# Differential Equations - Chapter 32 - 2nd Order Boundary Value Problems.

Definition: A boundary value problem in standard form is:

$$y'' + P(x)y' + Q(x)y = \phi(x)$$

and boundary conditions:

$$\left.\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) &= \gamma_2 \end{aligned}\right\}$$

$P(x), Q(x)$ continuous on $[a,b]$, $\alpha_1$ and $\beta_1$ not both zero, also $\alpha_2$ and $\beta_2$.

Homogenous: Problem is homogenous if both the differential equation and the boundary conditions are homogenous i.e $\phi(x) \equiv 0$ and $\gamma_1, \gamma_2 = 0$

Homogenous boundary-value problem has form:

$$y'' + P(x)y' + Q(x)y = 0$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0$$

If $P(x)$ and $Q(x)$ depend on arbitrary constant $\lambda$ has form:

$$y'' + P(x,\lambda) y' + Q(x,\lambda) y = 0$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0$$

Theorem: Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of:

$$y'' + P(x) y' + Q(x) y = 0$$

Solutions exist if:

$$\begin{vmatrix} \alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2' \\ \alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2(a) + \beta_2 y_2'(b) \end{vmatrix} = 0.$$

**Sturm Liouville:** A second order homogenous boundary-value problem of the form:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0 \qquad (*)$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0$$

where $p(x)$, $p'(x)$, $q(x)$ and $w(x)$ are continuous on $[a,b]$, and both $p(x)$ and $w(x)$ are positive on $[a,b]$.

Rewriting $(*)$ by dividing by $p(x)$:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y' + \lambda r(x)y = 0$$

**Sturm Liouville Properties:**

(a). Eigenvalues of Sturm-Liouville problems are real and non-negative

(b). Eigenvalues can be arranged into form strictly increasing infinite sequence

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

Further more $\lambda_n \to \infty$ as $n \to \infty$

(c). For each eigenvalue there exists one and only one linearly independent eigenfunction.

(d). Eigenfunctions $\{e_1(x), e_2(x), \dots\}$ satisfies the relation:

$$\int_a^b w(x)e_n(x)e_m(x)\,dx, \quad n \neq m.$$

**Example:** Solve $y'' + 2y' - 3y = 0$, $y(0) = 0$, $y'(1) = 0$

$P(x) = 2$, $Q(x) = -3$, $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$, $a = 0$.

General solution: $y = c_1 e^{-3x} + c_2 e^{x}$

Apply boundary condition: $c_1 = c_2 = 0$, hence solution is $y = 0$.

Two linearly independent solutions are: $y_1(x) = e^{-3x}$ and $y_2(x) = e^{x}$

$$\begin{vmatrix} 1 & 1 \\ -3e^{-3} & e \end{vmatrix} = e + 3e^{-3} \neq 0, \text{ hence only solution is } y(x) = 0$$