Calculus - Chapter 35 - Improper Integrals

Definition:

(a). $\int_a^{+\infty} f(x)\,dx = \lim_{c\to+\infty} \int_a^c f(x)\,dx$

(b). $\int_{-\infty}^{b} f(x)\,dx = \lim_{c\to-\infty} \int_c^b f(x)\,dx$

(c). $\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^{+\infty} f(x)\,dx + \int_{-\infty}^{b} f(x)\,dx$

Discontinuity: (a). If f is continuous on $[a,b]$ except that it is not continuous from the **right** at a, then

$$\int_a^b f(x)\,dx = \lim_{u\to a^+} \int_u^b f(x)\,dx$$

(b). If f is continuous on $[a,b]$ except that it is not continuous from the **left** at b, then

$$\int_a^b f(x)\,dx = \lim_{u\to b^-} \int_a^u f(x)\,dx$$

(c). If f is continuous on $[a,b]$ except at a point c in (a,b), then

$$\int_a^b f(x)\,dx = \lim_{u\to c^-} \int_a^u f(x)\,dx + \lim_{u\to c^+} \int_u^b f(x)\,dx$$

Example:

$$\int_1^{+\infty} 1/x^2\,dx = \lim_{c\to+\infty} \int_1^c (1/x^2)\,dx = \lim_{c\to+\infty} [-1/x]_1^c$$
$$= \lim_{c\to+\infty} -\left(\tfrac{1}{c} - 1\right)$$
$$= -(0-1)$$
$$= 1$$

Example:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{c\to+\infty} \int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty} \ln(x)\Big]_1^c$$
$$= \lim_{c\to+\infty} -(\ln c - 0)$$
$$= +\infty \quad \text{(divergent)}.$$

Example: Show that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges to $+\infty$ for $p\leq 1$:

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{c\to+\infty}\int_1^c \frac{1}{x^p}\,dx = \lim_{c\to\infty} \frac{1}{1-p}\frac{1}{x^{p-1}}\Big]_1^c$$

Assume $p>1$: $\lim_{c\to+\infty} \frac{1}{1-p}\left(\frac{1}{c^{p-1}}-1\right) = \frac{1}{p-1}(0-1) = \frac{1}{p-1}$.

Example:
$$\int_0^{+\infty} \frac{1}{x^2+4}\,dx = \lim_{c\to+\infty}\int_0^c \frac{1}{x^2+4}\,dx = \lim_{c\to+\infty} \frac{1}{2}\tan^{-1}(x/2)\Big]_0^c$$
$$= \lim_{c\to+\infty} \frac{1}{2}\left(\tan^{-1}(c/2)-0\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$
$$= \pi/4.$$

Example:
$$\int_0^{+\infty} e^{-x}\sin(x)\,dx = \lim_{c\to+\infty}\int_0^c e^{-x}\sin(x)\,dx$$
$$= \lim_{c\to\infty}\left[\left(-\tfrac{1}{2}e^{-x}(\sin(x)+\cos(x)\right)\right]_0^c$$
$$= \lim_{c\to\infty}\left[\left(-\tfrac{1}{2}e^{-c}(\sin c+\cos c)\right)+\tfrac{1}{2}\right]$$

As $c\to+\infty$, $e^{-c}\to 0$ while $\sin(c)$ and $\cos(c)$ oscillate.

Hence,
$$\lim_{c\to+\infty} e^{-c}(\sin(c)+\cos(c)) = 0$$
$$\therefore \int_0^{+\infty} e^{-x}\sin(x)\,dx = 1/2.$$

Example:
$$\int_3^{+\infty} \frac{dx}{x^2-1}\,dx = \lim_{c\to+\infty}\int_3^c \frac{dx}{x^2-1}$$
$$= \frac{1}{2}\lim_{c\to+\infty} \ln\frac{x-1}{x+1}\Big]_3^c$$
$$= \frac{1}{2}\lim_{c\to+\infty}\left(\ln\frac{c-1}{c+1}-\ln\frac{1}{2}\right)$$
$$= \frac{1}{2}\lim_{c\to+\infty}\left(\ln\frac{1-(1/c)}{1+(1/c)}-\ln\frac{1}{2}\right)$$
$$= \frac{1}{2}(\ln 1+\ln 2)$$
$$= \ln 2/2.$$