CHAPTER 4 - DERIVATIVES

The Concept and Definition of a Derivative

Let $P_0\ (x_0)$ be a point on the graph of $y = f(x)$. Let $P(x)$ be a nearby point on this same graph of the function f. Then the line through these two points is called a *secant line*. Its slope, m_s, is the difference quotient

$$m_s = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta y}{\Delta x} \tag{1}$$

where Δx and Δy are called the increments in x and y, respectively. Also this slope may be written

$$m_s = \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

where $h = x - x_0 = \Delta x$. See Figure 4.2.

We can imagine a sequence of lines formed as $h \to 0$. It is the limiting line of this sequence that is the natural one to be the tangent line to the graph at P_0.

To make this mode of reasoning precise, the limit (when it exists), is formed as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3a}$$

As indicated, this limit is given the name $f'(x_0)$. It is called the *derivative* of the function f at its domain value x_0. If this limit can be formed at each point of a subdomain of the domain of f, then f is said to be *differentiable* on that subdomain and a new function f' has been constructed.

Theorem: If f is differentiable at a domain value, then it is continuous at that value.

As indicated, the converse of this theorem is not true.

Right- and Left-Hand Derivatives

The *right-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_+(x_0) = \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} \tag{5}$$

if this limit exists. Note that in this case $h(= \Delta x)$ is restricted only to positive values as it approaches zero.

Similarly, the *left-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_-(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \tag{6}$$

if this limit exists. In this case h is restricted to negative values as it approaches zero.

A function f has a derivative at $x = x_0$ if and only if $f'_+(x_0) = f'_-(x_0)$.

Differentiability in an Interval

If a function has a derivative at all points of an interval, it is said to be *differentiable in the interval*. In particular, if f is defined in the closed interval $a \leq x \leq b$—i.e. $[a, b]$—then f is differentiable in the interval if and only if $f'(x_0)$ exists for each x_0 such that $a < x_0 < b$ and if both $f'_+(a)$ and $f'_-(b)$ exist.

If a function has a continuous derivative, it is sometimes called *continuously differentiable*.

Piecewise Differentiability

A function is called *piecewise differentiable* or *piecewise smooth* in an interval $a \leqq x \leqq b$ if $f'(x)$ is piecewise continuous. An example of a piecewise continuous function is shown graphically on Page 47.

An equation for the tangent line to the curve $y = f(x)$ at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{7}$$

Differentials

Let $\Delta x = dx$ be an increment given to x. Then

$$\Delta y = f(x + \Delta x) - f(x) \tag{8}$$

is called the *increment* in $y = f(x)$. If $f(x)$ is continuous and has a continuous first derivative in an interval, then

$$\Delta y = f'(x)\,\Delta x + \epsilon\Delta x = f'(x)dx + dx \tag{9}$$

where $\epsilon \to 0$ as $\Delta x \to 0$. The expression

$$dy = f'(x)dx \tag{10}$$

is called the *differential of y or f(x)* or the *principal part of* Δy. Note that $\Delta y \neq dy$, in general. However, if $\Delta x = dx$ is small, then dy is a close approximation of Δy (see Problem 4.11). The quantities dx (called the *differential of x*) and dy need not be small.

Because of the definitions given by Equations (8) and (10), we often write

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{11}$$

dx and dy are called differentials of x and y, respectively. Because the preceding linear equation is valid at every point in the domain of f at which the function has a derivative, the subscript may be dropped and we can write

$$dy = f'(x)\,dx$$

The following important observations should be made. $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, thus $\frac{dy}{dx}$ is not the same thing as $\frac{\Delta y}{\Delta x}$.

On the other hand, dy and Δy are related. In particular, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ means that for any $\epsilon > 0$ there exists $\delta > 0$ such that $-\varepsilon < \frac{\Delta y}{\Delta x} - \frac{dy}{dx} < \varepsilon$ whenever $|\Delta x| < \delta$. Now dx is an independent variable and the axes of x and dx are parallel; therefore, dx may be chosen equal to Δx. With this choice,

$$-\varepsilon\Delta x < \Delta y - dy < \varepsilon\Delta x$$

or

$$dy - \varepsilon\Delta x < \Delta y < dy + \varepsilon\Delta x$$

From this relation we see that dy is an approximation to Δy in small neighborhoods of x, dy is called the *principal part* of Δy.

The Differentiation of Composite Functions

Many functions are a composition of simpler ones. For example, if f and g have the rules of correspondence $u = x^3$ and $y = \sin u$, respectively, then $y = \sin x^3$ is the rule for a composite function $F = g(f)$. The domain of F is that subset of the domain of F whose corresponding range values are in the domain of g. The rule of composite function differentiation is called the *chain rule* and is represented by $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}[F'(x) = g'(u)f'(x)]$.

In the example,

$$\frac{dy}{dx} \equiv \frac{d(\sin x^3)}{dx} = \cos x^3(3x^2dx)$$

Implicit Differentiation

The rule of correspondence for a function may not be explicit. For example, the rule $y = f(x)$ is *implicit* to the equation $x^2 + 4xy^5 + 7xy + 8 = 0$. Furthermore, there is no reason to believe that this equation can be solved for y in terms of x. However, assuming a common domain (described by the independent variable x), the left-hand member of the equation can be construed as a composition of functions and differentiated accordingly. (The rules for differentiation are listed here for your review.)

In this example, differentiation with respect to x yields

$$2x + 4\left(y^5 + 5xy^4\frac{dy}{dx}\right) + 7\left(y + x\frac{dy}{dx}\right) = 0$$

Observe that this equation can be solved for $\frac{dy}{dx}$ as a function of x and y (but not of x alone).

Rules for Differentiation

If f, g, and h are differentiable functions, the following differentiation rules are valid.

1. $\frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$ (Addition rule)

2. $\frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\frac{d}{dx}\{Cf(x)\} = C\frac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\frac{d}{dx}\{f(x)g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ (Product rule)

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ (Quotient rule)

6. If $y = f(u)$ where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\frac{du}{dx} = f'\{g(x)\}g'(x) \tag{12}$$

Similarly, if $y = f(u)$ where $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \tag{13}$$

The results (*12*) and (*13*) are often called *chain rules* for differentiation of composite functions.

These rules probably are the most misused (or perhaps unused) rules in the application of the calculus.

7. If $y = f(x)$ and $x = f^{-1}(y)$, then dy/dx and dx/dy are related by

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{14}$$

8. If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \tag{15}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(x) + g(x)\} = df(x) + dg(x) = f'(x)dx + g'(x)dx = \{f'(x) + g'(x)\}dx$$

$$d\{f(x)g(x)\} = f(x)\mathrm{d}g(x) + df(x) = \{f(x)g'(x) + g(x)f'(x)\}dx$$

Derivatives of Elementary Functions

In the following we assume that u is a differentiable function of x; if $u = x$, $du/dx = 1$. The inverse functions are defined according to the principal values given in Chapter 3.

1. $\frac{d}{dx}(C) = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
4. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$
5. $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$
6. $\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$
7. $\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}$
8. $\frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}$
9. $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \quad a > 0, a \neq 1$
10. $\frac{d}{dx}\log_e u = \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
11. $\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}$
12. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$
13. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
16. $\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$
17. $\frac{d}{dx}\sec^{-1} u = \pm\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} + \text{ if } u > 1 \\ - \text{ if } u < -1 \end{cases}$
18. $\frac{d}{dx}\csc^{-1} u = \mp\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} - \text{ if } u > 1 \\ + \text{ if } u < -1 \end{cases}$
19. $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$
20. $\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$
21. $\frac{d}{dx}\tanh u = \sec h^2 u\frac{du}{dx}$
22. $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\frac{du}{dx}$
23. $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u\tanh u\frac{du}{dx}$
24. $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u\coth u\frac{du}{dx}$
25. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
26. $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
27. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$
28. $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$

14. $\dfrac{d}{dx}\cos^{-1}u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

15. $\dfrac{d}{dx}\tan^{-1}u = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$

29. $\dfrac{d}{dx}\operatorname{sech}^{-1}u = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$

30. $\dfrac{d}{dx}\operatorname{csch}^{-1}u = -\dfrac{1}{u\sqrt{u^2+1}}\dfrac{du}{dx}$

Higher-Order Derivatives

If $f(x)$ is differentiable in an interval, its derivative is given by $f'(x)$, y' or dy/dx, where $y = f(x)$. If $f'(x)$ is also differentiable in the interval, its derivative is denoted by $f''(x)$, y'' or $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$. Similarly, the nth derivative of $f(x)$, if it exists, is denoted by $f^{(n)}(x)$, $y^{(n)}$ or $\dfrac{d^ny}{dx^n}$, where n is called the order of the derivative. Thus, derivatives of the first, second, third, . . . orders are given by $f'(x), f''(x), f'''(x), \ldots.$

Computation of higher-order derivatives follows by repeated application of the differentiation rules given here.

Mean Value Theorems

These theorems are fundamental to the rigorous establishment of numerous theorems and formulas. (See Figure 4.5.)

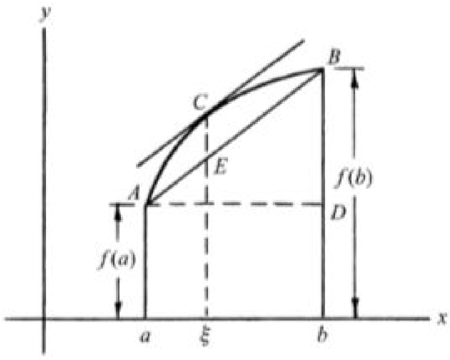

Figure 4.5

1. **Rolle's theorem.** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b) = 0$, then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

   Rolle's theorem is employed in the proof of the mean value theorem. It then becomes a special case of that theorem.

2. **The mean value theorem.** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \quad a < \xi < b \tag{16}$$

Rolle's theorem is the special case of this where $f(a) = f(b) = 0$.

The result (16) can be written in various alternative forms; for example, if x and x_0 are in (a, b), then

$$f(x) = f(x_0) + f'(\xi)(x - x_0) \quad \xi \text{ between } x_0 \text{ and } x \tag{17}$$

We can also write result (16) with $b = a + h$, in which case $\xi = a + \theta h$, where $0 < \theta < 1$.

The mean value theorem is also called the *law of the mean*.

3. **Cauchy's generalized mean value theorem.** If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad a < \xi < b \tag{18}$$

where we assume $g(a) \neq g(b)$ and $f'(x)$, $g'(x)$ are not simultaneously zero. Note that the special case $g(x) = x$ yields (16).

L'Hospital's Rules

If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$, where A and B are either both zero or both infinite, $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ is often called an *indeterminate* of the form 0/0 or ∞/∞, respectively, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. The following theorems, called *L'Hospital's rules*, facilitate evaluation of such limits.

1. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0$ for $x \neq x_0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \tag{19}$$

whenever the limit on the right can be found. In case $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

2. If $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = \infty$, the result (*19*) is also valid.

These can be extended to cases where $x \to \infty$ or $-\infty$, and to cases where $x_0 = a$ or $x_0 = b$ in which only one-sided limits, such as $x \to a+$ or $x \to b-$, are involved.

Limits represented by the *indeterminate forms* $0 \cdot \infty$, ∞^0, 0^0, 1^∞, and $\infty - \infty$ can be evaluated on replacing them by equivalent limits for which the aforementioned rules are applicable (see Problem 4.29).

Relative Extrema and Points of Inflection

Theorem Assume that x_1 is a number in an open set of the domain of f at which f' is continuous and f'' is defined. If $f'(x_1) = 0$ and $f''(x_1) \neq 0$, then $f(x_1)$ is a relative extreme of f. Specifically:

(a) If $f''(x_1) > 0$, then $f(x_1)$ is a relative minimum.

(b) If $f''(x_1) < 0$, then $f(x_1)$ is a relative maximum.

(The domain value x_1 is called a *critical value*.)

This theorem may be generalized in the following way. Assume existence and continuity of derivatives as needed and suppose that $f'(x_1) = f''(x_1) = \ldots f^{(2p-1)}(x_1) = 0$ and $f^{(2p)}(x_1) \neq 0$ (p a positive integer). Then:

(a) f has a relative minimum at x_1 if $f^{(2p)}(x_1) > 0$.

(b) f has a relative maximum at x_1 if $f^{(2p)}(x_1) < 0$.

(Notice that the order of differentiation in each succeeding case is two greater. The nature of the intermediate possibilities is suggested in the next paragraph.)

It is possible that the slope of the tangent line to the graph of f is positive to the left of P_1, zero at the point, and again positive to the right. Then P_1 is called a *point of inflection*. In the simplest case this point of inflection is characterized by $f'(x_1) = 0$, $f''(x_1) = 0$, and $f'''(x_1) \neq 0$.

Particle Motion

If x represents the distance of a particle from the origin and t signifies time, then $x = f(t)$ designates the position of a particle at time t. Instantaneous velocity (or speed in the one-dimensional case) is represented by $\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t)}{\Delta t}$ (the limiting case of the formula change in $\frac{\text{change in distance}}{\text{change in time}}$ in time for speed when the motion is constant). Furthermore, the instantaneous change in velocity is called *acceleration* and represented by $\frac{d^2x}{dt^2}$.

Newton's Method

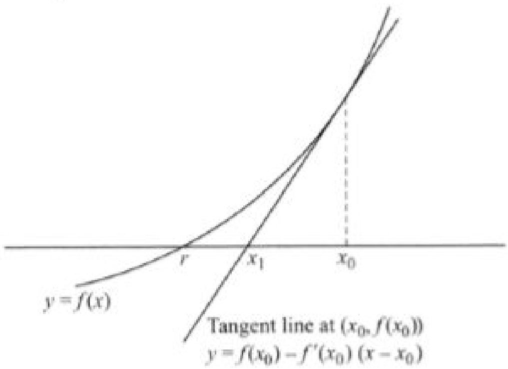

Figure 4.7

Suppose that f has as many derivatives as required. Let r be a real root of $f(x) = 0$; i.e., $f(r) = 0$. Let x_0 be a value of x near r—for example, the integer preceding or following r. Let $f'(x_0)$ be the slope of the graph of $y = f(x)$ at $P_0[x_0, f(x_0)]$. Let $Q_1(x_1, 0)$ be the x-axis intercept of the tangent line at P_0; then

$$\frac{0 - f(x_0)}{x - x_0} = f'(x_0)$$

where the two representations of the slope of the tangent line have been equated. The solution of this relation for x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with the tangent line to the graph at $P_1[x_1, f(x_1)]$ and repeating the process, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_1)}{f'(x_1)}$$

and, in general,

$$x_n = x_0 - \sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)}$$

Under appropriate circumstances, the approximation x_n to the root r can be made as good as desired.

Note: Success with Newton's method depends on the shape of the function's graph in the neighborhood of the root. There are various cases which have not been explored here.