Cheat Sheet

Bounded Function : $\exists m > 0$ s.t. $-M \leq f(x) \leq M \quad \forall x \in [a,b]$.

Continuity : Continuous if $\lim_{x \to a} f(x) = f(a)$

Fundamental Th. Calculus : $F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$ and $\int_a^b f(t)\,dt = F(b) - F(a)$

Additive Property : $\int_a^b [f(x) + g(x)] = \int_a^b f(x)dx + \int_a^b g(x)dx$

Homogenous : $\int_a^b k.f(x)dx = k\int_a^b f(x)dx$

Linearity : $\int_a^b [k_1 f(x) + k_2 g(x)] = k_1 \int_a^b f(x)dx + k_2 \int_a^b g(x)dx$

Signed Property : $f > 0$ in all $[a,b]$, $\int_a^b f(x)dx > 0$.

$f < 0$ in all $[a,b]$, $\int_a^b f(x) < 0$

Trig generalisations : $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

Comparison Property : if $f(x) \geq g(x)$ $x \in [a,b]$, $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Absolute Value property : $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\,dx$

Interchange limits : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Equal limits property : $\int_a^a f(x)dx = 0$.

Additivity : $c \in \mathbb{R}$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Bounded property : $m, M \in \mathbb{R}$, $m \leq f(x) \leq M$, $x \in [a,b]$, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Standard Forms : $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\int \tan x\,dx = -\ln|\cos x| + c$

$\int \cot x\,dx = \ln|\sin x| + c$

Integration by substitution :

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x)) \cdot g'(x)\,dx$$

Integration by parts :

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx \text{, and}$$

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx$$

Trig Formulas :

$$\sin A \cos B = \tfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \cos B = \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A \sin B = \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Trig integrals :

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x + \cot x| + c$$

Hyperbolic Functions :

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^{x} - e^{-x}}{2}.$$

Identies :

$$\cosh^2 x + \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

Sum / Difference Formulae :

$$\sinh(x \pm y) = \sinh x \cosh x \pm \cosh x \sinh x$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Double angle Formulae :

$$\sinh(2x) = 2 \sinh x \cosh x$$

$$\cosh(2x) = 1 + 2\sinh^2 x$$

$$\tan(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

Standard Hyperbolic Integrals :

$$\int \cosh x = \sinh x + c \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + c$$

$$\int \sinh x = \cosh x + c \qquad \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{sec}^2 h x\,dx = \tanh x + c \qquad \int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x$$

$$\int \coth x\,dx = \ln|\operatorname{sech} x| + c \qquad \int \tanh x\,dx = \ln|\cosh x| + c$$

$$\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + c$$

$$\int \operatorname{cosech} x\,dx = \ln|\tanh(x/2)| + c$$

Inverse Hyperbolic Functions:

$\sinh^{-1} x$	$\ln\left(x + \sqrt{x^2+1}\right)$
$\cosh^{-1} x$	$\ln\left(x + \sqrt{x^2-1}\right)$
$\tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
$\coth^{-1} x$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$
$\operatorname{sech}^{-1} x$	$\ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right)$
$\operatorname{cosech}^{-1} x$	$\ln\left(\frac{1 \pm \sqrt{1+x^2}}{x}\right)$

Derivatives:

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}, \quad x \in \mathbb{R}$

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \quad |x| < 1$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad |x| > 1$

$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

$\frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}} \quad x \neq 1.$

Integrals:

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a) = \ln\left(x+\sqrt{a^2+x^2}\right), \quad a>0$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) = \ln\left(x+\sqrt{x^2-a^2}\right), \quad 0<a<x$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}(x/a) = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right), \quad a>0, |x|<a.$$

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\coth^{-1}(x/a) = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right), \quad a>0, |x|>a.$$

Half Angle Substitution:

If $t = \tan(x/2)$

$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt.$

Bioche Rules:

$\pi \to -x$	$f(\sin(-x), \cos(-x)) \cdot (-dx)$	$t = \cos x$
$x \to \pi - x$	$f(\sin(\pi-x), \cos(\pi-x)) \cdot (-dx)$	$t = \sin x$
$x \to \pi + x$	$f(\sin(\pi+x), \cos(\pi+x)) \cdot (dx)$	$t = \tan x$

Note:

$\sin(-x) = -\sin x \qquad \sin(\pi-x) = \sin x \qquad \sin(\pi+x) = -\sin x$

$\cos(-x) = \cos x \qquad \cos(\pi-x) = -\cos x \qquad \cos(\pi+x) = -\cos x$

Special Substitutions :

$\int R(u, \sqrt{m^2 - x^2})\, dx \qquad u = m \sin\theta \qquad u = m\tanh t$

$\int R(u, \sqrt{m^2 + x^2})\, dx \qquad u = m\tan\theta \qquad u = m\sinh t$

$\int R(u, \sqrt{x^2 - m^2})\, dx \qquad u = m\sec\theta \qquad u = m\cosh t$

i.e integrals of form $\int R(x, \sqrt{ax^2 + bx + c})$ can be reduced to

$\int R(u, \sqrt{\pm u^2 \pm m^2})\, dx$

Improper :

$\deg P \geq \deg Q$ where $f(x) = \frac{P(x)}{Q(x)}$

Partial Fractions :

$$\frac{f(x)}{g(x)} = \sum_{i=1}^{m} \sum_{j=1}^{K_i} \frac{A_{ij}}{(a_i x + b_i)^j} + \sum_{i=1}^{n} \sum_{j=1}^{S_i} \frac{B_{ij}x + C_{ij}}{(a_i x^2 + \beta_i x + \gamma_i)^j}$$

$$\frac{A}{(ax+b)^m} \quad \text{and} \quad \frac{Bx + C}{(ax^2 + bx + c)} = \frac{Bx}{(ax^2+bx+c)^n} + \frac{C}{(ax^2+bx+c)^n}$$

Ostrogradsky's Method :

If $\deg P < \deg Q$, $\int \frac{P(x)}{Q(x)}\, dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}$,

$Q_1(x)$ greatest common factor of Q.

Useful Integrals :

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a), \quad a > 0, \ |x| < a$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|x + \sqrt{x^2 + a^2}|, \quad a > 0$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}|, \quad 0 < a < x$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(x/a), \quad a \neq 0$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right|, \quad a > 0, \ |x| < a$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right|, \quad a > 0, \ |x| > a.$$