(1.) $U_t + F(U)_x = 0$ $\qquad a = \sqrt{\gamma R T}$

$$U = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} \rho \\ \rho u \end{vmatrix}$$

$$F = \begin{vmatrix} f_1 \\ f_2 \end{vmatrix} = \begin{vmatrix} \rho u \\ \rho u^2 + \rho a^2 \end{vmatrix} = \begin{vmatrix} \rho u \\ \rho u^2 + \rho \gamma R T \end{vmatrix}$$

1a) $U = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}$. $F = \begin{vmatrix} \rho u \\ \rho u^2 + \rho a^2 \end{vmatrix} \Rightarrow \boxed{F = \begin{vmatrix} u_2 \\ u_2^2/u_1 + u_1 a^2 \end{vmatrix}}$

1b) $$A(U) = \partial F / \partial U = \begin{bmatrix} 0 & 1 \\ -\frac{u_2^2}{u_1^2} + a^2 & \frac{2u_2}{u_1} \end{bmatrix}$$

1c.) $|A - \lambda I| = 0 \Rightarrow$

$$\det \begin{bmatrix} -\lambda & 1 \\ -u^2 + a^2 & 2u - \lambda \end{bmatrix} = 0 \Rightarrow$$

$-\lambda(2u-\lambda) - (-u^2 + a^2) = 0$

$-2\lambda u + \lambda^2 + u^2 + a^2 - 0$

$(\lambda - u + a)(\lambda - u - a) = 0$

$$\boxed{\begin{aligned} \lambda_1 &= u - a \\ \lambda_2 &= u + a \end{aligned}}$$

1d) $$\begin{bmatrix} -u + a & 1 \\ -u^2 + a^2 & u + a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1(a-u) + x_2 = 0$

$x_1(a^2 - u^2) + x_2(a+u) = 0$

$x_2 = -x_1(a-u)$

$x_2 = \frac{-x_1(a^2-u^2)}{(a+u)}$;

$x_1(a-u) = \frac{x_1(a^2-u^2)}{(a+u)} \rightarrow x_1(a^2-u^2) = x_1(a^2-u^2)$

$x_1 = 1, \; x_2 = u - a$ $\qquad \boxed{K^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}}$

$$\begin{bmatrix} -u - a & 1 \\ -u^2 + a^2 & u - a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1(-u-a) + x_2 = 0$

$x_2 = -x_1(-u-a)$

$x_1(-u-a) = \frac{x_1(-u^2+a^2)}{(u-a)}$

$-x_1(-u^2+a^2) = x_2(u-a)$

$x_2 = \frac{-x_1(-u^2+a^2)}{(u-a)}$

$x_1 = 1, x_2 = u + a$ $\qquad \boxed{K^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix}}$

1e.) a) units of eigenvalues: speed, m/s

b) units of f_1, f_2: $f_1 = kg/m^2 \cdot s$, $f_2 = kg/m \cdot s^2$

c) units of U: $U_1 = kg/m^3$, $U_2 = kg/m^2 \cdot s$

d) Yes, the eigenvalues depend on the state vector because they include u, which is U_2/U_1.

e) λ_2 will always be positive as long as u is positive.

$\lambda_1 = u - a = Ma - a = a(M-1)$.

$\lambda_1 = a(m-1)$. We see that λ_1 is positive when $M>1$, and negative when $M<1$. (Assuming positive velocity).

```
clear all; close all; clc

g = 1.4;

load U.mat;

for i=1:size(U,1)
  W(i,:)=U2W(U(i,:),g);
  lambda(i,:)=eigsEuler(U(i,:),g);
  F(i,:)=fluxEuler(U(i,:),g);
end

fprintf('\n\n--- W ----');
for i=1:size(U,1)
  fprintf('\n %12.3e %12.3e %12.3e',W(i,1:3));
end
fprintf('\n');

fprintf('\n\n--- lambda ----');
for i=1:size(U,1)
  fprintf('\n %12.3e %12.3e %12.3e',lambda(i,1:3));
end
fprintf('\n');

fprintf('\n\n--- F ----');
for i=1:size(U,1)
  fprintf('\n %12.3e %12.3e %12.3e',F(i,1:3));
end
fprintf('\n');
```

```
--- W ----
    1.000e+00    7.500e-01    1.000e+00
    1.000e+00   -2.000e+00    4.000e-01
    1.250e-01    0.000e+00    1.000e-01
    5.992e+00   -6.196e+00    4.609e+01
    1.000e+00   -1.960e+01    1.000e-02


--- lambda ----
    1.933e+00    7.500e-01   -4.332e-01
   -1.252e+00   -2.748e+00   -2.000e+00
    0.000e+00    1.058e+00   -1.058e+00
   -9.478e+00   -2.915e+00   -6.196e+00
   -1.948e+01   -1.960e+01   -1.972e+01


--- F ----
    7.500e-01    1.563e+00    2.836e+00
   -2.000e+00    4.400e+00   -6.800e+00
    0.000e+00    1.000e-01    0.000e+00
   -3.713e+01    2.762e+02   -1.712e+03
   -1.960e+01    3.841e+02   -3.764e+03
```

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```
function W = U2W(U,g)

% I would comment what is being done, but the comments
% would be the same as the actual code i.e. for the first
% line below: The first entry in W is set equal to the
% first entry in U, which is rho.

W(1) = U(1);
W(2) = U(2)/U(1);
W(3) = (g-1) * (U(3)-((U(2)^2/U(1))/2));

end

function lambda = eigsEuler(U,g)

A = zeros(3,3);
p = (g-1)*(U(3)-((U(2)^2/U(1))/2)); % defining p to make code clear
a = sqrt(g*p/U(1));                 % defining a to make code clear
u = (U(2)/U(1));                    % defining u to make code clear

% defining the jacobian matrix based on known values: gamma, a, u

A(1,1) = 0;
A(1,2) = 1;
A(1,3) = 0;

A(2,1) = (g-3) * (u^2/2);
A(2,2) = (3-g) * u;
A(2,3) = g-1;

A(3,1) = (g-2) * (u^3/2) - (a^2*u)/(g-1);
A(3,2) = (3-2*g)*(u^2/2) + a^2/(g-1);
A(3,3) = g*u;

lambda = eig(A);

end

function F = fluxEuler(U,g)

p = (g-1)*(U(3)-((U(2)^2/U(1))/2)); % defining p to make code more clear
F(1) = U(2);
F(2) = ((U(2)^2)/U(1)) + p;
F(3) = (U(2)/U(1))*(U(3)+p);

end
```

```
Not enough input arguments.

Error in sumbit (line 8)
W(1) = U(1);
```

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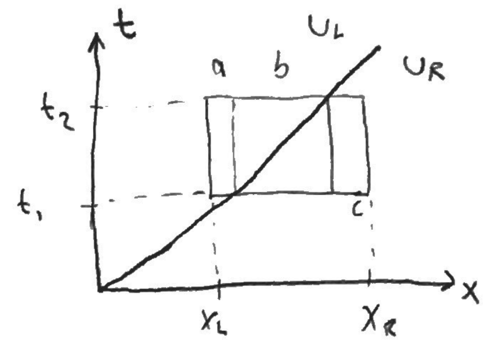
③ $$\underbrace{\int_{t_1}^{t_2}\int_{x_L}^{x_R} \frac{\partial}{\partial t} u(z,\tau)\,dz\,d\tau}_{①} + \underbrace{\int_{t_1}^{t_2}\int_{x_L}^{x_R} \frac{\partial}{\partial x} f(u(z,\tau))\,dz\,d\tau}_{②} = 0$$

$$① = \int_{t_1}^{t_2} \cancel{\frac{\partial}{\partial t}} \int_{x_L}^{x_R} u(z,\tau)\,dz\,d\tau = \left[\int_{x_L}^{x_R} u(z,\tau)\,dz\,d\tau\right]_{t_1}^{t_2} = \int_{x_L}^{x_R} u(z,t_2) - u(z,t_1)\,dz = ①$$

$$② = \int_{t_1}^{t_2} \underbrace{f(u(x_R,\tau))}_{\frac{1}{2}u_R^2} - \underbrace{f(u(x_L,\tau))}_{\frac{1}{2}u_L^2}\,d\tau$$



We see that ① is the difference between the top and bottom areas.

$$① = \left[(a+b)u_L + c\,u_R\right] - \left[a\,u_L + (b+c)u_R\right] = b u_L - b u_R.$$

$$b = s(t_2-t_1) = \frac{u_L+u_R}{2}(t_2-t_1) = \frac{1}{2}(u_L+u_R)(t_2-t_1). \quad ① = \frac{1}{2}(u_L-u_R)(u_L+u_R)(t_2-t_1)$$

$$② = \left(\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2\right)(t_2-t_1) = \frac{1}{2}(u_R^2-u_L^2)(t_2-t_1) = \frac{1}{2}(u_R+u_L)(u_R-u_L)(t_2-t_1)$$

$$= -\frac{1}{2}(u_L+u_R)(u_L-u_R)(t_2-t_1). \rightarrow ② = -①, \quad ①+② = 0 \quad \checkmark$$