Chapter 7 - Integration by parts

Theorem:

$$\int f(x)\, g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) g(x)\, dx$$

Proof:

Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

$\therefore \quad f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$

$\therefore \quad \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$

Compact:

$$\int (u)(dv) = uv - \int v\,du, \quad u(x) \text{ and } v(x).$$

Example:

$$\int x\cos x\, dx$$

$u = x, \quad du = dx$

$dv = \cos x\, dx, \quad v = \int \cos x\, dx = \sin x$

$\therefore \int x\cos x\, dx = x\sin x - \int \sin x\, dx$

$= x\sin x + \cos x + c$

Example:

$$\int x^2 e^{-x} dx$$

$u = x^2, \quad du = 2x$

$dv = e^{-x}, \quad v = \int e^{-x} dx = -e^{-x}$

$\int x^2 e^{-x} dx = x^2 \cdot (-e^{-x}) - \int (-e^{-x}) . 2x\, dx$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

But $\int x e^{-x} dx = -xe^{-x} - \int 1 . (-e^{-x}) dx = -xe^{-x} + e^{-x} + c$

$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2(-xe^{-x} + e^{-x})$

$= -(x^2 + 2x + 2)e^{-x} + D.$

Example:

$$\int_0^{\pi/2} x\sin 2x\, dx = \left[x \cdot \frac{-1}{2}\cos 2x\right]_0^{\pi/2} - \int_0^{\pi/2} \left(-\frac{\cos 2x}{2}\right) \cdot 1\, dx$$

$= \frac{\pi}{4} + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$= \frac{\pi}{4}.$

Example : $\int x^3 \ln x \, dx$, let $u = \ln x$ (easier to diff), $du = \frac{dx}{x}$

$dv = x^3$, $v = \frac{1}{4}x^4$.

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^4 \cdot \left(\frac{1}{x}\right) dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4} \cdot \frac{1}{4}x^4 + c$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{12}x^4 + c.$$

$$= x^4\left(\frac{3\ln x - 1}{12}\right) + c.$$

Example : $\int 2x \ln(x+12) \, dx$

$u = \ln(x+12)$, $du = \frac{1}{x+12} dx$

$dv = 2x\,dx$, $v = x^2$.

$$\int 2x \ln(x+12)\,dx = x^2 \ln(x+12) - \int x^2 \cdot \frac{1}{x+12} dx$$

$$= x^2 \ln(x+12) - \int \frac{(x^2 - 144) + 144}{x+12} dx$$

$$= x^2 \ln(x+12) - 144\int \frac{dx}{x+12} - \int \frac{(x+12)(x-12)}{x+12} dx$$

$$= (x^2 - 144)\ln(x+12) - \frac{x^2}{2} + 12x + c.$$

Example : using intermediate constant of integration v :

$v = \int 2x \, dx = x^2 - 144.$

$$\int 2x \ln(x+12)\,dx = (x^2 - 144)\ln(x+12) - \int \frac{x^2 - 144}{x+12} dx$$

$$= (x^2 - 144)\ln(x+12) - \int \frac{(x-12)(x+12)}{x+12} dx$$

$$= (x^2 - 144)\ln(x+12) - \frac{x^2}{2} + 12x + C.$$