Chapter 4 – Basic Principles of Integrals.

Comparison Property:

If $f(x) \geq g(x)$ $\forall x \in [a,b]$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Proof: assume $f(x) \geq g(x) \iff f(x) - g(x) \geq 0$. From signed property:

$$\int_a^b [f(x) - g(x)]\,dx \geq 0$$

$$\therefore \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \quad \text{or} \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Example:

Given that $\dfrac{\sin x}{x} > \cos x$ for $(0, \pi/2)$, show that $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\sin x}{x}\,dx > 1/2$.

Let $f(x) = \sin x / x$ and $g(x) = \cos x$. Since $\dfrac{\sin x}{x} > \cos x$ for $0 < x < \pi/2$, from comparison property for definite integrals:

$$\int_{\pi/6}^{\pi/2} \frac{\sin x}{x}\,dx > \int_{\pi/6}^{\pi/2} \cos x\,dx = \sin x \Big|_{\pi/6}^{\pi/2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Comparison Property 2:

If $g(x) \leq f(x) \leq h(x)$ $\forall x \in [a,b]$, $\displaystyle\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b h(x)\,dx$

Proof: as $f(x) \geq g(x)$, $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

also from 1st property, $\displaystyle\int_a^b h(x)\,dx \geq \int_a^b f(x)\,dx$