CHAPTER 2 - SEQUENCES

Definition of a Sequence

A sequence is a set of numbers $u_1, u_2, u_3, \ldots$ in a definite order of arrangement (i.e., a *correspondence* with the natural numbers or a subset thereof) and formed according to a definite rule. Each number in the sequence is called a *term*; u_n is called the nth *term*. The sequence is called *finite* or *infinite* according as there are or are not a finite number of terms. The sequence $u_1, u_2, u_3, \ldots$ is is also designated briefly by $\{u_n\}$.

Limit of a Sequence

A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all integers $n > N$. In such case we write $\lim_{n\to\infty} u_n = l$.

EXAMPLE. If $u_n = 3 + 1/n = (3n + 1)/n$, the sequence is 4, 7/2, 10/3, . . . and we can show that $\lim_{n\to\infty} u_n = 3$.

If the limit of a sequence exists, the sequence is called *convergent*; otherwise, it is called *divergent*. A sequence can converge to only one limit; i.e., if a limit exists, it is unique. See Problem 2.8.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3, \ldots$ has a limit l if the successive terms get "closer and closer" to l. This is often used to provide a "guess" as to the value of the limit, after which the definition is applied to see if the guess is really correct.

Theorems on Limits of Sequences

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$

2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$

3. $\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = AB$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$ if $\lim_{n\to\infty} b_n = B \neq 0$

   If $B = 0$ and $A \neq 0$, $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist.

   If $B = 0$ and $A = 0$, $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ may or may not exist.

5. $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p = A^p$, for p = any real number if A^p exists.

6. $\lim_{n\to\infty} p^{a_n} = p^{\lim_{n\to\infty} a_n} = p^A$, for p = any real number if p^A exists.

Infinity

We write $\lim_{n\to\infty} a_n = \infty$ if for each positive number M we can find a positive number N (depending on M) such that $a_n > M$ for all $n > N$. Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ if for each positive number M we can find a positive number N such that $a_n < -M$ for all $n > N$. It should be emphasized that ∞ and $-\infty$ are not numbers and the sequences are not convergent. The terminology employed merely indicates that the sequences diverge in a certain manner. That is, no matter how large a number in absolute value that one chooses, there is an n such that the absolute value of a_n is greater than that quantity.

Bounded, Monotonic Sequences

If $u_n \leqq M$ for $n = 1, 2, 3, \ldots$, where M is a constant (independent of n), we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound*. If $u_n \geqq m$, the sequence is *bounded below* and m is called a *lower bound*.

If $m \leqq u_n \leqq M$ the sequence is called *bounded*. Often this is indicated by $|u_n| \leqq P$. Every convergent sequence is bounded, but the converse is not necessarily true.

If $u_{n+1} \geqq u_n$ the sequence is called *monotonic increasing*; if $u_{n+1} > u_n$ it is called *strictly increasing*. Similarly, if $u_{n+1} \leqq u_n$ the sequence is called *monotonic decreasing*, while if $u_{n+1} < u_n$ it is *strictly decreasing*.

Least Upper Bound and Greatest Lower Bound of a Sequence

A number $\underline{M}$ is called the *least upper bound* (l.u.b.) of the sequence $\{u_n\}$ if $u_n \leqq \underline{M}$, $n = 1, 2, 3, \ldots$ while at least one term is greater than $\underline{M} - \epsilon$ for any $\epsilon > 0$.

A number $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the sequence $\{u_n\}$ if $u_n \geqq \bar{m}$, $n = 1, 2, 3, \ldots$ while at least one term is less than $\bar{m} + \epsilon$ for any $\epsilon > 0$.

Compare with the definition of l.u.b. and g.l.b. for sets of numbers in general (see Page 6).

Limit Superior, Limit Inferior

A number $\bar{l}$ is called the *limit superior, greatest limit,* or *upper limit* (lim sup or $\overline{\lim}$) of the sequence $\{u_n\}$ if infinitely many terms of the sequence are greater than $\bar{l} - \epsilon$ while only a finite number of terms are greater than $\bar{l} + \epsilon$, where ϵ is any positive number.

A number $\underline{l}$ is called the *limit inferior, least limit,* or *lower limit* (lim inf or $\underline{\lim}$) of the sequence $\{u_n\}$ if infinitely many terms of the sequence are less than $\underline{l} + \epsilon$ while only a finite number of terms are less than $\underline{l} - \epsilon$, where ϵ is any positive number.

These correspond to least and greatest limiting points of general sets of numbers.

If infinitely many terms of $\{u_n\}$ exceed any positive number M, we define lim sup $\{u_n\} = \infty$. If infinitely many terms are less than $-M$, where M is any positive number, we define lim inf $\{u_n\} = -\infty$.

If $\lim_{n\to\infty} u_n = \infty$, we define lim sup $\{u_n\}$ = lim inf $\{u_n\} = \infty$.

If $\lim_{n\to\infty} u_n = -\infty$, we define lim sup $\{u_n\}$ = lim inf $\{u_n\} = -\infty$.

Although every bounded sequence is not necessarily convergent, it always has a finite lim sup and lim inf.

A sequence $\{u_n\}$ converges if and only if lim sup u_n = lim inf u_n is finite.

Nested Intervals

Consider a set of intervals $[a_n, b_n]$, $n = 1, 2, 3, \ldots$, where each interval is contained in the preceding one and $\lim_{n\to\infty} (a_n - b_n) = 0$. Such intervals are called *nested intervals*.

We can prove that to every set of nested intervals there corresponds one and only one real number. This can be used to establish the Bolzano-Weierstrass theorem of Chapter 1. (See Problems 2.22 and 2.23.)

Cauchy's Convergence Criterion

Cauchy's convergence criterion states that a sequence $\{u_n\}$ converges if and only if for each $\epsilon > 0$ we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p, q > N$. This criterion has the advantage that one need not know the limit l in order to demonstrate convergence.

Infinite Series

Let $u_1, u_2, u_3, \ldots$ be a given sequence. Form a new sequence $S_1, S_2, S_3, \ldots$ where

$$S_1 = u_1, S_2 = u_1 + u_2, S_3 = u_1 + u_2 + u_3, \ldots, + S_n = u_1 + u_2 + u_3 + \cdots + u_n, \ldots$$

where S_n, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n\}$.

The sequence $S_1, S_2, S_3, \ldots$ is symbolized by

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n$$

which is called an *infinite series*. If $\lim_{n\to\infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*; otherwise, the series is called *divergent*.