Exercise 2.8

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Let $\rho : G \to GL(V)$ be a linear representation of G, a finite group. Let $\chi_1, \ldots, \chi_h$ be all possible distinct characters of irreducible represntations of G. Note that the number of isomorphism classes of irreducible representations of a finite group G over the complex numbers is equal to the number of conjugacy classes of G, which is finite. For each $k = 1, \ldots, h$, pick W_k be an irreducible representation of G with character χ_k and denote its degree by n_k. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into irreducible representations. For $k = 1, \ldots, h$, denote by V_k the direct sum of those of the $U_1, \ldots, U_m$ which are isomorphic to W_k.

Let H_k be the vector space of linear mappings $h : W_k \to V$ such that $\rho_s h = h\rho_s$ for all $s \in G$. Each $h \in H_k$ maps W_k to V_k.

Proposition 0.1. *The dimension of H_k is equal to the number of times that W_k appears in V, i.e., to $\dim V_k / \dim W_k$.*

Proof. Note that in this case, $\rho_s h = h\rho_s$ for all $s \in G$ boils down to $\rho_s|V_k h = h\rho_s|V_k$ for all $s \in G$. Suppose $V_k = W_k$. Then by Shur's Lemma, any such h must be a homothety, so $\dim H_k = 1 = \dim V_k / \dim W_k$.

For the more general case, we can decompose $V_k = U_{k_1} \oplus \cdots \oplus U_{k_n}$ and apply Schur's Lemma on each U_k. Then $\dim H_k = n$ and $\dim V_k = \dim U_{k_1} + \cdots + \dim U_{k_n} = n \dim W_k$, so the claim follows. □

Proposition 0.2. *Let G act on $H_k \otimes W_k$ through the tensor product of the trivial representation of G on H_k and the given representation on W_k. Then the map*

$$F : \; H_k \otimes W_k \to V_k$$

defined by the formula

$$F\left(\sum h_\alpha \cdot w_\alpha\right) = \sum h_\alpha(w_\alpha)$$

is an isomorphism.

Proof. Since $\dim(H_k \otimes W_k) = dim(H_k)\dim(W_k) = \dim V_k$, it suffices to show that F is injective. Suppose $V_k = W_k$. Then $H_k \otimes W_k = span(h \cdot w)$ for some $w \cdot \alpha \neq 0_{H_k} \cdot 0_{W_k}$. By Shur's Lemma, h is a non-zero homothety, so if $F(c(h \cdot w)) = ch(w) = 0$ for some $c \in \mathbb{C}$, $c = 0$, proving F injective.

For general cases, we can decompose V into irreducible subrepresentations and apply the same argument. □

Proposition 0.3. *Let $(h_1, \ldots, h_k)$ be a basis for H_k and form the direct sum $W_k \times \cdots \times W_k$ of k copies of W_k. The system $(h_1, \ldots, h_k)$ clearly defines a linear mapping h of $W_k \times \cdots \times W_k$ into V_k. Show that this is an isomorphism.*

Proof. The linear map h is defined by $h(w_1, \ldots, w_k) := h_1(w_1) + \cdots h_k(w_k)$. This is surjective by Part B. □