Chapter 8 - Trigonometric Integrals.

Definition: Consider integrals of the form $\int \cos^n x\,dx$ or $\int \sin^n x\,dx$

Case $n=1$: $\int \sin x\,dx = -\cos x + c$.

case $n=2$: $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c$.

case $n=3$:
$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x \cdot \sin^2 x\,dx\\ &= \int \sin x(1-\cos^2 x)\,dx\\ &= \int \sin x\,dx - \int \sin x\cos^2 x\,dx\\ &= -\cos x - \int \sin x \cos^2 x\,dx\end{aligned}$$

Let $u = \cos x$, $du = -\sin x\,dx$

$$\begin{aligned}\int \sin^3 x\,dx &= -\cos x + \int u^2\,du.\\ &= -\cos x + \tfrac{1}{3}u^3 + c\\ &= -\cos x + \tfrac{1}{3}\cos^3 x + c.\end{aligned}$$

Case $n=4$:
$$\begin{aligned}\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx\\ &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx\\ &= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx\\ &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx\\ &= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)dx\\ &= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + c\end{aligned}$$

Definition: Consider integrals of the form $\int \cos^m x \cdot \sin^n x\,dx$

Case 1: Suppose m is odd: the subst $u = \sin x$ with $\sin^2 x + \cos^2 x = 1$ can be used.

Example: e.g. Find $\int \cos^3 x \sin^4 x\,dx$

$$\begin{aligned}\int \cos^3 x \sin^4 x\,dx &= \int \cos^2 x \sin^4 x \cos x\,dx\\ &= \int (1-\sin^2 x)\sin^4 x \cos x\,dx\end{aligned}$$

Let $u = \sin x$, $du = \cos x \, dx$:

$$\int \cos^3 x \sin^4 x \, dx = \int (1-u^2) u^4 \, du = \int (u^4 - u^6) \, du = u^5/5 - u^7/7 + c$$
$$= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + c.$$

Identities : $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$. are used to transform

$\int \cos^k(2x) \, dx$, $k \in \mathbb{N}$.

Example :

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \sin^2 x (\cos^2 x)^2 \, dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 \, dx \\
&= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x) \, dx. \\
&= \frac{x}{8} + \frac{\sin 2x}{16} - \frac{1}{8}\int \cos^2 2x \, dx - \frac{1}{8}\int \cos^3 2x \, dx
\end{aligned}$$

Consider $\int \cos^2 2x \, dx = \frac{1}{2}\int (1+\cos 4x) \, dx = \frac{x}{2} + \frac{\sin 4x}{8} + c_1$

and $\int \cos^3 2x \, dx = \int \cos^2 2x \cos 2x \, dx = \frac{1}{2}\int (1-\sin^2 2x) 2\cos 2x \, dx$

and using subst $u = \sin 2x$, $du = 2\cos 2x \, dx$

$$\int \cos^3 2x \, dx = \frac{1}{2}\int (1-u^2) \, du = \frac{u}{2} - \frac{u^3}{6} + c_2.$$
$$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + c_2, \text{ since } u = \sin 2x.$$

So
$$\int \sin^2 x \cos^4 x \, dx = \frac{x}{8} + \frac{\sin 2x}{16} - \frac{1}{8}\left(\frac{x}{2} + \frac{\sin 4x}{8} + c_1\right) - \frac{1}{8}\left(\frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + c_2\right)$$
$$= \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + c$$

Products of Multiple Angles:

Consider integrals of the form:

$$\int \cos mx \sin nx\,dx,\quad \int \cos mx \cos nx\,dx \text{ and } \int \sin mx \sin nx\,dx.$$

1. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
2. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
3. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

Example:

$$\int \cos 5x \cos 3x\,dx = \frac{1}{2}\int (\cos 2x + \cos 8x)\,dx = \frac{1}{4}\sin 2x + \frac{1}{16}\sin 8x + c$$

Tangent and cotangent:

Consider integrals of the form:

$$\int \tan^n x\,dx \text{ and } \int \cot^n x\,dx$$

Case $n=1$: $\int \tan x\,dx = \int \sin x/\cos x\,dx = -\ln|\cos x| + c$

Case $n=2$: $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + c.$

Case $n=3$: Let $u = \tan x$, $du = \sec^2 x\,dx$ thus:

$$\begin{aligned}
\int \tan^3 x\,dx &= \int \tan x \tan^2 x\,dx \\
&= \int \tan x(\sec^2 x - 1)\,dx \\
&= \int \tan x \sec^2 x\,dx - \int \tan x \\
&= \int \tan x \sec^2 x\,dx + \ln|\cos x| + c. \\
&= \int u\,du + \ln|\cos x| + c \\
&= u^2/2 + \ln|\cos x| + c \\
&= \frac{1}{2}\tan^2 x + \ln|\cos x| + c.
\end{aligned}$$

Sec and cosec: Consider integrals $\int \sec^n x \, dx$ or $\int \operatorname{cosec}^n x \, dx$

$$\int \sec x \, dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

giving $\int \sec x = \ln|\sec x + \tan x| + c$

Also $\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) dx = \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x} \, dx$

giving $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + c$

For squares: $\int \sec^2 x \, dx = \tan x + c$ and $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$.

Products of sec and tan: Consider integral $\int \sec^m x \tan^n x \, dx$

Example:

$$\begin{aligned} \int \sec^4 x \tan^3 x \, dx &= \int \sec^2 x \sec^2 x \tan^3 x \, dx \quad \text{as } m=4 \text{ is even.} \\ &= \int (1+\tan^2 x) \tan^3 x \sec^2 x \\ &= \int (\tan^3 x + \tan^5 x) \sec^2 x \, dx \end{aligned}$$

Let $u = \tan x$, $du = \sec^2 x \, dx$

$$\int \sec^4 x \tan^3 x \, dx = \int (u^5 + u^3) \, du = \frac{u^6}{6} + \frac{u^4}{3} + c = \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c.$$