Logic: Encoding, Reasoning and Proving

Due **21:00 on Friday March 25th, 2022**.
(As always MMS is definitive on deadline)

You are NOT required to format your solutions on computer. For example, clearly handwritten answers are acceptable. In this case you must scan or photograph your answers clearly and include in a single pdf which you upload to MMS.

1. **Truth Tables and Equivalent Formulae**

   Construct the truth table for each of the following of the following Boolean formulas. Use this truth table to determine the simplest possible Boolean formula that is equivalent to it. Show that the formula you give is equivalent by adding its value to each line of the truth table. Note that you do *not* need to give a proof that the formula you construct is the simplest possible.

   If a formula is a tautology then the simplest equivalent formula is **1** (for truth). If the formula is unsatisfiable then the simplest equivalent formula is **0** (for falsity).

   (a)
   $$A \wedge (B \wedge (\neg C \wedge \neg A))$$

   (b)
   $$A \wedge ((B \vee \neg B) \wedge A)$$

   (c)
   $$(A \vee (0 \vee B)) \wedge \neg(\neg B \wedge \neg A)$$

   (d)
   $$\neg(\neg(\neg A \vee B) \vee A) \vee A$$

2. **Proof and DPLL**

   Use the DPLL algorithm to give "reductio ad absurdum" proofs of the following conclusions from the given premises. For each part, convert the premises and the *negation* of the conclusion to CNF (conjunctive normal form) and then use DPLL to show that the resulting set of clauses is unsatisfiable.

   (a) Conclusion $R \rightarrow \neg P$ from premise $(Q \rightarrow \neg R) \wedge (\neg Q \rightarrow \neg P)$

(b) Conclusion $(P \to Q) \vee (R \to \neg S)$ from premise $(P \to \neg S) \vee (R \to Q)$

(c) Conclusion $\neg U$ from premise

$$(\neg(P \wedge \neg Q) \vee \neg(\neg S \wedge \neg R)) \wedge \neg(R \vee Q) \wedge (U \to (\neg R \to (\neg S \wedge P)))$$

3. **Encoding and Proof in a Puzzle Game**

In the game "Minesweeper", the board is a grid of squares each of which may contain a mine.[1] When you correctly mark a square as safe, you are told the number of adjacent (including diagonally) squares that contain mines. Known mines are indicated by a ×, with squares that are known safe show a number. For example, in the following layout the middle square in the top row is a known mine, while the 1 in the top right corner indicates that exactly one of its adjacent cells is a mine - in this case it is a diagonally adjacent cell. In this diagram, 8 cells marked A to H are of unknown status

A	B	×	3	1
3	C	×	×	3
1	D	E	×	×
1	2	F	G	H
0	1	1	1	0

You are to deduce, using formal logic, exactly which squares we can prove either to be mines or to be safe.

(a) First encode the following statements into propositional logic:

i. Exactly three of the cells A, B, C, D are mines
ii. Exactly one of the cells C, D is a mine
iii. Exactly one of the cells D is a mine
iv. Exactly two of the cells D, E, F are mines
v. Exactly one of the cells F is a mine
vi. Exactly one of the cells F, G are mines
vii. Exactly one of the cells F, G, H is a mine
viii. None of the cells G, H are mines

Hint: it might be useful to remember that "exactly two" for example means the same as "at least two and also no more than two".

(b) Convert the propositional logic statements you have created into Conjunctive Normal Form to give a set of clauses.

[1] For more about this game, see `https://en.wikipedia.org/wiki/Minesweeper_(video_game)`.

(c) Show how DPLL would solve this problem and what solution it would give you. Which cells are safe and which are mines?

(d) Describe how to use DPLL and a reductio ad absurdum proof (or proofs) to show that all the answers to part (c) are necessarily true. (If the DPLL procedure shares a lot of steps in common with your answer to (c), you may provide only any differences in this part, as long as it is completely clear how the DPLL search would proceed.)

Submission

The most appropriate thing to submit is a single PDF containing your solutions.

You are NOT required to format your solutions on computer. For example, clearly handwritten answers are acceptable. In this case you should scan or photograph your answers clearly and include in a single pdf which you upload to MMS.

If absolutely necessary to submit multiple files, submit these as a single zip file.

Marking Guidance

The submission will be marked according to the general mark descriptors at:

`https://studres.cs.st-andrews.ac.uk/CS2002/Assessment/descriptors.pdf`

For a mark 14-16 your submission should contain almost all correct and understandable solutions. For a mark of 17 all solutions need to be correct and with details. For a mark of 19 and higher the solutions need to be highly detailed and alternative approaches should be provided to show insight.

Lateness

The standard penalty for late submission applies (Scheme B: 1 mark per 8 hour period, or part thereof):

`http://info.cs.st-andrews.ac.uk/student-handbook/learning-teaching/assessment.html#lateness-penalties`

Good Academic Practice

I would remind you to ensure you are following the relevant guidelines on good academic practice as outlined at:

`https://studres.cs.st-andrews.ac.uk/Library/Handbook/academic.html#Good_Academic_Practice`