CHAPTER 2

Euclidean Space

The Willis Family Bridge provides a walkway from the main campus of Indiana University–Purdue University Fort Wayne to the Waterfield Campus Student Housing. The bridge was designed by Kurt Heidenreich and dedicated in 2003. The triangular design accommodates the need to cross Crescent Ave. as well an area of uneven terrain. The two angled pylons and four support cables suggest a diagram for the addition of force vectors; the ellipse at the top of the triangle is formed by the intersection of two circular cylinders.

Bridge suggested by Adam Coffman, Indiana University – Purdue University Fort Wayne (James E. Whitcraft)

We can think of algebra as the study of the properties of arithmetic on the real numbers. In linear algebra, we study the properties of arithmetic performed on objects called *vectors*. As we shall see shortly, one use of vectors is to compactly describe the set of solutions of a linear system, but vectors have many other applications as well. Section 2.1 gives an introduction to vectors, arithmetic with vectors, and the geometry of vectors. Section 2.2 and Section 2.3 describe important properties of sets of vectors.

2.1 Vectors

In this section we introduce vectors, which for the moment we can think of as a list of numbers. We start with a specific example of vectors that occur in plain sight, on packages of plant fertilizer. Fertilizer is sold in bags labelled with three numbers that indicate the amount of nitrogen (N), phosphoric acid (P_2O_5), and potash (K_2O) present. The mixture of these nutrients varies from one type of fertilizer to the next. For example, a bag of *Vigoro Ultra Turf* has the numbers "29–3–4," which means that 100 pounds of this fertilizer contains 29 pounds of nitrogen, 3 pounds of phosphoric acid, and 4 pounds of potash. Organizing these quantities vertically in a matrix, we have

$$\begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix}$$

This representation is an example of a *vector*. Using a vector provides a convenient way to record the amounts of each nutrient and also lends itself to compact forms of algebraic operations that arise naturally. For instance, if we want to know the amount of nitrogen, phosphoric acid, and potash contained in a ton (2000 pounds) of Ultra Turf, we just multiply each vector entry by 20. If we think of this as multiplying the vector by 20, then it is reasonable to represent this operation by

$$20\begin{bmatrix}29\\3\\4\end{bmatrix}$$

so that we have

$$20\begin{bmatrix}29\\3\\4\end{bmatrix}=\begin{bmatrix}20\cdot 29\\20\cdot\ \ 3\\20\cdot\ \ 4\end{bmatrix}=\begin{bmatrix}580\\60\\80\end{bmatrix}$$

Note that the "=" sign between the vectors means that the entries in corresponding positions are equal.

Another type of fertilizer, *Parker's Premium Starter*, has 18 pounds of nitrogen, 25 pounds of phosphoric acid, and 6 pounds of potash per 100 pounds, which is represented in vector form by

$$\begin{bmatrix}18\\25\\6\end{bmatrix}$$

If we mix together 100 pounds of each type of fertilizer, then we can find the total amount of each nutrient in the mixture by adding entries in each of the vectors. Thinking of this as adding the vectors, we have

$$\begin{bmatrix}29\\3\\4\end{bmatrix}+\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}29+18\\3+25\\4+\ \ 6\end{bmatrix}=\begin{bmatrix}47\\28\\10\end{bmatrix}$$

Vectors and $\mathbf{R}^n$

We formalize our notion of vector with the following definition.

Definition Vector

DEFINITION 2.1

A **vector** is an ordered list of real numbers $u_1, u_2, \ldots, u_n$ expressed as

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$$

Definition $\mathbf{R}^n$

or as $\mathbf{u}=(u_1, u_2, \ldots, u_n)$. The set of all vectors with n entries is denoted by $\mathbf{R}^n$.

Definition Component
Definition Column Vector
Definition Row Vector

Our convention will be to denote vectors using boldface, as in $\mathbf{u}$. Each of the entries $u_1, u_2, \ldots, u_n$ is called a **component** of the vector. A vector expressed in the vertical form is also called a **column vector**, and a vector expressed in horizontal form is also called a **row vector**. It is customary to express vectors in column form, but we will occasionally use row form to save space.

The fertilizer discussion provides a good model for how vector arithmetic works. Here we formalize the definitions.

DEFINITION 2.2

Definition Vector Arithmetic, Scalar, Euclidean Space

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$ given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Suppose that c is a real number, called a **scalar** in this context. Then we have the following definitions:

Equality: $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$.

Addition:
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Scalar Multiplication:
$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c \cdot u_1 \\ c \cdot u_2 \\ \vdots \\ c \cdot u_n \end{bmatrix}$$

The set of all vectors in $\mathbf{R}^n$, taken together with these definitions of addition and scalar multiplication, is called **Euclidean space**.

▶ Euclidean space is named for the Greek mathematician Euclid, the father of geometry. Euclidean space is an example of a *vector space*, discussed in Chapter 7.

Although vectors with negative components and negative scalars do not make sense in the fertilizer discussion, they do in other contexts and are included in Definition 2.2.

EXAMPLE 1 Suppose that

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -4 \\ 6 \\ -2 \\ 7 \end{bmatrix}$$

Find $\mathbf{u} + \mathbf{v}$, $-4\mathbf{v}$, and $2\mathbf{u} - 3\mathbf{v}$.

▶ Two vectors can be equal only if they have the same number of components. Similarly, there is no way to add two vectors that have a different number of components.

Solution The solutions to the first two parts are

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2-4 \\ -3+6 \\ 0-2 \\ -1+7 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \\ 6 \end{bmatrix}$$

$$-4\mathbf{v} = -4 \begin{bmatrix} -4 \\ 6 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -4(-4) \\ -4(\ \ 6) \\ -4(-2) \\ -4(\ \ 7) \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 8 \\ -28 \end{bmatrix}$$

The third computation has a slight twist because we have not yet defined the difference of two vectors. But subtraction works exactly as we would expect and follows from the natural interpretation that $2\mathbf{u} - 3\mathbf{v} = 2\mathbf{u} + (-3)\mathbf{v}$.

$$2\mathbf{u} - 3\mathbf{v} = 2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} -4 \\ 6 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2(\ \ 2) - 3(-4) \\ 2(-3) - 3(\ \ 6) \\ 2(\ \ 0) - 3(-2) \\ 2(-1) - 3(\ \ 7) \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 6 \\ -23 \end{bmatrix}$$

■

Many of the properties of arithmetic of real numbers, such as the commutative, distributive, and associative laws, carry over as properties of vector arithmetic. These are summarized in the next theorem.

THEOREM 2.3 **(ALGEBRAIC PROPERTIES OF VECTORS)** Let a and b be scalars, and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbf{R}^n$. Then

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$

(c) $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$

(d) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(e) $a(b\mathbf{u}) = (ab)\mathbf{u}$

(f) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(g) $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

▶ The **zero vector** is given by

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $-\mathbf{u} = (-1)\mathbf{u}$.

Proof Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Since the components of each vector are real numbers, we have

$$u_1 + v_1 = v_1 + u_1, \quad \ldots \quad u_n + v_n = v_n + u_n$$

Hence

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

which proves (a). For (b), suppose that a is a scalar. Since

$$a(u_1+v_1) = au_1 + av_1, \quad \ldots \quad a(u_n+v_n) = au_n + av_n$$

it follows that

$$a(\mathbf{u}+\mathbf{v}) = a\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} a(u_1+v_1) \\ a(u_2+v_2) \\ \vdots \\ a(u_n+v_n) \end{bmatrix} = \begin{bmatrix} au_1+av_1 \\ au_2+av_2 \\ \vdots \\ au_n+av_n \end{bmatrix} = a\mathbf{v}+a\mathbf{u}$$

Therefore (b) is true. Proofs of the remaining properties are left as exercises. ■

Vectors and Systems of Equations

Let's return to the fertilizer example from the beginning of this section. We have two different kinds, Vigoro and Parker's, with nutrient vectors given by

$$\text{Vigoro:} \quad \mathbf{v} = \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} \qquad \text{Parker's:} \quad \mathbf{p} = \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix}$$

By using vector arithmetic, we can find the nutrient vector for combinations of the two fertilizers. For example, if 500 pounds of Vigoro and 300 pounds of Parker's are mixed, then the total amount of each nutrient is given by

$$5\mathbf{v} + 3\mathbf{p} = 5\begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 145 + 54 \\ 15 + 75 \\ 20 + 18 \end{bmatrix} = \begin{bmatrix} 199 \\ 90 \\ 38 \end{bmatrix}$$

The sum $5\mathbf{v} + 3\mathbf{p}$ is an example of a *linear combination* of vectors.

DEFINITION 2.4

Definition Linear Combination

If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors and $c_1, c_2, \ldots, c_m$ are scalars, then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m$$

is a **linear combination** of the vectors. Note that it is possible for scalars to be negative or equal to zero.

EXAMPLE 2 If possible, find the amount of Vigoro and Parker's required to create a mixture containing 148 pounds of nitrogen, 131 pounds of phosphoric acid, and 38 pounds of potash.

Solution We formulate the problem in terms of a linear combination. Specifically, we need to find scalars x_1 and x_2 such that

▶ (1) is an example of a **vector equation**.

$$x_1\begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 148 \\ 131 \\ 38 \end{bmatrix} \tag{1}$$

Taking one component at a time, we see that this vector equation is equivalent to the system of equations

$$\begin{aligned} 29x_1 + 18x_2 &= 148 \\ 3x_1 + 25x_2 &= 131 \\ 4x_1 + 6x_2 &= 38 \end{aligned}$$

▶ The row operations used in (2) are (in order performed):

$$\begin{aligned} R_1 &\Leftrightarrow R_3 \\ -(3/4)R_1 + R_2 &\Rightarrow R_2 \\ -(29/4)R_1 + R_3 &\Rightarrow R_3 \\ (51/41)R_2 + R_3 &\Rightarrow R_3 \\ (2/41)R_2 &\Rightarrow R_3 \end{aligned}$$

The augmented matrix and echelon form are

$$\begin{bmatrix} 29 & 18 & 148 \\ 3 & 25 & 131 \\ 4 & 6 & 38 \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 38 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Back substitution gives the solution $x_1 = 2$ and $x_2 = 5$. ■

We were lucky in the mixture of components in Example 2. Had we needed 40 pounds of potash instead of 38 pounds, there is no combination that works (see Exercise 37).

Solutions as Linear Combinations

The solution to Example 2 can be expressed in the form of a vector,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Definition Vector Form

In fact, the general solution to any system of linear equations can be expressed as a linear combination of vectors, called the **vector form** of the general solution.

EXAMPLE 3 Express the general solution to the linear system

$$\begin{aligned} 2x_1 - 3x_2 + 10x_3 &= -2 \\ x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

in vector form.

Solution In Example 3 of Section 1.2, we found the general solution to this system. Separating the general solution into the constant term and the term multiplied by the parameter s_1, we have

$$\begin{aligned} x_1 &= 2 - 11s_1 = 2 - 11s_1 \\ x_2 &= 2 - 4s_1 = 2 - 4s_1 \\ x_3 &= s_1 \qquad\;\; = 0 + 1s_1 \end{aligned}$$

Thus the vector form of the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -11 \\ -4 \\ 1 \end{bmatrix}$$

where s_1 can be any real number. ■

A more complicated general solution arises in Example 4 of Section 1.2. There we found the general solution

$$\begin{aligned} x_1 &= -5 - 4s_1 + 2s_2 - 4s_3 = -5 - 4s_1 + 2s_2 - 4s_3 \\ x_2 &= s_1 = 0 + 1s_1 + 0s_2 + 0s_3 \\ x_3 &= -14 - 5s_3 = -14 + 0s_1 + 0s_2 - 5s_3 \\ x_4 &= s_2 = 0 + 0s_1 + 1s_2 + 0s_3 \\ x_5 &= 3 + s_3 = 3 + 0s_1 + 0s_2 + 1s_3 \\ x_6 &= s_3 = 0 + 0s_1 + 0s_2 + 1s_3 \end{aligned}$$

where s_1, s_2, and s_3 can be any real numbers. In vector form, the general solution is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -14 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -4 \\ 0 \\ -5 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

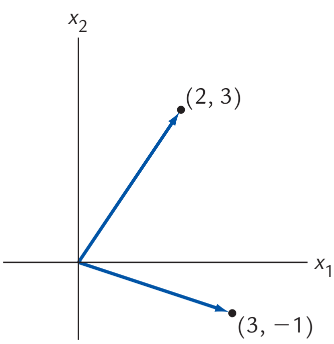

Figure 1 Vectors in $\mathbf{R}^2$.

Geometry of Vectors

Vectors have a geometric interpretation that is most easily understood in $\mathbf{R}^2$. We plot the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by drawing an arrow from the origin to the point (x_1, x_2) in the plane. For example, the vectors (2, 3) and (3, −1) are illustrated in Figure 1. Using an arrow to denote a vector suggests a direction, which is a common interpretation in physics and other sciences, and will frequently be useful for us as well. We call the end of the vector with the arrow the **tip**, and the end at the origin the **tail**.

Definition **Tip, Tail of Vector**

Note that the ordered pair for the point (x_1, x_2) looks the same as the row vector (x_1, x_2). The difference between the two is that vectors have an algebraic and geometric

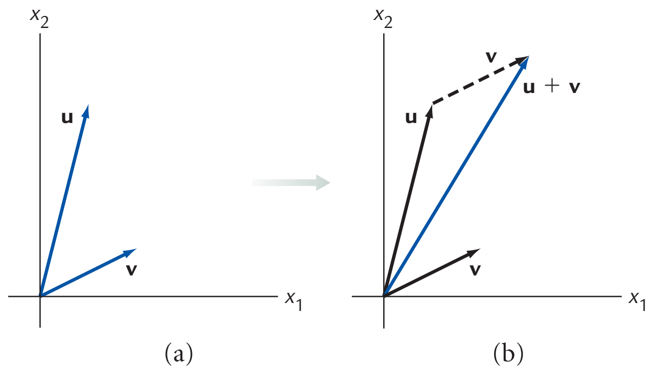

Figure 2 (a) The vectors $\mathbf{u}$ and $\mathbf{v}$. (b) The vector $\mathbf{u} + \mathbf{v}$.

structure that is not associated with points. Most of the time we focus on vectors, so use that interpretation unless the alternative is clearly appropriate.

There are two related geometric procedures for adding vectors.

1. The Tip-to-Tail Rule: Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. Translate the graph of $\mathbf{v}$, preserving direction, so that its tail is at the tip of $\mathbf{u}$. Then the tip of the translated $\mathbf{v}$ is at the tip of $\mathbf{u} + \mathbf{v}$.

Figure 2(a) shows vectors $\mathbf{u}$ and $\mathbf{v}$, and Figure 2(b) shows $\mathbf{u}$, the translated $\mathbf{v}$ (dashed), and $\mathbf{u} + \mathbf{v}$.

The Tip-to-Tail Rule makes sense from an algebraic standpoint. When we add $\mathbf{v}$ to $\mathbf{u}$, we add each component of $\mathbf{v}$ to the corresponding component of $\mathbf{u}$, which is exactly what we are doing geometrically. We also see in Figure 2(b) that we get to the same place if we translate $\mathbf{u}$ instead of $\mathbf{v}$.

The second rule follows easily from the first.

2. The Parallelogram Rule: Let vectors $\mathbf{u}$ and $\mathbf{v}$ form two adjacent sides of a parallelogram with vertices at the origin, the tip of $\mathbf{u}$, and the tip of $\mathbf{v}$. Then the tip of $\mathbf{u} + \mathbf{v}$ is at the fourth vertex.

Figure 3 illustrates the Parallelogram Rule. It is evident that the third and fourth sides of the parallelogram are translated copies of $\mathbf{u}$ and $\mathbf{v}$, which shows the connection to the Tip-to-Tail Rule.

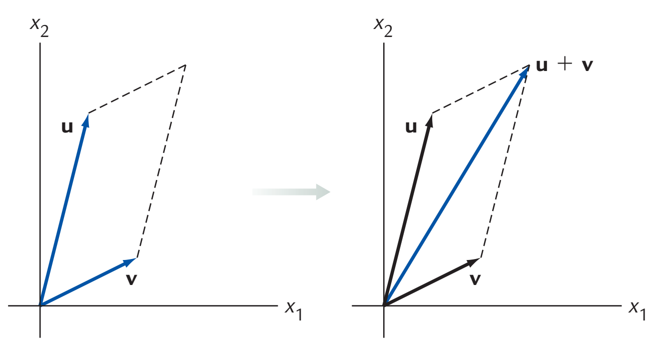

Figure 3 The Parallelogram Rule for vector addition.

Scalar multiplication and subtraction also have nice geometric interpretations.

Scalar Multiplication: If a vector $\mathbf{u}$ is multiplied by a scalar c, then the new vector $c\mathbf{u}$ points in the same direction as $\mathbf{u}$ when $c > 0$ and in the opposite direction when $c < 0$. The length of $c\mathbf{u}$ is equal to the length of $\mathbf{u}$ multiplied by $|c|$.

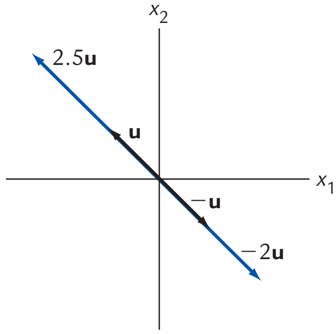

Figure 4 Scalar multiples of the vector **u**.

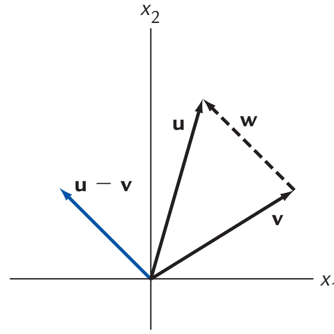

Figure 5 Subtracting vectors.

For example, $-2\mathbf{u}$ points in the opposite direction of $\mathbf{u}$ and is twice as long. (We will consider how to find the length of a vector later in the book.) A few examples of scalar multiples, starting with $\mathbf{u} = (-2, 3)$, are shown in Figure 4.

Subtraction: Draw a vector $\mathbf{w}$ from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$. Then translate $\mathbf{w}$, preserving direction and placing the tail at the origin. The resulting vector is $\mathbf{u} - \mathbf{v}$.

The subtraction procedure is illustrated in Figure 5; it is considered in more detail in Exercise 74.

EXERCISES

For Exercises 1–6, let

$$\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}$$

1. Compute $\mathbf{u} - \mathbf{v}$.

2. Compute $-5\mathbf{u}$.

3. Compute $\mathbf{w} + 3\mathbf{v}$.

4. Compute $4\mathbf{w} - \mathbf{u}$.

5. Compute $-\mathbf{u} + \mathbf{v} + \mathbf{w}$.

6. Compute $3\mathbf{u} - 2\mathbf{v} + 5\mathbf{w}$.

In Exercises 7–10, express the given vector equation as a system of linear equations.

7. $x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$

8. $x_1 \begin{bmatrix} -1 \\ 6 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ -11 \\ 3 \end{bmatrix}$

9. $x_1 \begin{bmatrix} -6 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$

10. $x_1 \begin{bmatrix} 2 \\ 7 \\ 8 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 1 \\ 6 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 5 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 5 \end{bmatrix}$

In Exercises 11–14, express the given system of linear equations as a vector equation.

11. $\begin{aligned} 2x_1 + 8x_2 - 4x_3 &= -10 \\ -x_1 - 3x_2 + 5x_3 &= 4 \end{aligned}$

12. $\begin{aligned} -2x_1 + 5x_2 - 10x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 7x_1 - 17x_2 + 34x_3 &= -16 \end{aligned}$

13. $\begin{aligned} x_1 - x_2 - 3x_3 - x_4 &= -1 \\ -2x_1 + 2x_2 + 6x_3 + 2x_4 &= -1 \\ -3x_1 - 3x_2 + 10x_3 &= 5 \end{aligned}$

14. $\begin{aligned} -5x_1 + 9x_2 &= 13 \\ 3x_1 - 5x_2 &= -9 \\ x_1 - 2x_2 &= -2 \end{aligned}$

In Exercises 15–18, the general solution to a linear system is given. Express this as a linear combination of vectors.

15. $\begin{aligned} x_1 &= -4 + 3s_1 \\ x_2 &= s_1 \end{aligned}$

16. $\begin{aligned} x_1 &= 7 - 2s_1 \\ x_2 &= -3 \\ x_3 &= s_1 \end{aligned}$

17. $\begin{aligned} x_1 &= 4 + 6s_1 - 5s_2 \\ x_2 &= s_2 \\ x_3 &= -9 + 3s_1 \\ x_4 &= s_1 \end{aligned}$

18. $\begin{aligned} x_1 &= 1 - 7s_1 + 14s_2 - s_3 \\ x_2 &= s_3 \\ x_3 &= s_2 \\ x_4 &= -12 + s_1 \\ x_5 &= s_1 \end{aligned}$

In Exercises 19–22, find three different vectors that are a linear combination of the given vectors.

19. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

20. $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \\ -13 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

21. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 9 \\ 6 \\ 11 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 5 \\ -5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 9 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 23–26, a vector equation is given with some unknown entries. Find the unknowns.

23. $-3\begin{bmatrix} a \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ b \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$

24. $4\begin{bmatrix} 4 \\ a \end{bmatrix} + 3\begin{bmatrix} -3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} b \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

25. $-\begin{bmatrix} -1 \\ a \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ b \end{bmatrix} = \begin{bmatrix} c \\ -7 \\ 8 \end{bmatrix}$

26. $-\begin{bmatrix} a \\ 4 \\ -2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 1 \\ b \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ c \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \\ d \end{bmatrix}$

In Exercises 27–30, determine if **b** is a linear combination of the other vectors. If so, write **b** as a linear combination.

27. $\mathbf{a}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

28. $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}$

29. $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix}$

30. $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$

Exercises 31–32 refer to Vigoro and Parker's fertilizers described at the start of the section. Determine the total amount of nitrogen, phosphoric acid, and potash in the given mixture.

31. 200 pounds of Vigoro, 100 pounds of Parker's.

32. 400 pounds of Vigoro, 700 pounds of Parker's.

Exercises 33–36 refer to Vigoro and Parker's fertilizers described at the start of the section. Determine the amount of each type required to yield a mixture containing the given amounts of nitrogen, phosphoric acid, and potash.

33. 112 pounds of nitrogen, 81 pounds of phosphoric acid, and 26 pounds of potash.

34. 285 pounds of nitrogen, 284 pounds of phosphoric acid, and 78 pounds of potash.

35. 123 pounds of nitrogen, 59 pounds of phosphoric acid, and 24 pounds of potash.

36. 159 pounds of nitrogen, 109 pounds of phosphoric acid, and 36 pounds of potash.

Exercises 37–40 refer to Vigoro and Parker's fertilizers described at the start of the section. Show that it is not possible to combine Vigoro and Parker's to obtain the specified mixture of nitrogen, phosphoric acid, and potash.

37. 148 pounds of nitrogen, 131 pounds of phosphoric acid, and 40 pounds of potash.

38. 100 pounds of nitrogen, 120 pounds of phosphoric acid, and 40 pounds of potash.

39. 25 pounds of nitrogen, 72 pounds of phosphoric acid, and 14 pounds of potash.

40. 301 pounds of nitrogen, 8 pounds of phosphoric acid, and 38 pounds of potash.

One 8.3 ounce can of Red Bull contains energy in two forms: 27 grams of sugar and 80 milligrams of caffeine. One 23.5 ounce can of Jolt Cola contains 94 grams of sugar and 280 milligrams of caffeine. In Exercises 41–44, determine the number of cans of each drink that when combined will contain the specified nerve-jangling combination of sugar and caffeine.

41. 148 grams sugar, 440 milligrams caffeine.

42. 309 grams sugar, 920 milligrams caffeine.

43. 242 grams sugar, 720 milligrams caffeine.

44. 457 grams sugar, 1360 milligrams caffeine.

One serving of Lucky Charms contains 10% of the percent daily values (PDV) for calcium, 25% of the PDV for iron, and 25% of the PDV for zinc. One serving of Raisin Bran contains 2% of the PDV for calcium, 25% of the PDV for iron, and 10% of the PDV for zinc. In Exercises 45–48, determine the number of servings of each cereal required to get the given mix of nutrients.

45. 40% of the PDV for calcium, 200% of the PDV for iron, and 125% of the PDV for zinc.

46. 34% of the PDV for calcium, 125% of the PDV for iron, and 95% of the PDV for zinc.

47. 26% of the PDV for calcium, 125% of the PDV for iron, and 80% of the PDV for zinc.

48. 38% of the PDV for calcium, 175% of the PDV for iron, and 115% of the PDV for zinc.

49. An electronics company has two production facilities, A and B. During an average week, facility A produces 2000 computer monitors and 8000 flat panel televisions, and facility B produces 3000 computer monitors and 10,000 flat panel televisions.

(a) Give vectors **a** and **b** that give the weekly production amounts at A and B, respectively.

(b) Compute 8**b**, and then describe what the entries tell us.

(c) Determine the combined output from A and B over a 6-week period.

(d) Determine the number of weeks of production from A and B required to produce 24,000 monitors and 92,000 televisions.

50. An industrial chemical company has three facilities A, B, and C. Each facility produces polyethylene (PE), polyvinyl chloride (PVC), and polystyrene (PS). The table below gives the daily production output (in metric tons) for each facility:

	Facility		
Product	A	B	C
PE	10	20	40
PVC	20	30	70
PS	10	40	50

(a) Give vectors **a**, **b**, and **c** that give the daily production amounts at each facility.

(b) Compute 20**c**, and describe what the entries tell us.

(c) Determine the combined output from all three facilities over a 2-week period. (Note: The facility does not operate on weekends.)

(d) Determine the number of days of production from each facility required to produce 240 metric tons of polyethylene, 420 metric tons of polyvinyl chloride and 320 metric tons of polystyrene.

Exercises 51–54 refer to the following: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors, and suppose that a *point mass* of $m_1, \ldots, m_k$ is located at the tip of each vector. The *center of mass* for this set of point masses is

$$\overline{\mathbf{v}} = \frac{m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k}{m}$$

where $m = m_1 + \cdots + m_k$.

51. Let $\mathbf{u}_1 = (3, 2)$ have mass 5kg, $\mathbf{u}_2 = (-1, 4)$ have mass 3kg, and $\mathbf{u}_3 = (2, 5)$ have mass 2kg. Graph the vectors, and then determine the center of mass.

52. Determine the center of mass for the vectors $\mathbf{u}_1 = (-1, 0, 2)$ (mass 4kg), $\mathbf{u}_2 = (2, 1, -3)$ (mass 1kg), $\mathbf{u}_3 = (0, 4, 3)$ (mass 2kg), and $\mathbf{u}_4 = (5, 2, 0)$ (mass 5kg).

53. Determine how to divide a total mass of 11kg among the vectors $\mathbf{u}_1 = (-1, 3)$, $\mathbf{u}_2 = (3, -2)$, and $\mathbf{u}_3 = (5, 2)$ so that the center of mass is $\left(13/11, 16/11\right)$.

54. Determine how to divide a total mass of 11kg among the vectors $\mathbf{u}_1 = (1, 1, 2)$, $\mathbf{u}_2 = (2, -1, 0)$, $\mathbf{u}_3 = (0, 3, 2)$, and $\mathbf{u}_4 = (-1, 0, 1)$ so that the center of mass is $\left(4/11, 5/11, 12/11\right)$.

FIND AN EXAMPLE For Exercises 55–62, find an example that meets the given specifications.

55. Two nonzero vectors **u** and **v** in $\mathbf{R}^3$ such that $\mathbf{u}+\mathbf{v} = (3, 2, -1)$.

56. Two nonzero vectors **u** and **v** in $\mathbf{R}^4$ such that $\mathbf{u} - \mathbf{v} = (4, -2, 0, -1)$.

57. Three nonzero vectors in $\mathbf{R}^3$ whose sum is the zero vector.

58. Three nonzero vectors in $\mathbf{R}^4$ whose sum is the zero vector.

59. Two vectors **u** and **v** in $\mathbf{R}^2$ that point in the same direction.

60. Two vectors **u** and **v** in $\mathbf{R}^2$ that point in opposite directions.

61. A linear system with two equations and two variables that has $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ as the only solution.

62. A linear system with two equations and three variables that has $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ as the general solution.

TRUE OR FALSE For Exercises 63–72, determine if the statement is true or false, and justify your answer.

63. If $\mathbf{u} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, then $-2\mathbf{u} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$.

64. If $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, then $\mathbf{u} - \mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

65. If **u** and **v** are vectors and c is a scalar, then $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$.

66. A vector can have positive or negative components, but a scalar must be positive.

67. If c_1 and c_2 are scalars and **u** is a vector, then $(c_1 + \mathbf{u})c_2 = c_1c_2 + c_2\mathbf{u}$.

68. The vectors $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$ point in opposite directions.

69. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $(-2, 1)$ are the same when both are considered as vectors.

70. The vector $2\mathbf{u}$ is longer than the vector $-3\mathbf{u}$.

71. The parallelogram rule for adding vectors only works in the first quadrant.

72. The difference $\mathbf{u} - \mathbf{v}$ is found by adding $-\mathbf{u}$ to **v**.

73. Prove each of the following parts of Theorem 2.3:

(a) Part (c).	**(b)** Part (d).	**(c)** Part (e).
(d) Part (f).	**(e)** Part (g).	**(f)** Part (h).

74. In this exercise we verify the geometric subtraction rule shown in Figure 5 by combining the identity $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$ and the Tip-to- Tail rule for addition. Draw a set of coordinate axes, and then sketch and label each of the following:

(a) Vectors **u** and **v** of your choosing.

(b) The vector $-\mathbf{v}$.

(c) The translation of $-\mathbf{v}$ so that its tail is at the tip of **u**.

(d) Using the Tip-to-Tail rule, the vector $\mathbf{u} + (-\mathbf{v})$.

Explain why the vector you get is the same as the one obtained using the subtraction rule shown in Figure 5.

In Exercises 75–76, sketch the graph of the vectors **u** and **v** and then use the Tip-to-Tail Rule to sketch the graph of $\mathbf{u} + \mathbf{v}$.

75. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

76. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

In Exercises 77–78, sketch the graph of the vectors $\mathbf{u}$ and $\mathbf{v}$ and then use the Parallelogram Rule to sketch the graph of $\mathbf{u}+\mathbf{v}$.

77. $\mathbf{u} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

78. $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

In Exercises 79–80, sketch the graph of the vectors $\mathbf{u}$ and $\mathbf{v}$, and then use the subtraction procedure shown in Figure 5 to sketch the graph of $\mathbf{u}-\mathbf{v}$.

79. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

80. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

In Exercises 81–82, find the solutions to the vector equation.

81. $x_1 \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

82. $x_1 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 2 \\ -3 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ -6 \end{bmatrix}$

2.2 Span

We open this section with a fictitious hypothetical situation. Imagine that you live in the two-dimensional plane $\mathbf{R}^2$ and have just purchased a new car, the VecMobile II. The VecMobile II is delivered at the origin (0, 0) and is a fairly simple vehicle. At any given time, it can be pointed in the direction of

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

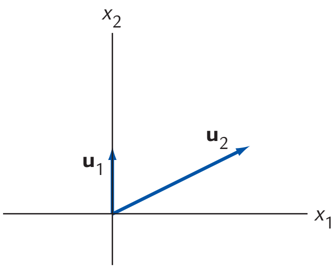

Figure 1 The VecMobile II vectors.

both shown in Figure 1. The VecMobile II can also go in forward or reverse.

Despite its simplicity, there are many places in $\mathbf{R}^2$ that we can go in the VecMobile II. For instance, the point (2, 3) can be reached by first traversing $2\mathbf{u}_1$, changing direction, and then traversing $\mathbf{u}_2$, as shown in Figure 2. The trip also could be made in the reverse order, first taking $\mathbf{u}_2$ and then $2\mathbf{u}_1$. Since we are traversing vectors in a "tip-to-tail" manner, the entire trip can be summarized by the sum

$$2\mathbf{u}_1 + \mathbf{u}_2 = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

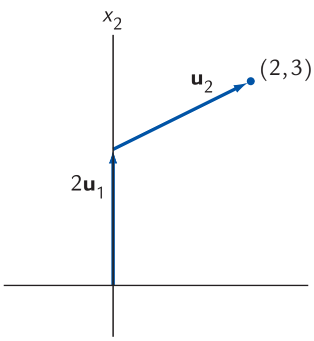

Figure 2 $2\mathbf{u}_1 + \mathbf{u}_2 = (2, 3)$.

Figure 3 depicts a more complicated path that arises from traversing $3\mathbf{u}_1$, then $-\mathbf{u}_2$, then $-2\mathbf{u}_1$, and finally $2\mathbf{u}_2$. This simplifies algebraically to

$$3\mathbf{u}_1 - \mathbf{u}_2 - 2\mathbf{u}_1 + 2\mathbf{u}_2 = \mathbf{u}_1 + \mathbf{u}_2$$

Any path taken in the VecMobile II can be similarly simplified, so that the set of all possible destinations can be expressed as

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2$$

where x_1 and x_2 can be any real numbers. This set of linear combinations is called the *span* of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

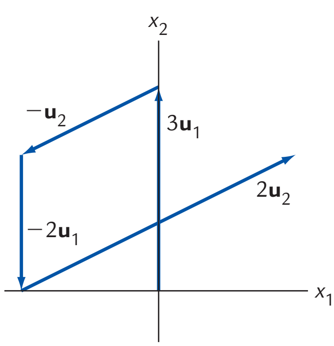

Figure 3 $3\mathbf{u}_1 - \mathbf{u}_2 - 2\mathbf{u}_1 + 2\mathbf{u}_2 = (2, 2)$.

Although the VecMobile II is simple, we can go anywhere within $\mathbf{R}^2$ by first selecting the multiple (positive or negative) of $\mathbf{u}_2$ to reach the horizontal position we desire and then using a multiple of $\mathbf{u}_1$ to adjust the vertical position. For example, to reach (6, 5), we start with $3\mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and then add $2\mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

$$3\mathbf{u}_2 + 2\mathbf{u}_1 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

EXAMPLE 1 Show algebraically that the VecMobile II can reach any position in $\mathbf{R}^2$.

Solution Suppose that we want to reach an arbitrary point (a, b). To do so, we need to find scalars x_1 and x_2 such that

$$x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

This vector equation translates into the system of equations

$$\begin{aligned} 2x_2 &= a \\ x_1 + x_2 &= b \end{aligned}$$

which has the unique solution $x_1 = b - a/2$ and $x_2 = a/2$. We now know exactly how to find the scalars x_1 and x_2 required to reach any point (a, b), and so we can conclude that the VecMobile II can get anywhere in $\mathbf{R}^2$. ■

The notion of span generalizes to sets of vectors in $\mathbf{R}^n$.

DEFINITION 2.5

Definition Span

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a set of vectors in $\mathbf{R}^n$. The **span** of this set is denoted $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and is defined to be the set of all linear combinations

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m$$

where $x_1, x_2, \ldots, x_m$ can be any real numbers.

If $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \mathbf{R}^n$, then we say that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ spans $\mathbf{R}^n$.

The VecMobile II in $\mathbf{R}^3$

Suppose that our world has expanded from $\mathbf{R}^2$ to $\mathbf{R}^3$. Happily, a VecMobile II model exists in $\mathbf{R}^3$. Like the $\mathbf{R}^2$ version, this vehicle only can move in two directions—in this case, that of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

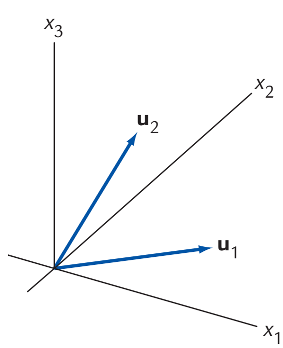

Figure 4 The VecMobile II (in $\mathbf{R}^3$) vectors.

shown in Figure 4. Following the reasoning given earlier, we know that it is possible to get to any location that can be described as a linear combination of the form

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2$$

That is, we can get anywhere within $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. This covers a lot of territory, but not all of $\mathbf{R}^3$.

EXAMPLE 2 Show that the VecMobile II cannot reach the point $(1, 0, 0)$.

Solution In order for the VecMobile II to reach $(1, 0, 0)$, there need to be scalars x_1 and x_2 that satisfy the equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

▶ The row operations used in (1) are (in order performed):

$$\begin{aligned} R_1 &\Leftrightarrow R_2 \\ -2R_1 + R_2 &\Rightarrow R_2 \\ -R_1 + R_3 &\Rightarrow R_3 \\ R_2 &\Leftrightarrow R_3 \\ -3R_2 + R_3 &\Rightarrow R_3 \end{aligned}$$

This is equivalent to the linear system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 0 \\ x_1 + 3x_2 &= 0 \end{aligned}$$

Transferring to an augmented matrix and reducing, we find

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The third row of the reduced matrix corresponds to the equation $0 = 1$. Thus the system has no solutions and hence the VecMobile II cannot reach the point $(1, 0, 0)$. ■

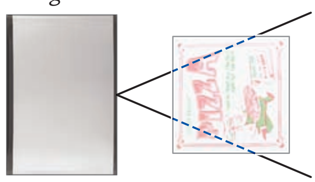

Figure 5 String vectors and pizza box.

Here is another way to visualize the span of two vectors in $\mathbf{R}^3$. Get two pieces of string, about 3 feet long each, and tie them both to some solid object (like a refrigerator). Get a friend to pull the strings tight and in different directions. These are your vectors.

Next get a light-weight flat surface (a pizza box works well) and gently rest it on the strings (see Figure 5). Think of the surface as representing a plane. Then the span of the two "string" vectors is the set of all vectors that lie within the plane. Note that no matter the angle of the strings, if you are doing this correctly it is possible to rest the surface on them.

Figure 6 shows a plane resting on $\mathbf{u}_1$ and $\mathbf{u}_2$. The span$\{\mathbf{u}_1, \mathbf{u}_2\}$ consists exactly of those vectors that are contained in the plane. Therefore, if $\mathbf{u}_3$ is contained in span$\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{u}_3$ will lie in the plane, as shown in Figure 7(a). On the other hand, if $\mathbf{u}_3$ is not contained in span$\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{u}_3$ will be outside the plane, as in Figure 7(b).

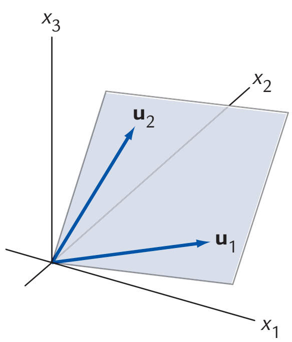

Figure 6 The plane is equal to span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

The VecMobile III in $\mathbf{R}^3$

Continuing with our hypothetical line of cars, consumers disappointed by the limitations of the VecMobile II in $\mathbf{R}^3$ can spend more money and buy the VecMobile III, which can be pointed in the directions given by vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{or} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

With the addition of $\mathbf{u}_3$, it is clear that the point $(1, 0, 0)$ can now be reached. But can we get to all points in $\mathbf{R}^3$?

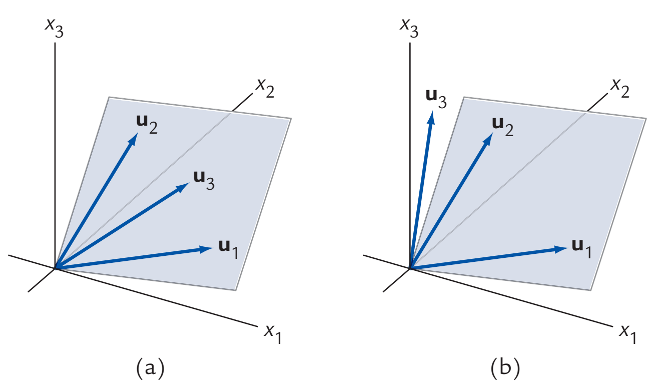

Figure 7 In (a), $\mathbf{u}_3$ is in span$\{\mathbf{u}_1, \mathbf{u}_2\}$. In (b), $\mathbf{u}_3$ is *not* in span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

EXAMPLE 3 Show that the VecMobile III can reach any point in $\mathbf{R}^3$.

Solution To reach an arbitrary point (a, b, c), we need to find scalars x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2}$$

The augmented matrix and equivalent reduced echelon form are

$$\begin{bmatrix} 2 & 1 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & 3 & 0 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & (3b - 2c) \\ 0 & 1 & 0 & (-b + c) \\ 0 & 0 & 1 & (a - 5b + 3c) \end{bmatrix} \tag{3}$$

▶ The row operations used in (3) are (in order performed):
$$\begin{aligned} R_1 &\Leftrightarrow R_2 \\ -2R_1 + R_2 &\Rightarrow R_2 \\ -R_1 + R_3 &\Rightarrow R_3 \\ R_2 &\Leftrightarrow R_3 \\ 3R_2 + R_3 &\Rightarrow R_3 \\ -2R_2 + R_1 &\Rightarrow R_1 \end{aligned}$$

We see from the reduced echelon form that the solution is

$$\begin{aligned} x_1 &= 3b - 2c \\ x_2 &= -b + c \\ x_3 &= a - 5b + 3c \end{aligned}$$

This shows that we can reach any point (a, b, c), and so $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbf{R}^3$. ■

The solution also shows how to get to any point. For example, if we want to get to the point $(-3, 1, 4)$, we substitute in $a = -3$, $b = 1$, and $c = 4$, which gives $x_1 = -5$, $x_2 = 3$, and $x_3 = 4$. Hence the linear combination we need is

$$-5\mathbf{u}_1 + 3\mathbf{u}_2 + 4\mathbf{u}_3$$

In Example 3, the vectors in (2) become the columns in the augmented matrix in (3),

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{array}{c} \begin{bmatrix} 2 & 1 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & 3 & 0 & c \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v} \end{array} \end{array} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v} \end{bmatrix}$$

The same is true in general. Given the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{v}$$

the augmented matrix for the corresponding linear system is

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m & \mathbf{v} \end{bmatrix}$$

where the columns are given by the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$, and $\mathbf{v}$.

THEOREM 2.6 Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$. Then $\mathbf{v}$ is an element of *span*$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ if and only if the linear system represented by the augmented matrix

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m & \mathbf{v} \end{bmatrix} \tag{4}$$

has a solution.

Proof The vector $\mathbf{v}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ if and only if there exist scalars $x_1, x_2, \ldots, x_m$ that satisfy

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{v}$$

This is true if and only if the corresponding linear system has a solution. As noted above, the linear system is equivalent to the augmented matrix (4), so the proof is complete. ■

How Many Vectors Are Needed to Span $\mathbf{R}^n$?

In the examples we have seen, a set of two vectors spanned $\mathbf{R}^2$ and a set of three vectors spanned $\mathbf{R}^3$. This suggests the following question.

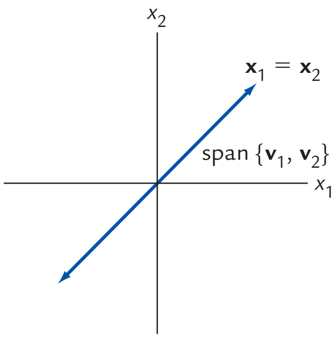

Figure 8 The span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbf{R}^2$ in Example 4.

EXAMPLE 4 Is it always true that a set of n vectors will span $\mathbf{R}^n$?

Solution Not always. For example, the span of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

is a line in $\mathbf{R}^2$ (shown in Figure 8) because $\mathbf{v}_2 = 2\mathbf{v}_1$, so is not all of $\mathbf{R}^2$.

A more subtle example is given by the set of vectors in $\mathbf{R}^3$,

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$$

Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are the vectors from the VecMobile II in $\mathbf{R}^3$. It is straightforward to verify that $\mathbf{u}_3$ is given by the linear combination

$$\mathbf{u}_3 = 2\mathbf{u}_1 - 3\mathbf{u}_2$$

Thus any vector that is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ can be written as a linear combination of just $\mathbf{u}_1$ and $\mathbf{u}_2$ by substituting

$$\begin{aligned} x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3(2\mathbf{u}_1 - 3\mathbf{u}_2) \\ &= (x_1 + 2x_3)\mathbf{u}_1 + (x_2 - 3x_3)\mathbf{u}_2 \end{aligned}$$

Therefore $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and since we have already shown (in Example 2) that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} \neq \mathbf{R}^3$, it follows that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \mathbf{R}^3$. ■

The preceding argument serves as a model for proving the next theorem.

THEOREM 2.7 Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{u}$ be vectors in $\mathbf{R}^n$. If $\mathbf{u}$ is in *span*$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, then

$$\text{span}\{\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}.$$

Proof Let $S_0 = \text{span}\{\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and $S_1 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$. We need to show that sets $S_0 = S_1$, which we do by showing that each is a subset of the other. First suppose that a vector $\mathbf{v}$ is in S_1. Then there exist scalars $a_1, \ldots, a_m$ such that

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m = 0\mathbf{u} + a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$$

Hence $\mathbf{v}$ is also in S_0, so S_1 is a subset of S_0.

Now suppose that $\mathbf{v}$ is in S_0. Then there exist scalars $b_0, b_1, \ldots, b_m$ such that $\mathbf{v} = b_0\mathbf{u} + b_1\mathbf{u}_1 + \cdots + b_m\mathbf{u}_m$. Since $\mathbf{u}$ is in S_1, there also exist scalars $c_1, \ldots, c_m$ such that $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m$. Then

$$\begin{aligned} \mathbf{v} &= b_0\left(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m\right) + b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_m\mathbf{u}_m \\ &= (b_0c_1 + b_1)\mathbf{u}_1 + (b_0c_2 + b_2)\mathbf{u}_2 + \cdots + (b_0c_m + b_m)\mathbf{u}_m \end{aligned}$$

Hence $\mathbf{v}$ is in S_1, so S_0 is a subset of S_1. Since S_0 and S_1 are subsets of each other, it follows that $S_0 = S_1$. ■

EXAMPLE 5 Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$. If $m < n$, can $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ span $\mathbf{R}^n$?

Solution The answer is no. If $m < n$, we can always construct $\mathbf{b}$ in $\mathbf{R}^n$ that is not in the span of the given vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. We illustrate the procedure for vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 7 \\ 4 \\ -6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

in $\mathbf{R}^4$, but it will work in general. Start by forming the matrix with only our vectors as columns,

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 0 \\ -1 & 4 & 2 \\ 2 & -6 & 5 \end{bmatrix}$$

Now perform the usual row operations needed to transform the matrix to echelon form, recording each operation along the way. We need only perform enough operations to introduce a row of zeroes on the bottom of the matrix, which must be possible because there are more rows than columns (see Exercise 54 in Section 1.2). The "Forward Operations" shown in the margin yields the transformation

▶ Forward Operations:

$$\begin{aligned} 2R_1 + R_2 &\Rightarrow R_2 \\ R_1 + R_3 &\Rightarrow R_3 \\ -2R_1 + R_4 &\Rightarrow R_4 \\ -R_2 + R_3 &\Rightarrow R_3 \\ R_3 &\Leftrightarrow R_4 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 0 \\ -1 & 4 & 2 \\ 2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The next step is to append a column to the right side of the echelon matrix,

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Viewing this new matrix as an augmented matrix, we see that the bottom row is equivalent to the equation $0 = 1$, so that the associated linear system has no solutions. If we now reverse the row operations used previously (as shown in the margin), the first three columns of the augmented matrix will be returned to their original form. This gives us

▶ Reverse Operations:

$$\begin{aligned} R_3 &\Leftrightarrow R_4 \\ R_2 + R_3 &\Rightarrow R_3 \\ 2R_1 + R_4 &\Rightarrow R_4 \\ -R_1 + R_3 &\Rightarrow R_3 \\ -2R_1 + R_2 &\Rightarrow R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 0 & 0 \\ -1 & 4 & 2 & 1 \\ 2 & -6 & 5 & 0 \end{bmatrix} \tag{6}$$

Since the system associated with the augmented matrix (5) has no solutions, the system associated with the equivalent augmented matrix (6) also has no solutions. But this system is represented by the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{b} \quad \text{where} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus this vector equation can have no solutions, and so it follows that $\mathbf{b}$ is *not* in the span of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Therefore this set of vectors does not span $\mathbf{R}^4$. ■

As long as $m < n$, the argument described above generalizes to any set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ in $\mathbf{R}^n$. Theorem 2.8 summarizes the main results of this subsection.

THEOREM 2.8 Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a set of vectors in $\mathbf{R}^n$. If $m < n$, then this set does not span $\mathbf{R}^n$. If $m \geq n$, then the set might span $\mathbf{R}^n$ or it might not. In this case, we cannot say more without additional information about the vectors.

The proof of this theorem is left as an exercise.

The Equation $Ax = b$

By now we are comfortable with translating back and forth between vector equations and linear systems. Here we give new notation that will be used for a variety of purposes, including expressing linear systems in a compact form.

Let A be the matrix with columns $\mathbf{a}_1 = \begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 8 \\ 6 \\ -1 \end{bmatrix}$. That is,

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 5 & 6 \\ 7 & -1 \end{bmatrix}$$

Also, let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then the product of the matrix A and the vector $\mathbf{x}$ is defined to be

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

Thus $A\mathbf{x}$ is a linear combination of the columns of A, with the scalars given by the components of $\mathbf{x}$. Now take it a step farther and let

$$\mathbf{b} = \begin{bmatrix} 18 \\ 31 \\ 3 \end{bmatrix}$$

Then $A\mathbf{x} = \mathbf{b}$ is a compact form of the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, which in turn is equivalent to the linear system

$$\begin{aligned} 10x_1 + 8x_2 &= 18 \\ 5x_1 + 6x_2 &= 31 \\ 7x_1 - x_2 &= 3 \end{aligned}$$

Below is the general formula for multiplying a matrix by a vector.

DEFINITION 2.9 Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be vectors in $\mathbf{R}^n$. If

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \tag{7}$$

then $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m$.

▶ Remember: The product $A\mathbf{x}$ only is defined when the number of columns of A equals the number of components (entries) of $\mathbf{x}$.

EXAMPLE 6 Find A, $\mathbf{x}$, and $\mathbf{b}$ so that the equation $A\mathbf{x} = \mathbf{b}$ corresponds to the system of equations

$$\begin{aligned} 4x_1 - 3x_2 + 7x_3 - x_4 &= 13 \\ -x_1 + 2x_2 + 6x_4 &= -2 \\ x_2 - 3x_3 - 5x_4 &= 29 \end{aligned}$$

Solution Translating the system to the form $A\mathbf{x} = \mathbf{b}$, the matrix A will contain the coefficients of the system, the vector $\mathbf{x}$ has the variables, and the vector $\mathbf{b}$ will contain the constant terms. Thus we have

$$A = \begin{bmatrix} 4 & -3 & 7 & -1 \\ -1 & 2 & 0 & 6 \\ 0 & 1 & -3 & -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13 \\ -2 \\ 29 \end{bmatrix}$$

■

EXAMPLE 7 Suppose that

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -6 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Find the following (if they exist):

(a) $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

(b) The system of equations corresponding to $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{v}_2$.

Solution

(a) In order for $A\mathbf{v}_1$ to exist, the number of columns of A must equal the number of components of $\mathbf{v}_1$. Since this is *not* the case, $A\mathbf{v}_1$ does not exist.

On the other hand, $\mathbf{v}_2$ has two components, so $A\mathbf{v}_2$ exists. We have

$$A\mathbf{v}_2 = -6\mathbf{a}_1 + 4\mathbf{a}_2 = -6 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -4 \end{bmatrix}$$

(b) Since $\mathbf{x}$ has two components, $A\mathbf{x}$ exists. Moreover, $A\mathbf{x}$ and $\mathbf{b}$ both have three components, so the equation $A\mathbf{x} = \mathbf{b}$ is defined and corresponds to the system

$$\begin{aligned} x_1 + 3x_2 &= -1 \\ -2x_1 + x_2 &= 2 \\ - x_2 &= 5 \end{aligned}$$

For the second part, $A\mathbf{x}$ exists and has three components, but $\mathbf{v}_2$ has only two components, so $A\mathbf{x} = \mathbf{v}_2$ is undefined. ■

We close this section with a theorem that ties together several closely related ideas.

THEOREM 2.10 Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and $\mathbf{b}$ be vectors in $\mathbf{R}^n$. Then the following statements are equivalent. That is, if one is true, then so are the others, and if one is false, then so are the others.

(a) $\mathbf{b}$ is in span$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

(b) The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$ has at least one solution.

(c) The linear system corresponding to $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{b} \end{bmatrix}$ has at least one solution.

(d) The equation $A\mathbf{x} = \mathbf{b}$, with A and $\mathbf{x}$ given as in (7), has at least one solution.

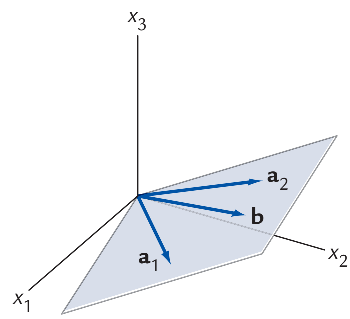

Figure 9 span{$\mathbf{a}_1$, $\mathbf{a}_2$} includes **b**, so (8) has a solution.

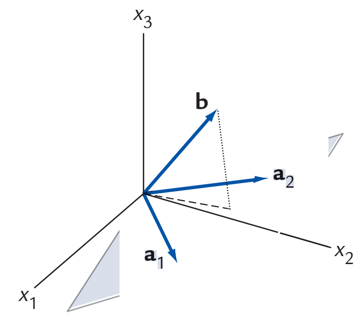

Figure 10 span{$\mathbf{a}_1$, $\mathbf{a}_2$} does not include **b**, so (8) does not have a solution.

The theorem follows directly from the definitions, so the proof is left as an exercise. Although this result is not hard to arrive at, it is important because it explicitly states the connection between these different formulations of the same basic idea.

As a quick application, note that the vector **b** in Figure 9 is in span{$\mathbf{a}_1$, $\mathbf{a}_2$}, so by Theorem 2.10 it follows that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{8}$$

has at least one solution. On the other hand, if $\mathbf{a}_1$, $\mathbf{a}_2$, and **b** are as shown in Figure 10, then (8) has no solutions.

EXERCISES

For Exercises 1–6, find three vectors that are in the span of the given vectors.

1. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$; $\mathbf{u}_2 = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$

2. $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

4. $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix}$

5. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -4 \\ 0 \\ 7 \end{bmatrix}$

6. $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 12 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

For Exercises 7–12, determine if **b** is in the span of the other given vectors. If so, write **b** as a linear combination of the other vectors.

7. $\mathbf{a}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ -15 \end{bmatrix}$

8. $\mathbf{a}_1 = \begin{bmatrix} 10 \\ -15 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -30 \\ 45 \end{bmatrix}$

9. $\mathbf{a}_1 = \begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$

10. $\mathbf{a}_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -6 \\ 9 \\ 2 \end{bmatrix}$

11. $\mathbf{a}_1 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -10 \\ -8 \\ 7 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -4 \\ 2 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix}$

In Exercises 13–16, find A, $\mathbf{x}$, and $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ corresponds to the given linear system.

13.
$$\begin{aligned} 2x_1 + 8x_2 - 4x_3 &= -10 \\ -x_1 - 3x_2 + 5x_3 &= 4 \end{aligned}$$

14.
$$\begin{aligned} -2x_1 + 5x_2 - 10x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 7x_1 - 17x_2 + 34x_3 &= -16 \end{aligned}$$

15.
$$\begin{aligned} x_1 - x_2 - 3x_3 - x_4 &= -1 \\ -2x_1 + 2x_2 + 6x_3 + 2x_4 &= -1 \\ -3x_1 - 3x_2 + 10x_3 &= 5 \end{aligned}$$

16. $$\begin{aligned} -5x_1 + 9x_2 &= 13 \\ 3x_1 - 5x_2 &= -9 \\ x_1 - 2x_2 &= -2 \end{aligned}$$

In Exercises 17–20, find an equation involving vectors that corresponds to the given linear system.

17. $$\begin{aligned} 5x_1 + 7x_2 - 2x_3 &= 9 \\ x_1 - 5x_2 - 4x_3 &= 2 \end{aligned}$$

18. $$\begin{aligned} 4x_1 - 5x_2 - 3x_3 &= 0 \\ 3x_1 + 4x_2 + 2x_3 &= 1 \\ 6x_1 - 13x_2 + 7x_3 &= 2 \end{aligned}$$

19. $$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 5x_4 &= 12 \\ -5x_2 + 7x_3 + 3x_4 &= 6 \\ 3x_1 + 8x_2 + 2x_3 - x_4 &= 2 \end{aligned}$$

20. $$\begin{aligned} 4x_1 - 9x_2 &= 11 \\ 2x_1 + 4x_2 &= 9 \\ x_1 - 7x_2 &= 2 \end{aligned}$$

In Exercises 21–24, determine if the columns of the given matrix span $\mathbf{R}^2$.

21. $\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 4 & -12 \\ 2 & 6 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & -1 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 2 & 7 \end{bmatrix}$

In Exercises 25–28, determine if the columns of the given matrix span $\mathbf{R}^3$.

25. $\begin{bmatrix} 3 & 1 & 0 \\ 5 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 2 & 8 \\ -2 & 3 & 7 \\ 3 & -1 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 1 & -3 & 5 \\ 1 & 4 & 2 & 6 \\ 0 & 3 & 3 & 3 \end{bmatrix}$

28. $\begin{bmatrix} -4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 8 \\ 5 & -1 & 1 & -4 \end{bmatrix}$

In Exercises 29–34, a matrix A is given. Determine if the system $A\mathbf{x} = \mathbf{b}$ (where $\mathbf{x}$ and $\mathbf{b}$ have the appropriate number of components) has a solution for all choices of $\mathbf{b}$.

29. $A = \begin{bmatrix} 3 & -4 \\ 4 & 2 \end{bmatrix}$

30. $A = \begin{bmatrix} -9 & 21 \\ 6 & -14 \end{bmatrix}$

31. $A = \begin{bmatrix} 8 & 1 \\ 0 & -1 \\ -3 & 2 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 1 & 0 & 5 \end{bmatrix}$

33. $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ 5 & -4 & -3 \end{bmatrix}$

34. $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 2 \\ -5 & 3 & -9 \\ 3 & 0 & 9 \end{bmatrix}$

For Exercises 35–38, find a vector of matching dimension that is *not* in the given span.

35. $\text{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

36. $\text{span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}$

37. $\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

38. $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \right\}$

39. Find all values of h such that the vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ span $\mathbf{R}^2$, where

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} h \\ 6 \end{bmatrix}$$

40. Find all values of h such that the vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ span $\mathbf{R}^2$, where

$$\mathbf{a}_1 = \begin{bmatrix} -3 \\ h \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

41. Find all values of h such that the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbf{R}^3$, where

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} h \\ 8 \\ 10 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

42. Find all values of h such that the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbf{R}^3$, where

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ h \\ 7 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

FIND AN EXAMPLE For Exercises 43–50, find an example that meets the given specifications.

43. Four distinct nonzero vectors that span $\mathbf{R}^3$.

44. Four distinct nonzero vectors that span $\mathbf{R}^4$.

45. Four distinct nonzero vectors that do not span $\mathbf{R}^3$.

46. Four distinct nonzero vectors that do not span $\mathbf{R}^4$.

47. Two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbf{R}^3$ that span the set of all vectors of the form $\mathbf{v} = (v_1, v_2, 0)$.

48. Three vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ in $\mathbf{R}^4$ that span the set of all vectors of the form $\mathbf{v} = (0, v_2, v_3, v_4)$.

49. Two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbf{R}^3$ that span the set of all vectors of the form $\mathbf{v} = (v_1, v_2, v_3)$ where $v_1 + v_2 + v_3 = 0$.

50. Three vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ in $\mathbf{R}^4$ that span the set of all vectors of the form $\mathbf{v} = (v_1, v_2, v_3, v_4)$ where $v_1 + v_2 + v_3 + v_4 = 0$.

TRUE OR FALSE For Exercises 51–64, determine if the statement is true or false, and justify your answer.

51. If $m < n$, then a set of m vectors cannot span $\mathbf{R}^n$.

52. If a set of vectors includes $\mathbf{0}$, then it cannot span $\mathbf{R}^n$.

53. Suppose A is a matrix with n rows and m columns. If $n < m$, then the columns of A span $\mathbf{R}^n$.

54. Suppose A is a matrix with n rows and m columns. If $m < n$, then the columns of A span $\mathbf{R}^n$.

55. If A is a matrix with columns that span $\mathbf{R}^n$, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

56. If A is a matrix with columns that span $\mathbf{R}^n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbf{R}^n$.

57. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbf{R}^3$, then so does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

58. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ does not span $\mathbf{R}^3$, then neither does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

59. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ spans $\mathbf{R}^3$, then so does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

60. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ does not span $\mathbf{R}^3$, then neither does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

61. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}.$$

62. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \neq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}.$$

63. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}.$$

64. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \neq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}.$$

65. Which of the following sets of vectors in $\mathbf{R}^3$ can possibly span $\mathbf{R}^3$? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

66. Which of the following sets of vectors in $\mathbf{R}^3$ cannot possibly span $\mathbf{R}^3$? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

67. Prove that if c is a nonzero scalar, then $\operatorname{span}\{\mathbf{u}\} = \operatorname{span}\{c\mathbf{u}\}$.

68. Prove that if c_1 and c_2 are nonzero scalars, then $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$.

69. Suppose that S_1 are S_2 are two finite sets of vectors, and that S_1 is a subset of S_2. Prove that the span of S_1 is a subset of the span of S_2.

70. Prove that if $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbf{R}^2$, then $\operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\} = \mathbf{R}^2$.

71. Prove that if $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbf{R}^3$, then $\operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\} = \mathbf{R}^3$.

72. Suppose that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a subset of $\mathbf{R}^n$, with $m > n$. Prove that if $\mathbf{b}$ is in $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, then there are infinitely many ways to express $\mathbf{b}$ as a linear combination of $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$.

73. Prove Theorem 2.8.

74. Prove Theorem 2.10.

C For Exercises 75–78, determine if the claimed equality is true or false.

75. $\operatorname{span}\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\5\\4\end{bmatrix}, \begin{bmatrix}-2\\3\\0\end{bmatrix}\right\} = \mathbf{R}^3$

76. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\4\\-1\end{bmatrix}, \begin{bmatrix}4\\6\\-6\end{bmatrix}\right\} = \mathbf{R}^3$

77. $\operatorname{span}\left\{\begin{bmatrix}4\\0\\2\\3\end{bmatrix}, \begin{bmatrix}7\\-4\\6\\7\end{bmatrix}, \begin{bmatrix}1\\3\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\2\\0\\2\end{bmatrix}\right\} = \mathbf{R}^4$

78. $\operatorname{span}\left\{\begin{bmatrix}3\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}8\\5\\-9\\7\end{bmatrix}, \begin{bmatrix}1\\6\\2\\-3\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\5\end{bmatrix}\right\} = \mathbf{R}^4$

2.3 Linear Independence

The myology clinic at a university research hospital helps patients recover muscle mass lost due to illness. After a full evaluation, patients receive exercise training and each is given a nutritional powder that has the exact balance of protein, fat, and carbohydrates required to meet his or her needs. The nutritional powders are created by combining

	Brand			
	A	B	C	D
Protein	16	22	18	18
Fat	2	4	0	2
Carbohydrates	8	4	4	6

Table 1 **Nutritional Powder Brand Components (grams per serving)**

some or all of four powder brands that the clinic keeps in stock. The components for brands A, B, C, and D (in grams per serving) are shown in Table 1.

Stocking all four brands is expensive, as they have a limited shelf life and take up valuable storage space. The clinic would like to eliminate unnecessary brands, but it does not want to sacrifice any flexibility to create specialized combinations. Are all four brands needed, or can they get by with fewer?

We can solve this problem using vectors, but first we need to develop some additional concepts. Recall that in the previous section we noted the set of vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix} \tag{1}$$

are such that the third is a linear combination of the first two, with

$$\mathbf{u}_3 = 2\mathbf{u}_1 - 3\mathbf{u}_2 \tag{2}$$

Thus, in a sense, $\mathbf{u}_3$ depends on $\mathbf{u}_1$ and $\mathbf{u}_2$. We can also solve (2) for $\mathbf{u}_1$ or $\mathbf{u}_2$,

$$\mathbf{u}_1 = \frac{3}{2}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3 \qquad \text{or} \qquad \mathbf{u}_2 = \frac{2}{3}\mathbf{u}_1 - \frac{1}{3}\mathbf{u}_3$$

so each of the vectors is "dependent" on the others. Rather than separating out one particular vector, we can move all terms to one side of the equation, giving us

$$2\mathbf{u}_1 - 3\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$$

Definition **Linear Independence**

This brings us to the following important definition.

DEFINITION 2.11

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a set of vectors in $\mathbf{R}^n$. If the only solution to the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0}$$

is the trivial solution given by $x_1 = x_2 = \cdots = x_m = 0$, then the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is **linearly independent**. If there are nontrivial solutions, then the set is **linearly dependent**.

▶ To determine if a set of vectors is linearly dependent or independent, we almost always use the method illustrated in Example 1: Set the linear combination equal to **0** and find the solutions.

EXAMPLE 1 Determine if the set

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 4 \\ -2 \\ -3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -13 \\ 7 \\ 7 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 9 \\ -5 \end{bmatrix}$$

is linearly dependent or linearly independent.

Solution To determine if the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent or linearly independent, we need to find the solutions of the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$$

This is equivalent to the linear system

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 0 \\ 4x_1 - 13x_2 + x_3 &= 0 \\ -2x_1 + 7x_2 + 9x_3 &= 0 \\ -3x_1 + 7x_2 - 5x_3 &= 0 \end{aligned}$$

The corresponding augmented matrix and echelon form are

▶ The row operations used in (3) are (in order performed):

$$\begin{aligned} 4R_1 + R_2 &\Rightarrow R_2 \\ -2R_1 + R_3 &\Rightarrow R_3 \\ -3R_1 + R_4 &\Rightarrow R_4 \\ R_2 + R_3 &\Rightarrow R_3 \\ -2R_2 + R_4 &\Rightarrow R_4 \\ (-5/2)R_3 + R_4 &\Rightarrow R_4 \end{aligned}$$

$$\begin{bmatrix} -1 & 3 & -2 & 0 \\ 4 & -13 & 1 & 0 \\ -2 & 7 & 9 & 0 \\ -3 & 7 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & -2 & 0 \\ 0 & -1 & -7 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

Back substitution shows that the only solution is the trivial one, $x_1 = x_2 = x_3 = 0$. Hence the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. ■

EXAMPLE 2 Determine if the myology clinic described earlier can eliminate any of the nutritional powder brands with components given in Table 1.

Solution We start by determining if the nutrient vectors for the four brands

$$\mathbf{a} = \begin{bmatrix} 16 \\ 2 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 22 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 18 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 18 \\ 2 \\ 6 \end{bmatrix}$$

are linearly independent. To do so, we must find the solutions to the vector equation $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} + x_4\mathbf{d} = \mathbf{0}$. The augmented matrix of the equivalent linear system and the echelon form are

▶ The row operations used in (4) are (in order performed):

$$\begin{aligned} R_1 &\Leftrightarrow R_2 \\ -8R_1 + R_2 &\Rightarrow R_2 \\ -4R_1 + R_3 &\Rightarrow R_3 \\ (-6/5)R_2 + R_3 &\Rightarrow R_3 \\ (-5/22)R_3 &\Rightarrow R_3 \end{aligned}$$

$$\begin{bmatrix} 16 & 22 & 18 & 18 & 0 \\ 2 & 4 & 0 & 2 & 0 \\ 8 & 4 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 0 & 2 & 0 \\ 0 & -10 & 18 & 2 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix} \tag{4}$$

Back substitution leads to the general solution

$$x_1 = -\tfrac{1}{2}s, \qquad x_2 = -\tfrac{1}{4}s, \qquad x_3 = -\tfrac{1}{4}s, \qquad x_4 = s$$

which holds for all choices of s. Thus there exist nontrivial solutions, so the set of vectors is linearly dependent. Letting $s = 1$, we have $x_1 = -\frac{1}{2}$, $x_2 = -\frac{1}{4}$, $x_3 = -\frac{1}{4}$, and $x_4 = 1$, which gives us

$$-\tfrac{1}{2}\mathbf{a} - \tfrac{1}{4}\mathbf{b} - \tfrac{1}{4}\mathbf{c} + \mathbf{d} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{d} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{4}\mathbf{b} + \tfrac{1}{4}\mathbf{c}$$

Thus we obtain a serving of brand D by combining a $\frac{1}{2}$ serving of A, a $\frac{1}{4}$ serving of B, and a $\frac{1}{4}$ serving of C. Hence there is no need to stock brand D. ■

When working with a new concept, it can be helpful to start with simple cases. In this spirit, suppose that we have a set with one vector, $\{\mathbf{u}_1\}$. Is this set linearly independent? To check, we need to determine the solutions to the equation

$$x_1\mathbf{u}_1 = \mathbf{0} \tag{5}$$

At first glance, it seems that the only solution is the trivial one $x_1 = 0$, and for most choices of $\mathbf{u}_1$ that is true. Specifically, as long as $\mathbf{u}_1 \neq \mathbf{0}$, then the only solution is $x_1 = 0$ and the set $\{\mathbf{u}_1\}$ is linearly independent. But if it happens that $\mathbf{u}_1 = \mathbf{0}$, then the set is linearly dependent, because now there are nontrivial solutions to (5), such as

$$3\mathbf{u}_1 = \mathbf{0}$$

In fact, having $\mathbf{0}$ in *any* set of vectors always guarantees that the set will be linearly dependent.

THEOREM 2.12 Suppose that $\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is a set of vectors in $\mathbf{R}^n$. Then the set is linearly dependent.

Proof We need to determine if the vector equation

$$x_0\mathbf{0} + x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0}$$

has any nontrivial solutions. Regardless of the values of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, setting $x_0 = 1$ and $x_1 = x_2 = \cdots = x_m = 0$ gives us an easy (but legitimate) nontrivial solution, so that the set is linearly dependent. ■

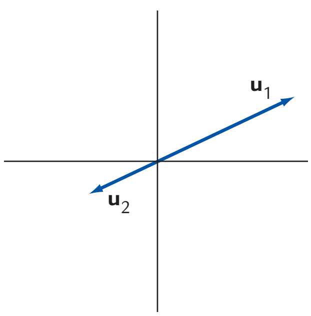

Figure 1 $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent vectors.

For a set of two vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$, we already know what happens if one of these is $\mathbf{0}$. Let's assume that both vectors are nonzero and ask the same question as above: Is this set linearly independent? As usual, we need to determine the nature of the solutions to

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{0} \tag{6}$$

If there is a nontrivial solution, then it must be that both x_1 and x_2 are nonzero. (Why?) In this case, we can solve (6) for $\mathbf{u}_1$, giving us

$$\mathbf{u}_1 = -\frac{x_2}{x_1}\mathbf{u}_2$$

Thus we see that the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent if and only if $\mathbf{u}_1$ is a scalar multiple of $\mathbf{u}_2$. Geometrically, the set is linearly dependent if and only if the two vectors point in the same (or opposite) direction. (See Figure 1.)

When trying to determine if a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ of three or more vectors is linearly dependent or independent, in general we have to find the solutions to

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0}$$

However, there is a special case where virtually no work is required.

THEOREM 2.13 Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is a set of vectors in $\mathbf{R}^n$. If $n < m$, then the set is linearly dependent.

In other words, if the number of vectors m exceeds the number of components n, then the set is linearly dependent.

Proof As usual when testing for linear independence, we start with the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0} \tag{7}$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ each have n components, this is equivalent to a homogeneous linear system with n equations and m unknowns. Because $n < m$, this system has more variables than equations, so there are infinitely many solutions (see Exercise 55 in Section 1.2). Therefore (7) has nontrivial solutions, and hence the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent. ■

This theorem immediately tells us that the myology clinic's set of four nutritional powder brands must not all be needed, because we can represent them by four vectors ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$) in $\mathbf{R}^3$ (protein, fat, carbohydrates). However, the theorem does not tell us which brand can be eliminated.

Note also that Theorem 2.13 tells us nothing about the case when $n \geq m$. In this instance, it is possible for the set to be linearly dependent, as in (1), where $n = 3$ and $m = 3$. Or the set can be linearly independent, as in Example 1, where $n = 4$ and $m = 3$.

Span and Linear Independence

It is common to confuse span and linear independence, because although they are different concepts, they are related. To see the connection, let's return to the earlier discussion about the set of two vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$. This set is linearly dependent exactly when $\mathbf{u}_1$ is a multiple of $\mathbf{u}_2$—that is, exactly when $\mathbf{u}_1$ is in span$\{\mathbf{u}_2\}$. This connection between span and linear independence holds more generally.

THEOREM 2.14 Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a set of vectors in $\mathbf{R}^n$. Then the set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.

Proof Suppose first that the set is linearly dependent. Then there exist scalars $c_1, \ldots, c_m$, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m = \mathbf{0}$$

To simplify notation, assume that $c_1 \neq 0$. Then we can solve for $\mathbf{u}_1$,

$$\mathbf{u}_1 = -\frac{c_2}{c_1}\mathbf{u}_2 - \cdots - \frac{c_m}{c_1}\mathbf{u}_m$$

which shows that $\mathbf{u}_1$ is in span$\{\mathbf{u}_2, \ldots, \mathbf{u}_m\}$. This completes the "forward" direction of the proof.

Now suppose that one of the vectors in the set is in the span of the remaining vectors—say, $\mathbf{u}_1$ is in span$\{\mathbf{u}_2, \ldots, \mathbf{u}_m\}$. Then there exist scalars $c_2, c_3, \ldots, c_m$ such that

$$\mathbf{u}_1 = c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m \tag{8}$$

Moving all terms to the left side in (8), we have

$$\mathbf{u}_1 - c_2\mathbf{u}_2 - \cdots - c_m\mathbf{u}_m = \mathbf{0}$$

Since the coefficient on $\mathbf{u}_1$ is 1, this shows the set is linearly dependent. This completes the "backward" direction, finishing the proof. ■

EXAMPLE 3 Give a linearly dependent set of vectors such that one vector is not a linear combination of the others. Explain why this does not contradict Theorem 2.14.

Solution Theorem 2.14 tells us that in a linearly dependent set, at least one vector is a linear combination of the other vectors. However, it does not say that *every* vector is a linear combination of the others, so what we seek does not contradict Theorem 2.14.

Let

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

By Theorem 2.13, this set must be linearly dependent because there are four vectors with three components. Theorem 2.14 says that one of the vectors is a linear combination of the others, and indeed we have $\mathbf{u}_3 = -2\mathbf{u}_1 + \mathbf{u}_2$. On the other hand, the equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{u}_4 \tag{9}$$

has corresponding augmented matrix and echelon form

▶ The row operations used in (10) are (in order performed):

$-2R_1 + R_3 \Rightarrow R_3$
$-2R_2 + R_3 \Rightarrow R_3$

$$\begin{bmatrix} -1 & 3 & 5 & 1 \\ 0 & -2 & -2 & 2 \\ -2 & 2 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 5 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{10}$$

From the echelon form it follows that (9) has no solutions, so that $\mathbf{u}_4$ is not a linear combination of the other vectors in the set. ■

Homogeneous Systems

In Section 2.2, we introduced the notation

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m$$

where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, and noted that any linear system can be expressed in the compact form

$$A\mathbf{x} = \mathbf{b}$$

The system $A\mathbf{x} = \mathbf{0}$ is a **homogeneous** linear system, introduced in Section 1.2. There we showed that homogeneous linear systems have either one solution (the trivial solution) or infinitely many solutions.

The next theorem shows that there is a direct connection between the number of solutions to $A\mathbf{x} = \mathbf{0}$ and whether the columns of A are linearly independent.

THEOREM 2.15 Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. The set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent if and only if the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution.

Proof Written as a vector equation, the system $A\mathbf{x} = \mathbf{0}$ has the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{0} \tag{11}$$

Thus if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then so does (11) and the columns of A are linearly independent. On the other hand, if $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, then so does (11) and the columns of A are linearly dependent. ■

Definition Nonhomogeneous, Associated Homogeneous System

If $\mathbf{b} \neq \mathbf{0}$, then the system $A\mathbf{x} = \mathbf{b}$ is **nonhomogeneous**, and the **associated homogeneous system** is $A\mathbf{x} = \mathbf{0}$. There is a close connection between the set of solutions to a nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ and the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, illustrated in the following example.

EXAMPLE 4 Find the general solution for the linear system

$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 7 \\ x_1 - 3x_2 - x_3 + 6x_4 &= 6 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 4 \end{aligned} \tag{12}$$

and the general solution for the associated homogeneous system.

Solution Applying our usual matrix and row reduction methods, we find that the general solution to (12) is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -5 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

The general solution to the associated homogeneous system was found in Example 7 of Section 1.2. It is

$$\mathbf{x} = s_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

For both solutions, s_1 and s_2 can be any real numbers. ■

Comparing the preceding solutions, we see that the only difference is the "constant" vector

$$\begin{bmatrix} 1 \\ 0 \\ -5 \\ 0 \end{bmatrix}$$

This type of relationship between general solutions occurs in all such cases. To see why, it is helpful to have the following result showing that the product $A\mathbf{x}$ obeys the distributive law.

THEOREM 2.16 Suppose that $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \end{bmatrix}$, and let $\mathbf{x} = (x_1, x_2, \dots, x_m)$ and $\mathbf{y} = (y_1, y_2, \dots, y_m)$. Then

(a) $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$

(b) $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y}$

Proof The results follow from the definition of the product and a bit of algebra. Starting with (a), we have

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{bmatrix}$$

so that

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \cdots + (x_m+y_m)\mathbf{a}_m \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m) + (y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_m\mathbf{a}_m) \\ &= A\mathbf{x} + A\mathbf{y} \end{aligned}$$

The proof of (b) is similar and left as an exercise. ■

Definition **Particular Solution**

Now let $\mathbf{x}_p$ be any solution to $A\mathbf{x} = \mathbf{b}$. We call $\mathbf{x}_p$ a **particular solution** to the system, and it can be thought of as any fixed solution to the system.

THEOREM 2.17

Let $\mathbf{x}_p$ be a particular solution to

$$A\mathbf{x} = \mathbf{b} \tag{13}$$

Then all solutions $\mathbf{x}_g$ to (13) have the form $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof Let $\mathbf{x}_p$ be a particular solution to (13), and suppose that $\mathbf{x}_g$ is any solution to the same system. Then

$$A(\mathbf{x}_g - \mathbf{x}_p) = A\mathbf{x}_g - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence if we let $\mathbf{x}_h = \mathbf{x}_g - \mathbf{x}_p$, then $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Solving for $\mathbf{x}_g$, we have

$$\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$$

so that $\mathbf{x}_g$ has the form claimed. ■

EXAMPLE 5 Find the general solution and solution to the associated homogeneous system for

$$\begin{aligned} 4x_1 - 6x_2 &= -14 \\ -6x_1 + 9x_2 &= 21 \end{aligned} \tag{14}$$

Solution Applying our standard solution procedures yields the vector form of the general solution to (14)

$$\mathbf{x}_g = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

which is a line when graphed in $\mathbf{R}^2$. The solutions to the associated homogeneous system are

$$\mathbf{x}_h = s \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

which is also a line when graphed in $\mathbf{R}^2$. If we let $\mathbf{x}_p = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then every value of $\mathbf{x}_g$ can be expressed as the sum of $\mathbf{x}_p$ and one of the homogeneous solutions $\mathbf{x}_h$. Thus the general solution $\mathbf{x}_g$ is a translation by $\mathbf{x}_p$ of the general solution $\mathbf{x}_h$ of the associated homogeneous system. The graphs are shown in Figure 2. ■

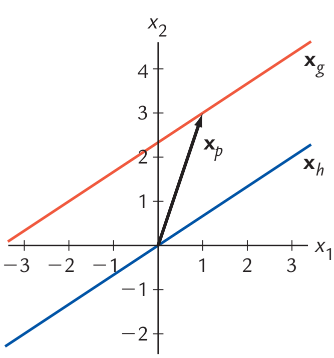

Figure 2 Graphs of $\mathbf{x}_g$, $\mathbf{x}_h$, and $\mathbf{x}_p$ from Example 5.

At the end of Section 2.2, Theorem 2.10 linked span with solutions to linear systems. Theorem 2.18 is similar in spirit, this time linking linear independence with solutions to linear systems.

THEOREM 2.18 Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and $\mathbf{b}$ be vectors in $\mathbf{R}^n$. Then the following statements are equivalent. That is, if one is true, then so are the others, and if one is false, then so are the others.

(a) The set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent.

(b) The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$ has at most one solution.

(c) The linear system corresponding to $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{b} \end{bmatrix}$ has at most one solution.

(d) The equation $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix}$, has at most one solution.

Proof The equivalence of (b), (c), and (d) is immediate from the definitions, so all that is needed to complete the proof is show that (a) and (b) are equivalent.

We start by showing that (a) implies (b). Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ be linearly independent, and suppose to the contrary that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$$

has more than one solution. Then there exist scalars $r_1, \ldots, r_m$ and $s_1, \ldots, s_m$ such that

$$\begin{aligned} r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_m\mathbf{a}_m &= \mathbf{b} \\ s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_m\mathbf{a}_m &= \mathbf{b} \end{aligned}$$

and so

$$r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_m\mathbf{a}_m = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_m\mathbf{a}_m$$

Moving all terms to one side and regrouping yields

$$(r_1 - s_1)\mathbf{a}_1 + (r_2 - s_2)\mathbf{a}_2 + \cdots + (r_m - s_m)\mathbf{a}_m = \mathbf{0}$$

Since $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent, each coefficient must be 0. Hence $r_1 = s_1$, $\ldots$, $r_m = s_m$, so there is just one solution to $x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$.

Proving that (b) implies (a) is easier. Since $\mathbf{b}$ can be any vector, we can set $\mathbf{b} = \mathbf{0}$. By (b) there is at most one solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{0}$$

Of course, there is the trivial solution $x_1 = \cdots = x_m = 0$, so this must be the only solution. Hence the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent, and (a) follows. ■

The Big Theorem – Version 1

Next, we present the first version of the Big Theorem. This theorem is Big for two reasons: (a) it is Big—as in important—because it will serve to tie together and unify many of the ideas that we shall be developing, and (b) subsequent versions will include more and more equivalent statements, making it Big in size.

THEOREM 2.19

▶ In all versions of the Big Theorem, n is both the number of vectors in $\mathcal{A}$ and the number of components in each vector. Thus A has n rows and n columns.

THE BIG THEOREM—VERSION 1 Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbf{R}^n$, and let $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$. Then the following are equivalent:

(a) $\mathcal{A}$ spans $\mathbf{R}^n$.

(b) $\mathcal{A}$ is linearly independent.

(c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^n$.

Proof We start by showing that (a) and (b) are equivalent. First suppose that $\mathcal{A}$ spans $\mathbf{R}^n$. If $\mathcal{A}$ is linearly dependent, then one of $\mathbf{a}_1, \dots, \mathbf{a}_n$—say, $\mathbf{a}_1$—is a linear combination of the others. Then by Theorem 2.7, it follows that

$$\operatorname{span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \operatorname{span}\{\mathbf{a}_2, \dots, \mathbf{a}_n\}.$$

But this implies that $\mathbf{R}^n = \operatorname{span}\{\mathbf{a}_2, \dots, \mathbf{a}_n\}$, contradicting Theorem 2.8. Hence it must be that $\mathcal{A}$ is linearly independent. This shows that (a) implies (b).

To show that (b) implies (a), we assume that $\mathcal{A}$ is linearly independent. Now, if $\mathcal{A}$ does not span $\mathbf{R}^n$, then there exists a vector $\mathbf{a}$ that is not a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$. Since $\mathcal{A}$ is linearly independent, it follows that the set $\{\mathbf{a}, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ of $n+1$ vectors is also linearly independent, contradicting Theorem 2.13. Hence $\mathcal{A}$ must span $\mathbf{R}^n$. Thus (b) implies (a), and therefore (a) is equivalent to (b).

Now suppose that (a) and (b) are both true. Then by Theorem 2.10, (a) implies that $A\mathbf{x} = \mathbf{b}$ has *at least* one solution for every $\mathbf{b}$ in $\mathbf{R}^n$. On the other hand, from Theorem 2.18 we know that (b) implies that $A\mathbf{x} = \mathbf{b}$ has *at most* one solution for every $\mathbf{b}$ in $\mathbf{R}^n$. This leaves us with only one possibility, that $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$ in $\mathbf{R}^n$, confirming (c) is true.

Finally, suppose that (c) is true. Since $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^n$, then in particular $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Appealing to Theorem 2.15, we conclude that $\mathcal{A}$ must be linearly independent and hence also spans $\mathbf{R}^n$. ■

EXAMPLE 6 Suppose that

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}, \quad \text{and} \quad A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$$

Show that the columns of A are linearly independent and span $\mathbf{R}^3$, and that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$.

Solution We start with linear independence, so we need to find the solutions to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0} \tag{15}$$

The corresponding augmented matrix and echelon form are

▶ The row operations used in (16) are (in order performed):

$$\begin{array}{r} -7R_1 + R_2 \Rightarrow R_2 \\ 2R_1 + R_3 \Rightarrow R_3 \\ R_2 \Leftrightarrow R_3 \\ 3R_2 + R_3 \Rightarrow R_3 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 7 & 0 & 2 & 0 \\ -2 & 1 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 0 & -21 & 0 \end{bmatrix} \tag{16}$$

From the echelon form it follows that (15) has only the trivial solution, so the columns of A are linearly independent.

Because we have three vectors and each has three components, the other questions follow immediately from the Big Theorem. Specifically, since $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly independent, the set must also span $\mathbf{R}^3$ and there is exactly one solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$ in $\mathbf{R}^3$. The Big Theorem and its successors to come are all very powerful. ■

For one more application of Theorem 2.19, let's return to the nutritional powder problem described at the beginning of the section.

EXAMPLE 7 Use the Big Theorem to show that stocking powder brands A, B, and C is efficient.

Solution We previously determined that the nutrient vectors **a**, **b**, **c**, and **d** are linearly dependent, and we concluded that brand D can be eliminated. For the remaining three brands, it can be verified that

$$x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{0}$$

has only the trivial solution, which tells us that **a**, **b**, and **c** are linearly independent. By Theorem 2.19, we can conclude the following:

- The vectors **a**, **b**, and **c** span all of $\mathbf{R}^3$. Therefore *every* vector in $\mathbf{R}^3$ can be expressed as a linear combination of these three vectors. (Note, though, that some combinations will require negative values of x_1, x_2, and x_3, which is not physically possible when combining powders.)
- Item (c) of the Big Theorem tells us that there is exactly one way to combine brands A, B, and C to create any blend with a specific combination of protein, fat, and carbohydrates. Thus stocking brands A, B, and C is efficient, in that there is no redundancy. ■

EXERCISES

For Exercises 1–6, determine if the given vectors are linearly independent.

1. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 6 \\ -15 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ -10 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \\ -13 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -8 \\ 2 \\ -19 \end{bmatrix}$

5. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$

6. $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 5 \\ -5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 7–12, determine if the columns of the given matrix are linearly independent.

7. $\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & -12 \\ 2 & 6 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 5 & -7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -5 \\ -1 & 2 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 1 & 0 \\ 5 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -4 & -7 & 1 \\ 0 & 0 & 3 \\ 5 & -1 & 1 \\ 8 & 2 & -4 \end{bmatrix}$

In Exercises 13–18, a matrix A is given. Determine if the homogeneous system $A\mathbf{x} = \mathbf{0}$ (where $\mathbf{x}$ and $\mathbf{0}$ have the appropriate number of components) has any nontrivial solutions.

13. $A = \begin{bmatrix} -3 & 5 \\ 4 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 12 & 10 \\ 6 & 5 \end{bmatrix}$

15. $A = \begin{bmatrix} 8 & 1 \\ 0 & -1 \\ -3 & 2 \end{bmatrix}$

16. $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ 5 & -4 & -3 \end{bmatrix}$

17. $A = \begin{bmatrix} -1 & 3 & 1 \\ 4 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 2 \\ -5 & 3 & -9 \\ 3 & 0 & 9 \end{bmatrix}$

In Exercises 19–24, determine by inspection (that is, with only minimal calculations) if the given vectors form a linearly dependent or linearly independent set. Justify your answer.

19. $\mathbf{u} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

20. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

21. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

23. $\mathbf{u} = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -7 \\ 1 \\ 12 \end{bmatrix}$

24. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 25–28, determine if one of the given vectors is in the span of the other vectors. (HINT: Check to see if the vectors are linearly dependent, and then appeal to Theorem 2.14.)

25. $\mathbf{u} = \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$

26. $\mathbf{u} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

27. $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -5 \\ 7 \\ -7 \end{bmatrix}$

28. $\mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

For each matrix A given in Exercises 29–32, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$. (HINT: the Big Theorem is helpful here.)

29. $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ -3 & 4 & 5 \end{bmatrix}$

30. $A = \begin{bmatrix} 3 & 4 & 7 \\ 7 & -1 & 6 \\ -2 & 0 & 2 \end{bmatrix}$

31. $A = \begin{bmatrix} 3 & -2 & 1 \\ -4 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & 7 \end{bmatrix}$

FIND AN EXAMPLE For Exercises 33–38, find an example that meets the given specifications.

33. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^4$.

34. Three linearly independent vectors in $\mathbf{R}^5$.

35. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^2$ that do not span $\mathbf{R}^2$.

36. Three distinct nonzero vectors in $\mathbf{R}^2$ such that any pair is linearly independent.

37. Three distinct nonzero linearly dependent vectors in $\mathbf{R}^3$ such that each vector is in the span of the other two vectors.

38. Four vectors in $\mathbf{R}^3$ such that no vector is a nontrivial linear combination of the other three. (Explain why this does not contradict Theorem 2.14.)

TRUE OR FALSE For Exercises 39–52, determine if the statement is true or false, and justify your answer.

39. If a set of vectors in $\mathbf{R}^n$ is linearly dependent, then the set must span $\mathbf{R}^n$.

40. If $m > n$, then a set of m vectors in $\mathbf{R}^n$ is linearly dependent.

41. If A is a matrix with more rows than columns, then the columns of A are linearly independent.

42. If A is a matrix with more columns than rows, then the columns of A are linearly independent.

43. If A is a matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

44. If A is a matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$.

45. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

46. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

47. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

48. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, then so is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

49. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

50. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

51. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

52. If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

53. Which of the following sets of vectors in $\mathbf{R}^3$ could possibly be linearly independent? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

54. Which of the following sets of vectors in $\mathbf{R}^3$ could possibly be linearly independent *and* span $\mathbf{R}^3$? Justify your answer.

(a) $\{\mathbf{u}_1\}$

(b) $\{\mathbf{u}_1, \mathbf{u}_2\}$

(c) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(d) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

55. Prove that if c_1, c_2, and c_3 are nonzero scalars and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors, then so is $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, c_3\mathbf{u}_3\}$.

56. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors, then so are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

57. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors, then so is $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$.

58. Prove that if $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly independent, then any nonempty subset of U is also linearly independent.

59. Prove that if a set of vectors is linearly dependent, then adding additional vectors to the set will create a new set that is still linearly dependent.

60. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set, then $\mathbf{w}$ is in span$\{\mathbf{u}, \mathbf{v}\}$.

61. Prove that two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if $\mathbf{u} = c\mathbf{v}$ for some scalar c.

62. Let A be an $n \times m$ matrix that is in echelon form. Prove that the nonzero rows of A, when considered as vectors in $\mathbf{R}^m$, are a linearly independent set.

63. Prove part (b) of Theorem 2.16.

64. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a linearly dependent set of nonzero vectors. Prove that some vector in the set can be written as a linear combination of a linearly independent subset of the remaining vectors, with the set of coefficients all nonzero and unique for the given subset. (HINT: Start with Theorem 2.14.)

In Exercises 65–66, suppose that the given vectors are direction vectors for a model of the VecMobile III (discussed in Section 2.2). Determine if there is any redundancy in the vectors and if it is possible to reach every point in $\mathbf{R}^3$.

65. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$

66. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix}$

C In Exercises 67–70, determine if the given vectors form a linearly dependent or linearly independent set.

67. $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix}$

68. $\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix}$

69. $\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \\ -3 \end{bmatrix}$

70. $\begin{bmatrix} 3 \\ 5 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 2 \\ 6 \end{bmatrix}$

C In Exercises 71–72, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$.

71. $A = \begin{bmatrix} 1 & -2 & 4 \\ 5 & -3 & -1 \\ -3 & -7 & -9 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

72. $A = \begin{bmatrix} 3 & -2 & 5 \\ 2 & 0 & -4 \\ -2 & 7 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

C In Exercises 73–74, determine if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^4$.

73. $A = \begin{bmatrix} 2 & 5 & -3 & 6 \\ -1 & 0 & 1 & -1 \\ 5 & 2 & -3 & 9 \\ 3 & -4 & 6 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

74. $A = \begin{bmatrix} 5 & 1 & 0 & 8 \\ -2 & 4 & 3 & 11 \\ -3 & 8 & 2 & 5 \\ 0 & 3 & -1 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$