Practical 01

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Q1.

Interpretation I: $\mathcal{D} = \{0, \mathbb{N}\}$.

a. $\exists y.\forall x.(y < x)$

This formula's semantics are: there exists some $y \in \mathcal{D}$, s.t. for all $x \in \mathcal{D}$, $(y < x)$

Given an arbitrary object $I[a] \in \mathcal{D}$, we h

b. $\exists x.\forall y.(y < x)$

The semantics of this formula are: There exists some $x \in \mathcal{D}$, s.t. for all $y \in \mathcal{D}$, $y < x$.

Given an arbitrary object $I[a] \in \mathcal{D}$, where $x = a$, we are left with the formula $\forall y.(y < a)$. The formula is true if a value of a exists which makes this true.

If $a = 0$, then we have $\forall y.(y < 0)$, which is false by counter example of $y = 1$.

For all $a \in \mathbb{N}$, the definition of $<$ means that if $y = a$, then $(y < a)$ is false for all values of a, as $(a < a)$ is false.

Therefore, by definition of $\mathcal{D}$, and $\forall$, the formula is false for all values of x, and so $\exists x.\forall y(y < x)$ is false by definition of $\exists$.

c. $\forall y.\forall x(y = x \lor y > x)$

The semantics of this formula are: for all $y \in \mathcal{D}$, for all $x \in \mathcal{D}$, $y == x$ or $y > x$.

Therefore, we can prove this to be false by counter example. If $x = 1$, and $y = 0$, then $y = x$ is false, and $y > x$ is false by the definition of $=$ and $<$. This means that $(y = x \lor y > x)$ is false, as $F \lor F = F$ by definition of $\lor$.

Therefore, we have an example x and y where the formula is false, and so $\forall y.\forall x.(y = x \lor y > x)$ is false by defintion of $\forall$.

d. $\forall x.(odd(x) \lor odd(x + 2))$

The semantics of this formula are: for all values of $x \in \mathcal{D}$, x is odd, or $x + 2$ is odd.

We can prove this to be false by counter example. If $x = 2$, then $odd(2)$ is false by definition of odd and $even$.

By definition of +, $2 + 2 = 4$. By definition of odd and $even$, $odd(4)$ is false and so $odd(2 + 2)$ is false and so by definition of $\vee$, $odd(2) \vee odd(2 + 2)$ is false.

By definition of $\forall$, $\forall x.(odd(x) \vee odd(x + 2))$ is false by counter example of $x = 2$.

Q2.

a.

P	$P \to P$
0	1
1	1

$P \to P$ is a tautology

b.

P	$\neg P$	$P \to \neg P$
0	1	1
1	0	0

$P \to \neg P$ is contingent

c.

P	$P \to P$	$\neg(P \to P)$
0	1	0
1	1	0

$\neg(P \to P)$ is contradictory

d.

P	Q	$\neg P$	$\neg Q$	$\neg P \to \neg Q$	$\neg(\neg P \to \neg Q)$	$P \vee \neg Q$	$(P \vee \neg Q) \wedge \neg(\neg P \to \neg Q)$
0	0	1	1	1	0	1	1
0	1	1	0	0	1	0	1
1	0	0	1	1	0	1	1
1	1	0	0	1	0	1	1

$(P \vee \neg Q) \wedge \neg(\neg P \to \neg Q)$ is a tautology

e.

P	Q	R	$P \wedge Q$	$P \wedge Q \rightarrow R$	$P \rightarrow R$	$Q \rightarrow R$	$(P \wedge Q \rightarrow R) \rightarrow (P \rightarrow R)$	$(P \wedge Q \rightarrow R) \rightarrow (P \rightarrow R) \rightarrow (Q \rightarrow R)$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	1	1
0	1	0	0	1	1	0	1	0
0	1	1	0	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	1	0
1	1	1	1	1	1	1	1	1

$(P \wedge Q \rightarrow R) \rightarrow (P \rightarrow R) \rightarrow (Q \rightarrow R)$ is contingent

f.

P	Q	R	$P \vee Q$	$P \rightarrow R$	$Q \rightarrow R$	$(P \rightarrow R) \wedge (Q \rightarrow R)$	$P \vee Q \rightarrow R$	$(P \vee Q \rightarrow R) \Leftrightarrow ((P \rightarrow R) \wedge (Q \rightarrow R))$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	1	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

$(P \vee Q \rightarrow R) \Leftrightarrow ((P \rightarrow R) \wedge (Q \rightarrow R))$ is tautological

g.

P	Q	R	S	$P \rightarrow Q$	$R \rightarrow S$	$P \vee R$	$Q \vee S$	$P \vee R \rightarrow Q \vee S$	$(P \rightarrow Q) \wedge (R \rightarrow S)$	$(P \rightarrow Q) \wedge (R \rightarrow S) \rightarrow (P \vee R \rightarrow Q \vee S)$
0	0	0	0	1	1	0	0	1	1	1
0	0	0	1	1	1	0	1	1	1	1
0	0	1	0	1	0	1	0	0	0	1
0	0	1	1	1	1	1	1	1	1	1
0	1	0	0	1	1	0	1	1	1	1
0	1	0	1	1	1	0	1	1	1	1
0	1	1	0	1	0	1	1	1	0	1
0	1	1	1	1	1	1	1	1	1	1
1	0	0	0	0	1	1	0	0	0	1
1	0	0	1	0	1	1	1	1	0	1
1	0	1	0	0	0	1	0	0	0	1
1	0	1	1	0	1	1	1	1	0	1
1	1	0	0	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1	1
1	1	1	0	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1	1	1	1

$(P \rightarrow Q) \wedge (R \rightarrow S) \rightarrow (P \vee R \rightarrow Q \vee S)$ is a tautology

Q3.

Prove: $\neg((P \wedge Q) \wedge ((P \vee (R \wedge P)) \wedge (\neg Q \wedge \neg P)) \wedge (\neg\neg P \vee (Q \vee \neg Q))) \Leftrightarrow \mathbb{T}$

Starting with:

$\neg((P \wedge Q) \wedge ((P \vee (R \wedge P)) \wedge (\neg Q \wedge \neg P)) \wedge (\neg\neg P \vee (Q \vee \neg Q)))$

Double negation rule $\neg\neg P = P$:

$\neg((P \wedge Q) \wedge ((P \vee (R \wedge P)) \wedge (\neg Q \wedge \neg P)) \wedge (P \vee (Q \vee \neg Q)))$

Complementation $Q \vee \neg Q = \mathbb{T}$:

$\neg((P \wedge Q) \wedge ((P \vee (R \wedge P)) \wedge (\neg Q \wedge \neg P)) \wedge (P \vee \mathbb{T}))$

Distributive law $P \vee (R \wedge P) = (P \vee R) \wedge (P \vee P)$:

$\neg((P \wedge Q) \wedge (((P \vee R) \wedge (P \vee P)) \wedge (\neg Q \wedge \neg P)) \wedge (P \vee \mathbb{T}))$

Idempotent law $P \vee P = P$:

$\neg((P \wedge Q) \wedge (((P \vee R) \wedge P) \wedge (\neg Q \wedge \neg P)) \wedge (P \vee \mathbb{T}))$

Associative law $(((P \vee R) \wedge P) \wedge (\neg Q \wedge \neg P)) = (((\neg Q \wedge \neg P) \wedge P) \wedge (P \vee R))$

$\neg((P \wedge Q) \wedge (((\neg Q \wedge \neg P) \wedge P) \wedge (P \vee R)) \wedge (P \vee \mathbb{T}))$

Associative law $((\neg Q \wedge \neg P) \wedge P) = ((P \wedge \neg P) \wedge \neg Q)$

$\neg((P \wedge Q) \wedge (((P \wedge \neg P) \wedge \neg Q) \wedge (P \vee R)) \wedge (P \vee \mathbb{T}))$

Complementation $P \wedge \neg P = \mathbb{F}$

$\neg((P \wedge Q) \wedge ((\mathbb{F} \wedge \neg Q) \wedge (P \vee R)) \wedge (P \vee \mathbb{T}))$

Domination law $P \vee \mathbb{T} = T$ and $(\mathbb{F} \wedge \neg Q) = \mathbb{F}$:

$\neg((P \wedge Q) \wedge (\mathbb{F} \wedge (P \vee R)) \wedge \mathbb{T})$

Domination law $(\mathbb{F} \wedge (P \vee R)) = \mathbb{F}$

$\neg((P \wedge Q) \wedge \mathbb{F} \wedge \mathbb{T})$

Domination law $(P \wedge Q) \wedge \mathbb{F} = \mathbb{F}$

$\neg(\mathbb{F} \wedge \mathbb{T})$

Domination law $\mathbb{F} \wedge \mathbb{T} = \mathbb{F}$

$\neg\mathbb{F}$

$\neg\mathbb{F} = \mathbb{T}$, therefore $\neg((P \wedge Q) \wedge ((P \vee (R \wedge P)) \wedge (\neg Q \wedge \neg P)) \wedge (\neg\neg P \vee (Q \vee \neg Q))) = \mathbb{T}$ giving us $T \Leftrightarrow T$.

T	$T \Leftrightarrow T$
0	0
1	1

QED.

Q4.

a. $P \to Q, Q \to R \vdash P \to R$

1	(1)	$P \to Q$	Premise
2	(2)	$Q \to R$	Premise
3	(3)	P	Assumption ($\to E, \to I$)
1,3	(4)	Q	$1, 3, \to E$
1,2,3	(5)	R	$2, 4, \to E$
1,2	(6)	$P \to R$	$3, 5, \to I$

QED.

b.

1	(1)	R	Assumption (for $\to I$)
2	(2)	P	Assumption (for $\to I$)
3	(3)	Q	Assumption (for $\to I$)
1	(4)	$P \to R$	$1, 2 \to I$
1	(5)	$Q \to R$	$1, 3 \to I$
1	(6)	$(P \to R) \wedge (Q \to R)$	$4, 5, \wedge I$
2	(7)	$P \vee Q$	$2, \vee I_1$
1	(8)	$(P \vee Q) \to R$	$1, 7, \to I$
	(9)	$((P \vee Q) \to R) \to ((P \to R) \wedge (Q \to R))$	$6, 8, \to I$

QED.

c.

1	(1)	R	Assumption (for $\to I$)
2	(2)	P	Assumption (for $\wedge I$)
3	(3)	Q	Assumption (for $\wedge I$)
2,3	(4)	$P \wedge Q$	$2, 3, \wedge I$
1	(5)	$(P \wedge Q) \to R$	$1, 4, \to I$
1	(6)	$Q \to R$	$1, 3, \to I$
1	(7)	$P \to (Q \to R)$	$2, 6, \to I$
	(8)	$(P \to (Q \to R) \to ((P \wedge Q) \to R)$	$5, 7, \to I$

QED.

d.

1	(1)	$P \vee Q$	Premise
2	(2)	P	Assumption (for $\vee E$)
3	(3)	Q	Assumption (for $\vee E$)
4	(4)	$\neg P \wedge \neg Q$	Assumption (for $\neg E$)
4	(5)	$\neg P$	$4, \wedge E$
2,4	(6)	**false**	$2, 5, \neg E$
2	(7)	$\neg(\neg P \wedge \neg Q)$	$4, 6, \neg I$
4	(8)	$\neg Q$	$4, \wedge E$
3,4	(9)	**false**	$3, 8, \neg E$
3	(10)	$\neg(\neg P \wedge \neg Q)$	$4, 9, \neg I$
1	(11)	$\neg(\neg P \wedge \neg Q)$	$1, 2, 3, 7, 10, \vee E$

QED.

Q5.

a. $\forall x.(P(x) \to Q) \vdash ((\exists x.P(x)) \to Q)$

1	(1)	$\forall x.(P(x) \to Q)$	Premise
1	(2)	$P(a) \to Q$	$1, \forall E[a/x]$
3	(3)	$P(a)$	Assumtion (for $\to E$)
1,3	(4)	Q	$2, 3, \to E$
3	(5)	$\exists x.P(x)$	$3, \exists I[a/x]$
1	(6)	$(\exists x.P(x) \to Q)$	$4, 5, \to I$

QED.

b. $((\exists x.P(x)) \to Q) \vdash \forall x.(P(x) \to Q)$

1	(1)	$(\exists x.P(x)) \to Q$	Premise
2	(2)	$\exists x.P(x)$	Assumption (for $\to E, \exists E$)
1,2	(3)	Q	$1, 2, \to E$
4	(4)	$P(a)$	Assumption (for $\exists E$)
2	(5)	$P(a)$	$2, 4, \exists E$
1	(6)	$P(a) \to Q$	$3, 5, \to I$
1	(7)	$\forall x.(P(x) \to Q)$	$6, \forall I$

QED.