Differential Equations - Chapter 30 - Gamma and Bessel Functions.

Definition: $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$, p any positive real number.

Rules: Consequently $\Gamma(1) = 1$ and $\Gamma(p+1) = p\Gamma(p)$ $(*)$

Also when $p = n$, positive integer,

$$\Gamma(n+1) = n!$$

$(*)$ can be rewritten: $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$

Note: $\Gamma(0)$ remains undefined because:

$$\lim_{p \to 0^+} \Gamma(p) = \lim_{p \to 0^+} \frac{\Gamma(p+1)}{p} = \infty, \text{ and}$$

$$\lim_{p \to 0^-} \Gamma(p) = \lim_{p \to 0^-} \frac{\Gamma(p+1)}{p} = -\infty.$$

Bessel Fn: Let $p \in \mathbb{R}$ and $p > 0$.

Bessel function of the first kind of order p, $J_p(x)$ is:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k+p}\, k!\, \Gamma(p+k+1)}$$

Differential Equation: $J_p(x)$ is a solution near the regular singular point $x=0$ of Bessel's differential equation of order p:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

Infinite Series Operations: Changing the dummy index:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} = \frac{1}{1!} + \frac{1}{2!} + \dots$$

Consider charge of variables $j = k+1$ or $k = j-1$ then

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)!} = \sum_{j=1}^{\infty} \frac{1}{j!}$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)!} = \sum_{j=2}^{\infty} \frac{1}{(j-1)!} = \sum_{j=2}^{\infty} \frac{1}{(k-1)!}, \text{ where } j = k+2$$

Example: Prove that $\Gamma(p+1)\, p\Gamma(p) > 0$:

$$\Gamma(p+1) = \int_0^{\infty} x^{(p+1)-1} e^{-x}\, dx = \lim_{r\to\infty} \left[-x^p e^{-x} \Big|_0^r + \int_0^r p x^{p-1} e^{-x}\, dx \right]$$
$$= \lim_{r\to\infty} (-r^p e^{-r} + 0) + p\int_0^{\infty} x^{p-1} e^{-x}\, dx$$
$$= p\Gamma(0)$$

note $\lim_{r\to\infty} r^p e^{-r} = 0$ as rewriting $r^p e^{-r}$ as r^p / e^r and L'Hôpitals rule.

Example: Prove $\Gamma(1) = 1$:

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} = \lim_{r\to\infty} \int_0^r e^{-x}\, dx$$
$$= \lim_{r\to\infty} -e^{-x}\Big|_0^r$$
$$= \lim_{r\to\infty} (-e^{-r} + 1)$$
$$= 1.$$

Example: Prove $\Gamma(n+1) = n!$

By induction, $\Gamma(1+1) = 1\Gamma(1) = 1(1) = 1!$
Assume $\Gamma(n+1) = n!$ and holds for $n=k$.

For $n = k+1$:

$$\Gamma((k+1)+1) = (k+1)\Gamma(k+1)$$
$$= (k+1)(k!)$$
$$= (k+1)!$$

Thus $\Gamma(n+1) = n!$ is true.