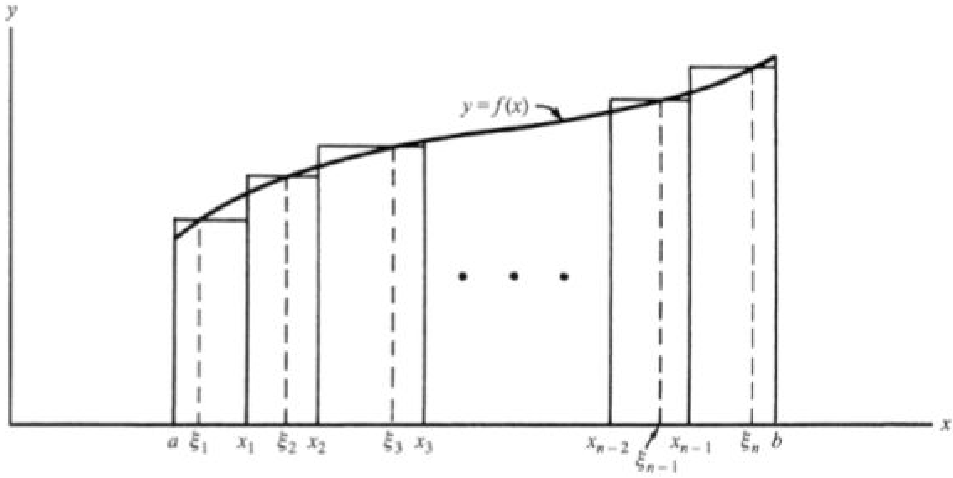
CHAPTER 5 - INTEGRALS

Introduction of the Definite Integral

Consider the area of the region bound by $y = f(x)$, the x axis, and the joining vertical segments (ordinates) $x = a$ and $x = b$. (See Figure 5.1.)



Subdivide the interval $a \leq x \leq b$ into n subintervals by means of the points $x_1, x_2, \ldots, x_{n-1}$, chosen arbitrarily. In each of the new intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ choose points $\xi_1, \xi_2, \ldots, \xi_n$ arbitrarily. Form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + f(\xi_3)(x_3 - x_2) + \cdots + f(\xi_n)(b - x_{n-1}) \tag{1}$$

By writing $x_0 = a$, $x_n = b$, and $x_k - x_{k-1} = \Delta x_k$, this can be written

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{2}$$

Geometrically, this sum represents the total area of all rectangles in Figure 5.1.

We now let the number of subdivisions n increase in such a way that each $\Delta x_k \to 0$. If, as a result, the sum (1) or (2) approaches a limit which does not depend on the mode of subdivision, we denote this limit by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_k)\, \Delta x_k \tag{3}$$

This is called the *definite integral of* $f(x)$ *between a and b*. In this symbol, $f(x)\,dx$ is called the *integrand* and $[a, b]$ is called the *range of integration*. We call a and b the limits of integration, a being the lower limit of integration and b the upper limit.

The limit (*3*) exists whenever $f(x)$ is continuous (or piecewise continuous) in $a \leq x \leq b$ (see Problem 5.31). When this limit exists we say that f is *Riemann integrable* or simply *integrable* in $[a, b]$.

Measure Zero

Theorem. If $f(x)$ is bounded in $[a, b]$, then a necessary and sufficient condition for the existence of $\int_a^b f(x)\,dx$ is that the set of discontinuities of $f(x)$ have measure zero.

Properties of Definite Integrals

If $f(x)$ and $g(x)$ are integrable in $[a, b]$, then

1. $\int_a^b \{f(x) \pm g(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
2. $\int_a^b Af(x)dx \qquad = A\int_a^b f(x)dx$ where A is any constant
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided $f(x)$ is integrable in $[a, c]$ and $[c, b]$
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
5. $\int_a^a f(x)dx = 0$
6. If in $a \leqq x \leqq b$, $m \leqq f(x) \leqq M$ where m and M are constants, then $m(b-a) \leqq \int_a^b f(x)\,dx \leqq M(b-a)$
7. If in $a \leqq x \leqq b, f(x) \leqq g(x)$, then $\int_a^b f(x)dx \leqq \int_a^b g(x)\,dx$
8. $\left|\int_a^b f(x)dx\right| \leqq \int_a^b |\,f(x)\,|\,dx$ if $a < b$

Mean Value Theorems for Integrals

Let f be continuous on the closed interval $a \leqq x \leqq b$. Assume the function is represented by the correspondence $y = f(x)$, with $f(x) > 0$. Insert points of equal subdivision, $a = x_0, x_1, \ldots, x_n = b$. Then all $\Delta x_k = x_k - x_{k-1}$ are equal and each can be designated by Δx. Observe that $b - a = n\,\Delta x$. Let ξ_k be the midpoint of the interval Δx_k and $f(\xi_k)$ the value of f there. Then the average of these functional values is

$$\frac{f(\xi_1)+\cdots+f(\xi_n)}{n} = \frac{[f(\xi_1)+\cdots+f(\xi_n)\,\Delta x}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(\xi_k)\,\Delta\xi_k$$

This sum specifies the average value of the n functions at the midpoints of the intervals. However, we may abstract the last member of the string of equalities (dropping the special conditions) and define

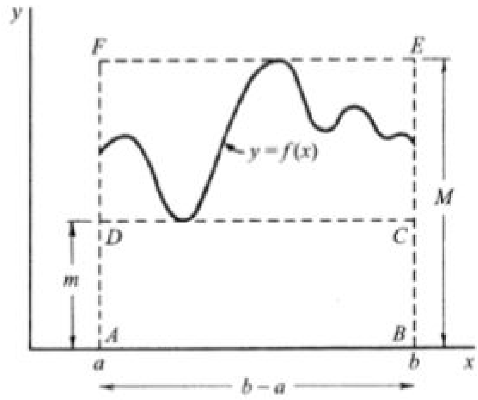
$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(\xi_k)\,\Delta\xi_k = \frac{1}{b-a}\int_a^b f(x)dx$$

as the average value of f on $[a, b]$.

$$m(b-a) \leqq \int_a^b f(x)\,dx \leqq M(b-a)$$

or

$$m \leqq \frac{1}{b-a}\int_a^b f(x)\,dx \leqq M$$



Since f is a continuous function on a closed interval, there exists a point $x = \xi$ in (a, b) intermediate to m and M such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

First Mean Value Theorem If $f(x)$ is continuous in $[a, b]$, there is a point ξ in (a, b) such that

$$\int_a^b f(x)\,dx = (b-a)f(\xi) \tag{4}$$

Generalized First Mean Value Theorem If $f(x)$ and $g(x)$ are continuous in $[a, b]$, and $g(x)$ does not change sign in the interval, then there is a point ξ in (a, b) such that

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)\,dx \tag{5}$$

This reduces to Equation (4) if $g(x) = 1$.

Connecting Integral and Differential Calculus

Definition Any function F such that $F'(x) = f(x)$ *is called an antiderivative, primitive,* or *indefinite integral* of f.

The antiderivative of a function is not unique. This is clear from the observation that for any constant c

$$(F(x) + c)' = F'(x) = f(x)$$

The following theorem is an even stronger statement.

Theorem Any two primitives (i.e., antiderivatives) F and G of f differ at most by a constant; i.e., $F(x) - G(x) = C$.

(See the problem set for the proof of this theorem.)

EXAMPLE. If $F'(x) = x^2$, then $F(x) = \int x^2 dx = \frac{x^3}{3} + c$ is an indefinite integral (antiderivative or primitive) of x^2.

The indefinite integral (which is a function) may be expressed as a definite integral by writing

$$\int f(x)dx = \int_c^x f(t)\,dt$$

The Fundamental Theorem of the Calculus

Part 1. Let f be integrable on a closed interval $[a, b]$. Let c satisfy the condition $a \leqq c \leqq b$, and define a new function

$$F(x) = \int_c^x f(t)\, dt \quad \text{if} \quad a \leqq x \leqq b$$

Then the derivative $F'(x)$ exists at each point x in the open interval (a, b), where f is continuous and $F'(x) = f(x)$. (See Problem 5.10 for proof of this theorem.)

Part 2. As in Part 1, assume that f is integrable on the closed interval $[a, b]$ and continuous in the open interval (a, b). Let F be any antiderivative so that $F'(x) = f(x)$ for each x in (a, b). If $a < c < b$, then for any x in (a, b)

$$\int_c^x f(t)\, dt = F(x) - F(c)$$

If the open interval on which f is continuous includes a and b, then we may write

$$\int_a^b f(x)\, dx = F(b) - F(a). \qquad \text{(See Problem 5.11)}$$

This is the usual form in which the theorem is used.

Generalization of the Limits of Integration

The upper and lower limits of integration may be variables. For example:

$$\int_{\sin x}^{\cos x} t\, dt = \left[\frac{t^2}{2}\right]_{\sin x}^{\cos x} = (\cos^2 x - \sin^2 x)/2$$

In general, if $F'(x) = f(x)$, then

$$\int_{u(x)}^{\upsilon(x)} f(t)\, dt = F[\upsilon(x)] = F[u(x)]$$

Change of Variable of Integration

If a determination of $\int f(x)\, dx$ is not immediately obvious in terms of elementary functions, useful results may be obtained by changing the variable from x to t according to the transformation $x = g(t)$. [This change of integrand that follows is suggested by the differential relation $dx = g'(t)\, dt$.] The fundamental theorem enabling us to do this is summarized in the statement

$$\int f(x)\, dx = \int f\{g(t)\}g'(t)\, dt \tag{6}$$

where, after obtaining the indefinite integral on the right, we replace t by its value in terms of x; i.e., $t = g^{-1}(x)$. This result is analogous to the chain rule for differentiation (see Page 76).

The corresponding theorem for definite integrals is

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f\{g(t)\}g'(t)\, dt \tag{7}$$

where $g(\alpha) = a$ and $g(\beta) = b$; i.e., $\alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$. This result is certainly valid if $f(x)$ is continuous in $[a, b]$ and if $g(t)$ is continuous and has a continuous derivative in $\alpha \leqq t \leqq \beta$.

Integrals of Elementary Functions

The following results can be demonstrated by differentiating both sides to produce an identity. In each case, an arbitrary constant c (which has been omitted here) should be added.

1. $\int u^n \, du = \dfrac{u^{n+1}}{n+1} \quad n \neq -1$

2. $\int \dfrac{du}{u} = \ln |u|$

3. $\int \sin u \, du = -\cos u$

4. $\int \cos u \, du = \sin u$

5. $\int \tan u \, du = \ln |\sec u|$
   $= -\ln |\cos u|$

6. $\int \cot u \, du = \ln |\sin u|$

7. $\int \sec u \, du = \ln |\sec u + \tan u|$
   $= \ln |\tan (u/2 + \pi/4)|$

8. $\int \csc u \, du = \ln |\csc u - \cot u|$
   $= \ln |\tan u/2|$

9. $\int \sec^2 u \, du = \tan u$

10. $\int \csc^2 u \, du = -\cot u$

11. $\int \sec u \tan u \, du = \sec u$

12. $\int \csc u \cot u \, du = -\csc u$

13. $\int a^u \, du = \dfrac{a^u}{\ln a} \quad a > 0, a \neq 1$

14. $\int e^u \, du = e^u$

15. $\int \sinh u \, du = \cosh u$

16. $\int \cosh u \, du = \sinh u$

17. $\int \tanh u \, du = \ln \cosh u$

18. $\int \coth u \, du = \ln |\sinh u|$

19. $\int \operatorname{sech} u \, du = \tan^{-1}(\sinh u)$

20. $\int \operatorname{csch} u \, du = -\coth^{-1} (\cosh u)$

21. $\int \operatorname{sech}^2 u \, du = \tanh u$

22. $\int \csc h^2 u \, du = -\coth u$

23. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u$

24. $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u$

25. $\int \dfrac{du}{\sqrt{s^2 - u^2}} = \sin^{-1} \dfrac{u}{a} \quad \text{or} \quad -\cos^{-1} \dfrac{u}{a}$

26. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln |u + \sqrt{u^2 \pm a^2}|$

27. $\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a} \tan^{-1} \dfrac{u}{a} \quad \text{or} \quad -\dfrac{1}{a} \cot^{-1} \dfrac{u}{a}$

28. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a} \ln \left| \dfrac{u-a}{u+a} \right|$

29. $\int \dfrac{du}{u\sqrt{a^2 \pm u^2}} = \dfrac{1}{a} \ln \left| \dfrac{u}{a + \sqrt{a^2 \pm u^2}} \right|$

30 $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \cos^{-1} \dfrac{a}{u} \quad \text{or} \quad \dfrac{1}{a} \sec^{-1} \dfrac{u}{a}$

31. $\int \sqrt{u^2 \pm a^2} \, du = \dfrac{u}{2} \sqrt{u^2 \pm a^2}$
    $\pm \dfrac{a^2}{2} \ln |u + \sqrt{u^2 \pm a^2}|$

32. $\int \sqrt{a^2 - u^2} \, du = \dfrac{u}{2} \sqrt{a^2 - u^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{u}{a}$

33. $\int e^{au} \sin bu \, du = \dfrac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$

34. $\int e^{au} \cos bu \, du = \dfrac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$

Special Methods of Integration

1. Integration by Parts Let u and υ be differentiable functions. According to the product rule for differentials,

$$d(u\upsilon) = u\,d\upsilon + \upsilon\,du$$

Upon taking the antiderivative of both sides of the equation, we obtain

$$u\upsilon = \int u\,d\upsilon + \int \upsilon\,du$$

This is the formula for integration by parts when written in the form

$$\int \upsilon\,dv = uv - \int \upsilon\,du \quad \text{or} \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)dx$$

where $u = f(x)$ and $\upsilon = g(x)$. The corresponding result for definite integrals over the interval $[a, b]$ is certainly valid if $f(x)$ and $g(x)$ are continuous and have continuous derivatives in $[a, b]$. See Problems 5.17 to 5.19.

2. Partial Fractions Any rational function $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, with the degree of $P(x)$ less than that of $Q(x)$, can be written as the sum of rational functions having the form $\frac{A}{(ax+b)^r}, \frac{Ax+B}{(ax^2+bx+c)^r}$ where $r = 1, 2, 3, \ldots$, which can always be integrated in terms of elementary functions.

3. Rational Functions of sin x and cos x These can always be integrated in terms of elementary functions by the substitution $\tan x/2 = u$ (see Problem 5.21).

4. Special Devices Depending on the particular form of the integrand, special devices are often employed (see Problems 5.22 and 5.23).

Improper Integrals

If the range of integration $[a, b]$ is not finite or if $f(x)$ is not defined or not bounded at one or more points of $[a, b]$, then the integral of $f(x)$ over this range is called an *improper integral*. By use of appropriate limiting operations, we may define the integrals in such cases.

EXAMPLE 1

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} = \lim_{M\to\infty} \tan^{-1} x \Big|_0^M = \lim_{M\to\infty} \tan^{-1} M = \pi/2$$

Numerical Methods for Evaluating Definite Integrals

Numerical methods for evaluating definite integrals are available in case the integrals cannot be evaluated exactly. The following special numerical methods are based on subdividing the interval $[a, b]$ into n equal parts of length $\Delta x = (b - a)/n$. For simplicity we denote $f(a + k\Delta x) = f(x_k)$ by y_k, where $k = 0, 1, 2, \ldots, n$. The symbol $\approx$ means "approximately equal." In general, the approximation improves as n increases.

1. Rectangular Rule

$$\int_a^b f(x)dx \approx \Delta x\{y_0 + y_1 + y_2 + \cdots + y_{n-1}\} \text{ or } \Delta x\{y_1 + y_2 + y_3 + \cdots + y_n\} \tag{8}$$

2. Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\} \tag{9}$$

This is obtained by taking the mean of the approximations in Equation (*8*). Geometrically, this replaces the curve $y = f(x)$ by a set of approximating line segments.

3. Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\} \tag{10}$$

This formula is obtained by approximating the graph of $y = g(x)$ by a set of parabolic arcs of the form $y = ax^2 + bx + c$. The correlation of two observations lead to Equation (*10*). First,

$$\int_{-h}^{h}[ax^2 + bx + c]dx = \frac{h}{3}[2ah^2 + 6c]$$

Arc Length

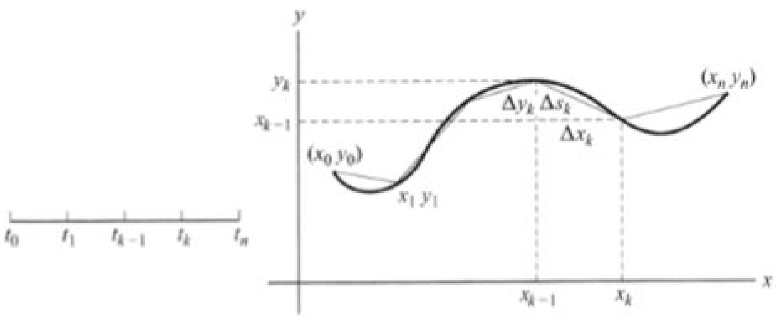


Figure 5.4

Geometrically, the measurement of the kth segment of the arc $0 \leqq t \leqq s$ is accomplished by employing the Pythagorean theorem; thus, the measure is defined by

$$\lim_{n\to\infty}\sum_{k=1}^{n}\{(\Delta x_k)^2 + (\Delta y_k)^2\}^{1/2}$$

or, equivalently,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right\}^{1/2}(\Delta x_k)$$

where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$.

Thus, the length of the arc of a curve in rectangular Cartesian coordinates is

$$L = \int_a^b \{[f'(t)^2] + [g'(t)]^2\}^{1/2}\, dt = \int\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2} dt$$

Upon changing the variable of integration from t to x we obtain the planar form

$$L = \int_{f(a)}^{f(b)} \left\{1 + \left[\frac{dy}{dx}\right]^2\right\}^{1/2}$$

(This form is appropriate only in the plane.)

The generic differential formula $ds^2 = dx^2 + dy^2$ is useful, in that various representations algebraically arise from it. For example,

$$\frac{ds}{dt}$$

expresses instantaneous speed.

Area

Let f and g be continuous functions whose graphs intersect at the graphical points corresponding to $x = a$ and $x = b$, $a < b$. If $g(x)$ ε $f(x)$ ε $f(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ is

$$A = \int_a^b \{g(x) - f(x)\}dx$$

If the functions intersect in (a, b), then the integral yields an algebraic sum. For example, if $g(x) = \sin x$ and $f(x) = 0$ then

$$\int_0^{2\pi} \sin x\, dx = \cos x \Big|_0^{2\pi} = 0$$

Volumes of Revolution

Disk Method Assume that f is continuous on a closed interval $a \leq x \leq b$ and that $f(x)$ ε 0. Then the solid realized through the revolution of a plane region R [bound by $f(x)$, the x axis, and $x = a$ and $x = b$] about the x axis has the volume

$$V = \pi \int_a^b [f(x)]^2\, dx$$

Shell Method Suppose f is a continuous function on $[a, b]$, a ε 0, satisfying the condition $f(x)$ ε 0. Let R be a plane region bounded by $f(x)$, $x = a$, $x = b$, and the x axis. The volume obtained by orbiting R about the y axis is

$$V = \int_a^b 2\pi x f(x) dx$$

Moment of Inertia Moment of inertia is an important physical concept that can be studied through its idealized geometric form. This form is abstracted in the following way from the physical notions of kinetic energy $K = 1/2\, m\upsilon^2$ and angular velocity $\upsilon = \omega r$ (m represents mass and υ signifies linear velocity). Upon substituting for υ,

$$K = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2}(mr^2)\omega^2$$

When this form is compared to the original representation of kinetic energy, it is reasonable to identify mr^2 as rotational mass. It is this quantity, $I = mr^2$, that we call the *moment of inertia*.