# Method of Undetermined Coefficients.

Sample Form: $\varphi(x)$ and all derivatives can be written in terms of linearly independent function $\{y_1(x), \dots, y_n(x)\}$.

$$y_p(x) = A_1 y_1(x) + \dots + A_n y_n(x).$$

Case 1: $\varphi(x) = P_n(x)$, assume a solution of the form:

$$y_p(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

Case 2: $\varphi(x) = ke^{\alpha x}$, where $k$ and $\alpha$ are known constants.

$$y_p(x) = Ae^{\alpha x}$$

Case 3: $\varphi(x) = k_1 \sin\beta x + k_2 \cos\beta x$

$$y_p(x) = A\sin\beta x + B\cos\beta x.$$

Example:

$$y'' - y' - 2y = 4x^2$$

$y_h = c_1 e^{-x} + c_2 e^{2x}$ where $\varphi(x) = 4x^2$

Assume $y_p = A_2 x^2 + A_1 x + A_0$.

$y_p' = 2A_2 x + A_1$, $y_p'' = 2A_2$

Subst: $2A_2 - (2A_2 x + A_1) - 2(A_2 x^2 + A_1 x + A_0) = 4x^2$

$\therefore$ $-2A_2 = 4$, $-2A_2 - 2A_1 = 0$, $2A_2 - A_1 - 2A_0 = 0$.

$\therefore$ $A_2 = -2$, $A_1 = 2$, $A_0 = -3$.

$\therefore$ $y_p = -2x^2 + 2x - 3$.

and general solution is:

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{2x} - 2x + 2x - 3$$

Example :

$$y'' - y' - 2y = e^{3x}$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}, \quad k = 1, \alpha = 3.$$

Assume that $y_p = Ae^{3x}$, $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$

Subst : $4Ae^{3x} = e^{3x}$ $\therefore A = 1/4.$

General solution :

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}$$

Example :

$$y'' - y' - 2y = \sin(2x)$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

Assume $y_p = A\sin 2x + B\cos 2x$, $y_p' = 2A\cos 2x - 2B\sin 2x$

, $y_p'' = -4A\sin 2x - 4B\cos 2x$

$\therefore A = -3/20, \; B = 1/20$

$$\therefore y_p = -\frac{3}{20}\sin(2x) + \frac{1}{20}\cos(2x)$$

General solution

$$y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{20}\sin(2x) + \frac{1}{20}\cos(2x)$$