Math 110BH Homework 2

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Problem 2.1.2

Proof. For all $s \in G$ and all $x \in X$, e_x is an eigenvector of $\rho(s)$ if and only if $\rho(s)e_x = e_x$, which is equivalent to $s * x = x$. Thus, $\chi_X(s) = tr(\rho_s)$ is the sum of the eigenvalues of ρ_s counted with geometric multiplicity (the total number of eigenvectors among $(e_x)_{x\in X}$), so $\chi_X(s) = |G(s)|$ as desired. $\square$

Problem 2.1.3

Proof. (Uniqueness) Suppose $\rho', \rho^{\#} : G \to GL(V')$ are two linear representations such that $\langle \rho_s x, \rho'_s x'\rangle = \langle x, \rho^t_s \rho^{\#}_s x'\rangle = \langle x, x'\rangle$ for all $s \in G, x \in V, x' \in V'$. Then for all $s \in G, x \in V, x' \in V'$, $\langle x, \rho^t_s \rho^{\#}_s x'\rangle = \langle x, \rho^t_s \rho'_s x'\rangle = \langle x, x'\rangle$ for all $s \in G, x \in V, x' \in V'$, so $\rho^t_s \rho'_s = \rho^t_s \rho^{\#}_s = Id_{V'}$. As a result $\rho^{\#} = \rho'_s = (\rho^t_s)^{-1} = (\rho_s^{-1})^t = (\rho_{s^{-1}})^t$ for all $s \in G$, and $\rho' = \rho^{\#}$ as desired.

(Existence) Define ρ' by $\rho'(s) := \rho(s^{-1})^t$ for all $s \in G$. $\square$