# Laplace Transform.

Definition: Let $f(x)$ be bounded for $0 \leq x \leq \infty$ and $s$ an arbitrary real variable. The Laplace transform of $f(x)$ is

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{\infty} e^{-sx} f(x)\, dx$$

for all values of $s$ for which the improper integral converges, when the limit exists:

$$\lim_{R \to \infty} \int_0^{R} e^{-sx} f(x)\, dx.$$

Properties: 1). If $\mathcal{L}\{f(x)\} = F(s)$ and $\mathcal{L}\{g(x)\} = G(s)$, then for any two constants:

$$\mathcal{L}\{c_1 f(x) + c_2 f_2(x)\} = c_1 \mathcal{L}\{f(x)\} + c_2 \mathcal{L}\{g(x)\}$$
$$= c_1 F(s) + c_2 G(s).$$

2). If $\mathcal{L}\{f(x)\} = F(s)$ then for any constant $a$ $\mathcal{L}\{e^{ax} f(x)\} = F(s-a)$.

3). If $\mathcal{L}\{f(x)\} = F(s)$, then for any positive integer $n$:

4). $$\mathcal{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

5). If $\mathcal{L}\{f(x)\} = F(s)$ and if $\lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{f(x)}{x}$ exists then:

$$\mathcal{L}\left\{\frac{1}{x} f(x)\right\} = \int_x^{\infty} F(t)\, dt$$

6). If $\mathcal{L}\{f(x)\} = F(s)$ then

$$\mathcal{L}\left\{\int_0^{x} f(t)\, dt\right. = \frac{1}{s} F(s)$$

7). If $f(x)$ is periodic with period $\omega$ i.e. $f(x+\omega) = f(x)$ then

$$\mathcal{L}\{f(x)\} \equiv \frac{\int_0^{\omega} e^{-sx} f(x)\, dx}{1 - e^{-\omega s}}$$

Functions of other independent variables:

The counterpart of $\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx$ for function of $t$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Example: Check if $\int_2^\infty 1/x^2\,dx$ converges:

$$\lim_{R\to\infty} \int_2^R 1/x^2\,dx = \lim_{R\to\infty} (-1/x)\Big|_2^R = \lim_{R\to\infty} \left(-\frac{1}{R}+\frac{1}{2}\right) = \frac{1}{2}.$$

Example: For which values of $s$ does the following converge? $\int_0^\infty e^{-sx}\,dx$

For $s=0$, $\int_0^\infty e^{-sx}\,dx = \int_0^\infty e^{-(0)x}\,dx = \lim_{R\to\infty} \int_0^R (1)\,dx = \lim_{R\to\infty} x\Big|_0^R = \lim_{R\to\infty} R = \infty$

hence integral diverges.

For $s \neq 0$, $\int_0^\infty e^{-sx}\,dx = \lim_{R\to\infty} \left[-\frac{1}{s}e^{-sx}\right]_{x=0}^{x=R} = \lim_{R\to\infty} \left(\frac{-1}{s}e^{-sR} + \frac{1}{s}\right)$

when $s<0$, $-sR > 0$ and limit is $\infty$ $\therefore$ integral diverges.

when $s>0$, $-sR<0$ and limit is $1/s$ $\therefore$ integral converges.

Example: Find Laplace transform of $f(x) = 1$

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-sx}(1)\,dx = 1/s \quad \text{for } s>0.$$

Example: Find the Laplace transform of $f(x) = x^2$

$$F(s) = \mathcal{L}\{x^2\} = \int_0^\infty e^{-sx} x^2\,dx = \lim_{R\to\infty} \int_0^R x^2 e^{-sx}\,dx$$

$$= \lim_{R\to\infty} \left[-\frac{x^2}{s}e^{-sx} - \frac{2x}{s^2}e^{-sx} - \frac{2}{s^3}e^{-sx}\right]_{x=0}^{x=R}$$

$$= \lim_{R\to\infty} \left(\frac{-R^2}{s}e^{-sR} - \frac{2R}{s^2}e^{-sR} - \frac{2}{s^3}e^{-sR} + \frac{2}{s^3}\right).$$

For $s<0$, $\lim_{R\to\infty}\left[-(R^2/s)e^{-sR}\right] = \infty$

For $s>0$, from L'Hôpital's Rule:

$$\lim_{R\to\infty}\left(-\frac{R^2}{s}e^{-sR}\right) = \lim_{R\to\infty}\left(\frac{-R^2}{se^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2R}{s^2e^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2}{s^3e^{sR}}\right) = 0.$$

Also, $\lim_{R\to\infty}\left[-(2/s^3)e^{-sR}\right] = 0$, $F(s) = 2/s^3$

For $s=0$, $\int_0^\infty e^{-sx}x^2\,dx = \int_0^\infty e^{-s(0)}x^2\,dx = \lim_{R\to\infty}\int_0^R x^2\,dx = \lim_{R\to\infty} R^3/3 = \infty$

So $\mathcal{L}\{x^2\} = 2/s^3$, $s>0$.

Example: Find Laplace transform $\mathcal{L}\{e^{ax}\}$

$$F(s) = \mathcal{L}\{e^{ax}\} = \int_0^\infty e^{-sx}e^{ax}\,dx = \lim_{R\to\infty}\int_0^R e^{(a-s)x}\,dx$$

$$= \lim_{R\to\infty}\left[\frac{e^{(a-s)x}}{a-s}\right]_{x=0}^{x=R} = \lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right] = \frac{1}{s-a} \text{ for } s>a.$$

Example: Find $\mathcal{L}\{\sin ax\}$

$$\mathcal{L}\{\sin ax\} = \int_0^\infty e^{-sx}\sin ax\,dx = \lim_{R\to\infty}\int_0^R e^{-sx}\sin ax\,dx$$

$$= \lim_{R\to\infty}\left[\frac{-se^{-sx}\sin ax}{s^2+a^2} - \frac{ae^{-sx}\cos ax}{s^2+a^2}\right]_{x=0}^{x=R}.$$

$$= \lim_{R\to\infty}\left[\frac{-se^{-sR}\sin(aR)}{s^2+a^2} - \frac{ae^{-sR}\cos(aR)}{s^2+a^2} + \frac{a}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2}, \quad s>0.$$