Chapter 6 - Integration by substitution.

Theorem : $$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(x)) \cdot g'(x)\, dx$$

Proof : $$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b) - g(a))$$

also $$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

$$\therefore \int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(x))\Big|_a^b = F(g(b) - g(a)).$$

Examples : $$\int_0^{\pi/2} \cos x \sin^3 x\, dx$$

Let $f(x) = x^3$ and $g(x) = \sin x$

$f(g(x)) = \sin(x^3)$ and $g'(x) = \cos x$.

$g(0) = \cos(0) = 0$

$g(\pi/2) = \cos(\pi/2) = 1$

$$\therefore \int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

as $$\int_0^{\pi/2} \cos x \sin^3 x\, dx = \int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 = 1/4.$$

Example : $\int_0^{\pi/2} \cos x \sin^3 x\, dx$ using "change of variable" method.

Let $u = \sin x$, $du = \cos x$

when $x = 0$, $u = 0$

$x = \frac{\pi}{2}$, u 1

$$\int_0^{\pi/2} \cos x \sin^3 x\, dx = \int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 = 1/4.$$

Example : $$\int_0^1 x\sqrt{1 - x^2}\, dx$$

Let $u = 1 - x^2$, $du = -2x\, dx$ $\therefore$ $dx = -\frac{du}{2x}$

when $x = 0$, $u = 1$

$x = 1$, $u = 0$

$$\therefore \int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_1^0 \sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = 1/3.$$

Example : $\displaystyle\int x^2(1+x)^{100}\,dx$

Let $u = 1+x$, $du = dx$ and $x = u-1$

$$\therefore \int x^2(1+x)^{100}\,dx = \int (u-1)^2 u^{100}\,du$$

$$= \int (u^{102} - 2u^{101} + u^{100})\,du$$

$$= \frac{1}{103}(1+x)^{103} - \frac{1}{51}(1+x)^{102} + \frac{1}{101}(1+x)^{101} + c$$

Example : $\displaystyle\int \frac{x^3}{\sqrt{x^4+4}}\,dx$

Let $u = x^4+4$, $du = 4x^3\,dx \Leftrightarrow dx = \dfrac{du}{4x^3}$

$$\int \frac{x^3}{\sqrt{x^4+4}}\,dx = \int \frac{x^3}{\sqrt{u}} \cdot \frac{du}{4x^3} = \frac{1}{4}\int \frac{du}{\sqrt{u}}\,du = \frac{1}{4}\cdot 2u^{1/2} + c = \frac{1}{2}\sqrt{x^4+1} + c.$$

Example : $\displaystyle\int \frac{x^2+1}{x^3+3x+8}\,dx$

Let $u = x^3+3x+8$, $du = (3x^2+3)\,dx \Leftrightarrow dx = \dfrac{du}{3(x^2+1)}$

$$\int \frac{x^2+1}{x^3+3x+8}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + c$$

$$= \frac{1}{3}\ln|x^3+3x+8| + c. \quad \left(\text{or } \frac{f'(x)}{f(x)} \Rightarrow \ln(u)\right).$$

Example : $\displaystyle\int xe^{x}\,dx = \frac{1}{2}e^{x^2} + c$

Because let $u = x^2$, $du = 2x\,dx \Leftrightarrow dx = \dfrac{du}{2x}$

$$\int xe^{x}\,dx = \int x \cdot e^{u}\,\frac{du}{2x}\,dx = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + c = \frac{1}{2}e^{x^2} + c.$$

Example : $\displaystyle\int \frac{dx}{x\ln x}$, let $u = \ln x$ $du = \dfrac{dx}{x} \Leftrightarrow dx = x\,du.$

$$\int \frac{x\,du}{xu} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$$

Example: $\int \frac{x^3}{x^8+9}\,dx = \int \frac{x^3}{(x^4)^2+9}\,dx$

Let $u = x^4$, $du = 4x^3\,dx \Leftrightarrow dx = \frac{du}{4x^3}$.

$$\int \frac{x^3}{x^8+9}\,dx = \int \frac{\cancel{x^3}}{u^2+9}\cdot\frac{du}{4\cancel{x^3}} = \frac{1}{4}\int\frac{du}{u^2+9} = \frac{1}{4}\cdot\frac{1}{3}\tan^{-1}(u/3)+c.$$

$$= \frac{1}{12}\tan^{-1}(x^4/3)+c.$$

Example: $\int \frac{dx}{1+\sqrt{x}}$

Let $x = u^2$, $dx = 2u\,du$

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u}{1+u}\,du = 2\int \frac{(1+u)-1}{1+u}\,du = 2\int du - 2\int \frac{du}{1+u}$$

$$= 2u - 2\ln|1+u| + c$$

$$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + c.$$

Example: $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}$

Let $u = e^x$, $\frac{du}{dx} = e^x$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-u^2}}\cdot\frac{du}{e^x} = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u)+c = \sin^{-1}(e^x)+c.$$

Example: $\int \frac{dx}{1+\sin^2 x}\,dx = \int \frac{dx}{1+\sin^2 x+(\sin^2 x+\cos^2 x)} = \int \frac{dx}{\cos^2 x + 2\sin^2 x} = \int \frac{dx}{\cos^2 x(1+2\tan^2 x)}$

$$= \int \frac{\sec^2 x}{1+2\tan^2 x}\,dx$$

Let $u = \sqrt{2}\tan x$, $du = \sqrt{2}\sec^2 x$

$$\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{2}}\tan^{-1}u + c = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c.$$