CHAPTER 13 - FOURIER SERIES

Periodic Functions

A function $f(x)$ is said to have a *period* T or to be *periodic* with period T if for all x, $f(x + T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the *least period* or simply *the period* of $f(x)$.

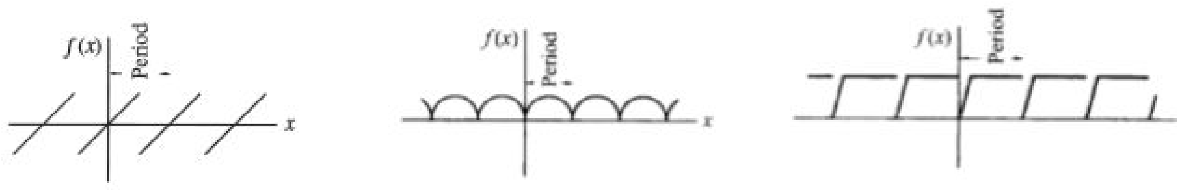
EXAMPLE 1. The function $\sin x$ has periods 2π, 4π, 6π, . . . , since $\sin(x + 2\pi)$, $\sin(+ 4\pi)$, $\sin(x + 6\pi)$, . . . all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

EXAMPLE 2. The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

EXAMPLE 3. The period of $\tan x$ is π.

EXAMPLE 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figure 13.1(a), (b), and (c).



(a) (b) (c)

Fourier Series

Let $f(x)$ be defined in the interval $(-L, L)$ and outside of this interval by $f(x + 2L) = f(x)$; i.e., $f(x)$ is $2L$-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on $(-L, L)$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

To correlate the coefficients with the expansion, see the following Examples 1 and 2.

Orthogonality Conditions for the Sine and Cosine Functions

Notice that the Fourier coefficients are integrals. These are obtained by starting with the series (1) and employing the following properties called orthogonality conditions:

$$\text{(a)} \quad \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n$$

$$\text{(b)} \quad \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n \tag{3}$$

$$\text{(c)} \quad \int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0. \text{ Where } m \text{ and } n \text{ assume any positive integer values.}$$

Dirichlet Conditions

Suppose that

1. $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$.
2. $f(x)$ is periodic outside $(-L, L)$ with period $2L$.
3. $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (1) with Fourier coefficients converges to

(a) $f(x)$ if x is a point of continuity

(b) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right- and left-hand limits of $f(x)$ at x and represent $\lim_{\epsilon \to 0+} f(x+\epsilon)$ and $\lim_{\epsilon \to 0+} f(x-\epsilon)$, respectively. For a proof, see Problems 13.18 through 13.23.

The conditions 1, 2, and 3 imposed on $f(x)$ are *sufficient* but not necessary, and are generally satisfied in practice. There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* ensure convergence of a Fourier series.

Odd and Even Functions

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus, x^3, $x^5 - 3x^3 + 2x$, $\sin x$, and $\tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus, x^4, $2x^6 - 4x^2 + 5$, $\cos x$, and $e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Figure 13.1(a) and (b) are odd and even, respectively, but that of Figure 13.1(c) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant, which we shall consider a cosine term) can be present.

Half Range Fourier Sine or Cosine Series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0, L) [which is half of the interval ($-L$, L), thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, ($-L$, 0). In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L} \displaystyle\int_{-L}^{L} f(x) \sin \dfrac{n\pi x}{L}\, dx & \text{for } \textit{half range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L} \displaystyle\int_{0}^{L} f(x) \cos \dfrac{n\pi x}{L}\, dx & \text{for } \textit{half range consine series} \end{cases} \tag{4}$$

Parseval's Identity

If a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions, then

$$\frac{1}{L}\int_{-L}^{L} \{f(x)\}^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{5}$$

(See Problem 13.13.)

Complex Notation for Fourier Series

Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta, \; e^{-i\theta} = \cos\theta - i\sin\theta \tag{6}$$

where $i = \sqrt{-1}$ (see Problem 11.48), the Fourier series for $f(x)$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{7}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx \tag{8}$$

In writing the equality (7), we are supposing that the Dirichlet conditions are satisfied and, further, that $f(x)$ is continuous at x. If $f(x)$ is discontinuous at x, the left side of (7) should be replaced by $\dfrac{(f(x+0) + f(x-0))}{2}$.

Orthogonal Functions

Two vectors **A** and **B** are called *orthogonal* (perpendicular) if $\mathbf{A} \cdot \mathbf{B} = 0$ or or $A_1B_1 + A_2B_2 + A_3B_3 = 0$, where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Although not geometrically or physically evident, these ideas can be generalized to include vectors with more than three components. In particular, we can think of a function—say, $A(x)$—as being a vector with an *infinity of components* (i.e., an *infinite dimensional vector*), the value of each component being specified by substituting a particular value of x in some interval (a, b). It is natural in such case to define two functions, $A(x)$ and $B(x)$, as *orthogonal* in (a, b) if

$$\int_a^b A(x)B(x)dx = 0 \tag{9}$$

A vector **A** is called a *unit vector* or *normalized vector* if its magnitude is unity, i.e., if $\mathbf{A} \cdot \mathbf{A} = A^2 = 1$. Extending the concept, we say that the function $A(x)$ is *normal* or *normalized* in (a, b) if

$$\int_a^b \{A(x)\}^2 dx = 1 \tag{10}$$

From this, it is clear that we can consider a set of functions $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots$, having the properties

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0 \quad m \neq n \tag{11}$$

$$\int_a^b \{\phi_m(x)\}^2 dx = 1 \quad m = 1, 2, 3, \ldots \tag{12}$$

In such case, each member of the set is orthogonal to every other member of the set and is also normalized. We call such a set of functions an *orthonormal set*.

Equations (11) and (12) can be summarized by writing

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = \delta_{mn} \tag{13}$$

where δ_{mn}, called *Kronecker's symbol*, is defined as 0 if $m \neq n$ and 1 if $m = n$.

Just as any vector **r** in three dimensions can be expanded in a set of mutually orthogonal unit vectors **i**, **j**, **k** in the form $\mathbf{r} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, so we consider the possibility of expanding a function $f(x)$ in a set of orthonormal functions, i.e.,

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x) \quad a \leq x \leq b \tag{14}$$

As we have seen, Fourier series are constructed from orthogonal functions. Generalizations of Fourier series are of great interest and utility from both theoretical and applied viewpoints.