Calculus - Chapter 27 - Taylor/Maclaurin Series

Definition. Let f function infinitely differentiable at $x=c$, $f^{(n)}(c)$ exist $\forall n>0$.

Taylor series about c:

$$\sum_{n=0}^{+\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots, \quad a_n = f^{(n)}(c)/n!$$

Definition: Maclaurin series is Taylor series about 0:

$$\sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

Example: $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$

Since $f^{(4)}(x) = \sin x$, further derivatives cycle.

$\sin 0 = 0$, $\cos 0 = 1$, $f^{(2k)}(0) = 0$ and $a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!}$

Maclauren series: $\displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.

Example: $f(x) = \dfrac{1}{1-x}$, $f'(x) = \dfrac{1}{(1-x)^2}$, $f''(x) = \dfrac{2}{(1-x)^3}$, $f'''(x) = \dfrac{3 \cdot 2}{(1-x)^4}$, $f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$

Hence $a_n = \dfrac{f^{(n)}(0)}{n!} = 1 \quad \forall n$

$\therefore \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{+\infty} x^n$, $|x| < 1$.

Theorem: If $f(x) = \sum_{n=0}^{+\infty} b_n(x-c)^n$, for some $x \neq c$, then Taylor Series for f, $b_n = \dfrac{f^{(n)}(c)}{n!} \forall n$

In particular if $f(x) = \sum_{n=0}^{+\infty} b_n x^n$ for some $x \neq 0$, is MacLauren series for f.

Assume $f(x) = \sum_{n=0}^{+\infty} b_n(x-c)^n$

$f'(x) = \sum_{n=0}^{+\infty} n b_n (x-c)^{n-1}$ in interval of convergence of $\sum_{n=0}^{+\infty} b_n(x-c)^n$

$\therefore f'(c) = b_1$.

$f''(x) = \sum_{n=0}^{+\infty} n(n-1) b_n (x-c)^{n-2} \quad \therefore f''(c) = 2b_2, \quad \therefore b_2 = f''(c)/2!$

$f'''(x) = \sum_{n=0}^{+\infty} n(n-1)(n-2) b_n (x-c)^{n-3} \quad \therefore f'''(c) = 3! b_3, \quad \therefore b_3 = f'''(c)/3!$

$\therefore b_n = f^{(n)}(c)/n! \quad \forall n \geq 0$.

Example: $\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}, \ |x|<1.$

Example: if $f(x) = \frac{1}{1-x}$, find $f^{(47)}(0)$.

$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \ |x|<1$, the coefficient of x^n, namely 1, is equal to $f^{(n)}(0)/n!$

So for $n=47$, $1 = \frac{f^{(47)}(0)}{47!}$ $\therefore f^{(47)}(0) = 47!$

Taylor's Formula with Remainder: Let f have $(n+1)$st derivative $f^{(n+1)}$ exists in (α,β). Also assume $c, x \in (\alpha,\beta)$, then $\exists x^*$ between c and x s.t.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(c)}}{k!}(x-c)^k + R_n(x)$$

$\therefore R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1}$ is the remainder (or error)

Application of $R_n(x)$: Prove that $\lim_{n\to+\infty} R_n(x) = 0$

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

If $\lim_{n\to+\infty} R_n(x) = 0$ then $f(x) = \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{+\infty} \frac{f^{(k)}(c)}{k!}(x-c)$

i.e $f(x) \equiv$ taylor series.

Example: Prove $\sin x$ is equal to its Maclauren Series.

$f(x) = \sin x$, then $f^{(n)}(x)$ is either $\sin x, \cos x, -\sin x, -\cos x$ $\therefore |f^{(n)}(x)| \geq 1$.

So $|R_n(x)| = \left| \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1} \right| \leq \frac{|(x-c)^{n+1}|}{(n+1)!}$

$\lim_{n\to+\infty} \frac{|(x-c)^{n+1}|}{(n+1)!} = 0$ hence $\lim_{n\to+\infty} R_n(x) = 0$

$\therefore \sin x = \sum_{n=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

Example: Approximate values of functions or integrals, approximate e to 4 decimal places with error less than 0.00005.

$e^x = \sum_{n=0}^{+\infty} x^n/n!$

$e = e^1 = \sum_{n=0}^{+\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$

$= 1 + \sum_{n=0}^{+\infty} \frac{1}{2^n} = 1 + \frac{1}{1-(1/2)} = 1 + 2 = 3.$ $\therefore e < 3.$

$R_n(1) < 0.00005$

$|R_n(1)| = \left| \frac{f^{(n+1)}(x^*)}{(n+1)!} \right|$ where $0 < x^* < 1$

Since $D_x(e^x) = e^x \ \forall x$ so $f^{(n+1)}(x^*)$.

Since e^x is increasing function, $e^{x^*} < e^1 < = e < 3$

$|R_1(1)| < \frac{3}{(n+1)!} \leq 0.00005 \iff 60{,}000 \leq (n+1)!$

For $n < 8$, above holds, we can use sum $\sum_{n=0}^{8} \frac{1}{n!} \sim 1.7183$.

Binomial Series: $(1+x)^r = \sum_{n=1}^{+\infty} \frac{r(r-1)(r-2)\dots(r-n+1)}{n!} x^n, \quad |x| < 1$

$= 1 + rx + \frac{r(r-1)}{2!} x^2 + \dots$

Note if r is positive integer k, then coefficients of x^n for $n > k$ are 0

$(1+x)^k = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} x^n$

Example: $\sqrt{1+x} = (1+x)^{1/2} = 1 + (1/2)x + \frac{(1/2)(-1/2)}{2!} x^2 + \dots$

Example: $\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + (-1/2)(-x) + \dots = 1 + \sum_{n=1}^{+\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} x^n$

Theorem: If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, $|x| < R_1$ and $g(x) = \sum_{n=0}^{+\infty} b_n x^n$, $|x| < R_2$ then $f(x)g(x) = \sum_{n=0}^{+\infty} c_n x^n$ for $|x| < \min(R_1, R_2)$ where

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$