Separable First-Order Differential Equations.

General solution: $A(x)\,dx + B(y)\,dy = 0 \Rightarrow \int A(x)\,dx + \int B(y)\,dy = c$

Solutions to Initial Value Problem: $A(x)\,dx + B(y)\,dy = 0, \quad y(x_0) = y_0.$

$$\int_{x_0}^{x} A(x)\,dx + \int_{y_0}^{y} B(y)\,dy = 0$$

Reduction of Homogenous Eqns: Homogenous differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be transferred into a seperable equation by making substitution

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Example: Solve $x\,dx - y^2\,dy = 0$

$A(x) = x, \quad B(y) = -y^2$

$$\int x\,dx + \int(-y^2)\,dy = c \Rightarrow x^2/2 - y^3/3 = c$$

$$\Rightarrow y = \left(\tfrac{3}{2}x^2 + k\right)^{1/3}, \quad k = -3c.$$

Example:

$$y' = y^2x^3 \Rightarrow x^3\,dx - (1/y^2)\,dy = 0$$

$$\Rightarrow \int x^3\,dx + \int(-1/y^2)\,dy = c$$

$$\Rightarrow y = \frac{-4}{x^4 + k}, \quad k = -4c.$$

Example:

$$dy = 2t(y^2+9)\,dt \Rightarrow \frac{dy}{y^2+9} - 2t\,dt = 0$$

$$\Rightarrow \int \frac{dy}{y^2+9} - \int 2t\,dt = 0.$$

$$\Rightarrow \tfrac{1}{3}\arctan(y/3) - t^2 = c$$

$$\Rightarrow y = 3\tan(3t^2 + k)$$

Example : $\frac{dx}{dt} = x^2 - 2x + 2 \Rightarrow \int \frac{dx}{x^2-2x+2} - \int dt = 0$.

$$\Rightarrow \int \frac{dx}{(x-1)^2+1} - \int dt = 0$$

$$\Rightarrow \arctan(x-1) - t = 0$$

$$\Rightarrow x = 1 + \tan(t+c)$$

Example : $e^x dx - y dy = 0, \quad y(0) = 1$

$$\int e^x dx + \int (-y) dy = c \quad \Rightarrow y^2 = 2e^x + k$$

$$\Rightarrow y = \sqrt{2e^x - 1}$$ using boundary condition

Example : $y' = \frac{y+x}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{xv+x}{x}$

$$\Rightarrow x\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{1}{x}dx - dv = 0$$

$$\Rightarrow \int \frac{1}{x}dx - \int dv = c$$

$$\Rightarrow v = \ln|x| - c$$

$$\Rightarrow v = \ln|kx|$$

Example : $y' = \frac{2y^4+x^4}{xy^3}$

Let $f(x,y) = \frac{2y^4+x^4}{xy^3}$ where $f(tx,ty) = \frac{2(ty)^4+(tx)^4}{(tx)(ty)^3} = \frac{t^4}{t^4} \cdot \frac{2y^4+x^4}{xy^3}$

$\therefore$ Eqn is homogenous.

$$v + x\frac{dv}{dx} = \frac{2(xv)^4+x^4}{x(xv)^3}$$

$$x\frac{dv}{dx} = \frac{v^4+1}{v^3} \quad \text{or} \quad \frac{1}{x}dx - \frac{v^3}{v^4+1}dv = 0.$$

$$\int \frac{1}{x}dx - \int \frac{v^3}{v^4+1}dv = c$$

$$\Rightarrow |x| - \frac{1}{4}\ln(v^4+1) = c$$

$$\Rightarrow v^4 + 1 = (kx)^4$$

$$\Rightarrow y^4 = c_1 x^3 - x^4 \quad (c_1 = k).$$