0	1	0	5	-1	146
x_2	0	1	1	0	1	0	-1	1	14
x_5	0	-2	0	0	1	1	-3	2	2
x_3	0	1	0	1	-1	0	2	0	8

4.5.5 Addition of a New Constraint

It is easy to determine whether a new constraint has an impact (i.e., without yet evaluating the actual measure of the impact). We simply evaluate the constraint at the present basic feasible solution ($\mathbf{X}^*$). If the constraint is satisfied, there is no impact and we go no further. However, if the constrains is violated at $\mathbf{X}^*$, the constraint will have an impact on the solution and we proceed to the next phase of our analysis.

Having determined that a new constraint will affect the present solution, we proceed to incorporate this new constraint into the previous final tableau. The slack variable for the new constraint will enter the basis and a new row must be added to the tableau. However, when adding this new row to the tableau, we must return the tableau to canonical form by "eliminating" the coefficients of any variables in the new constraint that are basic in the previous final tableau. We accomplish this through simple matrix row operations that are illustrated via the following example.

Example 4.16. Addition of a new constraint

Returning again to our problem of Example 4.10, let us evaluate the impact of adding the following constraint:

$$x_1 + x_2 + x_3 \leq 20$$

Examining the optimal tableau, we see that $x_1^* = 0, x_2^* = 14$, and $x_3^* = 8$. Thus, $x_1 + x_2 + x_3 = 22$ at the current optimal solution, the new constraint is violated and will have an impact on the solution. We first rewrite the constraint as

$$x_1 + x_2 + x_3 + x_7 = 20$$

where x_7 is a zero-cost slack variable. Now adding this constraint to the optimal tableau, we have the following table. Note however, that adding the constraint in this manner destroys the canonical form of the tableau. That is, x_2 and $x_3$are basic variables, but the columns of the tableau associated with x_2 and x_3 no longer correspond to columns of the identity. However, if it is a straightforward task to restore the tableau to canonical form using simplex matrix operations. This is done by multiplying constraint rows and 3 by -1 and adding them to the new constraint row.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	3	0	0	1	0	5	0	146
x_2	0	1	1	0	1	0	-1	0	14
x_5	0	-2	0	0	1	1	-3	0	2
x_3	0	1	0	1	-1	0	2	0	8
x_7	0	1	1	1	0	0	0	1	20
z	1	3	0	0	1	0	5	0	146
x_2	0	1	1	0	1	0	-1	0	14
x_5	0	-2	0	0	1	1	-3	0	2
x_3	0	1	0	1	-1	0	2	0	8
x_7	0	-1	0	0	0	0	-1	1	-2
z	1	0	0	0	1	0	2	3	140
x_2	0	0	1	0	1	0	-2	1	12
x_5	0	0	0	0	1	1	5	-2	6
x_3	0	0	0	1	-1	0	1	1	6
x_1	0	1	0	0	0	0	1	-1	2

4.6 Parametric Programming

We now consider the impact over a range of variation in model structure, but we only examine these systematic changes in the cost vector $\mathbf{c}$ and the right-hand-side vector $\mathbf{b}$. Evaluation of other parameters, over a range, is also possible but tends to be much more involved. Fortunately, for the majority of practical cases, the c_j's and the b_i's are the data of major interest.

4.6.1 Systematic Variation of the Cost Vector c

We consider first an evaluation of the impact on the original problem solution of a systematic variation of the objective function coefficients. This is modeled by perturbing the original objective coefficients via a scalar parameter t and a perturbation vector $\mathbf{c}$'. The resulting model can be written as follows:

$$\text{maximize } z = \mathbf{cX} + t\mathbf{c}'\mathbf{X} = (\mathbf{c} + t\mathbf{c}')\mathbf{X}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

To help fix the basic idea, consider that a company's profit is a linear function of its sales of three products. Letting x_1, x_2, and x_3 be the number of units of products 1,2, and 3 sold respectively, we have

$$z = \mathbf{cX} = c_1x_1 + c_2x_2 + c_3x_3$$

where c_1, c_2, and c_3 are associated per unit profits. Now, let us consider a single parameter that we identify simply as t. Such a parameter might represent calendar time, inflation rate, or it might simply be used to investigate a relationship between the profits of x_1, x_2, and x_3. For example, an increase in unit profit of one product may result in a decrease in unit profit of another. If the profit per product can be reasonably expected to vary linearly with this parameter, our method of analysis is appropriate. As an illustration, examine the following revised objective function:

$$\begin{aligned} z &= (c_1 + t)x_1 + (c_2 - 2t)x_2 + c_3x_3 \\ &= (c_1, c_2, c_3)\mathbf{X} + t(1 \ -2 \ 0)\mathbf{X} \\ &= \mathbf{cX} + t\mathbf{c}'\mathbf{X} \end{aligned}$$

where

$$\begin{aligned} \mathbf{c} &= (c_1, c_2, c_3) \\ \mathbf{c}' &= (1 \ -2 \ 0) \\ \mathbf{X} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

Reading the function defined above, we see that the profit of x_1 increases linearly with t. The profit of product x_2, on the other hand decreases, and it does so as a function of two times t. Finally, the profit for x_3 is unaffected by t.

Note that because we are examining systematic changes in the $\mathbf{c}$ vector, only primal optimality(dual feasibility) will be affected. That is, changes in $\mathbf{c}$ have no effect on primal feasibility. Let us begin our analysis of this problem by deriving the optimality conditions for a linear program in the standard form

$$\text{maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

are given simply by

$$z_j - c_j = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j \geq 0, \quad \text{for all } j$$

Now, note that in the perturbed problem, we have simply replaced $\mathbf{c}$ by $\mathbf{c} + t\mathbf{c}'$. That is, each c_j has been replaced by $c_j + tc'_j$. Thus, by substitution,

$$(\mathbf{c}_B + t\mathbf{c}'_B)\mathbf{B}^{-1}\mathbf{a}_j - (c_j + tc'_j) \geq 0, \quad \text{for all } j$$

But by rearranging terms simplifies to

$$(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j) + t(\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{a}_j - c'_j) = (z_j - c_j) + t(z'_j - c'_j) \geq 0, \quad \text{for all } j$$

Thus, the optimality conditions of the perturbed problem combines the $z_j - c_j$ values computed with the original costs with those computed using the perturbation costs (i.e., $z'_j - c'_j$). Also note that the objective value of the perturbed problem is given by

$$z = (\mathbf{c}_B + t\mathbf{c}'_B)\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + t\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b}$$

We will handle this information in the tableau by adding an additional row to the top of the tableau. Finally, note that, as expected, the optimality conditions and objective value for the perturbed problem reduce to the original unperturbed problem when $t = 0$.

Parametric programming procedure: c vector (maximization problem)

1. Set the parameter $t = 0$ and find an optimal solution to the original problem.
2. Add an additional top row to the optimal tableau containing the $z'_j - c'_j$, which are computed using $z'_j - c'_j = \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{a}_j - c'_j$. The contribution to the objective value is given by $\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b}$.
3. Determine the parameter range over which the tableau is optimal by examining the optimality conditions

   $$(z_j - c_j) + t(z'_j - c'_j) \geq 0, \quad \text{for all } j$$

   Let this range be given by $l \leq t \leq u$, where l is the lower bound and u is the upper bound on the parameter t. (Note that the values of l and u need not be finite.)
4. If l is finite, determine which nonbasic variable has $(z_j - c_j) + t(z'_j - c'_j) = 0$ when $t = l$. Enter this variable into the basis by performing a primal simplex pivot. This will possibly result in a new tableau that is optimal for additional values of t.

   Similarly, if u is finite, determine which nonbasic variable has $(z_j - c_j) + t(z'_j - c'_j) = 0$ when $t = u$. Enter this variable into the basis by performing a primal simplex pivot. This will possibly result in a new tableau that is optimal for additional values of t.
5. Repeat steps 3 and 4 until all appropriate ranges of parameter have been investigated.

Example 4.17. Parametric programming: c vector

maximize $z = x_1 + 4x_2 + t(-x_1 + x_2) = (1 - t)x_1 + (4 + t)x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 6 \\ x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Considering the original problem with $t = 0$ and denoting the respective slack variables by x_3, x_4 and x_5 yield the optimal tableau shown below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0	1	3	18
x_3	0	0	0	1	-2	1	2
x_1	0	1	0	0	1	-1	2
x_2	0	0	1	0	0	1	4

Note that $\mathbf{c}$ and $\mathbf{c}'$ are given by:

$$\begin{aligned}\mathbf{c} &= (1\ 4\ 0\ 0\ 0)\\ \mathbf{c}' &= (1\ -1\ 0\ 0\ 0)\end{aligned}$$

Now add an additional top row to the optimal tableau containing $z'_j - c'_j = \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{a}_j - c'_j$ with objective value given by $\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b}$. The updated tableau is shown below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
		0	0	0	1	-2	-2
z	1	0	0	0	1	3	18
x_3	0	0	0	1	-2	1	2
x_1	0	1	0	0	1	-1	2
x_2	0	0	1	0	0	1	4

Examining this table, we see that the present solution is $\mathbf{X} = (2, 4, 2, 0, 0)$ and $z = 18-2t$. Now we use the optimality condition $(z_j - c_j) + t(z'_j - c'_j) \geq 0$ to determine for what range of the parameter t the current solution is optimal. The optimality conditions are

$$\begin{aligned}(z_4 - c_4) + t(z'_4 - c'_4) &= 1 + t \geq 0\\ (z_5 - c_5) + t(z'_5 - c'_5) &= 3 - 2t \geq 0\end{aligned}$$

which result in

$$-1 \leq t \leq \frac{3}{2}$$

Thus, the current tableau is optimal for $-1 \leq t \leq \frac{3}{2}$. Note that $(z_4 - c_4) + t(z'_4 - c'_4) = 0$ when $t = -1$. Therefore, an alternative optimal solution exists for this tableau, which may be found by entering x_4 via a primal simplex pivot. The departing variable is x_1 and the new tableau is shown below.

	z	x_1	x_2	x_3	x_4	x_5	RHS
		-1	0	0	0	-1	-4
z	1	-1	0	0	0	4	16
x_3	0	2	0	1	0	-1	6
x_4	0	1	0	0	1	-1	2
x_2	0	0	1	0	0	1	4

From this table, we see that this is optimal if

$$\begin{aligned}-1 - t &\geq 0\\ 4 - t &\geq 0\end{aligned}$$

That is, for $-\infty \leq t \leq -1, \mathbf{X} = (0, 4, 6, 2, 0)$ and $z = 16 - 4t$. Note that because we have already examined the bound $t = -1$, no finite bounds remain to be examined for the above tableau.

We now examine the case of $t = \frac{3}{2}$. In this case, we perform a primal pivot entering x_5 into the basis. The following table shows the resulting tableau.

	z	x_1	x_2	x_3	x_4	x_5	RHS
		0	0	2	-3	0	2
z	1	0	0	-3	7	0	12
x_5	0	0	0	1	-2	1	2
x_1	0	1	0	1	-1	0	4
x_2	0	0	1	-1	2	0	2

Examining this table, we find that $\mathbf{X} = (4, 2, 0, 0, 2)$ and $z = 12 + 2t$ for $\frac{3}{2} \leq t \leq \frac{7}{3}$. This results from the optimality conditions

$$\begin{aligned} -3 + 2t &\geq 0 \\ 7 - 3t &\geq 0 \end{aligned}$$

There are two finite bounds on the parameter t at this point, but we have already examined $t = \frac{3}{2}$. Thus, we look at $t = \frac{7}{3}$. For this case, we use a primal pivot to enter x_4 in the above table.

	z	x_1	x_2	x_3	x_4	x_5	RHS
		0	$\frac{3}{2}$	$\frac{1}{2}$	0	0	5
z	1	0	$-\frac{7}{2}$	$\frac{1}{2}$	0	0	5
x_5	0	0	1	0	0	1	4
x_1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5
x_4	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	1

This indicates optimality if

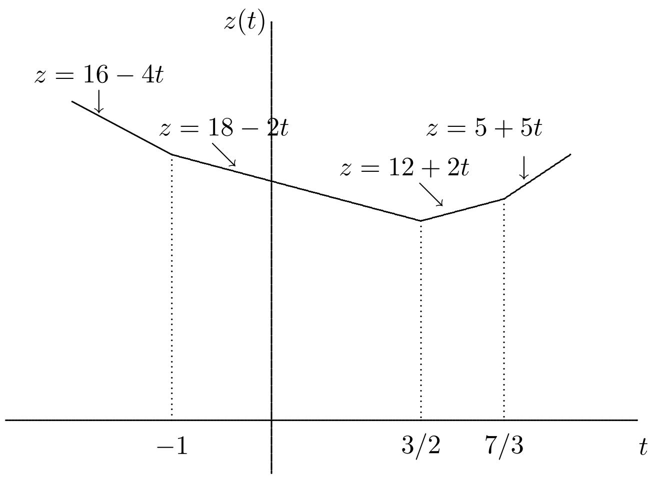
$$\begin{aligned} -\frac{7}{2} + \frac{3}{2}t &\geq 0 \\ \frac{1}{2} + \frac{1}{2}t &\geq 0 \end{aligned}$$

This results in the solution $\mathbf{X} = (5, 0, 0, 1, 4)$ and $z = 5 + 5t$ for $\frac{7}{3} \leq t \leq \infty$.

At this point no finite bounds on t remain to be examined. Thus we may summarize the results for the entire range of t (from $-\infty$ to ∞), as shown in the following table. The results can also be viewed graphically as shown below.

RESULTS OF EXAMPLE 4.17

Range of t	Optimal Solution	Optimal objective
$-\infty \leq t \leq -1$	$\mathbf{X} = (0, 4, 6, 2, 0)$	$z = 16 - 4t$
$-1 \leq t \leq \frac{3}{2}$	$\mathbf{X} = (2, 4, 2, 0, 0)$	$z = 18 - 2t$
$\frac{3}{2} \leq t \leq \frac{7}{3}$	$\mathbf{X} = (4, 2, 0, 0, 2)$	$z = 12 + 2t$
$\frac{7}{3} \leq t \leq \infty$	$\mathbf{X} = (5, 0, 0, 1, 4)$	$z = 5 + 5t$



Despite the rather trivial nature of this problem, it does serve to illustrate the general procedure that is to be carried out for any problem in which variations in the cost vector $\mathbf{c}$ are to be analyzed. The use of such an analysis can yield significant information to the organization. Consider for example, the problem investigated in Example 4.17. Let us assume that at present, the value of t is $-\frac{1}{2}$. At such a value, the optimal solution is to produce 4 units of x_2 and none of x_1. Thus, the "production line" for x_1 need not be built. However, if our forecasters predict a likelihood in the near future of t exceeding -1, we should plan for the future production of x_1 because it enters the optimal solution at that time. Furthermore, if it is anticipated that t will exceed $\frac{7}{3}$, note that it is optimal to produce 5 units of x_1 and to stop production of x_2.

4.6.2 Systematic Variation of the Right-Hand-Side Vector b

The only other systematic variation to be considered is that associated with the right-hand-side values of the constraints. Because a change in $\mathbf{b}$ is equivalent, in terms of the dual to a change in $\mathbf{c}$, the process used to analyze a systematic variation in $\mathbf{b}$ is the dual of that process used to analyze a systematic variation in $\mathbf{c}$. Consider the following model in which the vector $\mathbf{b}$ is perturbed using a vector $\mathbf{b}'$ and the scalar parameter t.

$$\text{maximize } z = \mathbf{cX}$$

$$\text{subject to}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

are given simply by

$$\mathbf{X}_B = \beta = \mathbf{B}^{-1}\mathbf{b} \geq 0$$

Noting that we have simply replaced $\mathbf{b}$ by $(\mathbf{b}+t\mathbf{b}')$, the primal feasibility conditions for the perturbed problem are given by

$$\mathbf{X}_B = \mathbf{B}^{-1}(\mathbf{b}+t\mathbf{b}') = \mathbf{B}^{-1}\mathbf{b} + t\mathbf{B}^{-1}\mathbf{b}' = \beta + t\beta' \geq 0$$

Also note that the objective value of the perturbed problem is given by

$$z = \mathbf{c}_B\mathbf{B}^{-1}(\mathbf{b}+t\mathbf{b}') = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + t\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}' = \mathbf{c}_B\beta + t\mathbf{c}_B\beta' \geq 0$$

This additional information will be displayed in the tableau by adding an additional right-hand-side column (RHS') to the tableau. Also observe that the feasibility conditions and objective value for the perturbed problem reduce to the original unperturbed problem when $t = 0$.

Parametric programming procedure: b vector (maximization problem)

1. Set the parameter $t = 0$ and find an optimal solution to the original problem.
2. Add an additional right-hand-side column to the optimal tableau containing the objective value $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}'$ and the vector $\beta' = \mathbf{B}^{-1}\mathbf{b}'$
3. Determine the parameter range over which the tableau is primal feasible by examining the conditions

   $$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} + t\mathbf{B}^{-1}\mathbf{b}' = \beta + t\beta' \geq 0$$

   Let this range be given by $l \leq t \leq u$, where l is the lower bound and u is the upper bound on the parameter t. (Note that the values of l and u need not be finite.)
4. If l is finite, determine which basic variable has $x_{B,i}$ has $\beta_i + t\beta_i' = 0$ when $t = l$. Perform a dual simplex pivot, choosing this variable as the departing variable. This will possibly result in a new tableau that is feasible (and optimal) for additional values of t.
5. Repeat steps 3 and 4 until all appropriate ranges of parameter have been investigated.

Example 4.18. Parametric programming: b vector

maximize $z = x_1 + 4x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 6 + t \\ x_2 &\leq 4 - t \\ x_1, x_2 &\geq 0 \end{aligned}$$

For this example, note that

$$\mathbf{b} = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix}$$
$$\mathbf{b}' = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Resource 2 (associated with constraint 2) is being increase at a rate that is directly proportional to t whereas resource 3 is being decreased at a rate proportional to t. Thus, the combined total amount of resources 2 and 3 remains the same, and we may be investigating a trade-off between these two resources. The question to answered is: What are optimal solutions and associated objective value as these resources change?

This is the same basic example as that of Example 4.17. Computing the additional right-hand-side column results in the following table.

	z	x_1	x_2	x_3	x_4	x_5	RHS	RHS'
z	1	0	0	0	1	3	18	-2
x_3	0	0	0	1	-2	1	2	-3
x_1	0	1	0	0	1	-1	2	2
x_2	0	0	1	0	0	1	4	-1

Examining this table, we see that the present solution is given by

$$\begin{aligned} \mathbf{X}_B &= \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} = \beta + t\beta' = \begin{pmatrix} 2-3t \\ 2+2t \\ 4-t \end{pmatrix} \\ z &= 18-2t \end{aligned}$$

Thus, feasibility (and hence optimality) will occur when

$$\begin{aligned} x_3 &= 2-3t \geq 0 \\ x_1 &= 2+2t \geq 0 \\ x_2 &= 4-t \geq 0 \end{aligned}$$

or, equivalently, when $-1 \leq t \leq \frac{2}{3}$.

We are now ready for step 4 of the process. Arbitrarily, we shall first examine the lower bound on $t(t=-1)$. When $t=-1, x_1$ is zero in the tableau above and thus, using the dual simplex, we choose x_1 as the departing variable to obtain the tableau shown below.

	z	x_1	x_2	x_3	x_4	x_5	RHS	RHS'
z	1	3	0	0	4	0	24	4
x_3	0	1	0	1	-1	0	4	-1
x_5	0	-1	0	0	-1	1	-2	-2
x_2	0	1	1	0	1	0	6	1

The basis in this tableau is feasible only when

$$\begin{aligned} x_3 &= 4-t \geq 0 \\ x_5 &= -2-2t \geq 0 \\ x_2 &= 6+t \geq 0 \end{aligned}$$

which reduces to the condition $-6 \leq t \leq -1$.

At this point, observe that x_2 would be chosen as the departing variable when considering the lower limit of $t=-6$ in the above table. However, note that we cannot use the dual simplex to obtain a new feasible basis as we did earlier because there is no possible dual pivot for the row associate with x_2. This result should not be surprising. Look at the original problem formulation. Notice in constraint 2 that the resources associated with this constraint go negative at $t < -6$ (i.e., se run out of resources). This fact is what causes us to be unable to pivot on the x_2 row, and it is a logical result when interpreted from a physical point of view. As a result, the problem is infeasible for $-\infty \leq t < -6$.

Thus, we have now examined the parameter t from $-\infty$ to $\frac{2}{3}$. We may next examine t at its upper limit of $\frac{2}{3}$.

	z	x_1	x_2	x_3	x_4	x_5	RHS	RHS'
z	1	0	0	0	1	3	18	-2
x_3	0	0	0	1	-2	1	2	-3
x_1	0	1	0	0	1	-1	2	2
x_2	0	0	1	0	0	1	4	-1

In the table which is shown again here for convenience, we see that x_3 becomes zero at $t = \frac{2}{3}$, and a dual pivot in the x_3 row results in the following table.

	z	x_1	x_2	x_3	x_4	x_5	RHS	RHS'
z	1	0	0	$\frac{1}{2}$	0	$\frac{7}{2}$	19	$-\frac{7}{2}$
x_4	0	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	-1	$\frac{3}{2}$
x_1	0	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	3	$\frac{1}{2}$
x_2	0	0	1	0	0	1	4	-1

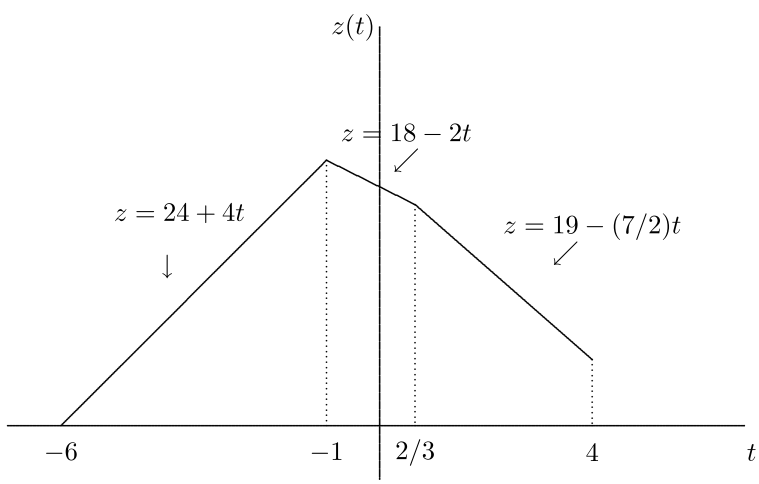
The basis is feasible when

$$
\begin{aligned}
x_4 &= -1 + \left(\frac{3}{2}\right) t \geq 0 \\
x_1 &= 3 + \left(\frac{1}{2}\right) t \geq 0 \\
x_2 &= 4 - t \geq 0
\end{aligned}
$$

which results in the range $\frac{2}{3} \leq t \leq 4$. It should be noted that the problem is infeasible for $t > 4$, because there is no possible dual pivot in the x_2 row of the above table.

RESULTS OF EXAMPLE 4.18

Range of t	Optimal Solution	Optimal objective
$-\infty \le t < -6$	Problem is infeasible.	
$-6 \le t \le -1$	$x_1 = 0$ $x_2 = 6 + t$ $x_3 = 4 - t$ $x_4 = 0$ $x_5 = -2 - 2t$	$z = 24 + 4t$
$-1 \le t \le \frac{2}{3}$	$x_1 = 2 + 2t$ $x_2 = 4 - t$ $x_3 = 2 - 3t$ $x_4 = 0$ $x_5 = 0$	$z = 18 - 2t$
$\frac{2}{3} \le t \le 4$	$x_1 = 3 + \left(\frac{1}{2}\right) t$ $x_2 = 4 - t$ $x_3 = 0$ $x_4 = -1 + 3 + \left(\frac{3}{2}\right) t$ $x_5 = 0$	$z = 19 - \left(\frac{7}{2}\right) t$
$4 \le t \le \infty$	Problem is infeasible.	



Note that the maximum of this concave piecewise linear function occurs at $t = -1$. Thus, given a choice, we should choose $t = -1$ because this results in the maximum value of the objective. Note that this choice of $t = -1$ corresponds to $(6 + t) = 6 - 1 = 5$ units of resource 2 and $(4 - t) - 4 + 1 = 5$ units of resource 3.

4.6.3 Resource Values and Ranges

Recall that the ith dual variable y_i, is associated with the ith primal constraint. However, not only is there an "association", but there is also an economic interpretation. That is, the value of y_i represents the change in the objective and may be obtained for every extra unit of resource i, *within a specified range of values for resource i.* Mathematically, y_i represents the rate of change of the objective as long as the current basis remains optimal. The same ideas that were used in parametric programming may be used to determine this range.

Example 4.19. Ranges for b_i

A chemical company produces two products. Product 1 known as "StripEasy," is a paint and finish remover for tough refinishing jobs (and is quite thick). Product 2, sold under the brand name "RenewIt," is less vicious and intended for easier refinishing (or to be used after one application of StripEasy). Every can of StripEasy returns a profit of \$0.50, and the return for RenewIt is \$0.30 per can.

Because these products can only be shipped out at the end of each week, the total amount produced must fit within the company warehouse, which has a capacity of 400,000 cubic feet. Each container (for either product) consumes 2 cubic feet of warehouse space.

Both products are produced through the same system of pipes, vessels, and processors. Production rates are 2000 and 3000 cans per hour for StripEasy and RenewIt, respectively, with a total of 120 hours per week available.

Marketing has determined that under the present market state and advertising level, the maximum weekly amounts that can be sold are 250,000 cans of StripEasy and 300,000 cans of RenewIt, Finally, because of a previous contract, the company must furnish at least 60,000 can per week of RenewIt to a particular customer.

Letting x_1 be the number of cans per week of StripEasy and x_2 be the number of cans of RenewIt, we may formulate this problem as the following linear program.

Find x_1 and x_2 so as to

maximize $z = 0.5x_1 + 0.3x_2$ (Weekly profit in \$)

subject to

$$\begin{array}{rcll} 2x_1 + 2x_2 & \leq & 400000 & \text{(Warehouse capacity limit)} \\ 3x_1 + 2x_2 & \leq & 720000 & \text{(Weekly production limit)} \\ x_1 & \leq & 250000 & \text{(Maximum weekly demand for } x_1\text{)} \\ x_2 & \leq & 300000 & \text{(Maximum weekly demand for } x_2\text{)} \\ x_2 & \geq & 60000 & \text{(Contractual obligation for } x_2\text{)} \\ x_1, x_2 & \geq & 0 & \end{array}$$

The dual problem is given by

minimize $Z = 400000y_1 + 720000y_2 + 250000y_3 + 300000y_4 + 60000y_5$

subject to

$$\begin{array}{rcl} 2y_1 + 3y_2 + y_3 & \geq & 0.5 \\ 2y_1 + 2y_2 + y_4 + y_5 & \geq & 0.3 \\ y_1, y_2, y_3, y_4 & \geq & 0 \\ y_5 & \leq & 0 \end{array}$$

The final phase-two tableau containing the optimal, weekly production program is given in the table below. Note that the tableau contains the column associated with the artificial variable x_8.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	0	0	0.25	0	0	0	0.2	-0.2	85000
x_1	0	1	0	$\frac{1}{2}$	0	0	0	1	-1	140000
x_4	0	0	0	$-\frac{3}{2}$	1	0	0	-1	1	180000
x_5	0	0	0	$-\frac{1}{2}$	0	1	0	-1	1	110000
x_6	0	0	0	0	0	0	1	1	-1	240000
x_2	0	0	1	0	0	0	0	-1	1	60000

We see that the optimal primal solution is

$$\begin{aligned} z^* &= \$85000 \\ x_1^* &= \$140000 \\ x_2^* &= \$60000 \\ x_3^* &= 0 \\ x_4^* &= \$180000 \\ x_5^* &= \$110000 \\ x_6^* &= \$240000 \\ x_7^* &= 0 \end{aligned}$$

Also, the optimal basis inverse can be read from the tableau as

$$\mathbf{B}^{-1} = (\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -1 \\ -\frac{3}{2} & 1 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The optimal dual solution is

$$\begin{aligned} Z^* &= \$85000 \\ y_1^* &= 0.25 \\ y_2^* &= 0 \\ y_3^* &= 0 \\ y_4^* &= 0 \\ y_5^* &= -0.2 \end{aligned}$$

Now, if one simply stops at this point and tells the company that its best plan is to produce 150,000 cans of StripEasy and 50,000 cans of RenewIt per week (for a weekly profit of $90,000), he or she has failed to exploit the full potential of the simplex method. Organizations are interested not just in the optimal plan for the present organizational status, they also (and often of more significance) wish to know what they should do in order to grow, increase profits, and so forth. Such information is contained within the simplex tableau.

Notice first that the only nonzero dual decision variables are $y_1 = 0.25$ and $y_5 = -0.2$, associated with the first constraint and the fifth constraint. This means that under the present program, we may increase profit by $0.25 for every extra cubic foot of warehouse space. Similarly we may increase profit by $0.20 for every unit less of product x_2 that we ship, per our contract, to our customer. Or, equivalently, shipping an additional unit of x_2 to our customer decreases profit by $0.20.

All other dual decision variables are zero, implying that for the present basis, the worth of any additional resources 2,3, or 4 is zero. This is obvious as the slack variables associated with these resources are positive-valued in the optimal solution.

With such information, we have a portion of what we need to answer such questions as the following:

1. What is a reasonable price to pay for additional warehouse space?
2. What is a reasonable penalty to pay for not completely satisfying the contract for 60,000 cans of x_2?

The partial answer to question 1 is a maximum of \$0.25 per cubic foot of warehouse space rented, per week. The partial answer to question 2 is a maximum of \$0.20 for every can of x_2 below the 60,000 units that we have contracted.

To find complete answer to both questions, we must determine the ranges of either b_1 or b_5 over which the values of y_1 and y_5 remain constant.

Because we only want to consider changes in the right-hand-side of constraint 1, the analysis can be done by perturbing $\mathbf{b}$ vector as follows:

$$\mathbf{b} + t\mathbf{b}' = \begin{pmatrix} 400000 \\ 720000 \\ 250000 \\ 300000 \\ 60000 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Computing β', we find that β' is simply the first column of $\mathbf{B}^{-1}$, that is,

$$\beta' = (\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -1 \\ -\frac{3}{2} & 1 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we can find the range of t for which the current basis remains optimal by examining

$$\mathbf{X}_B = \beta + t\beta' = \begin{pmatrix} 140000 \\ 180000 \\ 110000 \\ 240000 \\ 60000 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \geq \mathbf{0}$$

This results in the following feasibility conditions:

$$\begin{aligned} x_1 &= 140000 + \left(\frac{1}{2}\right) t \geq 0 \\ x_4 &= 180000 - \left(\frac{3}{2}\right) t \geq 0 \\ x_5 &= 110000 - \left(\frac{1}{2}\right) t \geq 0 \\ x_6 &= 240000 + 0t \geq 0 \\ x_2 &= 60000 + 0t \geq 0 \end{aligned}$$

and this is true if

$$-280000 \leq t \leq 120000$$

We see that this means that as long as b_1 lies between $(400000 - 280000) = 120000$ and $(400000 + 120000) = 520000$ cubic feet, and no other change is made in the original problem data, the present basis is optimal. Thus, we may consider renting up to 120,000 extra cubic feet of warehouse space before its associated shadow price changes from \$0.25 cubic foot per week.

For example, if we are given the opportunity of renting, say, 100,000 cubic feet of space at a rate of \$0.15 per cubic foot per week, we should take it and expect an increase in weekly profit of

$$(0.25 - 0.15)100000 = \$10000$$

In a similar manner, we can determine ranges for the remaining b_i's. For example, to determine the range for b_5 for which the current basis remains optimal, we would perturb the right-hand-side vector $\mathbf{b}$ as follows:

$$\mathbf{b} + t\mathbf{b}' = \begin{pmatrix} 400000 \\ 720000 \\ 250000 \\ 300000 \\ 60000 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Upon examining the feasibility conditions $\mathbf{X}_B = \beta + t\beta'$, we find that the current basis remains feasible (and optimal) if $-60000 \leq t \leq 140000$. Examining the fifth constraint, we see that the current basis is still optimal if b_5 lies between 0 and 200,000. This means that from a physical point of view, our maximum, per unit weekly penalty of \$0.20 per can holds true all the way down to a total elimination of this contract. Thus, if the customer is willing to modify the contract, for a penalty of \$5,000 per week, so as only to hold the company to 25,000 cans per week of RenewIt. We should accept this, as our total weekly profit increase by

$$0.20(60,0000 - 25,000) - 5000 = \$2000$$

RANGES FOR THE b_i OF EXAMPLE 4.19

Resource	Current Value	Allowable decrease before basis change	Allowable increase before basis change
b_1	400,000	280,000	120,000
b_2	720,000	180,000	∞
b_3	250,000	110,000	∞
b_4	300,000	240,000	∞
b_5	60,000	60,000	140,000

4.6.4 Objective Coefficients and Ranges

We examine each objective coefficient c_j to determine a range of values over which the current basis remains optimal.

Example 4.20. Ranges for c_j

Consider again the linear programming model defined by

maximize $z = 0.5x_1 + 0.3x_2$ (Weekly profit in \$)

subject to

$$
\begin{aligned}
2x_1 + 2x_2 &\leq 400000 && \text{(Warehouse capacity limit)} \\
3x_1 + 2x_2 &\leq 720000 && \text{(Weekly production limit)} \\
x_1 &\leq 250000 && \text{(Maximum weekly demand for } x_1\text{)} \\
x_2 &\leq 300000 && \text{(Maximum weekly demand for } x_2\text{)} \\
x_2 &\geq 60000 && \text{(Contractual obligation for } x_2\text{)} \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The optimal solution for this problem is given below.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	0	0	0.25	0	0	0	0.2	-0.2	85000
x_1	0	1	0	$\frac{1}{2}$	0	0	0	1	-1	140000
x_4	0	0	0	$-\frac{3}{2}$	1	0	0	-1	1	180000
x_5	0	0	0	$-\frac{1}{2}$	0	1	0	-1	1	110000
x_6	0	0	0	0	0	0	1	1	-1	240000
x_2	0	0	1	0	0	0	0	-1	1	60000

Now suppose, that we wish to determine the range of c_1, the objective coefficient of x_1, over which the current basis remains optimal. This can be done by doing a simple parametric analysis of the cost vector.

We only want to consider changes in the objective coefficient of x_1, so we can proceed with the analysis by perturbing the $\mathbf{c}$ vector as follows:

$$\mathbf{c} + t\mathbf{c}' = (0.5\ 0.3\ 0\ 0\ 0\ 0\ 0\ 0) + t(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

We are only changing the $\mathbf{c}$ vector, so only dual feasibility (primal optimality) will be affected. Thus, the current basis will remain optimal provided that

$$(z_j - c_j) + t(z'_j - c'_j) = (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j) + t(\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{a}_j - c'_j) \geq 0, \qquad \text{for all } j$$

Note that $\mathbf{c}'_B = (1\ 0\ 0\ 0\ 0)$. Of course, $(z_j - c_j) + t(z'_j - c'_j) = 0$ for all basic variables, therefore we only need to examine the optimality conditions for the nonbasic variables x_3 and x_7 .

$$
\begin{aligned}
(z_3 - c_3) + t(z'_3 - c'_3) &= 0.25 + t\left[(1\ 0\ 0\ 0\ 0)\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} - 0\right] = 0.25 - \left(\frac{1}{2}\right)t \geq 0 \\
(z_7 - c_7) + t(z'_7 - c'_7) &= 0.2 + t\left[(1\ 0\ 0\ 0\ 0)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - 0\right] = 0.2 + t \geq 0
\end{aligned}
$$

which simplify to $t \geq -0.2$. Observing that the current value of c_1 is 0.50, we see that as long as $c_1 \geq (0.50 - 0.20) = 0.30$ and the remaining problem data do not change, the current basis remains optimal. Thus, the unit of profit of x_1 would need to drop below \$0.30 before we would stop producing StripEasy.

In a similar manner, it can be determined that the current basis will remain optimal as long as $c_2 \leq 0.5$.

RANGES FOR THE c_j OF EXAMPLE 4.20

Cost Coefficient	Current Value	Allowable decrease before basis change	Allowable increase before basis change
c_1	0.5	0.2	∞
c_2	0.3	∞	0.2

EXERCISES.

1. Consider the following problem

   maximize $z = 3x_1 + 2x_2$

   subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 11 \\ x_1 - 3x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) Solve this problem graphically.

   (b) Write the canonical dual and solve the dual graphically. Compare the optimal objective values of the two problems.

2. Write the canonical dual of the following problem.

   maximize $z = 4x_1 - 3x_2 + 5x_3$

   subject to

$$\begin{aligned} -x_1 + x_2 &\leq 8 \\ x_1 + 2x_2 + x_3 &\leq 30 \\ 2x_1 - x_2 - 2x_3 &\leq -6 \\ x_1 + x_2 + 2x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

3. Consider the following problem

   maximize $z = x_1 + 2x_2 - 3x_3$

   subject to

$$\begin{aligned} -3x_1 + x_2 + 2x_3 &= 8 \\ 2x_1 + 4x_2 + 3x_3 &\geq 20 \\ x_1 &\geq 0 \\ x_2 &\leq 0 \\ x_3 & \quad \text{unrestricted} \end{aligned}$$

   (a) Without transforming the given problem, write the general dual.

(b) Transform the given problem into the canonical form. Write the canonical dual of this transformed problem and verify that it is equivalent to the general dual from part (a).

4. Write the general dual of the following problem.

minimize $z = 3x_1 + 2x_2 - 4x_3$

subject to

$$\begin{aligned} 5x_1 - 7x_2 + x_3 &\geq 12 \\ x_1 - x_2 + 2x_3 &= 18 \\ 2x_1 - x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

5. Write the dual of the following problems.

(a) minimize $z = \mathbf{c}\mathbf{X}$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{X} &= \mathbf{b} \\ \mathbf{D}\mathbf{X} &\geq \mathbf{d} \\ \mathbf{E}\mathbf{X} &\leq \mathbf{g} \\ \mathbf{X} &\geq 0 \end{aligned}$$

(b) maximize $z = \mathbf{c}\mathbf{X} + \mathbf{d}\mathbf{Y}$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{X} + \mathbf{D}\mathbf{Y} &= \mathbf{b} \\ \mathbf{E}\mathbf{X} + \mathbf{F}\mathbf{Y} &\leq \mathbf{g} \\ \mathbf{X} &\geq 0 \\ \mathbf{X} & \quad \text{unrestricted} \end{aligned}$$

6. Consider the following problem.

maximize $z = 3x_1 + 10x_2 + 5x_3 + 11x_4 + 6x_5 + 14x_6$

subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 4x_4 + 2x_5 + 5x_6 &= 42 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

(a) Write the dual problem.

(b) Solve the dual problem by inspection.

7. Consider the following problem.

maximize $z = x_1 + 2x_2 - 9x_3 + 8x_4 - 36x_5$

subject to

$$\begin{aligned} 2x_2 - x_3 + x_4 - 3x_5 &\leq 40 \\ x_1 - x_2 + 2x_4 - 2x_5 &\leq 10 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

(a) Write the dual problem and solve it graphically.

(b) Using complementary slackness and the optimal dual solution found in part (a), find an optimal solution to the primal problem.

8. Use the concepts of duality and complementary slackness to show that $(x_1, x_2, x_3, x_4) = (10, 0, 16, 6)$ is an optimal solution to the following problem.

maximize $z = x_1 + 2x_2 + 5x_3 + x_4$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &\leq 20 \\ -x_1 + x_2 + x_3 + x_4 &\leq 12 \\ 2x_1 + x_2 + x_3 - x_4 &\leq 30 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

9. Consider the following resource allocation problem and the accompanying optimal tableau, where x_3, x_4, x_5, and x_6 are the slack variables for constraints 1 through 4, respectively,

maximize $z = 2x_1 + 3x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 16 && \text{(Resource 1)} \\ x_1 &\leq 10 && \text{(Resource 2)} \\ x_2 &\leq 6 && \text{(Resource 3)} \\ 5x_1 + 6x_2 &\leq 60 && \text{(Resource 4)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	$\frac{3}{4}$	0	0	$\frac{1}{4}$	27
x_4	0	0	0	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	4
x_2	0	0	1	$\frac{5}{4}$	0	0	$-\frac{1}{4}$	5
x_5	0	0	0	$-\frac{5}{4}$	0	1	$\frac{1}{4}$	1
x_1	0	1	0	$-\frac{3}{2}$	0	0	$\frac{1}{2}$	6

(a) Write the dual problem in standard equality form.

(b) From the foregoing tableau, specify the optimal primal and optimal dual solutions.

(c) What is the shadow price of each resource? If you could acquire an additional unit of one of the resources, which resource would you choose? Why?

10. Consider the following problem.

maximize $z = x_1 + 4x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 24 \\ -x_1 + x_2 &\leq 4 \\ 3x_1 + 5x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Solve this problem graphically.

(b) Write the dual problem in standard equality form. Using complementary slackness and the optimal primal solution found in part (a), find an optimal solution to the dual problem.

11. Consider the following problem.

maximize $z = 3x_1 - x_2 + 6x_3$

subject to

$$\begin{aligned} 5x_1 + x_2 + 4x_3 &\leq 42 \\ 2x_1 - x_2 + 2x_3 &\leq 18 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Write the dual problem.

(b) Solve the primal problem by the primal simplex algorithm. Identify both the optimal primal and optimal dual solutions from the final tableau.

(c) At each iteration in part (b), identify the dual solution and indicate which dual constraints are violated. Also, at each iteration, identify the 2×2 primal basis matrix and the 3×3 dual basis matrix.

(d) Write the complementary slackness conditions and verify that these conditions are satisfied by the optimal solutions found in part (b).

12. Consider the following problem.

minimize $z = 6x_1 + 24x_2 + 16x_3$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 2 \\ -x_1 + 3x_2 + x_3 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Solve this problem by the dual simplex method.

(b) Solve this problem by the two-phase method and the primal simplex method.

13. Consider the following problem.

maximize $z = -8x_1 - x_2 - x_3$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 20 \\ -2x_1 + x_2 - x_3 &\geq 12 \\ x_1 + x_3 &\leq 7 \\ x_1 - x_2 + x_3 &\geq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Use the dual simplex method to show that this problem is infeasible.

(b) What does this indicate about the dual problem?

14. Suppose that the problem to maximize $\mathbf{cX}$ subject to $\mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$, has a finite optimal solution. Let $\mathbf{d}$ be an arbitrary vector in E^m. Show that if the problem to maximize $\mathbf{cX}$ subject to $\mathbf{AX} = \mathbf{d}, \mathbf{X} \geq \mathbf{0}$, has a feasible solution, then it has a finite optimal solution.

15. Solve the following problem by the dual simplex method.

    maximize $z = -8x_1 - 2x_2 - 12x_3$

    subject to

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &\geq -1 \\ x_1 - 2x_2 + 2x_3 &\geq 6 \\ -x_2 + x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

16. Solve the following linear programming problem by the dual simplex method.

    maximize $z = 3x_1 + x_2 + 2x_3$

    subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 8 \\ x_1 - 4x_2 + 2x_3 &\geq 20 \\ 4x_1 + 6x_2 + 2x_3 &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

17. Consider the following problem.

    maximize $z = 2x_1 + 3x_2$

    subject to

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

    (a) Solve this problem by the artificial constraint method.

    (b) Plot the results graphically.

18. Consider the following problem.

    maximize $z = 2x_1 + 6x_2$

    subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ -x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

    (a) Solve this problem by the artificial constraint method.

    (b) Plot the results graphically.

19. Consider the following optimal tableau for a maximization problem with ($\leq$) constraints. Let x_4, and x_5 be the respective slack variables in the first and second constraints.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{7}{3}$	d
x_3	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	14
x_1	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	16

(a) Specify $\mathbf{B}^{-1}$ corresponding to this tableau.

(b) What is the rate of change of the objective with respect to the x_5 (i.e. $\partial z/\partial x_5$)?

(c) What is the rate of change of x_3 with respect to the x_2 (i.e. $\partial x_3 \partial x_2$)?

(d) What is the rate of change of the objective with respect to the right-hand-side of the first constraint (i.e. $\partial z/\partial b_1$)?

(e) Find the optimal objective value d.

20. Consider the following resource-allocation problem and the accompanying optimal tableau (x_5,x_6, and x_7 are the respective slack variables).

$$\text{maximize } z = 15x_1 + 8x_2 + 10x_3 + 12x_4 \qquad \text{(Profit \$)}$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_4 &\leq 20 \quad \text{(Resource 1)} \\ x_1 + x_2 + x_3 + x_4 &\leq 54 \quad \text{(Resource 2)} \\ 2x_1 + x_3 + x_4 &\leq 36 \quad \text{(Resource 3)} \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	9	0	0	2	4	0	10	440
x_2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	10
x_6	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1	8
x_3	2	0	1	1	0	0	1	36

(a) Write the dual problem and specify the optimal dual solution from the foregoing tableau.

(b) What are the shadow prices of the resources? If you were to choose between increasing the amount of resource 1, 2 or 3, which one would you choose and why?

(c) Suppose that the coefficient of x_4 in the objective function changes from 12 to 16. Use sensitivity analysis to find the new optimal solution.

(d) Suppose that the available amount of resource 1 changes from 20 to 40.Use sensitivity analysis to find the new optimal solution.

(e) If the constraint $x_1 \geq 10$ is added to the problem, use sensitivity analysis to find the new optimal solution.

(f) Suppose that the constraint $3x_1 + 2x_2 + 2x_3 + x_4 \leq 80$ is added to the problem. Use sensitivity analysis to find the new optimal solution.

(g) Suppose that a new product is proposed with objective coefficient 16 and consumption vector $(1\ 2\ 1)^T$. Use sensitivity analysis to find the new optimal solution.

21. Three products, A, B, and C, are made using two manufacturing processes. The unit production times in hours are given in the accompanying table.

Product	Process 1	Process 2
1	2	2
2	2	1
3	1	3

The time available for Process 1 is 36 hours, and for Process 2 is 48 hours. Products A,B,C sell for \$9, \$8, and \$6, respectively. Let x_1, x_2 and x_3 represent the amounts of Products 1, 2, 3, respectively, and let x_4 and x_5 be the slack variables for the respective process constraints. Then the following tableau gives the optimal product mix.

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	$\frac{1}{4}$	0	$\frac{15}{4}$	$\frac{3}{4}$	171
x_1	1	$\frac{5}{4}$	0	$\frac{3}{4}$	$-\frac{1}{4}$	15
x_3	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	6

(a) Suppose the unit profit of Product 1 changes from \$9 to \$8. Use sensitivity analysis to find the new optimal solution.

(b) Suppose there is the additional requirement that at least 10 units of Product 3 must be produced.Use sensitivity analysis to find the new optimal solution.

(c) For what ranges of values on the total time available for Process 1 will the current basis remain optimal?

(d) If an additional hour of Process 1 could be purchased for \$3, would it be profitable to do so? Explain.

(e) Find the range of values on the unit of profit of Product 2 for which the current basis remains optimal.

(f) If the time available for Process 2 changes from 48 to 32 hours, use sensitivity analysis to find the new optimal solution.

22. Consider the following problem.

maximize $z = 20x_1 + 12x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 24 \\ x_1 + x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Solve this problem by the simplex method.

(b) Consider the parameterized objective to maximize $z = (20x_1 + 12x_2) + t(2x_1 + 3x_2)$. Find optimal solutions for all values of t.

23. Consider the problem of Exercise 22. Suppose that the vector

$$\mathbf{b} = \begin{pmatrix} 24 \\ 15 \end{pmatrix}$$

is replaced by

$$\begin{pmatrix} 24 \\ 15 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find optimal solutions for all values of t.

24. Consider the following resource-allocation problem and the accompanying optimal tableau (x_5,x_6, and x_7 are the respective slack variables).

$$\begin{aligned} &\text{maximize } z = 4x_1 + 6x_2 + 5x_3 \qquad \text{(Profit \$)} \\ &\text{subject to} \\ &\qquad 3x_1 + x_2 + 2x_3 \leq 64 \quad \text{(Resource 1)} \\ &\qquad x_1 + x_2 + x_3 \leq 20 \quad \text{(Resource 2)} \\ &\qquad x_1 + 2x_2 + 3x_3 \leq 30 \quad \text{(Resource 3)} \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	0	3	0	2	2	100
x_4	0	0	3	1	-5	2	24
x_2	0	1	2	0	-1	1	10
x_1	1	0	-1	0	2	-1	10

(a) Determine the range on the unit profit of Product 1 for which the current basis remains optimal.

(b) A new product is proposed. It is estimated that each unit of this new product consumes 4 units of Resource 1, 2 units of Resource 2 and 1 unit of Resource 3. What should be the unit profit on this new product for it to be profitable to manufacture?

(c) What is the range on the available units of Resource 3 (b_3) for which the current basis remains optimal?

(d) Suppose that an additional unit of resource 3 could be acquired for a total cost of $6.00. Would it be profitable to do so?

(e) Suppose that the production requirements for Product 3 are modified so that each unit of Product 3 now requires 2 units of Resource 1, 1 unit of Resource 2, and 1 unit of Resource 3. Use sensitivity analysis to find the new optimal solution.

25. Consider Exercise 24. Find optimal solutions for all values of t based on the objective function to maximize $z = (4x_1 + 6x_2 + 5x_3) + t(x_1 - x_3)$.

26. Consider Exercise 24. Find optimal solutions for all values of t based on the parameterized resource vector

$$\begin{pmatrix} 64 \\ 20 \\ 30 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$