Matrices and Vectors

General Matrix:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix}$$

Vector: A matrix with one column or one row

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Scalar Multiplication:

$$\lambda A = \lambda [a_{ij}] = [\lambda a_{ij}].$$

Product:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (i = 1, \dots, r;\ j = 1, \dots, p)$$

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA.$$

But $AB \neq BA$ in general

Powers:

$$A^n = \underbrace{AA \cdots A}_{n} \qquad I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \qquad AI = IA = A.$$

Calculus:

$$A = [a_{ij}], \quad \frac{dA}{dt} = \left[\frac{da_{ij}}{dt}\right], \quad \int_a^b A\,dt = \left[\int_c^b a_{ij}\,dt\right] \text{ and } \int A\,dt = \left[\int a_{ij}\,dt\right]$$

Characteristic Equation:

$$\det(A - \lambda I) = 0$$

Cayley-Hamilton Theorem:

$$\det(A - \lambda I) = b_n \lambda^n + b_{n-1} \lambda^{n-1} + \dots + b_2 \lambda^2 + b_1 \lambda + b_0 = 0.$$

then

$$b_n A^n + b_{n-1} A^{n-1} + \dots + b_2 A^2 + b_1 A + b_0 = 0.$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad AB = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Example: $\underset{\sim}{A} = \begin{bmatrix} t^2+1 & e^{2t} \\ \sin t & 45 \end{bmatrix}$ $\quad \dfrac{dA}{dt} = \begin{bmatrix} \frac{d}{dt}(t^2+1) & \frac{d}{dt}(e^{2t}) \\ \frac{d}{dt}(\sin t) & \frac{d}{dt}(45) \end{bmatrix} = \begin{bmatrix} 2t & 2e^{2t} \\ \cos t & 0 \end{bmatrix}.$

Example: $\underset{\sim}{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ $\quad \dfrac{dx}{dt} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$

Example: Eigenvalues of $\underset{\sim}{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$\underset{\sim}{A} - 2\underset{\sim}{I} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + (-\lambda)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$$

$$\det(\underset{\sim}{A} - 2\underset{\sim}{I}) = \det\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - (3)(4) = \lambda^2 - 3\lambda - 10$$

$\lambda_1 = 5,\ \lambda_2 = -2.$

Example: $\underset{\sim}{A} = \begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix}$ $\quad \det(\underset{\sim}{A} - \lambda\underset{\sim}{I}) = \det\begin{bmatrix} 4-\lambda & 1 & 0 \\ -1 & 2-\lambda & 0 \\ 2 & 1 & -3-\lambda \end{bmatrix}$

$$= (-3-\lambda)\left[(4-\lambda)(2-\lambda) - (1)(-1)\right]$$

characteristic eqn is: $(-3\lambda-\lambda)(\lambda-3)(\lambda-3) = 0.$

Eigenvalues: $\lambda_1 = -3,\ \lambda_2 = 3,\ \lambda_3 = 3.$