Calculus – Chapter 49 – Differentiability. Chain Rule.

Total differentiable:

$z = f(x,y)$, $\Delta x, \Delta y$ any numbers

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x(x,y)\Delta x + f_y(x,y)\Delta y \quad (*)$$

Note if $z = f(x,y) = x$ then $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 0$ and $dz = \Delta x \therefore dx = \Delta x$

Similarly $dy = \Delta y$, hence $(*)$ becomes

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

Extende to 3 variables:

If $u = f(x,y,z)$,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= f_x(x,y,z) + f_y(x,y,z) + f_z(x,y,z).$$

Example:

Let $z = x\cos y - 2x^2 + 3$

$$\frac{\partial z}{\partial x} = \cos y - 4x, \quad \frac{\partial z}{\partial y} = -x\sin x \quad \therefore dz = (\cos y - 4x)dx - (x\sin y)dy.$$

Differentiable:

$z = f(x,y)$ is differentiable at (a,b) if functions ϵ_1, ϵ_2 exists s.t.

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

and $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1 = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2 = 0.$

Normaly written $\Delta z = dz + \epsilon_1\Delta x + \epsilon_2\Delta y$.

$z = f(x,y)$ is differentiable on set A if it is differentiable at each point of A

Theorem:

Assume $f(x,y)$ is such that f_x, f_y are continuous in an open set A.
Then f is differentiable in A.

Example:

$$z = f(x,y) = \sqrt{9-x^2-y^2}, \quad f_x = \frac{-x}{\sqrt{9-x^2-y^2}} \text{ and } f_y = \frac{-y}{\sqrt{9-x^2-y^2}}$$

In disk $x^2+y^2 < 9$, evaluate change Δz as we move from $(1,2) \to (1.03, 2.01)$

So $\Delta x = 0.03$, $\Delta y = 0.01$, $dz = f_x(1,2)\Delta x + f_y(1,2)\Delta y = -\frac{1}{2}(0.03) + \frac{-2}{2}(0.01) = -0.02$

Chain Rule : Let $z = f(x,y)$, $x = g(t)$ and $y = h(t)$, then $z = f(g(t), h(t))$ and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof : Let $\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

Then $\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$

Let $\Delta t \to 0$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + O(\Delta x) + O(\Delta y) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example : Let $z = xy + \sin x$, $x = t^2$, $y = \cos t$.

$\frac{\partial z}{\partial x} = y + \cos x$, $\frac{\partial z}{\partial y} = x$, $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -\sin t$

As function of t, $z = t^2 \cos t + \sin(t^2)$.

$$\frac{dz}{dt} = (y + \cos x)\,2t + x(-\sin t) = (\cos t + \cos(t^2))\,2t - t^2 \sin t.$$

Chain Rule (2→2) : $z = f(x,y)$, $x = g(t,s)$, $y = h(t,s)$, then $z = f(g(t,s), h(t,s))$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \text{ and } \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Example : $z = e^x \sin y$, $x = ts^2$, $y = t + 2s$:

$\frac{\partial z}{\partial x} = e^x \sin y$, $\frac{\partial x}{\partial t} = s^2$, $\frac{\partial z}{\partial y} = e^x \cos y$, $\frac{\partial y}{\partial t} = 1$.

Implicit Differentiation : $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Example : $xy + yz^3 + xz = 0$

Let $F(x,y,z) = xy + yz^3 + xz$

Since $F_z = xy + 3yz^2$

$F_x = y + z$

$F_y = x + z^3$

$\therefore \dfrac{\partial z}{\partial x} = -\dfrac{y+z}{x+3yz^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{x+z^3}{x+3yz^2}$.