# CHAPTER 1 - NUMBERS

## Operations with Real Numbers

If $a$, $b$, $c$ belong to the set $R$ of real numbers, then:

1. $a + b$ and $ab$ belong to $R$ — Closure law
2. $a + b = b + a$ — Commutative law of addition
3. $a + (b + c) = (a + b) + c$ — Associative law of addition
4. $ab = ba$ — Commutative law of multiplication
5. $a(bc) = (ab)c$ — Associative law of multiplication
6. $a(b + c) = ab + ac$ — Distributive law
7. $a + 0 = 0 + a = a$, $1 \cdot a = a \cdot 1 = a$

   0 is called the *identity with respect to addition*; 1 is called the *identity with respect to multiplication*.

8. For any $a$ there is a number $x$ in $R$ such that $x + a = 0$.
   $x$ is called the *inverse of a with respect to addition* and is denoted by $-a$.

9. For any $a \neq 0$ there is a number $x$ in $R$ such that $ax = 1$.
   $x$ is called the *inverse of a with respect to multiplication* and is denoted by $a^{-1}$ or $1/a$.

Convention: For convenience, operations called subtraction and division are defined by $a - b = a + (-b)$ and $\frac{a}{b} = ab^{-1}$, respectively.

These enable us to operate according to the usual rules of algebra. In general, any set, such as $R$, whose members satisfy the preceding is called a *field*.

## Inequalities

If $a$, $b$, and $c$ are any given real numbers, then:

1. Either $a > b$, $a = b$ or $a < b$ — Law of trichotomy
2. If $a > b$ and $b > c$, then $a > c$ — Law of transitivity
3. If $a > b$, then $a + c > b + c$
4. If $a > b$ and $c > 0$, then $ac > bc$
5. If $a > b$ and $c < 0$, then $ac < bc$

   **EXAMPLES.** $3 < 5$ or $5 > 3$; $-2 < -1$ or $-1 > -2$; $x \leqq 3$ means that $x$ is a real number which may be 3 or less than 3.

## Absolute Value of Real Numbers

1. $|ab| = |a|\ |b|$ or $|abc \ldots m| = |a|\ |b|\ |c| \ldots |m|$
2. $|a+b| \leqq |a| + |b|$ or $|a+b+c+\cdots+m| \leqq |a| + |b| + |c| + \cdots |m|$
3. $|a-b| \geqq |a| - |b|$

**EXAMPLES.** $|-5| = 5$, $|+2| = 2$, $\left|-\frac{3}{4}\right| = \frac{3}{4}$, $|-\sqrt{2}| = \sqrt{2}$, $|0| = 0$.

The distance between any two points (real numbers) $a$ and $b$ on the real axis is $|a-b| = |b-a|$.

## Exponents and Roots

The product $a \cdot a \ldots a$ of a real number $a$ by itself $p$ times is denoted by $a^p$, where $p$ is called the *exponent* and $a$ is called the *base*. The following rules hold:

1. $a^p \cdot a^q = a^{p+q}$
2. $\dfrac{a^p}{a^q} = a^{p-q}$
3. $(a^p)^r = a^{pr}$
4. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$

## Logarithms

If $a^p = N$, $p$ is called the *logarithm* of $N$ to the base $a$, written $p = \log_a N$. If $a$ and $N$ are positive and $a \neq 1$, there is only one real value for $p$. The following rules hold:

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$
3. $\log_a M^r = r \log_a M$

## Point Sets, Intervals

A set of points (real numbers) located on the real axis is called a *one-dimensional point set*.

The set of points $x$ such that $a \leqq x \leqq b$ is called a *closed interval* and is denoted by $[a, b]$. The set $a < x < b$ is called an *open interval*, denoted by $(a, b)$. The sets $a < x \leqq b$ and $a \leqq x < b$, denoted by $(a, b]$ and $[a, b)$, respectively, are called *half-open* or *half-closed* intervals.

The symbol $x$, which can represent any number or point of a set, is called a *variable*. The given numbers $a$ or $b$ are called *constants*.

## Neighborhoods

The set of all points $x$ such that $|x-a| < \delta$, where $\delta > 0$, is called a $\delta$ *neighborhood* of the point $a$. The set of all points $x$ such that $0 < |x-a| < \delta$, in which $x = a$ is excluded, is called a *deleted* $\delta$ *neighborhood* of $a$ or an open ball of radius $\delta$ about $a$.

## Limit Points

A *limit point, point of accumulation*, or *cluster point* of a set of numbers is a number $l$ such that every deleted $\delta$ neighborhood of $l$ contains members of the set; that is, no matter how small the radius of a ball about $l$, there are points of the set within it. In other words, for any $\delta > 0$, however small, we can always find a member $x$ of the set which is not equal to $l$ but which is such that $|x - l| < \delta$. By considering smaller and smaller values of $\delta$, we see that there must be infinitely many such values of $x$.

A finite set cannot have a limit point. An infinite set may or may not have a limit point. Thus, the natural numbers have no limit point, while the set of rational numbers has infinitely many limit points.

A set containing all its limit points is called a *closed set*. The set of rational numbers is not a closed set, since, for example, the limit point $\sqrt{2}$ is not a member of the set (Problem 1.5). However, the set of all real numbers $x$ such that $0 \leqq x \leqq 1$ is a closed set.

## Bounds

If for all numbers $x$ of a set there is a number $M$ such that $x \leqq M$, the set is *bounded above* and $M$ is called an *upper bound*. Similarly if $x \geqq m$, the set is *bounded below* and $m$ is called a *lower bound*. If for all $x$ we have $m \leqq x \leqq M$, the set is called *bounded*.

If $\underline{M}$ is a number such that no member of the set is greater than $\underline{M}$ but there is at least one member which exceeds $\underline{M} - \epsilon$ for every $\epsilon > 0$, then $\underline{M}$ is called the *least upper bound* (l.u.b.) of the set. Similarly, if no member of the set is smaller than $\bar{m} + \epsilon$ for every $\epsilon > 0$, then $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the set.

## Bolzano-Weierstrass Theorem

The Bolzano-Weierstrass theorem states that every bounded infinite set has at least one limit point. A proof of this is given in Problem 2.23.

## Algebraic and Transcendental Numbers

A number $x$ which is a solution to the *polynomial equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

where $a_0 \neq 0, a_1, a_2, \ldots, a_n$ are integers and $n$ is a positive integer, called the *degree* of the equation, is called an *algebraic number*. A number which cannot be expressed as a solution of any polynomial equation with integer coefficients is called a *transcendental number*.

## The Complex Number System

Equations such as $x^2 + 1 = 0$ have no solution within the real number system. Because these equations were found to have a meaningful place in the mathematical structures being built, various mathematicians of the late nineteenth and early twentieth centuries developed an extended system of numbers in which there were solutions. The new system became known as the *complex number system*. It includes the real number system as a subset.

We can consider a complex number as having the form $a + bi$, where $a$ and $b$ are real numbers called the *real* and *imaginary parts*, and $i = \sqrt{-1}$ is called the *imaginary unit*. Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$. We can consider real numbers as a subset of the set of complex numbers with $b = 0$. The complex number $0 + 0i$ corresponds to the real number 0.

The *absolute value* or *modulus* of $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$. The *complex conjugate* of $a + bi$ is defined as $a - bi$. The complex conjugate of the complex number $z$ is often indicated by $\overline{z}$ or $z^*$.

## Polar Form of Complex Numbers

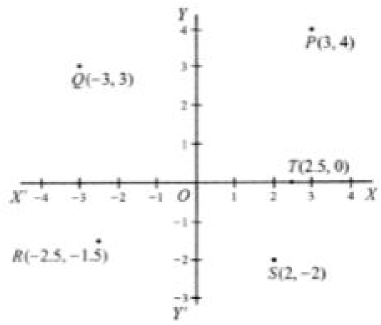


Figure 1.2

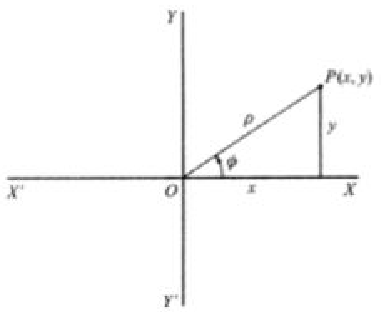


Figure 1.3

Since a complex number $x + iy$ can be considered as an ordered pair $(x, y)$, we can represent such numbers by points in an $xy$ plane called the *complex plane* or *Argand diagram*. Referring to Figure 1.3, we see that $x = \rho \cos \phi$, $y = \rho \sin \phi$, where $\rho = \sqrt{x^2 + y^2} = |x + iy|$ and $\phi$, called the *amplitude* or *argument*, is the angle which line $OP$ makes with the positive $x$ axis $OX$. It follows that

$$z = x + iy = \rho(\cos \phi + i \sin \phi) \tag{2}$$

called the *polar form* of the complex number, where $\rho$ and $\phi$ are called *polar coordinates*. It is sometimes convenient to write cis $\phi$ instead of $\cos \phi + i \sin \phi$.

If $z_1 = x_1 + iy_i = \rho_1 (\cos \phi_1 + i \sin \phi_1)$ and $z_2 = x_2 + iy_2 = \rho_2(\cos\phi_2 + i \sin \phi_2)$ and by using the addition formulas for sine and cosine, we can show that

$$z_1 z_2 = \rho_1 \rho_2 \{\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)\} \tag{3}$$

$$\frac{z_1}{z_2} = \frac{p_1}{p_2}\{\cos(\phi_1 - \phi_2) + i \sin(\phi_1 - \phi_2)\} \tag{4}$$

$$z^n = \{\rho(\cos \phi + i \sin \phi)\}^n = \rho^n(\cos n\phi + i \sin n\phi) \tag{5}$$

where $n$ is any real number. Equation (5) is sometimes called *De Moivre's theorem*. We can use this to determine roots of complex numbers. For example, if $n$ is a positive integer,

$$\begin{aligned} z^{1/n} &= \{p(\cos \phi + i \sin \phi)\}^{1/n} \\ &= p^{1/n}\left\{\cos\left(\frac{\phi + 2k\pi}{n}\right) + i \sin\left(\frac{\phi + 2k\pi}{n}\right)\right\} \qquad k = 0, 1, 2, 3, \ldots, n-1 \end{aligned} \tag{6}$$

from which it follows that there are in general $n$ different values of $z^{1/n}$. In Chapter 11 we will show that $e^{i\phi} = \cos \phi + i \sin \phi$ where $e = 2.71828\ldots$. This is called *Euler's formula*.

## Mathematical Induction

The principle of *mathematical induction* is an important property of the positive integers. It is especially useful in proving statements involving all positive integers when it is known, for example, that the statements are valid for $n = 1, 2, 3$ but it is *suspected* or *conjectured* that they hold for all positive integers. The method of proof consists of the following steps:

1. Prove the statement for $n = 1$ (or some other positive integer).
2. Assume the statement is true for $n = k$, where $k$ is any positive integer.
3. From the assumption in 2, prove that the statement must be true for $n = k + 1$. This is part of the proof establishing the induction and may be difficult or impossible.
4. Since the statement is true for $n = 1$ (from Step 1) it must (from Step 3) be true for $n = 1 + 1 = 2$ and from this for $n = 2 + 1 = 3$, and so on, and so must be true for all positive integers. (This assumption, which provides the link for the truth of a statement for a finite number of cases to the truth of that statement for the infinite set, is called the *axiom of mathematical induction.*)