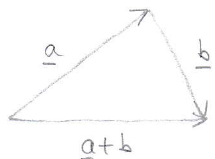
Vector Calculus (Springer)

Vector : Physical quantity with both magnitude and direction ($\underline{u}$)

Scalar : Physical quantity with magnitude only.

Vector Addition :



Vector Components : Suppose vector $\underline{a}$ is drawn from (x_1, y_1, z_1) to (x_2, y_2, z_2), components are :

$$a_1 = x_2 - x_1$$
$$a_2 = y_2 - y_1$$
$$a_3 = z_2 - z_1$$

Can be written in form $\underline{a} = (a_1, a_2, a_3)$

Unit Vectors : Introduce unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ which lie on x, y, z axes resp.

$$a = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3.$$

Hence
$$\underline{a}+\underline{b} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3 + b_1\underline{e}_1 + b_2\underline{e}_2 + b_3\underline{e}_3$$
$$= (a_1+b_1)\underline{e}_1 + (a_2+b_2)\underline{e}_2 + (a_3+b_3)\underline{e}_3.$$

Equivalent to letting $c_1 = a_1 + b_1$ etc.. and $\underline{c} = \underline{a} + \underline{b}$.

Magnitude : $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

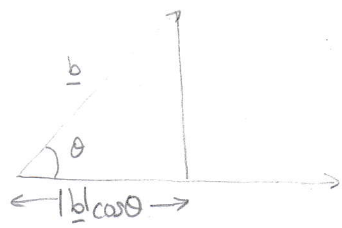
Position Vector : $\underline{r} = (x, y, z)$.

Dot product : Scalar quantity $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$.

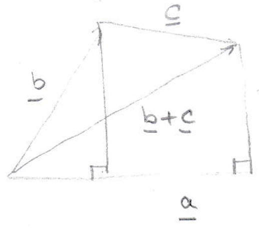
Commutative : $\underline{a}.\underline{b} = \underline{b}.\underline{a}$.

Orthogonal : $\underline{a}$ and $\underline{b}$ are perpendicular (orthogonal) then $\underline{a}\cdot\underline{b} = 0$.

Note : $|\underline{b}|\cos\theta$ is component of $\underline{b}$ in the direction of $\underline{a}$.



Distributive : $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a} + \underline{c}$.



Dotproduct Derivation :

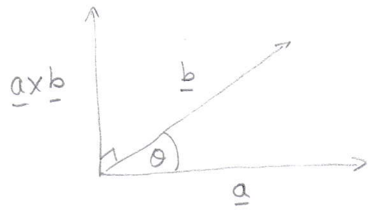
$$e_1 . e_1 = 1, \quad e_2 . e_2 = 1, \quad e_3 . e_3 = 1$$
$$e_1 . e_2 = 0, \quad e_2 . e_3 = 0, \quad e_3 . e_1 = 0.$$

$$a \cdot b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \cdot (b_1 e_1 + b_2 e_2 + b_3 e_3)$$
$$= a_1 b_1 e_1 \cdot e_1 + a_2 b_2 e_2 . e_2 + a_3 b_3 e_3 . e_3$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example : $a = (1, 1, 2)$ and $b = (2, 3, 2)$

$$a . b = 1\times 2 + 1\times 3 + 2\times 2 = 9.$$

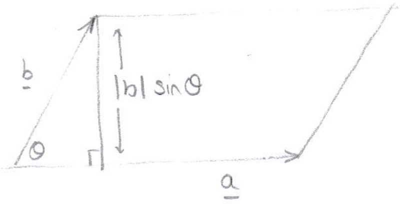
Cross Product : A vector quantity $a \times b = |a||b| \sin\theta\, u$



Not commutative : Due to right-hand rule, $a \times b \neq - b \times a$

Parallel : $a \times b = 0$ if a and b are parallel.

Note : $a \times a = 0$.



Distributive : $a \times (b \times c) = a \times b + a \times c$.

Note :

$$e_1 \times e_1 = 0 \qquad e_1 \times e_2 = e_3$$
$$e_2 \times e_2 = 0 \qquad e_2 \times e_3 = e_1$$
$$e_3 \times e_3 = 0 \qquad e_3 \times e_1 = e_2$$

Cross Product :

$$a \times b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \times (b_1 e_1 + b_2 e_2 + b_3 e_3)$$
$$= (a_2 b_3 - a_3 b_2) e_1 + (a_3 b_1 - a_1 b_3) e_2 + (a_1 b_2 - a_2 b_1) e_3.$$

$$\therefore \quad a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example : Cross product of $(1, 3, 0)$ and $(2, -1, 1)$ is

$$(1, 3, 0) \times (2, -1, 1) = (3 - 0, 0 - 1, -1 - 6) = (3, -1, -7).$$

Example : A unit vector perpendicular to $(1, 0, 1)$ and $(0, 1, 1)$.

A perpendicular vector is $(1, 0, 1) \times (0, 1, 1) = (-1, -1, 1)$

A perpendicular unit vector is $(-1, -1, 1)$ divided by magnitude $= (-1, -1, 1)/\sqrt{3}$.

Scalar Triple Product: $\underline{a} \cdot (\underline{b} \times \underline{c}) = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_3 c_1 - a_3 b_1 c_2 - a_3 b_2 c_1$

Interchargable: $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \cdot \underline{c}$,

$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

Determinant Form: $\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (often written $[a, b, c]$).

Example: Scalar triple product of $(1,2,1)$, $(0,1,1)$ and $(2,1,0)$:

First, $(0,1,1) \times (2,1,0) = (-1,2,-2)$

Second, $(1,2,1) \cdot (-1,2,-2) = 1$.

Note: If three vectors lie in a plane, scalar triple product is zero.

If $\underline{a}, \underline{b}, \underline{c}$ lie in a plane, $\underline{b} \times \underline{c}$ is perpendicular to plane hence perpendicular to $\underline{a}$.

The dot product of perpendicular vectors is also zero hence $\underline{a} \cdot \underline{b} \times \underline{c} = 0$.

Since $\underline{b} \times \underline{c} = (b_2 c_3 - b_3 c_2)\underline{e}_1 + (b_3 c_1 - b_1 c_3)\underline{e}_2 + (b_1 c_2 - b_2 c_1)\underline{e}_3$.

$$
\begin{aligned}
[\underline{a} \times (\underline{b} \times \underline{c})]_1 &= a_2(b_1 c_2 - b_2 c_1) - a_3(b_3 c_1 - b_1 c_3) \\
&= b_1(a_2 c_2 + a_3 c_3) - c_1(a_2 b_2 + a_3 b_3) \\
&= b_1(a_1 c_1 + a_2 c_2 + a_3 c_3) - c_1(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= b_1\, \underline{a} \cdot \underline{c} - c_1\, \underline{a} \cdot \underline{b}.
\end{aligned}
$$

Hence $\underline{a} \times (\underline{b} \times \underline{c}) = (a \cdot c)\underline{b} - (a \cdot b)\underline{c}$

Note: $(\underline{a} \times \underline{b}) \times \underline{c} = -\underline{c} \times (\underline{a} \times \underline{b}) = -(\underline{c} \cdot \underline{b})\underline{a} + (\underline{c} \cdot \underline{a})\underline{b}$

Example:

$$
\begin{aligned}
(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) &= \underline{a} \cdot (\underline{b} \times (\underline{c} \times \underline{d})) \\
&= \underline{a} \cdot ((\underline{b} \cdot \underline{d})\underline{c} - (\underline{b} \cdot \underline{c})\underline{d}) \\
&= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})
\end{aligned}
$$

Line Integrals : $\int_C \underline{F}.d\underline{r} = \int \underline{F}.\frac{d\underline{r}}{dt}\,dt$

Example : $x = t,\ y = t,\ z = 2t^2$ and $0 \leq t \leq 1$

Let $\underline{F} = (t, t, 2t^2)$

$$\frac{d\underline{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (1, 1, 4t).$$

$$\int_C \underline{F}.d\underline{r} = \int_0^1 (t, t, 2t)\cdot(1, 1, 4t)\,dt = \int_0^1 2t + 8t^2\,dt = 3.$$

Example : $x = \cos\theta,\ y = \sin\theta,\ z = 0$ where $0 \leq \theta \leq 2\pi$. $\oint$ = closed curve.

$$\oint \underline{F}.d\underline{r} = \int_0^{2\pi} (\sin\theta, \cos\theta, 0)\cdot(-\sin\theta, \cos\theta, 0)\,d\theta$$

$$= \int_0^{2\pi} -\sin^2\theta + \cos^2\theta$$

$$= \int_0^{2\pi} \cos 2\theta$$

$$= \left[\sin 2\theta / 2\right]_0^{2\pi}$$

$$= 0.$$

Conservative Field : $\oint_C \underline{F}.d\underline{r} = 0$

Line integral depends only on the endpoints $\int_{C_1} \underline{F}.d\underline{r} = \int_{C_2} \underline{F}.d\underline{r}$

C_1, C_2 are any two curves that have the same endpoints but different paths.

Let C be closed curve that starts from point A and follows curve C_1 to point B and then follows curve C_2 in the reverse direction to return to A, then :

$$\oint_C \underline{F}.d\underline{r} = \int_{C_1} \underline{F}.d\underline{r} - \int_{C_2} \underline{F}.d\underline{r}$$

Other line integral forms:

$\int_C \phi \, d\underline{r}$ and $\int_C \underline{F} \times d\underline{r}$ where ϕ is a scalar field and $\underline{F}$ is a vector field.

Example:

Evaluate $\int_C x+y^2 \, d\underline{r}$ where C is the parabola $y=x^2$ in the plane $z=0$, connecting points $(0,0,0)$ and $(1,1,0)$

Let $x=t$, $y=t^2$, $z=0$ where $0 \leq t \leq 1$

So $d\underline{r} = (1, 2t, 0)\, dt$

$$\int_C x+y^2 \, d\underline{r} = \int_0^1 (t+t^4)(1,2t,0)\, dt$$

$$= \underline{e}_1 \left(\int_0^1 t+t^4 \, dt \right) + \underline{e}_2 \left(\int_0^1 2t^2 + 2t^5 \, dt \right)$$

$$= 0.7\underline{e}_1 + \underline{e}_2 .$$

Example:

$\int_C \underline{F} \times d\underline{r}$ where $\underline{F}$ is vector field $(y,x,0)$ and C is the curve $y = \sin x$, $z=0$, between $x=0$ and $x=\pi$

Write as $x=t$, $y=\sin t$, $z=0$, $0 \leq t \leq \pi$

Then $\underline{F} = (\sin t, t, 0)$ and $d\underline{r} = (1, \cos t, 0) dt$

So $\underline{F} \times d\underline{r} = (0, 0, \sin t \cos t - t) dt$ and integral is

$$\int_C \underline{F} \times d\underline{r} = \underline{e}_3 \int_0^{\pi} \sin t \cos t - t \, dt$$

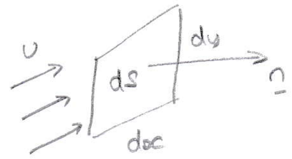
$$= \tfrac{1}{2}[\sin^2 t - t^2]_0^{\pi} \, \underline{e}_3$$

$$= -\pi^2/2\underline{e}_3$$

Surface Integrals :

The total flux across the surface S is :

$$Q = \iint_S \underline{u}.\underline{n}\, dS = \iint_S \underline{u}.\underline{n}\, dx dy$$



Surface Integral closed surface :

$$\oint\!\!\oint_S \underline{u}.\underline{n}\, dS$$

~~Example~~: Evaluation of Surface integrals :

Let S be square surface $0 \leq x \leq 1,\ 0 \leq y \leq 1$:

$$Q = \iint_S U_0(x,y)\, dS = \int_0^1\!\!\int_0^1 U_0(x,y)\, dx dy.$$

$$= \int_0^1 \left(\int_0^1 U_0(x,y)\, dx \right) dy.$$

Example :

Let $U_0(x,y) = (x - x^2)(y - y^2)$

$$Q = \int_0^1 \int_0^1 (x-x^2)(y-y^2)\, dx dy$$

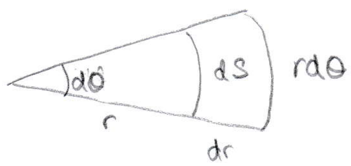
$$= \int_0^1 \left[x^2/2 - x^3/3 \right]_0^1 (y-y^2) dy.$$

$$= \int_0^1 1/6\, (y-y^2)\, dy.$$

$$= 1/36$$

Circular Surface :

Suppose radius of surface S is 1 and $U_0 = 1 - r^2$



$$Q = \iint_S U_0\, dS = \int_0^1 \int_0^{2\pi} (1-r^2) r\, d\theta\, dr$$

$$= \int_0^1 2\pi (1-r^2) r\, dr$$

$$= 2\pi \left[r^2/2 - r^4/4 \right]_0^1$$

$$= \pi/2.$$

Curved Surface:

Surface written as two parameters v and w, so position vector $\underline{r}$ on surface is

$$\underline{r} = \underline{r}(v, w)$$

For small change in value of v to $v + dv$

Vector $\underline{r}(v+dv, w)$ also lies on the surface.

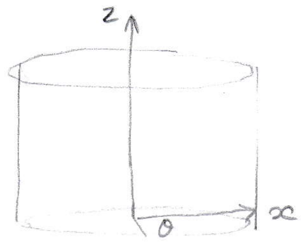
$\underline{r}(v+dv, w) - \underline{r}(v, w) = (\partial \underline{r} / \partial v)$ and similarly for $(\partial \underline{r} / \partial w)$

$$\iint_S \underline{u} \cdot \underline{n} \, dS = \iint_S \underline{u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w} \, dv\,dw.$$

Example:

Consider $\underline{u} = (x, z, -y)$ over surface of cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 1$.

$\underline{r} = (x, y, z) = (\cos\theta, \sin\theta, 0)$, $\frac{\partial \underline{r}}{\partial z} = (0, 0, 1)$



and $\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial z} = (\cos\theta, \sin\theta, 0)$.

Value of surface integral is

$$\iint_S \underline{u} \cdot \underline{n} \, dS = \int_0^1 \int_0^{2\pi} (\cos\theta, z, -\sin\theta) \cdot (\cos\theta, \sin\theta, 0) \, d\theta \, dz.$$

$$= \int_0^1 \int_0^{2\pi} \cos^2\theta + z\sin\theta \, d\theta \, dz$$

$$= \int_0^1 \pi \, dz = \pi$$

Other forms of integrals:

$\iint f \, dS$ where f is a scalar field

$\iint_S f \underline{n} \, dS$ and $\iint_S \underline{v} \times \underline{n} \, dS$ where $\underline{v}$ is a vector field.

Example:

If S is the entire x, y plane, evaluate $I = \iint_S e^{-x^2 - y^2} \, dS$

transforming into polar coordinates.

In polar coordinates (r, θ), $x^2 + y^2 = r^2$ and $dS = r\,d\theta\,dr$

The range of the variables to cover whole plane are $0 \leq r \leq \infty$ and $0 \leq \theta \leq 2\pi$.

$$I = \int_0^\infty \int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = \int_0^\infty 2\pi e^{-r^2} r\,dr = \pi\left[-e^{-r^2}\right]_0^\infty = \pi.$$

Volume Integrals :

Let volume V be divided into N small pieces with volumes δV_i, $i = 1, \ldots, N$ called volume elements.

Suppose object of volume V has density ρ.

If ρ is constant, the mass M of the object is simply $M = \rho V$

Suppose that the object has a density function of position $\rho = \rho(\underline{r})$

Mass M_i of volume element at position $\underline{r}_i$ is $M = \rho(\underline{r}_i)\delta V_i$

Total mass of all volume elements is

$$M = \sum_{i=1}^{N} \rho(r_i)\,\delta V_i$$

The volume integral is, as limit of sum as $N \to \infty$:

$$\iiint_V \rho\,dV = \lim_{N \to \infty} \sum_{i=1}^{N} \rho(\underline{r}_i)\,\delta V_i$$

Example :

Cube $0 \leq x, y, z \leq 1$ has variable density $\rho = 1 + x + y + z$.

The mass of the cube is :

$$M = \iiint_V \rho\,dV = \int_0^1\int_0^1\int_0^1 1 + x + y + z\;dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[x + x^2/2 + xy + xz\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 (3/2 + y + z)\,dy\,dz$$

$$= \int_0^1 \left[3y/2 + y^2/2 + yz\right]_0^1 dz$$

$$= \int_0^1 (2 + z)\,dz$$

$$= \left[2z + z^2/2\right]_0^1$$

$$= 5/2.$$

Curved Surface:

Surface written as two parameters v and w, so position vector $\underline{r}$ on surface is

$\underline{r} = \underline{r}(v, w)$

For small change in value of v to $v + dv$

Vector $\underline{r}(v+dv, w)$ also lies on the surface.

$\underline{r}(v+dv, w) - \underline{r}(v, w) = (\partial \underline{r} / \partial v)$ and similarly for $(\partial \underline{r} / \partial w)$

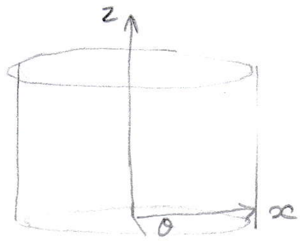
$$\iint_S \underline{u} \cdot \underline{n} \, dS = \iint_S \underline{u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w} \, dv \, dw.$$

Example:

Consider $\underline{u} = (x, z, -y)$ over surface of cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 1$.

$\underline{r} = (x, y, z) = (\cos\theta, \sin\theta, 0)$, $\frac{\partial \underline{r}}{\partial z} = (0, 0, 1)$

and $\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial z} = (\cos\theta, \sin\theta, 0)$.



Value of surface integral is

$$\iint_S \underline{u} \cdot \underline{n} \, dS = \int_0^1 \int_0^{2\pi} (\cos\theta, z, -\sin\theta) \cdot (\cos\theta, \sin\theta, 0) \, d\theta \, dz.$$

$$= \int_0^1 \int_0^{2\pi} \cos^2\theta + z \sin\theta \, d\theta \, dz$$

$$= \int_0^1 \pi \, dz = \pi$$

Other forms of integrals:

$\iint f \, dS$ where f is a scalar field

$\iint_S f \underline{n} \, dS$ and $\iint_S \underline{v} \times \underline{n} \, dS$ where $\underline{v}$ is a vector field.

Example:

If S is the entire x,y plane, evaluate $I = \iint_S e^{-x^2 - y^2} \, dS$

transforming into polar coordinates.

Partial Derivative: $$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y, z) - f(x,y,z)}{\delta x}$$

Second derivies: $\partial^2 f / \partial x^2$ and mixed derivatives $\partial^2 f / \partial y \partial x$.

Note that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Example: $f(x,y,z) = x^2 + xy\sin z - yz$

$\dfrac{\partial f}{\partial x} = 2x + y\sin z$, $\dfrac{\partial f}{\partial y} = x\sin z - z$ and $\dfrac{\partial f}{\partial z} = xy\cos z - y$.

$\dfrac{\partial f}{\partial y}$ wrt x gives: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \sin z$.

Similarly, $\dfrac{\partial f}{\partial x}$ wrt y gives $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \sin z$.

Taylor Series for more than one variable:

$f(x) = f(a) + (x-a)\dfrac{df}{dx}(a) + \dfrac{(x-a)^2}{2!}\dfrac{d^2 f}{dx^2}(a) + \dots$ where $\delta x = (x-a)$ is a small pertubation and $\delta f = f(x) - f(a)$

For functions of two independent variables:

$$\delta f = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \frac{(\delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\partial y)^2}{2!}\frac{\partial^2 f}{\partial y^2} + \delta x \delta y \frac{\partial^2 f}{\partial x \partial y} + \dots$$

Taylor Series three variables: $$\delta f = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z} + \dots$$

Example: $f(x,y,z) = 2x + (1+y)\sin z$ at $x = 0.1, y = 0.2, z = 0.3$.

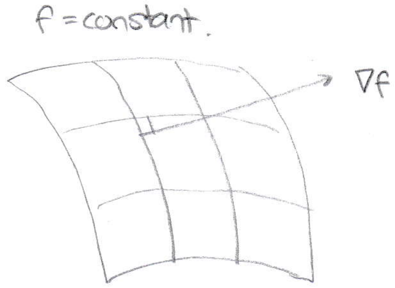
$\dfrac{\partial f}{\partial x} = 2$, $\dfrac{\partial f}{\partial y} = \sin z = 0$ and $\dfrac{\partial f}{\partial z} = (1+y)\cos z = 1$ at $(0,0,0)$ where f takes a value of 0.

Taylor expansion is $\delta f = 2\delta x + \delta z + \dots$

At point $(0.1, 0.2, 0.3)$, $f \approx 0.5$ (0.5546 is exact figure using formula).

Gradient of Scalar Field:

Gradient of scalar field grad$f$ = ∇f.



$$\nabla f = \frac{\partial f}{\partial x}\underline{e}_1 + \frac{\partial f}{\partial y}\underline{e}_2 + \frac{\partial f}{\partial z}\underline{e}_3$$

Consider small change of position vector $\underline{r}$ to $\underline{r}+d\underline{r}$

This results in a small change in the value of the scalar field f from f to $f+df$, then:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(dx, dy, dz)$$

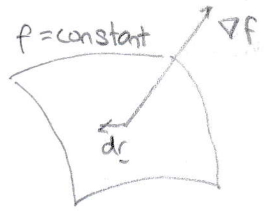
$$= \nabla f \cdot d\underline{r}.$$

Magnitude:

$|\nabla f| = \frac{df}{ds}$, which is the rate of change of f with position along the normal.

To find the rate of f in the direction of the unit vector $\underline{u}$, set $d\underline{r} = \underline{u}\,ds$ where ds is the distance along $\underline{u}$.

Then $df = \nabla f \cdot \underline{u}\, ds$



Directional Derivative:

$$\frac{df}{ds} = \nabla f \cdot \underline{u}$$

This is the rate of change of f in direction of the unit vector $\underline{u}$, and is the directional derivative.

Also written as

$$\frac{df}{ds} = |\nabla f|\cos\theta,$$ where θ is the angle between ∇f and unit vector $\underline{u}$

Del operator:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Example:

Find unit normal $\underline{n}$ to surface $x^2+y^2-z=0$ at point $(1,1,2)$.

$\nabla f = (2x, 2y, -1)$ so at $(1,1,2)$ $\nabla f = (2,2,1)$

To find unit normal, divide by magnitude $(2^2+2^2+1^2)^{1/2} = 3$

so $\underline{n} = \nabla f / |\nabla f| = (2/3, 2/3, -1/3)$

Theorem: Suppose F is a vector field related to scalar field ϕ by $\underline{F} = \nabla\phi$. Then F is conservative if $\nabla\phi$ exists everywhere in domain D.

Proof: Suppose $\underline{F} = \nabla\phi$.

Line integral $\underline{F}$ along a curve C connecting two points A and B is

$$\int_C \underline{F}\cdot d\underline{r} = \int_C \nabla\phi\cdot d\underline{r} = \int_C d\phi = [\phi]_A^B = \phi(B) - \phi(A).$$

Because result depends only on endpoints A and B thus $\underline{F}$ is conservative.

Example: Show that $\underline{F} = (2x+y, x, 2z)$ is conservative.

$$\frac{\partial\phi}{\partial x} = 2x+y, \quad \frac{\partial\phi}{\partial y} = x, \quad \frac{\partial\phi}{\partial z} = 2z$$

F is conservative everywhere if it can be written as gradient of scalar field ϕ.

Integrating first equation:

$$\phi = x^2 + xy + h(y,z)$$

2nd equation forces partial derivative of h wrt y to be zero, so that h only depends on z.

3rd equation yields $dh/dz = 2z$ so $h(z) = z^2 + c$

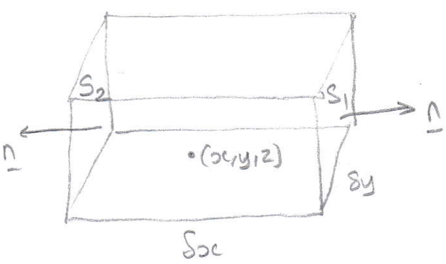
All three equations are satisfied by the potential function $\phi = x^2 + xy + z^2$.

Divergence: Div and curl are two ways of differentiating a vector field.

Divergence of a vector field $\underline{u}$ is a scalar field.

It's value at point P is defined by:

$$\text{div}\,\underline{u} = \lim_{\delta V\to 0} \frac{1}{\delta V} \oiint_{\delta S} \underline{u}\cdot\underline{n}\,dS.$$



$$\iint_{S_1} \underline{u}\cdot\underline{n}\,dS \approx u_1(x+\delta x/2, y, z)\,\delta y\,\delta z$$

$$\iint_{S_2} \underline{u}\cdot\underline{n}\,dS \approx -u_1(x-\delta x/2, y, z)\,\delta y\,\delta z.$$

$$\iint_{S_1+S_2} \underline{u}\cdot\underline{n}\,dS \approx \left(u_1(x+\tfrac{\delta x}{2}, y, z) - u_1(x-\tfrac{\delta x}{2}, y, z)\right)\delta y\,\delta z$$

$$\approx \frac{\partial u_1}{\partial x}\,\delta x\,\delta y\,\delta z.$$

There are six similar contributions made from all six sides to give:

$$\operatorname{div} \underline{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u_1, u_2, u_3) = \nabla \cdot \underline{u}$$

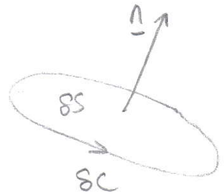
Example: Find the divergence of a vector field $\underline{u} = \underline{r}$.

$\underline{u} = (x, y, z)$ so $\operatorname{div} \underline{u} = 1 + 1 + 1 = 3$.

Laplacian:
$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

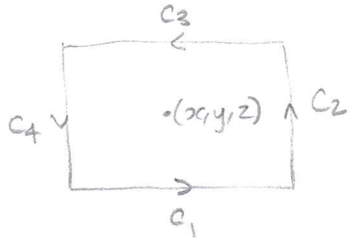
Curl: The curl of a vector field $\underline{u}$ is a vector field.

Its component in the direction of the unit vector $\underline{n}$ is



$$\underline{n} \cdot \operatorname{curl} \underline{u} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{u} \cdot d\underline{r}.$$

Consider C_1 section of line integral which has centre at point $(x, y - \delta y/2, z)$



$$\int_{C_1} \underline{u} \cdot d\underline{r} \approx u_1(x, y - \delta y/2, z)\, \delta x$$

$$\int_{3} \underline{u} \cdot d\underline{r} \approx -u_1(x, y + \delta y/2, z)\, \delta x$$

$$\int_{C_1 + C_3} \underline{u} \cdot d\underline{r} \approx (u_1(x, y - \delta y/2, z) - u_1(x, y + \delta y/2, z))\, \delta x \approx -\frac{\partial u_1}{\partial y} \delta y \delta x$$