Calculus - Chapter 42 - Infinite Sequences.

Examples: (a) $\langle \frac{1}{n} \rangle$ is the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$

(b) $\langle (\frac{1}{2})^n \rangle$ is the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$

(c) $\langle n^2 \rangle$ is the sequence $1, 4, 9, 16, \dots, n^2$

Limits: $\lim_{n\to\infty} S_n = L$ if any $\varepsilon > 0$ $\exists n_0 > 0$ s.t. whenever $n \geq n_0$ we have $|S_n - L| < \varepsilon$

If $\lim_{n\to\infty} S_n = L$ then sequence $\langle S_n \rangle$ converges to L.

Example: $\langle \frac{1}{n} \rangle$ is convergent since $\lim_{n\to\infty} \frac{1}{n} = 0$

To prove, let n_0 be smallest positive integer greater than $1/\varepsilon$, so $1/\varepsilon < n_0$.

Hence, if $n \geq n_0$ then $n > 1/\varepsilon$, therefore $1/n < \varepsilon$.

Thus if $n \geq n_0$, $|1/n - 0| < \varepsilon$

This proves $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example: $\langle 2n \rangle$ is a divergent sequence, since $\lim_{n\to\infty} 2n \neq L$

$\lim_{n\to\infty} S_n = +\infty$ iff for any number c, no matter how large $\exists n_0 > 0$ s.t. whenever $n \geq n_0$ we have $n \geq n_0$ we have $S_n < c$

So $\lim_{n\to\infty} S_n = \infty$ if $\lim_{n\to\infty} |S_n| = +\infty$

Bounded: $\langle S_n \rangle$ is bounded above if $\exists c$ s.t. $S_n \leq c$ $\forall n$ and $\langle S_n \rangle$ is bounded below if $\exists b$ s.t. $b \leq S_n$ $\forall n$.

$\langle S_n \rangle$ is bounded if it is bounded above and below

$\langle S_n \rangle$ is bounded iff $\exists d$ s.t. $|S_n| \leq d$ $\forall n$.

Example: $\langle 2n \rangle$ is bounded below (by 0, for example) but not bounded above

$\langle (-1)^n \rangle$ is bounded, $|(-1)^n| \leq 1$ $\forall n$.

Theorem: Every convergent sequence is bounded.
The converse is false e.g. $\langle (-1)^n \rangle$ is bounded but not convergent.

Theorem: Assume $\lim_{n\to\infty} s_n = c$ and $\lim_{s\to\infty} t_n = d$.

(a). $\lim_{n\to\infty} k = k$

(b). $\lim_{n\to\infty} k s_n = kc$

(c) $\lim_{n\to\infty} (s_n + t_n) = c + d$.

(d). $\lim_{n\to\infty} (s_n - t_n) = c - d$.

(e). $\lim_{n\to\infty} (s_n t_n) = cd$.

(f). $\lim_{n\to\infty} (s_n / t_n) = c/d$, provided $d \neq 0$ and $t_n \neq 0$ $\forall n$.

Theorem: If $\lim_{n\to\infty} s_n = \infty$ and $s_n \neq 0$ $\forall n$, then $\lim_{n\to\infty} 1/s_n = 0$.

Theorem: (a). If $|a| > 1$ then $\lim_{n\to+\infty} a^n = \infty$
In particular, if $a > 0$ then $\lim_{n\to+\infty} a^n = +\infty$

(b). If $|r| < 1$, $\lim_{n\to+\infty} r^n = 0$

Theorem: $\lim_{n\to+\infty} s_n = L = \lim_{n\to+\infty} u_n$ and $\exists m$ st. $s_n \leq t_n \leq u_n$ $\forall n \geq m$ then $\lim_{n\to+\infty} t_n = L$

Corollary: if $\lim_{n\to+\infty} u_n = 0$ $\exists m$ st. $|t_n| \leq |u_n|$ $\forall n \geq m$ then $\lim_{n\to+\infty} t_n = 0$

Example: $\lim_{n\to\infty} (-1)^n \frac{1}{n^2} = 0$.
Proof: note that $\left| (-1)^n \frac{1}{n^2} \right| \leq \frac{1}{n}$ and $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Theorem: Assume f function continuous at c and assume $\lim_{n\to+\infty} s_n = c$, where all terms s_n are in the domain f, then $\lim_{n\to+\infty} f(s_n) = f(c)$

Monotonic:

(a). $\langle S_n \rangle$ is non-decreasing if $S_n \leq S_{n+1}$ $\forall n$.

(b). $\langle S_n \rangle$ is increasing if $S_n < S_{n+1}$ $\forall n$.

(c). $\langle S_n \rangle$ is non-decreasing if $S_n \geq S_{n+1}$ $\forall n$

(d). $\langle S_n \rangle$ is decreasing if $S_n > S_{n+1}$ $\forall n$

(e). Sequence is monotonic if it is either non-decreasing or non-increasing.

Examples: (a). $1, 1, 2, 2, 3, 3, \ldots$ is non-decreasing but not increasing.

(b). $-1, -1, -2, -2, -3, -3 \ldots$ is non-increasing but not decreasing.

Theorem: Every bounded monotonic sequence is convergent.

Example: Consider $S_n = \dfrac{3n}{4n+1}$, then $S_{n+1} = \dfrac{3(n+1)}{4(n+1)+1} = \dfrac{3n+3}{4n+5}$

So $S_{n+1} - S_n = \dfrac{3}{(4n+5)(4n+1)} > 0$, Since $4n+5 > 0$ and $4n+1 > 0$

Example: $S_n = \dfrac{3n}{4n+1}$, $\dfrac{S_{n+1}}{S_n} = \dfrac{12n^2 + 15n + 3}{12n^2 + 15n} > 1$ $\left(\text{e.g. show } \dfrac{S_{n+1}}{S_n} > 1\right)$

Note: Find a differentiable function $f(x)$ s.t. $f(n) = S_n$ $\forall n$, and show that $f'(x) > 0$ $\forall x \geq 1$

Example: Consider $S_n = \dfrac{3n}{4n+1}$, let $f(x) = \dfrac{3x}{4n+1}$, $f'(x) = \dfrac{3}{(4n+1)^2} > 0$ $\forall x$.