

PY2010 Intermediate Logic
UNIVERSITY OF ST. ANDREWS
Assessed Logic Exercises - wk.05

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1. Explain why the following are true in the semantics of classical propositional logic.

(a) $q \vee \neg q \models_c ((p \supset q) \supset p) \supset p$

Show: For all interpretations of v where $v(q \vee \neg q) = 1$: $v(((p \supset q) \supset p) \supset p) = 1$.

Assume for reductio that for some v , $v(((p \supset q) \supset p) \supset p) \neq 1$. i.e. $v(((p \supset q) \supset p) \supset p) = 0$

To begin, we know by the semantic clause for \vee that $v(q \vee \neg q) = 1$ iff $v(q) = 1$ or $v(\neg q) = 1$.

By \neg semantics, we know that $v(\neg q) = 1$ when $v(q) = 0$. Therefore $v(q \vee \neg q) = 1$ iff $v(q) = 1$ or $v(q) = 0$, which means that $v(q \vee \neg q) = 1$ is always the case.

Next, by the semantics of \supset , we know $v(((p \supset q) \supset p) \supset p) = 0$ iff $v((p \supset q) \supset p) = 1$ and $v(p) = 0$.

Therefore, because $v((p \supset q) \supset p) = 1$ iff $v(p \supset q) = 0$ or $v(p) = 1$.

If $v(p) = 1$, then we have a contradiction as $v(p) = 1$. By the $\neg \supset$ semantic clause, $v(p \supset q) = 0$ iff $v(p) = 1$ and $v(q) = 0$.

However, this also means that $v(p) = 0$ at the same time that $v(p) = 1$ which cannot be, since v is a function.

So, there is no v such that $v(((p \supset q) \supset p) \supset p) = 0$ at the same time that $v(q \vee \neg q) = 1$.
QED

(b) $\neg\neg\neg(p \wedge q) \models_c \neg p \vee \neg q$

Show: For all interpretations of v , where $v(\neg\neg\neg(p \wedge q)) = 1$, $v(\neg p \vee \neg q) = 1$

Assume, for some v , $v(\neg\neg\neg(p \wedge q)) = 1$.

By the \neg semantic clause, $v(\neg\neg\neg(p \wedge q)) = 1$ iff $v(\neg\neg(p \wedge q)) = 0$. Then by the same clause, $v(\neg\neg(p \wedge q)) = 0$ iff $v(\neg(p \wedge q)) = 1$. Finally the same clause again means that $v(\neg(p \wedge q)) = 1$ iff $v(p \wedge q) = 0$.

With the \wedge semantic clause, we know that $v(p \wedge q) = 0$ iff $v(p) = 0$ or $v(q) = 0$.

Using the \neg semantics, we know that $v(p) = 0$ iff $v(\neg p) = 1$, and $v(q) = 0$ iff $v(\neg q) = 1$.

Finally, using the \vee semantic clause, we know that $v(\neg p \vee \neg q) = 1$ iff $v(p) = 0$ and $v(q) = 0$ and so for all interpretations of v where $v(\neg\neg\neg(p \wedge q)) = 1$, $v(\neg p \vee \neg q) = 1$.

QED

$$(a) \quad q \supset r \vdash_{NC} (p \wedge q) \supset r$$

1	(1)	$q \supset r$	Ass(Premise)
2	(2)	q	Ass($\supset E$)
1,2	(3)	r	1,2 $\supset E$
4	(4)	p	Ass($\wedge I$)
2,4	(5)	$p \wedge q$	2,4 $\wedge I$
1	(6)	$(p \wedge q) \supset r$	3,5 $\supset I$

hint: use the definition $p \equiv q \equiv (p \supset q) \wedge (q \supset p)$

1	(1)	$(\neg p \wedge \neg q) \vee (p \wedge q)$	Ass(Premise)
2	(2)	$\neg p \wedge \neg q$	Ass($\vee E$)
2	(3)	$\neg p$	$2 \wedge E$
2	(4)	$\neg q$	$2 \wedge E$
	(5)	$\neg p \supset \neg q$	$3,4 \supset I$
	(6)	$\neg q \supset \neg p$	$3,4 \supset I$
	(7)	$\neg p \equiv \neg q$	$5,6 \wedge I$
8	(8)	$p \wedge q$	Ass($\vee E$)
8	(9)	p	$8 \wedge E$
8	(10)	q	$8 \wedge E$
	(11)	$p \supset q$	$9,10 \supset I$
	(12)	$q \supset p$	$9,10 \supset I$
	(13)	$p \equiv q$	$11,12 \wedge I$
1	(14)	$p \equiv q$	$1,2,7,8,13 \vee E$

(a) $p \vee (\neg p \wedge (p \vee q)) \vdash_{TC} p \vee q$

$$\neg(p \vee q) \quad \checkmark$$

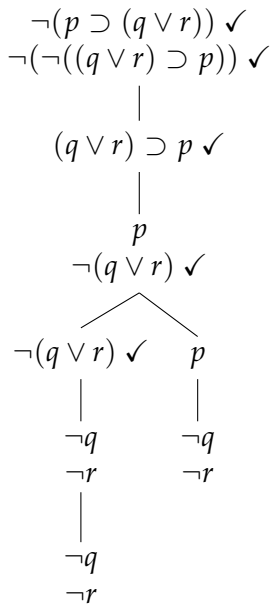
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$$\neg p$$
$$\neg q$$
$$\begin{array}{c}
 q \\
 \swarrow \quad \searrow \\
 p \quad \neg p \wedge (p \vee q) \checkmark \\
 \otimes \quad |
 \end{array}$$

③

$$p$$
$$p \vee q$$

$$(b) \neg(p \supset (q \vee r)) \vdash_{TC} \neg((q \vee r) \supset p)$$



Counter-model: Let $v(q) = 0, v(r) = 0$, and $v(p) = 1$.

The \supset semantic clause means that $v(\neg(p \supset (q \vee r))) = 1$ iff $v(p) = 1$ and $v(q \vee r) = 0$, and the semantic clause for \vee means that $v(q \vee r) = 0$ iff $v(q) = 0$ and $v(r) = 0$ and so the premises are satisfied.

Again, the semantic clause for \supset means that $v(\neg((q \vee r) \supset p)) = 1$ iff $v(q \vee r) = 1$ and $v(p) = 0$.

As $v(p) = 0$, it means that $v(\neg((q \vee r) \supset p)) = 0$, meaning the conclusion is false, despite the premises being true. QED

4. Explain why the following are true in the semantics of the various modal logics.

(a) $\Diamond p \models_K \neg \Box \neg p$

Take any interpretation with world w where $v_w(\Diamond p) = 1$.

The semantics of \Diamond mean that $v_w(\Diamond p) = 1$ iff at some x such that wRx , $v_x(p) = 1$.

Through the double negation rule, we can write this as $v_x(\neg \neg p) = 1$.

Due to the semantics of \Diamond , $v_w(\Diamond \neg \neg p) = 1$ iff at some x such that wRx , there is $v_x(\neg \neg p) = 1$. Therefore, with $v_w(\Diamond \neg \neg p) = 1$, the semantics of \Diamond mean that this can be written as $v_w(\neg \Box \neg p) = 1$. QED

(b) $\Box \Box \Box p \models_{K\sigma\tau} \Box p$

For reductio, assume an arbitrary $\langle W, R, v \rangle$ with reflexive and transitive R such that $v_w(\Box \Box \Box p) = 1$, and $v_w(\Box p) = 0$.

Next, the semantics of \Box dictate that $v_w(\Box \Box \Box p) = 1$ iff at ALL x such that wRx with reflexive and transitive R , $v_x(\Box \Box p) = 1$.

Again, the \Box semantics mean that $v_x(\Box \Box p) = 1$ iff at ALL y such that xRy with reflexive and transitive R , $v_y(\Box p) = 1$.

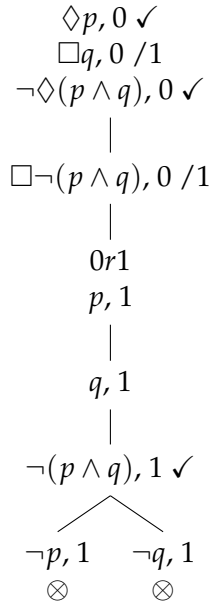
Finally, we know that $v_y(\Box p) = 1$ iff at ALL z such that yRz with reflexive and transitive R , $v_z(p) = 1$.

However, as R is transitive, because wRx , and xRy , and yRz , we have wRy , and wRz

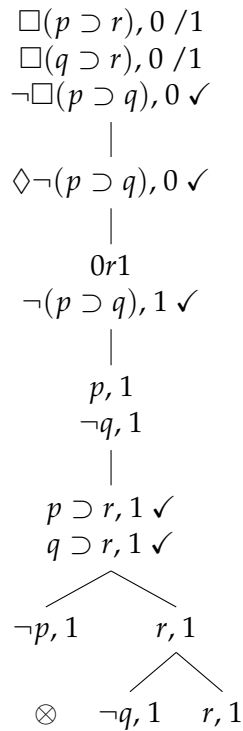
Therefore, $v_z(p) = 1$ would contradict the fact that $v_w(\Box p) = 0$. So there cannot be y such that wRy where $v_y(\Box p) = 0$, so $v_y(\Box p) = 1$ at all y such that wRy so $v_w(\Box p) = 1$. QED

5. Check the following using tableaux for the various modal logics. Give a countermodel if the argument is invalid. (Make sure you explain why the interpretation you give works)

(a) $\Diamond p, \Box q \vdash_{TK} \Diamond(p \wedge q)$



(b) $p \supset r, q \supset r \vdash_{TK_{\rho\tau}} p \supset q$



Counter-model:

$$W = \{w_0, w_1\}$$

$$R = \{\langle w_0, w_1 \rangle\}$$

$$v_{w_1}(q) = 0$$

$$v_{w_1}(r) = 1$$

$$v_{w_1}(p) = 1$$

This countermodel works as $v_{w_1}(q \supset r) = 1$ iff $v_{w_1}(q) = 0$ or $v_{w_1}(r) = 1$ which is the case. So, because w_0 only has relation 0r1, the \Box semantics are that $v_w(\Box(q \supset r)) = 1$ iff at all worlds x such that wRx $v_x(q \supset r) = 1$, which is the case for w_0 , and so $v_{w_0}(\Box(q \supset r)) = 1$. Also, $v_{w_1}(p \supset r) = 1$ iff $v_{w_1}(p) = 0$ or $v_{w_1}(r) = 1$, and we have $v_{w_1}(r) = 1$ so $v_{w_1}(p \supset r) = 1$.

Similarly to before, as w_0 only has relation 0r1, the \Box semantics are that $v_w(\Box(p \supset r)) = 1$ iff at all worlds x such that wRx $v_x(p \supset r) = 1$, which is the case for w_0 , and so $v_{w_0}(\Box(p \supset r)) = 1$.

Finally, as we have $v_{w_1}(p) = 1$, and $v_{w_1}(q) = 0$. The semantics for \supset mean that $v_{w_1}(p \supset q) = 1$ iff $v_{w_1}(p) = 0$ or $v_{w_1}(q) = 1$, meaning that $v_{w_1}(p \supset q) = 0$, which can be written $v_{w_1}(\neg(p \supset q)) = 1$.

Therefore, the \Diamond rule states that $v_w(\Diamond\neg(p \supset q)) = 1$ iff at some x such that wRx $v_x(\neg(p \supset q)) = 1$. As we have 0r1 and $v_{w_1}(\neg(p \supset q)) = 1$, we can write $v_{w_0}(\Diamond\neg(p \supset q)) = 1$.

The semantics of \Diamond means $v_{w_0}(\Diamond\neg(p \supset q)) = 1$ can be written $v_{w_0}(\neg\Box(p \supset q)) = 1$, which can be written as $v_{w_0}(\Box(p \supset q)) = 0$, which is our conclusion.

This constructs a world in which all of the premises: $\Box(p \supset r)$, and $\Box(q \supset r)$ are true in world 0, whilst the conclusion $\Box(p \supset q)$ is false also at world 0. QED