Calculus – Chapter 8 – Continuity

Definition: A function f is defined to be continuous at x_0 if:

(a). $f(x_0)$ is defined

(b). $\lim_{x \to x_0} f(x) = f(x_0)$

(c). $\lim_{x \to x_0}$ exists.

Example: $f(x) = x^2 + 1$ is continuous at 2 since $\lim_{x \to 2} f(x) = 5 = f(2)$

$f(x) = \sqrt{4 - x^2}$ is not continuous at 3 because $f(3)$ is not defined.

$f(x) = 1/(x-2)$ is discontinuous at 2 because $f(2)$ is not defined and also because $\lim_{x \to 2} f(x)$ does not exist.

$f(x) = (x^2 - 4)/(x-2)$ is discontinuous at $x = 2$ because $f(2)$ is not defined.

However, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} (x+2) = 4$ so that $\lim_{x \to x_0}$ exists.

This is a "removable" discontinuity.

Removable: Discontinuity of function f at x_0 is removable when $f(x_0)$ is defined and changing the value of the function at x_0 produces a function that is continuous at x_0.

e.g. $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$, $\lim_{x \to 2} f(x) = 4$ but $f(2) = 0$

However setting the value at $x = 2$ to 4 then the function is continuous.

Example: $f(x) = |x|/x$, $x \neq 0$ is discontinuous at $x = 0$ as $f(0)$ is not defined

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$ and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1$

Thus $\lim_{x \to 0^-} \neq \lim_{x \to 0^+}$ so not removable.

Rational Function: Every rational function $H(x) = f(x)/g(x)$ is continuous on all set of points where $g(x) \neq 0$.

e.g. $H(x) = x/(x^2-1)$ is continous at all points except 1 and -1.

Intermediate Value Theorem: f is continuous on $[a,b]$ and $f(a) \neq f(b)$ then for any number c between $f(a)$ and $f(b)$, there is at least one number x_0 in the open interval (a,b) for which $f(x_0) = c$.

Corollary: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one root in the open interval (a,b) and therefore the graph of f crosses the x-axis at least once between a and b.

Extreme Value Theorem: If f is continuous on $[a,b]$ then f takes on a least value m and a greatest value of M on the interval.

Theorem: If f is continuous at c and $f(c) > 0$ then $\exists \delta > 0$ s.t. whenever

$c - \delta < x < c + \delta$ then $f(x) > 0$.