CHAPTER 7 - VECTORS

Algebraic Properties of Vectors

1.	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law for addition
2.	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	Associative law for addition
3.	$m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$	Associative law for multiplication
4.	$(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$	Distributive law
5.	$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$	Distributive law

Components of a Vector

The sum or resultant of $A_1\mathbf{i}$, $A_2\mathbf{j}$, and $A_3\mathbf{k}$ is the vector $\mathbf{A}$, so that we can write

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \tag{1}$$

The magnitude of $\mathbf{A}$ is

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{2}$$

In particular, the *position vector* or *radius vector* $\mathbf{r}$ from O to the point $(x, y, z$ is written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3}$$

and has magnitude $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Dot, Scalar, or Inner Product

The dot or scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cdot \mathbf{B}$ (read: $\mathbf{A}$ dot $\mathbf{B}$) is defined as the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the cosine of the angle between them. In symbols,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta, \qquad 0 \leqq \theta \leqq \pi \tag{4}$$

1.	$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$	Commutative law for dot products
2.	$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$	Distributive Law

3. $m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m$, where m is a scalar
4. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$, $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then $\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$

The equivalence of this component form the dot product with the geometric definition 4 following from the law of cosines. (See Figure 7.8).

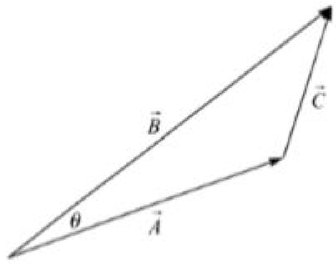


Figure 7.8

In particular,

$$|\mathbf{C}|^2 = |\mathrm{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

Since $\mathbf{C} = \mathbf{B} - \mathbf{A}$, its components are $B_1 - A_1$, $B_2 - A_2$, $B_3 - A_3$ and the square of its magnitude is

$$|\mathbf{B}|^2 + |\mathbf{A}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3$$

When this representation for $|\mathbf{C}^2|$ is placed in the original equation and cancellations are made, we obtain

$$A_1B_1 + A_2B_2 + A_3B_3 = |\mathbf{A}|\,|\mathbf{B}|\cos\theta.$$

Cross or Vector Product

The cross or vector product of **A** and **B** is a vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ (read: **A** cross **B**). The magnitude of $\mathbf{A} \times \mathbf{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between them. The direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of **A** and **B**, and such that **A**, **B**, and **C** form a right-handed system. In symbols,

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta\mathbf{u},\ 0 \leqq \theta \leqq \pi \tag{5}$$

where **u** is a unit vector indicating the direction of $\mathbf{A} \times \mathbf{B}$. If $\mathbf{A} = \mathbf{B}$ or if **A** is parallel to **B**, then $\sin\theta = 0$ and $\mathbf{A} \times \mathbf{B} = 0$.

The following laws are valid:

1. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ (Commutative law for cross products fails)
2. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Distributive Law
3. $m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m$, where m is a scalar

Also, the following consequences of the definition are important:

4. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathrm{A} \times \mathrm{B} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

The equivalence of this component representation (5) and the geometric definition may be seen as follows. Choose a coodinate system such that the direction of the x-axis is that of **A** and the xy plane is the plane of the vectors **A** and **B**. (See Figure 7.9.) Then

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ A_1 & 0 & 0 \\ B_1 & B_2 & 0 \end{vmatrix}=A_1B_2\mathrm{k}=|\mathbf{A}|\,|\mathbf{B}|\text{ sine }\theta\; k$$

Since this choice of coordinate system places no restrictions on the vectors **A** and **B**, the result is general and thus establishes the equivalence.

6. $|\mathbf{A}\times\mathbf{B}|$ = the area of a parallelogram with sides **A** and **B**.
7. If $\mathbf{A}\times\mathbf{B}=0$ and neither **A** nor **B** is a null vector, then **A** and **B** are parallel.

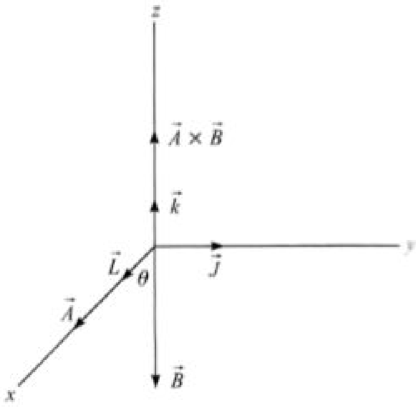


Figure 7.9

Triple Products

Dot and cross multiplication of three vectors, **A**, **B**, and **C** may produce meaningful products of the form $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The following laws are valid:

1. $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}\neq\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ in general
2. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ = volume of a parallelepiped having **A**, **B**, and **C** as edges, or the negative of this volume according as **A**, **B**, and **C** do or do not form a right-handed handed system. If $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$, $\mathbf{B}=B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}$ and $\mathbf{C}=C_1\mathbf{i}+C_2\mathbf{j}+C_3\mathbf{k}$, then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{6}$$

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (Associative law for cross products fails)
4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

Axiomatic Approach to Vector Analysis

1. $\mathbf{A}=\mathbf{B}$ if and only if $A_1=B_1, A_2=B_2, A_3=B_3$
2. $\mathbf{A}+\mathbf{B}=(A_1+B_1, A_2+B_2, A_3+B_3)$
3. $\mathbf{A}-\mathbf{B}=(A_1-B_1, A_2-B_2, A_3-B_3)$
4. $\mathbf{0}=(0,0,0)$
5. $m\mathbf{A}=m(A_1,A_2,A_3)=(mA_1, mA_2, mA_3)$

In addition, two forms of multiplication are established.

6. $\mathbf{A}\cdot\mathbf{B}=A_1B_1+A_2B_2+A_3B_3$
7. Length or magnitude of $\mathrm{A}=|\mathrm{A}|=\sqrt{\mathrm{A}\cdot\mathrm{A}}=\sqrt{A_1^2+A_2^2+A_3^2}$
8. $\mathbf{A}\times\mathbf{B}=(A_2B_3-A_3B_2, A_3B_1-A_1B_3, A_1B_3, A_1B_2-A_2B_1)$

Limits, Continuity, and Derivatives of Vector Functions

1. The vector function represented by $\mathbf{A}(u)$ is said to be *continuous* at u_0 if, given any positive number δ, we can find some positive number δ such that $|\mathbf{A}(u) - \mathbf{A}(u_0)| < \delta$ whenever $|u - u_0| < \delta$. This is equivalent to the statement $\lim_{u \to u_0} \mathbf{A}(u) = \mathbf{A}(u_0)$.
2. The derivative of $\mathbf{A}(u)$ is defined as

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u + \Delta u) - \mathbf{A}(u)}{\Delta u}$$

provided this limit exists. In case $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2\, A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$; then

$$\frac{d\mathbf{A}}{du} = \frac{dA_1}{du}\mathbf{i} + \frac{dA_2}{du}\mathbf{j} + \frac{dA_3}{du}\mathbf{k}$$

Higher derivatives such as $d^2\mathbf{A}/du^2$, etc., can be similarly defined.

3. If $\mathbf{A}(x, y, z) = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}$; then

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x} dx + \frac{\partial \mathbf{A}}{\partial y} dy + \frac{\partial \mathbf{A}}{\partial z} dz$$

is the *differential* of $\mathbf{A}$.

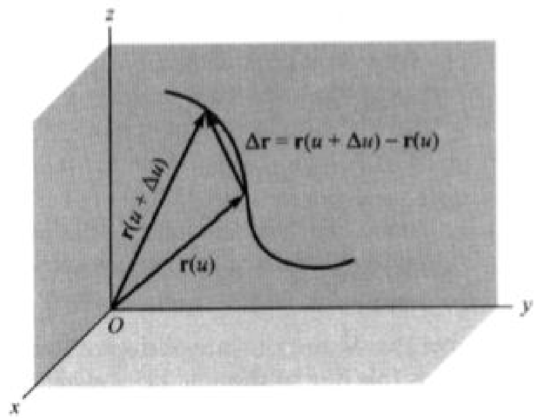
4. Derivatives of products obey rules similar to those for scalar functions. However, when cross products are involved, the order is important. Some examples are

(a) $\dfrac{d}{du}(\phi \mathbf{A}) = \phi \dfrac{d\mathbf{A}}{du} + \dfrac{d\phi}{du}\mathbf{A}.$

(b) $\dfrac{\partial}{\partial y}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \dfrac{\partial \mathbf{B}}{\partial y} + \dfrac{\partial \mathbf{A}}{\partial y} \cdot \mathbf{B}$

(c) $\dfrac{\partial}{\partial z}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \dfrac{\partial \mathbf{B}}{\partial z} + \dfrac{\partial \mathbf{A}}{\partial z} \times \mathbf{B}$ (maintain the order of $\mathbf{A}$ and $\mathbf{B}$)

Geometric Interpretation of a Vector Derivative



$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{8}$$

is the *velocity* with which the terminal point of $\mathbf{r}$ describes the curve. We have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{T} = \upsilon\mathbf{T} \tag{9}$$

from which we see that the magnitude of $\mathbf{v}$ is $\upsilon = ds/dt$. Similarly,

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} \tag{10}$$

is the *acceleration* with which the terminal point of $\mathbf{r}$ describes the curve. These concepts have important applications in *mechanics* and *differential geometry*.

An intuitive understanding of these entities begins with examination of the differential of a scalar field, i.e.,

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{dy}dy + \frac{\partial\Phi}{\partial z}dz$$

Now suppose the function Φ is constant on a surface S and that C; $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ is a curve on S. At any point of this curve, $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dx}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$ lies in the tangent plane to the surface. Since this statement is true for every surface curve through a given point, the differential $d\mathbf{r}$ spans the tangent plane. Thus, the triple $\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}$ represents a vector perpendicular to S. With this special geometric characteristic in mind we define

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

to be the *gradient of the scalar field* Φ.

Gradient, Divergence, and Curl

Consider the vector operator ∇ (*del*) defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{11}$$

Then if $\phi(x, y, z)$ and $\mathbf{A}(x, y, z)$ have continuous first partial derivatives in a region (a condition which is in many cases stronger than necessary), we can define the following.

1. **Gradient.** The *gradient* of ϕ is defined by

$$\begin{aligned}\text{grad}\,\phi = \nabla\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \\ &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\end{aligned} \tag{12}$$

2. **Divergence.** The *divergence* of $\mathbf{A}$ is defined by

$$\begin{aligned}\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned} \tag{13}$$

3. **Curl.** The *curl* of **A** is defined by

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 & A_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_1 & A_2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_1 & A_2 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\mathbf{k}$$

Formulas Involving ∇

If the partial derivatives of **A**, **B**, U, and V are assumed to exist, then

1. $\nabla(U + V) = \nabla U + \nabla V$ or grad $(U + V)$ = grad u + grad V
2. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$ or div $(\mathbf{A} + \mathbf{B})$ + div $\mathbf{A}$ + div $\mathbf{B}$
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ or curl $(\mathbf{A} + \mathbf{B})$ = curl $\mathbf{A}$ + curl $\mathbf{B}$
4. $\nabla \cdot (U\mathbf{A}) = (\nabla U) \cdot \mathbf{A} + U(\nabla \cdot \mathbf{A})$
5. $\nabla \times (U\mathbf{A}) = (\nabla U) \times \mathbf{A} + U(\nabla \times \mathbf{A})$
6. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
7. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$
8. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$
9. $\nabla.(\nabla U) \equiv \nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$ is called the *Laplacian* of U.

   and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Lapacian operator*.
10. $\nabla \times (\nabla U) = 0$. The curl of the gradient of U is zero.
11. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. The divergence of the curl of **A** is zero.
12. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Vector Interpretation of Jacobians and Orthogonal Curvilinear Coordinates

The transformation equations

$$x = f(u_1, u_2, u_3), \qquad y = g(u_1, u_2, u_3), \qquad z = h(u_1, u_2, u_3) \tag{15}$$

(where we assume that f, g, h are continuous, have continuous partial derivatives, and have a single-valued inverse) establish a one-to-one correspondence between points in an xyz and $u_1\, u_2 u_3$ rectangular coordinate system. In vector notation, the transformation (15) can be written

$$ds^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 + g_{33}(du_3)^2$$

where

$$g_{11} = \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_1}, \quad g_{22} = \frac{\partial \mathbf{r}}{\partial u_2} \cdot \frac{\partial \mathbf{r}}{\partial u_2}, \quad g_{33} = \frac{\partial \mathbf{r}}{\partial u_3} \cdot \frac{\partial \mathbf{r}}{\partial u_3}$$

$$dV = |g_{jk}|\ du_1\, du_2\, du_3 = |(h_1\, du_1\, \mathbf{e}_1) \cdot (h_2\, du_2\, \mathbf{e}_2) \times (h_3\, du_3\, \mathbf{e}_3)| = h_1 h_2 h_3\, du_1\, du_2\, du_3 \tag{20}$$

which can be written as

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1 du_2 du_3 = \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3 \tag{21}$$

where $\partial(x, y, z)/\partial(u_1, u_2, u_3)$ is the *Jacobian* of the transformation.

Gradient Divergence, Curl, and Laplacian in Orthogonal Curvilinear Coordinates

If Φ is a scalar function and $\mathbf{A} = A_1\, \mathbf{e}_1 + A_2\, \mathbf{e}_2 + A_3 \mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3, we have the following results.

1. $\nabla \Phi = \text{grad } \Phi = \dfrac{1}{h_1} \dfrac{\partial \Phi}{\partial u_1} \mathbf{e}_1 + \dfrac{1}{h_2} \dfrac{\partial \Phi}{\partial u_2} \mathbf{e}_2 + \dfrac{1}{h_3} \dfrac{\partial \Phi}{\partial u_3} \mathbf{e}_3$

2. $\nabla \cdot \mathbf{A} = \text{div } \mathbf{A} = \dfrac{1}{h_1 h_2 h_3} \left[\dfrac{\partial}{\partial u_1}(h_2, h_3 A_1) + \dfrac{\partial}{\partial u_2}(h_3 h_1 A_2) + \dfrac{\partial}{\partial u_3}(h_1 h_2 A_3) \right]$

3. $\nabla \times \mathbf{A} = \text{curl } \mathbf{A} = \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$

4. $\nabla^2 \Phi = \text{Laplacian of } \Phi = \dfrac{1}{h_1 h_2 h_3} \left[\dfrac{\partial}{\partial u_1} \left(\dfrac{h_2 h_3}{h_1} \dfrac{\partial \Phi}{\partial u_1} \right) + \dfrac{\partial}{\partial u_2} \left(\dfrac{h_3 h_1}{h_2} \dfrac{\partial \Phi}{\partial u_2} \right) + \dfrac{\partial}{\partial u_3} \left(\dfrac{h_1 h_2}{h_3} \dfrac{\partial \Phi}{\partial u_3} \right) \right]$

Special Curvilinear Coordinates

Cylindrical Coordinates (ρ, φ, z) See Figure 7.13.

Transformation equations:

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

where $\rho \geqq 0,\ 0 \leqq \phi < 2\pi,\ -\infty < z < \infty$.

Scale factors: $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$

Element of arc length: $ds^2 = d\rho^2 + \rho^2\, \rho^2\, d\phi^2 + dz^2$

Jacobian: $\dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$

Element of volume: $dV = \rho\, d\rho\, d\phi\, dz$

Laplacian:
$$\nabla^2 U = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2} + \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$
$$= \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} + \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

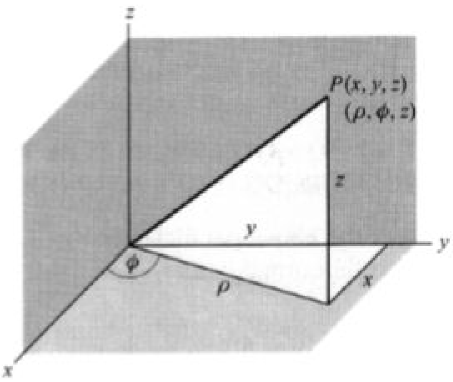


Figure 7.13

Note that corresponding results can be obtained for polar coordinates in the plane by omitting z dependence. In such case, for example, $ds^2 = d\rho^2 + \rho^2\, d\phi^2$, while the element of volume is replaced by the element of area, $dA = \rho\, d\rho\, d\phi$.

Spherical Coordinates (r, θ, φ) See Figure 7.14.

Transformation equations:

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

where $r \geqq 0,\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \phi < 2\pi$.

Scale factors: $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

Element of arc length: $ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

Jacobian: $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$

Element of volume: $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Laplacian:
$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

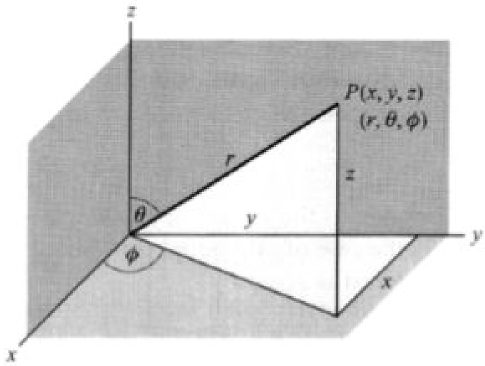


Figure 7.14

Other types of coordinate systems are possible.