Computation of e^{At}.

Definition: For square matrix $\underline{A}$, $e^{At} = \underline{I} + \frac{1}{1!}\underline{A}t + \frac{1}{2!}\underline{A}^2t^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\underline{A}^n t^n$

Theorem: $e^{At} = \alpha_{n-1}\underline{A}^{n-1}t^{n-1} + \alpha_{n-2}\underline{A}^{n-2}t^{n-2} + \ldots + \alpha_1\underline{A}t + \alpha_0\underline{I}$.

Example: If $\underline{A}$ has 2 rows and 2 columns. $e^{\underline{A}t} = \alpha_1\underline{A}t + \alpha_0\underline{I}$.

If $\underline{A}$ has 3 rows and 3 columns $e^{\underline{A}t} = \alpha_2\underline{A}^2t^2 + \alpha_1\underline{A}t + \alpha_0\underline{I}$.

Theorem: $r(\lambda) \equiv \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \ldots + \alpha_2\lambda^2 + \ldots + \alpha_1\lambda + \alpha_0$.

Then λ_i is an eigenvalue of $\underline{A}t$, $e^{\lambda_i} = r(\lambda_i)$

$$e^{\lambda_i} = \frac{d}{d\lambda} r(\lambda)\Big|_{\lambda=\lambda_i}, \quad e^{\lambda_i} = \frac{d^{k-1}}{dk^{k-1}} r(\lambda)\Big|_{\lambda=\lambda_i}.$$

Example: Let $\underline{A}$ be 4x4 and let $\lambda = 5t$ and $\lambda = 2t$ be eigenvalues of $\underline{A}t$ of multiplicities 3 and 1. Then $= 4$ and:

$$r(\lambda) = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$
$$r'(\lambda) = 3\alpha_3\lambda^2 + 2\alpha_2\lambda + \alpha_1$$
$$r''(\lambda) = 6\alpha_3\lambda + 2\alpha_2.$$

Since $\lambda = 5t$ is an eigenvalue of multiplicity 3, it follows:

$$e^{5t} = r(5t), \quad e^{5t} = r'(5t) \text{ and } e^{5t} = r''(5t)$$

Thus,

$$e^{5t} = \alpha_3(5t)^3 + \alpha_2(5t)^2 + \alpha_1(5t) + \alpha_0$$
$$e^{5t} = 3\alpha_3(5t)^2 + 2\alpha_2(5t)^2 + \alpha_1$$
$$e^{5t} = 6\alpha_3(5t) + 2\alpha_2$$

Also, since $\lambda = 2t$ is an eigenvalue of multiplicity 1, it follows that $e^{2t} = r(2t)$.

$$e^{2t} = \alpha_3(2t)^3 + \alpha_2(2t)^2 + \alpha_1(2t) + \alpha_0.$$

Example : Find $e^{\underline{A}t}$ for $\underline{A} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}$

Here $n=2$,

$$e^{\underline{A}t} = \alpha_1 \underline{A}t + \alpha_0 \underline{I} = \begin{bmatrix} \alpha_1 t + \alpha_0 & \alpha_1 t \\ 9\alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

$$r(\alpha) = \alpha_1 \lambda + \alpha_0$$

Eigenvalues of $\underline{A}t$ are $\lambda_1 = 4t$, $\lambda_2 = -2t$, which are both of multiplicity 1.

Subst into $e^{\lambda_i} = r(\lambda_i)$:

$$\left.\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0. \end{aligned}\right\} \quad \alpha_1 = \frac{1}{6t}\left(e^{4t} - e^{-2t}\right), \quad \alpha_2 = \frac{1}{3}\left(e^{4t} + 2e^{-2t}\right).$$