Reduction Formulae

Definition: $I_n = \int f(x,n)\,dx$, where n parameter and reduces it to integral of the form:

$I_k = \int f(x,k)\,dx$, where $k < n$.

Example: Suppose that $\int \sin^n x\,dx$, $n \geqslant 0$

(a). Show that $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} dx$

(b). Hence find $\int \sin^5 x\,dx$

(a).
$$\begin{aligned}
\int \sin^n x\,dx &= \int \sin x \cdot \sin^{n-1} x\,dx \\
&= -\cos x \cdot \sin^{n-1} x - \int (-\cos x)\cdot(n-1)\sin^{n-2} x \cdot \cos x\,dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x\,dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \\
&= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx
\end{aligned}$$

(b). Setting $n=5$:

$$\int \sin^5 x\,dx = -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x\,dx + C$$

Setting $n=3$:

$$\int \sin^3 x\,dx = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x\,dx = -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C$$

$$\therefore \int \sin^5 x\,dx = -\frac{1}{5}\cos x \sin^4 x - \frac{4}{15}\cos x \sin^2 x - \frac{8}{15}\cos x + C.$$

Example: Suppose that I_n is defined by $I_n = \int_0^{\pi/4} \tan^n x\,dx$, $n \geqslant 0$

(a). Find the value for I_0

(b). Show that $I_n = \frac{1}{n-1} - I_{n-2}$, $n \geqslant 2$

(c). Using the result in (b), evaluate $\int_0^{\pi/4} \tan^4 x\,dx$

(a). When $n=0$:

$$I_0 = \int_0^{\pi/4} (\tan x)^0\,dx = \int_0^{\pi/4} dx = x\Big|_0^{\pi/4} = \pi/4.$$

(b). As $\tan^2 x = \sec^2 x - 1$:

$$I_n = \int_0^{\pi/4} \tan^n x\,dx = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - I_{n-2}.$$

Let $u = \tan x$, $du = \sec^2 x\,dx$

$x = 0, u = 0$ and $x = \pi/4$, $u = 1$:

$$\int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx = \int_0^1 u^{n-2}\,du = \left[\frac{u^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$$

$$\therefore\ I_n = \frac{1}{n-1} - I_{n-2}.$$

(c). Let $n=4$ in reduction formula:

$$\int_0^{\pi/4} \tan^4 x\,dx = I_4 = \frac{1}{4-1} - I_2 = \frac{1}{3} - I_2.$$

Now let $n=2$:

$$I_2 = \frac{1}{2-1} - I_0 = 1 - \frac{\pi}{4}.$$

Example: Suppose that I_n is defined by $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$, $n \in \mathbb{N}$.

(a). Find the value of I_1

(b). Show that $I_{n+1} = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n} I_n$, $n \geq 1$

(c). Using the result from (b), evaluate $\int_0^1 \frac{dx}{(1+x^2)^3}dx$

(a) Setting $n=1$:

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}.$$

(b). Integrating by parts:

$$I_n = \left[\frac{x}{(1+x^2)^n}\right]_0^1 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{dx}{(1+x^2)^n} - 2n\int_0^1 \frac{dx}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}.$$

Rearranging:

$$I_{n+1} = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n}I_n.$$

(c). Set $n=2$:

$$I_3 = \frac{1}{16} + \frac{3}{4}I_2.$$

Set $n=1$:

$$I_2 = \frac{1}{4} + \frac{1}{2}I_1 = \frac{1}{4} + \frac{\pi}{8}.$$

$$\therefore \int_0^1 \frac{dx}{(1+x^2)^3} = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right) = \frac{1}{4} + \frac{3\pi}{32}.$$

Example: Let $I_n = \displaystyle\int_0^\pi \frac{1-\cos(nx)}{1-\cos x}\,dx, \; n = 0,1,2,\dots$

(a). Show that $I_{n+2} - 2I_{n+1} + I_n = 0$

(b). Evaluate I_0 and I_1

(c). Using induction, prove that $I_n = n\pi \;\; \forall n = 0,1,2,\dots$

(a). $$I_{n+2} - 2I_{n+1} + I_n = \int_0^\pi \frac{(1-\cos(n+2)x)}{1-\cos x}\,dx - 2\int_0^\pi \frac{(1-\cos(n+1)x)}{1-\cos x}\,dx + \int_0^\pi \frac{(1-\cos nx)}{1-\cos x}\,dx$$

$$= \int_0^\pi \frac{2\cos(n+1)x - \cos(n+2)x - \cos nx}{1-\cos x}\,dx$$

using $\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$

setting $\theta = nx + 2x$ and $\varphi = nx$:

$$\cos(n+2)x + \cos nx = 2\cos(n+1)x\cos x$$

Thus
$$I_{n+2} - 2I_{n+1} + I_n = \int_0^\pi \frac{2\cos(n+1)x - 2\cos(n+1)x\cos x}{1-\cos x}\,dx$$
$$= \int_0^\pi \frac{2\cos(n+1)x\cdot(1-\cos x)}{1-\cos x}\,dx$$
$$= \int_0^\pi 2\cos(n+1)x\,dx$$
$$= \frac{2}{n+1}\sin(n+1)x\Big|_0^\pi = 0.$$

(b). setting $n=0$:

$$I_0 = \int_0^\pi \frac{1-\cos(0\cdot x)}{1-\cos x}\,dx = 0.$$

setting $n=1$:

$$I_1 = \int_0^\pi \frac{1-\cos x}{1-\cos x} = \int_0^\pi dx = \pi.$$

when $n=0$, $I_0 = 0.\pi = 0$

$n=1$, $I_1 = 1.\pi = \pi$

Assume true for $n=k-1$ and $n=k$

$I_{k-1} = (k-1)\pi$ and $I_k = k\pi$.

Now prove for $n=k+1$, setting $n=k-1$:

$I_{k+1} - 2I_k + I_{k-1} = 0$

Thus :

$I_{k+1} = 2I_k - I_{k-1} = 2k\pi - (k-1)\pi = (k+1)\pi.$

So the statement is true for $n=k+1$,