CHAPTER 3 - FUNCTIONS, LIMITS AND CONTINUITY

Functions

A function is composed of a domain set, a range set, and a rule of correspondence that assigns exactly one element of the range to each element of the domain.

Graph of a Function

A function f establishes a set of ordered pairs (x, y) of real numbers. The plot of these pairs $[x, f(x)]$ in a co-ordinate system is the graph of f. The result can be thought of as a pictorial representation of the function.

For example, the graphs of the functions described by $y = x^2$, $-1 \leqq x \leqq 1$, and $y^2 = x$, $0 \leqq x \leqq 1$, $y \geqq 0$ appear in Figure 3.1.

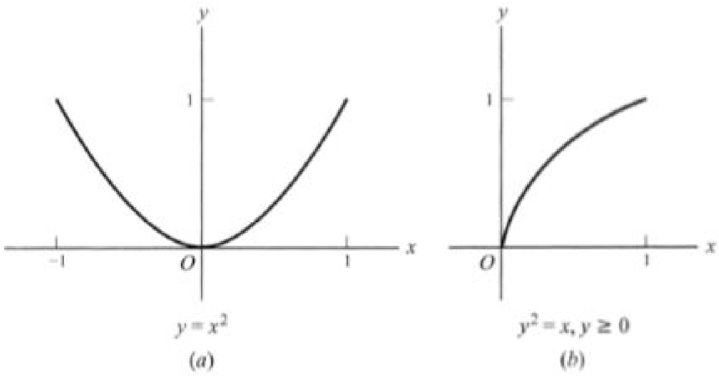


Figure 3.1

Bounded Functions

If there is a constant M such that $f(x) \leqq M$ for all x in an interval (or other set of numbers), we say that f is *bounded above* in the interval (or the set) and call M an *upper bound* of the function.

If a constant m exists such that $f(x) \geqq m$ for all x in an interval, we say that $f(x)$ is *bounded below* in the interval and call m a *lower bound.*

If $m \leqq f(x) \leqq M$ in an interval, we call $f(x)$ *bounded*. Frequently, when we wish to indicate that a function is bounded, we write $|f(x)| < P$.

If $f(x)$ has an upper bound, it has a *least upper bound* (l.u.b.); if it has a lower bound, it has a *greatest lower bound* (g.l.b.). (See Chapter 1 for these definitions.)

Monotonic Functions

A function is called *monotonic increasing* in an interval if for any two points x_1 and x_2 in the interval $x_1 < x_2$, $f(x_1) \leqq f(x_2)$. If $(f(x_1) < f(x_2)$, the function is called *strictly increasing*.

Similarly, if $f(x_1) \geqq f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is *monotonic decreasing,* while if $f(x_1) > f(x_2)$, it is *strictly decreasing*.

Inverse Functions, Principal Values

Suppose y is the range variable of a function f with domain variable x. Furthermore, let the correspondence between the domain and range values be one-to-one. Then a new function f^{-1}, called the *inverse function* of f, can be created by interchanging the domain and range of f. This information is contained in the form $x = f^{-1}(y)$.

If the domain and range elements of f are not in one-to-one correspondence (this would mean that distinct domain elements have the same image), then a collection of one-to-one functions may be created. Each of them is called a *branch*. It is often convenient to choose one of these branches, called the *principal branch*, and denote it as the inverse function f^{-1}. The range values of f that compose the principal branch, and hence the domain of f^{-1}, are called the *principal values*. (As will be seen in the section on elementary functions, it is common practice to specify these principal values for that class of functions.)

Maxima and Minima

The seventeenth-century development of the calculus was strongly motivated by questions concerning extreme values of functions. Of most importance to the calculus and its applications were the notions of *local extrema*, called the *relative maximum* and *relative minimum*.

If the graph of a function were compared to a path over hills and through valleys, the local extrema would be the high and low points along the way. This intuitive view is given mathematical precision by the following definition.

Definition If there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all x other than c in the interval, then $f(c)$ is a *relative maximum* of f. If $f(x) > f(c)$ for all x in (a, b) other than c, then $f(c)$ is a *relative minimum* of f. (See Figure 3.3.)

Definition If c is in the domain of f and for all x in the domain of the function $f(x) \leqq f(c)$; then $f(c)$ is an *absolute maximum* of the function f. If for all x in the domain $f(x) \geqq f(c)$, then $f(c)$ is an *absolute minimum* of f. (See Figure 3.3.)

Types of Functions

1. **Polynomial functions** have the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1}$$

where $a_0, \ldots, a_n$ are constants and n is a positive integer called the *degree* of the polynomial if $a_0 \neq 0$.

2. **Algebraic functions** are functions $y = f(x)$ satisfying an equation of the form

$$p_0(x)\, y^n + p_1(x)\, y^{n-1} + \cdots + p_{n-1}(x)\, y + p_n(x) = 0 \tag{2}$$

where $p_0(x), \ldots, p_n(x)$ are polynomials in x.

If the function can be expressed as the quotient of two polynomials, i.e., $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, it is called a *rational algebraic function*; otherwise, it is an *irrational algebraic function*.

3. **Transcendental functions** are functions which are not algebraic; i.e., they do not satisfy equations of the form of Equation (2).

Note the analogy with real numbers, polynomials corresponding to integers, rational functions to rational numbers, and so on.

Transcendental Functions

The following are sometimes called *elementary transcendental functions*.

1. **Exponential function:** $f(x) = a^x$, $a \neq 0, 1$. For properties, see Page 4.

2. **Logarithmic function:** $f(x) = \log_a x$, $a \neq 0, 1$. This and the exponential function are inverse functions. If $a = e = 2.71828\ldots$, called the *natural base of logarithms*, we write $f(x) = \log_e x = \text{In } x$, called the *natural logarithm* of x. For properties, see Page 4.

3. **Trigonometric functions** (also called *circular functions* because of their geometric interpretation with respect to the unit circle):

$$\sin x, \cos x, \tan x = \frac{\sin x}{\cos x}, \csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

The variable x is generally expressed in radians (π radians = 180º). For real values of x, $\sin x$ and $\cos x$ lie between –1 and 1 inclusive.

The following are some properties of these functions:

$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \text{sce}^2 x \qquad 1 + \cot^2 x = \csc^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sin(-x) = -\sin x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cos(-x) = \cos x$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \qquad \tan(-x) = -\tan x$

4. **Inverse trigonometric functions.** The following is a list of the inverse trigonometric functions and their principal values:

(*a*) $y = \sin^{-1} x$, $(-\pi/2 \leqq y \leqq \pi/2)$ (*d*) $y = \csc^{-1} x = \sin^{-1} 1/x$, $(-\pi/2 \leqq y \leqq \pi/2)$

(*b*) $y = \cos^{-1} x$, $(0 \leqq y \leqq \pi)$ (*e*) $y = \sec^{-1} x = \cos^{-1} 1/x$, $(0 \leqq y \leqq \pi)$

(*c*) $y = \tan^{-1} x$, $(-\pi/2 < y < \pi/2)$ (*f*) $y = \cot^{-1} x = \pi/2 - \tan^{-1} x$, $(0 < y < \pi)$

5. **Hyperbolic functions** are defined in terms of exponential functions as follows. These functions may be interpreted geometrically, much as the trigonometric functions but with respect to the unit hyperbola.

(*a*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (*d*) $\text{csch } x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(*b*) $\cosh x = \dfrac{e^x + e^{-x}}{2}$ (*e*) $\text{sec h } x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(*c*) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (*f*) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

The following are some properties of these functions:

$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \text{sech}^2 x \qquad \coth^2 x - 1 = \text{csch}^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \qquad \sinh(-x) = -\sinh x$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \qquad \cosh(-x) = \cosh x$

$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \qquad \tanh(-x) = -\tanh x$

6. **Inverse hyperbolic functions.** If $x = \sinh y$, then $y = \sinh^{-1} x$ is the *inverse hyperbolic sine* of x. The following list gives the principal values of the inverse hyperbolic functions in terms of natural logarithms and the domains for which they are real.

(*a*) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, all x

(*b*) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, $x \geqq 1$

(*c*) $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

(*d*) $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right)$, $x \neq 0$

(*e*) $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \leqq 1$

(*f*) $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x| > 1$

Limits of Functions

Let $f(x)$ be defined and single-valued for all values of x near $x = x_0$ with the possible exception of $x = x_0$ itself (i.e., in a deleted δ neighborhood of x_0). We say that the number l is the *limit of $f(x)$ as x approaches x_0* and write $\lim_{x \to x_0} f(x) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$. In such a case we also say that $f(x)$ approaches l as x approaches x_0 and write $f(x) \to l$ as $x \to x_0$.

In words, this means that we can make $f(x)$ arbitrarily close to l by choosing x sufficiently close to x_0.

Right- and Left-Hand Limits

In the definition of limit, no restriction was made as to how x should approach x_0. It is sometimes found convenient to restrict this approach. Considering x and x_0 as points on the real axis where x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0+$ and $x \to x_0-$.

If $\lim_{x \to x_0} + f(x) = l_1$ and $\lim_{x \to x_{0^-}} f(x) = l_2$, we call l_1 and l_{l2}, respectively, the *right- and left-hand limits* of f at x_0 and denote them by $f(x_0+)$ or $f(x_0 + 0)$ and $f(x_0-)$ or $f(x_0 - 0)$. The ϵ, δ definitions of limit of $f(x)$ as $x \to x_0+$ or $x \to x_0-$ are the same as those for $x \to x_0$ except for the fact that values of x are restricted to $x > x_0$ or $x < x_0$, respectively.

We have $\lim_{x \to x_0} f(x) = l$ if and only if $\lim_{x \to x_0+} f(x) = \lim_{x \to x_{0-}} fs(x) = l$.

Theorems on Limits

If $\lim\limits_{x \to x_0} f(x) = A$ and $\lim\limits_{x \to x_0} g(x) = B$, then

1. $\lim\limits_{x \to x_0} (f(x) + g(x)) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x) = A + B$

2. $\lim\limits_{x \to x_0} (f(x) - g(x)) = \lim\limits_{x \to x_0} f(x) - \lim\limits_{x \to x_0} g(x) = A - B$

3. $\lim\limits_{x \to x_0} (f(x)g(x)) = \left(\lim\limits_{x \to x_0} f(x)\right)\left(\lim\limits_{x \to x_0} g(x)\right) = AB$

4. $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)} = \dfrac{A}{B}$ if $B \neq 0$

Similar results hold for right- and left-hand limits.

Infinity

It sometimes happens that as $x \to x_0$, $f(x)$ increases or decreases without bound. In such case it is customary to write $\lim\limits_{x \to x_0} f(x) = +\infty$ or $\lim\limits_{x \to x_0} f(x) = -\infty$, respectively. The symbols $+\infty$ (also written ∞) and $-\infty$ are read "plus infinity" (or "infinity") and "minus infinity," respectively, but it must be emphasized that they are not numbers.

In precise language, we say that $\lim\limits_{x \to x_0} f(x) = \infty$ if for each positive number M we can find a positive number δ (depending on M in general) such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$. Similarly, we say that $\lim\limits_{x \to x_0} f(x) = -\infty$ if for each positive number M we can find a positive number δ such that $f(x) < -M$ whenever $0 < |x - x_0| < \delta$. Analogous remarks apply in case $x \to x_0 +$ or $x \to x_0 -$.

Frequently we wish to examine the behavior of a function as x increases or decreases without bound. In such cases it is customary to write $x \to +\infty$ (or ∞) or $x \to -\infty$, respectively.

We say that $\lim\limits_{x \to +\infty} f(x) = l$, or $f(x) \to l$ as $x \to +\infty$, if for any positive number ϵ we can find a positive number N (depending on ϵ in general) such that $|f(x) - l| < \epsilon$ whenever $x > N$. A similar definition can be formulated for $\lim\limits_{x \to -\infty} f(x)$.

Special Limits

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ $\qquad$ $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

2. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$ $\qquad$ $\lim\limits_{x \to 0+} (1 + x)^{1/x} = e$

3. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ $\qquad$ $\lim\limits_{x \to 1} \dfrac{x - 1}{\ln x} = 1$

Continuity

Let f be defined for all values of x near $x = x_0$ as well as at $x = x_0$ (i.e., in a δ neighborhood of x_0). The function f is called *continuous* at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Note that this implies three conditions which must be met in order that $f(x)$ be continuous at $x = x_0$:

1. $\lim_{x \to x_0} f(x) = l$ must exist.
2. $f(x_0)$ must exist; i.e., $f(x)$ is defined *at* x_0.
3. $l = f(x_0)$.

In summary, $\lim_{x \to x_0} f(x)$ is the value suggested for f at $x = x_0$ by the behavior of f in arbitrarily small neighborhoods of x_0. If, in fact, this limit is the actual value, $f(x_0)$, of the function at x_0, then f is continuous there.

Equivalently, if f is continuous at x_0, we can write this in the suggestive form $\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)$.

Points where f fails to be continuous are called *discontinuities* of f and f is said to be *discontinuous* at these points.

In constructing a graph of a continuous function, the pencil need never leave the paper, while for a discontinuous function this is not true, since there is generally a jump taking place. This is, of course, merely a characteristic property and not a definition of continuity or discontinuity.

Alternative to the preceding definition of continuity, we can define f as continuous at $x = x_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x_0) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Note that this is simply the definition of limit with $l = f(x_0)$ and removal of the restriction that $x \neq x_0$.

Right- and Left-Hand Continuity

If f is defined only for $x \geqq x_0$, the preceding definition does not apply. In such case we call f *continuous* (*on the right*) at $x = x_0$ if $\lim_{x \to x_{0+}} f(x) = f(x_0)$, i.e., if $f(x_0 +) = f(x_0)$. Similarly, f is *continuous* (*on the left*) at $x = x_0$ if $\lim_{x \to x_{0-}} f(x) = f(x)_0$, i.e., $f(x_0 -) = f(x_0)$. Definitions in terms of ϵ and δ can be given.

Continuity in an Interval

A function f is said to be *continuous in an interval* if it is continuous at all points of the interval. In particular, if f is defined in the closed interval $a \leqq x \leqq b$ or $[a, b]$, then f is continuous in the interval if and only if $\lim_{x \to x_0} f(x) = f(x_0)$ for $a < x_0 < b$, $\lim_{x \to x_{a+}} f(x) = f(a)$, and $\lim_{x \to b-} f(x) = f(b)$.

Theorems on Continuity

Theorem 1 If f and g are continuous at $x = x_0$, so also are the functions whose image values satisfy the relations $f(x)+g(x), f(x)-g(x), f(x)g(x)$, and $\dfrac{f(x)}{g(x)}$, the last only if $g(x_0) \neq 0$. Similar results hold for continuity in an interval.

Theorem 2 Functions described as follows are continuous in every finite interval: (*a*) all polynomials; (*b*) $\sin x$ and $\cos x$; and (*c*) a^x, $a > 0$.

Theorem 3 Let the function f be continuous at the domain value $x = x_0$. Also suppose that a function g, represented by $z = g(y)$, is continuous at y_0, where $y = f(x)$ (i.e., the range value of f corresponding to x_0 is a domain value of g). Then a new function, called a *composite function*, $f(g)$, represented by $z = g[f(x)]$, may be created which is continuous at its domain point $x = x_0$. (One says that *a continuous function of a continuous function is continuous*.)

Theorem 4 If $f(x)$ is continuous in a closed interval, it is bounded in the interval.

Theorem 5 If $f(x)$ is continuous at $x = x_0$ and $f(x_0) > 0$ [or $f(x_0) < 0$], there exists an interval about $x = x_0$ in which $f(x) > 0$ [or $f(x) < 0$].

Theorem 6 If a function $f(x)$ is continuous in an interval and either strictly increasing or strictly decreasing, the inverse function $f^{-1}(x)$ is single-valued, continuous, and either strictly increasing or strictly decreasing.

Theorem 7 If $f(x)$ is continuous in $[a, b]$ and if $f(a) = A$ and $f(b) = B$, then corresponding to any number C between A and B there exists at least one number c in $[a, b]$ such that $f(c) = C$. This is sometimes called the *intermediate value theorem*.

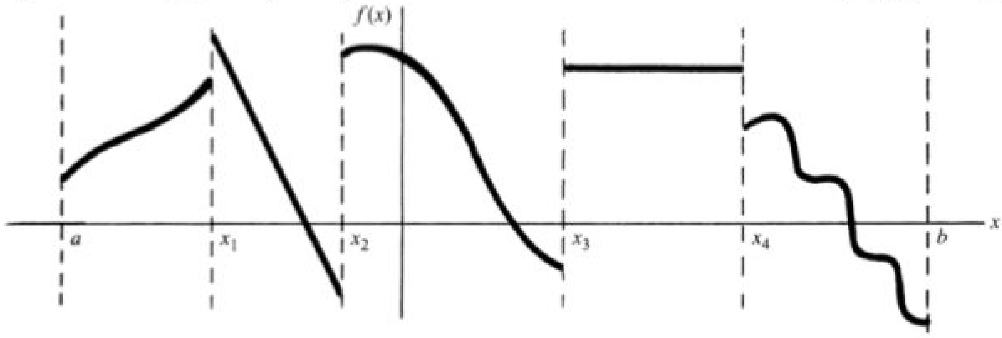
Theorem 8 If $f(x)$ is continuous in $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, there is at least one number c for which $f(c) = 0$ where $a < c < b$. This is related to Theorem 7.

Theorem 9 If $f(x)$ is continuous in a closed interval, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval. Furthermore, $f(x)$ assumes all values between m and M for one or more values of x in the interval.

Theorem 10 If $f(x)$ is continuous in a closed interval and if M and m are, respectively, the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of $f(x)$, there exists at least one value of x in the interval for which $f(x) = M$ or $f(x) = m$. This is related to Theorem 9.

Piecewise Continuity

A function is called *piecewise continuous* in an interval $a \leqq x \leqq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right- and left-hand limits. Such a function has only a finite number of discontinuities. An example of a function which is piecewise continuous in $a \leqq x \leqq b$ is shown graphically in Figure 3.4. This function has discontinuities at x_1, x_2, x_3, and x_4.



Uniform Continuity

Let f be continuous in an interval. Then, by definition, at each point x_0 of the interval and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point x_0) such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. If we can find δ for each ϵ which holds for all points of the interval (i.e., if δ depends *only* on ϵ and *not* on x_0), we say that f is *uniformly continuous* in the interval.

Alternatively, f is uniformly continuous in an interval if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where x_1 and x_2 are any two points in the interval.

Theorem If f is continuous in a *closed* interval, it is uniformly continuous in the interval.