Linear First-Order Differential Equations.

Method of Solution:

$$y' + p(x)y = q(x)$$

Integrating factor is $I(x) = e^{\int p(x)dx}$

Depends only on x and is independent of y.

Multiply both sides by $I(x)$:

$$I(x)y' + p(x)I(x)y = I(x)q(x)$$

Simpler procedure to solve is to rewrite:

$$\frac{d(yI)}{dx} = Iq(x)$$

Reduction of Bernoulli Equations:

$$y' + p(x)y = q(x)y^n, \quad n \in \mathbb{R}$$

Subst $z = y^{1-n}$ transforms into a linear eqn of unknown function $z(x)$.

Example:

$y' - 3y = 6$, $p(x) = -3$ and $q(x) = 6$.

$\Rightarrow \int p(x)\,dx = \int -3dx = -3x$ (integrating factor).

$\therefore I(x) = e^{\int p(x)dx} = e^{-3x}$.

Example:

$y' + y = \sin x$

Here $p(x) = 1 \Rightarrow I(x) = e^{\int I(x)dx} = e^{x}$

Multiply the differential eqn by $I(x)$:

$$e^{x}y' + e^{x}y = e^{x}\sin x \Leftrightarrow \frac{d}{dx}(ye^{x}) = e^{x}\sin x$$

Integrate both sides:

$$ye^{x} = \frac{1}{2}e^{x}(\sin x - \cos x) + c \quad \text{or} \quad y = ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$