Second-order Linear Homogeneous Differential Equations

Characteristic Equation:

$y'' + a_1 y' + a_0 y = 0$ where a_1, a_0 are constants.

$\lambda^2 + a_1\lambda + a_0 = 0$ is characteristic equation.

Case 1:

λ_1, λ_2 are real and distinct.

Two linearly independent solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$

General solution is: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Case 2:

$\lambda_1 = a + ib$, a complex number

The other root is $\lambda_2 = a - ib$

Two linearly independent solutions are $e^{(a+ib)x}$ and $e^{(a-ib)x}$

General solution is: $y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x}$

which is algebraically equivalent to

$$y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$$

Case 3:

$\lambda_1 = \lambda_2$, two linearly independent solutions are $e^{\lambda_1 x}$ and $xe^{\lambda_2 x}$.

General solution is:

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda_1 x}$$

Example:

$y'' - y' - 2y = 0$, characteristic eqn is $\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda+1)(\lambda-2) = 0$

Roots are $\lambda_1 = -1$, $\lambda_2 = 2$ are real and distinct.

Solution is $y = c_1 e^{-x} + c_2 e^{2x}$

Example:

$y'' - 7y' = 0 \Rightarrow \lambda^2 - 7\lambda = 0$ is characteristic eqn

$\Rightarrow \lambda_1 = 0, \lambda_2 = 7$ are real and distinct

$$y = c_1 e^{0x} + c_2 e^{7x} = c_1 + c_2 e^{7x}$$

Example : $y'' - 5y = 0$, characteristic eqn $\lambda^2 - 5 \Leftrightarrow (\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$.

$$\Rightarrow \qquad y = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}$$

$$e^{\lambda x} = \cosh \lambda x + \sinh \lambda x$$

and $e^{-\lambda x} = \cosh \lambda x - \sinh \lambda x$

$$\begin{aligned} y &= c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x} \\ &= c_1 (\cosh \sqrt{5}x + \sinh \sqrt{5}x) + c_2 (\cosh \sqrt{5}x - \sinh \sqrt{5}x) \\ &= (c_1 + c_2) \cosh \sqrt{5}x + (c_1 - c_2) \sinh \sqrt{5}x \\ &= k_1 \cos \sqrt{5}x + k_2 \sinh \sqrt{5}x \end{aligned}$$

Example : $\ddot{x} - 0.01x = 0$, characteristic eqn is $\lambda^2 - 0.01 = 0 \Leftrightarrow (\lambda - 0.1)(\lambda + 0.1) = 0$

$\lambda_1 = 0.1$, $\lambda_2 = -0.1$ are real and distinct.

General solution : $y = c_1 e^{0.1t} + c_2 e^{-0.1t}$

or $\qquad y = k_1 \cosh(0.1t) + k_2 \sinh(0.1t)$

Example : $y'' + 4y' + 5y = 0$, $\lambda^2 + 4\lambda + 5 = 0$, $\lambda = \dfrac{-(4) \pm \sqrt{(4)^2 - 4(5)}}{2} = -2 \pm i$.

Roots are complex, $a = -2$, $b = 1$:

$$y = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x)$$

Example : $y'' + 4y = 0$, $\lambda^2 + 4\lambda = 0$, $(\lambda - 2i)(\lambda + 2i) = 0$, $a = 0, b = 2$.

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Example : $y'' - 8y' + 16y = 0$, $\lambda^2 - 8\lambda + 16 = 0 \Leftrightarrow (\lambda - 4)^2 = 0$, $\lambda_1 = \lambda_2 = 4$.

$$y = c_1 e^{4x} + c_2 x e^{4x}.$$