Chapter 1 – Riemann Integral.

Boundedness: Function f is bounded if $\exists M > 0$ s.t. $-M \leq f(x) \leq M \ \forall x \in [a,b]$.

Partitions: $[a,b]$ subdivided into partitions $P = \{x_0, \dots, x_n\}$, length of interval $I = [a,b]$ is given by $b-a$.

Length of total interval:

$$\sum_{k=1}^{n} (x_k - x_{k-1}) = (x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})$$
$$= x_n - x_0$$
$$= b - a$$

Upper Darboux Sum: $M_k(f)$ maximum value of f on k-th sub-interval $[x_{k-1}, x_k]$ and $m_k(f)$ is minimum value on $[x_{k-1}, x_k]$

Upper Darboux sum $U(f:P)$ is:

$$U(f:P) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$

Lower Darboux Sum.

$$L(f:P) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1})$$

Riemann Integral: If f is bounded on $[a,b]$, $\exists I \in \mathbb{R}$ s.t.

$$L(f:P) \leq I \leq U(f:P)$$

$\forall P \in [a,b]$ then f is Riemann integrable on $[a,b]$, and has definite integral:

$$I = \int_a^b f(x)dx.$$

Example: Show that $f(x) = 1$ is Riemann integrable on $[a,b]$ and $\int_a^b f(x)dx = b-a$.

Let P partition of $[a,b]$ with endpoints: $\{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$.

As f is constant, $m_k(f) = M_k(f) = 1 \ \forall k \in [1,n]$, thus:

$$U = (f:P) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$
$$= \sum_{k=1}^{n} (x_k - x_{k-1})$$
$$= (x_1 - x_0) + \dots + (x_n - x_{n-1})$$
$$= x_n - x_0$$
$$= b - a.$$

Example:

Show $f(x) = x$ is Riemann integrable on $[a,b]$.

Let partition P of $[a,b]$ be $\{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$.

We have $m_k(f) = f(x_{k-1}) = x_{k-1}$

$$M_k(f) = f(x_k) = x_k \quad \forall k \in [1,n].$$

$$\therefore \quad U(f:P) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k(x_k - x_{k-1})$$

$$L(f:P) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n} x_{k-1}(x_k - x_{k-1})$$

$$\begin{aligned} U(f:P) - L(f:P) &= \sum_{k=1}^{n} [x_k(x_k - x_{k-1}) - x_{k-1}(x_k - x_{k-1})] \\ &= \sum_{k=1}^{n} (x_k - x_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (x_k - x_{k-1})^2 \end{aligned}$$

$$\begin{aligned} U(f:P) + L(f:P) &= \sum_{k=1}^{n} [x_k(x_k - x_{k-1}) + x_{k-1}(x_k - x_{k-1})] \\ &= \sum_{k=1}^{n} (x_k + x_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (x_k^2 - x_{k-1}^2). \\ &= (x_1^2 - x_0^2) + \dots + (x_n^2 - x_{n-1}^2) \\ &= x_n^2 - x_0^2 \\ &= b - a. \end{aligned}$$

$$\therefore \quad U(f:P) = \frac{b^2 - a^2}{2} + \frac{1}{2}\sum_{k=1}^{n} (x_k - x_{k-1})^2$$

$$L(f:P) = \frac{b^2 - a^2}{2} - \frac{1}{2}\sum_{k=1}^{n} (x_k - x_{k-1})^2$$

As size of subintervals approaches zero:

$$U(f:P) \leq (b^2 - a^2)/2$$

$$L(f:P) \geq (b^2 - a^2)/2.$$

By Riemann integral we must have $L(f:P) \leq U(f:P)$

$$U(f:P) = (b^2 - a^2)/2 = L(f:P)$$

Thus f is integrable and $\int_a^b f(x)\,dx = (b^2 - a^2)/2$

Continuity:

If f is defined on some open interval containing point a. Then f is continuous at the point a in the open interval if

$$\lim_{x \to a} f(x) = f(a)$$