Calculus - Chapter 23 - Definite Integrals.

Definition: $\sum_{j=k}^{n} f(j) = f(k) + f(k+1) + \ldots + f(n)$

Area under curve: Assume $f(x) \geq 0 \ \forall x \in [a,b]$.

Choose points $x_1, x_2, \ldots, x_{n-1}$ between a and b

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < b$$

Divide into n subintervals:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

Denote lengths of subintervals: $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$.

Hence if $1 \leq k \leq n$:

$$\Delta_k x = x_k - x_{k-1}$$

If $\Delta_k A$ is the area of the strip then:

$$A = \sum_{k=1}^{n} \Delta_k A$$

Select any point x^*_k in the k^{th} subinterval, hence total area is:

$$\sum_{k=1}^{n} f(x^*_k) \Delta_k x = f(x^*_1)\Delta_1 x + \ldots + f(x^*_k)\Delta_k x.$$

which converges to $\int_a^b f(x)\,dx$ as intervals become shorter.

Example: $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$, use $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and subdivide

$$\Delta_k x = (b-a)/n.$$

Then $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, and in general $x_k = a + k\Delta x$

$$\sum_{k=1}^{n} f(x^*_k)\Delta_k x = \sum_{k=1}^{n} x^*_k \Delta_k x = \sum_{k=1}^{n} (a + k\Delta x)\Delta x$$

$$= \sum_{k=1}^{n} (a\Delta x + k(\Delta x)^2) = \sum_{k=1}^{n} a\Delta x + \sum_{k=1}^{n} k(\Delta x)^2$$

$$= n(a\Delta x) + (\Delta x)^2 \sum_{k=1}^{n} k$$

$$= a(b-a) + \frac{1}{2}(b-a)^2 \frac{n+1}{n}$$

as $n \to \infty$, $\frac{1}{2}(b^2 - a^2)$. as $\frac{n+1}{n} \to 1$.