Calculus - Chapter 46 - Power Series.

Definition: An infinite series $\sum_{n=0}^{+\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$ is a power series.

Important case: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ is power series about 0.

Example: The power series about 0 $\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + \dots$ is a geometric series with ratio x. It converges for $|x| < 1$, with sum $1/(1-x)$.

Theorem: Assume $\sum_{n=0}^{+\infty} a_n(x-c)^n$ converges for $x_0 \neq c$, then it converges absolutely $\forall x$ s.t. $|x-c| < |x_0 - c|$ e.g. $\forall x$ closer to c than x_0.

Theorem: For power series $\sum_{n=0}^{+\infty} a_n(x-c)^n$, one of cases holds:

(a). converges $\forall x$, or

(b). converges $\forall x$ in an open interval $(c-R_1, c+R_1)$ around c but not outside the closed interval $[c-R_1, c+R_2]$, or

(c). it converges only for $x = c$

By interval of convergence of $\sum_{n=0}^{+\infty} a_n(x-c)^n$ we mean:

(a). $(-\infty, \infty)$ $\Rightarrow$ radius of convergence ∞

(b). $(c-R_1, c+R_1)$ R_1

(c). $\{c\}$ 0.

Example: Power series $\sum_{n=1}^{+\infty} (x-2)/n = (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \dots$ is power series about 2.

Ratio test: $|x_{n+1}/x_n| = \frac{(x-2)^{n+1}}{n+1} \Big/ \frac{(x-2)^n}{n} = \frac{n}{n+1}(x-2)$

Thus $\lim_{n\to\infty} |x_{n+1}/x_n| = x-2$, so series converges absolutely for $|x-2| < 1$

$\Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$ $\therefore$ convergence interval is $(1,3)$, radius of convergence is 1

Example: Power series $\sum_{n=0}^{+\infty} x^n/n! = 1 + x^2/2! + x^3/3! + \dots$ is a power series about 0.

Ratio test: $|S_{n+1}/S_n| = \dfrac{|x|^{n+1}}{(n+1)!} \Big/ \dfrac{|x|}{n!} = \dfrac{|x|}{n+1}$ so $\lim_{n\to\infty} |S_{n+1}/S_n| = 0$.

Series converges absolutely $\forall x$ $\therefore$ interval of convergence $(-\infty, \infty)$ and radius of convergence ∞.

Example: $\sum_{n=0}^{+\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \dots$ is power series about 0.

$$|S_{n+1}/S_n| = \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} = (n+1)|x| \quad \therefore \lim_{n\to\infty} |S_{n+1}/S_n| = +\infty$$

except when $x = 0$ $\therefore$ series converges only for $x = 0$ and interval of convergence $\{0\}$

Uniform Convergence: Let $\langle f_n \rangle$ be sequence of functions all defined on set A, and let f be a function defined on A. Then $\langle f_n \rangle$ converges uniformly to f on A if $\forall \varepsilon > 0$ $\exists m > 0$ st $\forall x \in A$ and $\forall n \geq m$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem: If a power series $\sum_{n=0}^{+\infty} a_n(x-c)^n$ converges for $x_0 \neq c$ and $d < |x_0 - c|$, then the sequence of partial sums $\langle S_k(x) \rangle$, where $S_k(x) = \sum_{n=0}^{k} a_n(x-c)^n$ converges uniformly to $\sum_{n=0}^{+\infty} a_n(x-c)^n$ on the interval $\forall x$ s.t. $|x-c| < d$

i.e the convergence is uniform on any interval strictly inside the interval of convergence.

Theorem: If $\langle f_n \rangle$ converges uniformly to f on a set A and $\forall f_n$ continuous on A, then f is continuous at A

Corollary: The function defined by a power series $\sum_{n=0}^{+\infty} a_n(x-c)^n$ is continuous at points within its interval of convergence.

Integration: $\int f(x)\,dx = \sum_{n=0}^{+\infty} a_n \dfrac{(x-c)^{n+1}}{n+1} + K$ for $|x-c| < R_1$, with convergence radius 1.

However, if a and b are in the interval of convergence then:

$$\int_a^b f(x)\,dx = \sum_{n=0}^{+\infty} \left(a_n \frac{(x-c)^{n+1}}{n+1} \right) \Bigg|_a^b$$

Differentiation: $f'(x) = \sum_{n=0}^{+\infty} na_n(x-c)^{n-1}$, $|x-c| < R$, $(*)$

Example: For $|x|<1$, $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n = 1+x+x^2+\dots+x^n$

By $(*)$

$$D_x\left(\frac{1}{1-x}\right) = 1+2x+3x^2+\dots+nx^{n-1}+\dots \quad |x|<1$$

$$= \sum_{n=1}^{+\infty} nx^{n-1}$$

$$= \sum_{n=0}^{+\infty} (n+1)x^n$$

Example: Replace x with $-x$: $\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-x)^n = \sum_{n=0}^{+\infty} (-1)^n x^n = 1-x+x^2+\dots$

By integral th : $\int \frac{dx}{1+x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} + k = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} + k$, $|x|<1$

$$\ln|1+x| = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x|<1$$

Ratio test proves series converges.

Replacing x by $x-1$: $\ln x = \sum_{n=1}^{+\infty} (-1)^{n-1}(x-1)^n/n$, $|x-1|<1 \Leftrightarrow 0<x<2$.

So $\ln x$ is definable power series in $(0,2)$.

Abel's Theorem: Assume $\sum_{n=0}^{+\infty} a_n(x-c)^n$ has finite interval of convergence $|x-c|<R$, if power series also converges at the right endpoint $b=c+R$, the $\lim_{n\to b^-} f(x)$ exists and equal to the sum of the power series at b, similarly for LHS : $a=c-R$,

Example: $\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^n}{n}$, $|x|<1$

At RHS, $x=1$, power series becomes convergent alternating harmonic series.

$$\sum_{n=1}^{+\infty} (-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$$

By Abel's theorem : the series is equal to $\lim_{x\to 1^-} \ln(1+x) = \ln 2$,

So $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

Example: $\frac{1}{1-x} = \sum_{n=0}^{+\infty} 1 + x + x^2 + \dots, \ |x| < 1$

Replacing with $-x^2$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots, \ |-x^2| < 1 \text{ eq to } |x| < 1$$

$$\tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + K, \ |x| < 1$$

$$= K + x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots$$

Let $x = 0$, $\tan^{-1}(0) = 0$ $\therefore K = 0$:

$$\tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots$$

At RHS endpoint $x=1$ of interval of convergence, the series becomes:

$$\sum_{n=0}^{+\infty} (-1)^n \frac{1}{2n+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots$$

$$\therefore 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots = \lim_{n \to 1^-} \tan^{-1}(x) = \tan^{-1} 1 = \pi/4.$$

Example: $\sum_{n=1}^{+\infty} x^n/n!$ converges $\forall x$

$$f'(x) = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = f(x)$$

Note $f(0) = 1$ $\therefore f(x) = e^x$

Thus

$$e^x = \sum_{n=0}^{+\infty} x^n/n!$$