Calculus - Chapter 43 - Infinite Series.

Notation: Sequence $\langle s_n \rangle$ has sum $\Sigma s_n = s_1 + \ldots + s_n$, where $s_1, \ldots, s_n$ are the terms.

Converge: If S is such a number that $\lim_{n \to +\infty} S_n = S$ then Σs_n is said to converge and S is the sum.

Diverge: If there is no such number S, Σs_n is said to diverge.

Geometric Series: Sequence $\langle ar^{n-1} \rangle$ has sum Σar^{n-1} with ratio r and first term a.
It's nth partial sum S_n is:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$(\times r) \quad rS_n = ar + ar^2 + \ldots + ar^n$$

$$(1-r)S_n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

If $|r| < 1$, $\lim_{n \to +\infty} r^n = 0 \therefore \lim_{n \to +\infty} S_n = a/(1-r)$
If $|r| > 1$, $\lim_{n \to +\infty} = \infty$

Theorem: Given Σar^{n-1}.
(a). If $|r| < 1$, the series converges and has sum $a/(1-r)$
(b). If $|r| > 1$, and $r \neq 0$, the series diverges to ∞.

Example: $\Sigma (1/2)^{n-1}$ with ratio $r = 1/2$ and first term $a = 1$. $(1 + \frac{1}{2} + \frac{1}{4} + \ldots)$
The series converges and has sum

$$\frac{1}{1-(\frac{1}{2})} = 2 \quad \therefore \sum_{n=1}^{+\infty} (1/2)^{n-1} = 2$$

Theorem: If $c \neq 0$ then Σcs_n converges iff Σs_n converges, e.g.

$$\sum_{n=1}^{+\infty} cs_n = c\sum_{n=1}^{+\infty} s_n$$

To obtain result, denote $T_n = cs_1 + cs_2 + \ldots + cs_n$, then $T_n = cS_n$

So $\lim_{n\to+\infty} T_n$ exists if $\lim_{n\to+\infty} S_n$ exists.

$\lim_{n\to+\infty} T_n = c\lim_{n\to+\infty} S_n$.

Theorem: Assume Σs_n and Σt_n both converge, sum is

$$\sum_{n=1}^{+\infty}(s_n+t_n) = \sum_{n=1}^{+\infty} s_n + \sum_{n=1}^{+\infty} t_n$$

To prove: the nth partial sum U_n of $\Sigma(s_n+t_n)$ is

$$S_n + T_n = \lim_{n\to+\infty} U_n = \lim_{n\to+\infty} S_n + \lim_{n\to+\infty} T_n.$$

Corollary: Assume Σs_n and Σt_n both converge, then $\sum_{n=1}^{+\infty}(s_n-t_n)$ also converges.

$$\sum_{n=1}^{+\infty}(s_n-t_n) = \sum_{n=1}^{+\infty} s_n - \sum_{n=1}^{+\infty} t_n.$$

Theorem: If Σs_n converges, then $\lim_{n\to+\infty} s_n = 0$ i.e $\lim_{n\to+\infty} S_n = S$

To prove: assume $\sum_{n=1}^{+\infty} s_n = S$, which means $\lim_{n\to+\infty} S_n = S$

we also have $\lim_{n\to+\infty} S_{n-1} = S$.

but $s_n = S_n - S_{n-1}$

$\therefore \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S-S = 0$.

Divergence Theorem: If $\lim_{n\to+\infty} s_n$ does not exist or $\lim_{n\to+\infty} s_n \neq 0$, then Σs_n diverges

Example: Series $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\cdots$ diverges

Proof: $s_n = \frac{n}{2n+1}$, and since $\lim_{n\to+\infty}\frac{n}{2n+1} = \frac{1}{2} \neq 0$