17. **Reduction of Linear Differential Equations of First-Order Equations.**

Example: Consider second-order differential equation:

$$t^4 \frac{d^2x}{dt^2} + (\sin t)\frac{dx}{dt} - 4x = \ln t \qquad (*)$$

So $$\frac{d^2x}{dt^2} = \frac{4}{t^4}x - \frac{\sin t}{t^4}\frac{dx}{dt} + \frac{\ln t}{t^4}.$$

Let $v = \frac{dx}{dt} = x' = \dot{x}$ and $v' = \frac{d^2x}{dt^2} = x'' = \ddot{x}$, then:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{4}{t^4} & -\frac{\sin t}{t^4} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\ln t}{t^4} \end{bmatrix}$$

because $\dot{x} = 0x + 1v$ and $\dot{v} = \frac{4}{t^4}x - \frac{\sin t}{t^4}v + \frac{\ln t}{t^4}$.

So $(*)$ can be expressed as $d\underline{x}(t)/dt = \underline{A}(t)\underline{x}(t) + \underline{f}(t)$.
If $x(0) = 5$ and $v(0) = -12$ in $(*)$ then these initial conditions are written as $x(0) = 5,\ v(0) = -12$.

Reduction of an n-order Equation: n first order matrix system:

$$b_n(t)\frac{d^nx}{dt^n} + b_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + b_1(t)\dot{x} + b_0(t)x = g(t)$$

$x(t_0) = c_0,\ \dot{x}(t_0) = c_1, \dots, x^{(n-1)}(t_0) = c_{n-1}$, with $b_n(t) \neq 0$, can be reduced to a first-order matrix system:

$$\dot{\underline{x}} = \underline{A}(t)\underline{x}(t) + \underline{f}(t), \qquad \underline{x}(t_0) = \underline{c}.$$

Method of reduction is as follows:

Step 1: $$\frac{d^nx}{dt^n} = a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \dots + a_1(t)\dot{x} + a_0(t)x + f(x)$$

where $a_j(t) = -b_j(t)/b_n(t)$ $(j = 0, 1, \dots, n-1)$ and $f(t) = g(t)/b_n(t)$.

Step 2 : Define n new variables (the same number as the order of the original differential equation)

$x_1(t), x_2(t), \dots, x_n(t)$ by the equations :

$$x_1(t) = x(t), \quad x_2(t) = \frac{dx(t)}{dt}, \quad x_3(t) = \frac{d^2x(t)}{dt^2}, \dots, x_n(t) = \frac{d^{n-1}x(t)}{dx^{n-1}}$$

The new variables are interrelated by the equations :

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= x_4(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \qquad (**) \end{aligned}$$

Step 3 : Express dx_n/dt in terms of new variables.

Proceed by frist differentiating the last equation of (**) to obtain:

$$\dot{x}_n(t) = \frac{d}{dt}\left[\frac{d^{n-1}x(t)}{dt^{n-1}}\right] = \frac{d^n x(t)}{dt^n}$$

Then,

$$\dot{x}_n(t) = a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + \dots + a_1(t)\dot{x}(t) + a_0(t)x(t) + f(t)$$

$$= a_{n-1}(t)x_n(t) + \dots + a_1(t)x_2(t) + a_0(t)x_1(t) + f(t)$$

For convenience :

$$\dot{x}_n(t) = a_0(t)x_1(t) + a_1(t)x_2(t) + \dots + a_{n-1}(t)x_n(t) + f(t).$$

Step 4:

This is a system of first-order differential equations in $x_1(t), x_2(t), \dots, x_n(t)$.
This is equivalent to the single matrix equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{f}(t), \quad \text{if we define:}$$

$$\underline{x}(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \underline{f}(t) \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

$$\underline{A}(t) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_n(t) & a_1(t) & a_2(t) & a_3(t) & \cdots & a_{n-1}(t) \end{bmatrix}$$

Step 5:

Define:

$$\underline{c} \equiv \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

The initial conditions can be given by the matrix (vector) equation $\underline{x}(t_0) = \underline{c}$.
This last equation is an immediate consequence of previous equations:

$$\underline{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_1) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \\ \vdots \\ x^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \equiv \underline{c}$$

Observe that if no initial conditions are prescribed, steps 1-4 by themselves reduce any linear differential

$$x(t_0) = c_0, \quad \dot{x}(t_0) = c_1, \; \dots, \; x^{(n-1)}(t_0) = c_{n-1}$$

to the matrix equation $\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{f}(t)$.