Special Probability Distributions

Binomial Distribution: Let p be probability than an event will happen in Bernoulli trial. Then $1-p$ is the probability it will not.

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-1} = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

Example: Probability of getting exactly 2 heads in 6 tosses of a fair coin:

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!\,4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = 15/64.$$

Binomial Expansion:

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1}p + \binom{n}{2} q^{n-2}p^2 + \dots + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

Properties:

Mean $\quad \mu = np$

Variance $\quad \sigma^2 = npq$

Standard Deviation $\quad \sigma = \sqrt{npq}$

Coefficient of Skewness $\quad \alpha_3 = \dfrac{q-p}{\sqrt{npq}}$

Coefficient of Kurtosis $\quad \alpha_4 = 3 + \dfrac{1-6pq}{npq}$

Moment generating function $\quad M(t) = (q + pe^t)^n$

Characteristic function $\quad \phi(\omega) = (q + pe^{i\omega})^n$.

Example: 100 tosses of a fair coin:

expected (mean) $= np = (100)\left(\frac{1}{2}\right) = 50$

standard deviation $= \sqrt{npq} = \sqrt{(100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = 5.$

Law of Large Numbers for Bernoulli Trials: Let X be random variable in n Bernoulli trails

X/n is the proportion of successes.

Then if p is the probability of **success** and ϵ is any **positive number**:

$$\lim_{n \to \infty} P\left(\left| \frac{X}{n} - p \right| \geq \epsilon \right) = 0.$$

Strong Law: $\lim_{n \to \infty} X/n = p$ i.e. X/n **converges** to p except in a negligible number of cases.

Normal Distribution: The Normal (Gaussian) distribution function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

μ = mean, σ = **standard distribution** (or variance σ^2).

Normal Distribution Function:

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} \, dv$$

Standardized Variable: Let Z be the **standardized** variable corresponding to X:

$$Z = \frac{X-\mu}{\sigma}$$

Standarized Normal Density: Placing $\mu = 0$, $\sigma = 1$:

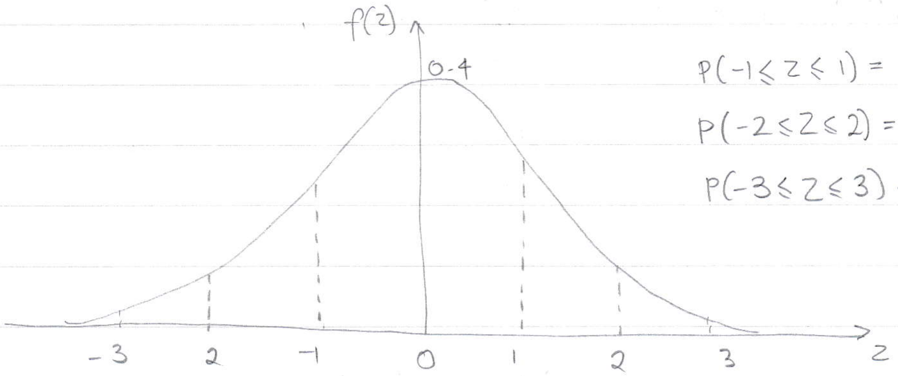
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The corresponding distribution function is:

$$F(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} \, du = \frac{1}{2} \int_{0}^{z} e^{-u^2/2} \, du.$$

Error function: $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$ and $F(z) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$

Standard Normal Curve:



$P(-1 \leq Z \leq 1) = 0.6827$

$P(-2 \leq Z \leq 2) = 0.9545$

$P(-3 \leq Z \leq 3) = 0.9973.$

Properties of Normal Distribution:

Mean	μ
Variance	σ^2
Standard Deviation	σ
Coefficient of skewness	$\alpha_3 = 0$
Coefficient of kurtosis	$\alpha_4 = 3$
Moment Generating Function	$M(t) = e^{\mu t + (\sigma^2 t^2/2)}$
Characteristic Function	$\phi(\omega) = e^{i\mu\omega - (\sigma^2\omega^2/2)}$

Note: If n is large enough and neither p nor q is too close to zero, the binomial distribution can be approximated by a normal distribution with standardised random variable given by:

$$Z = \frac{X - np}{\sqrt{npq}}$$

Proof: $\lim_{n \to \infty} P\left(a \leq \frac{X - np}{\sqrt{npq}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} du$

$(X - np)/\sqrt{npq}$ is asymptotically normal.

Poisson Distribution: Let X be random variable with probability function, and λ constant:

$$f(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, \ldots$$

Properties:

Mean	$\mu = \lambda$
Variance	$\sigma^2 = \lambda$
Standard Deviation	$\sigma = \sqrt{\lambda}$
Coefficient of skewness	$\alpha_3 = 1/\sqrt{\lambda}$
Coefficient of kurtosis	$\alpha_4 = 3 + (1/\lambda)$
Moment Generating Function	$M(t) = e^{\lambda(e^t - 1)}$
Characteristic Function	$\phi(w) = e^{\lambda(e^{iw} - 1)}$

Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be independent random variables that are identically distributed and have finite mean μ and variance σ^2.

$$S_n = X_1 + X_2 + \ldots + X_n \qquad (n = 1, 2, \ldots)$$

$$\lim_{n \to \infty} \left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} \, du.$$

Multinomial Distribution: Suppose events $A_1, A_2, \ldots, A_n$ are mutually exclusive with probabilities $p_1, p_2, \ldots, p_k$ where $P_1 + P_2 + \ldots + P_k = 1$. If $X_1, X_2, \ldots, X_n$ are the random variables respectively giving the number of times that $A_1, A_2, \ldots, A_k$ occur in a total of n trials, so that $X_1 + X_2 + \ldots + X_k = n$, then:

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k) = \frac{n}{n_1! n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$$

where $n_1 + n_2 + \ldots + n_k = n$, is the joint probability function for the random variables $X_1, \ldots, X_k$.