CHAPTER 10 - LINE INTEGRALS, SURFACE INTEGRALS AND INTEGRAL THEOREMS

Line Integrals

A curve C in three-dimensional space may be represented by parametric equations:

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t),\ a \leqq t \leqq b \tag{1}$$

or in vector notation:

$$\mathbf{x} = \mathbf{r}(t) \tag{2}$$

where

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

(see Figure 10.1).

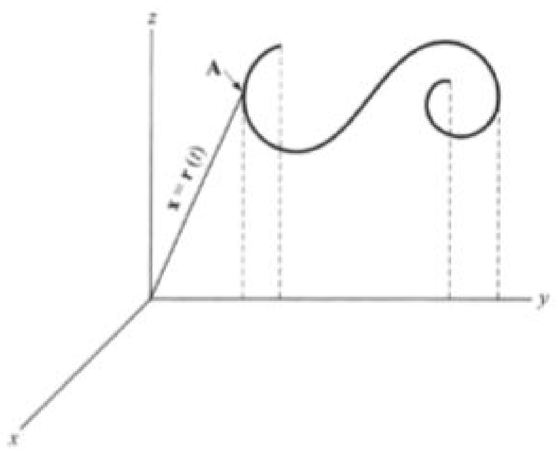

Figure 10.1

The integral

$$\int_C \mathbf{A} \cdot d\mathbf{r} \tag{3}$$

of a vector field $\mathbf{A}$ defined on a curve segment C is called a *line integral*. The integrand has the representation

$$A_1\,dx + A_2\,dy + A_3\,dz$$

obtained by expanding the dot product.

The scalar and vector integrals

$$\int_C \Theta(t)dt = \lim_{n\to\infty} \sum_{k=1}^{n} \Theta(\xi_k, \eta_k, \zeta_k)\Delta t_k \tag{4}$$

$$\int_C \mathbf{A}(t)dt = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{A}(\xi_k, \eta_k, \zeta_k)\Delta t_k \tag{5}$$

The following three basic ways are used to evaluate the line integral (3):

1. The parametric equations are used to express the integrand through the parameter t. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{A}\cdot\frac{d\mathbf{r}}{dt}\,dt$$

2. If the curve C is a plane curve (for example, in the xy plane) and has one of the representations $y = f(x)$ or $x = g(y)$, then the two integrals that arise are evaluated with respect to x or y, whichever is more convenient.

3. If the integrand is a perfect differential, then it may be evaluated through knowledge of the endpoints (that is, without reference to any particular joining curve). (See the section on independence of path on Page 246; also see Page 251.)

Evaluation of Line Integrals for Plane Curves

If the equation of a curve C in the plane $z = 0$ is given as $y = f(x)$, the line integral (2) is evaluated by placing $y = f(x)$, $dy = f'(x)\,dx$ in the integrand to obtain the definite integral

$$\int_{a_1}^{a_2} P\{x, f(x)\}dx + Q\{x, f(x)\}f'(x)dx \tag{6}$$

which is then evaluated in the usual manner.

Similarly, if C is given as $x = g,(y)$, then $dx = g'(y)\,dy$ and the line integral becomes

$$\int_{b_1}^{b_2} P\{g(y), y\}g'(y)dy + Q\{g(y), y\}dy \tag{7}$$

If C is given in parametric form $x = \phi(t)$, $y = \psi(t)$, the line integral becomes

$$\int_{t_1}^{t_2} P\{\phi(t), \psi(t)\}\phi'(t)dt + Q\{\phi(t), \psi(t)\}, \psi'(t)dt \tag{8}$$

where t_1 and t_2 denote the values of t corresponding to points A and B, respectively.

Combinations of these methods may be used in the evaluation. If the integrand $\mathbf{A}\cdot d\mathbf{r}$ is a perfect differential $d\Theta$, then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{(a,b)}^{(c,d)} d\Theta = \Theta(c, d) - \Theta(a, b) \tag{9}$$

Similar methods are used for evaluating line integrals along space curves.

Properties of Line Integrals Expressed for Plane Curves

Line integrals have properties which are analogous to those of ordinary integrals. For example:

1. $\int_C P(x, y)dx + Q(x, y)dy = \int_C P(x, y)dx + \int_C Q(x, y)dy$

2. $\int_{(a_1, b_1)}^{(a_2, b_2)} P\,dx + Q\,dy = -\int_{(a_2, b_2)}^{(a_1, b_1)} P\,dx + q\,dy$

   Thus, reversal of the path of integration changes the sign of the line integral.

3. $\int_{(a_1, b_1)}^{(a_2, b_2)} P\,dx + Q\,dy = \int_{(a_1, b_1)}^{(a_3, b_3)} P\,dx + Q\,dy + \int_{(a_3, b_3)}^{(a_2, b_2)} P\,dx + Q\,dy$

where (a_3, b_3) is another point on C.

Similar properties hold for line integrals in space.

Simple Closed Curves, Simply and Multiply Connected Regions

A *simple closed curve* is a closed curve which does not intersect itself anywhere. Mathematically, a curve in the xy plane is defined by the parametric equations $x = \phi(t)$, $y = \psi(t)$ where ϕ and ψ are single-valued and continuous in an interval $t_1 \leqq t \leqq t_2$. If $\phi(t_1) = \phi(t_2)$ and $\psi(t_1) = \psi(t_2)$, the curve is said to be *closed.* If $\phi(u) = \phi(\upsilon)$ and $\psi(u) = \psi(\upsilon)$ only when $u = \upsilon$ (except in the special case where $u = t_1$ and $\upsilon = t_2$), the curve is closed and does not intersect itself, and so is a simple closed curve. We shall also assume, unless otherwise stated, that ϕ and ψ are piecewise differentiable in $t_1 \leqq t \leqq t_2$.

If a plane region has the property that any closed curve in it can be continuously shrunk to a point without leaving the region, then the region is called *simple connected*; otherwise, it is called *multiply connected* (see Figure 10.2 and Page 127).

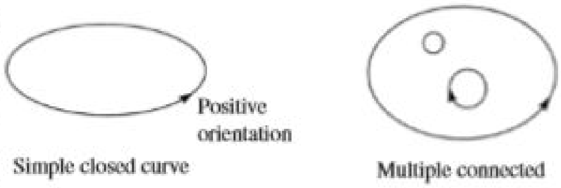

Figure 10.2

Green's Theorem in the Plane

This theorem is needed to prove Stokes's theorem (Page 251). Then it becomes a special case of that theorem.

Let P, Q, $\partial P/\partial y$, $\partial Q/\partial x$ be single-valued and continuous in a simple connected region $\mathfrak{R}$ bounded by a simple closed curve C. Then

$$\oint_C P\,dx + Q\,dy = \iint_{\mathfrak{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \tag{10}$$

where $\oint_C$ is used to emphasize that C is closed and that it is described in the positive direction.

This theorem is also true for regions bounded by two or more closed curves (i.e., multiply connected regions). See Problem 10.10.

Conditions for a Line Integral to Be Independent of the Path

The line integral of a vector field $\mathbf{A}$ is independent of path if its value is the same regardless of the (allowable) path from initial to terminal point. (Thus, the integral is evaluated from knowledge of the coordinates of these two points.)

For example, the integral of the vector field $\mathbf{A} = y\mathbf{i} + x\mathbf{j}$ is independent of path since

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_C y\,dx + x\,dy = \int_{x_1 y_1}^{x_2 y_2} d(xy) = x_2 y_2 - x_1 y_1$$

Thus, the value of the integral is obtained without reference to the curve joining P_1 and P_2.

This notion of the independence of path of line integrals of certain vector fields, important to theory and application, is characterized by the following three theorems.

Theorem 1 A necessary and sufficient condition that $\int_C \mathbf{A} \cdot d\mathbf{r}$ be independent of path is that there exists a scalar function Θ such that $\mathbf{A} = \nabla \Theta$.

Theorem 2 A necessary and sufficient condition that the line integral $\int_C \mathbf{A} \cdot d\mathbf{r}$ be independent of path is that $\nabla \times \mathbf{A} = \mathbf{0}$.

Theorem 3 If $\nabla \times \mathbf{A} = \mathbf{0}$, then the line integral of $\mathbf{A}$ over an allowable closed path is 0; i.e., $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$.

If C is a plane curve, then Theorem 3 follows immediately from Green's theorem, since in the plane case $\nabla \times \mathbf{A}$ reduces to

$$\frac{\partial A_1}{\partial y} = \frac{\partial A_2}{\partial x}$$

EXAMPLE. Newton's second law for forces is $\mathbf{F} = \dfrac{d(m\mathbf{v})}{dt}$, where m is the mass of an object and $\mathbf{v}$ is its velocity.

When $\mathbf{F}$ has the representation $\mathbf{F} = -\nabla\Theta$, it is said to be conservative. The previous theorems tell us that the integrals of conservative fields of force are independent of path. Furthermore, showing that $\nabla \times \mathbf{F} = \mathbf{0}$ is the preferred way of showing that $\mathbf{F}$ is conservative, since it involves differentiation, while demonstrating that Θ exists such that $\mathbf{F} = -\nabla\Theta$ requires integration.

Surface Integrals

Our previous double integrals have been related to a very special surface, the plane. Now we consider other surfaces. yet, the approach is quite similar. Surfaces can be viewed intrinsically, i.e., as non-Euclidean spaces: however, we do not do that. Rather, the surface is thought of as embedded in a three-dimensional Euclidean space and expressed through a two-parameter vector representation:

$$\mathbf{x} = \mathbf{r}(\upsilon_1, \upsilon_2)$$

While the purpose of the vector representation is to be general (that is, interpretable through any allowable three-space coordinate system), it is convenient to initially think in terms of rectangular Cartesian coordinates: therefore, assume

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and that there is a parametric representation

$$x = r(\upsilon_1, \upsilon_2),\ y = r(\upsilon_1, \upsilon_2),\ z = r(\upsilon_1, \upsilon_2) \tag{11}$$

The functions are assumed to be continuously differentiable.

The parameter curves υ_2 = const and υ_1 const establish a coordinate system on the surface (just as y = const and x = const form such a system in the plane). The key to establishing the surface integral of a function is the differential element of surface area. (For the plane, that element is $dA = dx, dy$.)

At any point P of the surface

$$d\mathbf{x} = \frac{\partial \mathbf{r}}{\partial \upsilon_1} d\upsilon_1 + \frac{\partial \mathbf{r}}{\partial \upsilon_2} d\upsilon_2$$

spans the tangent plane to the surface. In particular, the directions of the coordinate curves υ_2 = const and υ_1 = const are designated by $d\mathbf{x}_1 = \frac{\partial \mathbf{r}}{\partial \upsilon_1} d\upsilon_1$ and $d\mathbf{x}_2 \frac{\partial \mathbf{r}}{\partial \upsilon_2} d\upsilon_2$, respectively (see Figure 10.3).

The cross product

$$d\mathbf{x}_1\ x\ d\mathbf{x}_2 = \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} d\upsilon_1\ d\upsilon_2$$

is normal to the tangent plane at P, and its magnitude $\left|\frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2}\right|$ is the area of a differential coordinate parallelogram.

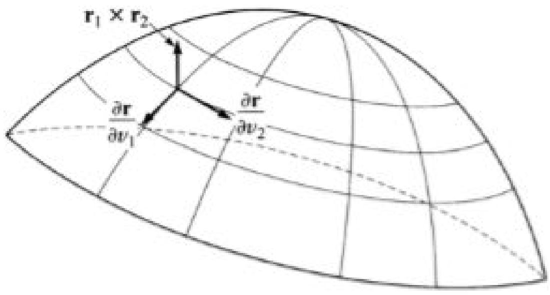

Figure 10.3

Definition The differential element of surface area is

$$dS = \left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right| d\upsilon_1\, d\upsilon_2 \tag{12}$$

For a function $\Theta(\upsilon_1, \upsilon_2)$ that is everywhere integrable on S,

$$\iint_S \Theta\, dS = \iint_S \Theta(\upsilon_1, \upsilon_2) \left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right| d\upsilon_1\, d\upsilon_2 \tag{13}$$

is the surface integral of the function Θ.

In general, the surface integral must be referred to three-space coordinates to be evaluated. If the surface has the Cartesian representation $z = f(x, y)$ and the identifications

$$\upsilon_1 = x,\ \upsilon_2 = y,\ z = f(\upsilon, \upsilon_2)$$

are made, then

$$\frac{\partial \mathbf{r}}{\partial \upsilon_1} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}, \quad \frac{\partial \mathbf{r}}{\partial \upsilon_2} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial \upsilon_2} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} = \mathbf{k} - \frac{\partial z}{\partial x}\mathbf{j} - \frac{\partial z}{\partial x}\mathbf{i}$$

Therefore,

$$\left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right| = \left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right]^{1/2}$$

Thus, the surface integral of Θ has the special representation

$$S = \iint_S \Theta(x, y, z) \left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right]^{1/2} dx\, dy \tag{14}$$

If the surface is given in the implicit form $F(x, y, z) = 0$, then the gradient may be employed to obtain another representation. To establish it, recall that, at any surface point P, the gradient ∇F is perpendicular (normal) to the tangent plane (and, hence, to S).

Therefore, the following equality of the unit vectors holds (up to sign):

$$\frac{\nabla F}{|\nabla F|} = \pm \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right) \Big/ \left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right| \tag{15}$$

[A conclusion of the theory of implicit functions is that from $F(x, y, z) = 0$ (and under appropriate conditions) there can be produced an explicit representation $z = f(x, y)$ of a portion of the surface. This is an existence statement. The theorem does not say that this representation can be explicitly produced.] With this fact in hand, we again let $\upsilon_1 = x$, $\upsilon_2 = y$, $z = f(\upsilon_1, \upsilon_2)$. Then

$$\nabla F = F_x \mathbf{i} + f_y \mathbf{j} + F_z \mathbf{k}$$

Taking the dot product of both sides of Equation (15), K yields

$$\frac{F_z}{|\nabla F|} = \pm \frac{1}{\left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right|}$$

The ambiguity of sign can be eliminated by taking the absolute value of both sides of the equation. Then

$$\left| \frac{\partial \mathbf{r}}{\partial \upsilon_1} \times \frac{\partial \mathbf{r}}{\partial \upsilon_2} \right| = \frac{|\nabla F|}{|F_z|} = \frac{[(F_x)^2 + (F_y)^2 + (F_z)^2]^{1/2}}{|F_z|}$$

and the surface integral of Θ takes the form

$$\iint_S \frac{[(F_x)^2 + (F_y)^2 + (F_z)^2]^{1/2}}{|F_z|} dx\, dy \tag{16}$$

The formulas (14) and (16) also can be introduced in the following nonvectorial manner.

Let S be a two-sided surface having projection $\Re$ on the xy plane, as in Figure 10.4. Assume that an equation for S is $z = f(x, y)$, where f is single-valued and continuous for all x and y in $\Re$. Divide $\Re$ into n subregions of area ΔA_p, $p = 1, 2, \ldots, n$, and erect a vertical column on each of these subregions to intersect S in an area ΔS_p.

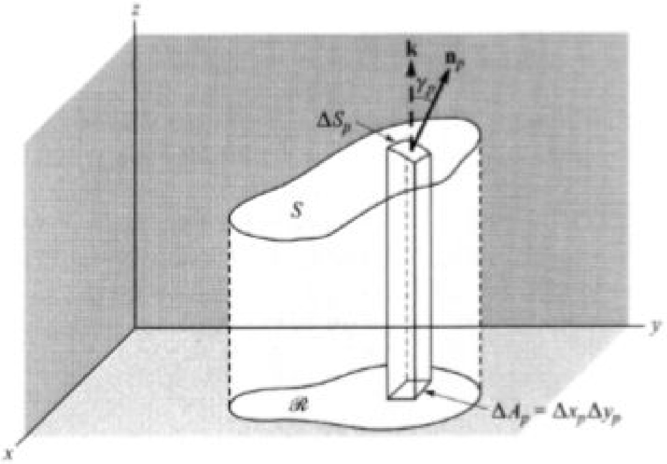

Figure 10.4

Let $\phi(x, y, z)$ be single-valued and continuous at all points of S. Form the sum

$$\sum_{p=1}^{n} \phi(\xi_p, \eta_p, \zeta_p)\Delta S_p \tag{17}$$

where (ξ_p, η_p, ζ_p) is some point of ΔS_p. If the limit of this sum as $n \to \infty$ in such a way that each $\Delta S_p \to 0$ exists, the resulting limit is called the *surface integral* of $\phi(x, y, z)$ over S and is designated by

$$\iint_S \phi(x, y, z) dS \tag{18}$$

Since $\Delta S_p = |\sec \gamma_p|\ \Delta A_p$ approximately, where γ_p is the angle between the normal line to S and the positive z axis, the limit of the sum (17) can be written

$$\iint_\Re \phi(x, y, z) |\sec \gamma| dA \tag{19}$$

The quantity $|\sec \gamma|$ is given by

$$|\sec \gamma| = \frac{1}{|\mathbf{n}_p \cdot \mathbf{k}|} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \tag{20}$$

Then, assuming that $z = f(x, y)$ has continuous (or sectionally continuous) derivatives in $\Re$, (19) can be written in rectangular form as

$$\iint_\Re \phi(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \tag{21}$$

In case the equation for S is given as $F(x, y, z) = 0$, (21) can also be written

$$\iint_{\Re} \phi(x, y, z) \frac{\sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}}{|F_z|} dx\,dy \tag{22}$$

The results (21) or (22) can be used to evaluate (18).

In the preceding we have assumed that S is such that any line parallel to the z axis intersects S in only one point. In case S is not of this type, we can usually subdivide S into surfaces $S_1, S_2, \ldots$ which are of this type. Then the surface integral over S is defined as the sum of the surface integrals over $S_1, S_2, \ldots$.

The results stated hold when S is projected onto a region $\Re$ on the xy plane. In some cases it is better to project S onto the yz or xz planes. For such cases, (18) can be evaluated by appropriately modifying (21) and (22).

The Divergence Theorem

Let **A** be a vector field that is continuously differentiable on a closed-space region V bounded by a smooth surface S. Then

$$\iiint_V \nabla \cdot \mathbf{A}\, dV = \iint_S \mathbf{A} \cdot \mathbf{n}\, dS \tag{23}$$

where **n** is an outwardly drawn normal.

If **n** is expressed through direction cosines, i.e., $\mathbf{n} = \mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma$, then Equation (23) may be written

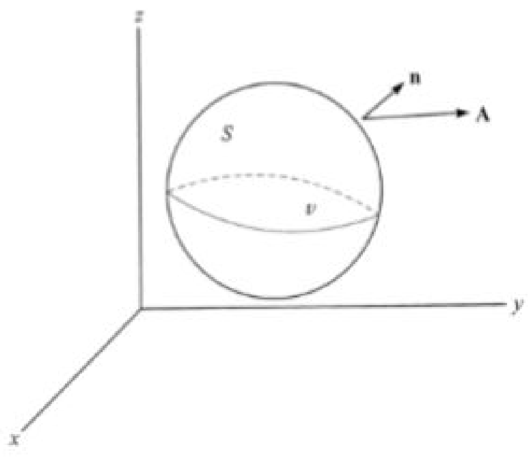

Figure 10.5

$$\iiint \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma)\, dS \tag{24}$$

The rectangular Cartesian component form of Equation (23) is

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1\, dy\,dz + A_2\, dz\,dx + A_3\, dx\,dy) \tag{25}$$

EXAMPLE. If **B** is the magnetic field vector, then one of Maxwell's equations of electromagnetic theory is $\nabla \cdot \mathbf{B} = 0$. When this equation is substituted into the left member of Equation (23), the right member tells us that the magnetic flux through a closed surface containing a magnetic field is zero. A simple interpretation of this fact results by thinking of a magnet enclosed in a ball. All magnetic lines of force that flow out of the ball must return (so that the total flux is zero). Thus, the lines of force flow from one pole to the other, and there is no dispersion.

Stokes's Theorem

Stokes's theorem establishes the equality of the double integral of a vector field over a portion of a surface and the line integral of the field over a simple closed curve bounding the surface portion. (See Figure 10.6.)

Suppose a closed curve C bounds a smooth surface portion S. If the component functions of $\mathbf{x} = \mathbf{r}(\upsilon_1, \upsilon_2)$ have continuous mixed partial derivatives, then for a vector field $\mathbf{A}$ with continuous partial derivatives on S

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S \mathbf{n} \cdot \nabla \times \mathbf{A}\, dS \tag{26}$$

where $\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ with α, β, and γ represeting the angles made by the outward normal $\mathbf{n}$ and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively.

Then the component form of Equation (26) is

$$\oint_C (A_1 dx + A_2 dy + A_3 dz) = \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] dS \tag{27}$$

If $\nabla \times \mathbf{A} = \mathbf{0}$, Stokes's theorem tells us that $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$. This is Theorem 3 on Page 233.

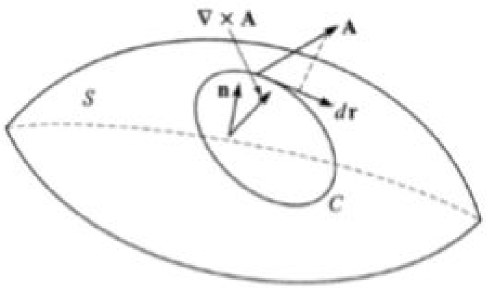

Figure 10.6