**Theorem:** The open interval $(0,1)$ is uncountable.

**Proof:** The interval $(0,1)$ includes the subset $\left\{\frac{1}{2^k} : k \in \mathbb{N}\right\}$, which is infinite. Thus, $(0,1)$ is infinite. Suppose $(0,1)$ is denumerable. Then, there is a function $f : \mathbb{N} \to (0,1)$ that is one-to-one and onto $(0,1)$. Now, we write the images of $f$, for each $n \in \mathbb{N}$, in their decimal form:

$$\begin{aligned}
f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\ldots \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\ldots \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\ldots \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\ldots \\
&\vdots \\
f(n) &= 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\ldots \\
&\vdots
\end{aligned}$$

Since some elements of $(0,1)$ have two different decimal representations (one with an infinite string of 9's and another one with an infinite string of 0's), we do not use representations that contain an infinite string of 9's. That is, for all $n \in \mathbb{N}$ we represent $f(n) = 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\ldots$ in such a way that there is no $k$ such that for all $i > k$, $a_{ni} = 9$.

Now let $b$ be the number $b = 0.b_1b_2b_3b_4b_5\ldots$, where $b_i = 5$ if $a_{ii} \neq 5$ and $b_i = 3$ if $a_{ii} = 5$. Because of the way $b$ has been constructed, we know that $b \in (0,1)$ and that $b$ has a unique decimal representation. However, for each natural number $n$, $b$ differs from $f(n)$ in the $n$th decimal place. Thus $b \neq f(n)$ for any $n \in \mathbb{N}$, which means $b$ does not belong to the range of $f$. Thus, $f$ is not onto $(0,1)$. This contradicts our assumptions. Therefore, $(0,1)$ is not denumerable. $\square$