Calculus - Chapter 52 - Relative Max/Min.

Definition:

Assume $z = f(x,y)$ has relative max (or min) at value $P_0(x_0, y_0, z_0)$.

Any plane through P_0 perpendicular to the xy plane cuts the surface in a curve having a relative max/min at P_0.

Directional derivative

$$\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \quad \text{of } z = f(x,y) \text{ must equal } 0 \text{ at } P_0.$$

In particular, when $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$,

$$\frac{\partial f}{\partial x} = 0$$

When $\theta = \pi/2$, $\sin\theta = 1$ and $\cos\theta = 0$ so that $\frac{\partial f}{\partial y} = 0$.

Theorem:

$z = f(x,y)$ has relative extremum at $P_0(x_0, y_0, z_0)$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist at (x_0, y_0)

then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at (x_0, y_0).

Theorem:

$z = f(x,y)$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, define $\Delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$

Assume $\Delta < 0$ at (x_0, y_0) then:

$$z = f(x,y) \text{ has } \begin{cases} \text{a relative minimum at } (x_0,y_0) \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} > 0 \\ \text{a relative maximum at } (x_0,y_0) \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0. \end{cases}$$

If $\Delta > 0$, there is neither a relative maximum or minimum at (x_0, y_0)

If $\Delta = 0$, no information.