Linear Differential Equations : Theory of solutions.

Linearly independent solution :

Set of functions $\{y_1(x), y_2(x), \dots, y_n(x)\}$ is linearly dependent on $a \leq x \leq b$ if there exists constants $c_1, c_2, \dots, c_n$ not all zero s.t.

$$c_1 y_1(x) + \dots + c_n y_n(x) = 0 \quad \text{on } a \leq x \leq b.$$

Example :

$\{x, 5x, 1, \sin x\}$ are linearly dependent on $[-1, 1]$.

$C_1 = -5$, $C_2 = 1$, $C_3 = 0$, $C_4 = 0$, not all zero.

e.g. $-5 \cdot x + 1 \cdot 5x + 0 \cdot 1 + 0 \cdot \sin x = 0$.

Wronskian :

Set of functions $\{z_1(x), \dots, z_n(x)\}$ on $a \leq x \leq b$, each function processes $n-1$ derivatives on this interval, is the determinant:

$$W(z_1, z_2, \dots, z_n) = \begin{vmatrix} z_1 & z_2 & \cdots & z_n \\ z_1' & z_2' & \cdots & z_n' \\ z_1'' & z_2'' & \cdots & z_n'' \\ \vdots & \vdots & & \vdots \\ z_1^{(n-1)} & z_2^{(n-1)} & \cdots & z_n^{(n-1)} \end{vmatrix}$$

Example :

Wronskian $\{e^x, e^{-x}\}$

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ \frac{de^x}{dx} & \frac{de^{-x}}{dx} \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^{-x}(e^x) = 2.$$