Lecture 10

date : 2 May 2022
week 10

PLANAR DYNAMICAL SYSTEMS (CONT.)

1. Phase portraits for linear planar systems
2. Phase portraits for planar systems using polar coordinates

1. $\dot{X}=AX$, $A\in \mathcal{M}_2(\mathbb{R})$ $\quad X=\begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^2$ $\quad \det A\neq 0$, $\lambda_1,\lambda_2\in\mathbb{C}$ the two eigenvalues of A

Recall from L9 :

$\det A\neq 0 \iff$ the only equil. of (1) is $\eta^*=0\in\mathbb{R}^2$

$\det A\neq 0 \iff \lambda_1\neq 0$ and $\lambda_2\neq 0$

DEF + THEOREM from L9 :

If $\lambda_1,\lambda_2\in\mathbb{R}$ and $\lambda_1\leq\lambda_2<0$ then $\eta^*=0_2$ is a NODE, global attractor, $\nexists$ global f. i.

$\lambda_1,\lambda_2\in\mathbb{R}$ and $0<\lambda_1\leq\lambda_2$ then $\eta^*=0_2$ is a NODE, global repeller, $\nexists$ global first integral

$\lambda_1,\lambda_2\in\mathbb{R}$ and $\lambda_1<0<\lambda_2$ then $\eta^*=0_2$ is a SADDLE, unstable, $\exists$ global first integral

$\lambda_{1,2}=\pm i\beta$ with $\beta\in\mathbb{R}^*$ then $\eta^*=0_2$ is a CENTER, STABLE, $\exists$ global first integral

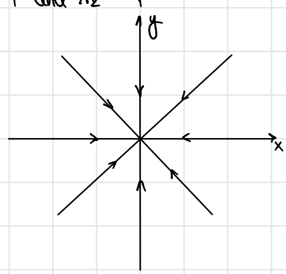
$\lambda_{1,2}=\alpha\pm i\beta$ with $\alpha,\beta\in\mathbb{R}^*$ then $\eta^*=0_2$ is a FOCUS, $\nexists$ global f. i. and global attractor when $\alpha<0$
global repeller when $\alpha>0$

TYPICAL PHASE PORTRAIT OF A NODE :

$$\begin{cases}\dot{x}=-x\\ \dot{y}=-y\end{cases}\qquad A=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$ dynamical matrix $\Rightarrow \lambda_1=-1$ and $\lambda_2=-1$

$\Rightarrow$ the equil. $\eta^*=0_2$ is a NODE, global attractor

From L8 we have the phase portrait :



TYPICAL PHASE PORTRAIT OF A SADDLE :

$$\begin{cases}\dot{x}=-x\\ \dot{y}=y\end{cases}\qquad A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$ $\lambda_1=-1$ and $\lambda_2=1\Rightarrow\eta^*=0_2$ is a SADDLE, unstable

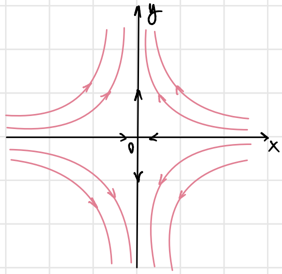
The flow : $\varphi(t,\eta_1,\eta_2)=\begin{pmatrix}\eta_1 e^{-t}\\ \eta_2 e^{t}\end{pmatrix}$, $\forall t\in\mathbb{R}$, $\forall\eta=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}^2$

A global first integral : $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x \cdot y$

The phase portrait : the level curves of H : $x \cdot y = c$, $c \in \mathbb{R}$; $x \cdot y = 1$, $y = \frac{1}{x}$

$f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2} < 0$

x	$-\infty$		0		∞
$\frac{1}{x}$	0	$\searrow$	$-\infty$ | ∞	$\searrow$	0



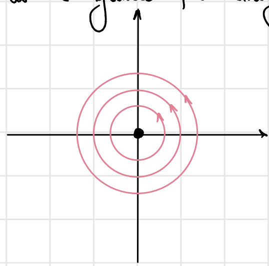
TYPICAL PHASE PORTRAIT OF A CENTER :

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \det(A - \lambda I_2) = 0 \iff \begin{vmatrix} -\lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \iff \lambda^2 + 1 = 0 \iff$$

$\iff \lambda_{1,2} = \pm i$ (Note that $Re(\pm i) = 0$)

Indeed $\eta^* = 0_2$ is a CENTER, STABLE, it is a global first integral from L8 : $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$

the phase portrait is :



TYPICAL PHASE PORTRAIT OF A FOCUS :

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases} \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 + 1 = 0 \iff 1 - \lambda = \pm i \iff \lambda_{1,2} = 1 \pm i$$

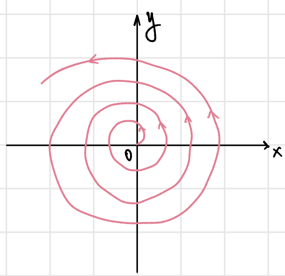
(note that $Re(1 \pm i) = 1 > 0$)

Then $\eta^* = 0_2$ is a FOCUS, global repeller

The flow is : $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^t \cos t - \eta_2 e^t \sin t \\ \eta_1 e^t \sin t + \eta_2 e^t \cos t \end{pmatrix}$, $\forall t \in \mathbb{R}$, $\forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$

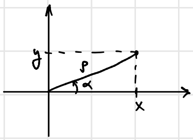
particular case : $\eta_2 = 0$

$\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^t \cos t \\ \eta_1 e^t \sin t \end{pmatrix}$, $t \in \mathbb{R}$



parametric equations $\begin{cases} x = \eta_1 e^t \cos t \\ y = \eta_1 e^t \sin t \end{cases}$, $t \in \mathbb{R}$

$$\begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases} \qquad \begin{aligned} &\rho^2 = x^2+y^2 = \eta_1^2 e^{2t} \Rightarrow \rho(t) = |\eta_1| e^{t} \\ &\tan\theta = \tan t \Rightarrow \theta(t) = t, \quad t \in \mathbb{R} \end{aligned}$$



2. PHASE PORTRAITS USING POLAR COORDINATES:

$$(1)\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

to transform system (1) to polar coordinates means to consider new unknowns as $(\rho(t), \theta(t))$ instead of $(x(t), y(t))$ related

by $$(2)\begin{cases} \rho(t)^2 = x(t)^2 + y(t)^2 \\ \tan\theta(t) = \dfrac{y(t)}{x(t)} \end{cases}$$

Step 1: Take the derivative w.r.t. t in (2)

$$(3)\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$$

Step 2: Replace in (3):

$\dot{x} = f_1(x,y)$

$\dot{y} = f_2(x,y)$

Step 3: Then replace

$x = \rho\cos\theta$

$y = \rho\sin\theta$

Come back to $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$ We transform it to polar coordinates.

$$\begin{cases} \rho\dot{\rho} = x(x-y) + y(x+y) \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x(x-y) + y(x+y)}{x^2} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho} = x^2+y^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x^2+y^2}{x^2} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho} = \rho^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\rho^2}{\rho^2\cos^2\theta} \end{cases} \Rightarrow \begin{cases} \dot{\rho} = \rho \\ \dot{\theta} = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \rho(t) = \rho_0 e^{t} \\ \theta(t) = t + \theta_0 \end{cases}$$ the same

$\dot{\rho} > 0 \Rightarrow \rho \uparrow \Rightarrow$ depart from O_2

$\dot{\rho} < 0 \Rightarrow \rho \downarrow \Rightarrow$ approach O_2

$\dot{\rho} = 0 \Rightarrow \rho$ is constant along an orbit $\Rightarrow$ the orbit is a circle centered in O_2

$\dot{\theta} > 0 \Rightarrow \theta \nearrow \Rightarrow$ contraclockwise rotation

$\dot{\theta} < 0 \Rightarrow \theta \searrow \Rightarrow$ clockwise rotation

$\dot{\theta} = 0 \Rightarrow \theta$ is constant $\Rightarrow$ the orbit lie on a line spanning to O_2

Exercise:

$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = -x + y(1-x^2-y^2) \end{cases}$$

a) Check that $\varphi(t,1,0) = (\cos t, \sin t)$, $\forall t \in \mathbb{R}$

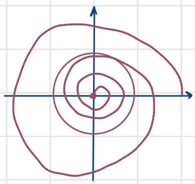
b) Pass to polar coordinates and represent the p.p. Reading the p.p. specify the stability of the equil. $(0,0)$. There is an attractor.

Recall that, by def, $\varphi(t,1,0)$ is the sol of the IVP

$$(*) \begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

We have to replace $x = \cos t$, $y = \sin t$ in $(*)$ and show that we obtain valid relations.

b) $$\begin{cases} \dot{\rho} = \rho(1-\rho^2) \\ \dot{\theta} = 1 \end{cases}$$



$(0,0)$ is equil. point

$$\underbrace{\rho(1+\rho)}_{>0}(1-\rho)$$