CHAPTER 14 - FOURIER INTEGRALS

Fourier integrals are generalizations of Fourier series. The series representation $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\}$ of a function is a periodic form on $-\infty < x < \infty$ obtained by generating the coefficients from the function's definition on the least period $[-L, L]$. If a function defined on the set of all real numbers has no period, then an analogy to Fourier integrals can be envisioned as letting $L \to \infty$ and replacing the integer valued index n by a real valued function α. The coefficients a_n and b_n then take the form $A(\alpha)$ and $B(\alpha)$. This mode of thought leads to the following definition. (See Problem 14.8.)

The Fourier Integral

Let us assume the following conditions on $f(x)$:

1. $f(x)$ satisfies the Dirichlet conditions (Page 350) in every finite interval $(-L, L)$.
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges; i.e., $f(x)$ is absolutely integrable in $(-\infty, \infty)$.

Then *Fourier's integral theorem* states that the Fourier integral of a function f is

$$f(x) = \int_0^{\infty} \{A(\alpha)\cos \alpha x + B(\alpha)\sin \alpha x\}\, d\alpha \tag{1}$$

where

$$\begin{Bmatrix} A(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos \alpha x\, dx \\ B(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin \alpha x\, dx \end{Bmatrix} \tag{2}$$

$A(\alpha)$ and $B(\alpha)$ with $-\infty < \alpha < \infty$ are generalizations of the Fourier coefficients a_n and b_n. The right-hand side of Equation (1) is also called a *Fourier integral expansion of f*. (Since Fourier integrals are improper integrals, a review of Chapter 12 is a prerequisite to the study of this chapter.) The result (*1*) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, we must replace $f(x)$ by $\frac{f(x+0)+f(x-0)}{2}$, as in the case of Fourier series. Note that these conditions are sufficient but not necessary.

In the generalization of Fourier coefficients to Fourier integrals, a_0 may be neglected, since whenever $\int_{-\infty}^{\infty} f(x)\, dx$ exists,

$$|a_0| = \left|\frac{1}{L}\int_{-L}^{L} f(x)\, dx\right| \to 0 \qquad as \qquad L \to \infty$$

Equivalent Forms of Fourier's Integral Theorem

Fourier's integral theorem can also be written in the forms

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha \tag{3}$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}d\alpha\int_{-\infty}^{\infty} f(u)e^{i\alpha u}du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(u-x)}du\,d\alpha \end{aligned} \tag{4}$$

where it is understood that if $f(x)$ is not continuous at x, the left side must be replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

These results can be simplified somewhat if $f(x)$ is either an odd or an even function, and we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\alpha x\,dx\int_0^{\infty} f(u)\cos\alpha u\,du \qquad \text{if } f(x) \text{ is even} \tag{5}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\alpha x\,dx\int_0^{\infty} f(u)\sin\alpha u\,du \qquad \text{if } f(x) \text{ is odd} \tag{6}$$

An entity of importance in evaluating integrals and solving differential and integral equations is introduced in the next paragraph. It is abstracted from the Fourier integral form of a function, as can be observed by putting Equation (4) in the form

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha x}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha u}f(u)du\right\}d\alpha$$

and identifying the parenthetic expression, as $F(\alpha)$. The following Fourier transforms result.

Fourier Transforms

From Equation (4) it follows that

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\alpha u}du \tag{7}$$

then

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}d\alpha \tag{8}$$

The function $F(\alpha)$ is called the *Fourier transform* of $f(x)$ and is sometimes written $F(\alpha) = \mathcal{F}\{f(x)\}$. The function $f(x)$ is the *inverse Fourier transform* of $F(\alpha)$ and is written $f(x) = \mathcal{F}^{-1}\{F(\alpha)\}$.

Note: The constants preceding the integral signs in Equations (7) and (8) were here taken as equal to $1/\sqrt{2\pi}$. However, they can be any constants different from zero so long as their product is $1/2\pi$. This is called the *symmetric form*. The literature is not uniform as to whether the negative exponent appears in Equation (7) or in (8).