PY2010 Intermediate Logic

UNIVERSITY OF ST. ANDREWS

Take Home Assessment

ID: 200007413

December 7, 2021

I hereby declare that the attached piece of written work is my own work and that I have not reproduced, without acknowledgement, the work of another.

1. (a) $\vdash_{TK\rho\sigma\tau} (\Box p \dashv \Box q) \vee (\Box q \dashv \Box p)$

$$\neg((\Box p \strictif \Box q) \vee (\Box q \strictif \Box p))\ ,0\ \checkmark$$

$$|$$

$$\neg(\Box p \strictif \Box q)\ ,0\ \checkmark$$
$$\neg(\Box q \strictif \Box p)\ ,0\ \checkmark$$

$$|$$

$$\neg\Box(\Box p \supset \Box q)\ ,0\ \checkmark$$
$$\neg\Box(\Box q \supset \Box p)\ ,0\ \checkmark$$

$$|$$

$$\Diamond\neg(\Box p \supset \Box q)\ ,0\ \checkmark$$
$$\Diamond\neg(\Box q \supset \Box p)\ ,0\ \checkmark$$

$$|$$

$$0r1$$
$$\neg(\Box p \supset \Box q)\ ,1\ \checkmark$$

$$|$$

$$\Box p\ ,1\ /3$$
$$\neg\Box q\ ,1$$

$$|$$

$$0r2$$
$$\neg(\Box q \supset \Box p)\ ,2\ \checkmark$$

$$|$$

$$\Box q\ ,2$$
$$\neg\Box p\ ,2$$

$$|$$

$$\Diamond\neg p\ ,2\ \checkmark$$

$$|$$

$$2r3$$
$$\neg p\ ,3$$
$$1r0,\ 0r3,\ 1r3$$

$$|$$

$$p\ ,3$$
$$\otimes$$

(b) $\vdash_{TK\rho} \Diamond(p \supset q) \equiv (\Box p \supset \Diamond q)$

$\neg(\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q)),0\ \checkmark$

$\Diamond(p \supset q),0\ \checkmark \qquad \Box p \supset \Diamond q,0\ \checkmark$

$\neg(\Box p \supset \Diamond q),0\ \checkmark \qquad \neg\Diamond(p \supset q),0\ \checkmark$

$\Box p,0\ /1 \qquad \Box\neg(p \supset q),0\ /1$

$\neg\Diamond q,0\ \checkmark \qquad \neg\Box p,0\ \checkmark \qquad \Diamond q,0\ \checkmark$

$\Box\neg q,0\ /1 \qquad \Diamond\neg p,0\ \checkmark \qquad 0r1$

$q,1$

$0r1 \qquad 0r1 \qquad \neg(p \supset q),1\ \checkmark$

$p \supset q,1\ \checkmark \qquad \neg p,1$

$p,1 \qquad \neg(p \supset q),1\ \checkmark \qquad p,1$

$\neg q,1 \qquad \neg q,1$

$\neg p,1 \qquad q,1 \qquad p,1 \qquad \otimes$

$\otimes \qquad \otimes$

$\neg q,1$

$\otimes$

(c) $\Box\forall x(\neg Fx \vee Gx) \vdash_{TCK} \forall x(\Diamond\neg Fx \vee \Box Gx)$

$\Box\forall x(\neg Fx \vee Gx)$,0 /1
$\neg\forall x(\Diamond\neg Fx \vee \Box Gx)$,0 ✓

$\exists x\neg(\Diamond\neg Fx \vee \Box Gx)$,0 ✓

$\neg(\Diamond\neg Fa \vee \Box Ga)$,0 ✓

$\neg\Diamond\neg Fa$,0 ✓
$\neg\Box Ga$, 0 ✓

$\Box\neg\neg Fa$, 0 /1
$\Diamond\neg Ga$, 0 ✓

$0r1$
$\neg Ga$, 1
$\neg\neg Fa$, 1 ✓

Fa, 1

$\forall x(\neg Fx \vee Gx)$,1 /a

$\neg Fa \vee Ga$,1

$\neg Fa$,1 $\quad$ Ga ,1
$\otimes$ $\quad$ $\otimes$

2. (a) $(p \wedge q) \vee r, \neg r \vdash_{NC} p$

1	(1)	$(p \wedge q) \vee r$	Ass(premise)
2	(2)	$\neg r$	Ass(premise)
3	(3)	$(p \wedge q)$	Ass($\vee E$)
3	(4)	p	3, $\wedge E$
5	(5)	r	Ass($\vee E$)
2,5	(6)	$\perp$	2,5, $\neg E$
7	(7)	$\neg p$	Ass($\neg I$)
2,5	(8)	$\neg\neg p$	6,7, $\neg I$
2,5	(9)	p	8, DN
1,2	(10)	p	1,3,4,5,9, $\vee E$

(b) $\exists x(Fx \vee Gx) \vdash_{NC} \exists xFx \vee \exists xGx$

1	(1)	$\exists x(Fx \vee Gx)$	Ass(premise)
2	(2)	$Fx \vee Gx$	Ass($\exists E$)
3	(3)	Fx	Ass($\vee E$)
3	(4)	$\exists Fx$	3, $\exists I$
3	(5)	$\exists Fx \vee \exists Gx$	4, $\vee I$
6	(6)	Gx	Ass($\vee E$)
6	(7)	$\exists Gx$	6, $\exists I$
6	(8)	$\exists Fx \vee \exists Gx$	7, $\vee I$
2	(9)	$\exists Fx \vee \exists Gx$	2,3,5,6,8, $\vee E$
1	(10)	$\exists Fx \vee \exists Gx$	1,2,9, $\exists E$

(c) $\neg p \wedge \neg q \vdash_{NI} \neg (p \vee q)$

1	(1)	$\neg p \wedge \neg q$	Ass(premise)
1	(2)	$\neg p$	1, $\wedge E$
1	(3)	$\neg q$	1, $\wedge E$
4	(4)	$p \vee q$	Ass($\neg I$)
5	(5)	p	Ass($\vee E$)
1,5	(6)	$\perp$	2,5, $\neg E$
7	(7)	q	Ass($\vee E$)
1,7	(8)	$\perp$	3,7, $\neg E$
1,4	(9)	$\perp$	4,5,6,7,8 $\vee E$
1	(10)	$\neg (p \vee q)$	4,9 $\neg I$

3. (a) $p \vee \neg p \vdash \neg(q \wedge \neg q)$ - check in both K_3 and LP

i. $p \vee \neg p \vdash_{K_3} \neg(q \wedge \neg q)$

$p \vee \neg p$, + ✓
$\neg(q \wedge \neg q)$, - ✓

$\neg q \vee \neg\neg q$, - ✓

$\neg q$, -
$\neg\neg q$, - ✓

q, -

p, + $\qquad$ $\neg p$, +

Counter Model:
$v(p) = 1, v(q) = n$
$D \in \{1\}$
As $v(p) = 1$, then the semantics for $\neg$ mean that $v(\neg p) = 0$.
Then, the semantics of $\vee$ mean that $v(p \vee \neg p) = 1$ because the maximum of $v(p) = 1$ and $v(\neg p) = 0$ is 1. Therefore, as $1 \in D$, we have a case where the premises are true.
Next, with $v(q) = n$, then the semantics for $\neg$ mean that $v(\neg q) = n$.
Then, the semantics for $\wedge$ means $v(q \wedge \neg q)$ is the minimum of $v(q)$ and $v(\neg q)$ which are both n and so $v(q \wedge \neg q) = n$.
Then, the semantics for $\neg$ mean that $v(\neg(q \wedge \neg q)) = n$ as $v(q \wedge \neg q) = n$.
Therefore, we have a scenario where the premises are all satisfied, but the conclusion has value n and $n \notin D$ meaning it is not satisfied. Therefore, $p \vee \neg p \nvdash_{K_3} \neg(q \wedge \neg q)$

QED.

ii. $p \vee \neg p \vdash_{LP} \neg(q \wedge \neg q)$

$p \vee \neg p$, + ✓
$\neg(q \wedge \neg q)$, - ✓

$\neg q \vee \neg\neg q$, - ✓

$\neg q$, -
$\neg\neg q$, - ✓

q, -
⊗

(b) $\vdash_{L_3} (p \wedge \neg p) \rightarrow (q \vee \neg q)$

$(p \wedge \neg p) \rightarrow (q \vee \neg q)$, - ✓

$p \wedge \neg p$, + ✓	$\neg(q \vee \neg q)$, + ✓
$q \vee \neg q$, -	$\neg(p \wedge \neg p)$, -
p, +	$\neg q \wedge \neg\neg q$, + ✓
$\neg p$, +	$\neg q$, +
⊗	$\neg\neg q$, + ✓
	q, +
	⊗

(c) $\vdash_{RM3} \neg p \rightarrow (p \rightarrow q)$

$\neg p \rightarrow (p \rightarrow q)$, - ✓

$\neg p$, +		$\neg(p \rightarrow q)$, + ✓
$p \rightarrow q$, - ✓		$\neg\neg p$, - ✓
p, +	$\neg q$, +	p, -
q, -	$\neg p$, -	p, +
	⊗	$\neg q$, +
		⊗

Counter Model:
$v(q) = 0, v(p) = b$
$D \in \{1, b\}$
As there are no premises, they are automatically satisfied.
Then, because $v(p) = b$, the semantics for $\neg$ mean that $v(\neg p) = b$.

Then, with $v(p) = b$, and $v(q) = 0$, the semantics for $\rightarrow$ mean that $v(p \rightarrow q) = 0$.
Then, the semantics of $\rightarrow$ again mean that $v(\neg p \rightarrow (p \rightarrow q)) = 0$ because $v(\neg p) = b$ and $v(p \rightarrow q) = 0$.
Therefore, as $0 \notin D$, we have a scenario where all the premises are satisfied, but the conclusion is not, and so $\not\models_{RM3} \neg p \rightarrow (p \rightarrow q)$. QED.

4. n/a

5. (a) $\neg p \models_{L_{\aleph}} p \rightarrow q$
SHOW: for an arbitrary v, when $v(\neg p) = 1$, $v(p \rightarrow q) = 1$
PROOF: Assume for some arbitrary v, $v(\neg p) = 1$.
Then, by the semantics of $\neg$, $v(p) = 0$.
Since the values in $L_{\aleph}$ are linearly ordered, we know that either (i) $v(p) > v(q)$, or (ii) $v(p) \leq v(q)$.
However, as $v(p) = 0$, it cannot be that (i) $v(p) > v(q)$, as there are no values smaller than 0 and so $v(p)$ cannot be strictly greater than $v(q)$. Therefore $v(p) \leq v(q)$.
The semantics of $\rightarrow$ mean that when $v(p) \leq v(q)$, $v(p \rightarrow q) = 1$. QED.

(b) $\models_G \neg(q \vee \neg q) \rightarrow p$
SHOW: for any arbitrary v, $v(\neg(q \vee \neg q) \rightarrow p) = 1$
PROOF: Since the semantic values in G are linearly ordered, We know that for any v. $0 \leq v(q) \leq 1$.
The semantics for $\neg$ give us two options, either: (i) $v(q) = 0$ giving $v(\neg q) = 1$; or (ii) $v(q) > 0$ giving $v(\neg q) = 0$.

i. With $v(q) = 0$ and $v(\neg q) = 1$, we know by the semantics of $\vee$ that $v(q \vee \neg q) = 1$
From this, the semantics of $\neg$ mean that $v(\neg(q \vee \neg q)) = 0$ because $0 > 1$ and so $v(q \vee \neg q) > 0$.
ii. With $v(q) > 0$ and $v(\neg q) = 0$, we know that $v(\neg q) \leq v(q)$ because there are no potential values of $v(q)$ smaller than 0. So, by the semantics of $\vee$, $v(q \vee \neg q) > 0$.
From this, the semantics of $\neg$ mean that $v(\neg(q \vee \neg q)) = 0$.

Therefore, from both potential values for $v(p)$ with (i) and (ii), we get $v(\neg(q \vee \neg q)) = 0$.
We also know that $0 <= v(p) <= 1$. No matter what the value of $v(p)$ is, $v(\neg(q \vee \neg q)) = 0$ and so $v(\neg(q \vee \neg q)) <= v(p)$ as there are no values smaller than 0 on the linear scale.
Therefore, by the semantics of $\rightarrow$, $v(\neg(q \vee \neg q) \rightarrow p) = 1$ for all v. QED.