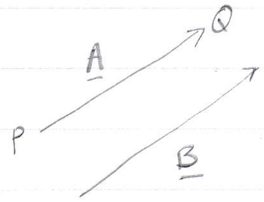
VECTORS.



Introduction

Magnitude $|\overrightarrow{PQ}|$, $|\underline{A}|$

$\underline{A} = \underline{B}$ if they have the same magnitude and direction.

Vector with same magnitude and opposite direction is $-\underline{A}$.

Difference of vectors is $\underline{A} - \underline{B}$, or $\underline{A} + (-\underline{B})$

Properties

$\underline{A} + \underline{B} = \underline{B} + \underline{A}$

$\underline{A} + (\underline{B} + \underline{C}) = (\underline{A} + \underline{B}) + \underline{C}$.

$m(n\underline{A}) = (mn)\underline{A} = n(m\underline{A})$

$(m+n)\underline{A} = m\underline{A} + n\underline{A}$

$m(\underline{A} + \underline{B}) = m\underline{A} + m\underline{B}$.

Linear independence

$\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_p$ is linearly independent that $a_1\underline{A}_1 + a_2\underline{A}_2 + \ldots + a_p\underline{A}_p = \underline{0}$

Only if $a_1 = a_2 = \ldots = a_p = 0$.

Magnitude

$A = |\underline{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

Position Vector

$r = x\underline{i} + y\underline{j} + z\underline{k}$ from O to (x, y, z).

magnitude $r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$.

Dot, Scalar, Inner Product

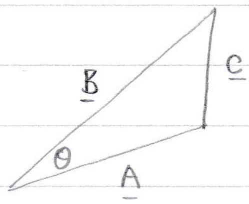
Scalar (dot) product is $\underline{A} \cdot \underline{B} = AB\cos\theta$. $0 \leq \theta \leq \pi$ (scalar, not a vector!)

$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$

if $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$, $\underline{B} = B_1\underline{i} + B_2\underline{j} + B_3\underline{k}$ then

$\underline{A} \cdot \underline{B} = A_1B_1 + A_2B_2 + A_3B_3$.



$$|\underline{C}|^2 = |\underline{A}|^2 + |\underline{B}|^2 - 2|\underline{A}||\underline{B}|\cos\theta.$$

$|\underline{B}|^2 + |\underline{A}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3)$ is magnitude

$$A_1B_1 + A_2B_2 + A_3B_3 = |\underline{A}||\underline{B}|\cos\theta.$$

Cross (vector) Product.

$$\underline{A} \times \underline{B} = AB\sin\theta\,\underline{u}, \quad 0 \leq \theta \leq \pi$$

$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$$

$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j}.$$

if $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$, $\underline{B} = B_1\underline{i} + B_2\underline{j} + B_3\underline{k}$.

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.$$

Triple Products

$$\underline{A} \times (\underline{B} \times \underline{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\underline{A} \times (\underline{B} \times \underline{C}) \neq (\underline{A} \times \underline{B}) \times \underline{C}.$$

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}.$$

$$(\underline{A} \times \underline{B}) \times \underline{C} = (\underline{A} \cdot \underline{C}) - (\underline{B} \cdot \underline{C})\underline{A}.$$

Rules

$$\underline{A} \cdot \underline{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\underline{A} \times \underline{B} = (A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1)$$

n-dim component

$$\underline{A}\,(A_1, A_2, \ldots, A_n).$$

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Limits, Continuity, Derivatives

$\underline{A}(u)$ is continuous at u_0 if given any positive number δ, we can find some positive number δ such that $|\underline{A}(u) - \underline{A}(u_0)| < \delta$ whenever $|u - u_0| < \delta$.

This is equivalent to

$$\lim_{u \to u_0} \underline{A}(u) = \underline{A}(u_0).$$

Derivative $\underline{A}(u)$ is

$$\frac{dA}{du} = \lim_{\Delta u \to 0} \frac{\underline{A}(u+\Delta u) - \underline{A}(u)}{\Delta u}$$

if $\underline{A}(u) = A_1(u)\underline{i} + A_2(u)\underline{j} + A_3(u)\underline{k}$ then

$$\frac{dA}{du} = \frac{dA_1}{du}\underline{i} + \frac{dA_2}{du}\underline{j} + \frac{dA_3}{du}\underline{k}$$

if $\underline{A}(x,y,z) = A_1(x,y,z)\underline{i} + A_2(x,y,z)\underline{j} + A_3(x,y,z)\underline{k}$ then

$$d\underline{A} = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz.$$

Rules

(a). $\frac{d}{du}(\phi\underline{A}) = \phi\frac{d\underline{A}}{du} + \frac{d\phi}{du}\underline{A}$

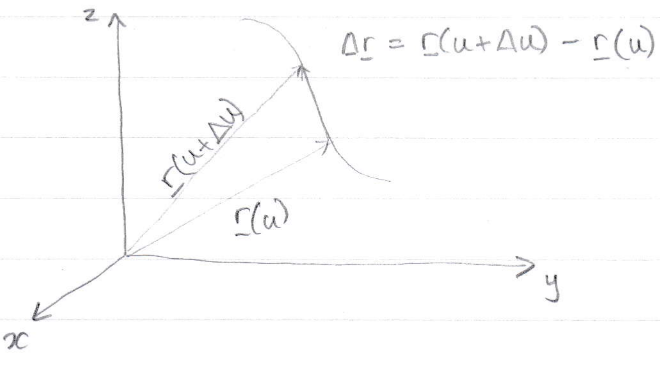
(b). $\frac{\partial}{\partial y}(\underline{A}\cdot\underline{B}) = \underline{A}\cdot\frac{\partial \underline{B}}{\partial y} + \frac{\partial \underline{A}}{\partial y}\cdot\underline{B}.$

(c). $\frac{\partial}{\partial z}(\underline{A}\times\underline{B}) = \underline{A}\times\frac{\partial \underline{B}}{\partial z} + \frac{\partial \underline{A}}{\partial z}\times\underline{B}.$

Tangent Vector

$\underline{r}$ is the vector joining origin O to (x,y,z), the position vector.
parametric equations $x = x(u)$, $y = y(u)$, $z = z(u)$.
arc length $ds^2 = d\underline{r}\cdot d\underline{r}$, thus

$$\frac{d\underline{r}}{ds} = \underline{T}$$



unit tangent vector

$$\frac{d\underline{r}}{dt} = \underline{v}$$ is the velocity with which the terminal point of $\underline{r}$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\underline{T} = v\underline{T}.$$

from which we see the magnitude of $\underline{v}$ is $v = ds/dt$, similarly

$$\frac{d^2\underline{r}}{dt^2} = \underline{a}.$$ is acceleration.

Differential of scalar field

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

Suppose function ϕ is constant on surface S and

$$C;\ x = f_1(t),\ y = f_2(t),\ z = f_3(t)$$

is a curve on S.

At any point on this curve $\frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$ lies in the tangent plane on the surface.

Thus the triple $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ is a vector perpendicular to S.

We define $\nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$. the gradient of the scalar field

"del"

Consider vector operator ∇ (del) defined by:

$$\nabla \equiv \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z},$$ $\phi(x,y,z)$ and $\underline{A}(x,y,z)$.

Laplacian.

9. $\nabla(\nabla U) \equiv \nabla^2 U \equiv \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}$, the Laplacian.

Laplacian operator

and $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, the Laplacian operator.

10. $\nabla \times (\nabla U) = 0$, the curl of the gradient of U is zero.

11. $\nabla \cdot (\nabla \times A) = 0$, the divergence of the curl of $\underline{A}$ is zero.

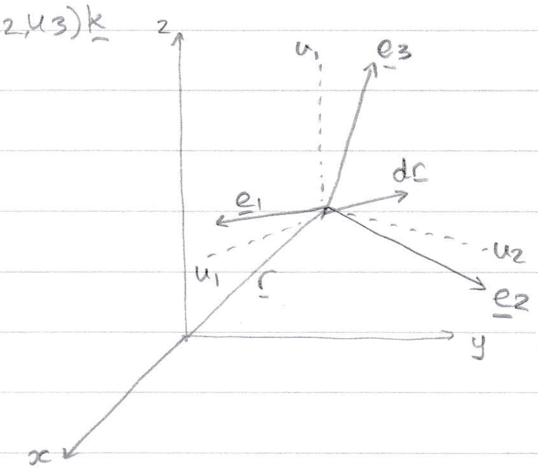
12. $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$.

Vector Interpretation

$x = f(u_1, u_2, u_3)$, $y = g(u_1, u_2, u_3)$, $z = h(u_1, u_2, u_3)$

one-to-one correspondance between xyz and $u_1 u_2 u_3$ in vector notation:

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$= f(u_1, u_2, u_3)\,\underline{i} + g(u_1, u_2, u_3)\,\underline{j} + h(u_1, u_2, u_3)\,\underline{k}$$



Curvilinear System

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3$$

Basis vectors

$\dfrac{\partial \underline{r}}{\partial u_1}, \dfrac{\partial \underline{r}}{\partial u_2}, \dfrac{\partial \underline{r}}{\partial u_3}$ is a vector basis.

Differential form of arc length

$$ds^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 + g_{33}(du_3)^2, \text{ where}$$

$$g_{11} = \frac{\partial \underline{r}}{\partial u_1} \cdot \frac{\partial \underline{r}}{\partial u_1}, \quad g_{22} = \frac{\partial \underline{r}}{\partial u_2} \cdot \frac{\partial \underline{r}}{\partial u_2}, \quad g_{33} = \frac{\partial \underline{r}}{\partial u_3} \cdot \frac{\partial \underline{r}}{\partial u_3}.$$

The vector $\partial \underline{r}/\partial u_1$ is the tangent to the u_1 coordinate curve at P.

Writing $\partial \underline{r}/\partial u_1 = h_1 \underline{e}_1$ etc.

$$d\underline{r} = h_1 du_1 \underline{e}_1 + h_2 du_2 \underline{e}_2 + h_3 du_3 \underline{e}_3.$$

gradient

$$\text{grad}\,\phi = \nabla\phi = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right)\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}.$$

$$= \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}.$$

divergence

$$\text{div}\,\underline{A} = \nabla . \underline{A} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \cdot (A_1\underline{i} + A_2\underline{j} + A_3\underline{k})$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Curl

$$\text{curl}\,\underline{A} = \nabla \times \underline{A} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \times (A_1\underline{i} + A_2\underline{j} + A_3\underline{k})$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 & A_1 \end{vmatrix} - \underline{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_1 & A_2 \end{vmatrix} + \underline{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_1 & A_2 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\underline{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\underline{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\underline{k}.$$

Formulas

1. $\nabla(U+V) = \nabla U + \nabla V$ or $\text{grad}(u+v) = \text{grad}\,u + \text{grad}\,V$
2. $\nabla\cdot(\underline{A}+\underline{B}) = \nabla\cdot\underline{A} + \nabla\cdot\underline{B}$ or $\text{div}(\underline{A}+\underline{B}) + \text{div}\underline{A} + \text{div}\underline{B}$.
3. $\nabla\times(\underline{A}+\underline{B}) = \nabla\times\underline{A} + \nabla . \underline{B}$ or $\text{curl}(\underline{A}+\underline{B}) = \text{curl}\underline{A} + \text{curl}\underline{B}$.
4. $\nabla\cdot(U\underline{A}) = (\nabla U)\cdot\underline{A} + U(\nabla\cdot\underline{A})$.
5. $\nabla\times(U\underline{A}) = (\nabla U)\times\underline{A} + U(\nabla\times\underline{A})$
6. $\nabla\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot(\nabla\times\underline{A}) - \underline{A}\cdot(\nabla\times\underline{B})$
7. $\nabla\times(\underline{A}\times\underline{B}) = (\underline{B}\cdot\nabla)\underline{A} - \underline{B}(\nabla\cdot\underline{A}) - (\underline{A}\cdot\nabla)\underline{B} + \underline{A}(\nabla\cdot\underline{B})$
8. $\nabla(\underline{A}\cdot\underline{B}) = (\underline{B}\cdot\nabla)\underline{A} - \underline{B}(\nabla\cdot\underline{A}) - (\underline{A}\cdot\nabla)\underline{B} + \underline{A}(\nabla\cdot\underline{B})$

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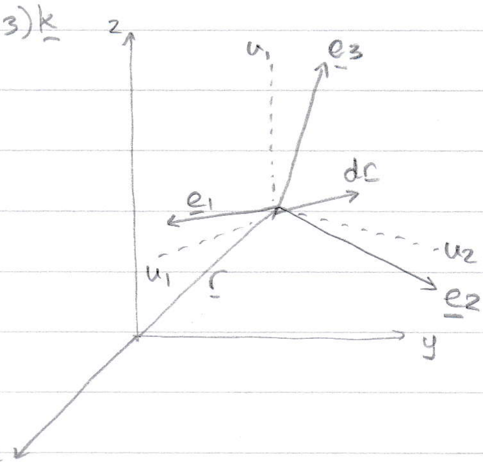

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The vector $\partial \underline{r}/\partial u_1$ is the tangent to the u_1 coordinate curve at P.

Writing $\partial \underline{r}/\partial u_1 = h_1 \underline{e}_1$ etc.

$$d\underline{r} = h_1 du_1 \underline{e}_1 + h_2 du_2 \underline{e}_2 + h_3 du_3 \underline{e}_3.$$

h_1, h_2, h_3 are scale factors

$\underline{e}_1, \underline{e}_2, \underline{e}_3$ are mutually perpendicular at any point P, orthogonal.

arc length

$$ds^2 = d\underline{r} \cdot d\underline{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Orthogonal Curvilear Coordinates

1. $$\nabla \phi = \text{grad}\, \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\underline{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\underline{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\underline{e}_3$$

2. $$\nabla \cdot \underline{A} = \text{div}\, \underline{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

3. $$\nabla \times \underline{A} = \text{curl}\, \underline{A} = \frac{1}{h_1 h_2 h_3}\begin{bmatrix} h_1 \underline{e}_1 & h_2 \underline{e}_2 & h_3 \underline{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{bmatrix}$$

4. $$\nabla^2 \phi = \text{Laplacian of } \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right.\right.$$

Cylindrical

Transformation equations: $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z,\ \rho \geq 0,\ 0 \leq \phi \leq 2\pi,\ -\infty < z <$

Scale factors: $h_1 = 1,\ h_2 = \rho,\ h_3 = 1.$

Element of arc length: $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$

Jacobian: $\dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$

Element of volume: $dV = \rho\, d\rho\, d\phi\, dz.$

Laplacian:

$$\nabla^2 U = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2} + \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} + \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$