Calculus – Chapter 7 – Limits.

Definition: A limit of $f(x)$ as x approaches a:

$$\lim_{x \to a} f(x) = A \qquad \text{e.g. } \lim_{x\to 3} x^2 = 9.$$

Formal: $\lim_{x\to a} f(x) = A$ iff given positive number ϵ, however small, there exists a positive number δ such that whenever $0 < |x-a| < \delta$, then $|f(x) - A| < \epsilon$

Example: $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$ although $\frac{x^2-4}{x-2}$ is not defined when $x=2$.

Since $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$, we see $\frac{x^2-4}{x-2}$ approaches 4 as x approaches 2.

Example: $\lim_{x\to 2}(4x-5) = 3$.

Let $\epsilon > 0$ be chosen. Produce some $\delta > 0$ s.t. whenever $0 < |x-2| < \delta$ then $|(4x-5) - 3| < \epsilon$

First note $|(4x-5)-3| = |4x-8| = 4|x-2|$.

If we take δ to be $\epsilon/4$ then whenever $0 < |x-2| < \delta$, $|(4x-5)-3| < 4|x-2| < 4\delta$

Right/Left Limits: By $\lim_{x\to a^-} f(x) = A$, means f is defined in some open interval (c,a) and $f(x)$ approaches A as x approaches a through values less than a.

Similarly $\lim_{x\to a^+} f(x) = A$.

$g(x) = \sqrt{1/x}$ is defined only for $x>0$ i.e $\lim_{x\to 0^-}\sqrt{1/x}$ doesn't exist.

Theorems:

1. If $f(x) = c$ then $\lim_{x\to a} f(x) = c$.
2. $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, $\lim_{x\to a} c\cdot f(x) = c\lim_{x\to a} f(x) = cA$.
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$.
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A.B$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$, if $B \neq 0$

6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$ if $\sqrt[n]{A}$ is defined.

Infinity:

$\lim_{x \to a} f(x) = +\infty$ iff for any $M > 0$ $\exists \delta > 0$ s.t. whenever $0 < |x-a| < \delta$ then $f(x) > M$

Similarly for $\lim_{x \to a} f(x) = -\infty$

Examples: (a). $\lim_{x \to 0} 1/x^2 = +\infty$ (b). $\lim_{x \to 1} \frac{-1}{(x-1)^2} = -\infty$ (c). $\lim_{x \to 0} 1/x = \infty$

note $\lim_{x \to 0^+} 1/x = +\infty$ and $\lim_{x \to 0^-} = -\infty$

Note that $x \to +\infty$, given any $\epsilon > 0$ $\exists N$ s.t. $x > N$, then $|f(x) - A| < \epsilon$