CHAPTER 6 - PARTIAL DERIVATIVES

Functions of Two or More Variables

We use the notation $f(x, y)$, $F(x, y)$, etc., to denote the value of the function at (x, y) and write $z = f(x, y)$, $z = F(x, y)$, etc. We also sometimes use the notation $z = z(x, y)$, although it should be understood that in this case z is used in two senses, namely, as a function and as a variable.

Neighborhoods

The set of all points (x, y) such that $|x - x_0| < \delta$, $|y - y_0| < \delta$ where $\delta > 0$ is called a *rectangular δ neighborhood* of (x_0, y_0); the set $0 < |x - x_0| < \delta$, $0 < |y - y_0| < \delta$, which excludes (x_0, y_0), is called a *rectangular deleted δ neighborhood* of (x_0, y_0). Similar remarks can be made for made for other neighborhoods; e.g., $(x - x_0)^2 + (y - y_0)^2 < \delta^2$ is a *circular δ neighborhood* of (x_0, y_0). The term *open ball* is used to designate this circular neighborhood. This terminology is appropriate for generalization to more dimensions. Whether neighborhoods are viewed as circular or square is immaterial, since the descriptions are interchangeable. Simply notice that given an open ball (circular neighborhood) of radius δ there is a centered square whose side is of length less than $\sqrt{2}\,\delta$ that is interior to the open ball, and, conversely, for a square of side δ there is an interior centered circle of radius less than $\delta/2$. (See Figure 6.1.)

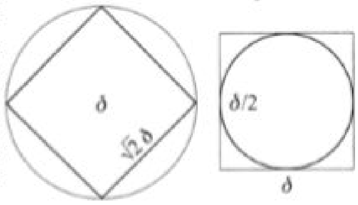

Figure 6.1

A point (x_0, y_0) is called a *limit point, accumulation point,* or *cluster point* of a point set S if every deleted δ neighborhood of (x_0, y_0) contains points of S. As in the case of one-dimensional point sets, every bounded infinite set has at least one limit point (the Bolzano-Weierstrass theorem; see Chapter 1). A set containing all its limit points is called a *closed set*.

Regions

A point P belonging to a point set S is called an *interior point* of S if there exists a deleted δ neighborhood of P all of whose points belong to S. A point P not belonging to S is called an *exterior point* of S if there exists a deleted δ neighborhood of P all of whose points do not belong to S. A point P is called a *boundary point* of S if every deleted δ neighborhood of P contains points belonging to S and also points not belonging to S.

If any two points of a set S can be joined by a path consisting of a finite number of broken line segments all of whose points belong to S, then S is called a *connected set*. A *region* is a connected set which consists of interior points or interior and boundary points. A *closed region* is a region containing all its boundary points. An *open region* consists only of interior points. The complement of a set S in the xy plane is the set of all points in the plane not belonging to S. (See Figure 6.2.)

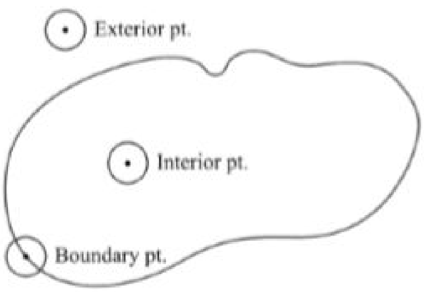

Figure 6.2

Examples of some regions are shown graphically in Figure 6.3(a), (b), and (c) . The rectangular region of Figure 6.1(a), including the boundary, represents the sets of points $a \leqq x \leqq b$, $c \leqq y \leqq d$ which is a natural extension of the closed interval $a \leqq x \leqq b$ for one dimension. The set $a < x < b$, $c < y < d$ corresponds to the boundary being excluded.

In the regions of Figure 6.3(a) and (b), any *simple closed curve* (one which does not intersect itself anywhere) lying inside the region can be shrunk to a point which also lies in the region. Such regions are called *simply connected regions*. In Figure 6.3(c), however, a simple closed curve $ABCD$ surrounding one of the "holes" in the region cannot be shrunk to a point without leaving the region. Such regions are called *multiply connected regions*.

Limits

Let $f(x, y)$ be defined in a deleted δ neighborhood of (x_0, y_0) [i.e., $f(x, y)$ may be undefined at (x_0, y_0)]. We say that l is the *limit* of $f(x,y)$ as x approaches x_0 and y approaches y_0 [or (x, y) approaches (x_0, y_0)] and write $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = l$ [or $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l$] if for any positive number δ we can find some positive number δ [depending on δ and (x_0, y_0), in general] such that $|f(x, y) - l| < \delta$ whenever $0 < |x - x_0| < \delta$ and $0 < |y - y_0| < \delta$.

Note that in order for $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ to exist, it must have the same value regardless of the approach of (x, y) to (x_0, y_0). It follows that if two different approaches give different values, the limit cannot exist (see Problem 6.7). This implies, as in the case of functions of one variable, that if a limit exists it is unique.

Iterated Limits

The *iterated limits* $\lim\limits_{x\to x_0}\left\{\lim\limits_{y\to y_0} f(x, y)\right\}$ and $\lim\limits_{y\to y_0}\left\{\lim\limits_{x\to x_0} f(x, y)\right\}$ [also denoted by $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x, y)$ and $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x, y)$, respectively] are not necessarily equal. Although they must be equal if $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$ is to exist, their equality does not guarantee the existence of this last limit.

Continuity

Let $f(x, y)$ be defined in a δ neighborhood of (x_0, y_0) [i.e., $f(x, y)$ must be defined *at* (x_0, y_0) as well as near it]. We say that $f(x, y)$ is *continuous* at (x_0, y_0) if for any positive number δ we can find some positive number δ [depending on δ and (x_0, y_0) in general] such that $|f(x, y) - f(x_0, y_0)| < \delta$ whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$, or, alternatively, $(x - x_0)^2 + (y - y_0)^2 < \delta^2$.

Note that three conditions must be satisfied in order for $f(x, y)$ to be continuous at (x_0, y_0):

1. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l$; i.e., the limit exists as $(x, y) \to (x_0, y_0)$.
2. $f(x_0, y_0)$ must exist; i.e., $f(x, y)$ is defined at (x_0, y_0).
3. $l = f(x_0, y_0)$.

If desired, we can write this in the suggestive form $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = f(\lim\limits_{x\to x_0} x, \lim\limits_{y\to y_0} y)$.

If a function is not continuous at a point (x_0, y_0), it is said to be *discontinuous* at (x_0, y_0), which is then called a *point of discontinuity*. If, as in the preceding example, it is possible to redefine the value of a function at a point of discontinuity so that the new function is continuous, we say that the point is a *removable discontinuity* of the old function. A function is said to be *continuous in a region* $\mathfrak{R}$ of the xy plane if it is continuous at every point of $\mathfrak{R}$.

Uniform Continuity

In the definition of continuity of $f(x, y)$ at (x_0, y_0), δ depends on δ and also (x_0, y_0) in general. If in a region $\mathfrak{R}$ we can find a δ which depends only on δ but not on any particular point (x_0, y_0) in $\mathfrak{R}$ (i.e., the same δ will work for *all* points in $\mathfrak{R}$), then $f(x, y)$ is said to be *uniformly continuous* in $\mathfrak{R}$. As in the case of functions of one variable, it can be proved that a function which is continuous in a closed and bounded region is uniformly continuous in the region.

Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the *partial derivative* of the function with respect to the variable. Partial derivatives of $f(x, y)$ with respect to x and y are denoted by $\frac{\partial}{\partial x}\left[\text{or } f_x, f_x(x, y), \left.\frac{\partial f}{\partial x}\right|_y\right]$ and $\frac{\partial f}{\partial y}\left[\text{or } f_y, f_y(x, y), \left.\frac{\partial f}{\partial y}\right|_x\right]$, respectively, the latter notations being used when needed to emphasize which variables are held constant.

By definition,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \qquad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{1}$$

when these limits exist. The derivatives evaluated at the particular point (x_0, y_0) are often indicated by $\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$ and $\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$, respectively.

Higher-Order Partial Derivatives

If $f(x, y)$ has partial derivatives at each point (x, y) in a region, then $\partial f/\partial x$ and $\partial f/\partial y$ are themselves functions of x and y, which may also have partial derivatives. These second derivatives are denoted by are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy} \tag{2}$$

If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ and the order of differentiation is immaterial; otherwise they may not be equal (see Problems 6.13 and 6.41).

Differentials

Let $\Delta x = dx$ and $\Delta y = dy$ be increments given to x and y, respectively. Then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \Delta f \tag{3}$$

is called the *increment* in $z = f(x, y)$. If $f(x, y)$ has continuous first partial derivatives in a region, then

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \epsilon_1\, dx + \epsilon_2 dy = \Delta f \tag{4}$$

where ϵ_1 and ϵ_2 approach zero as Δx and Δy approach zero (see Problem 6.14). The expression

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{or} \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{5}$$

is called the *total differential* or simply the *differential* of z or f, or the *principal part* of Δz or Δf. Note that $\Delta z \neq dz$ in general. However, if $\Delta x = dx$ and $\Delta y = dy$ are "small," then dz is a close approximation of Δz (see

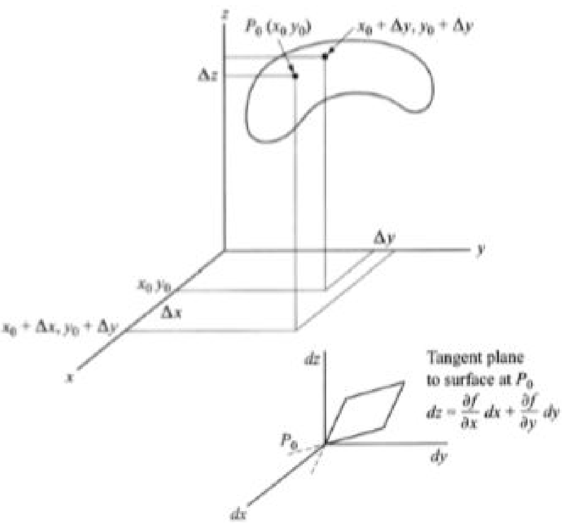

Figure 6.4

If f is such that Δf (or Δz) can be expressed in the form of Equation (4) where ϵ_1 and ϵ_2 approach zero as Δx and Δy approach zero, we call f *differentiable* at (x, y). The mere existence of f_x and f_y does not in itself guarantee differentiability; however, continuity of f_x and f_y does (although this condition happens to be slightly stronger than necessary). In case f_x and f_y are continuous in a region $\Re$, we say that f is *continuously differentiable* in $\Re$.

Theorems on Differentials

In the following, we assume that all functions have continuous first partial derivatives in a region $\Re$; i.e., the functions are continuously differentiable in $\Re$.

1. If $z = f(x_1, x_2, \ldots, x_n)$, then
$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n \tag{6}$$
regardless of whether the variables $x_1, x_2, \ldots, x_n$ are independent or dependent on other variables (see Problem 6.20). This is a generalization of result in Equation (*5*). In Euqtion (*6*) we often use z in place of f.

2. If $f(x_1, x_2, \ldots, x_n) = c$, a constant, then $df = 0$. Note that in this case $x_1, x_2, \ldots, x_n$ cannot all be independent variables.

3. The expression $P(x, y)dx + Q(x, y)dy$, or, briefly, $P\,dx + Q\,dy$, is the differential of $f(x, y)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. In such case, $P\,dx + Q\,dy$ is called an *exact differential.*
Note: Observe that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ implies that $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$.

4. The expression $P(x, y, z)\,dx + Q(x, y, z)dy + R(x, y, z)dz$, or, briefly, $P\,dx + Q\,dy + R\,dz$, is the differential of $f(x, y, z)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$. In such case, $P\,dx + Q\,dy + R\,dz$ is called an *exact differential*.

Differentiation of Composite Functions

Let $z = f(x, y)$ where $x = g(r, s)$, $y = h(r, s)$ so that z is a function of r and s. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{7}$$

In general, if $u = F(x_1, \ldots, x_n)$ where $x_1 = f_1(r_1, \ldots, r_p), \ldots, x_n = f_n(r_1, \ldots, r_p)$, then

$$\frac{\partial u}{\partial r_k} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_k} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial u}{\partial r_k} \qquad k = 1, 2, \ldots, p \tag{8}$$

If, in particular, $x_1, x_2, \ldots, x_n$ depend on only one variable s, then

$$\frac{du}{ds} = \frac{\partial u}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial u}{\partial x_2}\frac{dx_2}{ds} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{ds} \tag{9}$$

These results, often called *chain rules*, are useful in transforming derivatives from one set of variables to another.

Higher derivatives are obtained by repeated application of the chain rules.

Euler's Theorem on Homogeneous Functions

A function represented by $F(x_1, x_2, \ldots, x_2, \ldots, x_n)$ is called *homogeneous of degree p* if, for all values of the parameter λ and some constant p, we have the identity

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p F(x_1, x_2, \ldots, x_n) \tag{10}$$

Implicit Functions

In general, an equation such as $F(x, y, z) = 0$ defines one variable—say, z—as a function of the other two variables x and y. Then z is sometimes called an *implicit function* of x and y, as distinguished from an *explicit function f*, where $z = f(x, y)$, which is such that $F[x, y, f(x, y)] = 0$.

Differentiation of implicit functions requires considerable discipline in interpreting the independent and dependent character of the variables and in distinguishing the intent of one's notation. For example, suppose that in the implicit equation $F[x, y, f(x, z)] = 0$, the independent variables are x and y and that $z = f(x, y)$. In order to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we initially write the following [observe that $F(x, t, z)$ is zero for all domain pairs (x, y); i.e., it is a constant]:

$$0 = dF = F_x\, dx + F_y\, dy + F_z\, dz$$

and then compute the partial derivatives F_x, F_y, F_z as though y, y, z constituted an independent set of variables. At this stage we invoke the dependence of z on x and y to obtain the differential form $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. Upon substitution and some algebra (see Problem 6.30), the following results are obtained:

$$\frac{\partial f}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$$

Jacobians

If $F(u, \upsilon)$ and $G(u, \upsilon)$ are differentiable in a region, the *Jacobian determinant*, or the *Jacobian*, of F and G with respect to u and υ is the second-order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,\upsilon)} = \begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial \upsilon} \\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial \upsilon} \end{vmatrix} = \begin{vmatrix} F_u & F_\upsilon \\ G_u & G_\upsilon \end{vmatrix} \tag{12}$$

Similarly, the third-order determinant

$$\frac{\partial(F,G,H)}{\partial(u,\upsilon,w)} = \begin{vmatrix} F_u & F_\upsilon & F_w \\ G_u & G_\upsilon & G_w \\ H_u & H_\upsilon & H_w \end{vmatrix}$$

is called the Jacobian of F, G, and H with respect to u, υ, and w. Extensions easily made.

Partial Derivatives Using Jacobians

Jacobians often prove useful in obtaining partial derivatives of implicit functions. Thus, for example, given the simultaneous equations

$$F(x, y, u, \upsilon) = 0, \qquad G(x, y, u, \upsilon) = 0$$

we may, in general, consider u and υ as functions of x and y. In this case, we have (see Problem 6.31)

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,\upsilon)}}{\dfrac{\partial(F,G)}{\partial(u,\upsilon)}}, \quad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,\upsilon)}}{\dfrac{\partial(F,G)}{\partial(u,\upsilon)}}, \quad \frac{\partial \upsilon}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,\upsilon)}}, \quad \frac{\partial \upsilon}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,\upsilon)}}$$

The ideas are easily extended. Thus, if we consider the simultaneous equations

$$F(u, \upsilon, w, x, y) = 0, \qquad G(u, \upsilon, w, x, y)\, y) = 0, \qquad H(u, \upsilon, w, x, y) = 0$$

we may, for example, consider u, υ, and w as functions of x and y. In this case,

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,\upsilon,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,\upsilon,w)}}, \quad \frac{\partial w}{\partial y} = -\frac{\dfrac{\partial(F,G,H)}{\partial(u,\upsilon,y)}}{\dfrac{\partial(F,G,H)}{\partial(u,\upsilon,w)}}$$

with similar results for the remaining partial derivatives (see Problem 6.33).

Theorems on Jacobians

In the following we assume that all functions are continuously differentiable.

1. A necessary and sufficient condition that the equations $F(u, \upsilon, x, y, z) = 0$ and $G(u, \upsilon, x, y, z) = 0$ can be solved for u and υ (for example) is that $\dfrac{\partial(F, G)}{\partial(u, \upsilon)}$ is not identically zero in a region $\Re$.

   Similar results are valid for m equations in n variables, where $m < n$.

2. If x and y are functions of u and υ while u and υ are functions of and s, then (see Problem 6.43)

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, \upsilon)} \frac{\partial(u, \upsilon)}{\partial(r, s)} \tag{13}$$

   This is an example of a *chain rule* for Jacobians. These ideas are capable of generalization (see Problems 6.107 and 6.109, for example).

3. If $u = f(x, y)$ and $\upsilon = g(x, y)$, then a necessary and sufficient condition that a functional relation of the form $\phi(u, \upsilon) = 0$ exists between u and υ is that $\dfrac{\partial(u, \upsilon)}{\partial(x, y)}$ be identically zero Similar results. hold for n functions of n variables.

Transformations

The set of equations

$$\begin{cases} x = F(u, \upsilon) \\ y = F(u, \upsilon) \end{cases} \tag{14}$$

defines, in general, a *transformation* or *mapping* which establishes a correspondence between points in the $u\upsilon$ and xy planes. If to each point in the $u\upsilon$ plane there corresponds one and only one the xy plane, and conversely, we speak of a *one-to-one* transformation or mapping. This will be so if F and G are continuously differentiable, with Jacobian not identically zero in a region. In such case (which we assume unless otherwise stated), Equations (14) are said to define a *continuously differentiable transformation* or *mapping*.

The words *transformation* and *mapping* describe the same mathematical concept in different ways. A *transformation* correlates one coordinate representation of a region of space with another. A *mapping* views this correspondence as a correlation of two distinct regions.

Under the transformation (14) a closed region $\Re$ of the xy plane is, in general, mapped into a closed region $\Re'$ of the $u\upsilon$ plane. Then if ΔA_{xy} and $\Delta A_{u\upsilon}$ denote, respectively, the areas of these regions, we can show that

$$\lim \frac{\Delta A_{xy}}{\Delta A_{u\upsilon}} = \left|\frac{\partial(x, y)}{\partial(u, \upsilon)}\right| \tag{15}$$

where lim denotes the limit as ΔA_{xy} (or $\Delta A_{u\upsilon}$) approaches zero. The Jacobian on the right of Equation (15) is often called the *Jacobian of the transformation* (14).

If we solve transformation (14) for u and υ in terms of x and y, we obtain the transformation $u = f(x, y)$, $\upsilon = g(x, y)$, often called the *inverse transformation* corresponding to (14). The Jacobians $\dfrac{\partial(u, \upsilon)}{\partial(x, y)}$ and $\dfrac{\partial(x, y)}{\partial(u, \upsilon)}$ of these transformations are reciprocals of each other (see Problem 6.43). Hence, if one Jacobian is different from zero in a region, the inverse exists and is not zero.

Curvilinear Coordinates

Rectangular Cartesian coordinates in the Euclidean plane or in three-dimensional space were mentioned at the beginning of this chapter. Other coordinate systems, the coordinate curves of which are generated from families that are not necessarily linear, are useful. These are called *curvilinear coordinates.*

Mean Value Theorem

If $f(x, y)$ is continuous in a closed region and if the first partial derivatives exist in the open region (i.e., excluding boundary points). then

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k) \qquad 0 < \theta < 1 \qquad (16)$$

This is sometimes written in a form in which $h = \Delta x = x - x_0$ and $k = \Delta y = y - y_0$.