Integrating Inverse Functions.

Definite integral of inverse function: Let f be strictly monotonic with continuous derivative on interval $[a,b]$:

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a).$$

Proof:

$$\int_a^b f(x)dx = xf(x)\Big|_a^b - \int_a^b xf'(x)dx$$

$$= bf(b) - af(a) - \int_a^b xf'(x)dx$$

Set $y = f(x)$, $dy = f'(x)\,dx$ and $x = f^{-1}(y)$.

When $x = b$, $y = f(b)$ and when $x = a$, $y = f(a)$

Thus:

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy.$$

or

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$$

Example:

Inverse sine function on $[0,1]$:

$f^{-1}(x) = \sin^{-1}x$ so that $f(x) = \sin x$ and $a = 0$, $b = \pi/2$

$f(0) = 0$ and $f(b) = f(\pi/2) = 1$.

$$\int_0^1 \sin^{-1}x\,dx = \pi/2 - \int_0^{\pi/2} \sin x\,dx = \pi/2 - 1$$

Example:

$$\int_0^2 \left(\sqrt{1+x^3} + \sqrt[3]{x+2x}\right)dx$$

Let $f(x) = \sqrt{1+x^3}$. Since $f'(x) = \dfrac{x^2}{3\sqrt{1+x^3}} > 0$ for $x \in [0,2]$.

$f^{-1}(x) = \sqrt[3]{x^2-1}$

Set $a = 0$ so that $f(a) = f(0) = 1$ and $b = 2$ so that $f(b) = f(2) = 3$.

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$$

$$\int_0^2 \sqrt{1+x^3}\,dx + \int_1^3 \sqrt[3]{x^2-1}\,dx = 2\cdot 3 - 0\cdot 1 = 6$$

In second integral, setting $x = u+1$ we have $x = u+1$, $dx = du$

$x = 1, u = 0$ and $x = 3, u = 2$:

$$\int_0^2 \sqrt{1+x^3}\,dx + \int_0^2 \sqrt[3]{(u+1)^2 - 1}\,du = 6$$

or $$\int_0^2 \left(\sqrt{1+x^3} + \sqrt[3]{x^2+2x} \right) dx = 6$$

Indefinite Integral of Inverse function :

$$\int f(x)\,dx = xf(x) - \int^{f(x)} f^{-1}(u)\,du.$$

Proof : Integrating by parts :

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

as $f^{-1}(f(x)) = x$ the integral can be written as :

$$\int f(x)\,dx = xf(x) - \int f^{-1}(f(x))\,f'(x)\,dx$$

Setting $u = f(x)$, $du = f'(x)\,dx$ gives :

$$\int f(x)\,dx = xf(x) - \int f^{-1}(u)\,du.$$

Example : Find $\int (\cos^{-1}x)^2\,dx$

Let $f(x) = (\cos^{-1}x)^2$

As f is strictly decreasing function on its domain, its inverse will exist.

$$f^{-1}(x) = \cos\sqrt{x}$$

$$\int f(x)\,dx = xf(x) - \int f^{-1}(u)\,du \quad \text{where } u = f(x).$$

$$\int (\cos^{-1}x)^2\,dx = x(\cos^{-1}x)^2 - \int \cos\sqrt{u}\,du.$$

$$\int \cos\sqrt{u}\,du = 2\int t\cos t\,dt = 2t\sin t - 2\int \sin t\,dt, \quad u = t^2$$

$$= 2t\sin t + 2\cos t + c$$
$$= 2\sqrt{u}\sin\sqrt{u} + 2\cos\sqrt{u} + c$$

$$\therefore \int (\cos^{-1}x)^2\,dx = x(\cos^{-1}x)^2 - 2\sqrt{1-x^2}\,\cos^{-1}x - 2x + c.$$