PY2010 Intermediate Logic
University of St. Andrews
Assessed Logic Exercises - wk.05

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1. Explain why the following are true in the semantics of classical propositional logic.

   (a) $q \vee \neg q \models_C ((p \supset q) \supset p) \supset p$

   Show: For all interpretations of v where $v(q \vee \neg q) = 1$: $v(((p \supset q) \supset p) \supset p) = 1$.

   Assume for reductio that for some $v, v(((p \supset q) \supset p) \supset p) \neq 1$. i.e. $v(((p \supset q) \supset p) \supset p) = 0$

   To begin, we know by the semantic clause for $\vee$ that $v(q \vee \neg q) = 1$ iff $v(q) = 1$ or $v(\neg q) = 1$.

   By $\neg$ semantics, we know that $v(\neg q) = 1$ when $v(q) = 0$. Therefore $v(q \vee \neg q) = 1$ iff $v(q) = 1$ or $v(q) = 0$, which means that $v(q \vee \neg q) = 1$ is always the case.

   Next, by the semantics of $\supset$, we know $v(((p \supset q) \supset p) \supset p) = 0$ iff $v((p \supset q) \supset p) = 1$ and $v(p) = 0$.

   Therefore, because $v((p \supset q) \supset p) = 1$ iff $v(p \supset q) = 0$ or $v(p) = 1$.

   If $v(p) = 1$, then we have a contradiction as $v(p) = 1$. By the $\neg \supset$ semantic clause, $v(p \supset q) = 0$ iff $v(p) = 1$ and $v(q) = 0$.

   However, this also means that $v(p) = 0$ at the same time that $v(p) = 1$ which cannot be, since v is a function.

   So, there is no v such that $v(((p \supset q) \supset p) \supset p) = 0$ at the same time that $v(q \vee \neg q) = 1$. QED

   (b) $\neg\neg\neg(p \wedge q) \models_C \neg p \vee \neg q$

   Show: For all interpretations of v, where $v(\neg\neg\neg(p \wedge q)) = 1$, $v(\neg p \vee \neg q) = 1$

   Assume, for some $v, v(\neg\neg\neg(p \wedge q)) = 1$.

   By the $\neg$ semantic clause, $v(\neg\neg\neg(p \wedge q)) = 1$ iff $v(\neg\neg(p \wedge q)) = 0$. Then by the same clause, $v(\neg\neg(p \wedge q)) = 0$ iff $v(\neg(p \wedge q)) = 1$. Finally the same clause again means that $v(\neg(p \wedge q)) = 1$ iff $v(p \wedge q) = 0$.

   With the $\wedge$ semantic clause, we know that $v(p \wedge q) = 0$ iff $v(p) = 0$ or $v(q) = 0$.

   Using the $\neg$ semantics, we know that $v(p) = 0$ iff $v(\neg p) = 1$, and $v(q) = 0$ iff $v(\neg q) = 1$.

   Finally, using the $\vee$ semantic clause, we know that $v(\neg p \vee \neg q) = 1$ iff $v(p) = 0$ and $v(q) = 0$ and so for all interpretations of v where $v(\neg\neg\neg(p \wedge q)) = 1, v(\neg p \vee \neg q) = 1$. QED

2. Give a proof of each of the following using natural deduction.

   (a) $q \supset r \vdash_{NC} (p \wedge q) \supset r$

1	(1)	$q \supset r$	Ass(Premise)
2	(2)	q	Ass($\supset E$)
1,2	(3)	r	$1,2 \supset E$
4	(4)	p	Ass($\wedge I$)
2,4	(5)	$p \wedge q$	2,4$\wedge I$
1	(6)	$(p \wedge q) \supset r$	$3,5 \supset I$

   (b) $(\neg p \wedge \neg q) \vee (p \wedge q) \vdash_{NC} p \equiv q$

   hint: use the definition $p \equiv q :\equiv (p \supset q) \wedge (q \supset p)$

1	(1)	$(\neg p \wedge \neg q) \vee (p \wedge q)$	Ass(Premise)
2	(2)	$\neg p \wedge \neg q$	Ass($\vee E$)
2	(3)	$\neg p$	2 $\wedge E$
2	(4)	$\neg q$	2 $\wedge E$
	(5)	$\neg p \supset \neg q$	3,4 $\supset I$
	(6)	$\neg q \supset \neg p$	3,4 $\supset I$
	(7)	$\neg p \equiv \neg q$	5,6 $\wedge I$
8	(8)	$p \wedge q$	Ass($\vee E$)
8	(9)	p	8 $\wedge E$
8	(10)	q	8 $\wedge E$
	(11)	$p \supset q$	9,10 $\supset I$
	(12)	$q \supset p$	9,10 $\supset I$
	(13)	$p \equiv q$	11,12 $\wedge I$
1	(14)	$p \equiv q$	1,2,7,8,13 $\vee E$

3. Check the following using tableaux for classical propositional logic. Give a countermodel if the argument is invalid. (Make sure you explain why the interpretation you give 'works

   (a) $p \vee (\neg p \wedge (p \vee q)) \vdash_{TC} p \vee q$

$p \vee (\neg p \wedge (p \vee q))$ ✓

$\neg(p \vee q)$ ✓

$\neg p$

$\neg q$

p $\neg p \wedge (p \vee q)$ ✓

$\otimes$

p

$p \vee q$

$\otimes$

   (b) $\neg(p \supset (q \vee r)) \vdash_{TC} \neg((q \vee r) \supset p)$

$$\neg(p \supset (q \vee r)) \checkmark$$
$$\neg(\neg((q \vee r) \supset p)) \checkmark$$
$$|$$
$$(q \vee r) \supset p \checkmark$$
$$|$$
$$p$$
$$\neg(q \vee r) \checkmark$$

$$\begin{array}{cc} & \diagup \quad \diagdown & \\ \neg(q \vee r) \checkmark & p \\ | & | \\ \neg q & \neg q \\ \neg r & \neg r \\ | & \\ \neg q & \\ \neg r & \end{array}$$

Counter-model: Let $v(q) = 0, v(r) = 0$, and $v(p) = 1$.

The $\supset$ semantic clause means that $v(\neg(p \supset (q \vee r))) = 1$ iff $v(p) = 1$ and $v(q \vee r) = 0$, and the semantic clause for $\vee$ means that $v(q \vee r) = 0$ iff $v(q) = 0$ and $v(r) = 0$ and so the premises are satisfied.

Again, the semantic clause for $\supset$ means that $v(\neg((q \vee r) \supset p)) = 1$ iff $v(q \vee r) = 1$ and $v(p) = 0$.

As $v(p) = 0$, it means that $v(\neg((q \vee r) \supset p)) = 0$, meaning the conclusion is false, despite the premises being true. QED

4. Explain why the following are true in the semantics of the various modal logics.

(a) $\Diamond p \models_K \neg\Box\neg p$

Take any interpretation with world w where $v_w(\Diamond p) = 1$.

The semantics of $\Diamond$ mean that $v_w(\Diamond p) = 1$ iff at some x such that wRx, $v_x(p) = 1$.

Through the double negation rule, we can write this as $v_x(\neg\neg p) = 1$.

Due to the semantics of $\Diamond$, $v_w(\Diamond\neg\neg p) = 1$ iff at some x such that wRx, there is $v_x(\neg\neg p) = 1$. Therefore, with $v_w(\Diamond\neg\neg p) = 1$, the semantics of $\Diamond$ mean that this can be written as $v_w(\neg\Box\neg p) = 1$. QED

(b) $\Box\Box\Box p \models_{K\rho\tau} \Box p$

For reductio, assume an arbitrary $\langle W, R, v\rangle$ with reflexive and transitive R such that $v_w(\Box\Box\Box p) = 1$, and $v_w(\Box p) = 0$.

Next, the semantics of $\Box$ dictate that $v_w(\Box\Box\Box p) = 1$ iff at ALL x such that wRx with reflexive and transitive R, $v_x(\Box\Box p) = 1$.

Again, the $\Box$ semantics mean that $v_x(\Box\Box p) = 1$ iff at ALL y such that xRy with reflexive and transitive R, $v_y(\Box p) = 1$.

Finally, we know that $v_y(\Box p) = 1$ iff at ALL z such that yRz with reflexive and transitive R, $v_z(p) = 1$.

However, as R is transitive, because wRx, and xRy, and yRz, we have wRy, and wRz

Therefore, $v_z(p) = 1$ would contradict the fact that $v_w(\Box) = 0$. So there cannot be y such that wRy where $v_y(\Box) = 0$, so $v_y(\Box) = 1$ at at all y such that wRy so $v_w(\Box) = 1$. QED

5. Check the following using tableaux for the various modal logics. Give a countermodel if the argument is invalid. (Make sure you explain why the interpretation you give works

   (a) $\Diamond p, \Box q \vdash_{TK} \Diamond(p \wedge q)$

   $\Diamond p, 0 \checkmark$
   $\Box q, 0 \ /1$
   $\neg\Diamond(p \wedge q), 0 \checkmark$
   $|$
   $\Box\neg(p \wedge q), 0 \ /1$
   $|$
   $0r1$
   $p, 1$
   $|$
   $q, 1$
   $|$
   $\neg(p \wedge q), 1 \checkmark$

   $\neg p, 1 \qquad \neg q, 1$
   $\otimes \qquad\quad \otimes$

   (b) $p \strictif r, q \strictif r \vdash_{TK\rho\tau} p \strictif q$

   $\Box(p \supset r), 0 \ /1$
   $\Box(q \supset r), 0 \ /1$
   $\neg\Box(p \supset q), 0 \checkmark$
   $|$
   $\Diamond\neg(p \supset q), 0 \checkmark$
   $|$
   $0r1$
   $\neg(p \supset q), 1 \checkmark$
   $|$
   $p, 1$
   $\neg q, 1$
   $|$
   $p \supset r, 1 \checkmark$
   $q \supset r, 1 \checkmark$

   $\neg p, 1 \qquad\qquad r, 1$

   $\otimes \qquad \neg q, 1 \qquad r, 1$

   Counter-model:
   $W = \{w_0, w_1\}$
   $R = \{\langle w_0, w_1 \rangle\}$
   $v_{w_1}(q) = 0$

$v_{w_1}(r) = 1$
$v_{w_1}(p) = 1$
This countermodel works as $v_{w_1}(q \supset r) = 1$ iff $v_{w_1}(q) = 0$ or $v_{w_1}(r) = 1$ which is the case.
So, because w_0 only has relation 0r1, the $\Box$ semantics are that $v_w(\Box(q \supset r)) = 1$ iff at all worlds x such that wRx $v_x(q \supset r) = 1$, which is the case for w_0, and so $v_{w_0}(\Box(q \supset r)) = 1$.
Also, $v_{w_1}(p \supset r) = 1$ iff $v_{w_1}(p) = 0$ or $v_{w_1}(r) = 1$, and we have $v_{w_1}(r) = 1$ so $v_{w_1}(p \supset r) = 1$.
Similarly to before, as w_0 only has relation 0r1, the $\Box$ semantics are that $v_w(\Box(p \supset r)) = 1$ iff at all worlds x such that wRx $v_x(p \supset r) = 1$, which is the case for w_0, and so $v_{w_0}(\Box(p \supset r)) = 1$.
Finally, as we have $v_{w_1}(p) = 1$, and $v_{w_1}(q) = 0$. The semantics for $\supset$ mean that $v_{w_1}(p \supset q) = 1$ iff $v_{w_1}(p) = 0$ or $v_{w_1}(q) = 1$, meaning that $v_{w_1}(p \supset q) = 0$, which can be written $v_{w_1}(\neg(p \supset q)) = 1$.
Therefore, the $\Diamond$ rule states that $v_w(\Diamond\neg(p \supset q)) = 1$ iff at some x such that wRx $v_x(\neg(p \supset q) = 1$. As we have 0r1 and $v_{w_1}(\neg(p \supset q)) = 1$, we can write $v_{w_0}(\Diamond\neg(p \supset q)) = 1$.
The semantics of $\Diamond$ means $v_{w_0}(\Diamond\neg(p \supset q)) = 1$ can be written $v_{w_0}(\neg\Box(p \supset q)) = 1$, which can be written as $v_{w_0}(\Box(p \supset q)) = 0$, which is our conclusion.
This constructs a world in which all of the premises: $\Box(p \supset r)$, and $\Box(q \supset r)$ are true in world 0, whilst the conclusion $\Box(p \supset q)$ is false also at world 0. QED