CHAPTER 8 - PARTIAL DERIVATIVES

Applications to Geometry

1. Tangent Plane to a Surface Let $F(x, y, z) = 0$ be the equation of a surface S such as that shown in Figure 8.1. Assume that F, and all other functions in this chapter are continuously differentiable unless otherwise indicated. Suppose we wish to find the equation of a tangent plane to S at the point $P(x_0, y_0, z_0)$. A vector normal to S at this point is $\mathbf{N}_0 = \nabla F\big|_P$, the subscript P indicating that the gradient is to be evaluated at the point $P(x_0, y_0, z_0)$.

If $\mathbf{r}_0$ and $\mathbf{r}$ are tangent, the vectors drawn, respectively, from O to $P(x_0, y_0, z_0)$ and $Q(x, y, z)$ on the tangent plane, the equation of the plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N}_0 = (\mathbf{r} - \mathbf{r}_0) \cdot \nabla F\big|_P = 0 \tag{1}$$

since $\mathbf{r} - \mathbf{r}_0$ is perpendicular to $\mathbf{N}_0$.

In rectangular form this is

$$\left.\frac{\partial F}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_P (y - y_0) + \left.\frac{\partial F}{\partial z}\right|_P (z - z_0) = 0 \tag{2}$$

In case the equation of the surface is given in orthogonal curvilinear coordinates in the form $F(u_1, u_2, u_3) = 0$, the equation of the tangent plane can be obtained using the result on Page 172 for the gradient in these coordinates. See Problem 8.4.

2. Normal Line to a Surface. Suppose we require equations for the normal line to the surface S at $P(x_0, y_0, z_0)$, i.e., the line perpendicular to the tangent plane of the surface at P. If we now let $\mathbf{r}$ be the vector drawn from O in Figure 8.1 to any point (x, y, z) on the normal $\mathbf{N}_0$, we see that $\mathbf{r} - \mathbf{r}_0$ is collinear with $\mathbf{N}_0$, and so the required condition is

$$(\mathbf{r} - \mathbf{r}_0 \times \mathbf{N}_0 = (\mathbf{r} - \mathbf{r}_0) \times \times \nabla F\big|_P = \mathbf{0} \tag{3}$$

By expressing the cross product in the determinant form

$$\begin{vmatrix} i & j & k \\ x - x_0 & y - y_0 & z - z_0 \\ F_x|_P & F_y|_P & F_z|_P \end{vmatrix}$$

we find that

$$\frac{x - x_0}{\left.\frac{\partial F}{\partial x}\right|_P} = \frac{y - y_0}{\left.\frac{\partial F}{\partial y}\right|_P} = \frac{z - z_0}{\left.\frac{\partial F}{\partial z}\right|_P} \tag{4}$$

Setting each of these ratios equal to a parameter (such as t or u) and solving for x, y, and z yields the *parametric equations* of the normal line.

The equations for the normal line can also be written when the equation of the surface is expressed in orthogonal curvilinear coordinates. [See Problem 8.1(b).]

3. Tangent Line to a Curve Let the parametric equations of curve C of Figure 8.2 be $x = f(u)$, $y = g(u)$, $z = h(u)$, where we shall suppose, unless otherwise indicated, that f, g, and h are continuously differentiable. We wish to find equations for the tangent line to C at the point $P(x_0, y_0, z_0)$ where $u = u_0$.

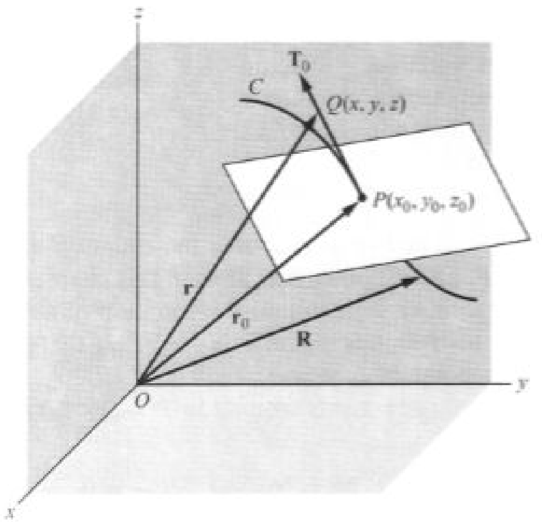

Fig. 8.2

If $\mathbf{R} = f(u)\mathbf{i} + g(u)\mathbf{j} + h(u)\mathbf{k}$, then a vector tangent to C at the point P is given by $\mathbf{T}_0 = \left.\frac{d\mathbf{R}}{du}\right|_P$. If $\mathbf{r}_0$ and $\mathbf{r}$ denote the vectors drawn respectively from O to $P(x_0, y_0, z_0)$ and $Q(x, y, z)$ on the tangent line, then $\mathbf{r} - \mathbf{r}_0$ is collinear with $\mathbf{T}_0$. Thus,

$$(\mathbf{r}-\mathbf{r}_0)\times\mathbf{T}_0 = (\mathbf{r}-\mathbf{r}_0)\times\left.\frac{d\mathbf{R}}{du}\right|_P = 0 \tag{5}$$

In rectangular form this becomes

$$\frac{x-x_0}{f'(u_0)} = \frac{y-y_0}{g'(u_0)} = \frac{z-z_0}{h'(u_0)} \tag{6}$$

The parametric form is obtained by setting each ratio equal to u.

If the curve C is given as the intersection of two surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$, observe that $\nabla F \times \nabla G$ has the direction of the line of intersection of the tangent planes; therefore, the corresponding equations of the tangent line are

$$\frac{x-x_0}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_P} = \frac{y-y_0}{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_P} = \frac{z-z_0}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_P} \tag{7}$$

Note that the determinants in Equation (7) are Jacobians. A similar result can be found when the surfaces are given in terms of orthogonal curvilinear coordinates.

4. Normal Plane to a Curve Suppose we wish to find an equation for the normal plane to curve C at $P(x_0, y_0, z_0)$ in Figure 8.2 (i.e., the plane perpendicular to the tangent line to C at this point). Letting $\mathbf{r}$ be the vector from O to any point (x, y, z) on this plane, it follows that $\mathbf{r} - \mathbf{r}_0$ is perpendicular to $\mathbf{T}_0$. Then the required equation is

$$(\mathbf{r}-\mathbf{r}_0).\mathbf{T}_0 = (\mathbf{r}-\mathbf{r}_0).\left.\frac{d\mathbf{R}}{du}\right|_P = 0 \tag{8}$$

When the curve has parametric equations $x = f(u)$, $y = g(u)$, $z = h(u)$ this becomes

$$f'(u_0)(x-x_0) + g'(u_0)(y-y_0) + h'(u_0)(z-z_0) = 0 \tag{9}$$

Furthermore, when the curve is the intersection of the implicitly defined surfaces

$$F(x, y, z) = 0 \text{ and } G(x, y, z) = 0$$

then

$$\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_P (x-x_0) + \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_P (y-y_0) + \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_P (z-z_0) = 0 \tag{10}$$

5. Envelopes Solutions of differential equations in two variables are geometrically represented by one-parameter families of curves. Sometimes such a family characterizes a curve called an *envelope*.

For example, the family of all lines (see Problem 8.9) one unit from the origin may be represented by $x \sin \alpha - y \cos \alpha - 1 = 0$, where α is a parameter. The envelope of this family is the circle $x^2 + y^2 = 1$.

If $\phi(x, y, z) = 0$, is a one-parameter family of curves in the xy plane, there may be a curve E which is tangent at each point to some member of the family and such that each member of the family is tangent to E. If E exists, its equation can be found by solving simultaneously the equations

$$\phi(x, y, \alpha) = 0, \ \phi(_\alpha(x, y, \alpha) = 0 \tag{11}$$

and E is called the *envelope* of the family.

The result can be extended to determine the envelope of a one-parameter family of surfaces $\phi(x, y, z, \alpha)$. This envelope can be found from

$$\phi(x, y, z, \alpha) = 0, \ \phi_\alpha(x , y, z, \alpha) = 0 \tag{12}$$

Extensions to two- (or more) parameter families can be made.

Directional Derivatives

Suppose $F(x, y, z)$ is defined at a point (x, y, z) on a given space curve C. Let $F(x + \Delta x, y + \Delta y, z + \Delta z)$ be the value of the function at a neighboring point on C and let Δs denote the length of arc of the curve between those points. Then

$$\lim_{\Delta s \to 0} \frac{\Delta F}{\Delta s} = \lim_{\Delta s \to 0} \frac{F(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - F(x,\ y,\ z)}{\Delta s} \tag{13}$$

if it exists, is called the *directional derivative* of F at the point (x, y, z) along the curve C and is given by

$$\frac{dF}{ds} = \frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} \tag{14}$$

In vector form this can be written

$$\frac{dF}{ds} = \left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = \nabla F \cdot \frac{dr}{ds} = \nabla F \cdot \mathbf{T} \tag{15}$$

from which it follows that the directional derivative is given by the component of ∇F in the direction of the tangent to C.

Thus, the maximum value of the directional derivative is given by $|\nabla F|$ and these maxima occur in directions normal to the surfaces $F(x, y, z) = c$ is (where c is any constant), which are sometimes called *equipotential surfaces* or *level surfaces*.

Differentiation Under the Integral Sign

Let

$$\phi(\alpha) = \int_{u_1}^{u_2} f(x, \alpha)\, dx \qquad a \leqq \alpha \leqq b \tag{16}$$

where u_1 and u_2 may depend on the parameter α. Then

$$\frac{d\phi}{d\alpha} = \int_{u_2}^{u_1} \frac{\partial f}{\partial \alpha} dx + f(u_2, \alpha)\frac{du_2}{d\alpha} - f(u_1, \alpha)\frac{du_1}{d\alpha} \tag{17}$$

for $a \leqq \alpha \leqq b$, if $f(x, \alpha)$ and $\partial f/\partial \alpha$ are continuous in both x and α in some region of the $x\alpha$ plane including $u_1 \leqq x \leqq u_2$, $a \leqq \alpha \leqq b$ and if u_1 and u_2 are continuous and have continuous derivatives for $a \leqq \alpha \leqq b$.

In case u_1 and u_2 are constants, the last two terms of Equation (*17*) are zero.

The result (*17*), called *Leibniz's rule*, is often useful in evaluating definite integrals (see Problems 8.15 and 8.29).

Integration Under the Integral Sign

If $\phi(\alpha)$ is defined by Equation (*16*) and $f(x, \alpha)$ is continuous in x and α in a region including $u_1 \leqq x \leqq u_2$, $a \leqq x \leqq b$, then if u_1 and u_2 are constants,

$$\int_a^b \phi(\alpha)\,d\alpha = \int_a^b \left\{ \int_{u_1}^{u_2} f(x,\alpha)\,dx \right\} d\alpha = \int_{u_1}^{u_2} \left\{ \int_a^b f(x,\alpha)\,d\alpha \right\} dx \tag{18}$$

The result is known as *interchange of the order of integration* or *integration under the integral sign*. (See Problem 8.18.)

Maxima and Minima

To determine the exact nature of the function at a critical point, P_0, $g''(x_0)$) had to be examined.

$$g''(x_0) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \text{ implied } \begin{cases} \text{counterclockwise rotation (relative minimum)} \\ \text{a clockwise rotation (relative maximum)} \\ \text{need for further investigation} \end{cases}$$

This section describes the n ecessary and sufficient conditions for relative extrema of functions of two variables. Geometrically, we think of surfaces S represented by $z = f(x, y)$. If at a point $P_0(x_0, y_0)$ then $f_x(x, y_0) = 0$, means that the curve of intersection of S and the plane $y = y_0$ has a tangent parallel to the x axis. Similarly, $f_y(x_0, y_0) = 0$ indicates that the curve of intersection of S and the cross section $x = x_0$ has a tangent parallel the y axis. (See Problem 8.20.)

Thus,

$$f_x(x, y_0) = 0, f_y(x_0, y) = 0$$

are necessary conditions for a relative extrema of $z = f(x, y)$ at P_0; however, they are not sufficient because there are directions associated with a rotation through 360° that have not been examined. Of course, no differentiation between relative maxima and relative minima has been made. (See Figure 8.4.)

A very special form $f_{xy} - f_x f_y$, invariant under plane rotation and capable of characterizing the roots of a quadratic equation $Ax^2 + 2Bx + C = 0$, allows us to form sufficient conditions for relative extrema. (See Problem 8.21.)

A point (x_0, y_0) is called a *relative maximum point* or *relative minimum point* of $f(x, y)$ respectively according as $f(x_0 + h, y_0 + k) < f(x_0, y_0)$ or $f(x_0 + h, y_0 + k) > f(x_0, y_0)$ for all h and k such that $0 < |h| < \delta$, $0 < |k| < \delta$ where δ is a sufficiently small positive number.

A necessary condition that a differentiable function $f(x, y)$ have a relative maximum or minimum is

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0 \tag{19}$$

If (x_0, y_0) is a point (called a *critical point*) satisfying Equations (19) and if Δ is defined by

$$\Delta = \left\{ \left(\frac{\partial^2 f}{\partial x^2} \right)\left(\frac{\partial^2 f}{\partial y^2} \right) - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right\} \Bigg|_{(x_0, y_0)} \tag{20}$$

then

1. (x_0, y_0) is a relative maximum point if $\Delta > 0$ and $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(x_0, y_0)} < 0 \left(\text{or} \quad \left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(x_0, y_0)} < 0 \right)$.
2. (x_0, y_0) is a relative minimum point of $\Delta > 0$ and $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(x_0, y_0)} > 0 \left(\text{or} \quad \left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(x_0, y_0)} > 0 \right)$.
3. (x_0, y_0) is neither a relative maximum nor a relative minimum point if $\Delta < 0$. If $\Delta < 0$, < 0, (x_0, y_0) is sometimes called a *saddle point*.
4. No information is obtained if $\Delta = 0$ (in such case further investigation is necessary).

Method of Lagrange Multipliers for Maxima and Minima

A method for obtaining the relative maximum or minimum values of a function $F(x, y, z)$ subject to a *constraint condition* $\phi(x, y, z) = 0$, consists of the formation of the auxiliary function

$$G(x, y, z) \equiv F(x, y, z) + \lambda\phi(x, y, z) \tag{21}$$

subject to the conditions

$$\frac{\partial G}{\partial x} = 0, \quad \frac{\partial G}{\partial y} = 0, \quad \frac{\partial G}{\partial z} = 0 \tag{22}$$

which are necessary conditions for a relative maximum or minimum. The parameter λ, which is independent of x, y, z, is called a *Lagrange multiplier*.

The conditions in Equation (22) are equivalent to $\nabla G = 0$, and, hence, $0 = \nabla F + \lambda \nabla \phi$.

Geometrically, this means that ∇F and $\nabla\phi$ are parallel. This fact gives rise to the method of Lagrange multipliers in the following way.

Let the maximum value of F on $\phi(x, y, z) = 0$ be A and suppose it occurs at $P_0(x_0, y_0, z_0)$. (A similar argument can be made for a minimum value of F.) Now consider a family of surfaces $F(x, y, z) = C$.

The member $F(x, y, z) = A$ passes through P_0, while those surface $F(x, y, z) = B$ with $B < A$ do not. (This choice of a surface, i.e., $f(x, y, z) = A$, geometrically imposes the condition $\phi(x, y, z) = 0$ on F.) Since at P_0 the condition $0 = \nabla F + \lambda \nabla \phi$ tells us that the gradients of $F(x, y, z) = A$ and $\phi(x, y, z)$ are parallel, we know that the surfaces have a common tangent plane at a point that is maximum for F. Thus, $\nabla G = 0$ is a necessary condition for a relative maximum of F at P_0. Of course, the condition is not sufficient. The critical point so determined may not be unique and it may not produce a relative extremum.

The method can be generalized. If we wish to find the relative maximum or minimum values of a function $F(x_1, x_2, x_3, \ldots, x_n)$ subject to the *constraint conditions* $\phi(x_1, \ldots, x_n) = 0$, $\phi_2(x_1, \ldots, x_n) = 0, \ldots, \phi_k(x_1, \ldots, x_n) = 0$, we form the we form the auxiliary function

$$G(x_1, x_2, \ldots, x_n) \equiv F + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_k\phi_k \tag{23}$$

subject to the (necessary) conditions

$$\frac{\partial G}{\partial x_1} = 0, \quad \frac{\partial G}{\partial x_2} = 0, \ldots\ldots\ldots\ldots, \frac{\partial G}{\partial x_n} \equiv 0 \tag{24}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$, which are independent of $x_1, x_2, \ldots, x_n$, are the *Lagrange multipliers*.