

CHAPTER 3 - IMPROPER INTEGRALS

Definition of an Improper Integral

The integral $\int_a^b f(x) dx$ is called an *improper integral* if

1. $a = -\infty$ or $b = \infty$ or both; i.e., one or both integration limits is infinite.
2. $f(x)$ is unbounded at one or more points of $a \leq x \leq b$. Such points are called *singularities* of $f(x)$.

Improper Integrals of the First Kind (Unbounded Intervals)

If f is integrable on the appropriate domains, then the indefinite integrals $\int_a^x f(t) dt$ and $\int_x^a f(t) dt$ (with variable upper and lower limits, respectively) are functions. Through them we define three forms of the improper integral of the first kind.

Definition

(a) If f is integrable on $a \leq x < \infty$, then $\int_a^\infty f(x) dx = \lim_{x \rightarrow \infty} \int_a^x f(t) dt$.

(b) If f is integrable on $-\infty < x \leq a$, then $\int_{-\infty}^a f(x) dx = \lim_{x \rightarrow -\infty} \int_x^a f(t) dt$.

(c) If f is integrable on $-\infty < x < \infty$, then

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx \\ &= \lim_{x \rightarrow -\infty} \int_x^a f(t) dt + \lim_{x \rightarrow \infty} \int_a^x f(t) dt \end{aligned}$$

In (c) it is important to observe that

$$\lim_{x \rightarrow -\infty} \int_x^a f(t) dt + \lim_{x \rightarrow \infty} \int_a^x f(t) dt$$

and

$$\lim_{x \rightarrow \infty} \left[\int_{-x}^a f(t) dt + \int_a^x f(t) dt \right]$$

are not necessarily equal.

This can be illustrated with $f(x) = xe^{x^2}$. The first expression is not defined, since neither of the improper integrals (i.e., limits) is defined, while the second form yields the value 0.

Convergence or Divergence of Improper Integrals of the First Kind

Let $f(x)$ be bounded and integrable in every finite interval $a \leq x \leq b$. Then we define

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx \quad (1)$$

where b is a variable on the positive real numbers.

The integral on the left is called *convergent* or *divergent* according as the limit on the right does or does not exist. Note that $\int_a^\infty f(x) dx$ bears close analogy to the infinite series $\sum_{n=1}^{\infty} u_n$, where $u_n = f(n)$, while $\int_a^b f(x) dx$ corresponds to the partial sums of such infinite series. We often write M in place of b in Equation (1).

Similarly, we define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx \quad (2)$$

where a is a variable on the negative real numbers. And we call the integral on the left convergent or divergent according as the limit on the right does or does not exist.

Special Improper Integrals of the First Kind

1. **Geometric or exponential integral** $\int_a^{\infty} e^{t-x} dx$, where t is a constant, converges if $t > 0$ and diverges if $t \leq 0$. Note the analogy with the geometric series if $r = e^{-t}$ so that $e^{-tx} = r^x$.
2. **The p integral of the first kind** $\int_a^{\infty} \frac{dx}{x^p}$, where p is a constant and $a > 0$, converges if $p > 1$ and diverges if $p \leq 1$. Compare with the p series.

Convergence Tests for Improper Integrals of the First Kind

The following tests are given for cases where an integration limit is ∞ . Similar tests exist where an integration limit is $-\infty$ (a change of variable $x = -y$ then makes the integration limit ∞). Unless otherwise specified, we assume that $f(x)$ is continuous and thus integrable in every finite interval $a \leq x \leq b$.

1. **Comparison test for integrals with nonnegative integrands.**

(a) *Convergence.* Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_a^{\infty} g(x) dx$ converges. Then if $0 \leq f(x) \leq g(x)$ for all $x \geq a$, $\int_a^{\infty} f(x) dx$ also converges.

EXAMPLE. Since $\frac{1}{e^x + 1} \leq \frac{1}{e^x} = e^{-x}$ and $\int_a^{\infty} e^{-x} dx$ converges, $\int_0^{\infty} \frac{dx}{e^x + 1}$ also converges.

(b) *Divergence.* Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_a^{\infty} g(x) dx$ diverges. Then if $f(x) \leq g(x)$ for all $x \geq a$, $\int_a^{\infty} f(x) dx$ also diverges.

EXAMPLE. Since $\frac{1}{\ln x} > \frac{1}{x}$ for $x \geq 2$ and $\int_2^{\infty} \frac{dx}{x}$ diverges (p integral with $p = 1$), $\int_2^{\infty} \frac{dx}{\ln x}$ also diverges.

2. **Quotient test for integrals with nonnegative integrands.**

(a) If $f(x) \geq 0$ and $g(x) \geq 0$, and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = A \neq 0$ or ∞ , then $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ either both converge or both diverge.

(b) If $A = 0$ in (a) and $\int_a^{\infty} g(x) dx$ converges, then $\int_a^{\infty} f(x) dx$ converges.

(c) If $A = \infty$ in (a) and $\int_a^{\infty} g(x) dx$ diverges, then $\int_a^{\infty} f(x) dx$ diverges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $g(x) = 1/x^p$, we have, from known facts about the p integral, the following theorem.

Theorem 1 Let $\lim_{x \rightarrow \infty} x^p f(x) = A$. Then

(i) $\int_a^{\infty} f(x) dx$ converges if $p > 1$ and A is finite.

(ii) $\int_a^{\infty} f(x) dx$ diverges if $p \leq 1$ and $A \neq 0$ (A may be infinite).

3. **Series test for integrals with nonnegative integrands.** $\int_a^\infty f(x)dx$ converges or diverges according as $\sum u_n = f(n)$, converges or diverges.
4. **Absolute and conditional convergence.** $\int_a^\infty f(x)dx$ is called *absolutely convergent* if $\int_a^\infty |f(x)|dx$ converges. If $\int_a^\infty f(x)dx$ converges but $\int_a^\infty |f(x)|dx$ diverges, then $\int_a^\infty f(x)dx$ is called *conditionally convergent*.

Theorem 2 If $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE 1. $\int_a^\infty \frac{\cos x}{x^2+1}dx$ is absolutely convergent and thus convergent, since

$$\int_0^\infty \left| \frac{\cos x}{x^2+1} \right| dx \leq \int_0^\infty \frac{dx}{x^2+1} \text{ and } \int_0^\infty \frac{dx}{x^2+1} \text{ converges.}$$

EXAMPLE 2. $\int_0^\infty \frac{\sin x}{x}dx$ converges (see Problem 12.11), but $\int_0^\infty \left| \frac{\sin x}{x} \right| dx$ does not converge (see Problem 12.12). Thus, $\int_0^\infty \frac{\sin x}{x}dx$ is conditionally convergent.

Any of the tests used for integrals with nonnegative integrands can be used to test for absolute convergence.

Improper Integrals of the Second Kind

If $f(x)$ becomes unbounded only at the endpoint $x = a$ of the interval $a \leq x \leq b$, then we define

$$\int_a^b f(x)dx = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b f(x)dx \quad (4)$$

and define it to be an improper integral of the second kind. If the limit on the right of Equation (4) exists, we call the integral on the left *convergent*; otherwise, it is *divergent*.

Similarly, if $f(x)$ becomes unbounded only at the endpoint $x = b$ of the interval $a \leq x \leq b$, then we extend the category of improper integrals of the second kind.

$$\int_a^b f(x)dx = \lim_{\epsilon \rightarrow 0^+} \int_a^{b-\epsilon} f(x)dx \quad (5)$$

Note: Be alert to the word *unbounded*. This is distinct from *undefined*. For example, $\int_a^1 \frac{\sin x}{x}dx = \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^1 \frac{\sin x}{x}dx$ is a proper integral, since $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$ and, hence, is bounded as $x \rightarrow 0$ even though the function is undefined at $x = 0$. In such case the integral on the left of Equation (5) is called convergent or divergent according as the limit on the right exists or does not exist.

Finally, the category of improper integrals of the second kind also includes the case where $f(x)$ becomes unbounded only at an interior point $x = x_0$ of the interval $a \leq x \leq b$; then we define

$$\int_a^b f(x)dx = \lim_{\epsilon_1 \rightarrow 0^+} \int_a^{x_0-\epsilon_1} f(x)dx + \lim_{\epsilon_2 \rightarrow 0^+} \int_{x_0+\epsilon_2}^b f(x)dx \quad (6)$$

The integral on the left of Equation (6) converges or diverges according as the limits on the right exist or do not exist.

Extensions of these definitions can be made in case $f(x)$ becomes unbounded at two or more points of the interval $a \leq x \leq b$.

Cauchy Principal Value

It may happen that the limits on the right of Equation (6) do not exist when ϵ_1 and ϵ_2 approach zero independently. In such case it is possible that by choosing $\epsilon_1 = \epsilon_2 = \epsilon$ in (6), i.e., writing

$$\int_a^b f(x)dx = \lim_{\epsilon \rightarrow 0^+} \left\{ \int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right\} \quad (7)$$

the limit does exist. If the limit on the right of Equation (7) does exist, we call this limiting value the *Cauchy principal value* of the integral on the left. See Problem 12.14.

Special Improper Integrals of the Second Kind

1. $\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$.
2. $\int_a^b \frac{dx}{(b-x)^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

These can be called *p integrals of the second kind*. Note that when $p \leq 0$ the integrals are proper.

Convergence Tests for Improper Integrals of the Second Kind

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leq x \leq b$. Similar tests are available if $f(x)$ is unbounded at $x = b$ or at $x = x_0$ where $a < x_0 < b$.

1. **Comparison test** for integrals with nonnegative integrands.

(a) *Convergence.* Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)dx$ converges. Then if $0 \leq f(x) \leq g(x)$ for $a < x \leq b$, $\int_a^b f(x)dx$ also converges.

EXAMPLE. $\frac{1}{\sqrt{x^4-1}} < \frac{1}{\sqrt{x-1}}$ for $x > 1$. Then since $\int_1^5 \frac{dx}{\sqrt{x-1}}$ converges (p integral with $a = 1, p = \frac{1}{2}$),

$\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ also converges.

(b) *Divergence.* Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)dx$ diverges. Then if $f(x) \geq g(x)$ for $a < x \leq b$, $\int_a^b f(x)dx$ also diverges.

EXAMPLE. $\frac{\ln x}{(x-3)^4} > \frac{1}{(x-3)^4}$ for $x > 3$. Then since $\int_3^b \frac{dx}{(x-3)^4}$ diverges (p integral with $a = 3, p = 4$),

$\int_3^b \frac{\ln x}{(x-3)^4} dx$ also diverges.

2. **Quotient test** for integrals with nonnegative integrands.

(a) If $f(x) \geq 0$ and $g(x) \geq 0$ for $a < x \leq b$, and if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = A \neq 0$ or ∞ , then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ either both converge or both diverge.

(b) If $A = 0$ in (a), and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

(c) If $A = \infty$ in (a), and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

This test is related to the comparison test and is a very useful alternative to it. In particular, taking $g(x) = 1/(x-a)^p$ we have, from known facts about the p integral, the following theorems.

Theorem 3. Let $\lim_{x \rightarrow a^+} (x - a)^p f(x) = A$. Then

- (i) $\int_a^b f(x) dx$ converges if $p < 1$ and A is finite.
- (ii) $\int_a^b f(x) dx$ diverges if $p \geq 1$ and $A \neq 0$ (A may be infinite).

If $f(x)$ becomes unbounded only at the upper limit, these conditions are replaced by those in Theorem 4.

Theorem 4. Let $\lim_{x \rightarrow b^-} (b - x)^p f(x) = B$. Then

- (i) $\int_a^b f(x) dx$ converges if $p < 1$ and B is finite.
- (ii) $\int_a^b f(x) dx$ diverges if $p \geq 1$ and $B \neq 0$ (B may be infinite).

EXAMPLE 1. $\int_1^5 \frac{dx}{\sqrt{x^4 - 1}}$ converges, since $\lim_{x \rightarrow 1^+} (x - 1)^{1/2} \cdot \frac{1}{(x^4 - 1)^{1/2}} = \lim_{x \rightarrow 1^+} \sqrt{\frac{x - 1}{x^4 - 1}} = \frac{1}{2}$.

EXAMPLE 2. $\int_0^3 \frac{dx}{(3 - x)\sqrt{x^2 + 1}}$ diverges, since $\lim_{x \rightarrow 3^-} (3 - x) \cdot \frac{1}{(3 - x)\sqrt{x^2 + 1}} = \frac{1}{\sqrt{10}}$.

3. **Absolute and conditional convergence.** $\int_a^b f(x) dx$ is called *absolute convergent* if $\int_a^b |f(x)| dx$ converges. If $\int_a^b f(x) dx$ converges but $\int_a^b |f(x)| dx$ diverges, then $\int_a^b f(x) dx$ is called *conditionally convergent*.

Theorem 5. If $\int_a^b |f(x)| dx$ converges, then $\int_a^b f(x) dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE. $\left| \frac{\sin x}{\sqrt[3]{x - \pi}} \right| \leq \frac{1}{\sqrt[3]{x - \pi}}$ and $\int_{\pi}^{4\pi} \frac{dx}{\sqrt[3]{x - \pi}}$ converges (p integral with $a = \pi$, $p = \frac{1}{3}$), it follows that $\int_{\pi}^{4\pi} \left| \frac{\sin x}{\sqrt[3]{x - \pi}} \right| dx$ converges and thus $\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x - \pi}} dx$ converges (absolutely).

Any of the tests used for integrals with nonnegative integrands can be used to test for absolute convergence.

Improper Integrals Containing a Parameter, Uniform Convergence

Let

$$\phi(\alpha) = \int_a^{\infty} f(x, \alpha) dx \quad (8)$$

This integral is analogous to an infinite series of functions. In seeking conditions under which we may differentiate or integrate $\phi(\alpha)$ with respect to α , it is convenient to introduce the concept of convergence for integrals by analogy with infinite series.

We shall suppose that the integral (8) converges for $\alpha_1 \leq \alpha \leq \alpha_2$, or, briefly, $[\alpha_1, \alpha_2]$.

Definition. The integral (8) is said to be *uniformly convergent* in $[\alpha_1, \alpha_2]$ if for each $\epsilon > 0$ we can find a number N depending on ϵ but not on α , such that

$$\left| \phi(\alpha) - \int_a^u f(x, \alpha) dx \right| < \epsilon \quad \text{for all } u > N \text{ and all } \alpha \text{ in } [\alpha_1, \alpha_2]$$

This can be restated by noting that $\left| \phi(\alpha) - \int_a^u f(x, \alpha) dx \right| = \left| \int_u^{\infty} f(x, \alpha) dx \right|$, which is analogous in an infinite series to the absolute value of the remainder after N terms.

This definition and the properties of uniform convergence to be developed are formulated in terms of improper integrals of the first kind. However, analogous results can be given for improper integrals of the second and third kinds.

Special Tests for Uniform Convergence of Integrals

1. **Weierstrass M test.** If we can find a function $M(x) \geq 0$ such that

(a) $|f(x, \alpha)| \leq M(x)$ $\alpha_1 \leq \alpha \leq \alpha_2, x > a$

(b) $\int_a^\infty M(x) dx$ converges

then $\int_a^\infty f(x, \alpha) dx$ is uniformly and absolutely convergent in $\alpha_1 \leq \alpha \leq \alpha_2$.

2. **Dirichlet's test.** Suppose that

(a) $\psi(x)$ is a positive monotonic decreasing function which approaches zero as $x \rightarrow \infty$.

(b) $\left| \int_a^u f(x, \alpha) dx \right| < P$ for all $u > a$ and $\alpha_1 \leq \alpha \leq \alpha_2$.

Then the integral $\int_a^\infty f(x, \alpha) \psi(x) dx$ is uniformly convergent for $\alpha_1 \leq \alpha \leq \alpha_2$.

Theorems on Uniformly Convergent Integrals

Theorem 6. If $f(x, \alpha)$ is continuous for $x \in a$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and if $\int_a^\infty f(x, \alpha) dx$ is uniformly convergent for $\alpha_1 \leq \alpha \leq \alpha_2$, then $\phi(\alpha) = \int_a^\infty f(x, \alpha) dx$ is continuous in $\alpha_1 \leq \alpha \leq \alpha_2$. In particular, if α_0 is any point of $\alpha_1 \leq \alpha \leq \alpha_2$, we can write

$$\lim_{\alpha \rightarrow \alpha_0} \phi(\alpha) d\alpha = \lim_{\alpha \rightarrow \alpha_0} \int_a^\infty f(x, \alpha) dx = \int_a^\infty \lim_{\alpha \rightarrow \alpha_0} f(x, \alpha) dx \quad (9)$$

If α_0 is one of the endpoints, we use right- or left-hand limits.

Theorem 7. Under the conditions of Theorem 6, we can integrate $\phi(\alpha)$ with respect to α_1 to α_2 to obtain

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha) d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha) dx \right\} d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha) d\alpha \right\} dx \quad (10)$$

which corresponds to a change of the order of integration.

Theorem 8. If $f(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \in a$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and if $\int_a^\infty \frac{\partial f}{\partial \alpha} dx$ converges uniformly in $\alpha_1 \leq \alpha \leq \alpha_2$, then if a does not depend on α ,

$$\frac{d\phi}{d\alpha} = \int_a^\infty \frac{\partial f}{\partial \alpha} dx \quad (11)$$

If a depends on α , this result is easily modified (see Leibniz's rule, Page 198).

Laplace Transforms

The Laplace transform of a function $F(x)$ is defined as

$$f(s) = L\{F(x)\} = \int_0^\infty e^{-sx} F(x) dx \quad (12)$$

and is analogous to power series as seen by replacing $e^{-s} by t so that $e^{-sx} = t^x$. Many properties of power series also apply to Laplace transforms. Table 12.1, showing Laplace transforms, is useful. In each case, a is a real constant.$

TABLE 12.1

$F(x)$	$\mathcal{L}\{F(x)\}$
a	$\frac{a}{s} \quad s > 0$
e^{ax}	$\frac{1}{s-a} \quad s > a$
$\sin ax$	$\frac{a}{s^2 + a^2} \quad s > 0$
$\cos ax$	$\frac{s}{s^2 + a^2} \quad s > 0$
$x^n \quad n = 1, 2, 3, \dots$	$\frac{n!}{s^{n+1}} \quad s > 0$
$Y'(x)$	$s\mathcal{L}\{Y(x)\} - Y(0)$
$Y''(x)$	$s^2\mathcal{L}\{Y(x)\} - sY(0) - Y'(0)$

Linearity

The Laplace transform is a linear operator; i.e., $\zeta\{F(x) + G(x)\} = \zeta\{F(x)\} + \zeta\{G(x)\}$.

This property is essential for returning to the solution after having calculated in the setting of the transforms. (See the example that follows and the previously cited problems.)

Convergence

The exponential e^{-st} contributes to the convergence of the improper integral. What is required is that $F(x)$ does not approach infinity too rapidly as $x \rightarrow \infty$. This is formally stated as follows: If there is some constant a such that $|F(x)| \leq e^{ax}$ for all sufficiently large values of x , then $f(s) = \int_0^\infty e^{-sx} F(x) dx$ converges when $s > a$ and f has derivatives of all orders. (The differentiations of f can occur under the integral sign.)

Application

The feature of the Laplace transform that (when combined with linearity) establishes it as a tool for solving differential equations is revealed by applying integration by parts to $f(s) = \int_0^\infty e^{-st} F(t) dt$. By letting $u = F(t)$ and $dv = e^{-st} dt$, we obtain, after letting $x \rightarrow \infty$,

$$\int_0^\infty e^{-st} F(t) dt = \frac{1}{s} F(0) + \frac{1}{s} \int_0^\infty e^{-st} F'(t) dt$$

Conditions must be satisfied that guarantee the convergence of the integrals (for example, $e^{-st} F(t) \rightarrow 0$ as $t \rightarrow \infty$).

This result of integration by parts may be put in the form

$$(a) \quad \zeta\{F'(t)\} = s\zeta\{F(t)\} - F(0).$$

Repetition of the procedure combined with a little algebra yields

$$(b) \quad \zeta\{F''(t)\} = s^2\zeta\{F(t)\} - sF(0) - F'(0).$$

The Laplace representation of derivatives of the order needed can be obtained by repeating the process.