Basic Probability.

Sample Space. A set S that consists of all possible outcomes of a random experiment is the sample space, and each outcome is a sample point.
e.g dice roll is $\{1, 2, 3, 4, 5, 6\}$

If we toss a coin twice and use 0,1 to represent outcomes the sample space is
$\{(0,0), (1,0), (0,1), (1,1)$.

Finite Sample Space: Cantably infinite sample space, also discrete sample space.

Events: An event is a subset A of sample space S. e.g if we toss a coin twice and expect heads once, the subset is $\{(0,1), (1,0)\}$.

Empty set: The empty set $\emptyset$ is the impossible event.

Set operations:

1. $A \cup B$ either A or B (union)
2. $A \cap B$ both A and B (intersection)
3. A' not A (compliment of A).
4. $A - B = A \cap B$ A but not B, also $A' = S - A$

Mutually exclusive: A and B are disjoint and $A \cap B = \emptyset$.
e.g. say at least one head occurs is "A" and "B" is the second toss is tail.

$A = \{HT, TH, HH\}$, $B = \{HT, TT\}$

$\therefore$ $A \cup B = \{HT, TH, HH, TT\}$
$A \cap B = \{HT\}$.
$A' = \{TT\}$
$A - B = \{TH, HH\}$.

Estimate probability: If event can occur in h different ways, the probability of event is h/n

Frequency: If after n repititions of an experiment, where n is large, probability is h/n, the empirical probability.

Axioms:

1. For every event A is class C, $P(A) \geq 0$
2. For certain events S in C, $P(S) = 1$.
3. For mutually exclusive events A_1, A_2 in class C, $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Theorems:

1. $P(A_1) \leq P(A_2)$ and $P(A_2 - A_1) = P(A_2) - P(A_1)$
2. For every event A, $0 \leq P(A) \leq 1$.
3. $P(\phi) = 0$
4. If A' is the compliment of A then $P(A') = 1 - P(A)$.
5. If $A = A_1 \cup A_2 \cup \ldots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive then $P(A) = P(A_1) + P(A_2) + \ldots + P(A_n)$

   Also, if $A = S$ then

   $P(A_1) + P(A_2) + \ldots + P(A_n) = 1$
6. If A and B are any two events then $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

   More generally if A_1, A_2, A_3 are any three events then

$$\boxed{\begin{aligned} P(A_1 \cup A_2 \cup A_3) = {} & P(A_1) + P(A_2) + P(A_3) \\ & - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) \\ & + P(A_1 \cap A_2 \cap A_3). \end{aligned}}$$

7. For any events $P(A) = P(A \cap B) + P(A \cap B')$.

Example: 2 or 5 on roll of die: assign equal probabilities $P(1) = P(2) = \ldots P(6) = 1/6$

$$P(2 \cup 5) = P(2) + P(5) = 1/3.$$

Conditional Probability: $P(B|A)$ = "probability of B given A has occurred"

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}} \quad \text{or} \quad P(A \cap B) = P(A)P(B|A).$$

Example : Probability single toss of die is number less than 4 if (a). no other info and (b) it is given that the toss resulted in an odd number.

(a). Let B denote event $\{$ less than 4 $\}$

$$P(B) = P(1) + P(2) + P(3) = 1/2$$

(b). Let $A = \{$ odd number $\}$

$$P(A) = 1/2 \quad \text{and} \quad P(A \cap B) = 1/3$$

$$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = 2/3.$$

Therems :

1. For any three events A_1, A_2, A_3 :

$$\boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)}$$

2. $$\boxed{P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \ldots + P(A_n)P(A|A_n)}$$

Independent Events : If $P(B|A) = P(B)$ i.e B **not affected by** A then A and B are **independent events**.

e.g. $\boxed{P(A \cap B) = P(A)P(B)}$

A_1, A_2, A_3 are pairwise independent if $P(A_j \cap A_k) = P(A_j)P(A_k)$, $j \neq k$.

and $\boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)}$.

Bayes Theorem: Suppose $A_1, A_2, \ldots A_n$ are **mutually exclusive**. If A is any event we have :

$$\boxed{P(A_k | A) = \frac{P(A_k)P(A|A_k)}{\sum_{j=1}^{n} P(A_j)P(A|A_j)}}$$

i.e the probability of events $A_1, A_2, \ldots, A_n$ that cause A to occur.

Permutations : Given n distinct objects **arranged** in r **ways**, there are n ways of choosing the 1st object, $n-1$ ways of choosing the 2nd object and $n-r+1$ ways of choosing the rth:

$$\boxed{{}_nP_r = n(n-1)(n-2) \ldots (n-r+1)}$$

in the partialar case where $r = n$, $\quad {}_nP_r = n(n-1)(n-2)\cdots 1 = n!$

$$\boxed{{}_nP_r = n! \,/\, (n-r)!}$$

Example : Number of arrangements of 3 letters from A, B, C, D, E, F, G :

$${}_7P_3 = 7!/3! = 7.6.5 = 210.$$

Theorem : Suppose a sets S of n objects of which n_1 are of one type, n_2 of another, and n_k of the kth type, here $n = n_1 + n_2 + \dots + n_k$:

$$\boxed{{}_nP_{n_1, n_2, \dots, n_k} = n! \,/\, n_1!\, n_2! \dots n_k!}$$

Example : The number of different permutations of MISSISSIPPI which has 1M, 4Is, 4S and 2Ps :

$11! \,/\, 1!\,4!\,4!\,2! = 34{,}650.$

Combinations : A permutation is about order and arrangement, but combinatorics ignores order :

$$\boxed{\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{{}_nP_r}{r!}}$$

Also, $\boxed{\binom{n}{r} = \binom{n}{n-r} \text{ or } {}_nC_r = {}_nC_{n-r}}$

Example: ${}_8C_3 = \binom{8}{3} = 8.7.6 \,/\, 3! = 56.$

Binomial Coefficient :

$$\boxed{(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n.}$$

Example :

$$(x+y)^4 = x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Stirlings Approximation:

when n is large and direct evaluation of $n!$ is impractical:

$$\boxed{n! \sim \sqrt{2\pi n}\, n^n e^{-n} \quad \text{where } e = 2.71828\ldots}$$