CHAPTER 15 - GAMMA AND BETA FUNCTIONS

The Gamma Function

The gamma function may be regarded as a generalization of $n!$ (n-factorial), where n is any positive integer to $x!$, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.) Such an extension does not seem reasonable, yet, in certain ways, the gamma function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \tag{1}$$

meets the challenge. This integral has proved valuable in applications. However, because it cannot be represented through elementary functions, establishment of its properties takes some effort. Some of the important properties are outlined as follows.

The gamma function is convergent for $x > 0$. (See Problem 12.18.)

The fundamental property

$$\Gamma(x+1) = x\Gamma(x) \tag{2}$$

may be obtained by employing the technique of integration by parts to Equation (1). The process is carried out in Problem 15.1. From the form of Equation (2), the function $\Gamma(x)$ can be evaluated for all $x > 0$ when its values in the interval $1 \leqq x < 2$ are known. (Any other interval of unit length will suffice.) Table 15.1 and the graph in Figure 15.1 illustrate this idea.

Tables of Values and Graph of the Gamma Function

TABLE 15.1

N	$\Gamma(N)$
1.00	1.0000
1.10	0.9514
1.20	0.9182
1.30	0.8975
1.40	0.8873
1.50	0.8862
1.60	0.8935
1.70	0.9086
1.80	0.9314
1.90	0.9618
2.00	1.0000

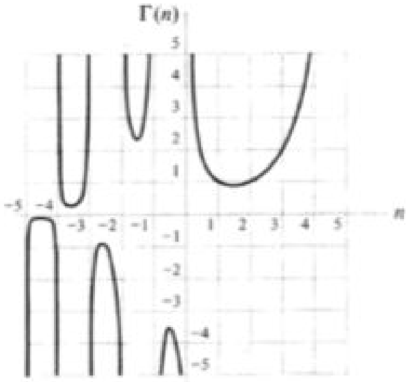

Figure 15.1

Equation (2) is a recurrence relationship that leads to the factorial concept. First observe that if $x = 1$, then Equation (1) can be evaluated and in particular,

$$\Gamma(1) = 1$$

From Equation (2)

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \ldots x(x-1)(x-2)\ldots(x-k)\Gamma(x-k)$$

If $x = n$, where n is a positive integer, then

$$\Gamma(n+1) = n(n-1)(n-2)\ldots 1 = n! \tag{3}$$

If x is a real number, then $x! = \Gamma(x+1)$ is defined by $\Gamma(x+1)$. The value of this identification is in intuitive guidance.

If the recurrence relation (2) is characterized as a differential equation, then the definition of $\Gamma(x)$ can be extended to negative real numbers by a process called *analytic continuation*. The key idea is that even though $\Gamma(x)$ as defined in Equation (1) is not convergent for $x < 0$, the relation $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ allows the meaning to be extended to the interval $-1 < x < 0$, and from there to $-2 < x < -1$, and so on. A general development of this concept is beyond the scope of this presentation; however, some information is presented in Problem 15.7.

The factorial notion guides us to information about $\Gamma(x + 1)$ in more than one way. In the eighteenth century, James Stirling introduced the formula (for positive integer values n)

$$\lim_{n\to\infty} \frac{\sqrt{2\pi}\, n^{n+1}e^{-n}}{n!} = 1 \tag{4}$$

This is called Stirling's formula, and it indicates that $n!$ *asymptotically* approaches $\sqrt{2\pi}\, n^{n+1}e^{-n}$ for large values of n. This information has proved useful, since $n!$ is difficult to calculate for large values of n.

There is another consequence of Stirling's formula. It suggests the possibility that for sufficiently large values of x,

$$x! = \Gamma(x+1) \approx \sqrt{2\pi}\, x^{x+1}e^{-x} \tag{5a}$$

(An argument supporting this is made in Problem 15.20.)

It is known that $\Gamma(x + 1)$ satisfies the inequality

$$\sqrt{2\pi}\, x^{x+1}e^{-x} < \Gamma(x+1) < \sqrt{2\pi}\, x^{x+1}e^{-x}\frac{1}{e^{12(x+1)}} \tag{5b}$$

Since the factor $\frac{1}{e^{12(x+1)}} \to 0$ for large values of x, the suggested value (5a) of $\Gamma(x + 1)$ is consistent with (5b).

An exact representation of $\Gamma(x + 1)$ is suggested by the following manipulation of $n!$. [It depends on $(n + k)! = (k + n)!$.]

$$n! = \lim_{k\to\infty}\frac{12\ldots n(n+1)+(n+2)\ldots(n+k)}{(n+1)(n+2)\ldots(n+k)} = \lim_{k\to\infty}\frac{k!k^n}{(n+1)\ldots(n+k)}\lim_{k\to\infty}\frac{(k+1)(k+2)\ldots(k+n)}{k^n}$$

Since n is fixed, the second limit is one; therefore, $n! = \lim_{k\to\infty}\frac{k!k^n}{(n+1)\ldots(n+k)}$. (This must be read as an infinite product.)

This factorial representation for positive integers suggests the possibility that

$$\Gamma(x+1) = x! = \lim_{k\to\infty}\frac{k!k^x}{(x+1)\ldots(x+k)} \qquad x \neq -1, -2, -k \tag{6}$$

Carl Friedrich Gauss verified this identification back in the nineteenth century.

This infinite product is symbolized by $\Pi(x,k)$; i.e., $\Pi(x,k) = \frac{k!k^x}{(x+1)\cdots(x+k)}$. It is called Gauss's function, and through this symbolism,

$$\Gamma(x+1) = \lim_{k\to\infty}\Pi(x,k) \tag{7}$$

The expression for $\frac{1}{\Gamma(x)}$ [which has some advantage in developing the derivative of $\Gamma(x)$] results as follows. Put Equation (6) in the form

$$\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)\ldots(1+x/k)} \qquad x \neq -\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}$$

Next, introduce

$$\gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{k} - \ln k$$

Then

$$\gamma = \lim_{k\to\infty}\gamma_k$$

is Euler's constant. This constant has been calculated to many places, a few of which are $\gamma \approx 0.57721566\ldots.$

By letting $k^x = e^{x \ln k} = e^{x[-\gamma_k+1+1/2+\cdots+1/k]}$, the representation (6) can be further modified so that

$$
\begin{aligned}
\Gamma(x+1) &= e^{-\gamma x} \lim_{k\to\infty} \frac{e^x}{1+x}\frac{e^{x/2}}{1+x/2}\cdots\frac{e^{x/k}}{1+x/k} = e^{-\gamma x}\prod_{k=1}^{\infty} e^{\gamma x}e^{x\ln k}/\left(1+\frac{x}{k}\right) \\
&= \prod_{k=1}^{\infty} k^x k!(k+x) = \lim_{k\to\infty}\frac{1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}x^x = \lim_{k\to\infty}\Pi(x,k)
\end{aligned}
\tag{8}
$$

Since $\Gamma(x+1) = x\Gamma(x)$,

$$
\frac{1}{\Gamma(x)} = xe^{\gamma x}\lim_{k\to\infty}\frac{1+x}{e^x}\frac{1+x/2}{e^{x/2}}\cdots\frac{1+x/k}{e^{x/k}} = xe^{\gamma x}\prod_{k=1}^{\infty}(1+x/k)e^{-x/k}
\tag{9}
$$

Another result of special interest emanates from a comparison of $\Gamma(x)\Gamma(1-x)$ with the well-known formula

$$
\frac{\pi x}{\sin \pi x} = \lim_{k\to\infty}\left\{\frac{1}{1-x^2}\frac{1}{1-(x/2)^2}\cdots\frac{1}{(1-x/k^2}\right\} = \prod_{k=1}^{\infty}\{1-(x/k^2\}
\tag{10}
$$

[See *Differential and Integral Calculus*, by R. Courant (translated by E. J. McShane), Blackie & Son Limited.]

$\Gamma(1-x)$ is obtained from $\Gamma(y) = \frac{1}{y}\Gamma(y+1)$ by letting $y = -x$; i.e.,

$$
\Gamma(-x) = -\frac{1}{x}\Gamma(1-x) \quad \text{or} \quad \Gamma(1-x) = -x\Gamma(-x)
$$

Now use Equation (8) to produce

$$
\Gamma(x)\Gamma(1-x) = \left(\left\{x^{-1}e^{-\gamma x}\lim_{k\to 1}\prod_{K=\infty}^{\infty}(1+/k)^{-1}e^{x/k}\right\}\right)\left(e^{\gamma x}\lim_{K=1}(1-x/k)^{-1}e^{-x/k}\right) = \frac{1}{x}\lim_{k\to\infty}\prod_{K=1}^{\infty}(1(x/k)^2)
$$

Thus,

$$
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi}, \quad 0 < x < 1
\tag{11a}
$$

Observe that Equation (11a) yields the result

$$
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
\tag{11b}
$$

Another exact representation of $\Gamma(x+1)$ is

$$
\Gamma(x+1) = \sqrt{2\pi}\,x^{x+1}e^{-x}\left\{1+\frac{1}{12x}+\frac{1}{288x^2}+\frac{139}{51840x^3}+\cdots\right\}
\tag{12}
$$

The method of obtaining this result is closely related to Stirling's asymptotic series for the gamma function. (See Problems 15.20 and 15.74.)

The *duplication* formula

$$
2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x)
\tag{13a}
$$

is also part of the literature. Its proof is given in Problem 15.24.

The duplication formula is a special case ($m = 2$) of the following product formula:

$$
\Gamma(x)\Gamma\left(x+\frac{1}{m}\right)\cdots\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(X+\frac{m-1}{m}\right) = m^{\frac{1}{2}-mx}(2\pi)^{\frac{m-1}{2}}\Gamma(mx)
\tag{13b}
$$

It can be shown that the gamma function has continuous derivatives of all orders. They are obtained by differentiating (with respect to the parameter) under the integral sign.

It helps to recall that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-y t}dt$ and that if $y = t^{x-1}$, then $\ln y = \ln t^{x-1} = (x-1)\ln t$.

Therefore; $\frac{1}{y}y' = \ln t.$

It follows that

$$\Gamma'(x) = \int_0^\infty t^{x-1}e^{-t}\ln t\,dt. \tag{14a}$$

This result can be obtained (after making assumptions about the interchange of differentiation with limits) by taking the logarithm of both sides of Equation (9) and then differentiating.

In particular,

$$\Gamma'(1) = -\gamma \ (\gamma \text{ is Euler's constant.}) \tag{14b}$$

It also may be shown that

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \left(\frac{1}{1}-\frac{1}{x}\right)+\left(\frac{1}{2}-\frac{1}{x+1}\right)+\cdots\left(\frac{1}{n}-\frac{1}{x+n-1}\right) \tag{15}$$

(See Problem 15.73 for further information.)

The Beta Function

The beta function is a two-parameter composition of gamma functions that has been useful enough in application to gain its own name. Its definition is

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \tag{16}$$

If $x \ge 1$ and $y \ge 1$, this is a proper integral. If $x > 0$, $y > 0$, and either or both $x < 1$ or $y < 1$, the integral is improper but convergent.

It is shown in Problem 15.11 that the beta function can be expressed through gamma functions in the following way

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{17}$$

Many integrals can be expressed through beta and gamma functions. Two of special interest are

$$\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta = \frac{1}{2}B(x,y) = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{18}$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \Gamma(p)\,\Gamma(p-1) = \frac{\pi}{\sin\pi P} \quad 0 < p < 1 \tag{19}$$

See Problem 15.17. Also see Page 391, where a classical reference is given. Finally, see Problem 16.38, where an elegant complex variable resolution of the integral is presented.

Dirichlet Integrals

If V denotes the closed region in the first octant bounded by the surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{a}\right)^q + \left(\frac{z}{c}\right)^r = 1$ and the coordinate planes, then if all the constants are positive,

$$\iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}dx\,dy\,dz = \frac{a^\alpha b^\beta c^\gamma}{pqr}\frac{\Gamma\left(\frac{\alpha}{p}\right)\Gamma\left(\frac{\beta}{b}\right)\Gamma\left(\frac{\gamma}{r}\right)}{\Gamma\left(1+\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\right)} \tag{20}$$

Integrals of this type are called *Dirichlet integrals* and are often useful in evaluating multiple integrals (see Problem 15.21).