

Intro

Based on fitting methods on the experimental patch-clamp data in [1], Hindmarsh and Rose came up with a 2D and then a 3D model, of which the analysis they do in [2]. At the time the model of reference was the Fitz-Hugh-Nagumo. Their innovation, in comparison with the latter, is that the recovery time of the spiking is much longer than the spike duration and the bursting properties of the new model. First we will see the intuition of their 2D modeling, then the explicit analysis of the phase plane, to visualize the properties of the 2D model, then the advantage that the introduction of the third variable adds to the properties of the model and finally a dimension one bifurcation analysis with respect to I and then a codimension two bifurcation analysis with respect to β and I .

Section 1. Brief intuition of the modeling

From [1] we have

$$\begin{aligned}\dot{v} &= \alpha(\beta r - f(v) + I) \\ \dot{r} &= \gamma(g(v) - \delta r)\end{aligned}$$

, where v : membrane potential, r : recovery variable, I : applied current, $\alpha, \beta, \gamma, \delta$: constants, $f(v_p) = I_{v_p}(0)$ the cubic function that fits $I_{v_p}(0)$, i.e. the current passing through the membrane at time $t=0$ of the clamp-patching of the membrane at potential v_p and $g(v_p) = f(v_p) - I_{v_p}(\infty)$, that fits their difference.

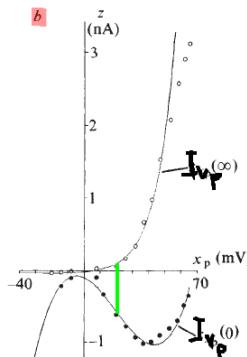


Fig. 1 Voltage-clamp data on which the model is based. In these

In order to take into account the difference of time scale between spike and recovery, they introduce a slow inward current, z , whose time constant is similar to that of the recovery process. The intuition behind this, is that a current, opposing to inward current that the patching of the membrane induces, slowly builds up and the two cancel out. Then, the overall current is $w = r + z$. Then the

$$\begin{aligned}\dot{v} &= \alpha(\beta(r+z) - f(v) + I) \\ \text{system } \dot{r} &= \gamma(g(v) - \delta r) \\ \dot{z} &= \gamma(h(v) - \delta z) \\ \text{becomes } \dot{v} &= \alpha(\beta w - f(v) + I) \\ \dot{w} &= \gamma(g(v) + h(v) - \delta w)\end{aligned}$$

We also need to change the time scale, s.t. $T = \gamma \delta t$, $x(T) = v(T)$, $y(T) = \frac{\alpha \beta}{\gamma \delta} w(t)$.

Then

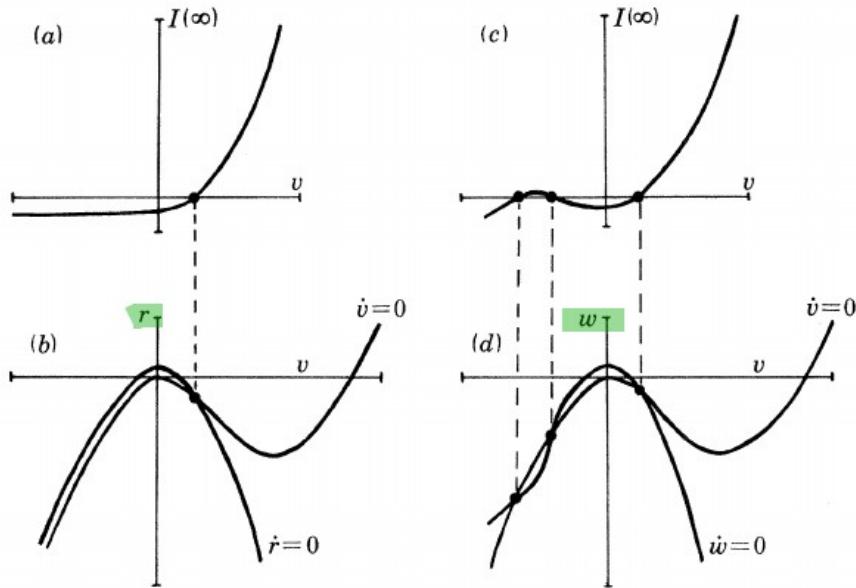
$$\frac{dt}{dT} = \frac{1}{\gamma \delta},$$

$$\dot{y} = \frac{dy}{dt} \frac{dt}{dT} = \frac{\alpha\beta}{\gamma\delta} \frac{dw}{dt} \frac{dt}{dT} = \frac{\alpha\beta}{\gamma\delta} \dot{w} \frac{1}{\gamma\delta} = \frac{\alpha\beta}{(\gamma\delta)^2} (\gamma(g(x) + h(x)) - \delta w) =$$

$$\frac{\alpha\beta}{(\gamma\delta)^2} (\gamma(g(x) + h(x)) - \gamma\delta w) = \frac{\alpha\beta}{\gamma} \delta^2 [\gamma(g(x) + h(x)) - \frac{(\gamma\delta)^2}{\alpha\beta} \gamma] = \frac{\alpha\beta}{\gamma} \delta^2 (g(x) + h(x)) - \gamma = \mathbf{G}(x) - y$$

$$\dot{x} = \dot{v} = \frac{dv}{dt} \frac{dt}{dT} = \alpha(\beta w - f(v) + I) \frac{1}{\gamma\delta} = \frac{\alpha\beta}{\gamma\delta} w - \frac{\alpha}{\gamma\delta} f(v) + \frac{\alpha}{\gamma\delta} I = y - \frac{\alpha}{\gamma\delta} (f(v) + I) = y - \mathbf{F}(x) .$$

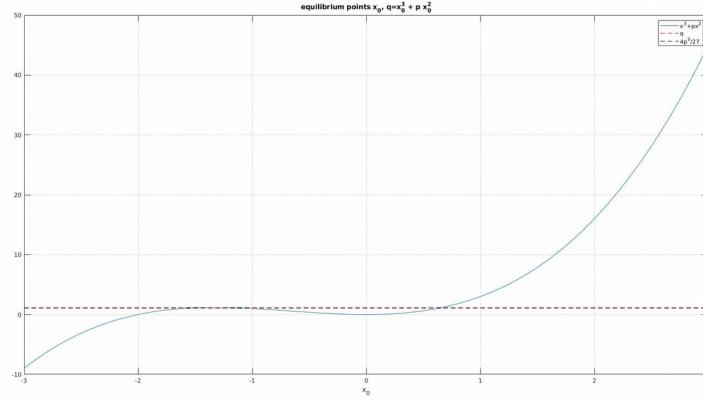
The analysis is for $F(x) = -\alpha x^3 + \beta x^2 + I$, $G(x) = c - \delta x^3$. The reason, why did they choose those functions, is the whole gist of this. In the figure below, by choosing these functions, they can “tune” the constants, α, β, δ, c , of the system, to resemble either the left column case or the right column case. In the left column case, there is **only one** equilibrium, that is an unstable spiral that leads all the phase points to a clockwise limit cycle (oscillation) around it. In the right column case, there are 3 equilibrium points. The **leftmost** equilibrium, is the stable node ($x = -1.618$) which defines the resting state membrane potential, the **rightmost** equilibrium ($x = 0.618$) is an unstable spiral that leads the phase plane surrounding it, to a limit cycle and the **middle** point ($x = -1$) is a saddle that with its eigenvector defines the separatrix that separates those two areas mentioned above. The two states have properties that a biologically plausible neuron should have and we check them in **section 3**.



The difference between the recovery and the spiking time scale, comes from the fact that the two nullclines are really close in the phase space left to the saddle, making the recovery phase much slower than the spiking phase (phase space right from the saddle).

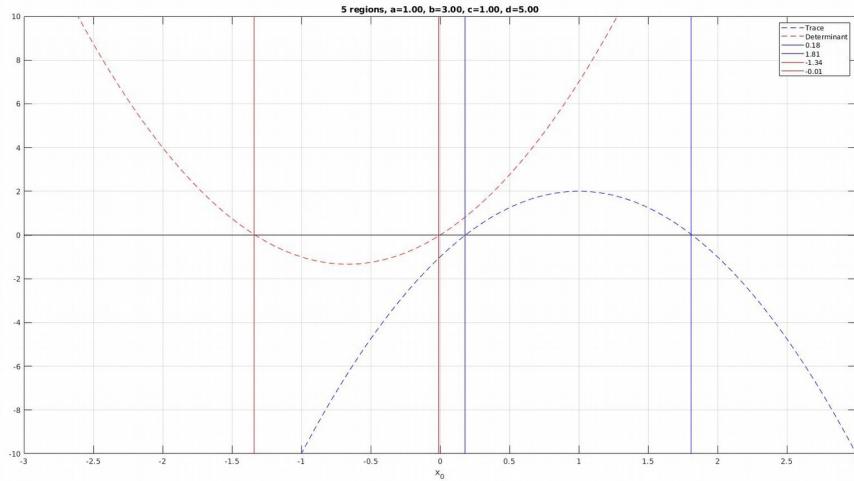
Section 2. Analysis of the equilibrium points – How to choose the constants α, β, δ, c ?

Suppose equilibrium point (x_0, y_0) s.t. $\dot{x} = \dot{y} = 0 \rightarrow x_0^3 + \frac{\delta - \beta}{\alpha} x_0^2 = \frac{y}{\gamma} \rightarrow x_0^3 + p x_0^2 - q = 0$, which in order to exist, the discriminant must be (+)ve $-4p^3(-q) - 27q^2 > 0 \rightarrow 4p^3q > 27q^2 \rightarrow 4p^3 > 27q$ [3]. Then more explicitly, $4(\delta - \beta)^3 > 27(\frac{\alpha}{\gamma})^2 \rightarrow 4(\delta - \beta)^3 > 27\alpha^2\gamma$ (**condition 1**).



Then the jacobian is $\begin{pmatrix} -3\alpha x_0^2 + 2\beta x_0 & 1 \\ -2\delta x_0 & -1 \end{pmatrix}$ and the trace and the determinants are
 $\text{trace} = -3\alpha x_0^2 + 2\beta x_0 - 1$
 $\det = 3\alpha x_0^2 + 2(\delta - \beta)x_0$

Now baring in mind the 2D flows we take the following cases from left to right (regions I-V):



region	values of x_0	sign of $\text{Tr}(A(x_0))$	sign of $\text{Det}(A(x_0))$	type of e.p.
I	$x_0 < -2(d-b)/3a$	-	+	stable node or spiral
II	$-2(d-b)/3a < x_0 < 0$	-	-	unstable saddle
III	$0 < x_0 < (b-D)/3a$	-	+	stable focus or spiral
IV	$(b-D)/3a < x_0 < (b+D)/3a$	+	+	unstable focus or spiral
V	$(b+D)/3a < x_0$	-	+	stable focus or spiral

If equilibrium point is to exist, then the trace must have a solution, that solution being

$$x_0^{(1),2} = \frac{-2\beta \pm \sqrt{((2\beta)^2 - 4(-3\alpha)(-1))}}{2(-3\alpha)} = \frac{-\beta \pm \sqrt{\beta^2 - 3\alpha}}{-3\alpha} \quad \text{with } \beta^2 - 3\alpha > 0 \rightarrow \beta^2 > 3\alpha . \text{ Then}$$

trace is >0 for $x_0 \in [x_0^{(1)}, x_0^{(2)}]$. If $\beta^2 = 3\alpha$ then trace is always negative. But we don't want this, because then we wouldn't have our unstable spiral (region IV). That does it for our second condition. We need $\beta^2 > 3\alpha$ (**condition 2**).

Finally given that x_0 is equilibrium then it must also satisfy $x_0^3 + p x_0^2 - q = 0$, with $p = \frac{\delta - \beta}{\alpha}$

and $q = \frac{\gamma}{\alpha}$ this combined with $x_0 \in [x_0^{(1)}, x_0^{(2)}]$ gives

$$\frac{\beta - \sqrt{\beta^2 - 3\alpha}}{3\alpha} < q < \frac{\beta + \sqrt{\beta^2 - 3\alpha}}{3\alpha} \rightarrow \frac{\beta - \sqrt{\beta^2 - 3\alpha}}{3\alpha} < \frac{\gamma}{\alpha} < \frac{\beta + \sqrt{\beta^2 - 3\alpha}}{3\alpha} \quad (\text{condition 3}).$$

The choice tha H,R. made was for $\alpha=1$, $\beta=3$, $\gamma=1$, $\delta=5$. And for this choice we can see in [4] (p.171-174) the worked exercise, that finds the equilibria and their stability. We have

$$\begin{aligned}\dot{x} &= y - x^3 + 3x^2 + I \\ \dot{y} &= 1 - 5x^2 - y\end{aligned}$$

then the equilibria are:

$$x^3 + 2x^2 - 1 = 0 \rightarrow (x^2 + x - 1)(x + 1) = 0 \rightarrow x_1 = -1, x_2 = \frac{-1 + \sqrt{5}}{2} = 0.618, x_3 = \frac{-1 - \sqrt{5}}{2} = -1.618$$

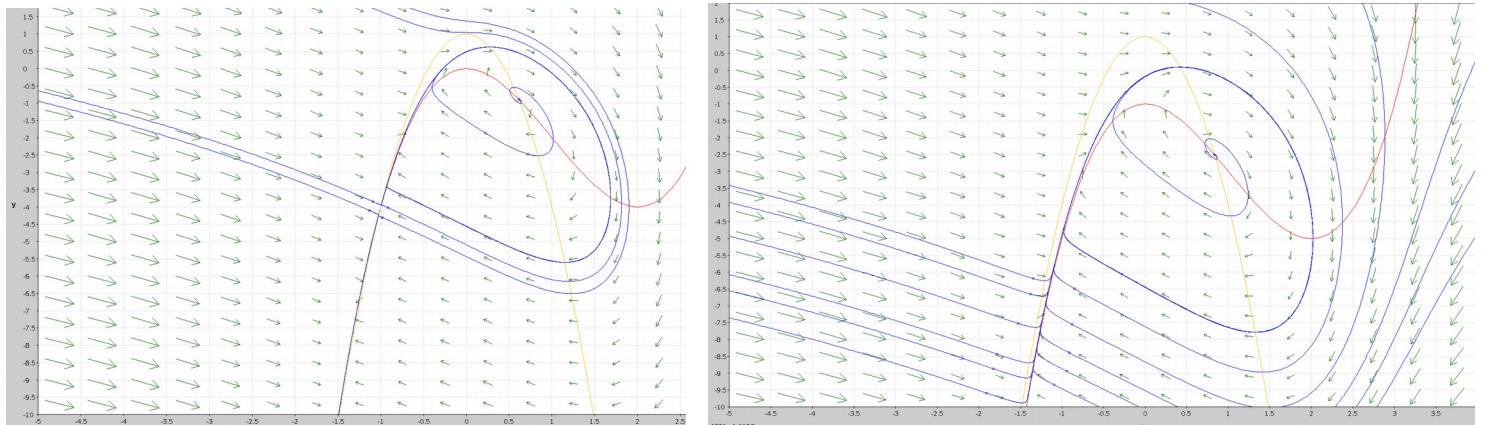
and respectively $y_1 = -4, y_2 = -13 + 5\sqrt{5} = -.91, y_3 = -13 - 5\sqrt{5} = -12.09$.

Linear analysis on jacobian gives

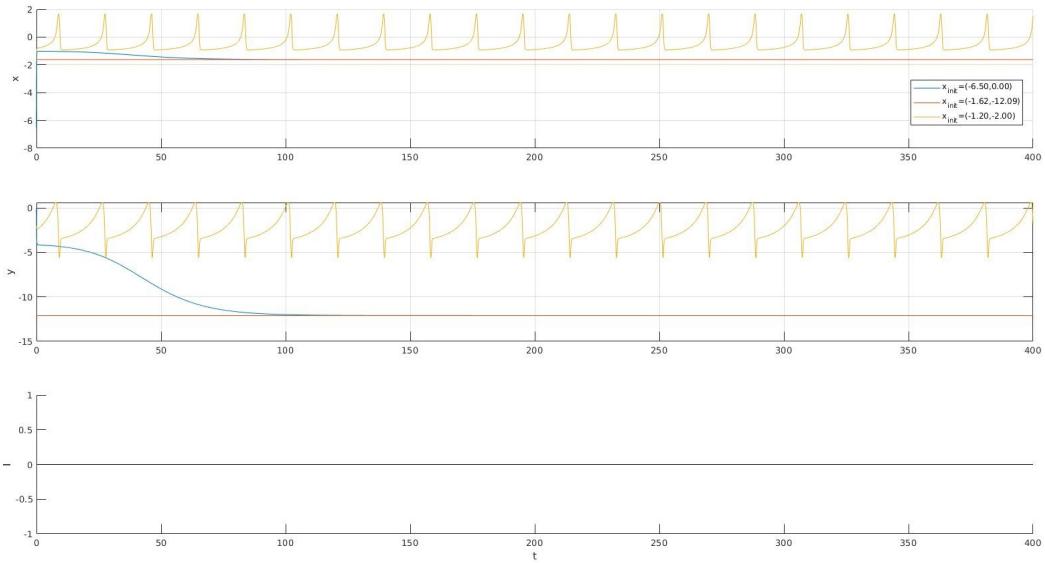
$$\begin{pmatrix} -3x^2 + 6x & 1 \\ -10x & -1 \end{pmatrix} \rightarrow \begin{cases} \text{trace} = -3x^2 + 6x - 1 \\ \det = 3x^2 + 4x \end{cases} \rightarrow \begin{cases} \text{tr}(x_1) = -10, \det(x_1) = -1 \rightarrow \text{saddle node} \\ \text{tr}(x_2) = 1.56, \det(x_2) = 3.62 \rightarrow \text{unstable spiral} \\ \text{tr}(x_3) = -18.6, \det(x_3) = 1.38 \rightarrow \text{stable node} \end{cases} .$$

Section 3. Properties

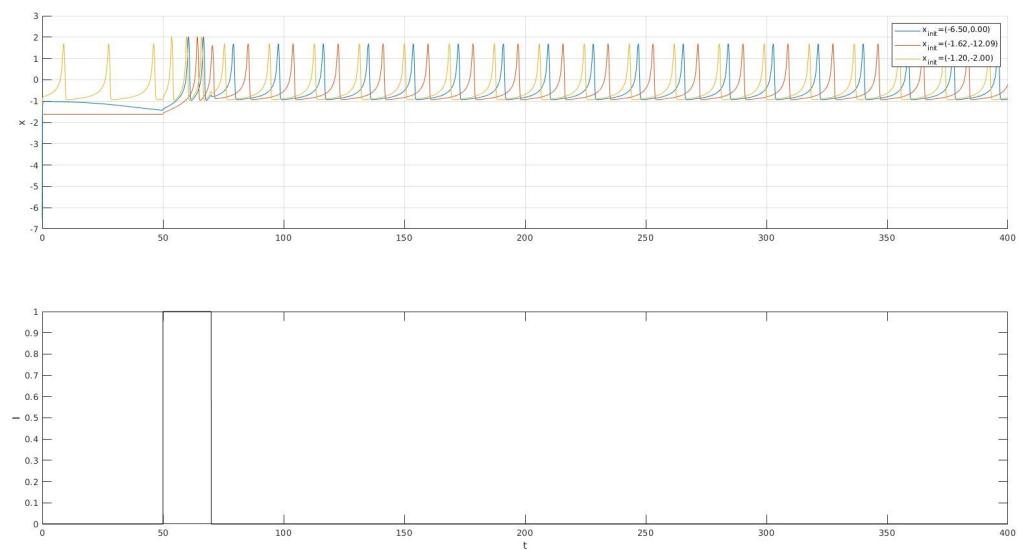
3.1. 2D model



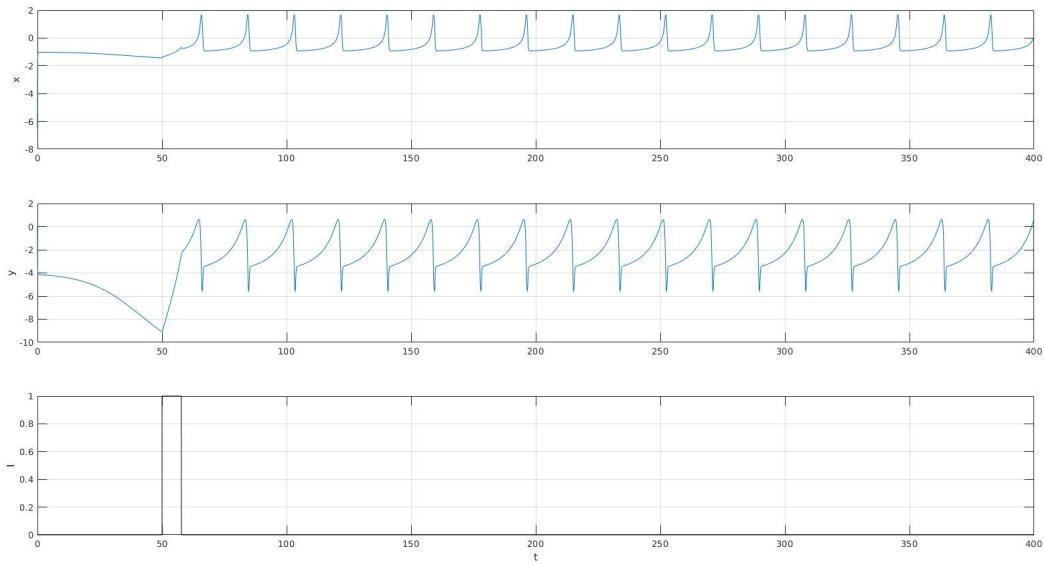
Notice the two phase planes for $I=0$ (left) and $I=1$ (right). For $\mathbf{I=0}$, we have 3 equilibria (from left to right: stable node, saddle node, unstable spiral). Notice the separatrix, that sends everything above it to the limit cycle and everything below it to the stable node, i.e. the resting membrane potential. For $\mathbf{I=1}$, the x -nullcline (red) gets pulled down, thus making the two leftmost equilibria, i.e. the stable node and saddle node disappear. The only equilibrium now is the unstable spiral that sends every phase point to the limit cycle surrounding it, in an endless oscillation.



(a) For $\mathbf{I=0}$, 3 different starting points. Notice that for phase point $\mathbf{x_3}$ above the separatrix, even without I , it oscillates endlessly. Phase point $\mathbf{x_1}$, the stable node itself $(-1.618, -12.09)$, does not move at all and phase point $\mathbf{x_2}$, which is below the separatrix, moves towards the stable node as well.



(b) For a **pulse** (I step=1) of predetermined window, the limit cycle is the only place to go. Therefore, all starting points, will oscillate, for at least the length of the I step. We get for **x_1** an oscillation that ends with the end of the step, and **x_2**, **x_3** keep oscillating. It just happens that, the window of the I step, was such that by the end of it, **x_2** and **x_3** phase points, are above the separatrix and **x_1** below, thus the first two ending in the limit cycle and the later, to the stable node.

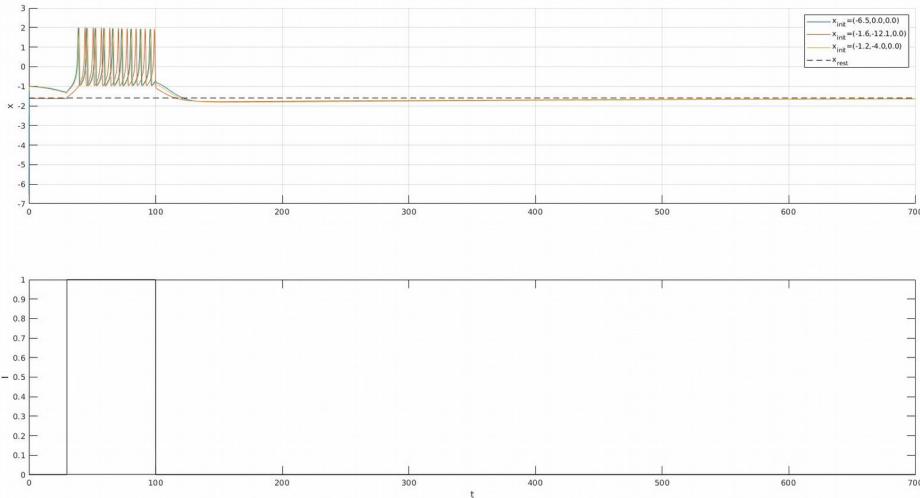


(c) Notice how **x_1**, that previously couldn't keep on oscillating after the end of the I step, now does. We can achieve this by making the I step length such that it ends when the phase point is still above the separatrix that will reappear by the end of the I step. (check **HIN84_2.m**).

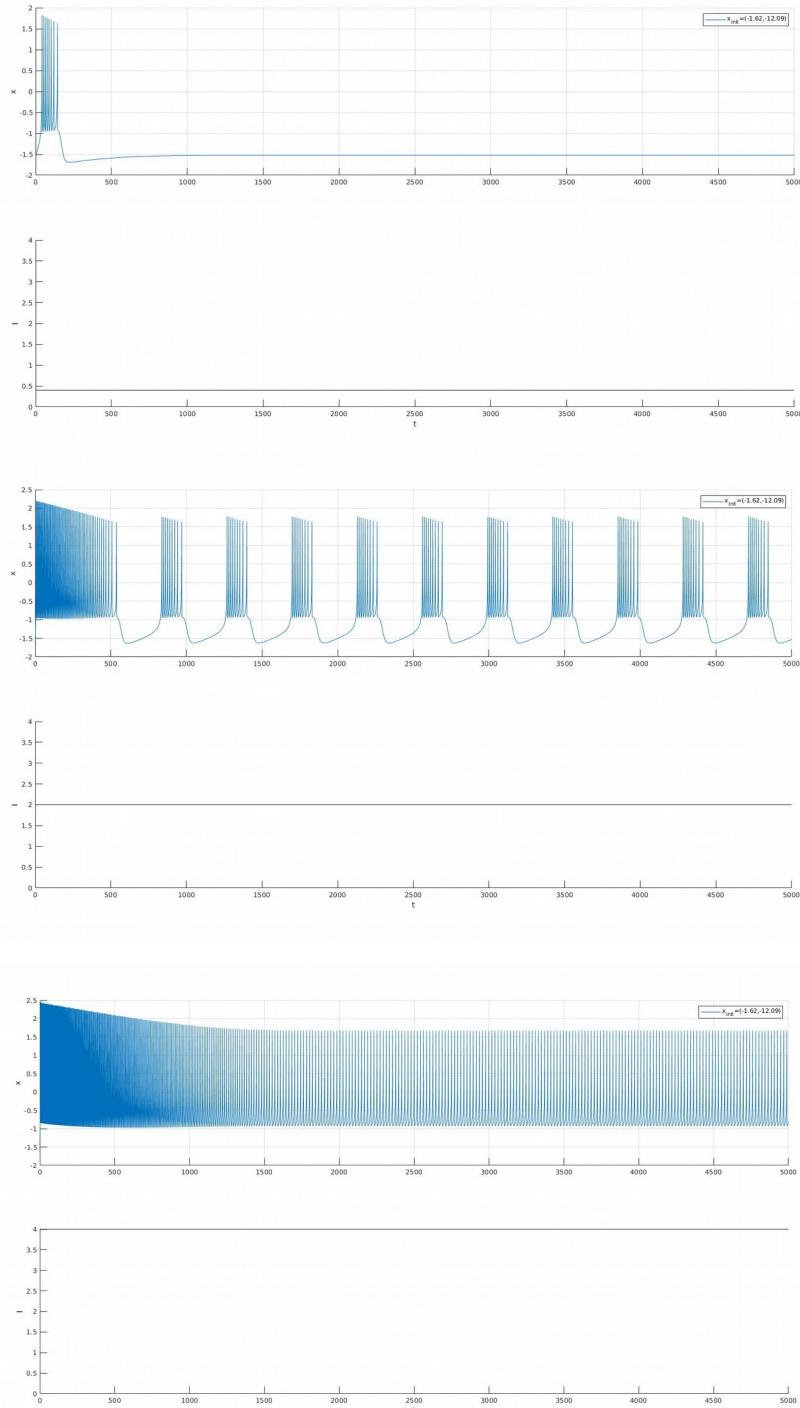
3.2. 3D model

Now on to the **3D** model (figures made by **HIN84_3.m**) ... The system is

$$\begin{aligned}\dot{x} &= y - \alpha x^3 + \beta x^2 + (I - z), \\ \dot{y} &= \gamma - \delta x^2 - y, \\ \dot{z} &= r(s(x - x_{rest}) - z)\end{aligned}$$

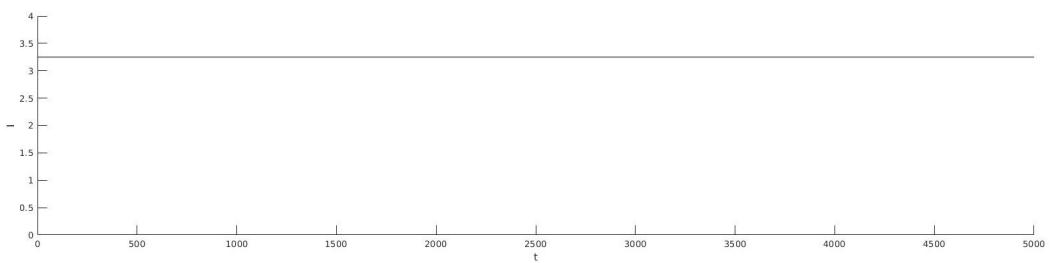
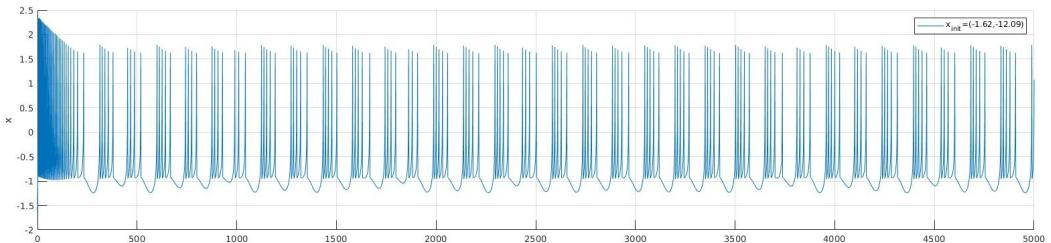


(d) For a **pulse** of predetermined window length, in the 2D model, the behavior of the phase point (limit cycle – continuation of oscillation /or stable node – back to resting state) would depend only on the location of the phase point, the exact time that the step would end. Now, with the 3D model, this behavior is controlled by the build-up of the slow current, z. The oscillation can't continue (this is called **adaptation**). (Constant values: $\alpha=1, \beta=3, \gamma=1, \delta=5, r=0.001, s=1$). Notice the jitter (initial dip) before rising to a peak, for all starting points.

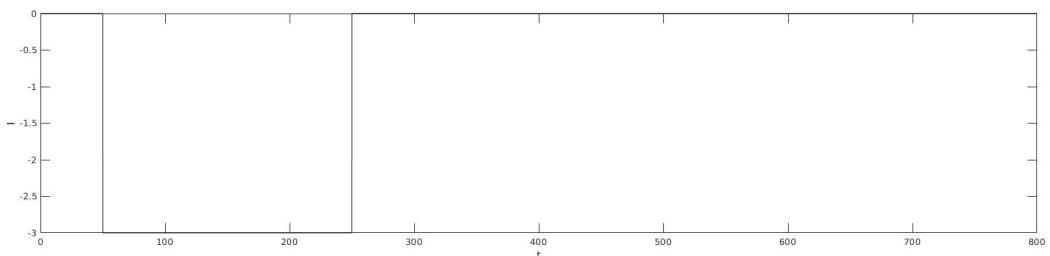
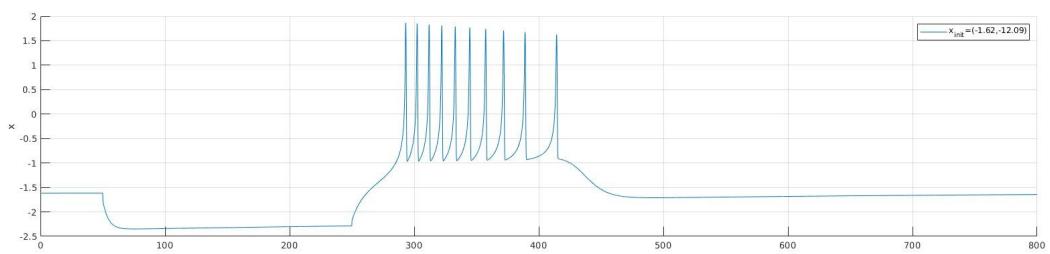


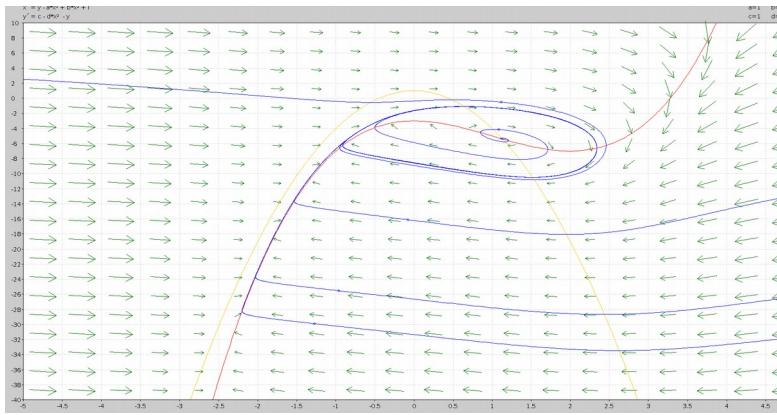
(e) For **constant I** (= .4, 2, 4 respectively top to bottom) we get different **bursting behaviors**, for the phase point starting from the stable node (resting). For **I=.4**, there is bursting for a while that then stops, as the slow current, z, builds up to cancel out I (the more z rises, the more the x-nullcline

gets pulled up and the stable node reappears). For **I=2**, the bursting continues, because after one bursting round is completed, the phase point hyperpolarizes, thus the slow recovery current, z , gets negative thanks to the $(x-x_{\text{rest}})$ term. Then the $(I-z)$ term of the \dot{x} equation increases again, enough to pull down the x -nullcline once more and let the phase point back into the limit cycle, in other words one more bursting round. Last, for **I=4**, there is always bursting, the slow recovery current, z , can not build up enough to cancel out I and thus the x -nullcline remains pulled down, leaving the phase plane with only one equilibrium, the unstable spiral that leads every phase point to the limit cycle. (constants: $\alpha=1$, $\beta=3$, $\gamma=1$, $\delta=5$, $r=0.001$, $s=4$).



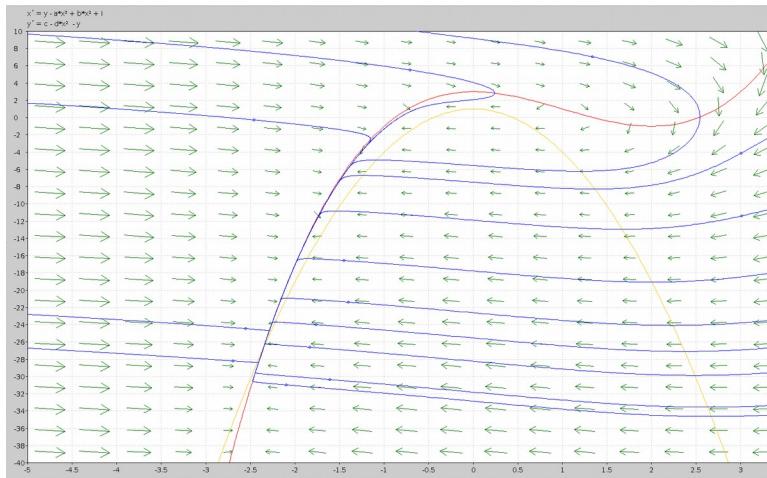
(f) Random structure: sequence of number of bursts: 5-4-5-4-4-3-5-5-5-5-4-3-5-5-5-4-..., for constant $I=3.25$.





$I = -3, z=0 \rightarrow (I-z) = -3$ (equivalent to running pplane with $I = -3$)

... $(I-z)$ increases ...

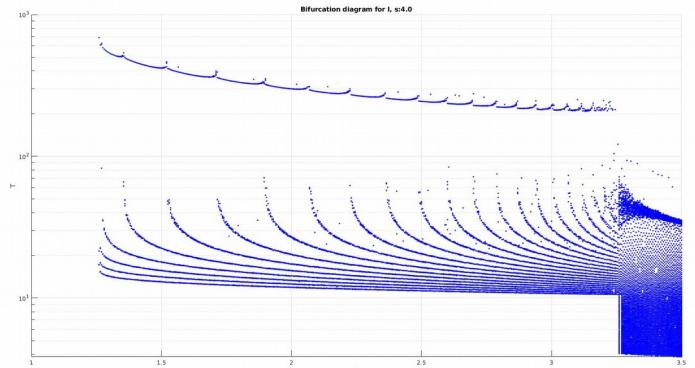
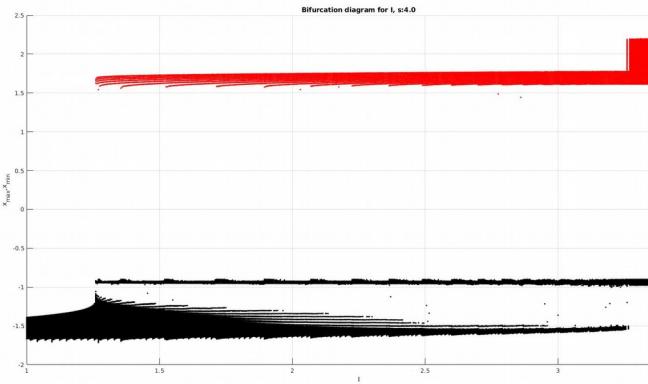
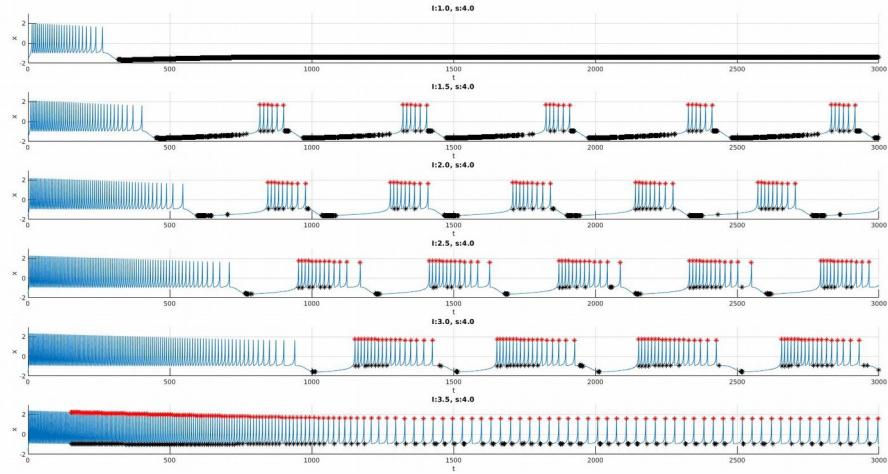


$I=0, z = -3 \rightarrow (I-z) = 3$ (equivalent to running pplane with $I=3$)

(g) For a negative pulse, we get **post-inhibitory rebound**. This happens because we hyperpolarize the neuron to -2.5 (it's evident in the 2D phase plane above, for a negative I ($=-3$), that I use without forgetting that we are inspecting the 3D model. We can do this because z is slow compared to x , and keeping that in mind, helps with the visualization). Then, when I is released back to 0, the slow recovery current, z , has already gone enough negative s.t. the term $(I-z)$ is increased enough to pull instantly the x -nullcline down and letting only the spiral and thus the limit cycle to survive.

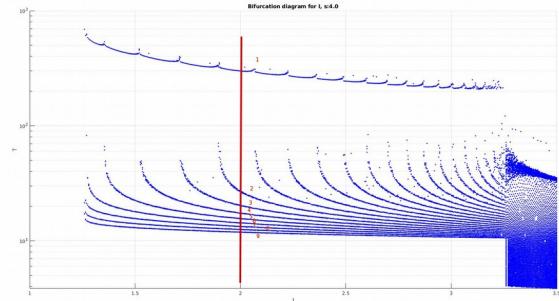
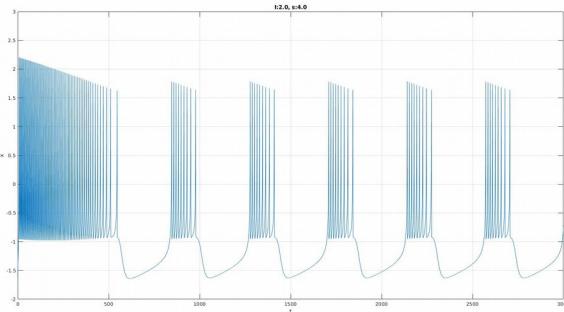
Section 4. Bifurcation analysis with respect to I

Here I plot the **time series** for different values of I , as well, the **bifurcation of the maximum x values** (red dots, i.e. bursting cycle peaks) and **minimum x values** (black dots, i.e. recovery phase lows), and the **bifurcation of the time distances between consecutive peaks** (let's call them periods T , the blue dots), for which i took inspiration from [5].



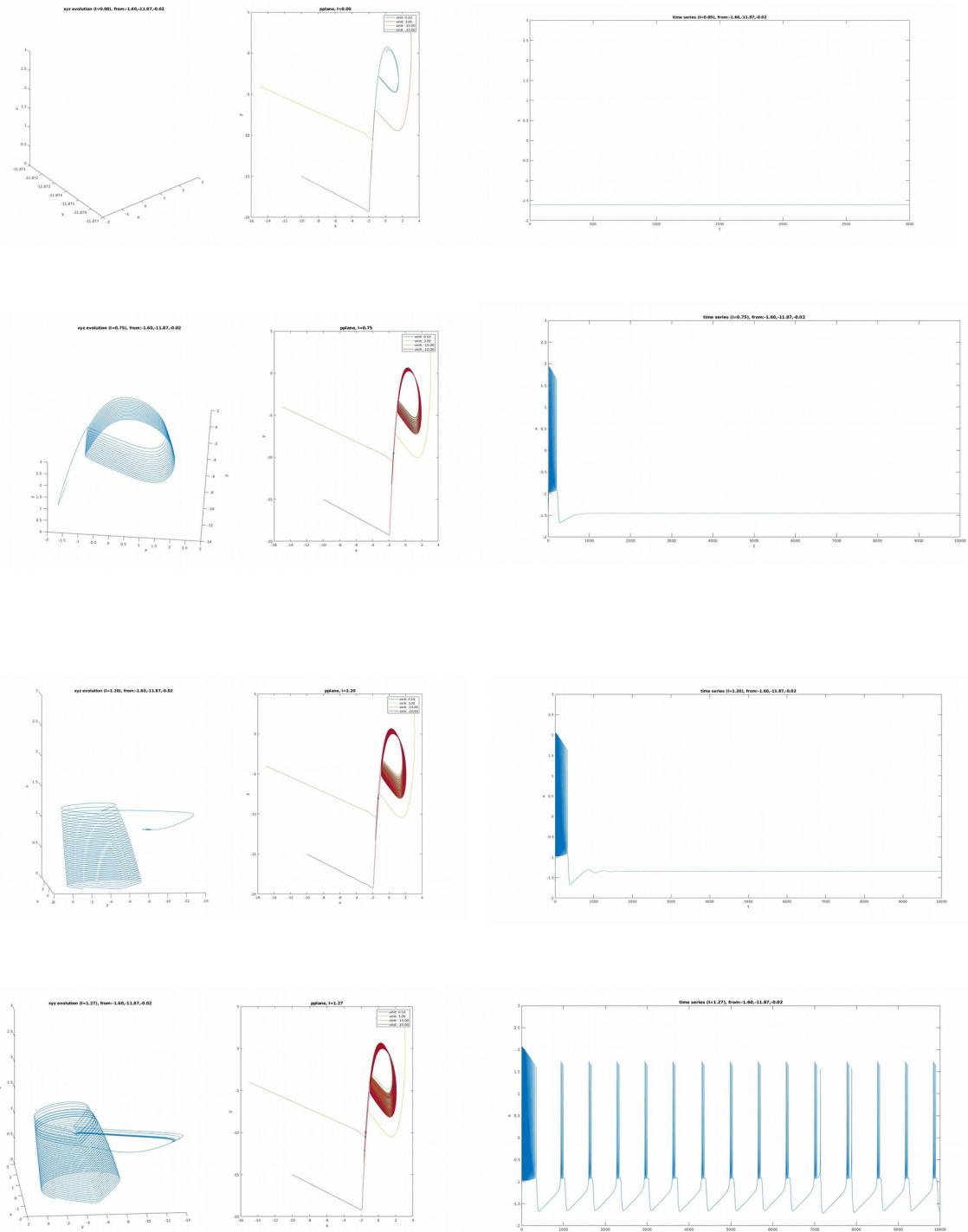
Notice for **$I < 1.3$** , the bursting limit cycles can't happen (black dots at equilibrium x , $x_{rest} = -1.6$, no peaks, i.e. no red dots, no blue dots). For **$1.3 < I < 3.2$** , there are bursting cycles and recovery cycles that take place, in an alternating fashion (black dots at -1.6 are the recovery cycle lows, black dots at -1 are the bursting cycle lows and red dots at ~ 1.6 are the bursting cycle peaks.).

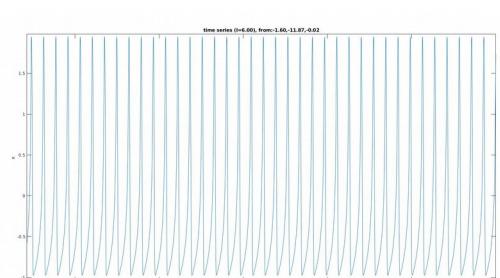
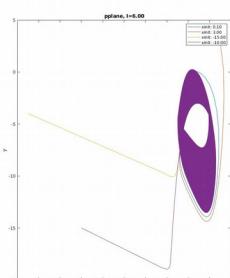
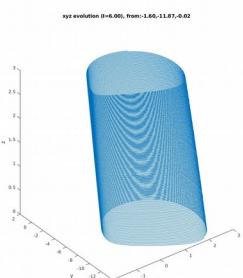
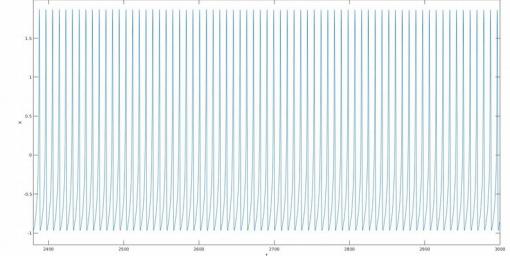
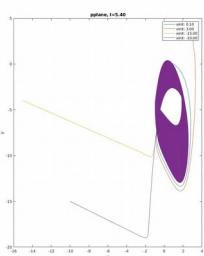
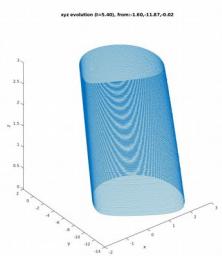
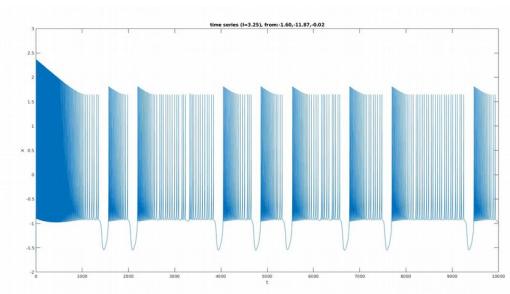
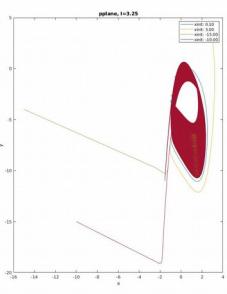
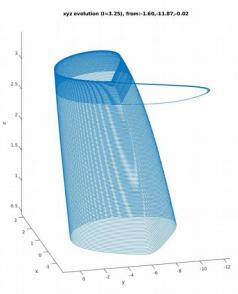
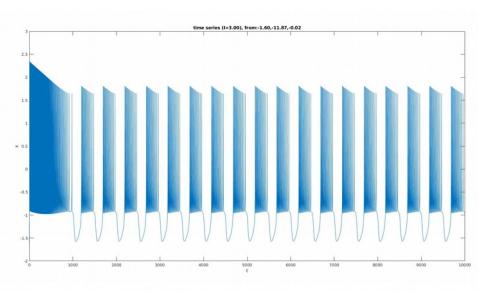
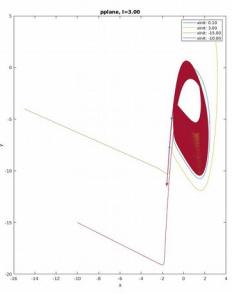
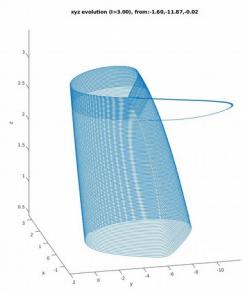
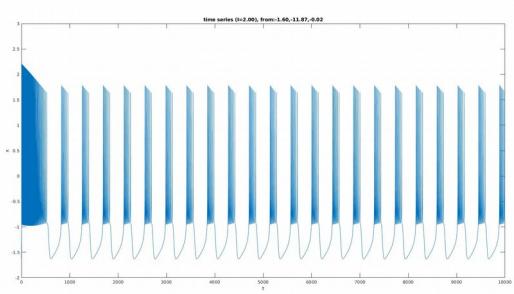
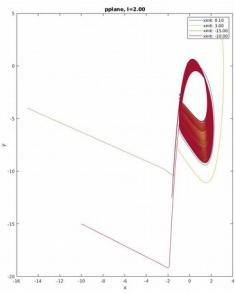
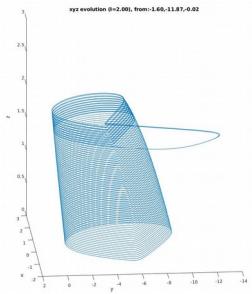
Notice also the form of the period, T bifurcation. Take for example $I=2$, where each bursting cycle has 9 peaks, there are 9 blue scatter point lines in this section. **8** T scatter points, which are the increasing time spaces between the consecutive peaks during a burst cycle and **1** scatter point for the time space between the last peak of a bursting cycle and the first peak of the next bursting cycle. Below I show this in zoom-in.



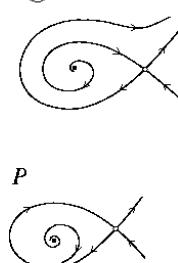
We can find the equilibrium point for $(b, I) = (3, 0)$, that is $x_{init} = (-1.60435320914467, -11.8748883483645, -0.0179916610514969)$, by drawing in Matcont an O_P curve, starting from a

close point, that we already suspect from the analysis we have already done. That is, starting point $(-1.6, -12, 0)$. With Matcont I also find the Hopf bifurcation points of this equilibrium, x_{init} , while changing I , by drawing a EP_EP line, with starting point the x_{init} . The Hopf bifurcations occur for $I_{H_1}=1.269866$, $I_{H_2}=5.3989871$, $I_{H_3}=6.2028149$.





We see in descending order: **I=0**: quiescence (one unstable spiral, one saddle that makes the homoclinic orbit, and one stable node) , **I=0.75** (one unstable spiral, one saddle (its homoclinic orbit becomes bigger) and one stable node), **I=1.2**: transient burst (limit point of cycle bifurcation , i.e. one limit cycle, that comes from the homoclinic orbit and one stable spiral), **I=1.27**: regular burst (after Hopf bifurcation, the stable spiral becomes stable limit cycle that joins the previously shaped limit cycle , thus giving a single limit cycle with a fast bursting phase, and a slow recovery phase) , **I=2**: regular burst, **I=3**: regular burst , **I=3.25**: irregular (non-periodic) burst, **I=5.4**: regular spike (after Hopf bifurcation, the limit cycle loses the slow recovery part, thus remaining a limit cycle, still, but with different morphology), **I=6**: regular spike. It helps to visualize the transition from I=0.75 to 1.2, before the first Hopf, as follows:



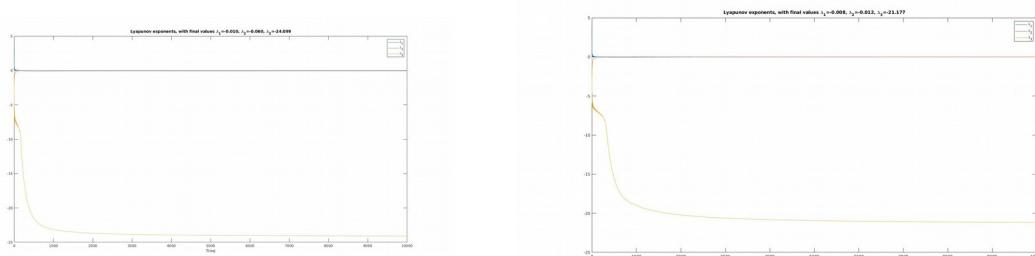
Section 5. Lyapunov analysis

Inspired by [6] I also plotted the Lyapunov coefficients. I used three ways. First I found the Lyapunov spectra, i.e. the 3 lambdas, $(\lambda_1, \lambda_2, \lambda_3)$, with **lyapunovspectra.m**, using the ortho-normalization method (Wolf's paper [7] section.3). I also found the dominant Lyapunov exponent, using the method which uses the divergence in time of two close phase points, by running **lyapunovdom.m** (two predetermined close enough starting phase points that are fed as inputs).

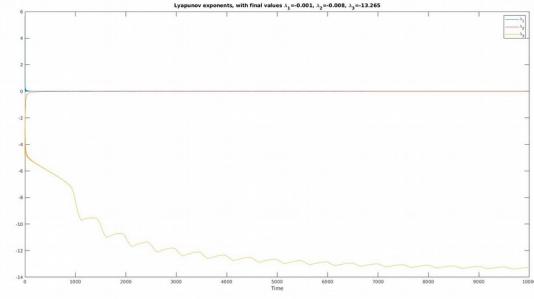
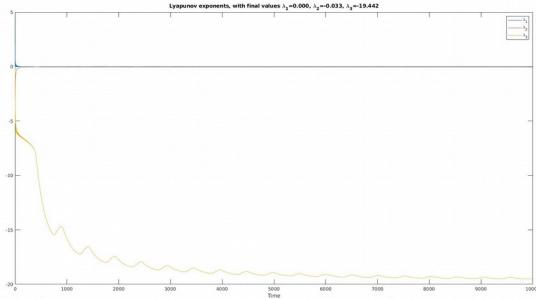
Note: **lyapunov3.m** (more sophisticated algorithm that investigates the divergence in time between a phase point and its closest neighbor, for a specific trajectory length, and then averages this among every phase point - Wolf's paper [7] section 5.1) fails.

All scripts can be executed inside **lyapunovs.m**.

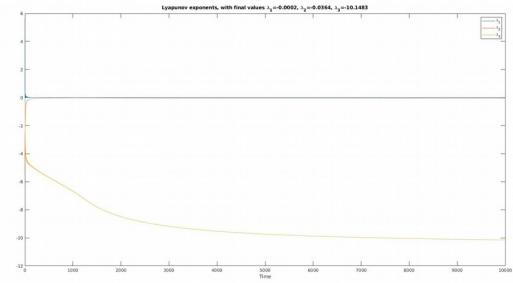
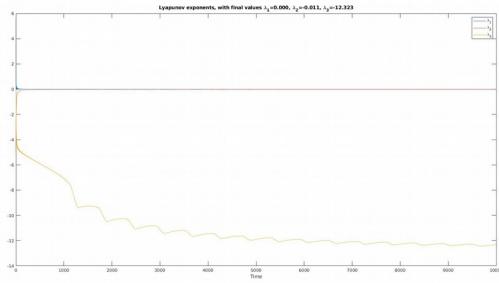
5.1. Lyapunov Spectra with **lyapunovspectra.m**



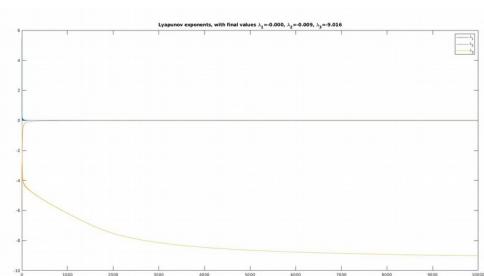
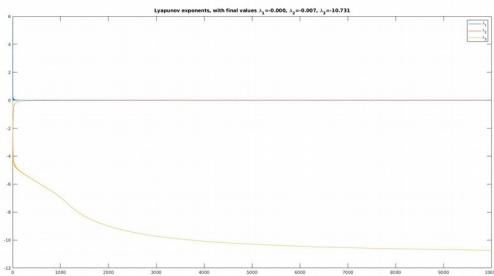
left: I=0.5 , righth I=1.1



I=1.5, 3



I=3.22, 3.25



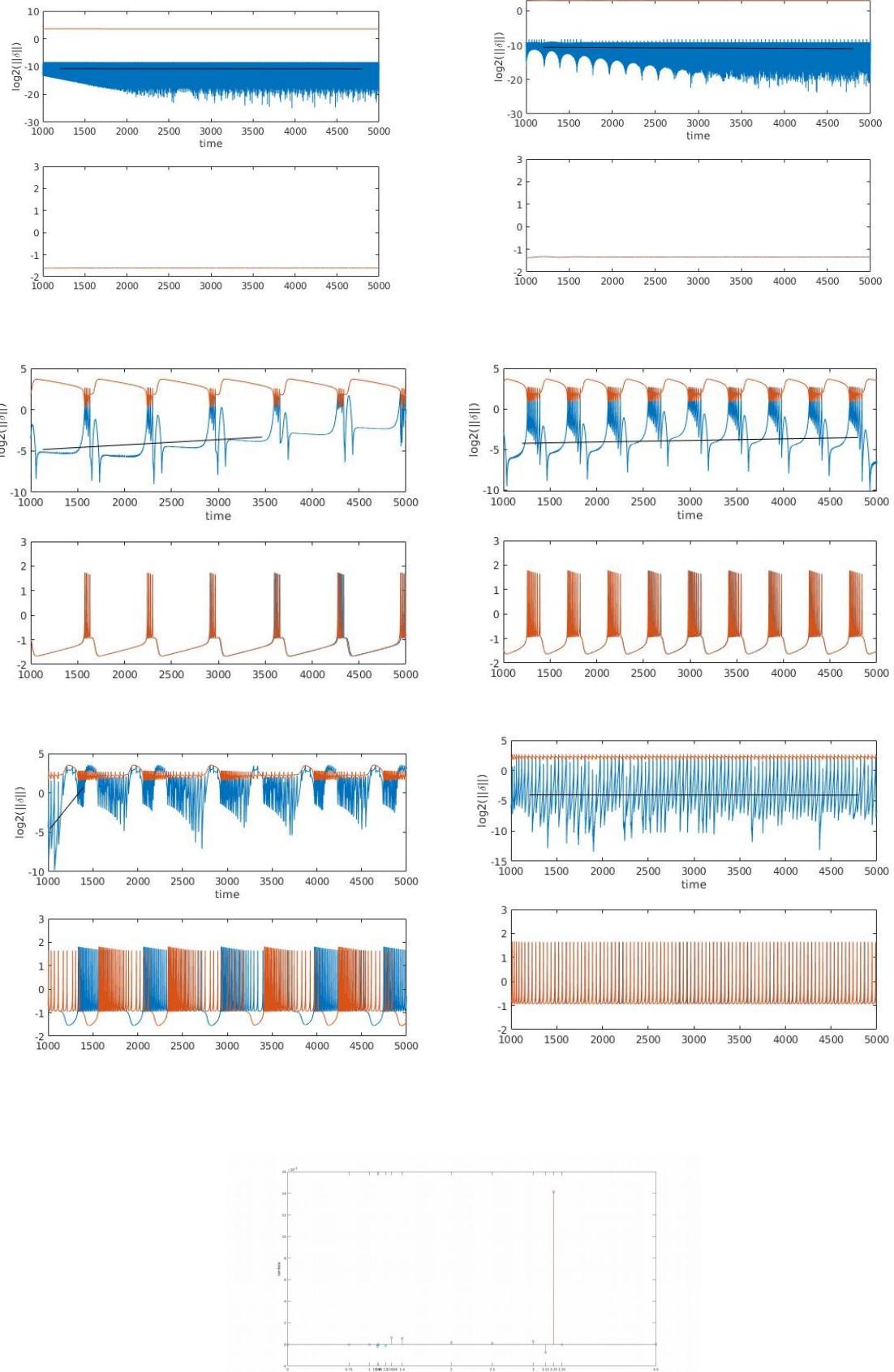
I=3.3, 4

Sequentially (top to bottom) : I=0, 0.5, 1, 1.1, 1.5, 3, 3.22, 3.25, 3.3, 4. Notice the form of the lyapunov exponents. I=0, 0.5, 1: (-,-,-) i.e. stable node, I=1.5, 3, 3.22: (0,-,-) i.e. limit cycle I=3.25 (+,0,-) i.e. strange attractor, I=3.3, 4: (0,-,-) i.e. limit cycle.

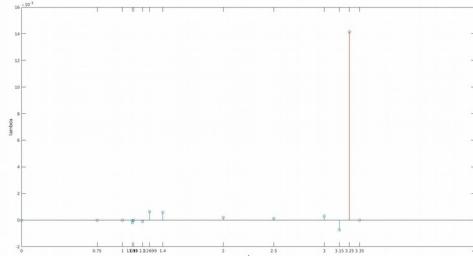
5.2. Lyapunov dominant coefficient, λ , with **lyapunovdom.m**

λ is calculated as the slope of delta difference between the two phase points until they overcome the threshold as this is defined by the instantaneous scale of the system. Transient is not taken into account.

Below the plots (delta,time) and (x,time), for values of $I = [0, 1.1, 1.269866, 1.4, 3.25, 4.5]$;



λ 's are close to zero for all I , except for **I=3.25**, where a strange attractor appears briefly.

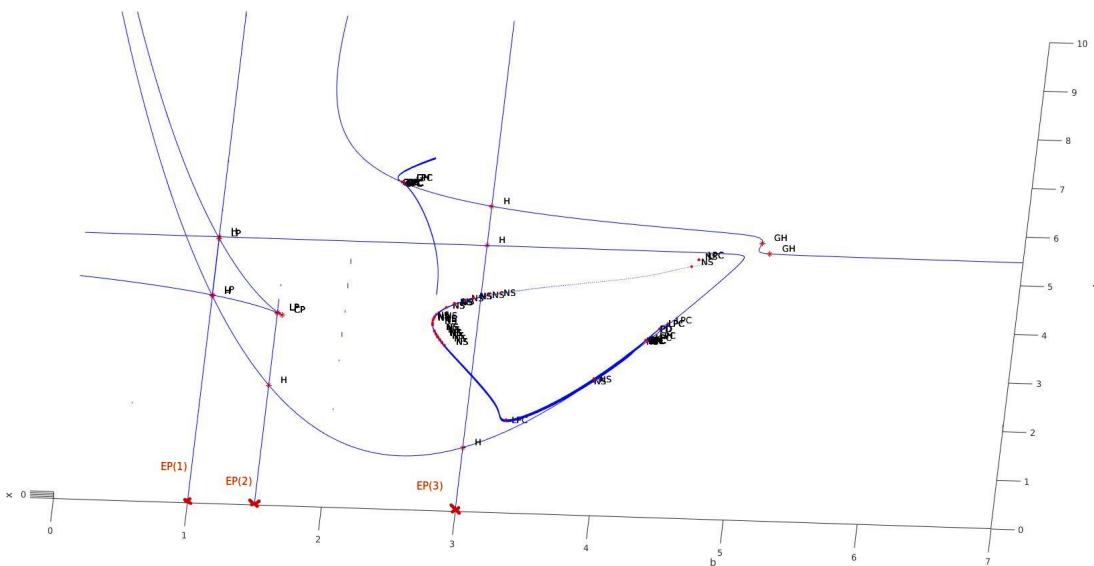


Section 6. Codimensional-two (b,I) bifurcation analysis with MATCONT

Finally I used MATCONT7p2, a tool that draws bifurcation curves based on continuation methods, and tried to replicate partly the fig.1 of [8]. Instead of the 2D parameter space (b,I) I did it in 3D (axes are x,I,b). Some basic steps are described in the appendix. The bifurcation point identification is implemented with the inspection of the eigenvalues: **Hopf (H)**: two conjugate eigenvalues with $\text{Re}(\lambda)=0$ (if first lyapunov coefficient $\lambda_1>0$, then **subcritical**, else if $\lambda_1<0$ **supercritical**).

Limit point (LP): Or Saddle-node bifurcation, i.e. one zero valued eigenvalue. **Limit point of Cycles (LPC)**:

two conjugate eigenvalues with $\text{Re}(\lambda)=0$ and $\lambda_1=0$. **Generalized Hopf (GH)**: Or Bautin bifurcation [9]. The center of the codimension-two parameter space (b,I) area, which begins from a stable spiral, that evolves through a super-, then a sub- critical bifurcation and then an LPC bifurcation back to its initial form, i.e. a stable spiral. **Neimar-Sacker (NS)**: A limit cycle gives birth to a torus.



I managed to replicate the following curves of the paper [8] :

T1: Limit point curve, starting form the Hopf bifurcation that occurs for EP(1), the equilibrium stable node for $(b,I)=(1,0)$, as I changes. In the area inside of T1 there are three equilibria: one stable node (or spiral) **and** one saddle **and** one limit cycle. Outside it only one: either one limit cycle/ **or** one stable node(or spiral).

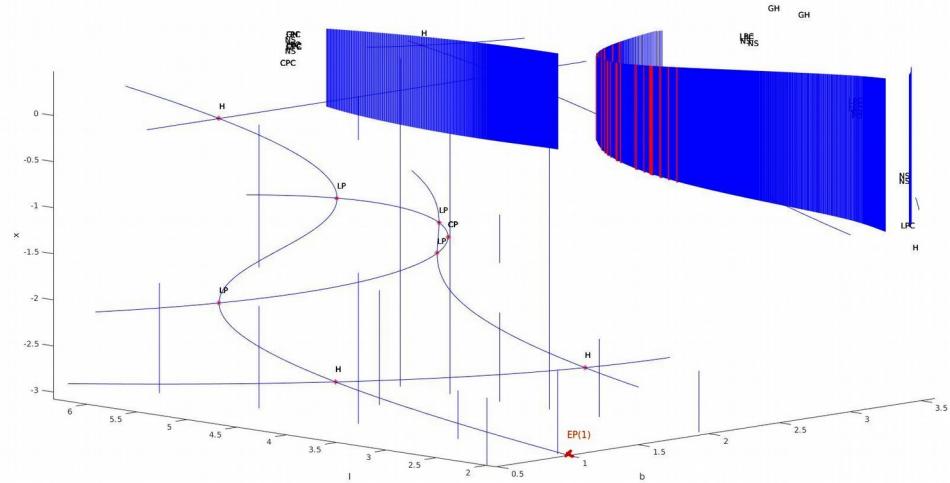
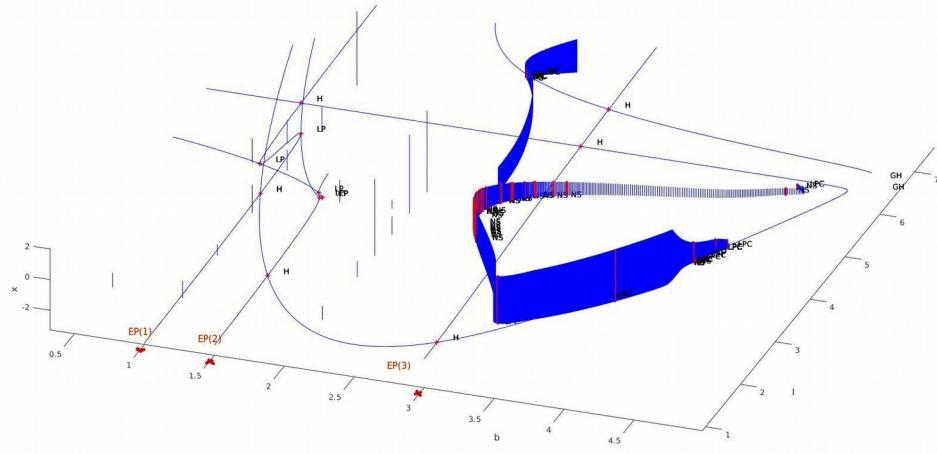
H1: Hopf curve, starting from the same Hopf point. On either side of the line, a limit cycle either appears or disappears through a Hopf bifurcation ($\text{Re}(\lambda)=0$).

H2: Hopf curve, starting from the Hopf point bifurcation that occurs for EP(3), i.e. the equilibrium stable node for $(b,I)=(3,0)$, as I changes.

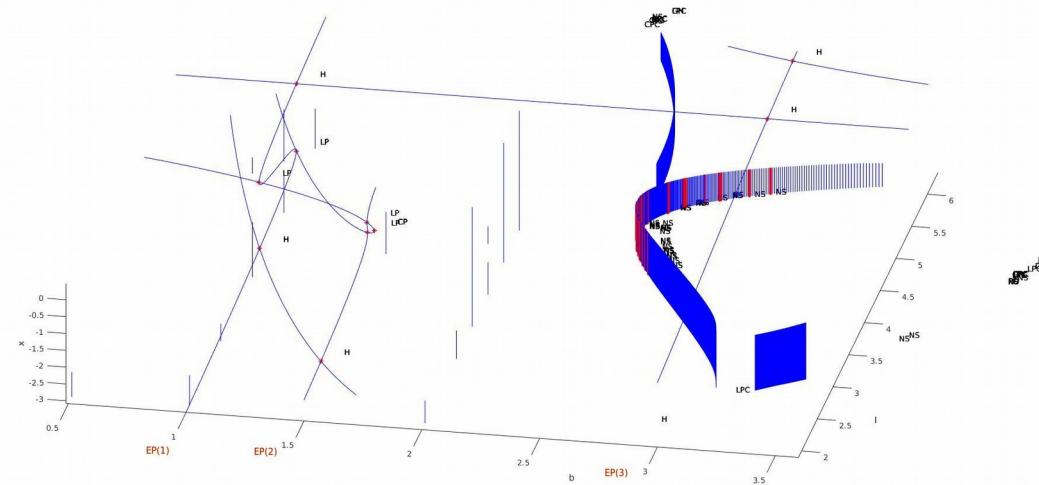
h2: Limit Cycle curve, starting from the GH which is discovered along the Hopf curve H1.

$t_0^{(1)}$: **Limit point of cycles** curve, starting from the GH which is discovered along the Hopf curve H2.

Note: In the figures below, ignore the vertical lines that were drawn, as I experimented with a couple of starting points, to draw the curves.



Az: -45 El: 16



Notice, the evolution of the phase space on the $b=3$ section, i.e. the line that starts from EP(3). That is the evolution we had studied in the previous sections. For $0 < I < 1.2698$, we have a stable node (at some value of I just before 1.2698 the stable node becomes stable spiral. I wasn't able to identify its curve). At $I=1.2698$ the stable spiral undergoes a Hopf bifurcation thus giving birth to the limit

cycle (slow recovery part between the burstings), which instantly joins the homoclinic orbit that starts and ends at the saddle (fast bursting phases). Then for $1.27 < I < 3.25$ nothing changes in the phase plane (bursting continues). Then for a brief space of I values around 3.25, we have chaotic behavior (irregular bursting). Then we have regular bursting again. Finally at $I \approx 6$ bursting stops and we have spiking.

Section 7. Discussion

Hodgkin and Huxley (HH) laid the mathematical groundwork for modern biological neural modeling in the 1940s and 1950s by developing a coupled set of differential equations describing the ionic basis of the action potential, and were able to accurately reproduce all the key biophysical properties of the action potential. While based on the HH model, the 3D Hindmarsh-Rose model (3DHR) depicts the *global behavior* of the neuron and its underlying operation is removed from the actual biological process. While it is a simpler model than the HH model, it is **more accurate** in modeling neurons seen in biology. The model for a single neuron must be both computationally simple, and capable of mimicking almost all the behaviors exhibited by real biological neurons. Dynamical behaviors of real neurons include (depend on biophysical parameters): **Quiescence**: the input to the neuron is below a certain threshold and the output reaches a stationary regime, **Bursting**: the output is made up of groups of two or more spikes (called bursts) separated by periods of inactivity, **Spiking**: the output is made up of a regular series of equally spaced spikes.

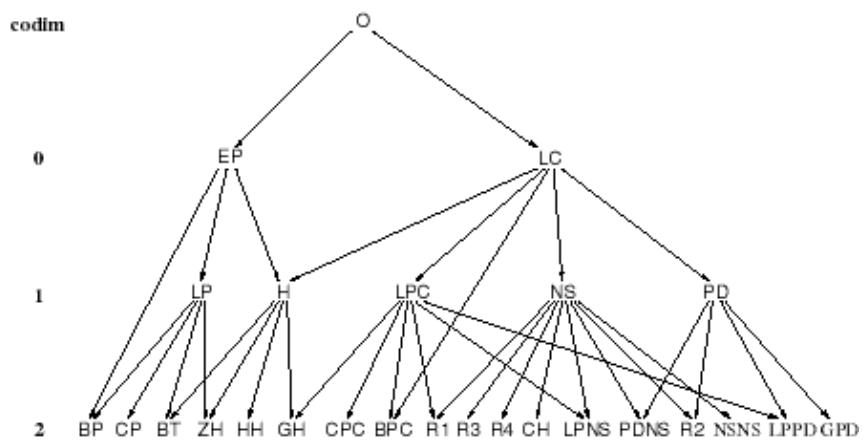
After applying the fixed parameters to the model, we explored its mathematical behavior with a range of input currents I . Nullcline plots with each varying input were used to investigate the presence of any bifurcation points, where two fixed points (intersections) of the model would converge into either a single or no fixed point. The fixed points (nodes) can be classified according to their behavior and can be analyzed using phase portraits close to each node. Our analysis revealed that a **bifurcation** took the system from having **three equilibria** (stable node, saddle, and unstable spiral \rightarrow quiescence, for $0 < I < 1.26$), to **two** (semi-stable limit cycle and stable limit cycle \rightarrow bursting, for $1.26 < I < 3.25$), **chaotic behavior** (briefly for $I \sim 3.25$) and finally **one** (stable limit cycle \rightarrow spiking, for $I > 5.39$).

Bibliography

- 1: Hindmarsh, Rose, A model of nerve impulse using two ODE's, 1982
- 2: Hindmarsh, Rose, A model of neuronal bursting using 3 coupled ODE's, 1984
- 3: , https://en.wikipedia.org/wiki/Cubic_equation,
- 4: Amir Fuchs, Non-linear Dynamics in Complex Systems,
- 5: J. M. GONZÁLEZ-MIRANDA, COMPLEX BIFURCATION STRUCTURES IN THE HINDMARSH-ROSE NEURON MODEL,
- 6: Paulo C. Rech, Dynamics of a neuron model in different two-dimensional parameter-spaces, 2011
- 7: Wolf, DETERMINING LYAPUNOV EXPONENTS FROM A TIME SERIES, 1985
- 8: Storace et al., The Hindmarsh–Rose neuron model: Bifurcation analysis and piecewise-linear approximations, 2008
- 9: , http://www.scholarpedia.org/article/Bautin_bifurcation, ,
- 10: Hil Meijer, Matcont Tutorial: ODE GUI version, 2016

Appendix

The intuition of MATCONT is that, for a specific (b,I) you can find an equilibrium point, EP, by using **init_P_O** (point orbit). Then draw an equilibrium to equilibrium curve, as I changes, on the 3d plot (x,I,b) , by using **init_EP_EP**, starting from that EP. Hopefully you discover some Hopfs. Then you can start building, based on these Hopfs, to draw the codimension-2 lines of the paper. For example you can use a Hopf bifurcation point $(x, I, b)_{\text{Hopf}}$, as a starting point to draw a Hopf-Hopf curve, **init_H_H** (i.e. from Hopf to Hopf as (b,I) both change), or even as a starting point to draw a Hopf to Limit Cycle curve, **init_H_LC**, to visualize how the limit cycle that is born at the Hopf point, evolves in the parameter space.



There are multiple useful tutorials on Matcont, on the web.

Matcont7p2 steps.

First extract matcont7p2.zip file. Then go to the directory where it is extracted, and in the matlab command window do : matcont. The GUI is up and running. First create the hindmarsh_rose model according to this tutorial [10] Now we are ready to construct the codimension-2 study window. Terminology of windows that will pop up and we will be using all the way: **Matcont GUI, Starter, Continuer, Control, 3D plot window, Numeric**. Keep them all the time, they are really usefull in understanding what you are doing.

Notes before the analytical steps:

- when **Control** finds a bifurcation it pauses. Click resume.
- if you draw a line, but you think it is short then change in **Continuer**, the field MaxNumPoints.
- the idea is always the same: Think of the type of line you want to draw. Then decide what point should be the starting point for this line, e.g. if you want to draw an EP (eqlbrm point) to EP line, the do Select/ initial_point / if you already have it, then choose from a diagram the equilibrium point but if you don't have it, then you should find one, using an orbit diagram first. Then, when you have selected the initial point, do Type/ initial_point/ Equilibrium/ choose in the field Initializer, one of the possible lines, that matcont can search, starting from an EP. Then do Compute/ forward (or backward) and check the (x,I,b) plot. The line starts shaping! There you have it.
- Some lines are essential for the steps, but mess up the 3d plot a bit. If you want a clean visualization, then do all the steps, keep the essential points (it's tricky because you have to save them explicitly in Select/ diagrams and change their names because the Data Browser can't handle more than two curves of each type, for example it will save EP_EP(1), EP_EP(2), but then it will

save next EP_EP to EP(2) ...) and then reopen a 3d plot, repeat the same steps, avoiding the ones that mess up the plot, by using the points that are already calculated in the curves that you have saved as mentioned above in the parenthesis.

Steps

3D codim-2 bifurcation plot (x,b,I)

0. **Open a (x,I,b) 3d plot.** Do: WindowOutput/ graphic/ 3d_plot/ Matcont/ Layout/ set Abscissa: b in range (0,7), Ordinate: I in range (0,10) Applicate: x in range (-4,4).

1. **Find equilibrium.** For the already studied model ($a=1, b=3, c=1, d=5, r=0.001, s=4, I=0, x_0=-1.6$), we know that the stable node is approx. at $(-1.6, -12, 0)$. Then do: (a) Type/initial_point/point (b) in the ‘Starter’ window that pops up set parameters: $t=0, x=-1.6, y=-12, z=0, a=1, b=3, g=1, d=5, r=0.001, s=4, I=0, x_0=-1.6$. Change in Integrator the field Interval to 20, s.t. it converges well (d) Compute/forward. A window pops up, called Control and reports the evolution landmarks. It should report ‘finished’ without anything else. Now the landmarks (initial point, bifurcations, ending point) along the orbit from our predetermined point to the convergence equilibrium point (i.e. the stable node) are saved in diagram curve P_O(1) and we will use it next. You can see the landmarks of this curve in Select/Diagram/O_P(1).

2. Using this equilibrium we **find the Hopf-bifurcations**, while changing I. We use this equilibrium as a starting point, and check its values, as I changes. We expect to find a couple of Hopfs along the way. That is, through the fold bifurcations (saddle-node bifurcations) between our point (which is the stable node), and the saddle node, they cancel out and simultaneously a limit cycle (bursting or spiking) pops up! Do: (a) Type/initial_point/Equilibrium. This informs MATCONT that we are about to feed it a starting point that is an equilibrium and that we want to plot an EP to EP line, that is, we want to know where this EP locates, each time one parameter changes (here we will choose I). Then (b) Select/ initial_point/ double click Diagram; diagram double click on curve P_O(1)/ click on last point/ select point. Now this equilibrium point is loaded as our starting point. Note: Double check that in MatCont GUI the field ‘Initializer’ still reports Equilibrium(init_EP_EP). Sometimes it resets to orbit. If it has changed, then just do again Type/ initial_point/ Equilibrium. Check in ‘Starter’ that the starting point is $(-0.832236961904908, -7.42406060237021, 0.00196871952273043)$. Check also parameter I, as the one changing parameter. (c) If you want to visualize also in (x,y) plane do: window/ graphic/ 2d_plot/ Matcont/ Layout. Ordinate: x in range (-3,3) and Abscissa: I in range (0,26). (d) If you want to see the eigenvalues to assert the Hopf bifurcations ($\text{Re}(\lambda)=0$), also do: Window/Output/ numeric (e) Compute/ forward. It should report 3 Hopfs and they are saved in curve EP_EP(1).

3. Now we will use these Hopfs to **construct H1, H2 lines** of the paper. Do: (a) Type/ initial_point/ Hopf. Set Initializer to Hopf(init_H_H). That is we inform MATCONT that we will start at a Hopf point, and we want to plot the Hopf-Hopf line, i.e. the line where all Hopf bifurcations happen. Then (b) Select/ initial_point/ double click on EP_EP(1) click on the first Hopf /select point. (c) From now on, we are beginning the construction of the main plot. (d) In Starter window check I and b, i.e. choose I and b as the changing parameters. (e) Change in Continuer the field MaxNumPoints to 800. we need it now. (f) Do: Compute/ forward and then Compute/ backward. Notice how we just made H1! Now, to plot H2, we do the same, but we choose as starting point the third Hopf point in EP_EP(1), as we did in (b) and then repeat steps (c-f).

4. Now we need a limit point to **construct T1 curve**. To find limit points we need a right combination of (I,b) that will have for a range of I (i.e. b section) one equilibrium, then three, then one again. By observation of the plot of the paper (a bit of cheating but anyway), we see that a

choice for (b,I) is (1, 5). Then run an orbit to find the EP line. Do: (a) Type/initial_point/orbit (b) fill in Starter t=0,x=-10,y=-10,z=0,a=1,b=1,g=1,d=5,r=0.001,s=4,x0= -1.6, I=5 and in the Continuer Interval =25 (c) WindowOutput/ graph/ 2d_plot/ Matcont/ Layout Ordinate: y in range (-5,-70) and Abscissa: x in range(-5,5). (d) Compute/ forward. Notice that it converges to (-3.4893526, -59.78703,-0.18009529), that is the stable node for those respective (b,I), and it is saved as the last point in P_O(2). Now use this equilibrium: (a) Select/ initial_point/ P_O(2)/ last point/ select point (b) Type/ initial_point/ Equilibrium (c) set Initializer to Equilibrium(init_EP_EP) (d) set in Continuer MaxNumPoints=850. We end up with the EP to EP curve, and along it there are our limit points. See them by doing: Select/ Diagram/ EP_EP(2). Now that we have our limit points choose one (say the first) to draw limit point to limit point curve. Do: (a) Select/ initial_point/ EP_EP(2)/ double click LP (choose the first)/ select point (b) Type/ initial_point/ Limit Point (c) set Initializer to Limit Point (init_LP_LP) (d) check in Starter I and b (e) set in Continuer MaxNumPoints to 300 (f) Compute/ forward and then backward. This curve contains all the (b,I) points where the limit points (zero eigenvalue) take place. Notice also the cusp that forms!

The rest is left as exercise, and needs experimentation.