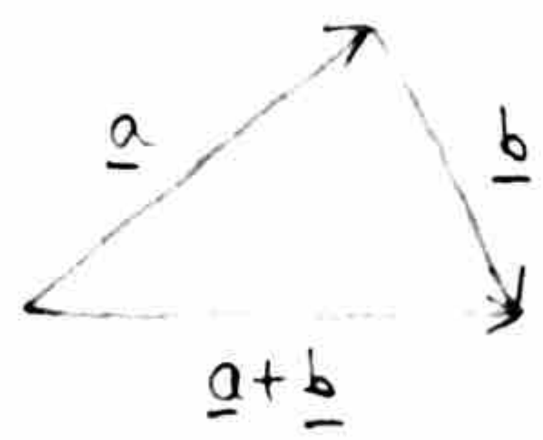
Vector Calculus (Springer)

Vector: Physical quantity with both magnitude and direction ($\underline{u}$)

Scalar: Physical quantity with magnitude only.

Vector Addition:



Vector Components: Suppose vector $\underline{a}$ is drawn from (x_1, y_1, z_1) to (x_2, y_2, z_2), components are:

$$a_1 = x_2 - x_1$$
$$a_2 = y_2 - y_1$$
$$a_3 = z_2 - z_1$$

Can be written in form $\underline{a} = (a_1, a_2, a_3)$

Unit Vectors: Introduce unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ which lie on x, y, z axes resp.

$$a = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3.$$

Hence
$$\underline{a}+\underline{b} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3 + b_1\underline{e}_1 + b_2\underline{e}_2 + b_3\underline{e}_3$$
$$= (a_1+b_1)\underline{e}_1 + (a_2+b_2)\underline{e}_2 + (a_3+b_3)\underline{e}_3.$$

Equivalent to letting $c_1 = a_1 + b_1$ etc.. and $\underline{c} = \underline{a} + \underline{b}$.

Magnitude: $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

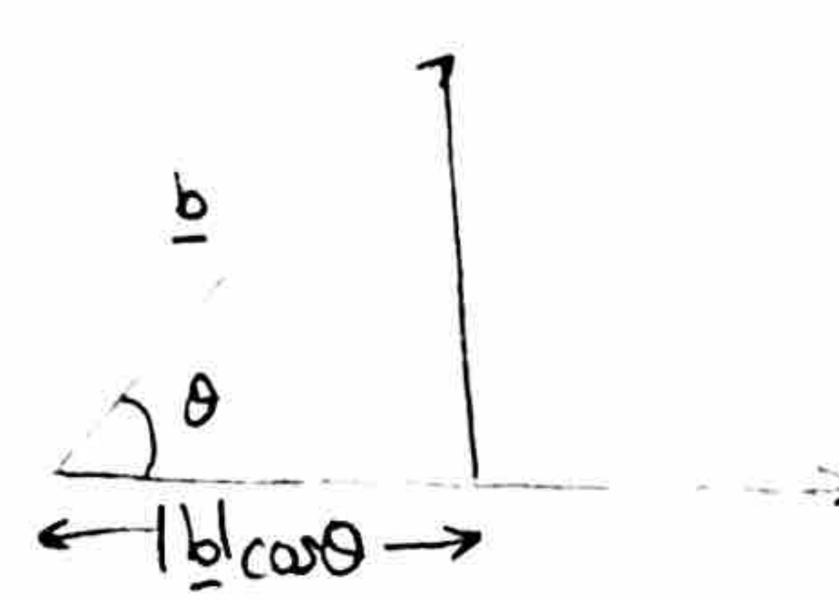
Position Vector: $\underline{r} = (x, y, z)$.

Dot product: Scalar quantity $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$.

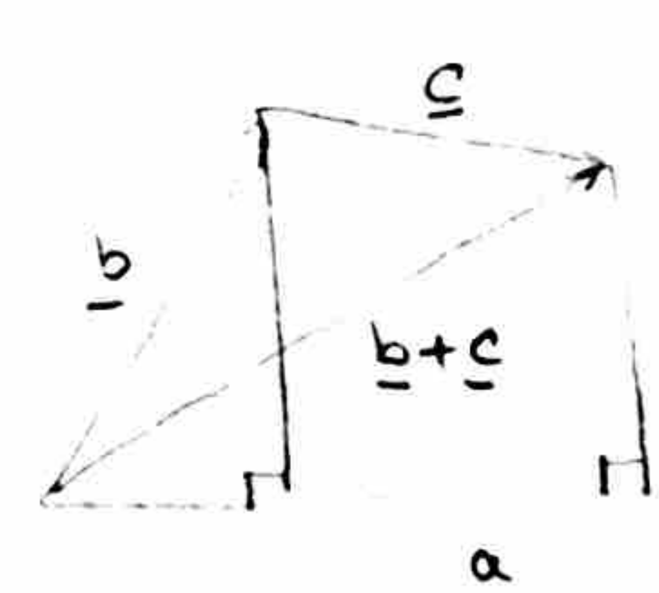
Commutative: $\underline{a}.\underline{b} = \underline{b}.\underline{a}$.

Orthogonal: $\underline{a}$ and $\underline{b}$ are perpendicular (orthogonal) then $\underline{a}.\underline{b} = 0$.

Note: $|\underline{b}|\cos\theta$ is component of $\underline{b}$ in the direction of $\underline{a}$.



Distributive: $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a} + \underline{c}$.



Dotproduct Derivation:

$\underline{e}_1 \cdot \underline{e}_1 = 1, \quad \underline{e}_2 \cdot \underline{e}_2 = 1, \quad \underline{e}_3 \cdot \underline{e}_3 = 1$

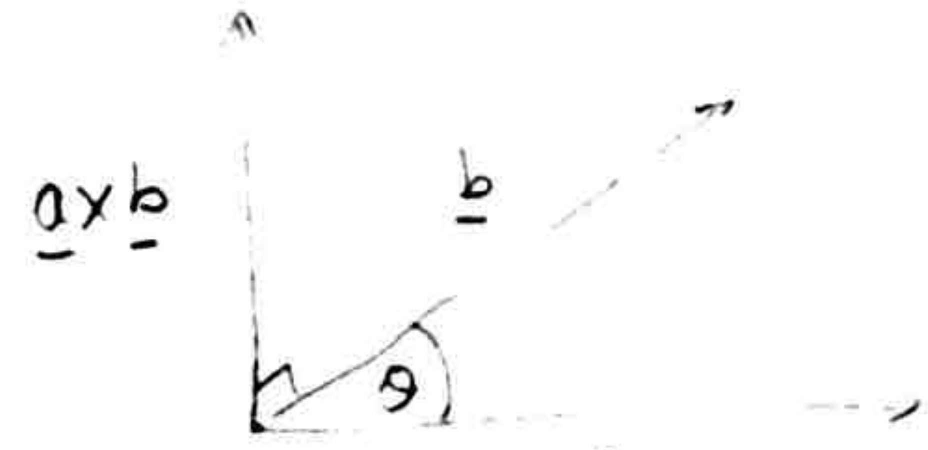
$\underline{e}_1 \cdot \underline{e}_2 = 0, \quad \underline{e}_2 \cdot \underline{e}_3 = 0, \quad \underline{e}_3 \cdot \underline{e}_1 = 0.$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3) \cdot (b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3) \\ &= a_1 b_1 \underline{e}_1 \cdot \underline{e}_1 + a_2 b_2 \underline{e}_2 \cdot \underline{e}_2 + a_3 b_3 \underline{e}_3 \cdot \underline{e}_3 \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

Example: $\underline{a} = (1,1,2)$ and $\underline{b} = (2,3,2)$

$\underline{a} \cdot \underline{b} = 1\times 2 + 1 \times 3 + 2 \times 2 = 9.$

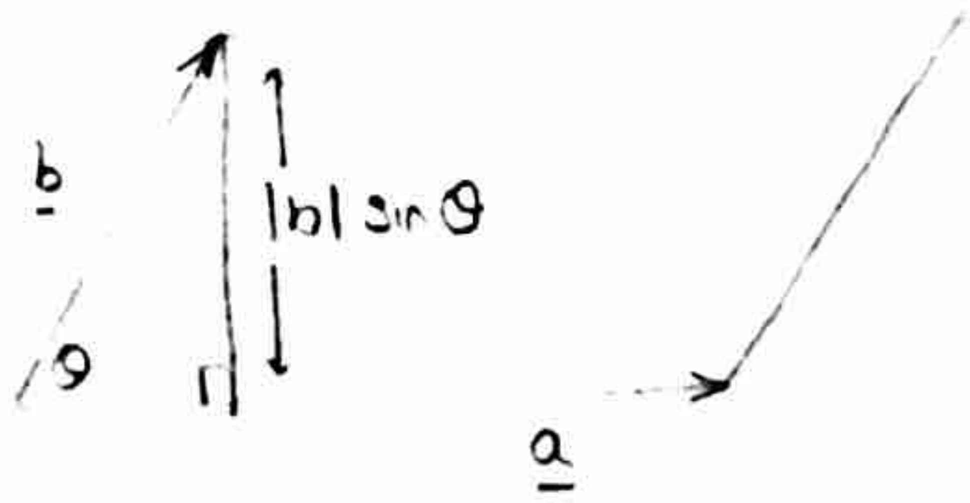
Cross Product: A vector quantity $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \underline{u}$



Not commutative: Due to right-hand rule, $\underline{a} \times \underline{b} \neq -\underline{b} \times \underline{a}$

Parallel: $\underline{a} \times \underline{b} = 0$ if $\underline{a}$ and $\underline{b}$ are parallel.

Note: $\underline{a} \times \underline{a} = 0$



Distributive: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

Note:

$\underline{e}_1 \times \underline{e}_1 = 0 \qquad \underline{e}_1 \times \underline{e}_2 = \underline{e}_3$

$\underline{e}_2 \times \underline{e}_2 = 0 \qquad \underline{e}_2 \times \underline{e}_3 = \underline{e}_1$

$\underline{e}_3 \times \underline{e}_3 = 0 \qquad \underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

Cross Product:

$$\begin{aligned} \underline{a} \times \underline{b} &= (a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3) \times (b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3) \\ &= (a_2 b_3 - a_3 b_2)\underline{e}_1 + (a_3 b_1 - a_1 b_3)\underline{e}_2 + (a_1 b_2 - a_2 b_1)\underline{e}_3. \end{aligned}$$

$$\therefore \quad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Cross product of $(1,3,0)$ and $(2,-1,1)$ is

$(1,3,0) \times (2,-1,1) = (3-0, 0-1, -1-6) = (3,-1,-7)$

Example: A unit vector perpendicular to $(1,0,1)$ and $(0,1,1)$.

A perpendicular vector is $(1,0,1) \times (0,1,1) = (-1,-1,1)$

A perpendicular <u>unit</u> vector is $(-1,-1,1)$ divided by magnitude $= (-1,-1,1)/\sqrt{3}$

Scalar Triple Product: $\underline{a}\cdot(\underline{b}\times\underline{c}) = a_1b_2c_3 - a_1b_3c_2 - a_2b_3c_1 - a_3b_1c_2 - a_3b_2c_1$

Interchangable: $\underline{a}\;\underline{b}\times\underline{c} = \underline{a}\times\underline{b}\cdot\underline{c}$,

$\underline{a}\cdot\underline{b}\times\underline{c} = \underline{b}\cdot\underline{c}\times\underline{a} = \underline{c}\cdot\underline{a}\times\underline{b}$

Determinant Form:

$$\underline{a}\cdot\underline{b}\times\underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ (often written $[a,b,c]$)

Example: Scalar triple product of $(1,2,1)$, $(0,1,1)$ and $(2,1,0)$.

First, $(0,1,1)\times(2,1,0) = (-1,2,-2)$

Second, $(1,2,1)\cdot(-1,2,-2) = 1$.

Note: If three vectors lie in a plane, scalar triple product is zero.

If $\underline{a},\underline{b},\underline{c}$ lie in a plane, $\underline{b}\times\underline{c}$ is perpendicular to plane hence perpendicular to $\underline{a}$.

The dot product of perpendicular vectors is also zero hence $\underline{a}\cdot\underline{b}\times\underline{c} = 0$.

Since $\underline{b}\times\underline{c} = (b_2c_3 - b_3c_2)\underline{e}_1 + (b_3c_1 - b_1c_3)\underline{e}_2 + (b_1c_2 - b_2c_1)\underline{e}_3$

$$\begin{aligned}[\underline{a}\times(\underline{b}\times\underline{c})]_1 &= a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ &= b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3) \\ &= b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ &= b_1\,\underline{a}\cdot\underline{c} - c_1\,\underline{a}\cdot\underline{b}.\end{aligned}$$

Hence $\underline{a}\times(b\times\underline{c}) = (a\cdot c)\underline{b} - (a\cdot b)\underline{c}$

Note: $(\underline{a}\times\underline{b})\times\underline{c} = -\underline{c}\times(\underline{a}\times\underline{b}) = -(\underline{c}\cdot\underline{b})\underline{a} + (\underline{c}\cdot\underline{a})\underline{b}$

Example:

$$\begin{aligned}(\underline{a}\times\underline{b})\cdot(\underline{c}\times\underline{d}) &= \underline{a}\cdot(\underline{b}\times(\underline{c}\times\underline{d})) \\ &= \underline{a}\cdot((\underline{b}\cdot\underline{d})\underline{c} - (\underline{b}\cdot\underline{c})\underline{d}) \\ &= (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d}) - (\underline{a}\cdot\underline{d})(\underline{b}\cdot\underline{c})\end{aligned}$$