Math 197 Week 3 Exercises

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Problem 2.2

Proof. For all $s \in G$ and all $x \in X$, e_x is an eigenvector of $\rho(s)$ if and only if $\rho(s)e_x = e_x$, which is equivalent to $s * x = x$. Thus, $\chi_X(s) = tr(\rho_s)$ is the sum of the eigenvalues of ρ_s counted with geometric multiplicity (the total number of eigenvectors among $(e_x)_{x \in X}$), so $\chi_X(s) = |G(s)|$ as desired. $\square$

Problem 2.3

Proof. (Uniqueness) Suppose $\rho', \rho^\# : G \to GL(V')$ are two linear representations such that $\langle \rho_s x, \rho'_s x' \rangle = \langle x, \rho_s^t \rho_s^\# x' \rangle = \langle x, x' \rangle$ for all $s \in G, x \in V, x' \in V'$. Then for all $s \in G, x \in V, x' \in V'$, $\langle x, \rho_s^t \rho_s^\# x' \rangle = \langle x, \rho_s^t \rho'_s x' \rangle = \langle x, x' \rangle$ for all $s \in G, x \in V, x' \in V'$, so $\rho_s^t \rho'_s = \rho_s^t \rho_s^\# = Id_{V'}$. As a result $\rho^\# = \rho'_s = (\rho_s^t)^{-1} = (\rho_s^{-1})^t = (\rho_{s^{-1}})^t$ for all $s \in G$, and $\rho' = \rho^\#$ as desired.

(Existence) Define ρ' by $\rho'(s) := \rho(s^{-1})^t$ for all $s \in G$. $\square$

Problem 2.4

Proof. Let $r, s \in G$. Then for all $f \in W$, $\rho_s \circ \rho_r f = \rho_{2,s} \circ \rho_{2,r} \circ f \circ \rho_{1,r}^{-1} \circ \rho_{1,s}^{-1} = \rho_{2,sr} \circ f \circ \rho_{1,sr}^{-1}$, so ρ defines a linear representation.

Since ρ is isomorphic to $\rho'_1 \otimes \rho_2$, $\chi_\rho = \chi_1^* \cdot \chi_2$. $\square$