Calculus - Chapter 44 - Series with Positive Terms.

Definition: A positive series is a series where all the terms of a series $\sum s_n$ are **positive**

$\langle S_n \rangle$ is an **increasing** sequence since $S_{n+1} = S_n + s_{n+1} > S_n$.

Theorem: Positive series $\sum s_n$ **converges** iff sequence of partial sums $\langle S_n \rangle$ is **bounded**.

Theorem: The **integral test** is : let $\sum s_n$ be a **positive series** and $f(x)$ is continuous, positive decreasing function $[1, +\infty)$ s.t. $f(n) = s_n \ \forall n > 0$, then:

$$\sum s_n \text{ converges iff } \int_1^{+\infty} f(x)dx \text{ converges.}$$

From the above theorem, $\int_1^n f(x)dx < s_1 + s_2 + \dots + s_{n-1} = S_{n-1}$

If $\sum s_n$ **converges** then $\langle S_n \rangle$ is **bounded**, so $\int_1^u f(x)\,dx$ will be **bounded** $\forall u \geq 1$,

therefore $\int_1^\infty f(x)dx$ converges

Conversely, we have $s_2 + s_3 + \dots + s_n < \int_1^n f(x)\,dx$, and therefore

$$S_n < \int_1^n f(x)dx$$

Thus if $\int_1^\infty f(x)\,dx$ **converges** then $S_n < \int_1^\infty f(x)\,dx + s$, so $\langle S_n \rangle$ will be **bounded**.

Hence, $\sum s_n$ **converges**.

Example: $\sum \frac{\ln(n)}{n}$ diverges

Proof: let $f(x) = \ln x / x$

$$\begin{aligned}\int_1^{+\infty} \frac{\ln x}{x}\,dx &= \lim_{u\to\infty} \int_1^u \frac{\ln(x)}{x}\,dx \\ &= \lim_{u\to\infty} \tfrac{1}{2}(\ln x)^2 \Big|_1^u \\ &= \lim_{u\to\infty} \tfrac{1}{2}((\ln u)^2 - 0) \\ &= +\infty \Rightarrow \text{diverges.}\end{aligned}$$

Example: $\sum \frac{1}{n^2}$ converges.

Proof $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{u\to\infty} \int_1^u \frac{1}{x^2}\,dx = \lim_{u\to\infty} -\left(\frac{1}{u} - 1\right) = 1 \Rightarrow$ converges.

Theorem: For Comparison Test: Let $\sum a_n$ and $\sum b_n$ be two positive series s.t. $\exists m > 0$ for which $a_k \leq b_k$ for all integers $k \geq m$, then:

(a). If $\sum b_n$ converges, so does $\sum a_n$

(b). If $\sum a_n$ diverges, so does $\sum b_n$.

To prove (a): Assume $\sum b_n$ converges. Let $B_n = b_1 + b_2 + \dots + b_n$

Then $A_n \leq B_n$, since $a_k \leq b_k$ $\forall k$.

Since $\sum b_n$ converges, it follows from the 1st theorem that $\langle B_n \rangle$ is bounded

Since $A_n \leq B_n$ $\forall n$, it follows $\langle A_n \rangle$ is bounded.

Example: $\sum \frac{1}{n^2+5}$ converges.

Proof: Let $a_n = \frac{1}{n^2+5}$ and $b_n = \frac{1}{n^2}$, then $a_n < b_n$ $\forall n$.

But $\frac{1}{n^2}$ converges, hence $\sum \frac{1}{n^2+5}$ converges.

Example: $\sum \frac{1}{3n+5}$ diverges.

Proof: let $a_n = \frac{1}{4n}$ and $b_n = \frac{1}{3n+5}$, so $a_n \leq b_n$ $\forall n \geq 5$

i.e. $\frac{1}{4n} \leq \frac{1}{3n+5} \iff 3n+5 \leq 4n$

Monotonic series $\frac{1}{n}$ diverges, hence $\frac{1}{4n}$ diverges and $\frac{1}{3n+5}$ diverges.

Theorem: Limit Comparison test: