Convolutions.

Definition: The convolution of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \int_0^x f(t)\, g(x-t)\, dt.$$

Theorems:

1. $f(x) * g(x) = g(x) * f(x)$

2. If $\mathcal{L}\{f(x)\} = F(s)$ and $\mathcal{L}\{g(x)\} = G(s)$, then

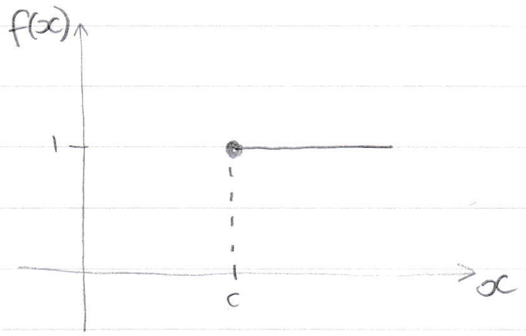
$$\mathcal{L}\{f(x) * g(x)\} = \mathcal{L}\{f(x)\}\,\mathcal{L}\{g(x)\} = F(s)\,G(s)$$

It follows from 1 and 2:

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(x) * g(x) = g(x) * f(x).$$

Unit step Function: Unit step function is

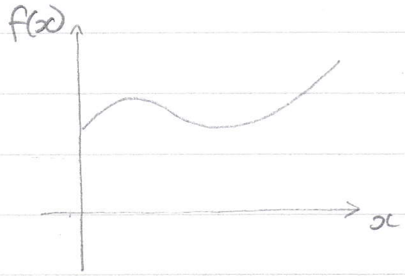
$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

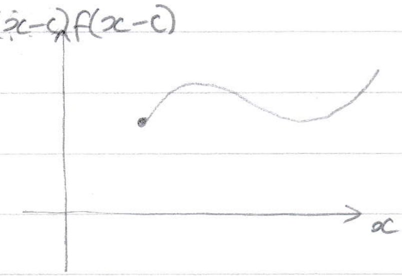


3. $\mathcal{L}\{u(x-c)\} = \frac{1}{s} e^{-cs}$

Translations: Given $f(x)$, $x \geq 0$:

$$u(x-c)f(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}.$$





4. If $F(s) = \mathcal{L}\{f(x)\}$ then

$$\mathcal{L}\{u(x-c)f(x-c)\} = e^{-cs}F(s)$$

Conversely,

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(x-c)f(x-c) = \begin{cases} 0 & x < 0 \\ f(x-c) & x \geq c \end{cases}$$

Example: Find $f(x) * g(x)$ when $f(x) = e^{3x}$ and $g(x) = e^{2x}$:

Here $f(t) = e^{3t}$, $g(x-t) = e^{2(x-t)}$ and

$$f(x) * g(x) = \int_0^x e^{3t} e^{2(x-t)} \, dt = \int_0^x e^{3t} e^{2x} e^{-2t} \, dt$$

$$= e^{2x} \int_0^x e^{t} \, dt = e^{2x} [e^t]_{t=0}^{t=x} = e^{2x}(e^x - 1) = e^{3x} - e^{2x}.$$