CHAPTER 11 - INFINITE SERIES

Definitions of Infinite Series and Their Convergence and Divergence

Definition The sum

$$S = \sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots \qquad (1)$$

is an *infinite series*. Its value, if one exists, is the limit of the sequence of partial sums $\{S_n\}$

$$S = \lim_{n\to\infty} S_n \qquad (2)$$

If there is a unique value, the series is said to *converge* to that *sum S*. If there is not a unique sum, the series is said to *diverge*.

Sometimes the character of a series is obvious. For example, the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ generated by the frog on the log surely converges, while $\sum_{n=1}^{\infty} n$ is divergent. On the other hand, the variable series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ raises questions.

This series may be obtained by carrying out the division $\frac{1}{1-x}$. If $-1 < x < 1$, the sum S_n yields an approximation to $\frac{1}{1-x}$ and Equation (2) is the exact value. The indecision arises for $x = -1$. Some very great mathematicians, including Leonhard Euler, thought that S should be equal to $\frac{1}{2}$, as is obtained by substituting -1 into $\frac{1}{1-x}$. The problem with this conclusion arises with examination of $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ and observation that appropriate associations can produce values of 1 or 0. Imposition of the condition of uniqueness for convergence puts this series in the category of divergent and eliminates such possibility of ambiguity in other cases.

Fundamental Facts Concerning Infinite Series

1. If Σu_n converges, then $\lim_{n\to\infty} u_n = 0$ (see Problem 2.26). The converse, however, is not necessarily true; i.e., if $\lim_{n\to\infty} u_n = 0$, Σu_n may or may not converge. It follows that if the nth term of a series does *not* approach zero, the series is divergent.
2. Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence.
3. Removal (or addition) of a *finite* number of terms from (or to) a series does not affect the convergence or divergence.

Special Series

1. **Geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$, where a and r are constants, converges to $S = \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geqq 1$. The sum of the first n terms is $S_n = \frac{a(1-r^n)}{1-r}$ (see Problem 2.25).
2. **The p series** $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$, where p is a constant, converges for $p > 1$ and diverges for $p \leqq 1$. The series with $p = 1$ is called the *harmonic series*.

Tests for Convergence and Divergence of Series of Constants

More often than not, exact values of infinite series cannot be obtained. Thus, the search turns toward information about the series. In particular, its convergence or divergence comes into question. The following tests aid in discovering this information.

1. **The comparison test** for series of nonnegative terms.

   (a) *Convergence.* Let $\upsilon_n \geqq 0$ for all $n > N$ and suppose that $\Sigma\upsilon_n$ converges. Then if $0 \leqq u_n \leqq \upsilon_n$ for all $n > N$, Σu_n also converges. Note that $n > N$ means *from some term onward*. Often, $N = 1$.

EXAMPLE. Since $\dfrac{1}{2^n+1} \leq \dfrac{1}{2^n}$ and $\sum \dfrac{1}{2^n}$ converges, $\sum \dfrac{1}{2^{n+1}}$ also converges.

   (b) *Divergence.* Let $\upsilon_n \geqq 0$ for all $n > N$ and suppose that $\Sigma\upsilon_n$ diverges. Then if $u_n \geqq \upsilon_n$ for all $n > N$, Σu_n also diverges.

EXAMPLE. Since $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ also diverges.

2. **The limit-comparison** or **quotient test** for series of nonnegative terms.

   (a) If $u_n \geqq 0$ and $\upsilon_n \geqq 0$ and if $\lim\limits_{n\to\infty} \dfrac{u_n}{\upsilon_n} = A \neq 0$ or ∞, then Σu_n and $\Sigma\upsilon_n$ either both converge or both diverge.

   (b) If $A = 0$ in (a) and $\Sigma\upsilon_n$ converges, then Σu_n converges.

   (c) If $A = \infty$ in (a) and $\Sigma\upsilon_n$ diverges, then Σu_n diverges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $\upsilon_n = 1/n^p$, we have the following theorems from known facts about the p series.

Theorem 1 Let $\lim\limits_{n\to\infty} n^p u_n = A$. Then

(i) Σu_n conyerges if $p > 1$ and A is finite.

(ii) Σu_n diverges if $p \leqq 1$ and $A \neq 0$ (A may be infinite).

EXAMPLES 1. $\sum \dfrac{n}{4n^3-2}$ converges since $\lim\limits_{n\to\infty} n^2 \cdot \dfrac{n}{4n^3-2} = \dfrac{1}{4}$.

2. $\sum \dfrac{\ln n}{\sqrt{n+1}}$ diverges since $\lim\limits_{n\to\infty} n^{1/2} \cdot \dfrac{\ln n}{(n+1)^{1/2}} = \infty$.

3. **Integral test** for series of non-negative terms.

   If $f(x)$ is positive, continuous, and monotonic decreasing for $x \geqq N$ and is such that $f(n) = u_n$, $n = N, N+1, N+2, \ldots$, then Σu_n converges or diverges according as $\int_N^\infty f(x)\,dx = \lim\limits_{M\to\infty} \int_n^M f(x)\,dx$ converges or diverges. In particular, we may have $N = 1$, as is often true in practice.

   This theorem borrows from Chapter 12, since the integral has an unbounded upper limit. (It is an improper integral. The convergence or divergence of these integrals is defined in much the same way as for infinite series.)

EXAMPLE: $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges since $\lim\limits_{M\to\infty} \int_1^M \dfrac{dx}{x^2} = \lim\limits_{M\to\infty} \left(1 - \dfrac{1}{M}\right)$ exists.

4. **Alternating series test.** An *alternating series* is one whose successive terms are alternately positive and negative.

An alternating series converges if the following two conditions are satisfied (see Problem 11.15).

(a) $|u_{n+1}| \leqq |u_n|$ for $n \geqq N$. (Since a fixed number of terms does not affect the convergence or divergence of a series, N may be any positive integer. Frequently it is chosen to be 1.)

(b) $\lim\limits_{n\to\infty} u_n = 0 \left(\text{or } \lim\limits_{n\to\infty} |u_n| = 0\right)$

EXAMPLE. For the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, we have $u_n = \frac{(-1)^{n-1}}{n}, |u_n| = \frac{1}{n}$, $|u_{n+1}| = \frac{1}{n+1}$. Then for $n \geqq 1$, $|u_{n+1}| \leqq |u_n|$. Also $\lim\limits_{n\to\infty} |u_n| = 0$. Hence, the series converges.

Theorem 2 The numerical error made in stopping at any particular term of a convergent alternating series which satisfies conditions (a) and (b) is less than the absolute value of the next term.

EXAMPLE. If we stop at the fourth term of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$, the error made is less than $\frac{1}{5} = 0.2$.

5. **Absolute and conditional convergence.** The series Σu_n is called *absolutely convergent* if $\Sigma |u_n|$ converges. If Σu_n converges but $\Sigma |u_n|$ diverges, then Σu_n is called *conditionally convergent*.

Theorem 3 If $\Sigma |u_n|$ converges, then Σu_n converges. In words, an absolutely convergent series is convergent (see Problem 11.17).

EXAMPLE 1. $\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \cdots$ is absolutely convergent and thus convergent, since the series of absolute values $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ converges.

EXAMPLE 2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, but $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges. Thus, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Any of the tests used for series with nonnegative terms can be used to test for absolute convergence. Also, tests that compare successive terms are common. Tests 6, 8, and 9 are of this type.

6. **Ratio test.** Let $\lim\limits_{n\to\infty} \left|\frac{u_{n-1}}{u_n}\right| = L$. Then the series Σu_n

(a) converges (absolutely) if $L < 1$.

(b) diverges if $L > 1$.

If $L = 1$ the test fails.

7. **The *n*th root test.** Let $\lim\limits_{n\to\infty} \sqrt[n]{|u_n|} = L$. Then the series Σu_n

(a) converges (absolutely) if $L < 1$

(b) diverges if $L > 1$.

If $L = 1$ the test fails.

8. **Raabe's test.** Let $\lim\limits_{n\to\infty} n\left(1 - \left|\frac{u_n + 1}{u_n}\right|\right) = L$. Then the series Σu_n

(a) converges (absolutely) if $L > 1$.

(b) diverges or converges conditionally if $L < 1$.

If $L = 1$ the test fails.

This test is often used when the ratio tests fails.

9. **Gauss's test.** If $\left|\frac{u_n+1}{u_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^q}$, where $|c_n| < P$ for all $n > N$ the sequence c_n is bounded, then the series Σu_n

   (a) converges (absolutely) if $L > 1$.

   (b) diverges or converges conditionally if $L \leqq 1$.

   This test is often used when Raabe's test fails.

Theorems on Absolutely Convergent Series

Theorem 4 (Rearrangement of terms.) The terms of an absolutely convergent series can be rearranged in any order, and all such rearranged series will converge to the same sum. However, if the terms of a conditionally convergent series are suitably rearranged, the resulting series may diverge or converge to *any* desired sum (see Problem 11.80).

Theorem 5 (Sums, differences, and products.) The sum, difference, and product of two absolutely convergent series is absolutely convergent. The operations can be performed as for finite series.

Infinite Sequences and Series of Functions, Uniform Convergence

We opened this chapter with the thought that functions could be expressed in series form. Such representation is illustrated by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^3}{5!} - + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-)!} + \cdots$$

where

$$\sin x = \lim_{n\to\infty} S_n. \quad \text{with} \quad S_1 = x, S_2 = x - \frac{x^3}{3!}, \cdots S_n = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}$$

Observe that until this section the sequences and series depended on one element, n. Now there is variation with respect to x as well. This complexity requires the introduction of a new concept called *uniform convergence*, which, in turn, is fundamental in exploring the continuity, differentiation, and integrability of series.

Let $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ be a sequence of functions defined in $[a, b]$. The sequence is said to converge to $F(x)$, or to have the limit $F(x)$ in $[a, b]$, if for each $\epsilon > 0$ and each x in $[a, b]$ we can find $N > 0$ such that $|u_n(x) - F(x)| < \epsilon$ for all $n > N$. In such case we write $\lim_{n\to\infty} u_n(x) = F(x)$. The number N may depend on x as well as ϵ. If it depends *only* on ϵ and not on x, the sequence is said to converge to $F(x)$ *uniformly* in $[a, b]$ or to be *uniformly convergent* in $[a, b]$.

The infinite series of functions

$$\sum_{n}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots \tag{3}$$

is said to be convergent in $[a, b]$ if the sequence of partial sums $\{S_n(x)\}$, $n = 1, 2, 3, \ldots$, where $S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$, is convergent in $[a, b]$. In such case we write $\lim_{n\to\infty} S_n(x) = S(x)$ and call $S(x)$ the *sum* of the series.

It follows that $\Sigma u_n(x)$ converges to $S(x)$ in $[a, b]$ if for each $\epsilon > 0$ and each x in $[a, b]$ we can find $N > 0$ such that $|S_n(x) - S(x)| < \epsilon$ for all $n > N$. If N depends *only* on ϵ and not on x, the series is called *uniformly convergent* in $[a, b]$.

Since $S(x) - S_n(x) = R_n(x)$, the remainder after n terms, we can equivalently say that $\Sigma u_n(x)$ is uniformly convergent in $[a, b]$ if for each $\epsilon > 0$ we can find N depending on ϵ but not on x such that $|R_n(x)| < \epsilon$ for all $n > N$ and all x in $[a, b]$.

These definitions can be modified to include other intervals besides $a \leqq x \leqq b$, such as $a < x < b$, and so on.

The domain of convergence (absolute or uniform) of a series is the set of values of x for which the series of functions converges (absolutely or uniformly).

EXAMPLE 1. Suppose $u_n = x^n/n$ and $-1/2 \leqq x \leqq 1$. Now think of the constant function $F(x) = 0$ on this interval. For any $\epsilon > 0$ and any x in the interval, there is N such that for all $n > N\,|u_n - F(x)| < \epsilon$, i.e., $|x^n/n| < \epsilon$. Since the limit does not depend on x, the sequence is uniformly convergent.

EXAMPLE 2. If $u_n = x^n$ and $0 \leqq x \leqq 1$, the sequence is not uniformly convergent because [think of the function $F(x) = 0$, $0 \leqq x < 1$, $F(1) = 1$]

$$\left|x^n - 0\right| < \varepsilon \text{ when } x^n < \varepsilon$$

thus

$$n \ln x < \ln \varepsilon$$

On the interval $0 \leqq x < 1$, and for $0 < \epsilon < 1$, both members of the inequality are negative; therefore, $n > \dfrac{\ln \varepsilon}{\ln x}$. Since $\dfrac{\ln \varepsilon}{\ln x} = \dfrac{\ln 1 - \ln \varepsilon}{\ln 1 - \ln x} = \dfrac{\ln (/\varepsilon)}{\ln (1/x)}$, it follows that we must choose N such that

$$n > N > \frac{\ln 1/\varepsilon}{\ln 1/x}$$

From this expression we see that $\epsilon \to 0$, then ln $\ln\dfrac{1}{\varepsilon} \to \infty$, and also as $x \to 1$ from the left $\ln\dfrac{1}{x} \to 0$ from the right; thus, in either case, N must increase without bound. This dependency on both ϵ and x demonstrates that the sequence is not uniformly convergent. For a pictorial view of this example, see Figure 11.1.

Special Tests for Uniform Convergence of Series

1. **Weierstrass M test.** If a sequence of positive constants $M_1, M_2, M_3, \ldots$ can be found such that in some interval
   (a) $|u_n(x)| \leqq M_n \qquad n = 1, 2, 3, \ldots$
   (b) ΣM_n converges

   then $\Sigma u_n(x)$ is uniformly and absolutely convergent in the interval.

   EXAMPLE. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly and absolutely convergent in $[0, 2\pi]$ since $\left|\dfrac{\cos nx}{n^2}\right| \leqq \dfrac{1}{n^2}$ and $\displaystyle\sum \frac{1}{n^2}$ converges.

   This test supplies a sufficient but not a necessary condition for uniform convergence, i.e., a series may be uniformly convergent even when the test cannot be made to apply.

   Because of this test, we may be led to believe that uniformly convergent series must be absolutely convergent, and conversely. However, the two properties are independent; i.e., a series can be uniformly convergent without being absolutely convergent, and conversely. See Problems 11.30 and 11.127.

2. **Dirichlet's test.** Suppose that
   (a) the sequence $\{a_n\}$ is a monotonic decreasing sequence of positive constants having limit zero.
   (b) there exists a constant P such that for $a \leqq x \leqq b\,|u_1(x) + u_2(x) + \cdots + u_n(x)| < P$, for all $n > N$.

   Then the series $a_1u_1(x) + a_2u_2\,(x) + \cdots = \displaystyle\sum_{n=1}^{\infty} a_nu_n(x)$ is uniformly convergent in $a \leqq x \leqq b$.

Theorems on Uniformly Convergent Series

Theorem 6 If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ are continuous in $[a, b]$ and if $\Sigma u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$.

Briefly, this states that a uniformly convergent series of continuous functions is a continuous function. This result is often used to demonstrate that a given series is not uniformly convergent by showing that the sum function $S(x)$ is discontinuous at some point (see Problem 11.30).

In particular, if x_0 is in $[a, b]$, then the theorem states that

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0)$$

where we use right- or left-hand limits in case x_0 is an endpoint of $[a, b]$.

Theorem 7 If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ are continuous in $[a, b]$ and if $\Sigma u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \tag{4}$$

or

$$\int_a^b \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \tag{5}$$

Briefly, a uniformly convergent series of continuous functions can be integrated term by term.

Theorem 8 If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ are continuous and have continuous derivatives in $[a, b]$ and if $\Sigma u_n(x)$ converges to $S(x)$ while $\Sigma u_n'(x)$ is uniformly convergent in $[a, b]$, then in $[a, b]$

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x) \tag{6}$$

or

$$\frac{d}{dx} \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x) \tag{7}$$

This shows conditions under which a series can be differentiated term by term.

Theorems similar to these can be formulated for sequences. For example, if $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ is uniformly convergent in $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b u_n(x)\,dx = \int_a^b \lim_{n \to \infty} u_n(x)\,dx \tag{8}$$

which is the analog of Theorem 7.

Power Series

A series having the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{9}$$

where $a_0, a_1, a_2, \ldots$ are constants, is called a *power series* in x. It is often convenient to abbreviate the series (9) as $\Sigma a_n x^n$.

In general, a power series converges for $|x| < R$ and diverges for $|x| > R$, where the constant R is called the *radius of convergence* of the series. For $|x| = R$, the series may or may not converge.

The interval $|x| < R$ or $-R < x < R$, with possible inclusion of endpoints, is called the *interval of convergence* of the series. Although the ratio test is often successful in obtaining this interval, it may fail, and in such cases, other tests may be used (see Problem 11.22).

The two special cases $R = 0$ and $R = \infty$ can arise. In the first case the series converges only for $x = 0$; in the second case it converges for all x, sometimes written $-\infty < x < \infty$ (see Problem 11.25). When we speak of a convergent power series, we shall assume, unless otherwise indicated, that $R > 0$.

Similar remarks hold for a power series of the form (9), where x is replaced by $(x - a)$.

Theorems on Power Series

Theorem 9 A power series converges uniformly and absolutely in any interval which lies *entirely within* its interval of convergence.

Theorem 10 A power series can be differentiated or integrated term by term over any interval lying entirely within the interval of convergence. Also, the sum of a convergent power series is continuous in any interval lying entirely within its interval of convergence.

This follows at once from Theorem 9 and the theorem on uniformly convergent series on Pages 284 and 285. The results can be extended to include endpoints of the interval of convergence by the following theorems.

Theorem 11 *Abel's theorem.* When a power series converges up to and including an endpoint of its interval of convergence, the interval of uniform convergence also extends so far as to include this endpoint. See Problem 11.42.

Theorem 12 *Abel's limit theorem.* If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, which may be an interior point or an endpoint of the interval of convergence, then

$$\lim_{x \to x_0} \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ \lim_{x \to x_0} a_n x^n \right\} = \sum_{n=0}^{\infty} a_n x_0^n \tag{10}$$

If x_0 is an endpoint, we must use $x \to x_0+$ or $x \to x_0-$ in Equation (10) according as x_0 is a left- or a right-hand endpoint.

This follows at once from Theorem 11 and Theorem 6 on the continuity of sums of uniformly convergent series.

Operations With Power Series

In the following theorems we assume that all power series are convergent in some interval.

Theorem 13 Two power series can be added or subtracted term by term for each value of x common to their intervals of convergence.

Theorem 14 Two power series, for example, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, can be multiplied to obtain $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 \tag{11}$$

the result being valid for each x within the common interval of convergence.

Theorem 15 If the power series $\sum_{n=0}^{\infty} a_n x^n$ is divided by the power series $\Sigma b_n x^n$ where $b_0 \neq 0$, the quotient can be written as a power series which converges for sufficiently small values of x.

Theorem 16 If $y = \sum_{n=0}^{\infty} a_n x^n$, then by substituting $x = \sum_{n=0}^{\infty} b_n y^n$, we can obtain the coefficients b_n in terms of a_n. This process is often called *reversion of series*.

Expansion of Functions in Power Series

This section gets at the heart of the use of infinite series in analysis. Functions are represented through them. Certain forms bear the names of mathematicians of the eighteenth and early nineteenth centuries who did so much to develop these ideas.

A simple way (and one often used to gain information in mathematics) to explore series representation of functions is to assume such a representation exists and then discover the details. Of course, whatever is found must be confirmed in a rigorous manner. Therefore, assume

$$f(x) = A_0 + A_1(x-c) + A_2(x-c)^2 + \cdots + A_n(x-c)^n + \cdots$$

Notice that the coefficients A_n can be identified with derivatives of f. In particular,

$$A_0 = f(c),\ A_1 = f'(c),\ A_2 = \frac{1}{2!} f''(c), \cdots, A_n = \frac{1}{n!} f^{(n)}(c), \cdots$$

This suggests that a series representation of f is

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!} f''(x-c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x-c)^n \cdots$$

A first step in formalizing series representation of a function f, for which the first n derivatives exist, is accomplished by introducing *Taylor polynomials* of the function.

$$P_0(x) = f(c) \qquad P_1(x) = f(c) + f'(c)(x-c),$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{1}{2!} f''(c)(x-c)^2,$$

$$P_n(x) = f(c) + f'(c)(x-c) + \cdots \frac{1}{n!} f^{(n)}(c)(x-c)^n \tag{12}$$

Taylor's Theorem

Let f and its derivatives $f', f'', \ldots, f^{(n)}$ exist and be continuous in a closed interval $a \le x \le b$ and suppose that $f^{(n+1)}$ exists in the open interval $a < x < b$. Then for c in $[a, b]$,

$$f(x) = P_n(x) + R_n(x)$$

where the remainder $R_n(x)$ may be represented in any of the three following ways.

For each n there exists ξ such that

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1} \qquad \text{(Lagrange form)} \tag{13}$$

(ξ is between c and x.)

(The theorem with this remainder is a mean value theorem. Also, it is called Taylor's formula.)

For each n there exists ξ such that

$$R_n(x) = \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n (x-c) \qquad \text{(Cauchy form)} \tag{14}$$

$$R_n(x) = \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\, dt \qquad \text{(Integral form)} \tag{15}$$

If all the derivatives of f exist, then the following form, without remainder, may be explored:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(c)(x-c)^n \tag{16}$$

This infinite series is called a Taylor series, although when $c = 0$, it can also be referred to as a MacLaurin series or expansion.

We might be tempted to believe that if all derivatives of $f(x)$ exist at $x = c$, the expansion shown here would be valid. This, however, is not necessarily the case, for although one can then *formally* obtain the series on the right of the expansion, the resulting series may not converge to $f(x)$. For an example of this see Problem 11.108.

Precise conditions under which the series converges to $f(x)$ are best obtained by means of the theory of functions of a complex variable. (See Chapter 16.)

The determination of values of functions at desired arguments is conveniently approached through Taylor polynomials.

EXAMPLE. The value of $\sin x$ may be determined geometrically for $0, \frac{\pi}{6}$, and an infinite number of other arguments. To obtain values for other real number arguments, a Taylor series may be expanded about any of these points. For example, let $c = 0$ and evaluate several derivatives there; i.e., $f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0, f'''(0) = -\cos 0 = -1, f^{iv}(0) = \sin 0 = 0, f^{v}(0) = \cos 0 = 1$.

Thus, the MacLaurin expansion to five terms is

$$\sin x = 0 + x - 0 - \frac{1}{3!}x^3 + 0 - \frac{1}{51}x^5 + \cdots$$

Since the fourth term is 0, the Taylor polynomials P_3 and P_4 are equal, i.e.,

$$P_3(x) = p_4(x) = x - \frac{x^3}{3!}$$

and the Lagrange remainder is

$$R_4(x) = \frac{1}{5!}\cos\xi \; x^5$$

Suppose an approximation of the value of sin .3 is required. Then

$$P_4(.3) = .3 - \frac{1}{6}(.3)^3 \approx .2945.$$

The accuracy of this approximation can be determined from examination of the remainder. In particular (remember $|\cos \xi| \leq 1$),

$$|R_4| = \left|\frac{1}{5!}\cos\xi \; (.3)^5\right| \leq \frac{1}{120}\frac{243}{10^5} < .000021$$

Thus, the approximation $P_4(.3)$ for sin .3 is correct to four decimal places.

Additional insight into the process of approximation of functional values results by constructing a graph of $P_4(x)$ and comparing it to $y = \sin x$. (See Figure 11.2.)

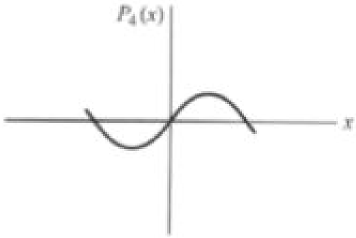

Figure 11.2

The roots of the equation are $0, \pm\sqrt{6}$. Examination of the first and second derivatives reveals a relative maximum at $x = \sqrt{2}$ and a relative minimum at $x = -\sqrt{2}$. The graph is a local approximation of the sin curve. The reader can show that $P_6(x)$ produces an even better approximation.

(For an example of series approximation of an integral, see the example that follows.)

Some Important Power Series

1. $\sin x \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots -\infty < x < \infty$

2. $\cos x \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \cdots -\infty < x < \infty$

3. $e^x \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots -\infty < x < \infty$

4. $\ln|1+x| \quad = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n-1} \frac{x^n}{n} + \cdots -1 < x \leq 1$

5. $\frac{1}{2}\ln\frac{|1+x|}{1-x} \quad = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots -1 < x < 1$

6. $\tan^{-1} x \quad = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots -1 \leq x \leq 1$

7. $(1+x)^p \quad = 1 + px + \frac{P(p-1)}{2!} x^2 + \cdots + \frac{P(p-1)\ldots(p-n+1)}{n!} x^n + \cdots$

This is the *binomial series*.

(a) If p is a positive integer or zero, the series terminates.

(b) If $p > 0$ but is not an integer, the series converges (absolutely) for $-1 \leqq x \leqq 1$.

(c) If $-1 < p < 0$, the series converges for $-1 < x \leqq 1$.

(d) If $p \leqq -1$, the series converges for $-1 < x < 1$.

For all p, the series certainly converges if $-1 < x < 1$.

EXAMPLE. Taylor's theorem applied to the series for e^x enables us to estimate the value of the integral $\int_0^1 e^{x^2}\,dx$. Substituting x^2 for x, we obtain

$$\int_0^1 e^{x^2}\,dx = \int_0^1 \left(1 + x + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{e^{\xi}}{5!} x^{10}\right) dx$$

where

$$p_4(x) = 1 + x + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \frac{1}{4!} x^8$$

and

$$R_4(x) = \frac{e^{\xi}}{5!} x^{10}, \qquad 0 < \xi < x$$

Then

$$\int_0^1 P_4(x)\,dx = 1 + \frac{1}{3} + \frac{1}{5(2!)} + \frac{1}{7(3!)} + \frac{1}{9(4!)} \approx 1.4618$$

$$\left|\int_0^1 R_4(x)\,dx\right| \leq \int_0^1 \left|\frac{e^{\xi}}{5!} x^{10}\right| dx \leq e\int_0^1 \frac{x^{10}}{5!}\,dx = \frac{e}{11.5} < .0021$$

Thus, the maximum error is less than .0021 and the value of the integral is accurate to two decimal places.

Special Topics

1. **Functions defined by series** are often useful in applications and frequently arise as solutions of differential equations. For example, the function defined by

$$J_p(x) = \frac{x^p}{2^p p!}\left\{1 - \frac{2}{2(2p+2)} + \frac{x^4}{2.4(2p+2)(2p+4)} - \dots\right\}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{p+2n}}{n!(n+p)!} \tag{16}$$

is a solution of *Bessel's differential equation* $x^2y'' + xy' + (x^2 - p^2)y = 0$ and is thus called a *Bessel function of order p*. See Problems 11.46 and 11.110 through 11.113.

Similarly, the *hypergeometric function*

$$F(a, b; c; x) = 1 + \frac{a \cdot b}{1 \cdot c}x + \frac{a(a+1)\,b\,(b+1)}{1 \cdot 2 \cdot c(c+1)}x^2 + \cdots \tag{17}$$

is a solution of *Gauss's differential equation* $x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0$.

These functions have many important properties.

2. **Infinite series of complex terms**, in particular, power series of the form $\sum_{n=0}^{\infty} a_n z^n$, where $z = x + iy$ and a_n may be complex and can be handled in a manner similar to real series.

Such power series converge for $|z| < R$; i.e., interior to a *circle of convergence* $x^2 + y^2 = R^2$, where R is the *radius of convergence* (if the series converges only for $z = 0$, we say that the radius of convergence R is zero; if it converges for all z, we say that the radius of convergence is infinite). On the boundary of this circle, i.e., $|z| = R$, the series may or may not converge, depending on the particular z.

Note that for $y = 0$ the circle of convergence reduces to the interval of convergence for real power series. Greater insight into the behavior of power series is obtained by use of the theory of functions of a complex variable (see Chapter 16).

3. **Infinite series of functions of two (or more) variables,** such as $\sum_{n=0}^{\infty} u_n(x, y)$, can be treated in a manner analogous to series in one variable. In particular, we can discuss power series in x and y having the form

$$a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \cdots$$

using double subscripts for the constants. As for one variable, we can expand suitable functions of x and y in such power series. In particular, the Taylor theroem may be extended as follows.

Taylor's Theorem (For Two Variables)

Let f be a function of two variables x and y. If all partial derivatives of order n are continuous in a closed region and if all the $(n+1)$ partial derivatives exist in the open region, then

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n \tag{18}$$

where

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \qquad 0 < \theta < 1$$

and where the meaning of the operator notation is as follows:

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f = hf_x + kf_y,$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 = h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}$$

and we formally expand $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n$ by the binomial theorem.

Note: In alternate notation $h = \Delta x = x - x_0$, $k = \Delta y = y - y_0$.
If $R_n \to 0$ as $n \to \infty$ then an unending continuation of terms produces the *Taylor series for* $f(x, y)$.
Multivariable Taylor series have a similar pattern.

4. **Double series.** Consider the array of numbers (or functions)

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots \\ u_{21} & u_{22} & u_{23} & \cdots \\ u_{31} & u_{32} & u_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Let $S_{mn} = \sum_{p=1}^{m}\sum_{q=1}^{n} u_{pq}$ be the sum of the numbers in the first m rows and first n columns of this array. If there exists a number S such that $\lim_{\substack{m\to\infty \\ n\to\infty}} S_{mn} = S$, we say that the double series $\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} u_{pq}$ *converges* to the *sum* S; otherwise, it *diverges*.

Definitions and theorems for double series are very similar to those for series already considered.

5. **Infinite products.** Let $P_n = (1 + u_1)(1 + u_1)(1 + u_2)(1 + u_3) \ldots (1 + u_n)$ denoted by $\prod_{k=1}^{n}(1 + u_k)$, where we suppose that $u_k \neq -1$, $k = 1, 2, 3, \ldots$. If there exists a number $P \neq 0$ such that $\lim_{n\to\infty} P_n = P$, we say that the *infinite product* $((1 + u_1)(1 + u_2)(1 + u_3) \ldots = \prod_{k=1}^{\infty}(1 + u_k)$, or, briefly, $\Pi(1 + u_k)$, converges to P; otherwise, it diverges.

If $\Pi(1 + |u_k|)$ converges, we call the infinite product $\Pi(1 + u_k)$ *absolutely convergent*. It can be shown that an absolutely convergent infinite product converges and that factors can in such cases be rearranged without affecting the result.

Theorems about infinite products can (by taking logarithms) often be made to depend on theorems for infinite series. Thus, for example, we have the following theorem.

Theorem A necessary and sufficient condition that $\Pi(1 + u_k)$ converge absolutely is that Σu_k converge absolutely.

6. **Summability.** Let $S_1, S_2, S_3, \ldots$ be the partial sums of a divergent series Σu_n. If the sequence $S_1, \frac{S_1 + S_2}{2}, \frac{S_1 + S_2 + S_3}{3} \cdots$ (formed by taking arithmetic means of the first n terms of $S_1, S_2, S_3, \ldots$) converges to S, we say that the series Σu_n is *summable* in the *Césaro sense*, or *C*-1 *summable* to S (see Problem 11.51).

If Σu_n converges to S, the Césaro method also yields the result S. For this reason, the Césaro method is said to be a *regular* method of summability.

In case the Césaro limit does not exist, we can apply the same technique to the sequence $S_1, \frac{S_1 + S_2}{3}, \frac{S_1 + S_2 + S_3}{3}, \cdots$ If the *C*-1 limit for this sequence exists and equals S, we say that Σu_k converges to S in the *C*-2 sense. The process can be continued indefinitely.