CHAPTER

1

Systems of Linear Equations

The New River Gorge Bridge near Fayetteville, West Virginia is the world's third longest steel single-span arch bridge, and one of the highest vehicular bridges at 876 feet above the ravine floor below. Like all arch bridges, the New River Gorge Bridge transfers its weight and loads onto a horizontal thrust restrained by the abutments on both sides. Before it was completed, travelers faced a 45-minute drive along a winding road to get from one side of the New River Gorge to the other. Now it takes less than a minute. The bridge is commemorated on West Virginia's state quarter as a monumental achievement in engineering.

Bridge suggested by Matt Clay, Allegheny College (Pat & Chuck Blackley/Alamy)

There are endless applications of linear algebra in the sciences, social sciences, and business, and many are included throughout this book. Chapter 1 begins our tour of linear algebra in territory that may be familiar, systems of linear equations. In the first two sections, we develop a systematic method for finding the set of solutions to a linear system. This method can be impractical for large linear systems, so in Section 1.3 we consider numerical methods for approximating solutions that can be applied to large systems. Section 1.4 focuses on applications.

1.1 Lines and Linear Equations

The goal of this section is to provide an introduction to systems of linear equations. The following example is a good place to start. Although not complicated, it contains the essential elements of other applications and also serves as a gateway to our treatment of more general systems of linear equations.

EXAMPLE 1 Fran is designing a solar hot water system for her home. The system works by circulating a mixture of water and propylene glycol through rooftop solar panels to absorb heat, and then through a heat exchanger to heat household water (Figure 1). The glycol is included in the mixture to prevent freezing during cold weather. Table 1 shows the percentage of glycol required for various minimum temperatures.

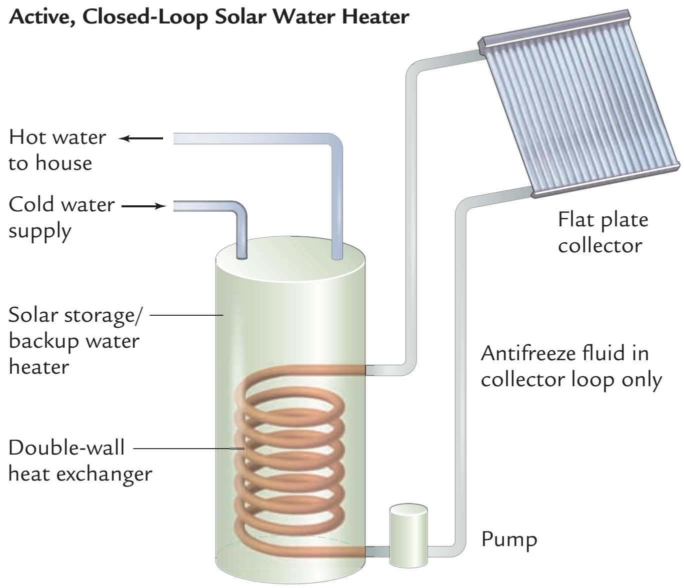


Figure 1 Schematic of a solar hot water system. (Source: U.S. Dept. of Energy).

Minimum Temp. (F)	Propylene Glycol Volume (%)
20	18
10	29
0	36
−10	42
−20	46
−30	50
−40	54
−50	57

Table 1 Percentage of Glycol Required to Prevent Freezing

The lowest the temperature ever gets at Fran's house is 0°F. Fran can purchase solutions of water and glycol that contain either 18% glycol or 50% glycol, which she will combine for her 300-liter system. How much of each type of solution is required?

Solution To solve this problem, we start by translating the given information into equations. Let x denote the required number of liters of the 18% solution, and y the required number of liters of the 50% solution. Since the system requires a total of 300 liters, it follows that

$$x + y = 300$$

To prevent freezing at 0°F, we must determine how much of each solution is needed for the mixture. From Table 1, we see that we need a 36% glycol mixture. Thus the total amount of glycol in the system must be $0.36(300) = 108$ liters. We will get $0.18x$ liters of glycol from the 18% solution and $0.50y$ liters of glycol from the 50% solution. This leads to a second equation,

$$0.18x + 0.50y = 108$$

Both $x + y = 300$ and $0.18x + 0.50y = 108$ are equations of lines. Figure 2 shows their graphs on the same set of axes. In our problem, we are looking for values of x and y that satisfy *both* equations, which means that the point with coordinates (x, y) will lie on the graph of *both* lines—that is, at the point of intersection of the two lines.

Instead of trying to determine the exact point of intersection from the graph, we use algebraic methods. Here are the two equations again,

$$\begin{aligned} x + y &= 300 \\ 0.18x + 0.50y &= 108 \end{aligned} \tag{1}$$

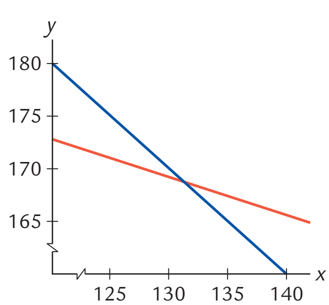


Figure 2 Graphs of $x + y = 300$ (blue) and $0.18x + 0.50y = 108$ (red) from Example 1.

We can "eliminate" x by multiplying the first equation by -0.18 and then adding it to the second equation,

$$\begin{array}{rrcrcr} & -0.18x & - & 0.18y & = & -54 \\ + & (\;\;0.18x & + & 0.50y & = & 108\;) \\ \hline \Rightarrow & & & 0.32y & = & 54 \end{array}$$

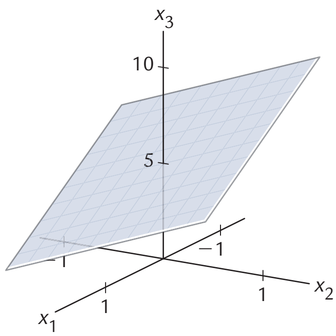


Figure 3 Graph of the solutions to $3x_1 - 2x_2 + x_3 = 5$.

Hence $y = 54/0.32 = 168.75$. Next, we substitute this back into the top equation in (1) to find x. Plugging in $y = 168.75$ gives

$$x + 168.75 = 300$$

which simplifies to $x = 131.25$. Writing our solution in the form (x, y), we have (131.25, 168.75). Referring back to Figure 2, we see that this looks like a plausible candidate for the point of intersection. We can check our answer by substituting the values $x = 131.25$ and $y = 168.75$ into the original pair of equations, to confirm that

$$131.25 + 168.75 = 300$$

and

$$0.18(131.25) + 0.50(168.75) = 108$$

This verifies that a combination of 131.25 liters of the 18% solution and 168.75 liters of the 50% solution should be used in the solar system. ■

Systems of Linear Equations

Definition **Linear Equation**

The equations in the preceding problem are examples of **linear equations**. In general, a linear equation has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \tag{2}$$

Definition **Solution of Linear Equation**

where $a_1, a_2, \cdots, a_n$ and b are constants and $x_1, x_2, \cdots, x_n$ are variables or unknowns. A **solution** $(s_1, s_2, \cdots, s_n)$ to (2) is an ordered set of n numbers (sometimes called an *n-tuple*) such that if we set $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$, then (2) is satisfied. That is, $(s_1, s_2, \cdots, s_n)$ is a solution to (2) if

$$a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$$

For example, $(-2, 5, 1, 13)$ is a solution to $3x_1 + 4x_2 - 7x_3 - 2x_4 = -19$, because

$$3(-2) + 4(5) - 7(1) - 2(13) = -19$$

Definition **Solution Set**

The **solution set** for a linear equation such as (2) consists of the set of all solutions to the equation. When the equation has two variables, the graph of the solution set is a line. In three variables, the graph of a solution set is a plane. (See Figure 3 for an example.) If $n \geq 4$, then the solution set of all points that satisfy equation (2) is called a **hyperplane**.

Definition **Hyperplane**

The set of two linear equations in (1) is an example of a *system of linear equations*. Other examples of systems of linear equations are

$$\begin{array}{rcrcrcr} -3x_1 & + & 5x_2 & - & x_3 & = & 4 \\ & - & x_2 & - & 9x_3 & = & -4 \\ 6x_1 & + & 4x_2 & - & 8x_3 & = & 11 \\ -5x_1 & - & 9x_2 & & & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcrcr} 4x_1 & - & 2x_2 & - & 8x_3 & + & 5x_4 & = & -1 \\ -x_1 & + & 7x_2 & & & + & 2x_4 & = & 13 \\ & & & & x_3 & - & 2x_4 & = & 5 \end{array} \tag{3}$$

Our standard practice is to write all systems of linear equations as shown above, aligning the variables vertically and with $x_1, x_2, \ldots$ appearing in order from left to right.

DEFINITION 1.1

Definition **System of Linear Equations**

▶ For brevity, we sometimes use "linear system" or "system" when referring to a system of linear equations.

A **system of linear equations** is a collection of equations of the form

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ \vdots & & \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{4}$$

When reading the coefficients, for a_{32} we say "a-three-two" instead of "a-thirty-two" because the "32" indicates that a_{32} is the coefficient from the third equation that is multiplied by x_2. For example, in the system on the right of (3) we have $a_{14} = 5$, $a_{22} = 7$, $a_{34} = -2$, and $a_{32} = 0$. Here $a_{32} = 0$ because there is no x_2 term in the third equation.

Definition **Solution for Linear System, Solution Set for a Linear System**

The system (4) has m equations with n unknowns. It is possible for m to be greater than, equal to, or less than n, and we will encounter all three cases. A **solution** to the linear system (4) is an n-tuple $(s_1, s_2, \cdots, s_n)$ that satisfies every equation in the system. The collection of all solutions to a linear system is called the **solution set** for the system.

In Example 1, there was exactly one solution to the linear system. This is not always the case.

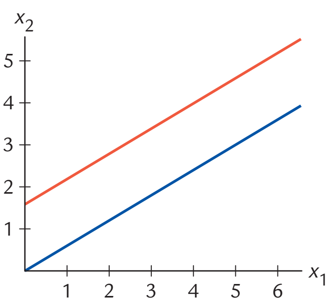


Figure 4 Graphs of $6x_1 - 10x_2 = 0$ (blue) and $-3x_1 + 5x_2 = 8$ (red) from Example 2.

EXAMPLE 2 Find all solutions to the system of linear equations

$$\begin{aligned} 6x_1 - 10x_2 &= 0 \\ -3x_1 + 5x_2 &= 8 \end{aligned} \tag{5}$$

Solution We will proceed as we did in Example 1, by eliminating a variable. This time we multiply the first equation by $\frac{1}{2}$ and then add it to the second,

$$\begin{array}{rrcl} & 3x_1 - 5x_2 & = & 0 \\ + & (-3x_1 + 5x_2 & = & 8\ ) \\ \hline & \Rightarrow \quad 0 & = & 8 \end{array}$$

The equation $0 = 8$ tells us that there are *no* solutions to the system. Why? Because if there were values of x_1 and x_2 that satisfied both the equations in (5), then we could plug them in, work through the above algebraic steps with these values in place, and *prove* that $0 = 8$, which we know is not true. So, it must be that our original assumption that there are values of x_1 and x_2 that satisfy (5) is false, and therefore we can conclude that the system has no solutions. ■

▶ This explanation gives an example of a mathematical proof technique called "proof by contradiction." You can read about this and other methods of proof in the appendix "Reading and writing proofs" posted on the text website. (See the Preface for the Web address.)

The graphs of the two equations in Example 2 are parallel lines (see Figure 4). Since the lines do not have any points in common, there cannot be values that satisfy both equations, confirming what we discovered algebraically.

Definition **Consistent Linear System, Inconsistent Linear System**

If a linear system has at least one solution, then we say that it is **consistent**. If not (as in Example 2), then it is **inconsistent**.

EXAMPLE 3 Find all solutions to the system of linear equations

$$\begin{aligned} 4x_1 + 10x_2 &= 14 \\ -6x_1 - 15x_2 &= -21 \end{aligned} \tag{6}$$

Solution This time, we multiply the first equation by $\frac{3}{2}$ and then add,

$$\begin{array}{rrcl} & 6x_1 + 15x_2 &=& 21 \\ + & (-6x_1 - 15x_2 &=& -21) \\ \hline & \Rightarrow \quad 0 &=& 0 \end{array}$$

Unlike Example 2, where we ended up with an equation that had no solutions, here we find ourselves with the equation $0 = 0$ that is satisfied by *any* choices of x_1 and x_2. This tells us that the relationship between x_1 and x_2 is the same in both equations. In this case we select one of the equations (either will work) and solve for x_1 in terms of x_2, which gives us

$$x_1 = \frac{7 - 5x_2}{2}$$

For *every* choice of x_2 there will be a corresponding choice of x_1 that satisfies the original system (6). Therefore there are infinitely many solutions. To avoid confusing variables with values satisfying the linear system, we describe the solutions to (6) by

$$\begin{aligned} x_1 &= (7 - 5s_1)/2 \\ x_2 &= s_1 \end{aligned} \tag{7}$$

Definition **Free Parameter, General Solution**

where s_1 is called a **free parameter** and can be any real number. This is known as the **general solution** because it gives all solutions to the system of equations.

We note that (7) is not the only way to describe the solutions. If we solve for x_2 instead of x_1, then we arrive at the formulation of the general solution

$$\begin{aligned} x_1 &= s_1 \\ x_2 &= (7 - 2s_1)/5 \end{aligned}$$

where, as before, s_1 is any real number. ■

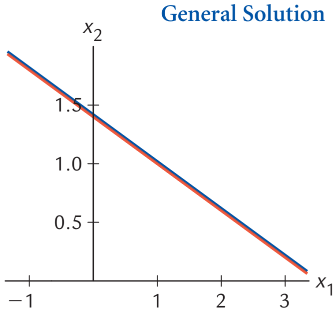


Figure 5 Graphs of $4x_1 + 10x_2 = 14$ (blue) and $-6x_1 - 15x_2 = -21$ (red) from Example 3.

Figure 5 shows the graphs of the two equations in (6). It looks like something is missing, but there is only one line because the two equations have the same graph. Since the graphs coincide, they have infinitely many points in common, which is consistent with our algebraic conclusion that there are infinitely many solutions to the system of linear equations.

In Examples 1–3, we have seen that a linear system can have a single solution, no solutions, or infinitely many solutions.

Figure 6 shows that our examples illustrate all possibilities for two lines, which are intersecting in exactly one point, being parallel and having no points in common, or coinciding and having infinitely many points of intersection. Thus it follows that a system of two linear equations with two variables can have zero, one, or infinitely many solutions.

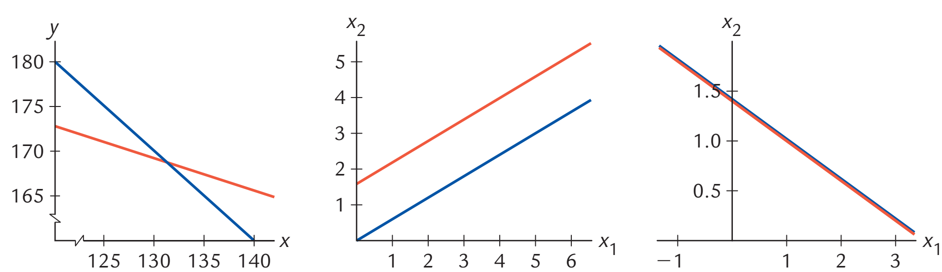
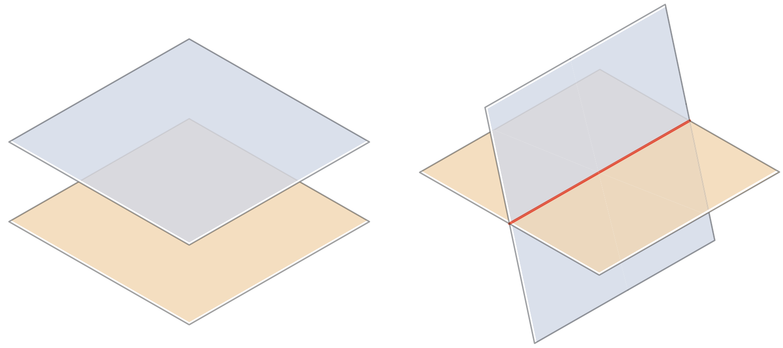


Figure 6 Graphs of equations in Examples 1–3.



(a) Parallel planes, no points in common.

(b) Planes intersect, infinitely many points in common.

Figure 7 Graphs of systems of two equations with three variables.

Now consider systems of linear equations with three variables. Recall that the graph of the solutions for each equation is a plane. To explore the solutions such a system can have, you can experiment by using a few pieces of cardboard to represent planes.

Starting with two pieces, you will quickly discover that the only two possibilities for the number of points of intersection is either none or infinitely many. (See Figure 7.)

This geometric observation is equivalent to the algebraic statement that a system of two linear equations in three variables has either no solutions or infinitely many solutions.

▶ A **theorem** is a mathematical statement that has been rigorously proved to be true. As we progress through this book, theorems will serve to organize our expanding body of linear algebra knowledge.

Now try three pieces of cardboard. There are more possible configurations, some shown in Figure 8.

This time, we see that the number of points of intersection can be zero, one, or infinitely many. (Note that this also held for a pair of lines.) In fact, this turns out to be true in general, not only for planes but also for solution sets in higher dimensions. The equivalent statement for systems of linear equations is contained in Theorem 1.2.

THEOREM 1.2 A system of linear equations has no solutions, exactly one solution, or infinitely many solutions.

We will prove this theorem at the end of the next section.

Finding Solutions: Triangular Systems

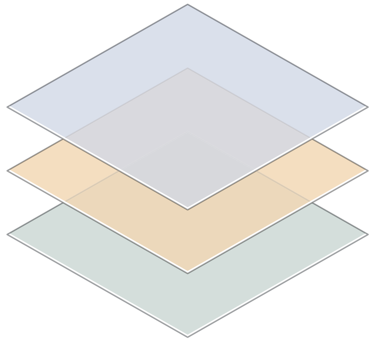
▶ In Section 1.2 we show how to generalize the results given here to find the solutions to *any* linear system.

Now that we know how many solutions a linear system can have, we turn to the problem of *finding* the solutions. For the remainder of this section, we concentrate on special types of linear systems.

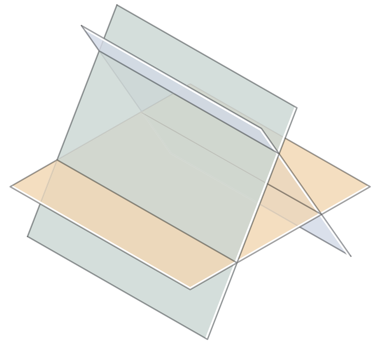
Consider the two systems below. Although not obvious, these systems have exactly the same solution set.

$$
\begin{aligned}
-2x_1 + 4x_2 + 11x_3 - 4x_4 &= 4 \\
3x_1 - 6x_2 - 15x_3 + 10x_4 &= 11 \\
2x_1 - 4x_2 - 10x_3 + 6x_4 &= 4 \\
-3x_1 + 7x_2 + 18x_3 - 13x_4 &= 1
\end{aligned}
\qquad
\begin{aligned}
x_1 - 2x_2 - 5x_3 + 3x_4 &= 2 \\
x_2 + 3x_3 - 4x_4 &= 7 \\
x_3 + 2x_4 &= -4 \\
x_4 &= 5
\end{aligned}
$$

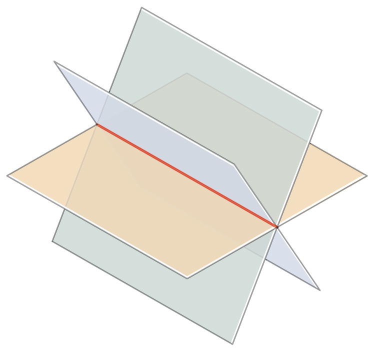
The one on the right looks easier to solve, so let's find its solutions.



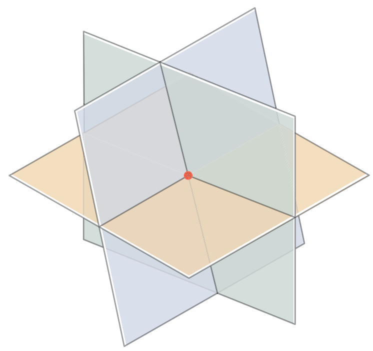
(a) Parallel planes, no points in common to all three.



(b) Planes intersect in pairs, no points in common to all three.



(c) Planes intersect in a line, infinitely many points in common.



(d) Planes intersect at a point, unique point in common.

Figure 8 Graphs of systems of three equations with three variables.

EXAMPLE 4 Find all solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 - 5x_3 + 3x_4 &= 2 \\ x_2 + 3x_3 - 4x_4 &= 7 \\ x_3 + 2x_4 &= -4 \\ x_4 &= 5 \end{aligned} \tag{8}$$

Definition Back Substitution

Solution The method that we use here is called **back substitution**. Looking at the system, we see that the easiest place to start is at the bottom. Since $x_4 = 5$, substituting this back (hence the name for the method) into the next equation up gives us

$$x_3 + 2(5) = -4$$

which simplifies to $x_3 = -14$. Now we know the values of x_3 and x_4. Substituting these back into the next equation up (second from the top) gives

$$x_2 + 3(-14) - 4(5) = 7$$

so that $x_2 = 69$. Finally, we substitute the values of x_2, x_3, and x_4 back into the top equation to get

$$x_1 - 2(69) - 5(-14) + 3(5) = 2$$

which simplifies to $x_1 = 55$. Thus this system of linear equations has one solution,

$$x_1 = 55, \quad x_2 = 69, \quad x_3 = -14, \quad x_4 = 5$$

■

Definition **Leading Variable**

In Example 4, each variable x_1, x_2, x_3, and x_4 appears as the first term of an equation. In a system of linear equations, a variable that appears as the first term in at least one equation is called a **leading variable**. Thus in Example 4 each of x_1, x_2, x_3, and x_4 is a leading variable. In the system

$$\begin{aligned} -4x_1 + 2x_2 - x_3 \qquad + 3x_5 &= 7 \\ - 3x_4 + 4x_5 &= -7 \\ x_4 - 2x_5 &= 1 \\ 7x_5 &= 2 \end{aligned} \tag{9}$$

x_1, x_4, and x_5 are leading variables, while x_2 and x_3 are not.

A key reason why the system in Example 4 is easy to solve is that every variable is a leading variable in exactly one equation. This feature is useful because as we back substitute from the bottom equation upward, at each step we are working with an equation that has only one remaining unknown variable.

Definition **Triangular Form, Triangular System**

The system in Example 4 is said to be in *triangular form*, with the name suggested by the triangular shape of the left side of the system. In general, a linear system is in **triangular form** (and is said to be a **triangular system**) if it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \ddots \qquad \vdots \quad &\quad \vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

where $a_{11}, a_{22}, \ldots, a_{nn}$ are all nonzero. It is straightforward to verify that triangular systems have the following properties.

PROPERTIES OF TRIANGULAR SYSTEMS

(a) Every variable of a triangular system is the leading variable of exactly one equation.

(b) A triangular system has the same number of equations as variables.

(c) A triangular system has exactly one solution.

Figure 9 Golden Gate Bridge. (Photo taken by John Holt.)

EXAMPLE 5 A bowling ball dropped off the Golden Gate bridge has height H (in meters) above the water at time t (in seconds after release time) given by $H(t) = at^2 + bt + c$, where a, b, and c are constants. Using ideas from calculus, it follows that the velocity is $V(t) = 2at + b$ and the acceleration is $A(t) = 2a$. At $t = 2$, it is known that the ball has height 47.4 m, velocity -19.6 m/s, and acceleration -9.8 m/s^2. (The velocity and acceleration are negative because the ball is moving in the negative direction.) What is the height of the bridge and when does the ball hit the water?

Solution We need to find the values of a, b, and c in order to answer these questions. At time $t = 2$, we have

$$47.4 = H(2) = 4a + 2b + c, \qquad -19.6 = V(2) = 4a + b, \qquad -9.8 = A(2) = 2a$$

This gives us the linear system

$$
\begin{aligned}
4a + 2b + c &= 47.4 \\
4a + b \quad &= -19.6 \\
2a \quad &= -9.8
\end{aligned}
\tag{10}
$$

▶ Our model ignores forces other than gravity. For falling objects, wind resistance can be significant. We chose to drop a bowling ball to reduce the effects of wind resistance to make the model more accurate.

Back substituting as usual, we find that

$$a = -4.9, \qquad b = 0, \qquad c = 67$$

so that the height function is $H(t) = -4.9t^2 + 67$. At time $t = 0$ the ball is just starting its descent, so the bridge has height $H(0) = 67$ meters. The ball strikes the water when $H(t) = 0$, which leads to the equation

$$-4.9t^2 + 67 = 0$$

The solution is $t = \sqrt{67/4.9} \approx 3.7$ seconds after the ball is released. ■

Finding Solutions: Echelon Systems

In the next example, we consider a linear system where each variable is a leading variable for *at most* one equation. Although this system is not quite triangular, it is close enough that the solutions still can be found using back substitution.

EXAMPLE 6 Find all solutions to the system of linear equations

$$
\begin{aligned}
2x_1 - 4x_2 + 2x_3 + x_4 &= 11 \\
x_2 - x_3 + 2x_4 &= 5 \\
3x_4 &= 9
\end{aligned}
\tag{11}
$$

Solution We find the solutions by back substituting, just like with a triangular system. Starting with the bottom equation yields $x_4 = 3$.

The middle equation has x_2 as the leading variable, but we do not yet have a value for x_3. We address this by setting $x_3 = s_1$, where s_1 is a free parameter. We now have values for both x_3 and x_4, which we substitute into the middle equation, giving us

$$x_2 - s_1 + 2(3) = 5$$

Thus $x_2 = -1 + s_1$. Substituting our values for x_2, x_3, and x_4 into the top equation, we have

$$2x_1 - 4(-1 + s_1) + 2s_1 + 3 = 11$$

which simplifies to $x_1 = 2 + s_1$. Therefore the general solution is

$$
\begin{aligned}
x_1 &= 2 + s_1 \\
x_2 &= -1 + s_1 \\
x_3 &= s_1 \\
x_4 &= 3
\end{aligned}
$$

Definition **Echelon Form, Echelon System**

where s_1 can be any real number. Note that each distinct choice for s_1 gives a new solution, so the system has infinitely many solutions. ■

▶ Note that all triangular systems are in echelon form.

The system (11) in Example 6 is in *echelon form* and is said to be an *echelon system*. Such systems have the properties given in the definition below.

DEFINITION 1.3

A linear system is in **echelon form** (and is called an **echelon system**) if

(a) Every variable is the leading variable of *at most* one equation.

(b) The system is organized in a descending "stair step" pattern so that the index of the leading variables increases from the top to bottom.

(c) Every equation has a leading variable.

For example, the systems (11) and (12) are in echelon form, but the system (9) is not, because x_4 is the leading variable of two equations.

Definition Free Variable

For a system in echelon form, any variable that is not a leading variable is called a **free variable**. For instance, x_3 is a free variable in Example 6. To find the general solution to a system in echelon form, we use the following two-step algorithm.

▶ For a system in echelon form, the total number of variables is equal to the number of leading variables plus the number of free variables.

(a) Set each free variable equal to a distinct free parameter.

(b) Back substitute to solve for the leading variables.

EXAMPLE 7 Find all solutions to the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_5 &= 7 \\ x_2 - 4x_3 + x_5 &= -2 \\ x_4 - 2x_5 &= 1 \end{aligned} \qquad (12)$$

Solution In this system x_3 and x_5 are free variables, so we set each equal to a free parameter

$$x_3 = s_1 \quad \text{and} \quad x_5 = s_2$$

It remains to determine the values of the leading variables. Substituting x_5 into the bottom equation, we have

$$x_4 - 2s_2 = 1$$

so that $x_4 = 1 + 2s_2$. Substituting our values for x_3 and x_5 into the next equation up gives

$$x_2 - 4s_1 + s_2 = -2$$

so that $x_2 = -2 + 4s_1 - s_2$. Finally, substituting in for x_2, x_3, and x_5 in the top equation, we have

$$x_1 + 2(-2 + 4s_1 - s_2) - s_1 + 3s_2 = 7$$

Hence $x_1 = 11 - 7s_1 - s_2$. Therefore the general solution is

$$\begin{aligned} x_1 &= 11 - 7s_1 - s_2 \\ x_2 &= -2 + 4s_1 - s_2 \\ x_3 &= s_1 \\ x_4 &= 1 + 2s_2 \\ x_5 &= s_2 \end{aligned}$$

where s_1 and s_2 can be any real numbers. ■

EXAMPLE 8 Find all solutions to the system of linear equations

$$\begin{aligned} x_1 - 4x_2 + x_3 + 5x_4 - x_5 &= -3 \\ - x_3 + 4x_4 + 3x_5 &= 8 \end{aligned} \tag{13}$$

Solution We see that x_2, x_4, and x_5 are free variables, so we set $x_2 = s_1$, $x_4 = s_2$, and $x_5 = s_3$, where s_1, s_2, and s_3 are free parameters.

Turning to the bottom equation, we substitute in our values for x_4 and x_5, yielding the equation

$$-x_3 + 4s_2 + 3s_3 = 8$$

so that $x_3 = -8 + 4s_2 + 3s_3$. Back substituting into the top equation gives us

$$x_1 - 4s_1 + (-8 + 4s_2 + 3s_3) + 5s_2 - s_3 = -3$$

which simplifies to $x_1 = 5 + 4s_1 - 9s_2 - 2s_3$. Therefore the general solution is

$$\begin{aligned} x_1 &= 5 + 4s_1 - 9s_2 - 2s_3 \\ x_2 &= s_1 \\ x_3 &= -8 + 4s_2 + 3s_3 \\ x_4 &= s_2 \\ x_5 &= s_3 \end{aligned}$$

where s_1, s_2, and s_3 can be any real numbers. ■

To sum up, there are two possibilities for a linear system in echelon form.

1. The system has no free variables. In this case, the system is also triangular and there is exactly one solution.
2. The system has at least one free variable. In this case, the general solution has free parameters and there are infinitely many solutions.

EXERCISES

In each exercise set, problems marked with Ⓒ are designed to be solved using a programmable calculator or computer algebra system.

1. Determine which of the points $(1, -2)$, $(-3, -3)$, and $(-2, -3)$ lie on the line $2x_1 - 5x_2 = 9$.

2. Determine which of the points $(1, -2, 0)$, $(4, 2, 1)$, and $(2, -5, 1)$ lie in the plane $x_1 - 3x_2 + 4x_3 = 7$.

3. Determine which of the points $(-1, 2)$, $(-2, 5)$, and $(1, -5)$ lie on both the lines $3x_1 + x_2 = -1$ and $-5x_1 + 2x_2 = 20$.

4. Determine which of the points $(3, 1)$, $(2, -4)$, and $(-4, 5)$ lie on both the lines $2x_1 - 5x_2 = 1$ and $-4x_1 + 10x_2 = -2$.

5. Determine which of the points $(1, 2, 3)$, $(1, -1, 1)$, and $(-1, -2, -6)$ satisfy the linear system

$$\begin{aligned} -2x_1 + 9x_2 - x_3 &= -10 \\ x_1 - 5x_2 + 2x_3 &= 4 \end{aligned}$$

6. Determine which of the points $(1, -2, -1, 3)$, $(-1, 0, 2, 1)$, and $(-2, -1, 4, -3)$ satisfy the linear system

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 3x_2 - x_4 &= -3 \end{aligned}$$

In Exercises 7–8, determine which of (a)–(d) form a solution to the given system for any choice of the free parameter(s). (HINT: All parameters of a solution must cancel completely when substituted into each equation.)

7. $-2x_1 + 3x_2 + 2x_3 = 6$

Note: This system has only one equation.

(a) $(-3 + s_1 + s_2, s_1, s_2)$

(b) $(-3 + 3s_1 + s_2, 2s_1, s_2)$

(c) $(3s_1 + s_2, 2s_1 + 2, s_2)$

(d) $(s_1, s_2, 3 - 3s_2/2 + s_1)$

8.
$$\begin{aligned} 3x_1 + 8x_2 - 14x_3 &= 6 \\ x_1 + 3x_2 - 4x_3 &= 1 \end{aligned}$$

(a) $(5 - 2s_1, 7 + 3s_1, s_1)$

(b) $(-5 - 5s_1, s_1, -(3 + s_1)/2)$

(c) $(10 + 10s_1, -3 - 2s_1, s_1)$

(d) $((6 - 4s_1)/3, s_1, -(5 - s_1)/4)$

In Exercises 9–14, find all solutions to the given system by eliminating one of the variables.

9.
$$\begin{aligned} 3x_1 + 5x_2 &= 4 \\ 2x_1 - 7x_2 &= 13 \end{aligned}$$

10.
$$\begin{aligned} -3x_1 + 2x_2 &= 1 \\ 5x_1 + x_2 &= -4 \end{aligned}$$

11.
$$\begin{aligned} -10x_1 + 4x_2 &= 2 \\ 15x_1 - 6x_2 &= -3 \end{aligned}$$

12.
$$\begin{aligned} -3x_1 + 4x_2 &= 0 \\ 9x_1 - 12x_2 &= -2 \end{aligned}$$

13.
$$\begin{aligned} 7x_1 - 3x_2 &= -1 \\ -5x_1 + 8x_2 &= 0 \end{aligned}$$

14.
$$\begin{aligned} 6x_1 - 3x_2 &= 5 \\ -8x_1 + 4x_2 &= 1 \end{aligned}$$

In Exercises 15–22, determine if the given linear system is in echelon form. If so, identify the leading variables and the free variables. If not, explain why not.

15.
$$\begin{aligned} x_1 - x_2 &= 7 \\ 7x_2 &= 0 \end{aligned}$$

16.
$$\begin{aligned} 6x_1 - 5x_2 &= 12 \\ -2x_1 + 7x_2 &= 0 \end{aligned}$$

17.
$$\begin{aligned} -7x_1 - x_2 + 2x_3 &= 11 \\ 6x_3 &= -1 \end{aligned}$$

18.
$$\begin{aligned} 3x_1 + 2x_2 + 7x_3 &= 0 \\ -3x_3 &= -3 \\ -x_2 - 4x_3 &= 13 \end{aligned}$$

19.
$$\begin{aligned} 4x_1 + 3x_2 - 9x_3 + 2x_4 &= 3 \\ 6x_2 + x_3 &= -2 \\ -5x_2 - 8x_3 + x_4 &= -4 \end{aligned}$$

20.
$$\begin{aligned} 2x_1 + 2x_3 &= 12 \\ 12x_2 - 5x_4 &= -19 \\ 3x_3 + 11x_4 &= 14 \\ -x_4 &= 3 \end{aligned}$$

21.
$$\begin{aligned} -2x_1 - 3x_2 + x_3 - 13x_4 &= 2 \\ 2x_3 &= -7 \end{aligned}$$

22.
$$\begin{aligned} -7x_1 + 3x_2 + 8x_4 - 2x_5 + 13x_6 &= -6 \\ -5x_3 - x_4 + 6x_5 + 3x_6 &= 0 \\ 2x_4 + 5x_5 &= 1 \end{aligned}$$

In Exercises 23–30, find the set of solutions for the given linear system. Note that some systems have only one equation.

23.
$$\begin{aligned} -5x_1 - 3x_2 &= 4 \\ 2x_2 &= 10 \end{aligned}$$

24.
$$\begin{aligned} x_1 + 4x_2 - 7x_3 &= -3 \\ -x_2 + 4x_3 &= 1 \\ 3x_3 &= -9 \end{aligned}$$

25. $-3x_1 + 4x_2 = 2$

26.
$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= 4 \\ -6x_3 &= -12 \end{aligned}$$

27.
$$\begin{aligned} x_1 + 5x_2 - 2x_3 &= 0 \\ -2x_2 + x_3 - x_4 &= -1 \\ x_4 &= 5 \end{aligned}$$

28. $2x_1 - x_2 + 6x_3 = -3$

29.
$$\begin{aligned} -2x_1 + x_2 + 2x_3 &= 1 \\ -3x_3 + x_4 &= -4 \end{aligned}$$

30.
$$\begin{aligned} -7x_1 + 3x_2 + 8x_4 - 2x_5 + 13x_6 &= -6 \\ -5x_3 - x_4 + 6x_5 + 3x_6 &= 0 \\ 2x_4 + 5x_5 &= 1 \end{aligned}$$

In Exercises 31–34, each linear system is not in echelon form but can be put in echelon form by reordering the equations. Write the system in echelon form, and then find the set of solutions.

31.
$$\begin{aligned} -5x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$

32.
$$\begin{aligned} -3x_3 &= -3 \\ -x_2 - 4x_3 &= 13 \\ 3x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

33.
$$\begin{aligned} 2x_2 + x_3 - 5x_4 &= 0 \\ x_1 + 3x_2 - 2x_3 + 2x_4 &= -1 \end{aligned}$$

34.
$$\begin{aligned} x_2 - 4x_3 + 3x_4 &= 2 \\ x_1 - 5x_2 - 6x_3 + 3x_4 &= 3 \\ -3x_4 &= 15 \\ 5x_3 - 4x_4 &= 10 \end{aligned}$$

35. For what value(s) of k is the linear system consistent?
$$\begin{aligned} 6x_1 - 5x_2 &= 4 \\ 9x_1 + kx_2 &= -1 \end{aligned}$$

36. For what value(s) of h is the linear system consistent?
$$\begin{aligned} 6x_1 - 8x_2 &= h \\ -9x_1 + 12x_2 &= -1 \end{aligned}$$

37. Find values of h and k so that the linear system has no solutions.
$$\begin{aligned} 2x_1 + 5x_2 &= -1 \\ hx_1 + 5x_2 &= k \end{aligned}$$

38. For what values of h and k does the linear system have infinitely many solutions?
$$\begin{aligned} 2x_1 + 5x_2 &= -1 \\ hx_1 + kx_2 &= 3 \end{aligned}$$

39. A system of linear equations is in echelon form. If there are four free variables and five leading variables, how many variables are there? Justify your answer.

40. Suppose that a system of five equations with eight unknowns is in echelon form. How many free variables are there? Justify your answer.

41. Suppose that a system of seven equations with thirteen unknowns is in echelon form. How many leading variables are there? Justify your answer.

42. A linear system is in echelon form. There are a total of nine variables, of which four are free variables. How many equations does the system have? Justify your answer.

FIND AN EXAMPLE For Exercises 43–50, find an example that meets the given specifications.

43. A linear system with three equations and three variables that has exactly one solution.

44. A linear system with three equations and three variables that has infinitely many solutions.

45. A linear system with four equations and three variables that has infinitely many solutions.

46. A linear system with three equations and four variables that has no solutions.

47. Come up with an application that has a solution found by solving an echelon linear system. Then solve the system to find the solution.

48. A linear system with two equations and two variables that has $x_1 = -1$ and $x_2 = 3$ as the only solution.

49. A linear system with two equations and three variables that has solutions $x_1 = 1$, $x_2 = 4$, $x_3 = -1$ and $x_1 = 2$, $x_2 = 5$, $x_3 = 2$.

50. A linear system with two equations and two variables that has the line $x_1 = 2x_2$ for solutions.

TRUE OR FALSE For Exercises 51–60, determine if the statement is true or false, and justify your answer.

51. A linear system with three equations and two variables must be inconsistent.

52. A linear system with three equations and five variables must be consistent.

53. There is only one way to express the general solution for a linear system.

54. A triangular system always has exactly one solution.

55. All triangular systems are in echelon form.

56. All systems in echelon form are also triangular systems.

57. A system in echelon form can be inconsistent.

58. A system in echelon form can have more equations than variables.

59. If a triangular system has integer coefficients (including the constant terms), then the solution consists of rational numbers.

60. A system in echelon form can have more variables than equations.

61. Referring to Example 1, suppose that the minimum outside temperature is 10°F. In this case, how much of each type of solution is required?

62. Referring to Example 1, suppose that the minimum outside temperature is −20°F. In this case, how much of each type of solution is required?

63. A total of 385 people attend the premiere of a new movie. Ticket prices are \$11 for adults and \$8 for children. If the total revenue is \$3974, how many adults and children attended?

64. For tax and accounting purposes, corporations depreciate the value of equipment each year. One method used is called "linear depreciation," where the value decreases over time in a linear manner. Suppose that two years after purchase, an industrial milling machine is worth \$800,000, and five years after purchase, the machine is worth \$440,000. Find a formula for the machine value at time $t \geq 0$ after purchase.

65. (Calculus required) Suppose that $f(x) = a_1e^{2x} + a_2e^{-3x}$ is a solution to a differential equation. If we know that $f(0) = 5$ and $f'(0) = -1$ (these are the *initial conditions*), what are the values of a_1 and a_2? (HINT: $f'(x) = 2a_1e^{2x} - 3a_2e^{-3x}$.)

66. An investor has \$100,000 and can invest in any combination of two types of bonds, one that is safe and pays 3% annually, and one that carries risk and pays 9% annually. The investor would like to keep risk as low as possible while realizing a 7% annual return. How much should be invested in each type of bond?

67. Degrees Fahrenheit (F) and Celsius (C) are related by a linear equation $C = aF + b$. Pure water freezes at 32°F and 0°C, and boils at 212°F and 100°C. Use this information to find a and b.

68. A 60-gallon bathtub is to be filled with water that is exactly 100°F. Unfortunately, the two sources of water available are 125°F and 60°F. When mixed, the temperature will be a weighted average based on the amount of each water source in the mix. How much of each should be used to fill the tub as specified?

69. This problem requires about 8 nickels, 8 quarters, and a sheet of 8.5-by-11-inch paper. The goal is to estimate the diameter of each type of coin as follows: Using trial and error, find a combination of nickels and quarters that, when placed side by side, extend the height (long side) of the paper. Then do the same along the width (short side) of the paper. Use the information obtained to write two linear equations involving the unknown diameters of each type of coin, then solve the resulting system to find the diameter for each type of coin.

70. The Bixby Creek Bridge is located along California's Big Sur coast and has been featured in numerous television commercials. Suppose that a bag of concrete is projected downward from the bridge deck at an initial rate of 5 meters per second. After 3 seconds, the bag is 25.9 meters from the Bixby Creek, has a velocity of −34.4 m/s, and has an acceleration of $-9.8\ \text{m/s}^2$. Use the model in Example 5 to find a formula for $H(t)$, the height at time t.

Bixby Creek Bridge. (Dennis Frates/Alamy)

C In Exercises 71–76, use computational assistance to find the set of solutions to the linear system.

71.
$$\begin{aligned} -4x_1 + 7x_2 &= -13 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

72. $$\begin{aligned} 3x_1 - 5x_2 &= 0 \\ -7x_1 - 2x_2 &= -2 \end{aligned}$$

73. $$\begin{aligned} 2x_1 - 5x_2 + 3x_3 &= 10 \\ 4x_2 - 9x_3 &= -7 \end{aligned}$$

74. $$\begin{aligned} -x_1 + 4x_2 + 7x_3 &= 6 \\ -3x_2 &= 1 \end{aligned}$$

75. $$\begin{aligned} -2x_1 - x_2 + 5x_3 + x_4 &= 20 \\ 3x_2 + 6x_4 &= 13 \\ -4x_3 + 7x_4 &= -6 \end{aligned}$$

76. $$\begin{aligned} 3x_1 + 5x_2 - x_3 - x_4 &= 17 \\ -x_2 - 6x_3 + 11x_4 &= 5 \\ 2x_3 + x_4 &= 11 \end{aligned}$$

1.2 Linear Systems and Matrices

Systems of linear equations arise naturally in many applications, but the systems rarely are in echelon form. For instance, consider the following projectile motion problem. Suppose that a cannon sits on a hill and fires a ball across a flat field below. The path of the ball is known to be approximately parabolic and so can be modeled by a quadratic function $E(x) = ax^2 + bx + c$, where E is the elevation (in feet) over position x, and a, b, and c are constants.

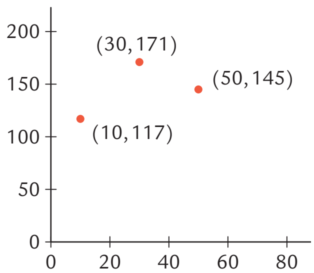


Figure 1 Positions and elevations $(x, E(x))$ of an airborne cannonball.

Figure 1 shows the elevation of the ball at three separate places. Since every point on its path is given by $(x, E(x))$, the data can be converted into three linear equations

$$\begin{aligned} 100a + 10b + c &= 117 \\ 900a + 30b + c &= 171 \\ 2500a + 50b + c &= 145 \end{aligned} \tag{1}$$

This system is not in echelon form, so back substitution is not easy to use here. We will return to this system shortly, after developing the tools to find a solution.

▶ Two linear systems are said to be **equivalent** if they have the same set of solutions.

The primary goal of this section is to develop a systematic procedure for transforming *any* linear system into a system that is in echelon form. The key feature of our transformation procedure is that it produces a new linear system that is in echelon form (hence solvable using back substitution) and has exactly the same set of solutions as the original system.

Elementary Operations

Definition Elementary Operations

We can transform a linear system using a sequence of **elementary operations**. Each operation produces a new system that is equivalent to the old one, so the solution set is unchanged. There are three types of elementary operations.

▶ The symbol $\sim$ indicates the transformation from one linear system to an equivalent linear system.

1. **Interchange the position of two equations.**

   This amounts to nothing more than rewriting the system of equations. For example, we exchange the places of the first and second equations in the following system.

$$\begin{aligned} 3x_1 - 5x_2 - 8x_3 &= -4 \\ x_1 + 2x_2 - 4x_3 &= 5 \\ -2x_1 + 6x_2 + x_3 &= 3 \end{aligned} \quad\sim\quad \begin{aligned} x_1 + 2x_2 - 4x_3 &= 5 \\ 3x_1 - 5x_2 - 8x_3 &= -4 \\ -2x_1 + 6x_2 + x_3 &= 3 \end{aligned}$$

2. **Multiply an equation by a nonzero constant.**

   For example, here we multiply the third equation by -2.

▶ Verifying that each elementary operation produces an equivalent linear system is left as Exercise 56.

$$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 5 \\ 3x_1 - 5x_2 - 8x_3 &= -4 \\ -2x_1 + 6x_2 + x_3 &= 3 \end{aligned} \quad\sim\quad \begin{aligned} x_1 + 2x_2 - 4x_3 &= 5 \\ 3x_1 - 5x_2 - 8x_3 &= -4 \\ 4x_1 - 12x_2 - 2x_3 &= -6 \end{aligned}$$

3. **Add a multiple of one equation to another.**

For this operation, we multiply one of the equations by a constant and then add it to another equation, replacing the latter with the result. For example, below we multiply the top equation by -4 and add it to the bottom equation, replacing the bottom equation with the result.

$$\begin{array}{rcrcrcr} x_1 &+& 2x_2 &-& 4x_3 &=& 5 \\ 3x_1 &-& 5x_2 &-& 8x_3 &=& -4 \\ 4x_1 &-& 12x_2 &-& 2x_3 &=& -6 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} x_1 &+& 2x_2 &-& 4x_3 &=& 5 \\ 3x_1 &-& 5x_2 &-& 8x_3 &=& -4 \\ &-& 20x_2 &+& 14x_3 &=& -26 \end{array}$$

The third operation may look familiar. It is similar to the method used in the first three examples of Section 1.1 to eliminate a variable. Note that this is exactly what happened here, with the lower left coefficient becoming zero, transforming the system closer to echelon form. This illustrates a single step of our basic strategy for transforming any linear system into a system that is in echelon form.

▶ Generic linear system

$$\begin{array}{ccccccc} a_{11}x_1 &+& a_{12}x_2 &+\cdots+& a_{1n}x_n &=& b_1 \\ a_{21}x_1 &+& a_{22}x_2 &+\cdots+& a_{2n}x_n &=& b_2 \\ a_{31}x_1 &+& a_{32}x_2 &+\cdots+& a_{3n}x_n &=& b_3 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 &+& a_{m2}x_2 &+\cdots+& a_{mn}x_n &=& b_m \end{array}$$

EXAMPLE 1 Find the set of solutions to the system of linear equations

$$\begin{array}{rcrcrcr} x_1 &-& 3x_2 &+& 2x_3 &=& -1 \\ 2x_1 &-& 5x_2 &-& x_3 &=& 2 \\ -4x_1 &+& 13x_2 &-& 12x_3 &=& 11 \end{array}$$

Solution We begin by focusing on the variable x_1 in each equation. Our goal is to transform the system to echelon form, so we want to eliminate the x_1 terms in the second and third equations. This will leave x_1 as the leading variable in only the top equation.

NOTE: Going forward, we identify coefficients using the notation for a generic system of equations introduced in Section 1.1 and shown again in the margin.

- **Add a multiple of one equation to another** (focus on x_1).

We need to transform a_{21} and a_{31} to 0. We do this in two parts. Since $a_{21} = 2$, if we take -2 times the first equation and add it to the second, then the resulting coefficient on x_1 will be $(-2) \cdot 1 + 2 = 0$, which is what we want.

$$\begin{array}{rcrcrcr} x_1 &-& 3x_2 &+& 2x_3 &=& -1 \\ 2x_1 &-& 5x_2 &-& x_3 &=& 2 \\ -4x_1 &+& 13x_2 &-& 12x_3 &=& 11 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} x_1 &-& 3x_2 &+& 2x_3 &=& -1 \\ & & x_2 &-& 5x_3 &=& 4 \\ -4x_1 &+& 13x_2 &-& 12x_3 &=& 11 \end{array}$$

The second part is similar. This time, since $(4) \cdot 1 - 4 = 0$, we multiply 4 times the first equation and add it to the third.

$$\begin{array}{rcrcrcr} x_1 &-& 3x_2 &+& 2x_3 &=& -1 \\ & & x_2 &-& 5x_3 &=& 4 \\ -4x_1 &+& 13x_2 &-& 12x_3 &=& 11 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} x_1 &-& 3x_2 &+& 2x_3 &=& -1 \\ & & x_2 &-& 5x_3 &=& 4 \\ & & x_2 &-& 4x_3 &=& 7 \end{array}$$

With these steps complete, the x_1 terms in the second and third equations are gone, exactly as we want.

Next, we focus on the x_2 coefficients. Since our goal is to reach echelon form, we do not care about the coefficient on x_2 in the top equation, so we concentrate on the second and third equations.

- **Add a multiple of one equation to another** (focus on x_2).
  Here we need to transform a_{32} to 0. Since $(-1)\cdot 1+1=0$, we multiply -1 times the second equation and add the result to the third equation.

▶ Using only the second and third equations avoids reintroducing x_1 into the third equation.

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 2x_3 & = & -1 \\ & & x_2 & - & 5x_3 & = & 4 \\ & & x_2 & - & 4x_3 & = & 7 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 2x_3 & = & -1 \\ & & x_2 & - & 5x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array}$$

The system is now in echelon (indeed, triangular) form, and using back substitution we can easily show that the solution (we know there is only one) is $x_1 = 50$, $x_2 = 19$, and $x_3 = 3$. To check our solution, we plug these values into the original system.

$$\begin{aligned} 1(50) - 3(19) + 2(3) &= -1 \\ 2(50) - 5(19) - 1(3) &= 2 \\ -4(50) + 13(19) - 12(3) &= 11 \end{aligned}$$

■

EXAMPLE 2 Find the set of solutions to the linear system 1 from the start of the section,

$$\begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ 900a & + & 30b & + & c & = & 171 \\ 2500a & + & 50b & + & c & = & 145 \end{array}$$

Solution We follow the same procedure as in the previous example.

- **Add a multiple of one equation to another** (focus on x_1).
  We need to transform a_{21} and a_{31} to 0. Since $a_{21} = 900$, we multiply the first equation by -9 and add it to the second, so that

$$\begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ 900a & + & 30b & + & c & = & 171 \\ 2500a & + & 50b & + & c & = & 145 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ 2500a & + & 50b & + & c & = & 145 \end{array}$$

The second part is similar. We multiply the first equation by -25 and add it to the third.

$$\begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ 2500a & + & 50b & + & c & = & 145 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ & - & 200b & - & 24c & = & -2780 \end{array}$$

- **Multiply an equation by a nonzero constant** (focus on x_2).
  Here we multiply the third equation by -0.3, so that the coefficients on b match up (other than sign).

$$\begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ & - & 200b & - & 24c & = & -2780 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ & & 60b & + & 7.2c & = & 834 \end{array}$$

- **Add a multiple of one equation to another** (focus on x_2).
  Thanks to the previous step, we need only add the second equation to the third to transform a_{32} to 0.

$$\begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ & & 60b & + & 7.2c & = & 834 \end{array} \quad \sim \quad \begin{array}{rcrcrcr} 100a & + & 10b & + & c & = & 117 \\ & - & 60b & - & 8c & = & -882 \\ & & & - & 0.8c & = & -48 \end{array}$$

The system is now in triangular form. Using back substitution, we can show that the solution is $a = 0.1$, $b = 6.7$, and $c = 60$, which gives us $E(x) = 0.1x^2 + 6.7x + 60$. Figure 2 shows a graph of the model together with the known points. ■

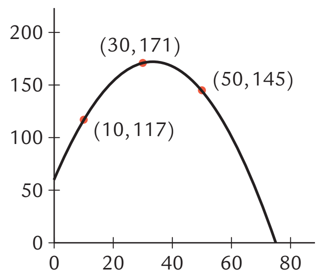


Figure 2 Cannonball data and the graph of the model.

Matrices and the Augmented Matrix

Definition Matrix

Definition Augmented Matrix

In the preceding examples, as we manipulated the equations the variables just served as placeholders. One way to simplify our work is by transferring the coefficients to a **matrix**, which for the moment we can think of as a rectangular table of numbers. When a matrix contains all the coefficients of a linear system, including the constant terms on the right side of each equation, it is called an **augmented matrix**. For instance, the system in Example 1 is transferred to an augmented matrix by

Linear System

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= -1 \\ 2x_1 - 5x_2 - x_3 &= 2 \\ -4x_1 + 13x_2 - 12x_3 &= 11 \end{aligned}$$

Augmented Matrix

$$\begin{bmatrix} 1 & -3 & 2 & -1 \\ 2 & -5 & -1 & 2 \\ -4 & 13 & -12 & 11 \end{bmatrix}$$

Definition Elementary Row Operations

The three elementary operations that we performed on equations can be translated into equivalent **elementary row operations** for matrices.[1]

ELEMENTARY ROW OPERATIONS

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another.

Definition Equivalent Matrices

Borrowing from the terminology for systems of equations, we say that two matrices are **equivalent** if one can be obtained from the other through a sequence of elementary row operations. Hence equivalent augmented matrices correspond to equivalent linear systems.

Definition Zero Row, Zero Column

When discussing matrices, the rows are numbered from top to bottom, and the columns are numbered from left to right. A **zero row** is a row consisting entirely of zeros, and a **nonzero row** contains at least one nonzero entry. The terms **zero column** and **nonzero column** are similarly defined.

In the examples that follow, we transfer the system of equations to an augmented matrix, but our goal is the same as before, to find an equivalent system in echelon form.

EXAMPLE 3 Find all solutions to the system of linear equations

$$\begin{aligned} 2x_1 - 3x_2 + 10x_3 &= -2 \\ x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Solution We begin by converting the system to an augmented matrix:

$$\begin{bmatrix} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{bmatrix}$$

[1]The plural of matrix is matrices.

▶ As with linear systems, we use the symbol $\sim$ to indicate that two matrices are equivalent.

- **Interchange rows** (focus on column 1).

  We focus on the first column of the matrix, which contains the coefficients of x_1. Although this step is not required, exchanging Row 1 and Row 2 will move a 1 into the upper left position and avoid the early introduction of fractions.

▶ We express this operation compactly as $R_1 \Leftrightarrow R_2$.

$$\begin{bmatrix} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & 4 \end{bmatrix}$$

- **Add a multiple of one row to another** (focus on column 1).

  To transform the system to echelon form, we need to introduce zeros in the first column below Row 1. This requires two operations. Focusing first on Row 2, since $(-2)(1) + 2 = 0$, we add -2 times Row 1 to Row 2 and replace Row 2 with the result.

▶ We express this operation compactly as $-2R_1 + R_2 \Rightarrow R_2$.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ -1 & 3 & 1 & 4 \end{bmatrix}$$

Focusing now on Row 3, since $(1)(1) + (-1) = 0$ we add 1 times Row 1 to Row 3 and replace Row 3 with the result.

▶ We express this operation compactly as $R_1 + R_3 \Rightarrow R_3$.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ -1 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

- **Add a multiple of one row to another** (focus on column 2).

  With the first column complete, we move down to the second row and to the right to the second column. Since $(-1)(1) + (1) = 0$, we add -1 times Row 2 to Row 3 and replace Row 3 with the result.

▶ We express this operation compactly as $-R_2 + R_3 \Rightarrow R_3$.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now extract the transformed system of equations from the matrix. The row of zeros indicates that one of the equations in the transformed system is $0 = 0$. Since any choice of values for the variables will satisfy $0 = 0$, this equation contributes no information about the solution set and so can be ignored. The new equivalent system is therefore

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 + 4x_3 &= 2 \end{aligned}$$

Back substitution can be used to show that the general solution is

$$\begin{aligned} x_1 &= 2 - 11s_2 \\ x_2 &= 2 - 4s_1 \\ x_3 &= s_1 \end{aligned}$$

where s_1 can be any real number. We can substitute into the original system to verify our solution.

$$\begin{aligned} 2(2 - 11s_1) - 3(2 - 4s_1) + 10s_1 &= 4 - 22s_1 - 6 + 12s_1 + 10s_1 = -2, \\ (2 - 11s_1) - 2(2 - 4s_1) + 3s_1 &= 2 - 11s_1 - 4 + 8s_1 + 3s_1 = -2, \\ -(2 - 11s_1) + 3(2 - 4s_1) + s_1 &= -2 + 11s_1 + 6 - 12s_1 + s_1 = 4 \end{aligned}$$

■

Gaussian Elimination

Definition **Gaussian Elimination**
Definition **Echelon Form**
Definition **Leading Term**

The procedure that we used in Example 3 is known as **Gaussian elimination**. The resulting matrix is said to be in **echelon form** (or **row echelon form**) and will have the properties given in Definition 1.4 below. In the definition, the **leading term** of a row is the leftmost nonzero term in that row, and a row of all zeros has no leading term.

DEFINITION 1.4

A matrix is in **echelon form** if

(a) Every leading term is in a column to the left of the leading term of the row below it.

(b) Any zero rows are at the bottom of the matrix.

▶ Gaussian elimination was originally discovered by Chinese mathematicians over 2000 years ago. It is named in honor of German mathematician Carl Friedrich Gauss, who independently discovered the method and introduced it to the West in the nineteenth century.

Note that the first condition in the definition implies that a matrix in echelon form will have zeros filling out the column below each of the leading terms. Examples of matrices in echelon form are

$$\begin{bmatrix} \mathbf{5} & 1 & -4 & 0 & 9 & 2 \\ 0 & \mathbf{2} & -3 & -6 & 7 & 31 \\ 0 & 0 & 0 & \mathbf{-2} & 4 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{-1} & 2 & 3 & -2 & 17 & 9 & 7 \\ 0 & 0 & \mathbf{9} & -6 & 26 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Definition **Pivot Positions**

For a matrix in echelon form, the **pivot positions** are those that contain a leading term. The entries in the pivot positions for the matrices in (2) are shown in boldface. The **pivot columns** are the columns that contain pivot positions, and a **pivot** is a nonzero number in a pivot position that is used during row operations to produce zeros.

Definition **Pivot Columns, Pivot**

In what follows, it will be handy to have a general matrix to refer to when talking about entries in specific positions. We adopt a notation similar to that for a general system of equations given in (4) of Section 1.1,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix}$$

In the previous example, the linear system had the same number of equations and variables. However, this is not required.

EXAMPLE 4 Find all solutions to the system of linear equations

$$\begin{aligned} 6x_3 \qquad + 19x_5 + 11x_6 &= -27 \\ 3x_1 + 12x_2 + 9x_3 - 6x_4 + 26x_5 + 31x_6 &= -63 \\ x_1 + 4x_2 + 3x_3 - 2x_4 + 10x_5 + 9x_6 &= -17 \\ -x_1 - 4x_2 - 4x_3 + 2x_4 - 13x_5 - 11x_6 &= 22 \end{aligned}$$

Solution The augmented matrix for this system is

$$\begin{bmatrix} 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{bmatrix}$$

- **Identify pivot position for Row 1.**

  Starting with the first column, we see that $a_{11} = 0$, which will not work for a pivot. However, there are nonzero terms down the first column, so we interchange Row 1 and Row 3 to place a 1 in the pivot position.

▶ The operation is $R_1 \Leftrightarrow R_3$.

$$\begin{bmatrix} 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{bmatrix}$$

- **Elimination.**

  Next, we need zeros down the first column below the pivot position. We already have $a_{31} = 0$, and we arrange for $a_{21} = 0$ and $a_{41} = 0$ by using the operations shown in the margin.

▶ The *elimination* steps are used to "eliminate" coefficients by transforming them to zero.

▶ The operations are

$$\begin{aligned} -3R_1 + R_2 &\Rightarrow R_2 \\ R_1 + R_4 &\Rightarrow R_4 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \end{bmatrix}$$

- **Identify pivot position for Row 2.**

  Moving down one row and to the right one column from a_{11}, we find $a_{22} = 0$. Since all the entries below a_{22} are also zero, interchanging with lower rows will not put a nonzero term in the a_{22} position. Thus a_{22} cannot be a pivot position, so we move to the right to the third column to determine if a_{23} is a suitable pivot position. Although a_{23} is also zero, there are nonzero terms below, so we interchange Row 2 and Row 4, putting a -1 in the pivot position.

▶ Do not be tempted to perform the operation $R_1 \Leftrightarrow R_2$. This will undo the zeros in the first column.

▶ The operation is $R_2 \Leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{bmatrix}$$

- **Elimination.**

  Down the remainder of the third column, we already have $a_{43} = 0$, so we need only introduce a zero at a_{33} by using the operation shown in the margin.

▶ The operation is $6R_2 + R_3 \Rightarrow R_3$.

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{bmatrix}$$

- **Identify pivot position for Row 3.**

  From the Row 2 pivot position, we move down one row and to the right one column to a_{34}. This entry is 0, as is the entry below, so interchanging rows will not yield an acceptable pivot. As we did before, we move one column to the right. Since $a_{35} = 1$ is nonzero, this becomes the pivot for Row 3.

- **Elimination.**

  We introduce a zero in the a_{45} position by using the operation shown in the margin.

▶ The operation is $4R_3 + R_4 \Rightarrow R_4$.

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Identify pivot position for Row 4.**

  Since Row 4 is the only remaining row and consists entirely of zeros, it has no pivot position. The matrix is now in echelon form, so no additional row operations are required. Converting the augmented matrix back to a linear system gives us

$$\begin{aligned} x_1 + 4x_2 + 3x_3 - 2x_4 + 10x_5 + 9x_6 &= -17 \\ - x_3 \qquad\quad - 3x_5 - 2x_6 &= 5 \\ x_5 - x_6 &= 3 \end{aligned}$$

Using back substitution, we arrive at the general solution

$$\begin{aligned} x_1 &= -5 - 4s_1 + 2s_2 - 4s_3 \\ x_2 &= s_1 \\ x_3 &= -14 - 5s_3 \\ x_4 &= s_2 \\ x_5 &= 3 + s_3 \\ x_6 &= s_3 \end{aligned}$$

where s_1, s_2, and s_3 can be any real numbers. ■

Gaussian elimination can be applied to any matrix to find an equivalent matrix that is in echelon form. If matrix A is equivalent to matrix B that is in echelon form, we say that B is an echelon form of A. Different sequences of row operations can produce different echelon forms of the same starting matrix, but all echelon forms of a given matrix will have the same pivot positions.

EXAMPLE 5 Use Gaussian elimination to find all solutions to the system of linear equations

$$\begin{aligned} x_1 + 4x_2 - 3x_3 &= 2 \\ 3x_1 - 2x_2 - x_3 &= -1 \\ -x_1 + 10x_2 - 5x_3 &= 3 \end{aligned}$$

Solution The augmented matrix for this system is

$$\begin{bmatrix} 1 & 4 & -3 & 2 \\ 3 & -2 & -1 & -1 \\ -1 & 10 & -5 & 3 \end{bmatrix}$$

- **Identify pivot position for Row 1, then elimination.**

  We have $a_{11} = 1$, so this is the pivot position for Row 1. We introduce zeros down the first column with the row operations shown in the margin.

▶ The operations are
$$\begin{aligned} -3R_1 + R_2 &\Rightarrow R_2 \\ R_1 + R_3 &\Rightarrow R_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & -3 & 2 \\ 3 & -2 & -1 & -1 \\ -1 & 10 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & -14 & 8 & -7 \\ 0 & 14 & -8 & 5 \end{bmatrix}$$ ■

- **Identify pivot position for Row 2, then elimination.**

▶ The operation is $R_2 + R_3 \Rightarrow R_3$.

$$\begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & -14 & 8 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Before continuing, let's consider what we have. We find ourselves with a matrix in echelon form, but when we translate the last row back into an equation, we get $0 = -2$, which clearly has no solutions. Thus this system has no solutions, and so is inconsistent. ■

The preceding example illustrates a general principle. When applying row operations to an augmented matrix, if at any point in the process the matrix has a row of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix} \tag{3}$$

where c is nonzero, then stop. The system is inconsistent.

▶ Gauss–Jordan elimination is named for the previously encountered C. F. Gauss, and Wilhelm Jordan (1842–1899), a German engineer who popularized this method for finding solutions to linear systems in his book on geodesy (the science of measuring earth shapes).

Gauss–Jordan Elimination

Let's return to the echelon form of the augmented matrix from Example 4,

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

After extracting the linear system from this matrix, we back substituted and simplified to find the general solution. We can make it easier to find the general solution by performing additional row operations on the matrix. Specifically, we do the following:

1. Multiply each nonzero row by the reciprocal of the pivot so that we end up with a 1 as the leading term in each nonzero row.
2. Use row operations to introduce zeros in the entries *above* each pivot position.

Picking up with our matrix, we see that the first and third rows already have a 1 in the pivot position. Multiplying the second row by -1 takes care of the remaining nonzero row.

▶ The operation is $-R_2 \Rightarrow R_2$.

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 1 & 0 & 3 & 2 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When implementing Gaussian elimination, we worked from left to right. To put zeros above pivot positions, we work from right to left, starting with the rightmost pivot, which in this case appears in the fifth column. Two row operations are required to introduce zeros above this pivot.

▶ The operations are

$$\begin{aligned} -3R_3 + R_2 &\Rightarrow R_2 \\ -10R_3 + R_1 &\Rightarrow R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 1 & 0 & 3 & 2 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 0 & 19 & -47 \\ 0 & 0 & 1 & 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we move up to the pivot in the second row, located in the third column. One row operation is required to introduce a zero in the a_{13} position.

▶ The operation is
$-3R_2 + R_1 \Rightarrow R_1$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 0 & 19 & -47 \\ 0 & 0 & 1 & 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & -2 & 0 & 4 & -5 \\ 0 & 0 & 1 & 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Naturally there are no rows above the pivot position in the first row, so we are done. Now when we extract the linear system, it has the form

▶ When using Gaussian and Gauss-Jordan elimination, do not yield to the temptation to alter the order of the row operations. Changing the order can result in a "circular" sequence of operations that lead nowhere.

$$\begin{aligned} x_1 + 4x_2 \qquad - 2x_4 \qquad + 4x_6 &= -5 \\ x_3 \qquad\qquad + 5x_6 &= -14 \\ x_5 - \ x_6 &= 3 \end{aligned}$$

Note that when the system is expressed in this form, the leading variables appear *only* in the equation that they lead. Thus during back substitution we need only plug in free parameters and then subtract to solve for the leading variables, simplifying the process considerably.

The matrix on the right in (4) is said to be in *reduced echelon form.*

DEFINITION 1.5

Definition Reduced Echelon Form

A matrix is in **reduced echelon form** (or **reduced row echelon form**) if

(a) It is in echelon form.

(b) All pivot positions contain a 1.

(c) The only nonzero term in a pivot column is in the pivot position.

Examples of matrices in reduced echelon form include

$$\begin{bmatrix} 0 & 1 & 0 & -2 & 0 & 0 & 17 \\ 0 & 0 & 1 & -6 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -3 & 0 & 0 & -7 & 21 \\ 0 & 0 & 1 & 0 & 2 & 13 \\ 0 & 0 & 0 & 1 & 5 & -9 \end{bmatrix}.$$

Definition Forward Phase, Backward Phase

Definition Gauss–Jordan Elimination

Transforming a matrix to reduced echelon form can be viewed as having two parts: The **forward phase** is Gaussian elimination, transforming the matrix to echelon form, and the **backward phase**, which completes the transformation to reduced echelon form. The combination of the forward and backward phases is referred to as **Gauss–Jordan elimination**. Although a given matrix can be equivalent to many different echelon form matrices, the same is not true of reduced echelon form matrices.

THEOREM 1.6

A given matrix is equivalent to a unique matrix that is in reduced echelon form.

The proof of this theorem is omitted.

EXAMPLE 6 Use Gauss–Jordan elimination to find all solutions to the system of linear equations

▶ Going forward, we omit detailed explanations and instead just show the row operations in the order performed.

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 \\ x_1 - \ x_2 - 2x_3 &= 1 \\ -x_1 + 3x_2 + 5x_3 &= 2 \end{aligned}$$

Solution The augmented matrix and row operations are shown below.

$$\begin{bmatrix} 1 & -2 & -3 & -1 \\ 1 & -1 & -2 & 1 \\ -1 & 3 & 5 & 2 \end{bmatrix} \begin{array}{c} -R_1+R_2 \Rightarrow R_2 \\ R_1+R_3 \Rightarrow R_3 \\ \sim \end{array} \begin{bmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} -R_2+R_3 \Rightarrow R_3 \\ \sim \end{array} \begin{bmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

That completes the forward phase, yielding a matrix in echelon form. Next, we implement the backward phase to transform the matrix to reduced echelon form.

$$\begin{bmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} -R_3+R_2 \Rightarrow R_2 \\ 3R_3+R_1 \Rightarrow R_1 \\ \sim \end{array} \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} 2R_2+R_1 \Rightarrow R_1 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The reduced echelon form is equivalent to the linear system

$$\begin{array}{rcl} x_1 & = & 2 \\ x_2 & = & 3 \\ x_3 & = & -1 \end{array}$$

We see immediately that the system has unique solution $x_1 = 2, x_2 = 3$, and $x_3 = -1$. ■

Homogeneous Linear Systems

A linear equation is **homogeneous** if it has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Definition Homogeneous Equation, Homogeneous System

Homogeneous linear systems are an important class of systems that are made up of homogeneous linear equations.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + \cdots + & a_{mn}x_n & = & 0 \end{array}$$

Note that all homogeneous systems are consistent, because there is always one easy solution, namely,

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_n = 0$$

Definition Trivial Solution, Nontrivial Solution

This is called the **trivial solution**. If there are additional solutions, they are called **nontrivial solutions**. We determine if there are nontrivial solutions in the usual way, using elimination methods.

EXAMPLE 7 Use Gauss–Jordan elimination to find all solutions to the homogeneous system of linear equations

$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 0 \\ x_1 - 3x_2 - x_3 + 6x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

Solution As the system is homogeneous, we know that it has the trivial solution. To find the other solutions, we load the system into an augmented matrix and transform to reduced echelon form.

$$\begin{bmatrix} 2 & -6 & -1 & 8 & 0 \\ 1 & -3 & -1 & 6 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & -3 & -1 & 6 & 0 \\ 2 & -6 & -1 & 8 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{bmatrix}$$

$$\underset{\sim}{\overset{-2R_1+R_2 \Rightarrow R_2}{R_1+R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & -2 & 8 & 0 \end{bmatrix}$$

$$\overset{2R_2+R_3 \Rightarrow R_3}{\sim} \begin{bmatrix} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2+R_1 \Rightarrow R_1}{\sim} \begin{bmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced echelon form. The corresponding linear system is

$$\begin{aligned} x_1 - 3x_2 \qquad + 2x_4 &= 0 \\ x_3 - 4x_4 &= 0 \end{aligned}$$

Back substituting yields the general solution

$$\begin{aligned} x_1 &= 3s_1 - 2s_2 \\ x_2 &= s_1 \\ x_3 &= 4s_2 \\ x_4 &= s_2 \end{aligned}$$

where s_1 and s_2 can be any real numbers. ■

Proof of Theorem 1.2

We are now in a position to revisit and prove Theorem 1.2 from Section 1.1. Recall the statement of the theorem.

THEOREM 1.2 A system of linear equations has no solutions, exactly one solution, or infinitely many solutions.

Proof We can take any linear system, form the augmented matrix, use Gaussian elimination to reduce to echelon form, and extract the transformed system. There are three possible outcomes from this process:

(a) The system has an equation of the form $0 = c$ for $c \neq 0$. In this case, the system has no solutions.

If (a) does not occur, then one of (b) or (c) must:

(b) The transformed system is triangular, and thus has no free variables and hence exactly one solution.

(c) The transformed system is not triangular, and so has one or more free variables and hence infinitely many solutions.

Homogeneous linear systems are even simpler. Since all such systems have the trivial solution, (a) cannot happen. Therefore a homogeneous linear system has either a unique solution or infinitely many solutions. ■

Computational Comments

- We can find the solutions to any system by using either Gaussian elimination or Gauss–Jordan elimination. Which is better? For a system of n equations with n unknowns, Gaussian elimination requires approximately $\frac{2}{3}n^3$ flops (i.e., arithmetic operations) and Gauss–Jordan requires about n^3 flops. Back substitution is slightly more complicated for Gaussian elimination than for Gauss–Jordan, but overall Gaussian elimination is more efficient and is the method that is usually implemented in computer software.

- When elimination methods are implemented on computers, to control round-off error they typically include an extra step called "partial pivoting," which involves selecting the entry having the largest absolute value to serve as the pivot. When performing row operations by hand, partial pivoting tends to introduce fractions and leads to messy calculations, so we avoided the topic. However, it is discussed in the next section.

▶ There are various similar definitions for what constitutes a "flop." Here we take a "flop" to be one arithmetic operation, either addition or multiplication. Counting flops gives a measure of algorithm efficiency.

EXERCISES

In each exercise set, problems marked with Ⓒ are designed to be solved using a programmable calculator or computer algebra system.

In Exercises 1–4, convert the given augmented matrix to the equivalent linear system.

1. $\begin{bmatrix} 4 & 2 & -1 & 2 \\ -1 & 0 & 5 & 7 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 1 & 0 \\ 13 & -3 & 6 \\ -11 & 7 & -5 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 12 & -3 & -9 & 17 \\ -12 & 5 & -3 & 11 & 0 \\ 6 & 8 & 2 & 10 & -8 \\ 17 & 0 & 0 & 13 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 2 \\ 5 & -7 \\ 3 & 0 \end{bmatrix}$

In Exercises 5–10, determine those matrices that are in echelon form, and those that are also in reduced echelon form.

5. $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 3 & 0 & 6 & -2 \\ 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 & 1 & -7 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 & 5 & -1 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & -3 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 0 & 9 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 11–14, the matrix on the right results after performing a single row operation on the matrix on the left. Identify the row operation.

11. $\begin{bmatrix} -2 & 1 & 0 \\ 13 & -3 & 6 \\ -11 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 0 \\ 13 & -3 & 6 \\ -11 & 7 & -5 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & -1 & 2 \\ -1 & 0 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 14 & 23 \\ -1 & 0 & 5 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & 9 & -2 & 3 \\ 6 & 7 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & 9 & -2 & 3 \\ -2 & -11 & 9 & -7 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & 9 & -2 & 3 \\ 6 & 7 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 6 & 7 & 5 & -1 \\ 4 & 9 & -2 & 3 \\ 2 & -1 & 3 & 0 \end{bmatrix}$

In Exercises 15–18, a single row operation was performed on the matrix on the left to produce the matrix on the right. Unfortunately, an error was made when performing the row operation. Identify the operation and fix the error.

15. $\begin{bmatrix} 3 & 7 & -2 \\ -1 & 4 & 3 \\ 5 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & -3 \\ 3 & 7 & -2 \\ 5 & 0 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -2 & -2 & 1 & 6 \\ 4 & -1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & -2 & 1 & 6 \\ 0 & 5 & 2 & 7 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 3 & -1 & 2 \\ -1 & -9 & 4 & 1 \\ 5 & 0 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & -2 & 4 \\ -1 & -9 & 4 & 1 \\ 5 & 0 & 7 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 7 & 2 & 0 \\ 0 & 4 & -8 & -3 \\ 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 & 0 \\ 0 & 4 & -8 & -3 \\ 1 & -14 & 0 & 1 \end{bmatrix}$

In Exercises 19–26, convert the given system to an augmented matrix and then find all solutions by reducing the system to echelon form and back substituting.

19. $\begin{aligned} 2x_1 + x_2 &= 1 \\ -4x_1 - x_2 &= 3 \end{aligned}$

20. $\begin{aligned} 3x_1 - 7x_2 &= 0 \\ x_1 + 4x_2 &= 0 \end{aligned}$

21. $\begin{aligned} -2x_1 + 5x_2 - 10x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 7x_1 - 17x_2 + 34x_3 &= -16 \end{aligned}$

22. $\begin{aligned} 2x_1 + 8x_2 - 4x_3 &= -10 \\ -x_1 - 3x_2 + 5x_3 &= 4 \end{aligned}$

23. $\begin{aligned} 2x_1 + 2x_2 - x_3 &= 8 \\ -x_1 - x_2 \phantom{{}+x_3} &= -3 \\ 3x_1 + 3x_2 + x_3 &= 7 \end{aligned}$

24. $\begin{aligned} -5x_1 + 9x_2 &= 13 \\ 3x_1 - 5x_2 &= -9 \\ x_1 - 2x_2 &= -2 \end{aligned}$

25. $\begin{aligned} 2x_1 + 6x_2 - 9x_3 - 4x_4 &= 0 \\ -3x_1 - 11x_2 + 9x_3 - x_4 &= 0 \\ x_1 + 4x_2 - 2x_3 + x_4 &= 0 \end{aligned}$

26. $\begin{aligned} x_1 - x_2 - 3x_3 - x_4 &= -1 \\ -2x_1 + 2x_2 + 6x_3 + 2x_4 &= -1 \\ -3x_1 - 3x_2 + 10x_3 \phantom{{}+2x_4} &= 5 \end{aligned}$

In Exercises 27–30, convert the given system to an augmented matrix and then find all solutions by transforming the system to reduced echelon form and back substituting.

27. $\begin{aligned} -2x_1 - 5x_2 &= 0 \\ x_1 + 3x_2 &= 1 \end{aligned}$

28. $\begin{aligned} -4x_1 + 2x_2 - 2x_3 &= 10 \\ x_1 \phantom{{}+2x_2} + x_3 &= -3 \\ 3x_1 - x_2 + x_3 &= -8 \end{aligned}$

29. $\begin{aligned} 2x_1 + x_2 \phantom{{}-x_3} &= 2 \\ -x_1 - x_2 - x_3 &= 1 \end{aligned}$

30. $\begin{aligned} -3x_1 + 2x_2 - x_3 + 6x_4 &= -7 \\ 7x_1 - 3x_2 + 2x_3 - 11x_4 &= 14 \\ x_1 \phantom{{}-3x_2+2x_3} - x_4 &= 1 \end{aligned}$

For each of Exercises 31–36, suppose that the given row operation is used to transform a matrix. Which row operation will transform the matrix back to its original form?

31. $5R_1 \Longrightarrow R_1$

32. $-2R_3 \Longrightarrow R_3$

33. $R_1 \Longleftrightarrow R_3$

34. $R_4 \Longleftrightarrow R_1$

35. $-5R_2 + R_6 \Longrightarrow R_6$

36. $-3R_1 + R_3 \Longrightarrow R_3$

FIND AN EXAMPLE For Exercises 37–42, find an example that meets the given specifications.

37. A matrix with three rows and five columns that is in echelon form, but not in reduced echelon form.

38. A matrix with six rows and four columns that is in echelon form, but not in reduced echelon form.

39. An augmented matrix for an inconsistent linear system that has four equations and three variables.

40. An augmented matrix for an inconsistent linear system that has three equations and four variables.

41. A homogeneous linear system with three equations, four variables, and infinitely many solutions.

42. Two matrices that are distinct yet equivalent.

TRUE OR FALSE For Exercises 43–50, determine if the statement is true or false, and justify your answer.

43. If two matrices are equivalent, then one can be transformed into the other with a sequence of elementary row operations.

44. Different sequences of row operations can lead to different echelon forms for the same matrix.

45. Different sequences of row operations can lead to different reduced echelon forms for the same matrix.

46. If a linear system has four equations and seven variables, then it must have infinitely many solutions.

47. If a linear system has seven equations and four variables, then it must be inconsistent.

48. Every linear system with free variables has infinitely many solutions.

49. Any linear system with more variables than equations cannot have a unique solution.

50. If a linear system has the same number of equations and variables, then it must have a unique solution.

51. Suppose that the echelon form of an augmented matrix has a pivot position in every column except the rightmost one. How many solutions does the associated linear system have? Justify your answer.

52. Suppose that the echelon form of an augmented matrix has a pivot position in every column. How many solutions does the associated linear system have? Justify your answer.

53. Show that if a linear system has two different solutions, then it must have infinitely many solutions.

54. Show that if a matrix has more rows than columns and is in echelon form, then it must have at least one row of zeros at the bottom.

55. Show that a homogeneous linear system with more variables than equations must have an infinite number of solutions.

56. Show that each of the elementary operations on linear systems (see pages 14–15) produces an equivalent linear system. (Recall two linear systems are equivalent if they have the same solution set.)

(a) Interchange the position of two equations.

(b) Multiply an equation by a nonzero constant.

(c) Add a multiple of one equation to another.

In Exercises 57–58 you are asked to find an *interpolating polynomial*, which is used to fit a function to a set of data.

57. Figure 3 shows the plot of the points (1, 4), (2, 7), and (3, 14). Find a polynomial of degree 2 of the form $f(x) = ax^2 + bx + c$ whose graph passes through these points.

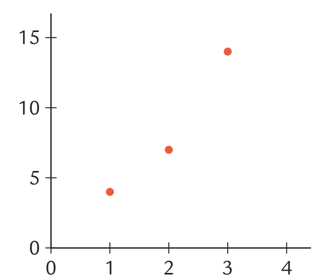


Figure 3 Exercise 57 data.

58. Figure 4 shows the plot of the points (1, 8), (2, 3), (3, 9), (5, 1), and (7, 7). Find a polynomial of degree 4 of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ whose graph passes through these points.

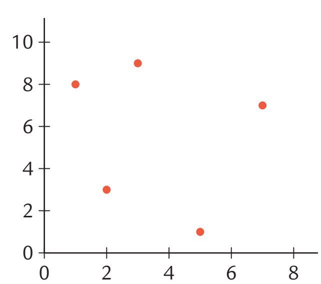


Figure 4 Exercise 58 data.

Exercises 59–60 refer to the cannonball scenario described at the start of the section. For each problem, the three ordered pairs are $(x, E(x))$, where x is the distance on the ground from the position of the cannon and $E(x)$ is the elevation of the ball. Find a model for the elevation of the ball, and use the model to determine where it hits the ground.

59. (20, 288), (40, 364), (60, 360)

60. (40, 814), (80, 1218), (110, 1311)

In Exercises 61–68, the given matrix is the augmented matrix for a linear system. Use technology to perform the row operations needed to transform the matrix to reduced echelon form, and then find all solutions to the system.

61. $\begin{bmatrix} 2 & 7 & -3 & 0 \\ -3 & 0 & 5 & 1 \\ -2 & 6 & -5 & 4 \end{bmatrix}$

62. $\begin{bmatrix} 11 & -5 & 0 & 0 \\ 2 & -3 & 8 & 0 \\ 7 & 3 & 3 & 0 \end{bmatrix}$

63. $\begin{bmatrix} 5 & -2 & 0 & 3 & 9 \\ 7 & 1 & 6 & 2 & -2 \\ 2 & 0 & -3 & 5 & 4 \end{bmatrix}$

64. $\begin{bmatrix} 9 & -2 & 0 & -4 & 6 \\ 0 & 7 & -1 & -1 & 3 \\ 8 & 12 & -6 & 5 & -8 \end{bmatrix}$

65. $\begin{bmatrix} 8 & -8 & 0 & -1 \\ 6 & 2 & -1 & 0 \\ 5 & 6 & -3 & 10 \\ -2 & 0 & -1 & -4 \end{bmatrix}$

66. $\begin{bmatrix} 5 & 3 & 7 & 5 \\ 4 & -3 & -2 & 0 \\ 0 & 3 & 17 & -2 \\ 4 & 7 & 8 & 12 \end{bmatrix}$

67. $\begin{bmatrix} 6 & 5 & 1 & 0 & -3 & 0 \\ 3 & -2 & -1 & 8 & 12 & 0 \\ -7 & 1 & 3 & 0 & 11 & 0 \\ 13 & 2 & 0 & -2 & -7 & 0 \end{bmatrix}$

68. $\begin{bmatrix} 2 & 1 & 0 & 0 & 3 & -5 & 7 \\ 0 & 5 & -1 & 8 & -1 & 4 & 0 \\ 3 & 11 & -9 & 1 & 6 & 0 & 13 \\ 7 & 0 & 5 & 5 & -3 & 2 & 11 \end{bmatrix}$

This section is optional and can be omitted without loss of continuity.

1.3 Numerical Solutions

In theory, the elimination methods developed in Section 1.2 can be used to find the solutions to *any* system of linear equations. And in practice, elimination methods work fine as long as the system is not too large. However, when implemented on a computer, elimination methods can lead to the wrong answer due to round-off error. Furthermore, for very large systems elimination methods may not be efficient enough to be practical. In this section we consider some shortcomings of elimination methods, and develop alternative solution methods.

"In theory, there is no difference between theory and practice. In practice, there is." —*Yogi Berra* (*Also attributed to computer scientist Jan L. A. van de Snepscheut and physicist Albert Einstein*.)

Round-off Error

No sensible person spends their day solving complicated systems of linear equations by hand—they use a computer. But while computers are fast, they have drawbacks, one being the round-off errors that can arise when using floating-point representations for numbers.

For example, suppose that we have a simple computer that has only four digits of accuracy. Using this computer and Gauss–Jordan elimination to solve the system

$$\begin{aligned} 7x_1 - 3x_2 + 2x_3 + 6x_4 &= 13 \\ -3x_1 + 9x_2 + 5x_3 - 2x_4 &= 9 \\ x_1 - 13x_2 - 3x_3 + 8x_4 &= -13 \\ 2x_1 \qquad\quad - x_3 + 3x_4 &= -6 \end{aligned}$$

yields the solution $x_1 = 2$, $x_2 = -0.999$, $x_3 = 3.998$, and $x_4 = -1.999$. This differs from the exact solution $x_1 = 2$, $x_2 = -1$, $x_3 = 4$, and $x_4 = -2$ because of round-off error occurring while performing row operations. The degree of error here is not too large, but this is a small system. Elimination methods applied to larger systems will require many more arithmetic operations, which can result in accumulation of round-off errors, even on a high-precision computer.

If the right combination of conditions exists, even small systems can generate significant round-off errors.

EXAMPLE 1 Suppose that we are using a computer with four digits of accuracy. Apply Gaussian elimination to find the solution to the system

$$\begin{aligned} 3x_1 + 1000x_2 &= 7006 \\ 42x_1 - 36x_2 &= -168 \end{aligned} \qquad (1)$$

The choice of four digits of accuracy does not restrict us to numbers less than 10,000. For instance, a number such as 973,400 can be represented as 9.734×10^5.

Solution We need only one row operation to put the system in triangular form. The exact computations are

$$\begin{bmatrix} 3 & 1000 & 7006 \\ 42 & -36 & -168 \end{bmatrix} \overset{-14R_1+R_2 \Rightarrow R_2}{\sim} \begin{bmatrix} 3 & 1000 & 7006 \\ 0 & -14036 & -98252 \end{bmatrix}$$

Since our computer only carries four digits of accuracy, the number $-14,036$ is rounded to $-14,040$ and $-98,252$ is rounded to $-98,250$. Thus the triangular system we end up with is

$$\begin{aligned} 3x_1 + 1000x_2 &= 7006 \\ -14,040x_2 &= -98,250 \end{aligned}$$

The notation :≈ means that rounding has occurred, and that the value on the right is being assigned to the indicated variable.

Solving for x_2, we get

$$x_2 = \frac{-98,250}{-14,040} :\approx 6.998$$

Back substituting to solve for x_1 gives us

$$x_1 = \frac{7006 - 1000(6.998)}{3} :\approx 2.667$$

The exact solution to the system is $x_1 = 2$ and $x_2 = 7$. Although the approximation for x_2 is fairly good, the approximation for x_1 is off by quite a bit. The source of the problem is that the coefficients in the equation

$$3x_1 + 1000x_2 = 7006$$

differ dramatically in size. During back substitution into this equation, the error in x_2 is magnified by the coefficient 1000 and only can be compensated for by the $3x_1$ term. But since the coefficient on this term is so much smaller, the error in x_1 is forced to be large. ■

Definition **Partial Pivoting**

One way to combat round-off error is to use **partial pivoting**, which adds a step to the usual elimination algorithms. With partial pivoting, when starting on a new column we first switch the row with the largest leading entry (compared using absolute values) to the pivot position before beginning the elimination process.

For instance, with the system (1) we interchange the position of the two rows because because $|42| > |3|$, which gives us

$$\begin{aligned} 42x_1 - 36x_2 &= -168 \\ 3x_1 + 1000x_2 &= 7006 \end{aligned}$$

This time the single elimination step is (shown with four digits of accuracy)

$$\begin{bmatrix} 42 & -36 & -168 \\ 3 & 1000 & 7006 \end{bmatrix} \overset{-\frac{1}{14}R_1+R_2\Rightarrow R_2}{\sim} \begin{bmatrix} 42 & -36 & -168 \\ 0 & 1003 & 7018 \end{bmatrix}$$

From this we have $x_2 = 7018/1003 = 6.997$, which is slightly less accurate than before. However, when we back substitute this value of x_2 into the equation

$$42x_1 - 36x_2 = -168$$

we get $x_1 = 1.997$, a much better approximation for the exact value of x_1.

When Gaussian and Gauss–Jordan elimination are implemented in computer software, partial pivoting is often used to help control round-off errors. It is also possible to implement **full pivoting**, where both rows and columns are interchanged to arrange for the largest possible leading coefficient. However, full pivoting is slower and so is employed less frequently than partial pivoting.

Definition **Full Pivoting**

Jacobi Iteration

It is not at all unusual for an application to yield a system of linear equations with thousands of equations and variables. In such a case, even if round-off error is controlled, elimination methods may not be efficient enough to be practical.

Here we turn our attention to a pair of related *iterative methods* that attempt to find the solution to a system of equations through a sequence of approximations. These methods do not suffer from the round-off problems described earlier, and in many cases they are faster than elimination methods. However, they only work on systems where the number of equations equals the number of variables, and sometimes they **diverge**—that is, they fail to reach the solution. In the cases where a solution is found, we say that the method **converges**.

Definition **Diverge, Converge**

Our first approximation method is called **Jacobi iteration**. We illustrate this method by using it to find the solution to the system

▶ Jacobi iteration is named for German mathematician Karl Gustav Jacobi (1804–1851).

$$\begin{aligned} 10x_1 + \ \ 4x_2 - \ \ x_3 &= \ \ \ 3 \\ 2x_1 + 10x_2 + \ \ x_3 &= -19 \\ x_1 - \ \ \ \ x_2 + 5x_3 &= \ \ -2 \end{aligned} \tag{2}$$

Step 1: Solve the first equation of the system for x_1, the second equation for x_2, and so on,

$$\begin{aligned} x_1 &= \ \ 0.3 - 0.4x_2 + 0.1x_3 \\ x_2 &= -1.9 - 0.2x_1 - 0.1x_3 \\ x_3 &= -0.4 - 0.2x_1 + 0.2x_2 \end{aligned} \tag{3}$$

Step 2: Make a guess at the values of x_1, x_2, and x_3 that satisfy the system. If we have no idea about the solution, then set each equal to 0,

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0$$

Step 3: Substitute the values for x_1, x_2, and x_3 into (3). This is Iteration 1, and it gives the updated values:

$$\text{Iteration 1:} \quad \begin{aligned} x_1 &= \ \ 0.3 - 0.4(0) + 0.1(0) = \ \ 0.3 \\ x_2 &= -1.9 - 0.2(0) - 0.1(0) = -1.9 \\ x_3 &= -0.4 - 0.2(0) + 0.2(0) = -0.4 \end{aligned}$$

Now repeat the process, substituting the new values for x_1, x_2, and x_3 into the equations in Step 1.

$$\text{Iteration 2:} \quad \begin{aligned} x_1 &= 0.3 - 0.4(-1.9) + 0.1(-0.4) = 1.02 \\ x_2 &= -1.9 - 0.2(0.3) - 0.1(-0.4) = -1.92 \\ x_3 &= -0.4 - 0.2(0.3) + 0.2(-1.9) = -0.84 \end{aligned}$$

We keep repeating this procedure until we have two consecutive iterations where each value differs from its predecessor by no more than the accuracy desired. Table 1 shows the outcome from the first nine iterations, each rounded to four decimal places. We see that the values have converged to $x_1 = 1$, $x_2 = -2$, and $x_3 = -1$, which is the exact solution to the system.

▶ Table values are rounded to four decimal places, and the rounded values are carried to the next iteration.

n	x_1	x_2	x_3
0	0	0	0
1	0.3000	−1.9000	−0.4000
2	1.0200	−1.9200	−0.8400
3	0.9840	−2.0200	−0.9880
4	1.0092	−1.9980	−1.0008
5	0.9991	−2.0018	−1.0014
6	1.0006	−1.9997	−1.0002
7	0.9999	−2.0001	−1.0001
8	1.0000	−2.0000	−1.0000
9	1.0000	−2.0000	−1.0000

Table 1 **Jacobi Iterations (*n* is the iteration number)**

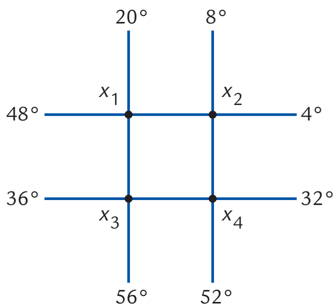


Figure 1 Grid Temperatures for Example 2.

EXAMPLE 2 Figure 1 gives a diagram of a piece of heavy wire mesh. Each of the eight wire ends has temperature held fixed as shown. When the temperature of the mesh reaches equilibrium, the temperature at each connecting point will be the average of the temperatures of the adjacent points and fixed ends. Determine the equilibrium temperature at the connecting points x_1, x_2, x_3, and x_4.

Solution The temperature of each connecting point depends in part on the temperature of other connecting points. For instance, since x_1 is adjacent to x_2, x_3, and the ends held fixed at 48° and 20°, its temperature at equilibrium will be the average

$$x_1 = \frac{x_2 + x_3 + 48 + 20}{4} = 0.25x_2 + 0.25x_3 + 17$$

Similarly, for the other connecting points we have the equations (after simplifying)

$$\begin{aligned} x_2 &= 0.25x_1 + 0.25x_4 + 3 \\ x_3 &= 0.25x_1 + 0.25x_4 + 23 \\ x_4 &= 0.25x_2 + 0.25x_3 + 21 \end{aligned}$$

Our four equations could be reorganized into the usual form of a linear equation and solved using elimination methods. But since each equation has one variable written in terms of the other variables, the problem sets up perfectly for Jacobi iteration. Starting with initial choices $x_1 = x_2 = x_3 = x_4 = 0$, the first two iterations are

$$\text{Iteration 1:} \quad \begin{aligned} x_1 &= 0.25(0) + 0.25(0) + 17 = 17 \\ x_2 &= 0.25(0) + 0.25(0) + 3 = 3 \\ x_3 &= 0.25(0) + 0.25(0) + 23 = 23 \\ x_4 &= 0.25(0) + 0.25(0) + 21 = 21 \end{aligned}$$

$$\text{Iteration 2:} \quad \begin{aligned} x_1 &= 0.25(3) + 0.25(23) + 17 = 23.5 \\ x_2 &= 0.25(17) + 0.25(21) + 3 = 12.5 \\ x_3 &= 0.25(17) + 0.25(21) + 23 = 32.5 \\ x_4 &= 0.25(3) + 0.25(23) + 21 = 27.5 \end{aligned}$$

Table 2 shows additional Jacobi iterations.

n	x_1	x_2	x_3	x_4
4	29.8750	18.1250	38.1250	33.8750
8	31.8672	19.8828	39.8828	35.8672
12	31.9917	19.9927	39.9927	35.9917
16	31.9995	19.9995	39.9995	35.9995
20	32.0000	20.0000	40.0000	36.0000
24	32.0000	20.0000	40.0000	36.0000

Table 2 Jacobi Iterations for Example 2

▶ To save space, only every fourth iteration is given in Table 2.

This suggests equilibrium temperatures of $x_1 = 32$, $x_2 = 20$, $x_3 = 40$, and $x_4 = 36$. Substituting these values into our four equations confirms that this is correct. ■

Gauss–Seidel Iteration

At each step of Jacobi iteration we take the values from the previous step and plug them into the set of equations, updating the values of all variables at the same time. We modify this approach with a variant of Jacobi iteration called Gauss–Seidel iteration. With this method, we always use the current value of each variable.

▶ We encountered C. F. Gauss earlier. Ludwig Philipp von Seidel (1821–1896) was a German mathematician. Interestingly, Gauss discovered the method long before Seidel but discarded it as worthless. Nonetheless, Gauss's name was attached to the algorithm along with that of Seidel, who independently discovered and published it after Gauss died.

To illustrate how Gauss–Seidel works, we use the system (2) considered before.

Step 1: As with Jacobi, we start by solving for x_1, x_2, and x_3,

$$\begin{aligned} x_1 &= 0.3 - 0.4x_2 + 0.1x_3 \\ x_2 &= -1.9 - 0.2x_1 - 0.1x_3 \\ x_3 &= -0.4 - 0.2x_1 + 0.2x_2 \end{aligned}$$

Step 2: We set initial values for x_1, x_2, and x_3. In the absence of any approximation for the solution, we use

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0$$

Step 3: For the first part of Iteration 1, we have (again as with Jacobi)

$$x_1 = 0.3 - 0.4(0) + 0.1(0) = 0.3$$

At this point the Jacobi and Gauss–Seidel methods begin to differ. To calculate the updated value of x_2, we use the most current variable values, which are $x_1 = 0.3$ and $x_3 = 0$.

$$x_2 = -1.9 - 0.2(0.3) - 0.1(0) = -1.96$$

We finish this iteration by updating the value of x_3, using the current values $x_1 = 0.3$ and $x_2 = -1.96$, so that

$$x_3 = -0.4 - 0.2(0.3) + 0.2(-1.96) = -0.852$$

Subsequent iterations proceed in the same way, always incorporating the most current variable values. The second iteration is

$$\text{Iteration 2:} \quad \begin{aligned} x_1 &= 0.3 - 0.4(-1.96) + 0.1(-0.852) = 0.9988 \\ x_2 &= -1.9 - 0.2(0.9988) - 0.1(-0.852) :\approx -2.0146 \\ x_3 &= -0.4 - 0.2(0.9988) + 0.2(-2.01456) :\approx -1.0027 \end{aligned}$$

As with Jacobi, we continue until reaching the point where two consecutive iterations yield values sufficiently close together. Table 3 gives the first six iterations of Gauss–Seidel applied to our system.

Note that Gauss–Seidel converged to the solution faster than Jacobi. Since Gauss–Seidel immediately incorporates new values into the computations, it seems reasonable to expect that it would converge faster than Jacobi. Most of the time this is true, but surprisingly not always—there are systems where Jacobi iteration converges more rapidly.

n	x_1	x_2	x_3
0	0	0	0
1	0.3000	−1.9600	−0.8520
2	0.9988	−2.0146	−1.0027
3	1.0056	−2.0008	−1.0013
4	1.0002	−1.9999	−1.0000
5	1.0000	−2.0000	−1.0000
6	1.0000	−2.0000	−1.0000

Table 3 Gauss–Seidel Iterations

▶ Example 3 is based on the work of Nobel Prize–winning economist Wassily Leontief (1906–1999). He divided the economy into 500 sectors in developing his input-output model.

EXAMPLE 3 Imagine a simple economy that consists of consumers and just three industries, which we refer to as A, B, and C. These industries have annual consumer sales of 60, 75, and 40 (in billions of dollars), respectively. In addition, for every dollar of goods A sells, A requires 10 cents of goods from B and 15 cents of goods from C to support production. (For instance, maybe B sells electricity and C sells shipping services.) Similarly, each dollar of goods B sells requires 20 cents of goods from A and 5 cents of goods from C, and each dollar of goods C sells requires 25 cents of goods from A and 15 cents of goods from B. What output from each industry will satisfy both consumer and between-industry demand?

Solution Let a, b, and c denote the total output from each of A, B, and C, respectively. The entire output for A is 60 for consumers, $0.20b$ for B, and $0.25c$ for C. Totaling this up yields the equation

$$a = 60 + 0.20b + 0.25c$$

Similar reasoning applied to industries B and C yields the equations

$$\begin{aligned} b &= 75 + 0.10a + 0.15c \\ c &= 40 + 0.15a + 0.05b \end{aligned}$$

Here we apply Gauss–Seidel iteration to find a solution. Since we are given the consumer demand for each industry, we take that as our starting point, initially setting $a = 60$, $b = 75$, and $c = 40$. For the first two iterations, we have

$$\begin{aligned} \text{Iteration 1:}\quad & a = 60 + 0.20(75) + 0.25(40) = 85 \\ & b = 75 + 0.10(85) + 0.15(40) = 89.5 \\ & c = 40 + 0.15(85) + 0.05(89.5) = 57.225 \\ \text{Iteration 2:}\quad & a = 60 + 0.20(89.5) + 0.25(57.225) :\approx 92.2063 \\ & b = 75 + 0.10(92.2063) + 0.15(57.225) :\approx 92.8044 \\ & c = 40 + 0.15(92.2063) + 0.05(92.8044) :\approx 58.4712 \end{aligned}$$

n	a	b	c
0	60	75	40
1	85	89.5	57.225
2	92.2063	92.8044	58.4712
3	93.1787	93.0885	58.6312
4	93.2755	93.1222	58.6474
5	93.2863	93.1257	58.6492
6	93.2875	93.1261	58.6494
7	93.2876	93.1262	58.6494
8	93.2876	93.1262	58.6494

Table 4 **Gauss–Seidel Iterations for Example 3**

Additional iterations are shown in Table 4, and suggest convergence to $a = 93.2876$, $b = 93.1262$, and $c = 58.6494$. These match the exact solution to four decimal places. ■

Convergence

Gaussian and Gauss–Jordan elimination are called *direct methods*, because they will always yield the solution in a finite number of steps (ignoring the potential problems brought about by round-off). On the other hand, as noted earlier, Jacobi and Gauss–Seidel are

n	x_1	x_2
0	0	0
1	6	11
2	−27	−55
3	171	341
4	−1017	−2035
5	6111	12221

Table 5 Gauss–Seidel Iterations

iterative methods and do not converge to a solution in all cases. For instance, applying Gauss–Seidel iteration starting at $x_1 = x_2 = 0$ to the system

$$\begin{aligned} x_1 + 3x_2 &= 6 \\ 2x_1 - x_2 &= 1 \end{aligned} \tag{4}$$

yields the sequence shown in Table 5. The values grow quickly in absolute value, and do not converge.

Definition **Diagonally Dominant**

One case where we are guaranteed convergence is if the coefficients of the system are **diagonally dominant**. This means that for each equation of the system, the coefficient a_{ii} (in equation i) along the diagonal has absolute value larger than the sum of the absolute values of the other coefficients in the equation. For example, the system

$$\begin{aligned} 7x_1 - 3x_2 + 2x_3 &= 6 \\ x_1 + 5x_2 - 2x_3 &= 1 \\ -3x_1 + x_2 - 6x_3 &= -4 \end{aligned} \tag{5}$$

is diagonally dominant because

$$\begin{aligned} |7| &> |-3| + |2| \\ |5| &> |1| + |-2| \\ |-6| &> |-3| + |1| \end{aligned}$$

On the other hand, the system

$$\begin{aligned} -2x_1 + x_2 - 9x_3 &= 0 \\ 6x_1 - x_2 + 4x_3 &= -12 \\ -x_1 + 4x_2 - x_3 &= 3 \end{aligned} \tag{6}$$

▶ Diagonal dominance is not required in order for the iterative methods to converge. There are instances where convergence occurs without diagonal dominance.

is not diagonally dominant as expressed, but reordering the equations to

$$\begin{aligned} 6x_1 - x_2 + 4x_3 &= -12 \\ -x_1 + 4x_2 - x_3 &= 3 \\ -2x_1 + x_2 - 9x_3 &= 0 \end{aligned} \tag{7}$$

makes it diagonally dominant.

EXAMPLE 4 Reverse the order of the equations in 4 to make the system diagonally dominant, and then find the solution using Gauss–Seidel iteration.

Solution Reversing the order of the equations gives us

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ x_1 + 3x_2 &= 6 \end{aligned}$$

which is diagonally dominant. Next, we solve for x_1 and x_2 (rounded to four decimal places),

$$\begin{aligned} x_1 &= 0.5 + 0.5x_2 \\ x_2 &= 2 - 0.3333x_1 \end{aligned}$$

Starting with $x_1 = 0$ and $x_2 = 0$, we have

Iteration 1: $x_1 = 0.5 + 0.5(0) = 0.5$
$x_2 = 2 - 0.3333(0.5) :\approx 1.8333$

Iteration 2: $x_1 = 0.5 + 0.5(1.8333) :\approx 1.4167$
$x_2 = 2 - 0.3333(1.4167) :\approx 1.5278$

n	x_1	x_2
0	0	0
1	0.5	1.8333
2	1.4167	1.5278
3	1.2639	1.5787
4	1.2894	1.5702
5	1.2851	1.5716
6	1.2858	1.5714
7	1.2857	1.5714
8	1.2857	1.5714

Table 6 Gauss–Seidel Iterations for Example 4

The first eight iterations are shown in Table 6, which shows convergence to $x_1 = 1.2857$ and $x_2 = 1.5714$. These match the exact solutions, which are $x_1 = 9/7$ and $x_2 = 11/7$. ■

Computational Comments

- Our iterative methods do not suffer from the round-off errors that can afflict elimination methods. Since the values from one iteration can be thought of as an "initial guess" for the next, there is no accumulation of errors. For the same reason, if a computation error is made, the result still can be used in the next iteration. By contrast, if a computation error is made when using elimination methods, the end result is almost always wrong.
- For a system of n equations with n unknowns, Jacobi and Gauss–Seidel both require about $2n^2$ flops per iteration. As mentioned earlier, Gauss–Seidel usually converges in fewer iterations than Jacobi, so Gauss–Seidel is typically the preferred method.
- If we ignore potential round-off issues and go solely by the number of flops, then Gaussian elimination requires about $2n^3/3$ flops, versus $2n^2$ flops per iteration for Gauss–Seidel. Hence, as long as Gauss–Seidel converges in fewer than $n/3$ iterations, this will be the more efficient method.
- The rate of convergence of our iterative methods is influenced by the degree of diagonal dominance of the system. If the diagonal terms are much larger than the others, then iterative methods generally will converge relatively quickly. If the diagonal terms are only slightly dominant, then although iterative methods eventually will converge, they can be too slow to be practical. There are other iterative methods besides those presented here that are designed to have better convergence properties.
- Iterative methods are particularly useful for solving **sparse systems**, which are linear systems where most of the coefficients are zero. The augmented matrix of such a system has mostly zero entries and is said to be a **sparse matrix**. Elimination methods applied to sparse systems have a tendency to change the zeros to nonzero terms, removing the sparseness.

▶ See *Matrix Computations* by G. Golub and C. Van Loan for a more extensive discussion of iterative methods and an explanation of why those described here work.

Definition Sparse System, Sparse Matrix

EXERCISES

In each exercise set, problems marked with Ⓒ are designed to be solved using a programmable calculator or computer algebra system.

In Exercises 1–4, use partial pivoting with Gaussian elimination to find the solutions to the given system.

1. $\begin{aligned} -2x_1 + 3x_2 &= 4 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$

2. $\begin{aligned} x_1 - 2x_2 &= -1 \\ -3x_1 + 7x_2 &= 5 \end{aligned}$

3. $\begin{aligned} x_1 + x_2 - 2x_3 &= -3 \\ 3x_1 - 2x_2 + 2x_3 &= 9 \\ 6x_1 - 7x_2 - x_3 &= 4 \end{aligned}$

4. $\begin{aligned} x_1 - 3x_2 + 2x_3 &= 4 \\ -2x_1 + 7x_2 - 2x_3 &= -7 \\ 4x_1 - 13x_2 + 7x_3 &= 12 \end{aligned}$

Ⓒ In Exercises 5–8, solve the system as given with Gaussian elimination with three significant digits of accuracy. Then solve the system again, incorporating partial pivoting.

5. $\begin{aligned} 2x_1 + 975x_2 &= 41 \\ 53x_1 - 82x_2 &= -13 \end{aligned}$

6. $\begin{aligned} 3x_1 - 813x_2 &= 32 \\ 71x_1 - 93x_2 &= -5 \end{aligned}$

7. $\begin{aligned} 3x_1 - 7x_2 + 639x_3 &= 12 \\ -2x_1 + 5x_2 + 803x_3 &= 7 \\ 56x_1 - 41x_2 + 79x_3 &= 10 \end{aligned}$

8. $\begin{aligned} 2x_1 - 5x_2 + 802x_3 &= -1 \\ -x_1 + 3x_2 - 789x_3 &= -8 \\ 40x_1 + 34x_2 + 51x_3 &= 19 \end{aligned}$

Ⓒ In Exercises 9–12, compute the first three Jacobi iterations for the given system, using 0 as the initial value for each variable. Then find the exact solution and compare.

9. $\begin{aligned} -5x_1 + 2x_2 &= 6 \\ 3x_1 + 10x_2 &= 2 \end{aligned}$

10. $\begin{aligned} 2x_1 - x_2 &= -4 \\ -4x_1 + 5x_2 &= 11 \end{aligned}$

11. $\begin{aligned} 20x_1 + 3x_2 + 5x_3 &= -26 \\ -2x_1 - 10x_2 + 3x_3 &= -23 \\ x_1 - 2x_2 - 5x_3 &= -13 \end{aligned}$

12. $\begin{aligned} -2x_1 \qquad + x_3 &= 5 \\ -x_1 + 5x_2 - x_3 &= 8 \\ 2x_1 - 6x_2 + 10x_3 &= 16 \end{aligned}$

C In Exercises 13–16, compute the first three Gauss–Seidel iterations for the given system, using 0 as the initial value for each variable. Then find the exact solution and compare.

13. The system given in Exercise 9.

14. The system given in Exercise 10.

15. The system given in Exercise 11.

16. The system given in Exercise 12.

In Exercises 17–20, determine if the given system is diagonally dominant. If not, then (if possible) rewrite the system so that it is diagonally dominant.

17. $\begin{aligned} 2x_1 - 5x_2 &= 7 \\ 3x_1 + 7x_2 &= 4 \end{aligned}$

18. $\begin{aligned} 4x_1 + 2x_2 - x_3 &= 13 \\ -2x_1 + 7x_2 + 2x_3 &= -9 \\ x_1 + 3x_2 - 5x_3 &= 6 \end{aligned}$

19. $\begin{aligned} 3x_1 + 6x_2 - x_3 &= 0 \\ -x_1 - 2x_2 + 4x_3 &= -1 \\ 7x_1 + 5x_2 - 3x_3 &= 3 \end{aligned}$

20. $\begin{aligned} -2x_1 + 6x_2 &= 12 \\ 5x_1 - x_2 &= -4 \end{aligned}$

C In Exercises 21–24, compute the first four Jacobi iterations for the system as written, with the initial value of each variable set equal to 0. Then rewrite the system so that it is diagonally dominant, set the value of each variable to 0, and again compute 4 Jacobi iterations.

21. $\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 - x_2 &= 1 \end{aligned}$

22. $\begin{aligned} x_1 - 3x_2 &= -2 \\ 3x_1 - x_2 &= 2 \end{aligned}$

23. $\begin{aligned} x_1 - 2x_2 + 5x_3 &= -1 \\ 5x_1 + x_2 - 2x_3 &= 8 \\ 2x_1 - 10x_2 + 3x_3 &= -1 \end{aligned}$

24. $\begin{aligned} 2x_1 + 4x_2 - 10x_3 &= -3 \\ 3x_1 - x_2 + x_3 &= 7 \\ -x_1 + 6x_2 - 2x_3 &= -6 \end{aligned}$

C In Exercises 25–28, compute the first four Gauss–Seidel iterations for the system as written, with the initial value of each variable set equal to 0. Then rewrite the system so that it is diagonally dominant, set the value of each variable to 0, and again compute four Gauss–Seidel iterations.

25. The system given in Exercise 21.

26. The system given in Exercise 22.

27. The system given in Exercise 23.

28. The system given in Exercise 24.

C In Exercises 29–30, the values from the first few Jacobi iterations are given for an unknown system. Find the values for the next iteration.

29.

n	x_1	x_2
0	0	0
1	1	−2
2	5	2
3	?	?

30.

n	x_1	x_2	x_3
0	0	0	0
1	−2	−1	1
2	−4	−4	5
3	−11	−4	5
4	?	?	?

C In Exercises 31–32, the values from the first few Gauss–Seidel iterations are given for an unknown system. Find the values for the next iteration.

31.

n	x_1	x_2
0	0	0
1	3	4
2	−5	−12
3	?	?

32.

n	x_1	x_2	x_3
0	0	0	0
1	3	4	12
2	7	−24	−76
3	−25	176	556
4	?	?	?

1.4 Applications of Linear Systems

▶ This section is optional. Although some applications presented here are referred to later, they can be reviewed as needed.

In this section we consider in depth some applications of linear systems. These are but a few of the many different possible applications that exist.

Traffic Flow

Arcata, on the northern coast of California, is a small college town with a central plaza (Figure 1). Figure 2 shows the streets surrounding and adjacent to the town's central plaza. As indicated by the arrows, all streets in the vicinity of the plaza are one-way.

Figure 1 The Arcata plaza. (Photo taken by Terrence McNally of Arcata Photo.)

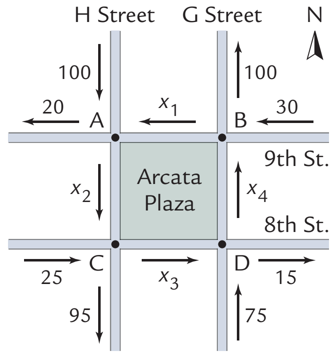


Figure 2 Traffic volumes around the Arcata plaza.

Traffic flows north and south on G and H streets, respectively, and east and west on 8th and 9th streets, respectively. The number of cars flowing on and off the plaza during a typical 15-minute period on a Saturday morning is also shown. Our goal is to find x_1, x_2, x_3, and x_4, the volume of traffic along each side of the plaza.

The four intersections are labeled A, B, C, and D. At each intersection, the number of cars entering the intersection must equal the number leaving. For example, the number of cars entering A is $100 + x_1$ and the number exiting is $20 + x_2$. Since these must be equal, we end up with the equation

$$\text{A:} \quad 100 + x_1 = 20 + x_2$$

Applying the same reasoning to intersections B, C, and D, we arrive at three more equations,

$$\begin{aligned} \text{B:} &\quad x_4 + 30 = x_1 + 100 \\ \text{C:} &\quad x_2 + 25 = x_3 + 95 \\ \text{D:} &\quad x_3 + 75 = x_4 + 15 \end{aligned}$$

Rewriting the equations in the usual form, we obtain the system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & & & = & -80 \\ x_1 & & & & & - & x_4 & = & -70 \\ & & x_2 & - & x_3 & & & = & 70 \\ & & & & x_3 & - & x_4 & = & -60 \end{array}$$

To solve the system, we populate an augmented matrix and transform to echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -80 \\ 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & 0 & 1 & -1 & -60 \end{bmatrix} \underset{\sim}{-R_1+R_2 \Rightarrow R_2} \begin{bmatrix} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & 0 & 1 & -1 & -60 \end{bmatrix}$$

$$\underset{\sim}{-R_2+R_3 \Rightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & -1 & -60 \end{bmatrix}$$

$$\underset{\sim}{R_3+R_4 \Rightarrow R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Back substitution yields the general solution

$$x_1 = -70 + s_1, \quad x_2 = 10 + s_1, \quad x_3 = -60 + s_1, \quad x_4 = s_1$$

where s_1 is a free parameter.

A moment's thought reveals why it makes sense that this system has infinitely many solutions. There can be an arbitrary number of cars simply circling the plaza, perhaps looking for a parking space. Note also that since each of x_1, x_2, x_3, and x_4 must be nonnegative, it follows that the parameter $s_1 \geq 70$.

The analysis performed here can be carried over to much more complex traffic questions, or to other similar settings, such as computer networks.

The BCS Ranking System

The BCS (Bowl Championship Series) is a system for ranking college football teams. The two teams ranked highest at the end of the regular season get to play in the national championship game. A "BCS Index" is calculated to rank the teams. The BCS Index is calculated by combining three team rankings from different sources:

USA: A survey of 62 college football coaches compiled by *USA Today*.

Harris: A survey of 114 college football experts compiled by *Harris Interactive*.

Computer: An average of computer-based rankings from various sources.

Table 1 gives the points awarded by each source to determine individual rankings together with the BCS Index. The higher the BCS Index, the higher the BCS ranking.

▶ A *conjecture* is the mathematical equivalent of an educated guess.

The information in Table 1 appeared frequently in the media, but the formula for computing the BCS Index rarely did. Our goal here is to deduce the BCS Index formula using linear algebra. To find our formula, we start by conjecturing that the BCS Index is a linear combination of the three components, so that each team's data will satisfy

$$x_1(\text{USA}) + x_2(\text{Harris}) + x_3(\text{Computer}) = \text{BCS Index}$$

for the right choice of x_1, x_2, and x_3. For example, using the data for Oklahoma gives us the equation

$$1482x_1 + 2699x_2 + 100x_3 = 0.9757$$

▶ This system can be solved using the methods presented in Section 1.2 or Section 1.3, but here a computer algebra system was used. There is a small amount of rounding in the solution.

Data from other schools can be used in the same way to obtain additional equations. Since we have three unknowns, we need three equations in order to find the values of x_1, x_2, and x_3. Taking the top three schools, we arrive at the linear system

$$\begin{aligned} 1482x_1 + 2699x_2 + 100x_3 &= 0.9757 \\ 1481x_1 + 2776x_2 + 89x_3 &= 0.9479 \\ 1408x_1 + 2616x_2 + 94x_3 &= 0.9298 \end{aligned}$$

Rank	Team	USA	Harris	Computer	BCS Index
1	Oklahoma	1482	2699	100	0.9757
2	Florida	1481	2776	89	0.9479
3	Texas	1408	2616	94	0.9298
4	Alabama	1309	2442	81	0.8443
5	Southern Cal	1309	2413	75	0.8208
6	Penn State	1193	2186	66	0.7387

Table 1 The 2008 Final Regular Season BCS Ranks

This system has unique solution

$$x_1 = 0.0002151, \quad x_2 = 0.0001195, \quad x_3 = 0.003344$$

This gives us the formula

$$\text{BCS Index} = 0.0002151(\text{USA}) + 0.0001195(\text{Harris}) + 0.003344(\text{Computer})$$

To check if we have the right formula, let's test it out on some schools not represented in our system, say, Alabama, Southern Cal, and Penn State.

$$\begin{aligned}
&\text{Alabama:} && 0.0002151(1309) + 0.0001195(2442) + 0.003344(81) = 0.8443\\
&\text{Southern Cal:} && 0.0002151(1309) + 0.0001195(2413) + 0.003344(75) = 0.8207\\
&\text{Penn State:} && 0.0002151(1193) + 0.0001195(2186) + 0.003344(66) = 0.7386
\end{aligned}$$

Other than small differences due to rounding, the formula checks out. Thus it is reasonable to assume that we have found the correct formula for the BCS index.

▶ At the time of this writing, a limited playoff system has been approved.

Finally, we note that the BCS index formula has been changed several times in the past, so it may no longer have this form. (See Exercise 32 for an older version that was more complicated.) However, if you obtain a version of Table 1 for the most recent college football season, you can probably perform the same analysis as we have here to find the current formula for the BCS index.

Planetary Orbital Periods

Most people are aware that the planets that are closer to the sun take a shorter amount of time to make one orbit around the sun than those that are farther out. Table 2 gives the average distance from the sun and the number of Earth days required to make one orbit.

Our goal here is to develop an equation that describes the relationship between the distance from the sun and the length of the orbital period. As a starting point, consider the scatter plot of the data given in Figure 3.

There seems to be a pattern to the data. The points do not lie on a line, but the curved shape suggests that for constants a and b, the data may come close to satisfying the equation

$$p = ad^b \tag{1}$$

where p is the orbital period and d is the distance from the sun. Here we proceed as in the BCS example, substituting data to create a system of equations to solve. However, before doing that, we note equation (1) is not linear in a and b, but it can be if we apply

Planet	Distance from Sun ($\times 10^6$ km)	Orbital Period (days)
Mercury	57.9	88
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687
Jupiter	778.6	4331
Saturn	1433.5	10747
Uranus	2872.5	30589
Neptune	4495.1	59800

Table 2 Planetary Orbital Distances and Periods

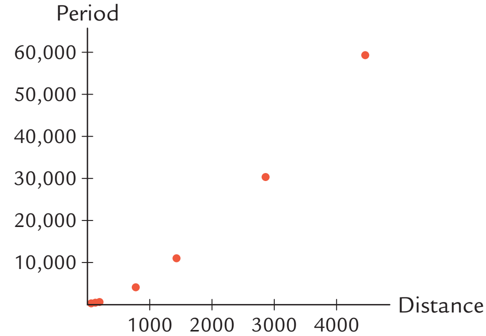


Figure 3 Orbital Distance vs. Orbit Period.

the logarithm function to both sides. This gives us

$$\begin{aligned} \ln(p) &= \ln(ad^b) \\ &= \ln(a) + b\ln(d) \end{aligned}$$

If we let $a_1 = \ln(a)$ and substitute the data from Mercury and Venus, we get the system of two equations and two unknowns

$$\begin{aligned} a_1 + b\ln(57.9) &= \ln(88) \\ a_1 + b\ln(108.2) &= \ln(224.7) \end{aligned}$$

The solution to this system is $a_1 \approx -1.60771$ and $b \approx 1.49925$. Since $a_1 = \ln(a)$, we have $a \approx e^{-1.60771} = 0.200346$, yielding the formula

$$p = (0.200346)d^{1.49925}$$

Table 3 gives the actual and predicted (using the above formula) orbital period for each planet.

Planet	Distance	Actual Period	Predicted Period
Mercury	57.9	88	87.9988
Venus	108.2	224.7	224.696
Earth	149.6	365.2	365.214
Mars	227.9	687	686.482
Jupiter	778.6	4331	4330.96
Saturn	1433.5	10,747	10,814.6
Uranus	2872.5	30,589	30,644.3
Neptune	4495.1	59,800	59,999.8

Table 3 Planetary Orbital Distances and Periods

The predictions are fairly good, suggesting that our formula is on the right track. However, the predictions become less accurate for those planets farther from the sun. Because we used the data for Mercury and Venus to develop our formula, perhaps this is not surprising. If instead we use Uranus and Neptune, we arrive at the formula

$$p = (0.20349)d^{1.497}$$

Table 4 shows this formula produces better predictions.

Planet	Predicted Period
Mercury	88.6
Venus	225.8
Earth	366.8
Mars	688.8
Jupiter	4334
Saturn	10,806
Uranus	30,589
Neptune	59,799

Table 4 Predicted Orbital Periods

A natural idea is to incorporate more data into our formula, by using more planets to generate a larger system of equations. Unfortunately, if we use more than two planets, we end up with a system that has no solutions. (Try it for yourself.) Thus there are limitations to what we can do with the tools we currently have available. In Chapter 8 we develop a more sophisticated method that allows us to use all of our data simultaneously to come up with a formula that provides a good estimate for a range of distances from the sun.

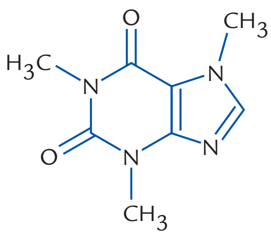


Figure 4 The caffeine molecule. (Source: Wikimedia Commons; Author: NEUROtiker)

Balancing Chemical Equations

A popular chemical among college students is caffeine, which has chemical composition $C_8H_{10}N_4O_2$. When heated and combined with oxygen (O_2), the ensuing reaction produces carbon dioxide (CO_2), water (H_2O), and nitrogen dioxide (NO_2). This chemical reaction is indicated using the notation

$$x_1C_8H_{10}N_4O_2 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O + x_5NO_2 \tag{2}$$

where the subscripts on the elements indicate the number of atoms. (No subscript indicates one atom.) Balancing the equation involves finding values for x_1, x_2, x_3, x_4, and x_5 so that the number of atoms of each element is the same before and after the reaction. Most chemistry texts describe a method of solution that is best described as trial and error. However, there is no need for a haphazard approach—we can use linear algebra.

In the reaction in 2, let's start with carbon. On the left side there are $8x_1$ carbon atoms, while on the right side there are x_3 carbon atoms. This yields the equation

$$8x_1 = x_3$$

For oxygen, we see that there are $2x_1 + 2x_2$ atoms on the left side and $2x_3 + x_4 + 2x_5$ on the right, producing another equation,

$$2x_1 + 2x_2 = 2x_3 + x_4 + 2x_5$$

Similar analysis on nitrogen and hydrogen results in two additional equations,

$$4x_1 = x_5 \quad \text{and} \quad 10x_1 = 2x_4$$

To balance the chemical equation, we must find a solution that satisfies all four equations. That is, we need to find the solution set to the linear system

$$\begin{array}{rcrcrcrcrcl}
2x_1 & + & 2x_2 & - & 2x_3 & - & x_4 & - & 2x_5 & = & 0 \\
4x_1 & & & & & & & - & x_5 & = & 0 \\
8x_1 & & & - & x_3 & & & & & = & 0 \\
10x_1 & & & & & - & 2x_4 & & & = & 0
\end{array}$$

The augmented matrix and row operations are

$$\begin{bmatrix} 2 & 2 & -2 & -1 & -2 & 0 \\ 4 & 0 & 0 & 0 & -1 & 0 \\ 8 & 0 & -1 & 0 & 0 & 0 \\ 10 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} -2R_1+R_2 \Rightarrow R_2 \\ -4R_1+R_3 \Rightarrow R_3 \\ -5R_1+R_4 \Rightarrow R_4 \\ \sim \end{array} \quad \begin{bmatrix} 2 & 2 & -2 & -1 & -2 & 0 \\ 0 & -4 & 4 & 2 & 3 & 0 \\ 0 & -8 & 7 & 4 & 8 & 0 \\ 0 & -10 & 10 & 3 & 10 & 0 \end{bmatrix}$$

$$\begin{array}{c} -2R_2+R_3 \Rightarrow R_3 \\ -\frac{5}{2}R_2+R_4 \Rightarrow R_4 \\ \sim \end{array} \quad \begin{bmatrix} 2 & 2 & -2 & -1 & -2 & 0 \\ 0 & -4 & 4 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & \frac{5}{2} & 0 \end{bmatrix}$$

Back substituting and scaling the free parameter gives the general solution

$$\begin{aligned} x_1 &= 2s_1 \\ x_2 &= 27s_1 \\ x_3 &= 16s_1 \\ x_4 &= 10s_1 \\ x_5 &= 8s_1 \end{aligned}$$

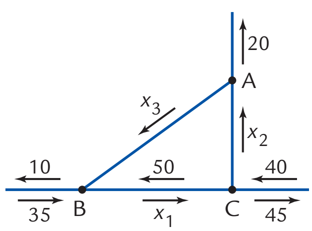
where s_1 can be any real number. Any choice of s_1 yields constants that balance our chemical equation, but it is customary to select the specific solution that makes each of the coefficients x_1, x_2, x_3, x_4, and x_5 integers that have no common factors. Setting $s_1 = 1$ accomplishes this, yielding the balanced equation

$$2C_8H_{10}N_4O_2 + 27O_2 \rightarrow 16CO_2 + 10H_2O + 8NO_2$$

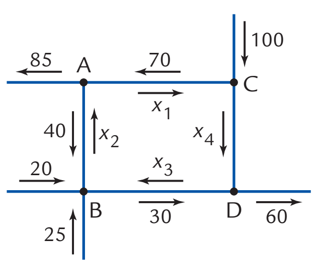
EXERCISES

In each exercise set, problems marked with **C** are designed to be solved using a programmable calculator or computer algebra system.

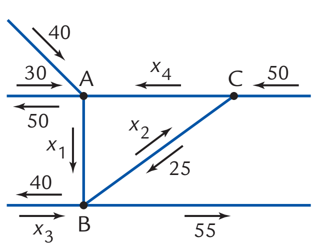
1. The volume of traffic for a collection of intersections is shown in the figure below. Find all possible values for x_1, x_2, and x_3. What is the minimum volume of traffic from C to A?



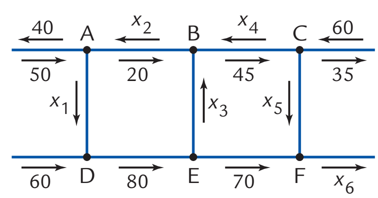
2. **C** The volume of traffic for a collection of intersections is shown in the figure below. Find all possible values for x_1, x_2, x_3, and x_4. What is the minimum volume of traffic from C to D?



3. **C** The volume of traffic for a collection of intersections is shown in the figure below. Find all possible values for x_1, x_2, x_3, and x_4. What is the minimum volume of traffic from C to A?



4. **C** The volume of traffic for a collection of intersections is shown in the figure below. Find all possible values for x_1, x_2, x_3, x_4, x_5, and x_6.



In Exercises 5–12, balance the given chemical equations.

5. Hydrogen burned in oxygen forms steam:

$H_2 + O \longrightarrow H_2O$

6. Hydrogen and nitrogen combine to form ammonia:

$H_2 + N_2 \longrightarrow NH_3$

7. Iron and oxygen combine to form iron oxide:

$Fe + O_2 \longrightarrow Fe_2O_3$

8. Sodium and water react to form sodium hydroxide (lye) and hydrogen:

$Na + H_2O \longrightarrow NaOH + H_2$

9. When propane burns in oxygen, it produces carbon dioxide and water:

$C_3H_8 + O_2 \longrightarrow CO_2 + H_2O$

10. When acetylene burns in oxygen, it produces carbon dioxide and water:

$C_2H_2 + O_2 \longrightarrow CO_2 + H_2O$

11. Potassium superoxide and carbon dioxide react to form potassium carbonate and oxygen:

$KO_2 + CO_2 \longrightarrow K_2CO_3 + O_2$

12. Manganese dioxide and hydrochloric acid combine to form manganese chloride, water, and chlorine gas:

$MnO_2 + HCl \longrightarrow MnCl_2 + H_2O + Cl_2$

In Exercises 13–16, find a model for planetary orbital period using the data for the given planets.

13. Earth and Mars.

14. Mercury and Uranus.

15. Venus and Neptune.

16. Jupiter and Saturn.

In Exercises 17–18, the data given provides the distance required for a particular type of car to stop when traveling at a variety of speeds. A reasonable model for braking distance is $d = as^k$, where d is distance, s is speed, and a and k are constants. Use the data in the table to find values for a and k, and test your model. (HINT: Methods similar to those used to find a model for planetary orbital periods can be applied here.)

17.

Speed (MPH)	10	20	30	40
Distance (Feet)	4.5	18	40.5	72

18.

Speed (MPH)	10	20	30	40
Distance (Feet)	20	80	180	320

When using partial fractions to find antiderivatives in calculus, we decompose complicated rational expressions into the sum of simpler expressions that can be integrated individually. In Exercises 19–22, the required decomposition is given. Find the values of the missing constants.

19. $\dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

20. $\dfrac{3x-1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

21. $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$

22. $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

23. The points $(1, 3)$ and $(-2, 6)$ lie on a line. Where does the line cross the x-axis?

24. The points $(2, -1, -2)$, $(1, 3, 12)$, and $(4, 2, 3)$ lie on a unique plane. Where does this plane cross the z-axis?

25. The equation for a parabola has the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$. Find an equation for the parabola that passes through the points $(-1, -10)$, $(1, -4)$, and $(2, -7)$.

26. C Find a polynomial of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

such that $f(0) = -3$, $f(1) = 2$, $f(3) = 5$, and $f(4) = 0$.

27. C Find a polynomial of the form

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e$$

such that $g(-2) = -17$, $g(-1) = 6$, $g(0) = 5$, $g(1) = 4$, and $g(2) = 3$.

C (Calculus required) In Exercises 28–29, find the values of the coefficients a, b, and c so the given conditions for the function f and its derivatives are met. (This type of problem arises in the study of differential equations.)

28. $f(x) = ae^x + be^{2x} + ce^{-3x}$; $f(0) = 2$, $f'(0) = 1$, and $f''(0) = 19$.

29. $f(x) = ae^{-2x} + be^x + cxe^x$; $f(0) = -1$, $f'(0) = -2$, and $f''(0) = 3$.

C In Exercises 30–31, a new "LAI" (for Linear Algebra Index) formula has been used to rank the eight college football teams shown. The new formula uses the same components as the 2008 BCS formula described earlier. Determine the formula for the LAI, and test it to be sure it is correct.

30.

Rank	Team	LAI
1	Oklahoma	0.9655
2	Florida	0.9652
3	Texas	0.9237
4	Alabama	0.8538
5	Southern Cal	0.8436
6	Penn State	0.7646
7	Utah	0.7560
8	Texas Tech	0.7522

31.

Rank	Team	LAI
1	Oklahoma	0.9895
2	Texas	0.9364
3	Florida	0.9204
4	Texas Tech	0.8285
5	Alabama	0.8284
6	Utah	0.8236
7	Southern Cal	0.7866
8	Penn State	0.7004

32. C The BCS ranking system was more complicated in 2001 than in 2008. The table below gives the BCS rankings at the end of the regular season. (A lower BCS Index gave a higher rank.)

Table headings:

- AP and USA gives the rank of each team in the two opinion polls of writers and coaches, respectively.
- SS stands for strength of schedule ranking, with 1 being the most challenging.
- L is the number of losses during the season.
- CA (Computer Average) is the average of computer rankings from various sources.
- QW (Quality Wins) gives a measurement of the number of victories over highly ranked teams.

Rank	Team	AP	USA	SS	L	CA	QW	BCS Index
1	Miami	1	1	18	0	1.00	0.1	2.62
2	Nebraska	4	4	14	1	2.17	0.5	7.23
3	Colorado	3	3	2	2	4.50	2.3	7.28
4	Oregon	2	2	31	1	4.83	0.4	8.67
5	Florida	5	5	19	2	5.83	0.5	13.09
6	Tennessee	8	8	3	2	6.17	1.6	14.69
7	Texas	9	9	33	2	6.67	1.2	17.79
8	Illinois	7	7	37	1	9.83	0.0	19.31
9	Stanford	11	11	22	2	7.83	1.3	20.41
10	Maryland	6	6	78	1	11.17	0.0	21.29
11	Oklahoma	10	10	36	2	9.00	0.9	21.54
12	Washington St	13	13	42	2	10.83	0.6	26.91
13	LSU	12	12	10	3	13.33	1.0	27.73
14	South Carolina	14	14	40	3	19.17	0.0	37.77
15	Washington	21	20	21	3	14.83	1.0	38.17

Find the 2001 BCS ranking formula. Test it for three schools not used to develop your formula to check for correctness. (HINT: To avoid a system with infinitely many solutions, include Washington among the schools used to develop the formula. Explain why including Washington will accomplish this.)