CALCULUS 3

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Contents

Part 1. Introduction

- In calculus 1 we learnt about transoformation as this form :

$$y = f(x), f : \mathbb{R}^1 \to \mathbb{R}^1$$

- In calculus 2 we learnt about transoformation as this form :

$$y = f(x_1, x_2, ..., x_m), f : \mathbb{R}^n \to \mathbb{R}^1$$

- In calculus 3 we will learn about transoformation in this form :

$$\substack{y_1 = f_1(x_1, x2, .., x_n) \\ \vdots \\ y_n = f_k(x_1, x2, .., x_n)}, f : \mathbb{R}^n \to \mathbb{R}^k$$

Part 2. Norms

Signs Convention

$\forall$ for all

$\exists$ exsists

Definition. $\delta_{i,j} = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\}$ kroner delta

Definition. column vector $= \begin{pmatrix} x^1 \\ \vdots \\ x^d \end{pmatrix}$

Definition. line vector $= \begin{pmatrix} x^1 & \cdots & x^d \end{pmatrix}$

Definition. $y^i = \sum_j A^i_j x^j$ - is the multiplication fo matrix with vector :

$$Mat_{mxd}(\mathbb{R}) = \begin{pmatrix} a^1_1 & \cdots & a^1_d \\ \vdots & \ddots & \vdots \\ a^m_1 & \cdots & a^m_d \end{pmatrix}$$

$$\sum_{j=1}^{d} a^1_j x^j = a^1_1 x^1 + a^1_2 x^2 + ...a^1_d x^d$$

Definition. euclidiean space on $\mathbb{R}$ is a vector space $Dim = d$ on $\mathbb{R}$ with inner product is a function $< \cdot, \cdot >: \mathbb{R}x\mathbb{R} \to \mathbb{R}$ and satisfies:

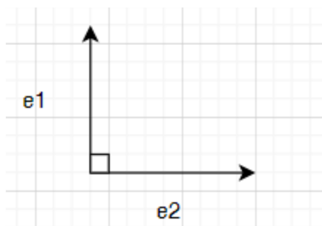
(1) $\forall x \in \mathbb{R}, < x, x > \geq 0 \Rightarrow x = 0 \Leftrightarrow < x, x >= 0$
(2) $< x, y >=< y, x >$
(3) $\lambda \in \mathbb{R}, < \lambda x, y >= \lambda < x, y >$
(4) $< x + y, z >=< x, x > + < y, x > \forall x, y, z \in \mathbb{R}.$

Definition. let $\{e_i\}_{i=1}^{d}$ base in $\mathbb{R}$ then $< \cdot, \cdot >$ defined as $g_{ij} =< e_i, e_j >$,$\forall 1 \leq i, j \leq d$

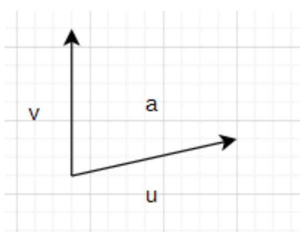
because if , $x = \sum_{i=1}^{d} x^i e_i$ when $\begin{pmatrix} x^1 \\ \vdots \\ x^d \end{pmatrix}$-coordination of x on base $\{e_i\}$ and $y = \sum_{j=1}^{d} y^j e_i$ when $\begin{pmatrix} y^1 \\ \vdots \\ y^d \end{pmatrix}$-coordination of y on base $\{e_i\}$.

$< x, y >=< \sum_{i=1}^{d} x^i e_i, \sum_{j=1}^{d} y^j e_i >= \sum_{i,j=1}^{d} x^i y^j < e_i, e_j >= \sum_{i,j=1}^{d} x^i y^j g_{ij}$

Example. $d = 2$, $||e_1|| = ||e_2|| = 1$,g_{ij} =$\delta_{i,j} = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\}$ kroner delta,$i, j = 1, 2$.



$< e_1, e_1 >= 1$,$g_{11} = g_{22} = 1$
$< e_2, e_2 >= 1$,$g_{12} = g_{21} = 0$
$< e_1, e_2 >= 0$



note : $< u, v >= ||u|| \cdot ||v|| \cdot cos\theta$

Exercise. if $g_{11}, g_{12} = 1, g_{21} = -1, g_{22} = 9$ define scalar product on $\mathbb{R}^2$?
Answer : no.

Definition. the "Standrad" base on $\mathbb{R}^d$is a base $\{e_i\}_{i=1}^{d}$is orthonormal : $g_{ij} = \delta_{ij}$i.e $< e_i, e_j >= 1$, $< e_i, e_j >= 0, \forall i \neq$j hence for

$$x = \sum x^i e_i, y = \sum y^j e_j, < x, y >= \sum_{i,}^{d} x^i y^j g_{ij} = \sum_{i,}^{d} x^i y^j \delta_{ij} = \begin{pmatrix} x^1 \\ \vdots \\ x^d \end{pmatrix}^t \begin{pmatrix} y^1 \\ \vdots \\ y^d \end{pmatrix}$$

(this defines a scalar product).

Definition. normal in $\mathbb{R}^d$ is a function $||\cdot||$:$\mathbb{R}^d \to \mathbb{R}$ and satisfies :

(1) $||x|| \geq 0, \forall x \in \mathbb{R}^d \wedge ||x|| = 0 \Leftrightarrow ||x|| + 0$.
(2) $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^d, ||\lambda x|| = |\lambda| \cdot ||x||$.
(3) $||x + y|| \leq ||x|| + ||y||$.

Definition. distance function (metrica) in $\mathbb{R}^d$ which defined as a normal as a function
$$d(\cdot,\cdot):\mathbb{R}^d x\mathbb{R}^d\to\mathbb{R}, d(x,y):=||x-y||$$

Definition. euclidean normal on $\mathbb{R}^d$ defined as $||x|| =< x,x >^{\frac{1}{2}}=_{standradbase} \sqrt{(x^1)^2+....+(x^d)^2}$

note: it stem from caushy-shwartz enequality $|<x,y>|\leq ||x||\cdot||y||$ i.e
$$|\sum_i x^i y^i|\leq\sqrt{\sum_i (x^i)^2}\cdot\sqrt{\sum_i (y^i)^2}$$
.

* exists other normals on $\mathbb{R}^d$ for instance :

The l^∞ normal which defined as :
$$||x||_\infty := max_{1\leq i\leq d}||x^i||$$

Example. Let $\zeta(\mathbb{R}^d,\mathbb{R}^m)$ a vector space of $L.T$, if we denote bases B_i on $\mathbb{R}^d,\mathbb{R}^m$as $\forall T$exists representaion matrix with respect to B_i $L_A:\mathbb{R}^d\to\mathbb{R}^m$, $A\in Mat(mxd)$.

$$\zeta(\mathbb{R}^d,\mathbb{R}^m)\cong\mathbb{R}^{mxd}, ||A||_{HS}:=\sqrt{\sum_{i,j}(a_i^j)^2}=\sqrt{Tr(A^t\cdot A)}(HS:Hilbert-Schmidt).$$

Example. The unit-ball **norm** is defined as :
$$||x||_p=(|x_1|^p+,....,|x_n|^p)^{\frac{1}{p}}$$

Remark. The $||\cdot||_\infty$ norm is defined as max between the vector elements, for example in calculus-1 we showed that :
$$\lim_{n\to\infty}=(x_1^p+x_2^p)=max\{x_1,x_2\}$$

So the intiuition is that we can generalize the follwing for vector with n demen- tions and not only 2 .

Fact. *for all $x\in\mathbb{R}^n$ we $\exists n\in\mathbb{N}$ S.T :*
$$||x||_\infty\leq||x||_1\leq n\cdot||x||_\infty$$

Proof. first we notice that :
$$||x||_2=\sqrt{x_1^2+,..,x_n^2}\leq\sqrt{n}\cdot max|x_i|$$
$$\Downarrow$$
$$||x|||_\infty\leq||x_2||\leq\sqrt{n}\cdot||x||_\infty$$
hence,
$$\frac{1}{n}\cdot||x||_1\leq||x||_2\leq\sqrt{n}\cdot||x_1||$$
We will keep the other direction for readers. □

Claim. $||x|| = ||x_1e_1+...+x_ne_n||$ is a continuous function in the n variables $x_1,...,x_n$.

Proof. Notice that :

$$||x|| - ||y|| \leq ||x-y|| = ||(x_1-y_1)e_1+, ..., (x_n-y_n)e_n||$$

$$\leq |x_1-y_1| \cdot ||e_1|| + .. + |x_n-y_n| \cdot ||e_n||$$

hence,

$$|||x|| - ||y||| \leq |x_1-y_1| \cdot ||e_1|| + ... < \epsilon, \epsilon > 0$$

So $||x||$ is comtinious function. □

Theorem. *for every two norms $||x||, ||x||^*$, $\exists m, M > 0$ S.T :*

$$m \cdot ||x|| \leq ||x||^* \leq M| \cdot |x||$$

Proof. We will show the following, by using Triangle-Inequality first

——**x+y**——$\leq ||x|| + ||y|| \rightarrow ||x+y|| - ||y|| \leq ||x||$**:**
$\Downarrow w = x+y \rightarrow x = w-y$**:**
——**w**$\| - ||y|| \leq ||w-y||, \forall x, y, w \in \mathbb{R}^n$**:**
we will write every vector in the Standard-Base $\{e_i\}_{i=1}^n$

$$x = x_1e_1 + ... + x_ne_n$$

Look at the set :

$$B = \{(x_1, .., x_n) | x_1^2 + ...x_n^2 = 1\}$$

Look at the :

$$h(x_1, .., x_n) = \frac{||x||^*}{||x||} = \frac{||x_1e_1 + ... + x_ne_n||^*}{||x_1e_1 + ... + x_ne_n||}$$

On the set B $||x|| \neq 0$ Since $\vec{0} \notin B$, Since $x_1^2 + .., x_n^2 = 1$ and $\vec{0}$ doesn't satisfy that.

Fact. *h is a continios function as a division of continuous funcions (stem from previous claim).*

Fact. *h is a continuous in a compactiv set so it has max value by weistrass-theorem.*

Corollary. *by fact two we conclude the $\exists m, M > 0$ s.t :*

$$0 < m \leq \frac{||x||^*}{||x||} \leq M, x \in B$$

Now we take random x(not necessarily in B), $R = \sqrt{x_1^2 + ..x_n^2}$ when

$$x = x_1e_1 + ... + x_ne_n = R(\frac{x_1}{R}e_1 + .. + \frac{x_n}{R}e_n)$$

But

$$(\frac{x_1}{R}e_1 + .. + \frac{x_n}{R}e_n) \in B$$

$$\Downarrow$$

$$\frac{||x||^*}{||x||} = \frac{R||\frac{x_1}{R}e_1 + .. + \frac{x_n}{R}e_n||^*}{R||\frac{x_1}{R}e_1 + .. + \frac{x_n}{R}e_n||}, \frac{x_1}{R}e_1 + .. + \frac{x_n}{R}e_n \in B$$

Hence,

$$0 < m \leq \frac{||x||^*}{||x||} \leq M, \forall x \neq 0$$

$$\Downarrow$$

$$m \cdot ||x|| \leq ||x||^* \leq M| \cdot |x||, \forall x$$

Q.E.D.

Exercise. show that $\forall A \in Mat_{nxk}(\mathbb{R})$, $\exists M$ S.T $\forall x \in \mathbb{R}^n$

$$||A\vec{x}||_{n-\mathbb{R}^k} \leq M \cdot ||x||_{in-\mathbb{R}^n}$$

Example. he proof for $||x||_\infty$:

$$||\vec{A}x|| = \underset{i=1,2..k}{Max}|\sum_{i=1}^{n} a_{ij}x_j| \leq n \cdot Max|a_{ij}| \cdot \underset{j=1,2,...n}{Max}|x_j|$$

when,

$$Max|a_{ij}| = M \in \mathbb{R}, \underset{j=1,2,...n}{Max}|x_j| = ||x||_\infty$$

.

Remark. we showed the exercise for $||x||_\infty$ **but** we also show it for every norm by the previous Threom.

Part 3. continuous Functions

Definition. Function $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous in p_0 if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon)$ S.T ;

$$||F(p) - F(p_0)||_{in-\mathbb{R}^k} < \epsilon$$

for all p $||p - p_0||_{in-\mathbb{R}^n} < \delta$

Remark. choosing norm doesn't change, if one good for any delta, for the other we choose works with other delta, for example $|| \cdot ||_\infty$:

$$\forall i, |F_i(x_1, .., x_n) - F_i(x_1^0, .., x_n^0)| < \epsilon$$

when :

$$\underset{j=1,..,n}{max}|x_j - x_j^0| < \delta$$

Corollary. *Function is continuous $\iff$ every one of it k elements continuous separately in it's n variables $x_1, .., x_n$.*

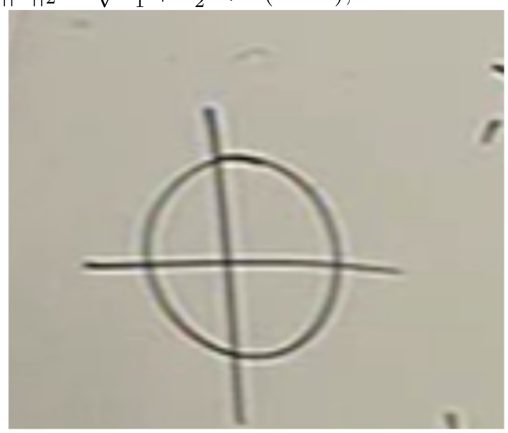
Part 4. Toplogy

Definition. A neigberhood of point p_0 is

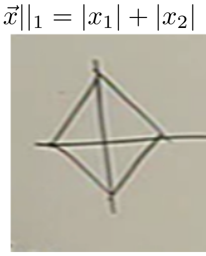
$$N = \{p : ||p - p_0|| < r\}$$

for example in the plane xy we can have different neighberhood depend on the norm :

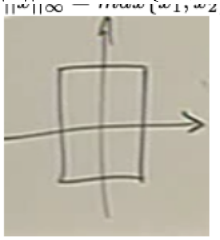
$||\vec{x}||_2 = \sqrt{x_1^2 + x_2^2} < r$ (circle),



$||\vec{x}||_1 = |x_1| + |x_2| < r$

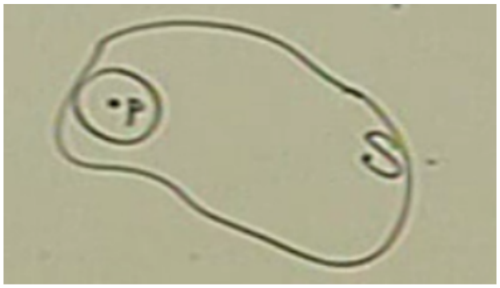


$||x||_\infty = max\{x_1, x_2\}$



Remark. every one of those neighberhood can obtain other by taking good bounder $M \in \mathbb{R}$.

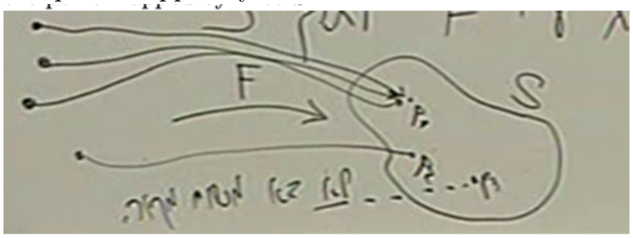
Definition. Given a set S, a point p_0 is interior point of S if $p_0 \in S$ and $\exists N$ neighberhood of p_0 S.T $N \subset S$.



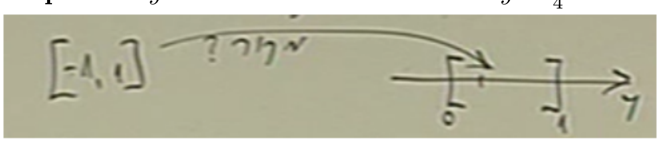
Definition. A set S is called open if $\forall p_0 \in S$, exist a neighberhood which is subset of S i.e $\forall p_0 \in S$, p_0 is interior point.

Example. $x^2 + y^2 < 1$ is open **BUT** $x^2 + y^2 \leq 1$ not opened since it's include the bounder.

Definition. Given a transformation F. the **source** of a set S by F (pre-image) is all the point mapped by F to S.



Example. $F : y = x^2$ what is the **source** of $y = \frac{1}{4}$ **?**



The (pre-Image) $x = \{\frac{1}{2}, -\frac{1}{2}\}$

Exercise. Does continuous function map open set to open set?

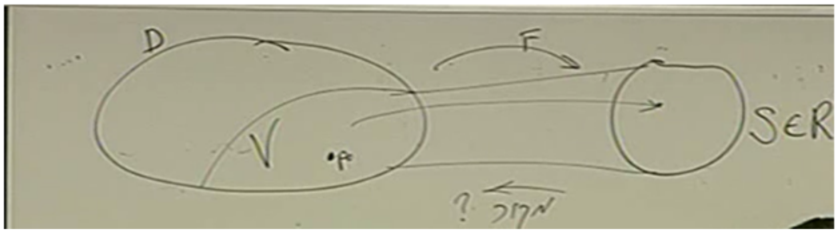
Solution. **not neccesserily** for example look at $F : y = x^2$, $(-1, 1) \to [0, 1)$
notice that $(-1, 1)$ is open set **BUT** $[0, 1)$ **is not open.**

Theorem. *let $D \subset \mathbb{R}^n$ be open set and a transformation $F : D \subset \mathbb{R}^n \to \mathbb{R}^k$ continuous.*
Then ***pre-image*** *of every open set in $\mathbb{R}^k$ is open in $\mathbb{R}^n$.*

proof.: let $S \subset \mathbb{R}^k$ open set the source $V = \{p \in D | F(p) \in S\}$

We need to show that ***V*** **OPEN SOURCE .** first we take a $p_0 \in V$ we will show that p_0 has a neighberhood in V.

that p_0 has a neighberhood in V.



p_0 is in the source of S, so p_0 is mapped to $q_0 \in S$ i.e $F(p_0) = q_0 \in S$.
Given that S **open** so q_0 has neighberhood $N \subset S$ when

$$N = \{q : ||q - q_0|| < \epsilon\} \subset S$$

F is continuous map, so $\forall \epsilon > 0$, $\exists \delta_1(\epsilon) > 0$ S.T

$$||F(p) - F(p_0)|| < \epsilon$$

when

$$||p - p_0|| < \delta_1, p \in D$$

Also D is open, $p_o \in D$, there is neighberhood $||p - p_0|| < \delta_2$ in D.
let

$$\delta = min\{\delta_1, \delta_2\}$$

then $||p - p_0|| < \delta$ is a neighberhood of p_0 , hence, it's image in S, So all of it in the **SOURCE** V**.**

Q.E.D.

reminder.: we showed that source of open set by continuous map is open set.

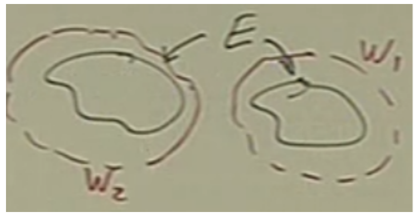
Theorem. *Given $F : D \subset \mathbb{R}^m \to \mathbb{R}^n$, D is open set.*
F continuous $\iff$ for every open set S, the source by F is open.

Remark. we showed the first direction, the other direction is exercise for readers.

Definition. A set E is **connected** if not possible to cover it by two open **disjont,open** sets W_1, W_2, in which every set obtain points ot E i.e

(1) $W_1 \cap W_2 = \emptyset$
(2) W_1, W_2 **open**
(3) $E \subset W_1 \cup W_2$
(4) $W_1 \cap E \neq \emptyset, W_2 \cap E \neq \emptyset$

Example. Our graph intiuition is :
if E is not-connected :



if E is connected then :

we notice that $W_1 \cap W_2$ can't be $\emptyset$, since there is intersections point on the boundries of each set.

so we can ask whether this point related to, W_1, W_2 if in both then intersection not empty, if not in both then it's not covered.

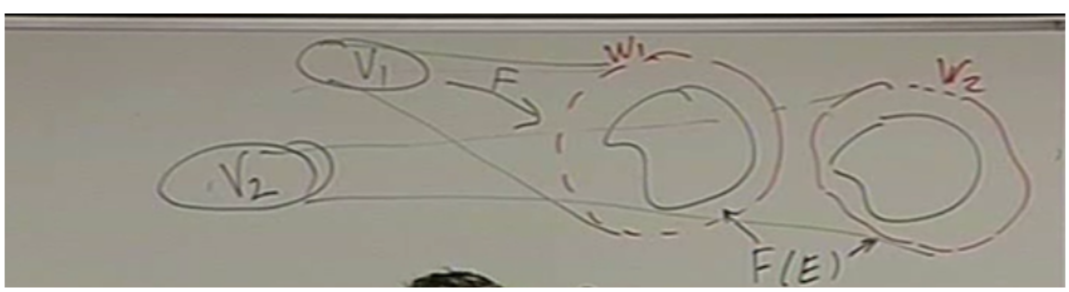
Theorem. *continuous function map connected set to connected set.*

proof.:

$E \subset \mathbb{R}^n$ connected, $F : E \subset \mathbb{R}^n \to F(E) \subset \mathbb{R}^k$, we need to show that $F(E)$ connected.

Assume toward conrtradiction that $F(E)$ is not connected, then $\exists W_1, W_2$ **open, disjoint** S.T $F(E) \subset W_1 \cup W_2$ Moreover, $W_i \cap F(E) \neq \emptyset, i = 1, 2$.

graph-intiuition.:



Denote the pre-images of W_1, W_2 as V_1, V_2 a source of open set (by previous-Theorem)

Since we show that **Source** of open set by continuous function is open, So V_1, V_2 are too.

Notice that $V_1 \cap V_2 = \emptyset$ (trivial-claim)**,** **Disjoint** Since not possible common points(assume toward contradiction thas it's not the case then F will map it to a common point **but** $W_1 \cap W_2 = \emptyset$ so it's contradiction).

Now, $V_1 \cup V_2$ obtain E, Since E is conected so by the defintion of connected set, assume without loose of generality that $E \subset V_1$,So, V_2 disjoint to E hence, W_2 **Disjoint** to E i.e

$$E \subset V_1, F(E) \subset W_1 = F(V_1) \wedge W_2 = F(V_2) \cap E = \emptyset$$

So we got contardiction to the fact that $W_i \cap F(E) \neq \emptyset, i = 1, 2 \Rightarrow$ $F(E)$ is connected.

Q.E.D.

Definition. A set called closed if it's obtain all it's limit point. for every point sequence of the set.

Theorem. *S is closed $\iff$ $\mathbb{R}^n \backslash S$ is open.*

Proof. we will show both Directions.

$\Rightarrow$

Let S open we will show that $\mathbb{R}^n \backslash S$ is closed.

Take sequence $\{p_k\}$ from S and assume that there is a limit point $\tilde{p}$.

We will show that $\tilde{p} \in \mathbb{R}^n \backslash S$ **Assume toward conrtradiction** that $\tilde{p} \notin \mathbb{R}^n \backslash S$.

then $\tilde{p} \in S$. S is open set therefore, for $\tilde{p}$ exist neighberhood

$$N = \{p : ||p - \tilde{p}|| < r\} \subset S$$

In N there is no point of $\mathbb{R}^n \backslash S$, explicitly, there is no point $\{p_k\}$ hence,

$$||p_k - \tilde{p}|| > r$$

So $\tilde{p}$ is not limit point, So we get contradiction $\Rightarrow \mathbb{R}^n \backslash S$ **closed.**

$\Leftarrow$

Assume that $\mathbb{R}^n \backslash S$ **closed** and we will show that S is **open**.

Take $\tilde{p} \in S$, in order to show that S **open** we need to fine neighberhood N of $\tilde{p}$ in S. **Assume toward conrtradiction** that any neighverhood of $\tilde{p}$ in S. for instance take :

$$||p - \tilde{p}|| < \frac{1}{k}$$

So in obtain at least one point $p_k \in \mathbb{R}^n \backslash S, p_k \neq \tilde{p}$, it's true $\forall k \in \mathbb{N}$, Hence, we get sequence of points

$$p_1, p_2, ... \in \mathbb{R}^n \backslash S, ||p_k - \tilde{p}|| < \frac{1}{k}$$

i.e $p_k \to \tilde{p}$,$p_k \in \mathbb{R}^n \backslash S$. it's contradiction that $\tilde{p} \in S \Rightarrow S$ is **open.** □

Remark. This Theorem require working with defintions, which make it easy.

Definition. A set S is bounded if it's obtained in a ball i.e $||x|| \leq R, \forall x \in S$.

Example. The infinite norm which defined as :

$$||x||_\infty = \max_{i=1,2...n} |x_i| \leq R$$

Corollary. *A set is bounded if each of it coordination is bounded.*

Theorem. *let F continuous function $F : \mathbb{R}^n \to \mathbb{R}^k$ if S closed and bounded then $F(S)$ is closed and bouned.*

Exercise. find **counter example** for the following (F continuous map) :

(1) S closed $\Rightarrow F(S)$ is closed.
(2) S bounded $\Rightarrow F(S)$ bouned

Solution 1. for example on $\mathbb{R}$, take $F(x) = \frac{1}{1+x^2}$, F maps the nutural number to a set which has 1 as limit point**, but** doesn't contain it. i.e $1 \notin F(\mathbb{N})$ so it's not closed.

Solution 2. for example in $\mathbb{R}$ take $(0,1)$ **bounded** set and consider $F(x) = \frac{1}{x}$ notice that we have that $F((0,1)) = \mathbb{R}^+$ i.e **not bounded.**

Theorem. *let F continuous function $F : \mathbb{R}^n \to \mathbb{R}^k$ if S closed and bounded then $F(S)$ is closed and bouned.*

Proof. $F : E \subset \mathbb{R}^n \to F(E) \subset \mathbb{R}^k$, E is closed. in order to show that $F(E)$ closed.

take a sequence of points $F(p_n), p_n \in E$, if subsequence $F(P_{n_k})$ converges to limit point q.

We need to show that $q \in F(E)$, first $p_{n_k} \in E$,and E is closed and bounded. So by boltzano-weirstrass

p_{n_k} has a converges subsequence $p_{n_{k_l}} \to p_0 \in E$, F a continioos map then $F(p_{n_{k_l}}) \to F(p_0) \in F(E)$.

In the other hand, $F(p_{n_{k_l}}) \to q$ hence, $q \in E \Rightarrow F(E)$ is closed.

Now we need to show $F(E)$ is bounded.

Notice that F continuous $\iff F_1(x_1,..,x_n),...,F_1(x_k,..,x_n)$ continuous

$F_i(x_1,..,x_n)$ is continuous on a closed and bounded set, and it has max, min by weirstrass-Theroem hence,

$$a_i \le F_i(x_1,..,x_n) \le b_i, i = 1,2...k$$

So $F(E)$ is obtained in a box, or a ball (since we can bound box by ball (norms-inequality))

Hence, $F(E)$ bounded by the definition. □

Part 5. **Differentiability**

Definition. $F(x,y)$ scalar function with 2 variables called deffrentiable in (x,y) if $\exists A, B \in \mathbb{R}$ S.T

$$F(x+h, y+k) = F(x,y) + Ah + Bk + R(h,k)$$

when $R(h,k)$ has order less the $\sqrt{h^2+k^2}$ i.e

$$\lim_{(h,k)\to(0,0)} \frac{R(h,k)}{\sqrt{h^2+k^2}} = 0$$

We can write it in other way :

$$\lim_{(h,k)\to(0,0)} \frac{R(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{F(x+h,y+k)-F(x,y)-Ah-Bk}{\sqrt{h^2+k^2}} = 0$$

Definition. $O(t)$ is the order/function with order less than t i.e when we devide it by t and let $t \to 0$ the lim is 0.

Remark. when we we write $R(t) = O(t)$ we mean :

$$\lim_{t\to 0} \frac{R(t)}{t} = 0$$

Remark. $O(t)$ doesn't mark only one function for example :

$$t^2 = O(t), t^3 = O(t), t^2 + t^3 = O(t), (when)t \to 0$$

$$\lim_{t\to 0} \frac{t^2}{t} = 0, \lim_{t\to 0} \frac{t^3}{t} = 0, (when)t \to 0$$

$$ln(t) = O(t), (when)t \to \infty$$

Which mean all those has order less than O(t), Since when we divede them all by t we get 0 (so they have less order)

Remark. propeties known from caculus-2.

if$F$ has a continuous partial deffrentials in $(x,y) \Rightarrow F(x,y)$ is deffrentiable.

if F deffrentiable then **NOT NECCESARILY** F has continuous partial deffrentials.

if F deffrentiable then partial deffrentials in (x,y).

if F has a partial deffrentials in (x,y) then **NOT NECCESARILY** F is continuous.

COUNTER EXAMPLES PROVIDED IN PREVIOUS COURSE.

Definition. $F : \mathbb{R}^n \to \mathbb{R}^k$ is called **deffrenial** in p if exist L.T L S.T :

$$F(\vec{p} + \triangle\vec{p}) = F(\vec{p}) + L\triangle\vec{p} + R(\triangle\vec{p})$$

when deffrential

$$\lim_{\vec{\triangle p}\to\vec{0}} \frac{||R(\triangle\vec{p})||}{||\triangle\vec{p}||} = 0$$

in matrix form we can write it as :

$$\underbrace{\begin{pmatrix} F_1(x_1+\triangle x_1, ..., x_n+\triangle x_n) \\ \vdots \\ F_k(x_1+\triangle x_1, ..., x_n+\triangle x_n) \end{pmatrix}}_{F(p+\triangle p)} = \begin{pmatrix} F_1(x_1, ..., x_n) \\ \vdots \\ F_k(x_1, ..., x_n) \end{pmatrix} + L \underbrace{\begin{pmatrix} \triangle x_1 \\ \vdots \\ \triangle x_n \end{pmatrix}}_{F(\vec{p})} + O(\sqrt{\triangle x_1^2 + ... + \triangle x_n^2})$$

when $L \in Mat_{kxn}(\mathbb{R})$ is a matrix in this form :

$$\begin{pmatrix} L_{11} & \cdots & L_{k1} \\ \vdots & \ddots & \vdots \\ L_{K1} & \cdots & L_{kn} \end{pmatrix}$$

Example. $n = 1, k = 1$.

$$F(x + \triangle x) = F(x) - L \cdot \triangle x + R(\triangle x)$$

$$\frac{R(\triangle x)}{\triangle x} = \underbrace{\frac{F(x + \triangle x) - F(x)}{\triangle x}}_{f'(x)} - L \underset{\triangle x \to 0}{\to} 0$$

Example. $n > 1, k = 1$.

$$F(x_1+\triangle x_1, ..., x_n+\triangle x_n) - F(x_1, .., x_n) - L_1 \triangle x_1 - ... - L_n \triangle x_n = O(\sqrt{\triangle x_1^2 + .. + \triangle x_n^2})$$

We will take $\triangle x_i \neq 0$ unique, and all the others $\triangle x_j = 0$ i.e

$$\frac{F(x_1 + \triangle x_1, .., x_i + \triangle x_i, ..) - F(x_1, .., x_n)}{\triangle x_i} - L_i \to 0$$

$$L_i = \frac{\partial F}{\partial x_i}$$

Theorem. *let $F : \mathbb{R}^n \to \mathbb{R}^k$ deffrentiable in $\vec{p}$ then*

$$\frac{\partial F_i}{\partial x_i}(\vec{p}) = L_{ij}$$

Proof. we can show this theorem by using the previous exmple on every element $F_1, .., F_k$:

$$F_i(x_1+\triangle x_1, ., x_j+\triangle x_j, .., x_n+\triangle x_n) - F_i(x_1, .., x_n) - \sum_{j=1}^{n} L_{ij} \triangle x_j + O(\sqrt{\triangle x_1^2 + .. + \triangle x_n^2})$$

We can take $\triangle x_j \neq 0$ unique, all the other $= 0$ and we will still with :

$$\frac{\partial F_i}{\partial x_i}|_p = L_{ij}$$

$\square$

Remark. Notice that $\sum\limits_{j=1}^{n} L_{ij} \triangle x_j$ when we change j we get all the function elements in the k demention as in case $k = 1, n > 1$, till we get $k = n$ (intiouition).

Definition. let $DF|_p = L$ is called the **deffrential** of F in point $\vec{p}$.

Theorem. *if $\frac{\partial F_i}{\partial x_j}$ exist and deffrentiable in a point $\vec{p}$.*

Remark. from now we will denote $DF(p) = L$, the $L.T$ called the deffreintial of F in $\vec{p}$.

property.:

if $F, G : \mathbb{R}^n \to \mathbb{R}^k$and both are deffrentiable in $\vec{p}$ then :

$$D(F + G)|_p = D(F)|_P + D(G)|_p$$

Remark. There is no deffreintial of multiplication since we can multiplicate every matrices.

Compostion-of-deffrentials::

let $F:\mathbb{R}^n \to \mathbb{R}^k$, G:$\mathbb{R}^n \to \mathbb{R}^l$ so the composition:

$$(\mathrm{G}\circ F)(\underbrace{\ p\ }_{\in \mathbb{R}^n}) \to G(F(p)) \in \mathbb{R}^l$$

Theorem. *if F deffrentiable in p and G deffrentiable in $F(p)$ then $G\circ F$ deffreniable in p and :*

$$D(G\circ F)|_p = DG|_{F(p)} \cdot DF|_p$$

when $DG|_{F(p)} \in Mat_{kxl}(\mathbb{R}), DF|_p \in Mat_{nxk}(\mathbb{R})$ a multiplication of linear operators/matrices.

Proof. (with no coordination)

$$(G\circ F)(p+\triangle p) = G(F(p+\triangle p)) = G(\underbrace{F(p)}_{q} + \underbrace{DF(p)}_{L} \cdot \triangle p + R_1(\triangle p)) = *$$

when

$$\frac{||R_1(\triangle p)||}{||\triangle p||} \to 0, \triangle p \to 0$$

$$* = G(q) + DG(q) \cdot (DF(p) \cdot \underbrace{\triangle \vec{p}}_{\triangle q} + R_1(\triangle p) + R_2(\triangle q)$$

when

$$\frac{||R_2(\triangle q)||}{||\triangle q||} \to 0, \triangle q \to 0$$

$$(G\circ F)(p+\triangle p) = (G\circ F)(p) + DG(q) \cdot \underbrace{DF}_{L.T}(p) \cdot \triangle p + (DG(q) \cdot R_1\underbrace{(\triangle p)}_{R_3} + R_2(\triangle q))$$

In order to show that $G\circ F$ deffrentiable and the deffreintial is $DG(q)\cdot DF(p)$ we ned to show that :

$$\frac{||R_3(\triangle p)||}{||\triangle p||} \to 0, \triangle p \to 0$$

we will show that how small is R_3.

$$R_3 = DG(q) \cdot R_1(\triangle p) + R_2(\triangle q), (when) \triangle p {\to} 0$$

EXERCISE. $\forall A$ matrix $\exists m \in \mathbb{R}$ S.T :

$$||A\vec{v}|| \leq m||\vec{v}||$$

Hence, by the exrcise we get that :

$$||DG(q) \cdot R_1(\triangle\vec{p})|| \leq m||R_1(\triangle p)||$$

remember that :

$$\frac{||R_1(\triangle p)||}{||\triangle p||} \to 0, \triangle p \to 0$$

So we can say that :

$||DG(q) \cdot R_1(\triangle\vec{p})|| \leq m||R_1(\triangle p)|| \leq m \cdot \epsilon \cdot ||\triangle\vec{p}||, (when)\triangle p \to 0, ||\triangle\vec{p}|| < \delta_1$

now for $R_2(\triangle q)$ we can do the same thing Since we know:

$$\frac{||R_2(\triangle q)||}{||\triangle q||} \to 0, \triangle q \to 0$$

So, we can say that :

$$\frac{||R_2(\triangle q)||}{||\triangle q||} \leq \epsilon \Rightarrow ||R_2(\triangle q)|| \leq \epsilon||\triangle q||, ||\triangle q|| < \delta_2$$

The relation beween $\triangle q$ to $\triangle p$

$$\triangle q = DF(p) \cdot \triangle p + R_1(\triangle p)$$

$$||\triangle q|| \leq ||DF(p) \cdot \triangle p|| + ||R_1(\triangle p)|| \leq \underset{\in\mathbb{R}}{M_2}||\triangle p|| + \epsilon \cdot ||\triangle p|| = (M_2 + \epsilon) \cdot ||\triangle p|| < \delta_2$$

in condition that $||\triangle p|| < \delta_2$, So in total :

$$||R_3(\triangle p)|| \leq M \cdot \epsilon \cdot ||\triangle p|| + \epsilon \cdot (M_2 + \epsilon) \cdot ||\triangle p||$$

$$\Rightarrow \frac{||R_3(\triangle p)||}{||\triangle p||} \leq \epsilon \cdot m + \epsilon \cdot (M_2 + \epsilon)$$

Smaller as we want when $\triangle\vec{p} \to \vec{0}$. □

Proof. (**with coordination**) □

$$\underbrace{\left\{ \begin{matrix} y_1 = F_1(x_1, x2, .., x_n) \\ \vdots \\ y_n = F_k(x_1, x2, .., x_n) \end{matrix}, \right\}}_{F}$$

$$\underbrace{\left\{ \begin{matrix} z_1 = G_1(y1, y2, .., y_n) \\ \vdots \\ z_n = G_k(y_1, y_2, .., y_n) \end{matrix}, \right\}}_{G}$$

Now notice that $G \circ F$:

$$\overbrace{\left\{\begin{array}{c} z_1=G_1(F_1(x_1,\dots,x_n),\dots,F_k(x_1,..,x_n)) \\ \vdots \\ z_l=G_l(F_1(x_1,\dots,x_n),\dots,F_k(x_1,..,x_n)) \end{array}\right\}}^{G\circ F},$$

Assumption : G has a continuous partial deffrentials in the compatable points. By the Chain-Rule :

$$\frac{\partial z_i}{\partial x_j} = \sum_{m=1}^{k} \frac{\partial z_i}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_j} = \sum_{m=1}^{k} \frac{\partial G_i}{\partial y_m} \cdot \frac{\partial F_m}{\partial x_j}$$

THIS is the element (i,j) of the multiplication matrices $(\frac{\partial G_i}{\partial y_m}), (\frac{\partial F_i}{\partial \text{x}_j})$

In this order we got element (i,j) we got $DG(q) \cdot DF(p)$.(multipliaction-of deffrentials)

Remark. we used the fact that G has a continuous partial deffrentials in order to use the Chain-Rule.

Part 6. Inverse Transformations

Introduction:

$f : (a,b) \to (\alpha,\beta)$, $f \in C^1$, The condition in order f has inversion, the condition in which $f(x)$ has inverse is $f'(x) \neq 0$ in every $x \in (a,b)$

In this case $f' > 0$ or $f' < 0$ in all the interval., in those **assumptions**, exist inverse function f^{-1}

and $f^{-1} \in C^1$.

The condition that $f' \neq 0$ is not important for the existance of f^{-1}.

Example. $y = x^3 \iff x = \sqrt[3]{y}$, **but tis function** $y'(0) = 0$ is not deffrentiable in $y = 0$.

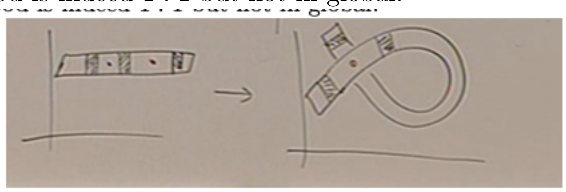
Moreover, if $f'(x) \neq 0$ then

$$\frac{d(f^{-1})}{dy}|_{y=y(x_0)} = \frac{1}{\frac{df}{dx}}|_{x_0}$$

Remark. There is big difference between $f : \mathbb{R}^1 \to \mathbb{R}^1$ to $\mathbb{R}^n \to \mathbb{R}^n$.

if $y = f(x)$ is $1:1$ in local neighberhood in every point in the domain, then it's a $1:1$ in global i.e all the domain **but** in more dementions it's not true.

Example. Notice that the photo elomentrate the fact that every local neighberhood is indeed $1:1$ but not in global.



The reason is that in $\mathbb{R}$ there is order $x < y < z$ but in $\mathbb{R}^n$ has no order we can't say that $\vec{V} > \vec{U}$

Theorem. *let $F:\mathbb{R}^n\to\mathbb{R}^n$ with a continuous partail deffrentials in a open set $D\subset\mathbb{R}^n$ ($F\in C^1(D)$)*

Assume that

$$J(p)=Det(DF(p))=|\frac{\partial F_i}{\partial x_j}|(p)=\frac{\partial(F_1,...,F_n)}{\partial(x_1,..,x_n)}(p)\neq 0$$

i.e (the L.T $DF(p)$ is invertable) then exist neigberhood (local way) of p i which F is injective.

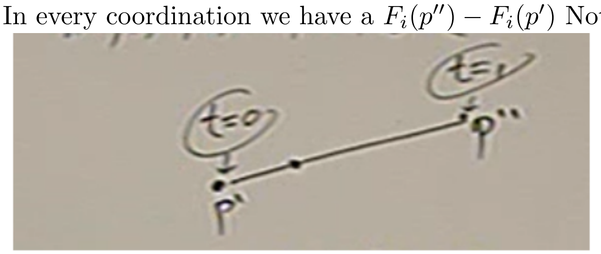
Proof. Notice that:

Choose a circle around p in which we choose two point $p'\neq p''$.

We need to show that $F(p')\neq F(p'')$, in condition we took a very small neigberhood.

$$F(p'')-F(p')=\begin{pmatrix}F_1(p'')-F_1(p')\\ \vdots\\ F_n(p'')-F_n(p')\end{pmatrix}$$

In every coordination we have a $F_i(p'')-F_i(p')$ Notice that :



So we can write :

$$p'+t(p''-p')=(1-t)p'+tp'',0\leq t\leq 1$$

Denote :

$$G_i(t)=F_i(p'+t(p''-p'))=F_i(x_1'+t(x_1''-x_1'),...,x_n'+t(x_n''-x_n'))$$

$$F(p'')-F(p')=\begin{pmatrix}F_1(p'')-F_1(p')\\ \vdots\\ F_n(p'')-F_n(p')\end{pmatrix}=\begin{pmatrix}G_1(1)-G_1(0)\\ \vdots\\ G_n(1)-G_n(0)\end{pmatrix}==\begin{pmatrix}G_1'(t_1)(1-0)\\ \vdots\\ G_n'(t_n)(1-0)\end{pmatrix}=*$$

$$**G_i(t)'=\sum_{j=1}^{n}\frac{\partial F_i}{\partial x_j}\cdot\underbrace{\frac{dx_j}{dt}}_{x_j''-x_j'}$$

$$* =_{using**} \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p'+t_1(p''+p')\cdot(x_1''-x_1')+....+\frac{\partial F_1}{\partial x_n}(p'+t_1(p''+p')\cdot(x_n''-x_n') \\ \vdots \\ \frac{\partial F_n}{\partial x_1}(p'+t_1(p''+p')\cdot(x_1''-x_1')+....+\frac{\partial F_n}{\partial x_n}(p'+t_1(p''+p')\cdot(x_n''-x_n') \end{pmatrix}$$

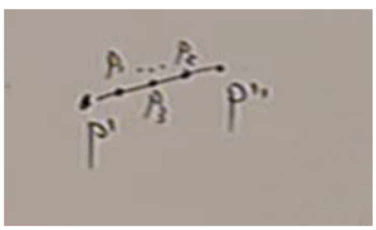
$$= \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p_1) & \cdots & \frac{\partial F_1}{\partial x_n}(p_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(p_n) & \cdots & \frac{\partial F_1}{\partial x_n}(p_n) \end{pmatrix} \begin{pmatrix} x_1''-x_1' \\ \vdots \\ x_n''-x_n' \end{pmatrix} = (\frac{\partial F_i}{\partial x_j}(p_i))(p''\vec{-}p')$$

So the target we need to show that

$$F(p'')-F(p') \neq 0$$

Define function

$$h(p_1,..,p_n)=det(\frac{\partial F_i}{\partial x_j}(p_i))_{i,j=1,..,n}$$

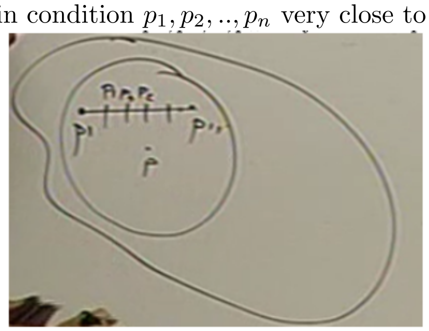


if $p_1=p_2=..=p_n=p$ then :

$$h(p_1,p_2,...,p_n)=det(\frac{\partial F_i}{\partial x_j})|_p=J(p)\neq 0$$

$h(p_1,p_2,...,p_n)$ is continuous funtion (as det) if $h(p_1,...,p_n)\neq 0$ Since continuity $h(p_1,p_2,..,p_n)\neq 0$

in condition $p_1,p_2,..,p_n$ very close to p.



□

Corollary. *$(\frac{\partial F_i}{\partial x_j}(p_i))$ is invertible matrix Since Det $\neq 0$ So $(\frac{\partial F_i}{\partial x_j}(p_i))(p''\vec{-}p')\neq 0$ Since assumme toward contradiction that $(\frac{\partial F_i}{\partial x_j}(p_i))(p''\vec{-}p')=\vec{0}$*

hence, we can multiplie by $(\frac{\partial F_i}{\partial x_j}(p_i))^{-1}$ *then we get that* $(\frac{\partial F_i}{\partial x_j}(p_i))^{-1} \cdot (\frac{\partial F_i}{\partial x_j}(p_i)) \cdot (p'' - p') = (\frac{\partial F_i}{\partial x_j}(p_i))^{-1} \cdot \vec{0} \Rightarrow I \cdot (p'' - p') = \vec{0} \Rightarrow p' = p''$

So it's contradiction to the fact the $p' \neq p''$ *as required.*

Theorem. *given* $F : \mathbb{R}^n \to \mathbb{R}^n$ *and continuous on a compactic set* S*,* F *is* $1:1$ *on* S*. then* F^{-1} *continuous on* $F(S)$ *.*

Proof. let $q \in F(S)$, and let $q_n \in F(S)$ s.t $q_n \to q$ in order to show that F^{-1} continuous, we will show

$$\underbrace{F^{-1}(q_n)}_{p_n \in S} \to \underbrace{F^{-1}(q)}_{p \in S}$$

we will show that $p_n \to p$ **ASSUME TOWARD CONTRADICTION** that $p_n \nrightarrow p$ i.e $p_n \to L \neq p$ or $\underset{n\to\infty}{lim}\, p_n$ doesn't exist.

anyway, there is a subsequence coverges to point $\tilde{p} \neq p, \tilde{p} \in S$ we will recall the subsequence in p_{n_k}.

$$p_{n_k} \to \tilde{p} \in S$$

$$F(\tilde{p}) = F(lim p_{n_k}) = lim F(p_{n_k}) = lim q_{n_q} = q$$

but $F(p) = q$ and $p \neq \tilde{p}$ So we get contradiction to $1:1$ of F. □

Theorem. *given* $F : \mathbb{R}^n \to \mathbb{R}^n$*, let* $F \in C^1(D)$ *continioud partial deffrentials from oorder 1. assume that* $J(P) \neq 0$

in a point p *of* D*.(hence* F *is* $1:1$ *in a specefic neighberhood of the point* p*). then :*

(1) F^{-1} has a continuous deffrential partial in a neighberhood of $q = F(p)$
(2) $DF^{-1}|_q = (DF|_q)^{-1}$

Proof. **first step:**

Given $J(p_0) \neq 0$, $F(p_0) = q_0$, take a adjacent point to q_o, $q_0 + \triangle q \in F(D)$.

$$\left\{ \begin{array}{c} F^{-1}(q + \triangle q) = p_0 + \triangle q \\ F^{-1}(q_1) = p_0 \end{array} \right\}$$

$$\triangle p = F^{-1}(q_0 + \triangle q) - F^{-1}(q_0)$$

step 2:

Given that F IS DEFFRENTIAL in p_0 i.e

$$F(P_0 + \triangle p) - F(p_0) = DF(p_0) \cdot \triangle p + R(\triangle p)$$

when :

$$\underset{\vec{\triangle p} \to \vec{0}}{lim} \frac{||R(\triangle p)||}{||\triangle p||} \to 0$$

As known $J(p_0) \neq 0$ hence, $DF(p_0)$, multiplication by inverse give us :

$$(DF(p_0))^{-1} \cdot \triangle q = \triangle p + (DF(p_0))^{-1} \cdot R(\triangle p) \Rightarrow \triangle p = (DF(p_0))^{-1} \cdot \triangle q - (DF(p_0))^{-1} \cdot R(\triangle p)$$

but

$$\triangle p = F^{-1}(q_0 + \triangle q) - F^{-1}(q_0)$$

Conclusion.
if we show that

$$\underset{\vec{\triangle q}\to\vec{0}}{lim}\frac{||-DF(p_0)^{-1}R(\triangle p)||}{||\triangle q||}=0$$

We can conclude that F^{-1} deffrential in q_0 and the deffrential $(DF(p_0))^{-1}$

$$DF^{-1}|_{q_0}=(DF(p_0))^{-1}$$

step 3:

$$-(DF(p_0))^{-1}R(\triangle p)$$

all the elements in the inverse matrix $(DF)^{-1}$ are continuous functions, and in a small neighberhood of p_0 they are all bounded.
we showed that :

$$||Av||_\infty\leq M||v||_\infty$$

And it's true for every normal we have that :

$$||Av||\leq M||v||$$

So we get that :

$$||-(DF(p_0))^{-1}R(\triangle p)||\leq M||R(\triangle p)||$$

$$\Downarrow$$

$$\frac{||-(DF(p_0))^{-1}R(\triangle p)||}{||\triangle q||}\leq\frac{M||R(\triangle p)||}{||\triangle q||}=M\cdot\underbrace{\frac{||R(\triangle q)||}{||\triangle p||}}_{<\epsilon}\cdot\frac{||\triangle p||}{||\triangle q||}$$

step 4:
we need to take care of :

$$\frac{||\triangle p||}{||\triangle q||}$$

remember that :

$$\triangle p=(DF(p_0))^{-1}\cdot\triangle q-(DF(p_0))^{-1}\cdot R(\triangle p)=(DF(p_0))^{-1}(\triangle q-R(\triangle p))$$

when :

$$\frac{||R(\triangle p)||}{||\triangle p||}\to 0\Rightarrow_{when(\triangle p)\to 0}\frac{||R(\triangle p)||}{||\triangle p||}<\epsilon$$

So by the Triangle-Inequality :

$$||\triangle p||\leq M[||\triangle q||+\underbrace{\epsilon\cdot||\triangle p||}_{||R(\triangle p)}]$$

$$\Downarrow$$

$$||\triangle p||(1-\epsilon M)\leq M\cdot||\triangle q||$$

it's true $\forall\epsilon > 0$ in condition that $\triangle p$ very small. (Let $\epsilon = \frac{1}{2M} \Rightarrow 1 - \epsilon M = 1 - \frac{1}{2} = \frac{1}{2}$) . hence,

$$\frac{||\triangle p||}{||\triangle q||} \leq \frac{M}{\frac{1}{2}} = 2M$$

So we showed that bounded then step 4 is true then step 2 is true then step 1 is true and we finished. □

Question 5.25

Given that $f : U \to \mathbb{R}^m$, $U \subset \mathbb{R}^d$ which satisfy :

$$||f(x) - f(y)|| \leq M \cdot ||x - y||^2, \forall x, y \in U$$

show that f is deffrentiable:

Proof. notice that we need to show that in p there is L.T L which satisfy :

$$*f(\vec{p} + \triangle\vec{p}) = f(\vec{p}) + L\triangle\vec{p} + R(\triangle\vec{p})$$

look at small neighberhood of p assume $N = \{p_0 : ||p - p_0|| < \epsilon\} \subset U$
take $p_0 \in N \subset U$ we know that in p we have that $||f(p) - f(p_0)|| \leq M \cdot ||p - p_0||^2$
Now we need to show $*$:

$$\frac{\partial f_i}{\partial x_j} = L_{ij} = \begin{pmatrix} L_{11} & \cdots & L_{1d} \\ \vdots & \ddots & \vdots \\ L_{m1} & \cdots & L_{md} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_d} \end{pmatrix}$$

$$f(\vec{p}+\triangle\vec{p}) = f(\vec{p}) + L\triangle\vec{p} + R(\triangle\vec{p}) \Rightarrow \frac{f(\vec{p} + \triangle\vec{p}) - f(\vec{p})}{\triangle\vec{p}} - \frac{R(\triangle\vec{p})}{\triangle\vec{p}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_d} \end{pmatrix}$$

Using defenition of partial :

$$\frac{\partial f_i}{\partial x_j} = \lim_{h\to 0} \frac{f_i(h + x_j) - f_i(x_j)}{h} \Rightarrow ||(Df)_{ij}(p)||$$

Notice that :

$$\lim_{\triangle\vec{p}\to 0} ||\frac{f_i(\triangle\vec{p} + \vec{p}) - f_i(\vec{p})}{\triangle\vec{p}}|| \leq \frac{M||(\triangle\vec{p} + \vec{p}) - (\vec{p})||^2}{||(\triangle\vec{p})||} = M||\triangle\vec{p}||$$

$$\forall f_i, 1 \leq i \leq m, 1 \leq j \leq d$$

hence, for $\triangle\vec{p} \to 0$ we get that :

$$M||\triangle\vec{p}|| = ||M\triangle\vec{p}|| = 0$$

(Since norm preserve multiplication by scalar) So we get that,

$$L_{ij} = Df = 0_{mxd}, 1 \leq i \leq m, 1 \leq j \leq d$$

Now we need to show that $Df = 0$ imply the defintion of deffrentiability,
First,

$$*f(\triangle\vec{p} + \vec{p}) = f(\vec{p}) + L\triangle\vec{p} + R(\triangle\vec{p})$$

$$\Downarrow Df = 0 = L$$

$$f(\triangle\vec{p}+\vec{p}) - f(\vec{p}) = 0 + R(\triangle\vec{p}) \Rightarrow ||R(\triangle\vec{p})|| \leq M||(\triangle\vec{p}+\vec{p}) - \vec{p}||^2$$

$$\Downarrow$$

$$\frac{||R(\triangle\vec{p})||}{||\triangle\vec{p}||} \leq M||\triangle\vec{p}|| \underset{\triangle\vec{p}\to 0}{\to} 0$$

So it satisfy the requiered. □

Theorem. *(The open transformation Theorem).*

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and D is open set, $F \in C^1(D)$. Assuming that $0 \neq$ J(p)=Det$(DF(p)), \forall p \in D$ then F map a open set D to open set $F(D)$.

Proof. We will spit that proof into steps.

Step 1:

In order to show that $F(D)$ is open set, we will take a point q_0 and we will show that it's has a nighberhood which is obtained in $F(D)$.

Notice that :

$q_0 \in F(D)$ So $q_0 = F(p_0), p_0 \in D$ given that D is open, hence there is neighberhood which is obtained in D recall it $N = \{p : ||p - p_0|| < r\} \subset D$ We already saw by the assumption that F is injective in small neighberhood of p_0 then we will take N small enough S.T F will bw injective on it's boundry ∂F, the boundry of N is $||p - p_0|| = r$, it's a closed and bounded set hence, $F(\partial N)$ is also closed and bounded Moreover, the image $F(\partial N)$ doesn't obtain q_0, Since p_0 mapped to q_0 and F in this neighberhood is injective.

Step 2:

we will build a neighberhood of q_0 which is obtained in $F(D)$. Since $F(\partial N)$ closed and bounded eexists a minimal ditsance $Dist(q_o, F(\partial N)) = d > 0$ (showed in the toutorials.

We will show that a circle of radius $\frac{1}{3}d$ around q_0 is obtianed in $F(D)$ Moreover, it will be the required neighberhood in order to show that $F(D)$ is opened.

Will we take in this circle a point q_1 S.T $||q_1 - q_0|| < \frac{1}{3}d$ and we will show that $q_1 \in F(D)$ i.e $p_1 \in D$ S.T $q_1 = F(p_1)$.

Step 3:

We will "solve" the equation $F(p_1) = q_1$ when $p_1 \in N$ Since we have no direct solution we will use a helping method.

We will look at $N \cup \partial N = \bar{N} = \{||p - p_0|| \leq r\}$ and we will search for

$$\underset{p\in\bar{N}}{min}||F(p) - q_1||_2$$

Target.

we will show that the min value is 0 then the as a collary we conclude that there is place in which F exactly get the value q_1.

Remark.

The existance of min stem from that fact that $\underset{p\in\bar{N}}{min}||F(p) - q_1||_2 = \phi$ is closed and bounded set and F is continuous Morover, norm is continuous as well hence, we get a continuous map as a composition so it has a min by Weierstrass-Theorem recall $p_1 \in \bar{N}$.

Remark.
We took in advance the euclidean norm for a reason.
Step 4:
The min is not on the boundry of N (∂N) **Assume toward contradinction** that it's on the boundry,

$$||F(p_1) - q_1|| = ||(F(p_1) - q_0) + (q_0 - q_1)|| \geq ||\underbrace{F(p_1) - q_0}_{\geq d}|| - ||\underbrace{q_0 - q_1}_{< \frac{1}{3}d}|| > \frac{2}{3}d$$

In other hand

$$||F(p_0) - q_1|| = ||q_1 - q_0|| < \frac{1}{3}d$$

So we got contradinction because we find a point which is more "qualified" to be the min hence, the min is not on the boundry. therefore, $\min_{\bar{N}}(||F(p) - q_1||_2)^2$ is in a interior point.

$$q_1 = (q_1, ..., q_n), p = (x_1, .., x_n)$$

So we are searching for the min,

$$min\{\sum_{i=1}^{n} (F_i(x_1, .., x_n) - q_i)^2\}$$

a condtion for that that all the deffrential partials are 0.

$$\sum_{i=1}^{n} 2(F_i(x_1, .., x_n) - q_i)\frac{\partial F_i}{\partial x_j} = 0, j = 1, .., n$$

We got a system of n equation with n variables, it's a system of homogonic linear equation with a solution Moreover, the Det of the system $det(\frac{\partial F_i}{\partial x_j}) \neq 0$ (in every point) hence, the only unique solution is when,

$$F_i(x_1, .., x_n) - q_i = 0, \forall i = 1, .., n$$

So we found solution,

$$F(p) = q_1$$

Collary.
We showed that $||q_1 - q_o|| < \frac{d}{3}$ is a img of specific p_1 is in the interiors of N so the img is open set.
Remark.
Now we know why we used that euclidean norm because it's easy to work with it derivative from other norms since it's simple. □

Part 7. The Inverse Functions Theorem

Theorem. *(The inverse Function Theorem).*

Given $F : \mathbb{R}^n \to \mathbb{R}^n$, with a partial deffrentials with order 1, $F \in C^1(D)$ in a domain D, and $J(p) \neq 0, \forall p \in D$. We take $p_0 \in D$, $q_0 = F(p_0)$ then exists nignberhood N of p_0 S.T :

- F map at N as $1:1$ to $F(N)$, $q_0 \in F(N)$.
- on $F(N)$ defined F^{-1}.
- F^{-1} have a continuous deffrential partials with order 1 in $F(N)$.
- $D(F^{-1})|_{q_0} = (DF|_{p_0})^{-1}$.

Example. Look at :

$$F : \left\{ \begin{matrix} y_1 = F_1(x_1, x_2, .., x_n) \\ \vdots \\ y_n = F_n(x_1, x_2, .., x_n) \end{matrix} \right\}$$

$$G : \left\{ \begin{matrix} x_1 = G_1(y_1, y_2, .., y_n) \\ \vdots \\ x_n = G_n(y_1, y_2, .., y_n) \end{matrix} \right\}$$

By proberty 4 of the Theorem $DF \cdot DF^{-1} = 1$

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=1,..,n} \left(\frac{\partial G_i}{\partial y_j}\right)_{i,j=1,..,n} = I$$

We know that :

$$det\left(\frac{\partial F_i}{\partial x_j}\right) = \frac{\partial(F_1, .., F_n)}{\partial(x_1, ..., x_n)}$$

$$\frac{\partial(F_1, .., F_n)}{\partial(x_1, ..., x_n)} \cdot \frac{\partial(G_1, .., G_n)}{\partial(y_1, ..., y_n)} = 1$$

$$\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)} \cdot \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} = 1$$

Part 8. The Implicit Functions Theorem

Abstract. Look at $y^2 - x^2 = 0$ in $(1,1)$ it's obvious that the solution is $y = x$

in neighberhood of $(2, -2)$ the solution is $y = -x$ but what in neighberhood of $(0,0)$ we can't extract $y$in good way.

now look at $x^2 + y^2 + 1 = 0$ we can't extrat y as well Since there is no solution y to this equation.

as collary in order to start talking about extracting a variable in function, we need to find a point which satisfy the implicit function.

Questions.

(1) existance?
(2) unique?
(3) which condition there is to a function when we exract a variable?

Example. Given $f(x,y,z)=0, f:\mathbb{R}^3\to\mathbb{R}^1$, for f there is continuous deffrentials partials with order 1 in $D\subset\mathbb{R}^3$.

Assuming there is point $(x_0,y_0,z_0)\in D$ S.T $f(x_0,y_0,z_0)=0$. Moreover, $\frac{\partial f}{\partial z}(x_0,y_0,z_0)\neq 0$.

hence, in those assumption exist neighberhood of (x_0,y_0) and in this neighberhood exist a unique function $\varphi(x,y)$ S.T $f(x,y,\varphi(x,y))=0$ in this neighberhood and $\varphi(x_0,y_0)=z_0$ in other word $z=\varphi(x,y)$.

Remark. Look at $f(x,y)=x^2-y^2=0$ and in $(x,y)=(0,0)$, here $\frac{\partial f}{\partial y}=-2y|_{(0,0)}=0$ and the Theorem is not valid.

Example. Look at $f(x,y)=x^2+y^2-1=0$ How we can extracy $y=y(x)$?

$$\frac{\partial f}{\partial y}=2y\neq 0,\forall y\neq 0$$

and in neighberhood of $(1,0)$ the Theorem requierments are invalid so there is no extraction/

Proof. (implicit Theorem for 3 variables) .

Define $F:\mathbb{R}^3\to\mathbb{R}^1$,

$$F:\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}u\\v\\w\end{pmatrix}\iff\left\{\begin{array}{c}u=x\\v=y\\w=f(x,y,z)\end{array}\right\}$$

$$F:\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}\to\begin{pmatrix}x_0\\y_0\\f(x_0,y_0,z_0)\end{pmatrix}=\begin{pmatrix}x_0\\y_0\\0\end{pmatrix}$$

Now out target is to use the inerse function Theorem on F Since we add it two coordination to satisfy that it's a transformation $\mathbb{R}^n\to\mathbb{R}^n$. So we need to check the jacobian in order to use the inverse functions Theorem in (x_0,y_0,z_0)

$$J(x_0,y_0,z_0)=\frac{\partial(F_1F_2F_3)}{\partial(x,y,z)}=\left|\begin{array}{ccc}1&0&0\\0&1&0\\f_x&f_y&f_z\end{array}\right|=\frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)}\neq 0$$

Now using the inverse function theorem, F is $1:1$ in (x_0,y_0,z_0) neighberhood and exist F^{-1} from $(x_0,y_0,0)$ to neighberhood (x_0,y_0,z_0) i.e

$$F^{-1}:(x_0,y_0,0)\to(x_0,y_0,z_0)$$

$$F^{-1}:\begin{pmatrix}u\\v\\w\end{pmatrix}\to\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}u\\v\\g(u,v,w)\end{pmatrix}$$

Also for $g(u,v,w)$ there is continuous partial deffrentials with order 1.

$$w=f(x,y,z)=f(u,v,g(u,v,w))$$

in neighberhood of $(x_0,y_0,0)$, it's a identity with u,v,w explicitly $(x,y,0)$

$$0 = f(x, y, \overbrace{g(x, y, 0)}^{\varphi(x,y)})$$

So we found a function in neighberhood of (x_0, y_0) a function $\varphi(x, y) = g(x, y, 0)$ which satisfy the implicit function

$$f(x, y, \varphi(x, y)) = 0$$

Moreover, $\varphi(x_0, y_0) = g(x_0, y_0, 0) = z_0$ and for φ there is continuous partial deffrentials as for g. $\square$

Example. $f(x, y, z) = 0$ a point $f(x_0, y_0, z_0) = 0, f_Z(x_0, y_0, z_0) \neq 0 \Rightarrow$ exist $z = \varphi(x, y)$ in neighberhood $(x_0, y_0), z_0 = (x_0, y_0)$.

Remember :

$$F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x, y, z) \end{pmatrix}$$

We show that $J_F = f_z \neq 0 \Rightarrow$ exist F^{-1} S.T :

$$F^{-1} : \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ g(u, v, w) \end{pmatrix}$$

$$w = f(x, y, z) = f(u, v, g(u, v, w))$$

in neighberhood of $(x, y, z) = (x_0, y_0, z_0)$ and in variables $(u, v, w) = (x_0, y_0, \underbrace{f(x_0, y_0, z_0)}_{0})$

we will take (u, v, w) in a neighberhood $(x_0, y_0, 0)$ the point $(x, y, 0)$, $f(x, y, g(x, y, 0)) = 0$ hence,

$$z = \varphi(x, y) = g(x, y, 0)$$

exactly satisfy that :

$$f(x, y, \varphi(x, y)) = 0$$

everything is in neighberhood of (x_0, y_0)

So we extracted z from $F(x, y, z) = 0$.

Unique

we defined

$$F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ f(x, y, z) \end{pmatrix}$$

explicitly,

$$F : \begin{pmatrix} x \\ y \\ \varphi(x, y) \end{pmatrix} \to \begin{pmatrix} x \\ y \\ f(x, y, \varphi(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

if there was other function $\tilde{\varphi}(x, y)$ which satisfy also $f(x, y, \tilde{\varphi}(x, y) = 0$ we will reset F on $\tilde{\varphi}$:

$$F : \begin{pmatrix} x \\ y \\ \tilde{\varphi}(x,y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x,y,\tilde{\varphi}(x,y)) \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

But F is injectice in the neighberhood of (x_0, y_0) hence,

$$\tilde{\varphi}(x,y) = \varphi(x,y)$$

Since F is C^1, stem also that $\varphi(x,y) = g(x,y,0)$ in neighberhood of (x_0, y_0).

$$f(x,y,\varphi(x,y)) = 0$$

the equality satisfied in neighberhood of (x_0, y_0), $(\varphi, f \in C^1)$ So by the chain rule

$$\frac{\partial}{\partial x} : \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 0 + \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial}{\partial y} : \frac{\partial f}{\partial x} \cdot 0 + \frac{\partial f}{\partial y} \cdot 1 + \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial y} = 0$$

hence, we get that :

$$\frac{\partial \varphi(x,y)}{\partial x} = -\frac{\frac{\partial f(x,y,\varphi(x,y))}{\partial x}}{\frac{\partial f(x,y,\varphi(x,y))}{\partial z}}$$

in (x_0, y_0) we have that :

$$\frac{\partial \varphi(x,y)}{\partial x} = -\frac{f_x(x_0,y_0,z_0)}{f_z(\underbrace{x_0,}_{x_0} y_0, z_0)}$$

$$\frac{\partial \varphi(x,y)}{\partial y} = -\frac{f_y(x_0,y_0,z_0)}{f_z(\underbrace{x_0,}_{y_0} y_0, z_0)}$$

Theorem. *(the implicit functions Theorem).*

given k functions in $k+n$ variables, $i = 1, 2, .., k$, $f_i : \mathbb{R}^{k+n} \to \mathbb{R}$ i.e $f : \mathbb{R}^{k+n} \to \mathbb{R}^k$.

$$k - variables \left\{ \begin{array}{c} f_1(x_1,x_2,..,x_n,y_1...,y_k)=0 \\ \vdots \\ f_k(\underbrace{x_1,x,..,x_n,y_1...,y_k}_{n+k=variables})=0 \end{array} \right\}$$

Given a point $(x_1, x2, .., x_n, y_1..., y_k) = (a_1, a_2, .., x_n, b_1..., b_k)$ which satisfy all the k variables.

Assuming that :

$$\frac{\partial(f_1, ..., f_k)}{\partial(y_1, ...y_k)}|_{(a_1,a_2,..,x_n,b_1...,b_k)} \neq 0$$

Then exist $(\varphi_1, ..., \varphi_k) = \varphi : \mathbb{R}^n \to \mathbb{R}^k$,

$$\left\{\begin{smallmatrix} y_1=\varphi_1(x_1,x_2,..,x_n) \\ \vdots \\ y_k=\varphi_k(x_1,x_2,..,x_n) \end{smallmatrix}\right\}$$

in neighberhood $(x_1,..,x_n)=(a_1,...,a_n)$ *which satisfy the system.*

$$i=1,...,k, f_i(x_1,..,x_n,\varphi_1(x_1,..,x_n),...,\varphi_k(x_1,..,x_n))=0$$

and implies,

$$i=1,...,k,\varphi_i(a_1,..,a_n)=b_i$$

Proof. Define F a map $\mathbb{R}^{n+k}\to\mathbb{R}^{n+k}$

$$F=\left\{\begin{matrix} u_1=x_1 \\ u_2=x_2 \\ \vdots \\ u_n=x_n \\ u_{n+1}=f_1(x_1,x_2,...,x_n,y_1,..,y_k) \\ \vdots \\ u_{n+k}=f_k(x_1,x_2,...,x_n,y_1,..,y_k) \end{matrix}\right\}$$

i.e for $(a_1,...a_n,b_1,...,b_k)\in\mathbb{R}^{n+k}$ F on it give us,

$$F(a_1,...a_n,b_1,...,b_k)=(a_1,...a_n,\underbrace{0,...,0}_{k-times})$$

$$J=det\begin{pmatrix} \begin{bmatrix} 1 & 0 & 0\cdots \\ 0 & \ddots & \cdots \\ 0 & 0 & 1 \end{bmatrix}_{nxn} & \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{kxk} \\ \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}_{kxn} & \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_k} \end{bmatrix}_{kxk} \end{pmatrix}=\frac{\partial(f_1,...f_k)}{\partial(y_1,...y_k)}|_{(a_1,...,a_n,b_1,...,b_k)}\neq 0$$

By the given F is injective in neighberhood of $(a_1,...,a_n,b_1,...,b_k)$ and exist F^{-1},

$$F^{-1}:(a_1,...a_n,\underbrace{0,...,0}_{k-times})\to(a_1,...,a_n,b_1,...,b_k)$$

Moreover $F^{-1}\in C^1$.

$$F^{-1}=\left\{\begin{matrix} x_1=u_1 \\ x_2=u_2 \\ \vdots \\ x_n=u_n \\ y_1=g_1(u_1,...,u_{n+k}) \\ \vdots \\ y_k=g_k(u_1,...,u_{n+k}) \end{matrix}\right\}$$

All in neighberhood of $(a_1, ...a_n, \underbrace{0, ..., 0}_{k-times})$.

Now we plugin into $F \circ F^{-1} = I$ i.e

$$(1.23). u_{n+i} = f_i(x_1, .., x_n, y_1, ..., y_k) = f_i(u_1, .., u_n, g_1(u_1, .., u_{n+k}), ..., g_k(u_1, ...,, u_{n+k})) = 0$$

$$\forall i = 1, 2..., k$$

Explicitly we choose in neighberhood if $(a_1, ..., a_n, 0, ..., 0)$ points in form of $(x_1, x_2, ..., x_n, 0, .., 0)$.

$$0 = f_i(x_1, ..x_n, g_1(x_1, .., x_n, 0, .., 0), .., g_1(x_1, .., x_n, 0, .., 0)) = 0$$

$$\forall i = 1, 2..., k$$

for $(x_1, ..., x_n)$ in neighberhood of $(a_1, .., a_n)$ hence,

$$\left\{ \begin{array}{c} y_1 = \varphi_1(x_1, .., x_n) = g_1(x_1, ..., x_n, 0, ..., 0) \\ \vdots \\ y_k = \varphi_k(x_1, .., x_n) = g_k(x_1, ..., x_n, 0, ..., 0) \end{array} \right\}$$

Moreover,

$$\left\{ \begin{array}{c} \varphi_1(a_1, .., a_n) = b_1 \\ \vdots \\ \varphi_k(a_1, .., a_n) = b_k \end{array} \right\}$$

So we success to extract $y_i = \varphi_i(x_1, .., x_n), i = 1, ..., k$ from the given sytem of k equations and $n+k$ variables in the given point neighberhood, we said that $F \in C^1$ hence, $F^{-1} \in C^1$ so $\varphi_i(x_1, .., x_n) = g_i(x_1, ..x_n, 0, ..., 0)$ is also in C^1.

as before ,

$$F : \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f_1(x_1, ..., x_n, y_1, .., y_n) \\ \vdots \\ f_k(x_1, ..., x_n, y_1, .., y_n) \end{pmatrix}$$

Explicitly,

$$F : \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \varphi_1(x_1, ..., x_n) \\ \vdots \\ \varphi_k(x_1, ..., x_n) \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We have zero by 1.23. now assuming that there is

$$(\varphi_1(x_1, ..., x_n), ..., \varphi_k(x_1, ..., x_n)) \neq (\tilde{\varphi}_1(x_1, ..., x_n), ..., \tilde{\varphi}_k(x_1, ..., x_n))$$

which satisfy the required system i.e

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \tilde{\varphi}_1(x_1,...,x_n) \\ \vdots \\ \tilde{\varphi}_k(x_1,...,x_n) \end{pmatrix} \to \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since F is injective in the local neighberhood we have $\tilde{\varphi}_i = \varphi_i, \forall i = 1, 2..., k$.
We will calculate $\frac{\partial \varphi_i}{\partial x_j}$:

$$f_i(x_1,...,x_m,\varphi_1(x_1,..,x_n),...,\varphi_k(x_1,..,x_n)), i = 1,2,...,k.$$

We will deffrentiate by $\frac{\partial}{\partial x_j}, j = 1, 2..., n$.

$$\frac{\partial f_i}{\partial x_1} \cdot \underbrace{\frac{\partial x_1}{\partial x_j}}_{0} + ... + \frac{\partial f_i}{\partial x_j} \cdot \cdot \underbrace{\frac{\partial x_j}{\partial x_j}}_{1} + ... \frac{\partial f_i}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_j} + \frac{\partial f_i}{\partial y_1} \cdot \frac{\partial \varphi_1}{\partial x_j} + ... + \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial \varphi_k}{\partial x_j} = 0$$

hence,

$$\sum_{l=1}^{k} \frac{\partial f_i}{\partial y_l} \cdot \frac{\partial \varphi_l}{\partial x_j} = -\frac{\partial f_i}{\partial x_j}$$

So it's a matrices multiplication,

$$(\frac{\partial f_i}{\partial y_l})_{kxk} \cdot (\frac{\partial \varphi_l}{\partial x_j})_{kxn} = -(\frac{\partial f_i}{\partial x_j})_{kxn}$$

We know that there is inverse to $(\frac{\partial f_i}{\partial y_l})_{kxk}$ so,

$$D\varphi = (\frac{\partial \varphi_l}{\partial x_j})_{kxn} = -(\frac{\partial f_i}{\partial y_l})^{-1}_{kxn}(\frac{\partial f_i}{\partial x_j})_{kxn}$$

So we found all $\frac{\partial \varphi_l}{\partial x_j}$.
Cramer Rule:
Solution for $A\vec{v} = b$ is given by

$$v_l = \frac{det\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \begin{matrix} \dots \\ \\ \dots \end{matrix} \underbrace{\begin{pmatrix} \vec{b} \end{pmatrix}}_{v_l} \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)}{det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}$$

In our case,

$$\frac{\partial\varphi_l}{\partial x_j}=\frac{det\left(\begin{pmatrix}\frac{\partial f_1}{\partial y_1}\\ \vdots\\ \frac{\partial f_k}{\partial y_2}\end{pmatrix}\begin{matrix}\cdots\\ \\ \cdots\end{matrix}\underbrace{\begin{pmatrix}-\frac{\partial f_1}{\partial x_j}\\ \vdots\\ -\frac{\partial f_k}{\partial x_j}\end{pmatrix}}_{l}\begin{pmatrix}\frac{\partial f_1}{\partial y_k}\\ \vdots\\ \frac{\partial f_k}{\partial y_{k1}}\end{pmatrix}\right)}{det(\frac{\partial f_i}{\partial y_l})}=\frac{\frac{\partial(f_1,\ldots,f_k)}{\partial(y_1,\ldots,x_j,y_{l+1},..,y_k)}}{\frac{\partial(f_1,\ldots,f_k)}{\partial(y_1,\ldots,y_k)}}$$

$\square$

A tangent space to a surface $z=f(x,y)$ if f deffrentiable in (x_0,y_0) then,

$$\triangle z=f(x,y)-f(x_0,y_0)=A(x-x_0)+B(y-y_0)+R$$

when

$$A=\frac{\partial f}{\partial x}(x_0,y_0),B=\frac{\partial f}{\partial y}(x_0,y_0),\triangle z=z-z_0$$

Definition. $z-z_0=f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)$ is called then <u>tangent space</u> to $z=f(x,y)$ in (x_0,y_0).

The surface which is written as implicit funtion $F(x,y,z)=0$ E.g unit ball in $\mathbb{R}^3 F(x,y,z)=x^2+y^2+z^2-r^2=0$ and there is point on it $F(x_0,y_0,z_0)$, and we assume the assumption of the implicit function Theorem i.e $F\in C^1, F(x_0,y_0,z_0)=0$ and at least one og the deffrential partials $\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\neq 0$ in (x_0,y_0,z_0) for instance $\frac{\partial F}{\partial z}|_{(x_0,y_0,z_0)}\neq 0\Rightarrow z=\varphi(x,y)$ in (x_0,y_0) neighberhood and we know that

$$\frac{\partial\varphi}{\partial x}|_{(x_0,y_0)}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}|_{(x_0,y_0)},\frac{\partial\varphi}{\partial y}|_{(x_0,y_0)}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}|_{(x_0,y_0)}$$

So now we can write the tangent space equation

$$z-z_0=\varphi_x(x_0,y_0)\cdot(x-x_0)+\varphi_y(x_0,y_0)\cdot(y-y_0)$$

$$\Downarrow$$

$$z-z_0=\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}\cdot(x-x_0)+\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}\cdot(y-y_0)$$

when we know that $F_z(x_0,y_0,z_0)\neq 0$ hence,

$$F_x(x_0,y_0,z_0)\cdot(x-x_0)+F_y(x_0,y_0,z_0)\cdot(y-y_0)+F_z(x_0,y_0,z_0)\cdot(z-z_0)=0$$

Symmetric in (x,y,z).

Example. $F(x,y,z)=x^2+y^2-z^2=0\Rightarrow z^2=x^2+y^2$ if we intersact it with the plane $y=0$ we get a cone and on it we have the point $(0,0,0)$, here $\{F_x,F_y,F_z\}=\{2x,2x,-2z\}$ in point $(0,0,0)$ which are all 0 in $(0,0,0)$ hence, there is no tangent space in $(0,0,0)$.

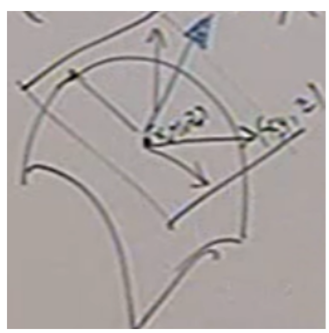
we notice that in $(0,0,0)$ we can't pass tangent space.

In the other hand we can see that for $F = x+y+z = 0$ in $(0,0,0)$ it has a tangent plane "itself" and indeend $\{F_x, F_y, F_z\} = \{1,1,1\} \neq 0$ but $G = (x+y+z)^7 = 0$ is the same plane and on it $(0,0,0)$ we have that $\{G_x, G_y, G_z\} = \{0,0,0\}$ but we do have a tangent space i.e the condtition that at least on of the deffrential partials $\neq 0$ is not neccessiraly but it's a enough condintion.

Given implicit function $F(x,y,z) = 0$ and point on it (x_0, y_0, z_0) so the equation of tangent plane in (x_0, y_0, z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$(F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0) + F_z(x_0,y_0,z_0))(x-x_0, y-y_0, z-z_0) = 0$$



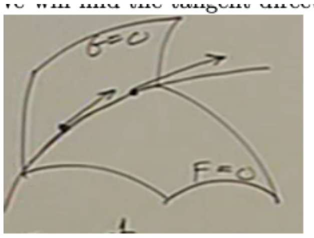
So we can see that the vector $(F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0), F_z(x_0,y_0,z_0))$ is normal to the surface So now we can define what gradiant is (the vector which is normal to surface).

Definition. $(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ = Gradiant of the function $F(x,y,z)$ marked by $\vec{\nabla} F, \vec{grad}(F)$.

Example. Given two surfaces

$$\left\{ \begin{array}{c} F(x,y,z,) = 0 \\ G(x,y,z,) = 0 \end{array} \right\}$$

We will find the tangent direction to the surface intersection curve.



Assumpotion: $F, G \in C^1$**,** (x_0, y_0, z_0) satisfy the system, assuming that $\frac{\partial(F,G)}{\partial(x,y)}|_{(x_0,y_0,z_0)} \neq 0$ we can extract x, y in neighberhood of z_0 i.e

$$\left\{ \begin{array}{c} x = x(z) \\ y = y(z) \end{array} \right\}$$

now look at

$$\left\{ \begin{array}{c} x = x(z) \\ y = y(z) \\ z = z \end{array} \right\}$$

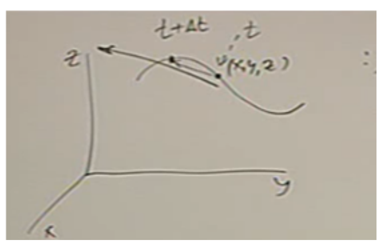
This remind as to the paramertic discribtion of curve,

$$\left\{ \begin{array}{l} x = x(t) \\ y = y(t) \quad , a \leq t \leq b \\ z = z(t) \end{array} \right\}$$

The tangent vector to the following curve in specific point t_0,

$$T = (\frac{dx}{dt}, \frac{dy}{dt}\frac{dz}{dt})|_{t_0}$$

Reminder:



$$(x(t+\triangle t), y(t+\triangle t), z(t+\triangle t)) - (x(t), y(t), z(t)) = (x(t+\triangle t) - x(t), y(t+\triangle t) - y(t), z(t+\triangle t) - z(t))$$

Deviding by $\triangle t$,

$$(\frac{x(t+\triangle t) - x(t)}{\triangle t}, \frac{y(t+\triangle t) - y(t)}{\triangle t}, \frac{z(t+\triangle t) - z(t)}{\triangle t})$$

The limit of the following vector is called the direction of the tangent vector.
Now in our case, we know that z is the parameter,

$$\left\{ \begin{array}{c} x = x(z) \\ y = y(z) \\ z = z \end{array} \right\}$$

Here the direction of tangent vector is

$$(\frac{dx}{dz}, \frac{dy}{dz}, 1)$$

when

$$\frac{dx}{dz} = -\frac{\frac{\partial(F,G)}{\partial(z,y)}}{\frac{\partial(F,G)}{\partial(x,y)}}|_{(x_0,y_0,z_0)} \frac{dy}{dz} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(x,y)}}|_{(x_0,y_0,z_0)}$$

So the tangent vector in the example is,

$$(\frac{dx}{dz}, \frac{dy}{dz}, 1) = (-\frac{\frac{\partial(F,G)}{\partial(z,y)}}{\frac{\partial(F,G)}{\partial(x,y)}}, \frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(x,y)}}, 1)$$

Now we can multiplie by $\frac{\partial(F,G)}{\partial(x,y)}$ but we still has $-$ So we can fix it by subsituted rows in Jacaobian i.e $\frac{\partial(F,G)}{\partial(z,y)} = -\frac{\partial(F,G)}{\partial(y,z)}$ hence the tangent vector is given by the following formula,

$$(\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(z,y)}) = \begin{vmatrix} i & j & k \\ Fx & F_y & F_Z \\ G_x & G_y & G_Z \end{vmatrix} = i\cdot \begin{vmatrix} F_y & F_Z \\ G_y & G_z \end{vmatrix} + j\cdot \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix} + k\cdot \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}$$

$$= (F_x, F_y, F_z)X(G_x, G_y, G_Z) = \vec{\nabla}F \cdot \vec{\nabla}G = \vec{N_1}x\vec{N_2}$$

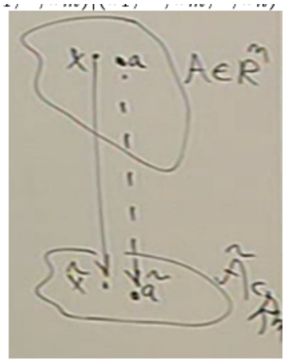
Part 9. **The open map Theorem**

Theorem. *(The open maps Theorem).*
$F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ A is open set, $F \in C^1$ and $rank(dF) = rank(\frac{\partial f_i}{\partial x_j})_{\substack{i=1,..,m \\ j=1,..,n}} = m$, for each point of the set then $F(A)$ opened in $\mathbb{R}^m$.

Proof. if $m = n$ then it's trivial. Since we have matrix nxn which is invertible hence, it's det$\neq 0$ then we finished. now we look at $m < n$.every point in $F(A)$ is in form $F(a)$ and we will show that for every point $F(a)$ there is neighberhood which is obtained in $F(A)$. By the given there is minor $mxm \neq 0$ and without loose of generality we assume that $det(\frac{\partial f_i}{\partial x_j})_{i,j=1,..,m} \neq 0$

$$\begin{array}{c} x_1 \to \\ \vdots \\ x_n \to \end{array} \begin{pmatrix} \begin{bmatrix} f_1 & \cdots \\ & \end{bmatrix} & & f_n \\ & & \end{pmatrix}$$

now take a point $a = (a_1, ..., a_m, .., a_n) \in \mathbb{R}^n$ now we look at the first m cooardination and we denote it by $\tilde{a} = (a_1, .., a_m)$, now every vector $x = (x_1, ..., x_m, .., x_n)$ and we denote $\tilde{x} = (x_1, .., x_m)$ so we can do the same for A we can define $\tilde{A} = \{\tilde{x} = (x_1, .., x_m) | (x_1, ..., x_m, .., x_n) \in A\}$



In neighberhood of $\tilde{a}$ in $\mathbb{R}^m$ we define $G : \mathbb{R}^m \to \mathbb{R}^m$ when,

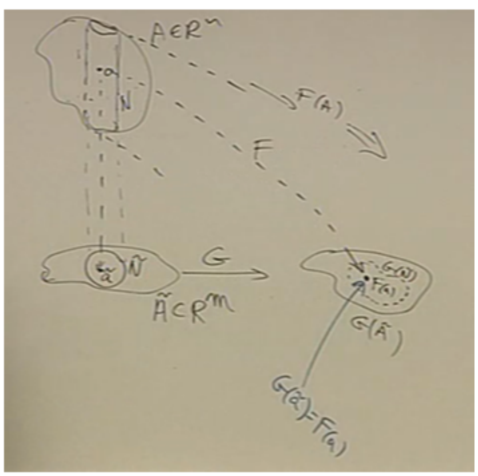
$$G(x_1, .., x_m) = F(x_1, .., x_m, a_{m+1}, .., a_n)$$

explicitly,

$$G(\tilde{a}) = G(a_1, .., a_n) = F(a_1, .., a_m, a_{m+1}, .., a_n) = F(a)$$

$$DG|_{\tilde{a}} = \underset{i,j=1,..,m}{\left(\frac{\partial g_i}{\partial x_j}\right)}|_{\tilde{a}} = \underset{i,j=1,..,m}{\left(\frac{\partial fi}{\partial x_j}\right)}|_{\tilde{a}} \neq 0$$

So we conclude that G is $1:1$ hence, for $\tilde{a} \in \mathbb{R}^m$ there is open neighberhood $\tilde{N} \subset \tilde{A}$ in which G is invertible.



Now define

$$N = \{x | (x_1, .., x_n) | \begin{matrix} (x_1, .., x_n) \in \tilde{N} \\ (x_1, ..., x_m, .., x_n) \in A \end{matrix} \}$$

$$F(A) \supseteq F(N) = \{F(x_1, ..., x_m, .., x_n) | x \in N\} \supseteq F(x_1, .., x_m, a_{m+1}, .., a_n) | \tilde{x} \in \tilde{N}\}$$

Notice that

$$\{F(x_1, .., x_m, a_{m+1}, .., a_n) | \tilde{x} \in \tilde{N}\} = \{G(\tilde{x}) | \tilde{x} \in \tilde{N}\} = G(\tilde{N})$$

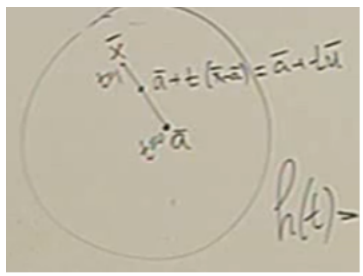
So we are finished. $\square$

Part 10. **Taylor Formula**

let $f:\mathbb{R}^n\to\mathbb{R}^n$, f has a continuous deffrential partial with otder $k+1$ in neighberhood of

$$a=(a_1,..,a_n),x=(x_1,..,x_n)$$

$$u=x-a=(x_1-a_1,..,x_n-a_n),u=(u_1,..,u_n)$$



$$h(t)=f(a+tu)=f(a_1+tu_1,...,a_n+tu_n)$$

AS in one variable we know that;

$$h(t)=h(0)+\frac{h'(0)}{1!}t^1+...+\frac{h^{(k)}(0)}{k!}t^k+R_k$$

(a_

$$R_k=\frac{h^{(k+1)}(c)}{(k+1)!}t^{k+1},0<c<t$$

Now notice that

$$h(0)=a$$

$$h'(t)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a+tu)\underbrace{\frac{dx_i}{dt}}_{u_i}=\sum_{i=1}^{n}f_{x_i}(a+tu)u_i=\vec{\nabla}f|_{a+tu}\cdot u$$

$$h'(0)=\sum\frac{\partial f}{\partial x_i}|_a\cdot u_i=\vec{\nabla}f(a)\cdot u$$

$$h"(t)=\frac{d}{dt}(\sum_{i=1}^{n}f_{x_i}(a+tu)\cdot u_i)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}u_ju_i$$

$$h"(0)=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(a)u_iu_j=u^tHu$$

This matrix is a quartar form i.e

$$H = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Since

$$f_{x_i x_j} = f_{x_j x_i}$$

We conclude the matrix is symmetric.

$$h^{(r)}(t) = \sum_{1 \le i_1, .., i_r \le n} \frac{\partial^t f(a+tu)}{\partial x_{i_1} \partial x_{i_2}, ..., \partial x_{i_r}} u_{i_1} u_{i_2} ... u_{i_r}$$

$$= \sum_{\substack{r_1 + r_2 + ... + r_n = r \\ 0 \le r_i}} \frac{r!}{r_1! r_2!, , , r_n!} \frac{\partial^t f(a+tu)}{\partial x_1^{r_1} \partial x_2^{r_2}, ..., \partial x_n^{r_n}} u_1^{r_1} ... u_n^{r_n}$$

using the mulitibinomial formula which is,

$$(u_1 + ... + u_n)^r = \sum_{r_1 + r_2 + ... + r_n = r} \frac{r!}{r_1! r_2!, , , r_n!} u_1^{r_1} ... u_n^{r_n}$$

$$(u_1 + u_2)^r = \sum_{0 \le r_2 \le r_2} \frac{r!}{r_1!(r - r_1)!} u_1^r u_2^{r - r_1}$$

So now we can use that and get

$$\sum_{\substack{r_1 + r_2 + ... + r_n = r \\ 0 \le r_i}} \frac{r!}{r_1! r_2!, , , r_n!} \frac{\partial^t f(a+tu)}{\partial x_1^{r_1} \partial x_2^{r_2}, ..., \partial x_n^{r_n}} u_1^{r_1} ... u_n^{r_n} = (u_1 \cdot \frac{\partial}{\partial x_1} + u_2 \cdot \frac{\partial}{\partial x_2} + .. + u_n \cdot \frac{\partial}{\partial x_n})^r f|_{(a+tu)}$$

Now we will back to f:

$$f(x) = f(a) + \frac{1}{1!} \sum_{i=1}^{n} \frac{\partial f(a)}{\partial x_i}(x_i - a_i) + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f(a)}{\partial x_i \partial x_j}(x_i - a_i)(x_j - a_j) + ...$$

Example. Multivariable function,

$$f(x,y) = f(a,b) + [\frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b)] + \frac{1}{2!}[f_{x,x}(a,b)(x-a)^2 + 2f_{x,y}(a,b)(x-a)(y-b)$$

$$+ f_{y,y}(a,b)(y-b)^2] + \frac{1}{3!}[..]....$$

Part 11. Extremum Problems

$f : A \subset \mathbb{R}^n \to \mathbb{R}^n$ for f there is local $\underline{\text{maximum}}$ $\tilde{x} \in A$ if for $\tilde{x}$ there is neighberhood which obtained in A S.T, $f(\tilde{x}) \geq f(x), \forall x \in U$. if $\tilde{x} \in A$ but not interior i.e $x \in \partial A$ then similar but $\forall x \in U \cap A$.

Theorem. *(A important condition for extremum).*
if $f \in C^1$ in set A and it has a Maxima Or minima in $\tilde{x}$ (interior point) then a neccessary condition that $\frac{\partial f}{\partial x_i}(\tilde{x}) = 0. i = 1, 2, ..., n$ i.e $\vec{\nabla} f(\tilde{x}) = (f_{x_1}, ..., f_{x_n})(\tilde{x}) = \vec{0}$.

Proof. recall $\tilde{x} = (\tilde{x_!}, \tilde{x_2} ..., \tilde{x_n})$, $g(x_1) = f(x_1, \overbrace{\tilde{x_2}, ..., \tilde{x_n}}^{parameters})$ then for $g(x_1)$ there is extremum for $x_1 = \tilde{x}_1$ and from calculus 1 for a function with single variable, a neccessary condition to extremum is $\frac{\partial g}{\partial x_1}|_{\tilde{x_1}} = 0$ and the same for $x_2, .., x_n$. □

Remark. a neccessary codition is not enough E.g $f(x, y, z) = x \cdot y \cdot z$ $f_x = yz$ $f_y = xz$ $f_z = xy$ now,

$$0 = f_x(0, 0, 0) = f_y(0, 0, 0) = f_z(0, 0, 0)$$

But in $(0, 0, 0)$ there is no extremum.

Definition. a point $\tilde{x}$ in which $\vec{\nabla} f(\tilde{x}) = 0$ is called stationary.

Theorem. *(a enough condition for extremun for maps $f : \mathbb{R}^2 \to \mathbb{R}$) .*
let $f : A \subset \mathbb{R}^2 \to \mathbb{R}^2$, A is open set, $\tilde{x} \in A$, f has a continuous deffreintial partials withb order 2, $\nabla f(\tilde{x}) = \vec{0}$.

$$A = f_{x_1 x_1}(\tilde{x}), B = f_{x_1 x_2}(\tilde{x}). C = f_{x_2 x_2}(\tilde{x})$$

$$\triangle = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$$

Then

$$H = \begin{vmatrix} f_{x_1 x_1} & f_{x_1 x_2} \\ f_{x_2 x_1} & f_{x_2 x_2} \end{vmatrix} (\tilde{x})$$

in those assumption,

(1) if $\triangle > 0$ then for $\tilde{x}$ there is local extremum. 1.1. if $A > 0$ then local minimum. 1.2. if $A < 0$ then local maximum.($A = 0$ not possible Since, $AC - B^2 > 0$).
(2) $\triangle < 0$ for f there is no extremum in $\tilde{x}$. (saddle point).
(3) $\triangle = 0$ we can't decide in those points.

Proof. Taylor formula for $f \in C^2$,

$$f(x) = f(a) + \frac{1}{1!}\sum_{i=1}^{n} \frac{\partial f(a)}{\partial x_i}(x_i - a_i) + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f(a)}{\partial x_i \partial x_j}(x_i - a_i)(x_j - a_j)$$

Now using the Taylor formula we get,

$$f(x) = f(\tilde{x}) + \frac{1}{1!}[\underbrace{\frac{\partial f}{\partial x_1}}_{0}(\tilde{x})(x_1 - \tilde{x}_1) + \underbrace{\frac{\partial f}{\partial x_2}}_{0}(\tilde{x})(x_2 - \tilde{x}_2)] + \frac{1}{2!}[f_{x_1x_1}(c)(x_1 - \tilde{x_1})^2$$

$$+2f_{x_1x_2}(c)(x_1 - \tilde{x}_1)(x_2 - \tilde{x}_2) + f_{x_2x_2}(c)(x_2 - \tilde{x}_2)^2]$$

when c is between x and $\tilde{x}$.

Now in case $1, 2$ we have that,

$$\triangle = f_{x_1x_1} \cdot f_{x_1x_2} - f^2_{x_1x_2}|_{\tilde{x}} \neq 0$$

by taking a small neighberhood U of $\tilde{x}$ in which,

$$sign(f_{x_1x_1} \cdot f_{x_1x_2} - f^2_{x_1x_2}) = sign(AC - B^2) \neq 0$$

Moreover,

$$sign(f_{x_1x_1}) - sign(A) \neq 0$$

$$f(x) - f(\tilde{x}) = \frac{1}{2}[\underbrace{\alpha}_{f_{x_1x_1}(c)}\underbrace{u^2}_{x_1 - \tilde{x}_1} + 2\beta uv + \gamma v^2]$$

Now in order to show that we have maxima or minima we need to show that $f(x) > f(\tilde{x})$ i.e we need to determine the sign of the Quartar Linear form $[\underbrace{\alpha}_{f_{x_1x_1}(c)}\underbrace{u^2}_{x_1 - \tilde{x}_1} + 2\beta uv + \gamma v^2]$.

$$[\alpha u^2 + 2\beta uv + \gamma v^2] = \alpha[u^2 + 2\frac{\beta}{\alpha}uv + \frac{\gamma}{\alpha}v^2]$$

when $f_{x_1x_1}(c)$ sign is the same as A, $\neq 0$.

$$[\alpha u^2 + 2\beta uv + \gamma v^2] = \alpha[u^2 + 2\frac{\beta}{\alpha}uv + \frac{\gamma}{\alpha}v^2] = \alpha[(u + \frac{\beta}{\alpha}v)^2 - \frac{\beta^2}{\alpha^2}v^2 + \frac{\gamma}{\alpha}v^2]$$

$$= \alpha[(u + \frac{\beta}{\alpha}v)^2 - \frac{\beta^2 - \alpha\gamma^2}{\alpha^2}v^2]$$

if $AC - B^2 > 0$ then $\alpha\gamma - \beta^2 > 0$ and,

CASE 1:

$A > 0 \Rightarrow \alpha > 0 \Rightarrow f(x) - f(\tilde{x}) > 0$ in neighberhood hence, we have minimum.if $\alpha < 0$ we have maximim since, $f(x) < f(\tilde{x})$.

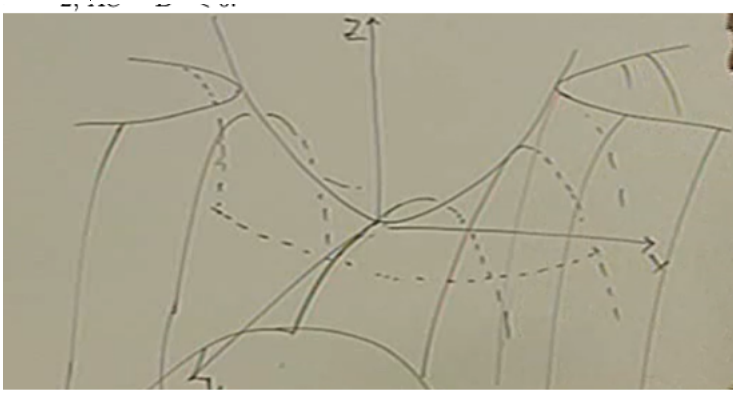
CASE 2:

$AC - B^2 < 0, \alpha\gamma - \beta^2 < 0$ i.e

$$(u + \frac{\beta}{\alpha}v)^2 - \frac{\beta^2 - \alpha\gamma^2}{\alpha^2}v^2$$

in this case, we can determine the sign Since, $E.g$ for the line $v = 0$ we still we $(u+\frac{\beta}{\alpha}v)^2 > 0$ and we can take the line $u = -\frac{\beta}{\alpha}v$, we still with $(negative{-}term)\cdot v^2 < 0$ So the collary is that we can't decide if it's maximum or minimum. $\square$

Example. look at $z = f(x,y) = x^2 - y^2$ $(0,0)$ is a stationary point, $A = 2, B = 0, C = -2, AC - B^2 < 0$.



This is a saddle point nor maximum or minimum.

Example. $f(x,y) = x^2y^7$ now notice that $(0,0)$ is a stationarty point $A = 0, B = 0, C = 0$ then $AC - B^2 = 0$ **BUT** $(0,0)$ is not extremum, now $f(x,y) = x^2y^8$ and $AC - B^2 = 0$ and $(0,0)$ is minimum.

in n variables:

$$x = (x_1, .., x_n)$$

$$\tilde{x} = (\tilde{x}_1, .., \tilde{x}_n)$$

$$f(x) = f(\tilde{x}) + \nabla f(x) \cdot (x - \tilde{x}) + \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(z)(x_i - \tilde{x}_i) \cdot (x_j - \tilde{x}_j)$$

$$\overbrace{R_2}$$

when z is a point between $\tilde{x}$ and x. if $\tilde{x}$ is a stationarty point then $\vec{\nabla} f|_{\tilde{x}} = 0$ hence,

$$f(x) - f(\tilde{x}) = Hu \cdot u = \vec{u^T} H \vec{u}$$

When H is the Hessian matrix calculated in a point z, Moreover, $\vec{u} = (x - \tilde{x})$.

Corollary. *f has minimum in the stationarty point $\tilde{x}$ if $\vec{u^T} H \vec{u}$ positive $\forall \vec{u}$ i.e*

$$H = (\frac{\partial^2 f}{\partial x_i \partial x_j})(z)$$

H is symmertic matrix, and those conditions are equivalent:

(1) $\vec{u^T} H \vec{u} > 0 \ \forall \vec{u} \neq 0$.
(2) $\lambda_1, .., \lambda_n$ are positive eigenvalues.
(3) Selbester: All possible minors are positive i.e

$$h_{11} > 0, \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} > 0, ..., \begin{vmatrix} h_{11} & & \\ & \ddots & \\ & & h_{nn} \end{vmatrix} > 0$$

Remark. <u>Max</u>: H is defined negative $\iff$ $-H$ defined positive. Since, f has maximum $\iff$ $-f$ has minimum.

Part 12. Extremum With Restrections

Find a critical points for $f(x,y)$ on the curve point $g(x,y) = 0$.

Example. Find a point on the surface $z = x^2 - y^2$ which is the most close to the point $(3,4,5)$.

$$\left\{ \begin{array}{c} (x-3)^2 + (y-4)^2 + (z-3)^2 \overset{?}{=} min(target - function). \\ x^2 - y^2 - z = 0(restriction). \end{array} \right\}$$

In general,

$$\left\{ \begin{array}{c} extr, f(x_1, .., x_n) \\ g(x_1, .., x_n) = 0 \end{array} \right\}$$

Example. find $\max\{x^2 + y^3 - 7xy\}$ in the set $x^2 + y^2 \leq 1$.

Solution. in the interior $x^2 + y^2 < 1$ we will search for a local maximum by,

$$\left\{ \begin{array}{c} (x^2 + y^2 - 7xy)_x = 0 \\ (x^2 + y^2 - 7xy)_y = 0 \end{array} \right\} Stationary - points, and - etc..$$

On the boundry,

$$\left\{ \begin{array}{c} max(x^2 + y^2 - 7xy) \\ x^2 + y^2 = 1 \end{array} \right\}$$

..

Part 13. Lagrange Multipliers Method

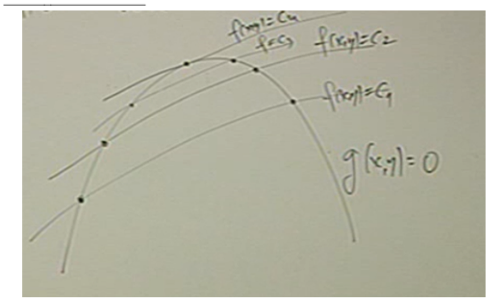
Theorem. *Given functions $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$, A is open set, $f, g \in C^1$ if to the following problem:*

$$\left\{ \begin{array}{c} extr, f(x_1, .., x_n) \\ g(x_1, .., x_n) = 0 \end{array} \right\}$$

There is extremum Maxima or Minima in $\tilde{x} \in A$ and if $\vec{\nabla} g(\tilde{x}) \neq 0$, the $\exists \lambda$ S.T

$$\left\{ \begin{array}{c} \frac{\partial}{\partial x_i}(f - \lambda g)(\tilde{x}) = 0, i = 1, 2.., n \\ g(\tilde{x}) = 0 \end{array} \right\}$$

"**<u>Geometric Motivation</u>**":



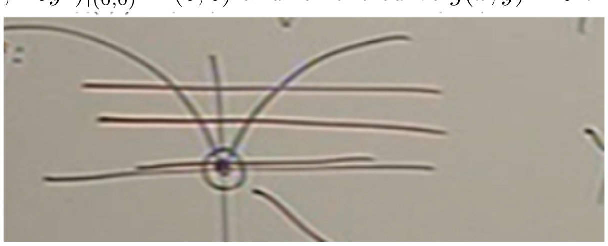
Now notice that we can use the Implicit Function Theorem hence, $g(x, y) = 0$ and we can extract $y = y(x) \Rightarrow \frac{dy}{dx} = -\frac{g_x}{g_y}$, $f(x, y) - C = 0 \Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$ Notice that they are tangent in the tangent point which is specious to be max or min i.e $-\frac{f_x}{f_y} = -\frac{g_x}{g_y}$ in the tangent point Denote $\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}|_{(\tilde{x}, \tilde{y})}$ So,

$$\left\{ \begin{array}{c} (f - \lambda g)_x = 0 \\ (f - \lambda g)_y = 0 \end{array} \right\}$$

Remark. Without the assumption $\vec{\nabla} g(\tilde{x}) \neq 0$ it's not true. for counter example:

$$\left\{ \begin{array}{c} f(x, y) = y \\ g(x, y) = x^2 - y^3 = 0 \end{array} \right\}$$

Our restriction is that $y = \sqrt[3]{x^2}$ as in the photo we can see that y get the minimal value in $(0, 0)$ but the method of lagrange multipliers is not working $\nabla g = (2x, -3y^2)|_{(0,0)} = (0, 0)$ and for the curve $g(x, y) = 0$ there is no tangent in $(0, 0)$.



Proof. Assuming that $extr f(x_1, .., x_n)$ is in $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)$ Moreover, $\bar{\nabla} g|_{\tilde{x}} \neq \bar{0}$.

Assuming without loose of generality that $\frac{\partial g}{\partial x_n}|_{\tilde{x}} \neq 0$ So we can extract x_n By the Implicit Function Theorem, we extract it from $g(x_1, .., x_n) = 0$ the variable

x_n Moreover, $x_n = h(x_1,..,x_{n-1})$ in the neighberhood of the following point i.e $g(x_1,..,x_{n-1},h(x_1,..,x_{n-1})) = 0$ in neighberhood of $(\tilde{x}_1,...,\tilde{x}_{n-1})$, now the target function, $u(x_1,..,x_{n-1}) = f(x_1,..,x_{n-1},h(x_1,..,x_{n-1})) \overset{?}{=} extr$ by the assumption for this there is extrmum in $(\tilde{x}_1,...,\tilde{x}_{n-1})$ hence,

$$\left\{ \begin{array}{c} \underset{i=1,2..,n-1}{\frac{\partial}{\partial x_i}u(\tilde{x})} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_n}\cdot\frac{\partial h}{\partial x_i}|_{\tilde{x}} = 0 \\ \frac{\partial g}{\partial x_i} + \frac{\partial g}{\partial x_n}\cdot\frac{\partial h}{\partial x_i}|_{\tilde{x}} = 0 \end{array} \right\} \Rightarrow \frac{\partial h}{\partial x_i} = -\frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_n}}$$

Now we plug in in the first equation and get that:

$$\left\{ \begin{array}{c} \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_n}\cdot(-\frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_n}})|_{\tilde{x}} = 0, i` = 1,2,..,n-1 \\ \frac{\partial g}{\partial x_i} + \frac{\partial g}{\partial x_n}\cdot\frac{\partial h}{\partial x_i}|_{\tilde{x}} = 0 \end{array} \right\}$$

Now denote $\lambda = -\frac{\frac{\partial f}{\partial x_n}}{\frac{\partial g}{\partial x_n}}$ hence,

$$\underset{i=1,2..,n-1}{\frac{\partial f}{\partial x_i}} - \underbrace{(\frac{\frac{\partial f}{\partial x_n}}{\frac{\partial g}{\partial x_n}}|_{\tilde{x}})}_{\lambda}\frac{\partial g}{\partial x_i}|_{\tilde{x}} = 0 \Rightarrow \frac{\partial}{\partial x_i}(f-\lambda g)(\tilde{x}) = 0, i = 1,2..,n-1$$

Notice that in case $i = n$ it's trivial and still working Since everything vanish

$$\frac{\partial f}{\partial x_n} - \underbrace{(\frac{\frac{\partial f}{\partial x_n}}{\frac{\partial g}{\partial x_n}})}_{\lambda}\frac{\partial g}{\partial x_n}|_{\tilde{x}} = \frac{\partial f}{\partial x_n} - \frac{\partial f}{\partial x_n} = 0$$

□

Example. find extr $Q(x_1,..,x_n) = \sum_{i=1}^{n} a_{ij}x_ix_j$ when ($a_{ij} = a_{ji}$ symmetric) under the restriction $x_1^2 + ...x_n^2 = 1$ i.e on the unit ball boundry.

Solution. Denote $g(x_1,..x_n) = \sum_{i=1}^{n} x_i^2 - 1 = 0$ Notice that Q continuous Since, by Weierstrass Theorem there is maxima and minima hence, we can find them by lagrange multipliers.

$$Q = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i\neq j} a_{ij}x_ix_j$$

$$\frac{\partial Q}{\partial x_k} = 2a_{kk}x_k + \sum(a_{kj}x_j + a_{ik}x_i) = 2[a_{kk}x_k + \sum_{j\neq k} a_{ij}x_j] = 2\sum_{j=1}^{n} a_{ji}x_j$$

$$[\frac{\partial Q}{\partial x_k} - \lambda g] = 2[\sum_{j=1}^{n} a_{ji}x_j - \lambda x_k]_{k=1,2,..,n} = 0 \Rightarrow \sum_{j=1}^{n} a_{ji}x_j = \lambda x_k, k = 1,2,..,n$$

it's identical to:

$$A\vec{x} = \lambda\vec{x}, \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix}$$

hence $\lambda = eigenvector, (x_1,..,x_n) = eigenvector, \sum x_i^2 = 1$ i.e it's not only a eigenvector also it's a unit vector. in the following point, the value of $Q(...)$ is

$$Q(\tilde{x}) = \sum_{i,j=1}^{n} a_{ij}x_ix_j = \sum_{i=1}^{n}(\underbrace{\sum_{j=1}^{n} a_{ij}x_j}_{\lambda x_i})x_i = \sum_{i=1}^{n} \lambda x_i^2 = \lambda(\underbrace{\sum_{i=1}^{n} x_i^2}_{1}) = \lambda$$

So the minima of Q defined as $minQ = min(\lambda_i)_{i=1}^k$ when $\{\lambda_i\}_{i=1}^k$ is the set of eigenvalues of A and the max defined in identical way i.e $maxQ = max(\lambda_i)_{i=1}^k$.

Part 14. Some Inequalities

Young Inequality: Given that $\frac{1}{p} + \frac{1}{q} = 1, \forall x, y \geq 0, p, q > 1$ Show that:

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy$$

Proof. We need to find $min(\frac{x^p}{p} + \frac{y^q}{q})$ under the restriction $xy = k$ $x, y > 0$. from Geometric propistion i'ts obvious the there is minimum on the line $xy = k$, we will search for it using the Lagrange Multipliers:

$$f - \lambda g = (\frac{x^p}{p} + \frac{y^q}{q}) - \lambda(xy - k)$$

In extremum point:

$$\left\{ \begin{array}{c} x^{p-1} - \lambda y = 0 \\ y^{q-1} - \lambda x = 0 \\ xy = k \end{array} \right\}$$

Hence,

$$x^p = \lambda xy, y^q = \lambda xy, x^p = y^q = \lambda xy = \lambda k \Rightarrow x = (\lambda k)^{\frac{1}{p}}, y = (\lambda k)^{\frac{1}{q}}$$

So the point $(x, y) = ((\lambda k)^{\frac{1}{p}}, (\lambda k)^{\frac{1}{q}})$ shoud be minimum and we got that:

$$k = xy = (\lambda k)^{\frac{1}{p}} \cdot (\lambda k)^{\frac{1}{q}} = \lambda^{\frac{1}{p}+\frac{1}{q}} \cdot k^{\frac{1}{p}+\frac{1}{q}} \Rightarrow \lambda = 1$$

$$\frac{x^p}{p} + \frac{y^q}{q} \geq min(\frac{x^p}{p} + \frac{y^q}{q}) = \frac{k}{p} + \frac{k}{q} = k(\frac{1}{p} + \frac{1}{q}) = k = xy$$

□

Remark. We should the folloing only in specefic line xy which is quartar of the plane since xy cover it, So notice that we required that $x, y > 0$ so indeed we should the following on the required part of the plane Moreover, for $x, y = 0$ it's trivial.

Holder Inequality: let $0 \leq u_1, .., u_n, v_1, .., v_n$ and define $x = \frac{u_i}{(\sum\limits_{j=1}^{n} u_j^p)^{\frac{1}{p}}}, y = \frac{v_i}{(\sum\limits_{j=1}^{n} u_j^q)^{\frac{1}{q}}}$ and we plug in into $\frac{x^p}{p} + \frac{y^q}{q} \geq xy$ hence we get that:

$$\frac{u_i}{(\sum\limits_{j=1}^{n} u_j^p)^{\frac{1}{p}}} \cdot \frac{v_i}{(\sum\limits_{j=1}^{n} u_j^q)^{\frac{1}{q}}} \leq \frac{1}{p} \cdot \underbrace{\frac{u_i^p}{(\sum\limits_{j=1}^{n} u_j^p)}}_{x^p} + \frac{1}{q} \cdot \underbrace{\frac{v_i^q}{(\sum\limits_{j=1}^{n} v_j^p)}}_{y^q}$$

Now we sum $i = 1, 2, .., n$

$$\frac{\sum u_i v_i}{(\sum\limits_{j=1}^{n} u_j^p)^{\frac{1}{p}} (\sum\limits_{j=1}^{n} v_j^q)^{\frac{1}{q}}} \leq \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = 1 \Rightarrow \sum u_i v_i \leq (\sum_{j=1}^{n} u_j^p)^{\frac{1}{p}} (\sum_{j=1}^{n} v_j^q)^{\frac{1}{q}}$$

Claim. Given $\frac{1}{p} + \frac{1}{q} = 1$ $p, q > 0$ then:

$$\sum_{i=1}^{n} u_i v_i \leq (\sum_{j=1}^{n} u_j^p)^{\frac{1}{p}} (\sum_{j=1}^{n} v_j^q)^{\frac{1}{q}}, u_i, v_i \geq 0$$

$$\vec{u} \cdot \vec{v} \leq ||\vec{u}||_p ||\vec{v}||_q, \frac{1}{p} + \frac{1}{q} = 1$$

Notice that for $p = 2, q = 2$ we get that Caushy-Schwartz Inequality.

Minkowski Inequality: Given $p > 1$,

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq_* \sum_{i=1}^{n} (|a_i| + |b_i|)|a_i + b_i|^{p-1} = \sum_{i=1}^{n} \underbrace{|a_i|}_{u_i} \cdot \underbrace{|a_i + b_i|^{p-1}}_{v_i} + \sum_{i=1}^{n} |b_i| \cdot |a_i + b_i|^{p-1}$$

$$\leq_{**} (\sum |a_i|^p)^{\frac{1}{p}})(\sum (|a_i + b_i|^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}}) + (\sum |b_i|^p)^{\frac{1}{p}})(\sum (|a_i + b_i|^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}})$$

Notice that :

$$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

hence,

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq_* (\sum |a_i|^p)^{\frac{1}{p}})(\sum(|a_i + b_i|^p)^{1-\frac{1}{p}}) + (\sum |b_i|^p)^{\frac{1}{p}})(\sum(|a_i + b_i|^p)^{1-\frac{1}{p}})$$

$$\Rightarrow \sum(|a_i + b_i|^p)^{\frac{1}{p}} \leq (\sum(|a_i|^p)^{\frac{1}{p}} + (\sum(|b_i|^p)^{\frac{1}{p}}$$

where in $*$ we used the triangle Inequality Moreover, in $**$ we used the holder Inequality.

Corollary. *We conclude that:*

$$\begin{Bmatrix} ||\vec{a} + \vec{b}||_p \leq ||\vec{a}||_p + ||\vec{b}||_p \\ p > 1 \end{Bmatrix}$$

Example. Find $max\{x^2 + y^2 - 12x + 16y\}$ into the set, $\begin{Bmatrix} x^2 + y^2 \leq 1 \\ 3x + y \geq 0 \end{Bmatrix}$.

Solution. We will search in steps:
Step 1:
We will find into the interior of the set,

$$\begin{Bmatrix} f_x : 2x - 12 = 0 \\ f_y : 2y + 16 = 0 \end{Bmatrix}$$

So $(6, -8)$ is not in the our domain hence, there is no local extremum.
Step 2:
We need to search for extremum on the boundy $x^2 + y^2 = 1$ i.e

$$\begin{Bmatrix} max\{x^2 + y^2 - 12x + 16y \\ g = x^2 + y^2 - 1 = 0 \end{Bmatrix}$$

We will solve it using Lagrange multipliers:

$$\begin{Bmatrix} 2x - 12 - \lambda \cdot 2x = 0 \\ 2y + 16 - \lambda \cdot 2y = 0 \\ x^2 + y^2 = 1 \end{Bmatrix}$$

Step 3:
maybe the max is on the line $3x + y = 0$ So it's again Lagrange multipliers problem.

$$\begin{Bmatrix} 2x - 12 - \lambda \cdot 3 = 0 \\ 2y + 16 - \lambda \cdot 1 = 0 \\ 3x + y = 0 \end{Bmatrix}$$

Step 4:
We will search of non deffrentiable points on the bounrdy (i.e the end) the values of the function, and between them we can identefy the global extremum.

Part 15. integration in n variables.

let D bounded set obtained in a box

$$B=\left\{(x_1,...,x_n)\}|\begin{array}{c}a_1\leq x_1\leq b_1\\ \vdots\\ a_n\leq x_n\leq b_n\end{array}\right\}$$

we can devide the box into small boxes i.e

$$B_{ijk...}=\left\{\begin{array}{c}x_{1,i}\leq x_1\leq x_{1,i+1}\\ x_{2,j}\leq x_2\leq x_{2,j+1}\\ \vdots\end{array}\right\}$$

.

(1) there is boxes which obtain interior point of D .§
(2) there is boxes which obtain the interior point of $\mathbb{R}^n\backslash D$.
(3) all the other point are a boundry point i.e in the ∂D.

Definition. we will define some trems.

(1) $\underline{S}(P,D)=$ sum of the boxes volume which are obtained in D.
(2) $\bar{S}(P,D)=$ sum of boxes volume which obtain D.

Remark. $\bar{S}\geq\underline{S}$ and we look into the $\bar{S}=\underset{P}{inf}\bar{S}(P,D),\underline{S}=\underset{P}{sup}\underline{S}(P,D)$.

Definition. if $\bar{S}=\underline{S}$ we will say the D has volume. and this constant is the volume of D, $V(D)$.

Definition. a set is called "volume 0" if $\forall\epsilon>0$ exist a finite set of boxes which cover that set and the sum of volumes $<\epsilon$.

Mark : ∂D is the boundry of the set D.

$$\bar{S}(P,\partial D)=\bar{S}(P,D)-\underline{S}(P,D)$$

$$inf\bar{S}(P,\partial D)=inf(\bar{S}-\underline{S})\geq inf\bar{S}(P,D)-sup\underline{S}(P,D)=\bar{V}(D)-\underline{V}(D)$$

Corollary. *if ∂D has a volume 0 then $\bar{V}(D)-\underline{V}(D)$ therfore, D has volume.*

other direction we assume that D has volume, $\bar{V}(D)=\underline{V}(D)$, $\bar{V}=\underset{P}{inf}\bar{S}(P,D)$ i.e $\forall\epsilon>0$ there is P_1 S.T:

$$\bar{V}(P_1,D)+\epsilon>\bar{S}(P_1,D)\geq\bar{V}(P_1,D)$$

$\underline{V}=\underset{P}{sup}\underline{S}$ i.e $'\forall\epsilon>0$ there is P_2 S.T $\underline{V}(D)\geq\underline{S}(P_2,D)>\underline{V}(D)-\epsilon$ for a partition $P_1\cup P_2$:

$$\bar{V}(D)+\epsilon > \bar{S}(P_1\cup P_2,D) \geq \bar{V}(D) = \underline{V}(D) \geq \underline{S}(P_1\cup P_2,D) > \underline{V}(D)-\epsilon$$

hence,

$$2\epsilon > \bar{S}(P_1\cup P_2) - \underline{S}(P_1\cup P_2,D) \geq 0$$

$$2\epsilon > \bar{S}(P_1\cup P_2,\partial D) \geq 0, \forall\epsilon > 0$$

hence, ∂D has volume 0.

Theorem. *D is bounded with volume $\iff$ it's boundry has volume 0 .*

Example. a continuous curve $y = f(x), a \leq x \leq b$ in $\mathbb{R}^2$ "volume 0".

Proof. f continoious in $[a, b] \Rightarrow f$ is a uniformly continuous i.e $\forall\epsilon > 0$ there is $\delta(\epsilon)$ S.T $u-v| < \delta$ then $|f(u)-f(v)| < \epsilon$. we will devide $[a, b]$ to $a = x_1 < x_2... < x_n = b$ S.T $x_{i+1} - x_i < \delta$:

$$\bar{S} = \sum(x_{i+1}-x_i)(\underbrace{maxf - minf}_{<\epsilon}) < \epsilon\cdot\sum(x_{i+1}-x_i) = \epsilon\cdot(b-a)..$$

it's small as we want.. □

<u>**Changing variables in n variables:**</u>

Reminder: changing variable in 1 dimension. $\int_a^b f(t)dt, f$ continuous. a plug in $t = \varphi(x), \varphi : [\alpha, \beta] \to [a, b], a = \varphi(\alpha), b = \varphi(\beta)$ then :

$$\left(\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)dt\right) = \int_a^b f(t)dt = \int_\alpha^\beta f(\varphi(x))\cdot\varphi'(x)dx$$

Remark. φ is not forced to be injective as described in the following photo.

/Users/georgesalman/Desktop/screenshots/Screen Shot 2021-04-30 at 13.12.55.png

Theorem. *given a open set $G \subset \mathbb{R}^n$ and a transformation $\varphi : G \subset \mathbb{R}^n \to \mathbb{R}^n$ S.T :*

$$\varphi : (x_1, .., x_n) \to (t_1, .., t_n)$$

$$t_i = \varphi_i(x_1, ..x_n), i = 1, 2..., n$$

With tho following assumptions:

(1) φ has a deffrential partials with order 1 in G.
(2) φ is injective when in one dimension it's not a neccesary condition.
(3) $J_\varphi(x) \neq 0$ for all $x \in G$.

let D be a closed set with a volume in G. and let $f(t) = f(t_1, .., t_n)$ be a continuous function on the image $\varphi(t)$ then:

$$\int\limits_{\varphi(D)} f(t_1, .., t_n) dt_1 ... dt_n = \int\limits_D f(\varphi(x_1, .., x_n)) \left| \frac{\partial(\varphi_1, ..., \varphi_n)}{\partial(x_1, .., x_n)} \right| dx_1 .... dx_n$$

A PRIVATE CASE:

$f(t_1, .., t_n) = 1$,

$$V(\varphi|D|) = \int\limits_{\varphi(D)} dt_1, .., dt_n = \int\limits_D |J_\varphi(x)| dx_1 .. dx_n$$

THE SIMPLEST CASE:

let $\varphi = L$ linear, $\varphi(x) = Lx$ then:

$$J_\varphi = J_L = \left| \left(\frac{\partial \varphi_i}{\partial x_j} \right) \right| = |(l_{ij})| = Det(L)$$

$$V(L(D)) = \int\limits_D |Det(L) dx_1 ... dx_n = |Det(L)| \cdot V(D)$$

Lemma. *for every transformation invertible L and for every box $B \subset \mathbb{R}^n$ exists, $V(L(B)) = |Det(L)| \cdot V(B)$.*

Proof. first remember that box volume = multipliaction of the edges.

Now by linear algebra we can write evely linear transformation L as elementary multiplaction $L_1, .., L_n$ of elementart matrices of different types: □

(1) multiplication of the first coordinate by constant i.e $(x_1, .., x_n) \to (\lambda x_1, x_2, .., x_n)$ by the matrix $\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

(2) adding coordinate to other i.e $(x_1, .., x_n) \to (x_1 + x_2, x_2, .., x_n)$ by the following matrix $\begin{pmatrix} 1 & 1 & 0 & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

(3) Changing coordinate order i.e $(x_1, .., x_n) \to (x_2, x_1, .., x_n)$ by the matrix $\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$.

In a matrix reduction $L \cdot (L_1 \cdot L_2 .... L_n) = I$ now we need to know what all of those following operations affect the box volume.

/Users/georgesalman/Desktop/Screen Shot 2021-04-30 at 18.18.19.png

this operation multplicate the volume by λ Moreover,

$$Det\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \lambda.$$

/Users/georgesalman/Desktop/Screen Shot 2021-04-30 at 18.19.19.png

by this operttion the Volume preserve, and $det(...) = 1$.

/Users/georgesalman/Desktop/Screen Shot 2021-04-30 at 18.35.32.png

The volume preserved Moreover, $|det(...)| = +1$.
Now we will summarize:

$$V(L(B)) = V(L_1(L_2(....L_n(B)...))) = |Det(L_1)|V(L_2...L_n(B))$$

$$= .... = |Det(L_1)| \cdot |Det(L_2)|....|Det(L_n)| \cdot V(B) =_* Det(L_1 \cdots L_n) \cdot V(B)$$

$$= |Det(L)| \cdot V(B)$$

Where in * we use determinant proberty of transformation multiplaction.

Remark. we already saw that if B is a box and L is $L.T$ then $V(L(B)) = |Det(L)| \cdot V(B)$. now if D a graph with volunme and φ is a s map as mentioned then our targe to show that if D has volume $\Rightarrow \varphi(D)$ has volume too.

Remark. if a set D has volume then ∂D has to be with volume 0.

Lemma. *G a open set in $\mathbb{R}^n, \varphi \in C^1(G), E$ is a closed and bounded set, then if for E there is "volume 0" then for $\varphi(E)$ there is volume* 0.

Proof. By the definition $\forall \epsilon > 0$ there is finite number of boxes E.g $B_1, B_2, ...$ which cover E and it's volume sum $< \epsilon$, now we take in a box $B_k$from those boxes two point x, y and we connect them by a slope i.e $(1-t)x + ty, 0 \le t \le 1$ and denote $h(t) = \varphi((1-t)x + ty)$ hence,

$$\varphi(y) - \varphi(x) = h(1) - h(0) = h'(c^*)(1-0)$$

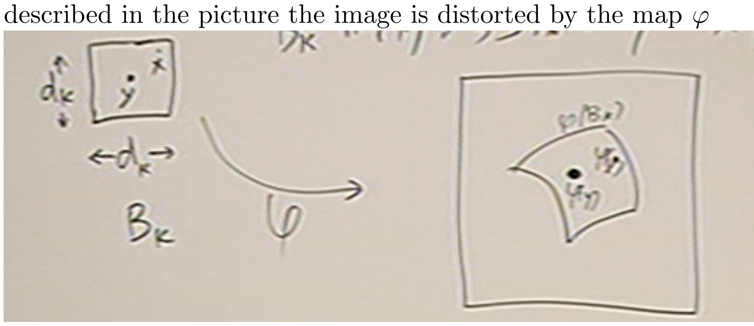
$$h(t) = \varphi((1-t)x_1 + ty_1, (1-t)x_2 + ty_2, ...)$$

$h'(c^*) = \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_j}|_{t=c^*}(y_j - x_j)$

$$||\varphi(y)-\varphi(x)||_\infty = \underset{i}{max}|\varphi_i(y)-\varphi_i(x)| = max\sum_{j=1}^{n}\frac{\partial\varphi_i}{\partial x_j}|_{t=c^*}(y_j-x_j)$$

$$\le n\cdot \underset{\cup B_k}{max}|\frac{\partial\varphi_i}{\partial x_j}|\cdot max|y_j-x_j| = M\cdot||y-x||_\infty$$

Now we take the center of box B_k assuming y and x is a point in the box, as described in the picture the image is distorted by the map φ



hence, the image of B_k is obtained in a box with edge Md_k so the volume of all the new boxes imply,

$$\sum(Md_k)^n = M^n\underbrace{\sum d_k^n}_{<\epsilon} \le M^n\epsilon$$

hence, $\varphi(E)$ has a "volume 0" as required $\square$

Lemma. *G open set in $\mathbb{R}^n, \varphi:\mathbb{R}^n\to\mathbb{R}^n, \varphi\in C^1, \varphi$ is $1:1$ on G and $J_\varphi\neq 0$ in every point, if D is a closed and bounded set with a volume which is obtaind in G, then $\varphi(D)$ has volume.*

Proof. D has volume $\iff$ ∂D has volume $0 \Rightarrow \varphi(\partial D)$ has volume 0. In our assumptions the open maps Theorem is valid Since, the interior of D is mapped to the interior of $\varphi(D)$ and the boundry ∂D is mapped to the boundry of $\varphi(D)$ (look at φ^{-1}) hence, $\varphi(\partial D)=\partial(\varphi(D))$ those are with volume 0 therefore, $\varphi(D)$ has volume. $\square$

Remark. if we asssume toward contradiction that one point x assuming without loose of generality interior point then it should be mapped to interior in $\varphi(D)$ otherwise, we can look at $\varphi^{-1}(x)$ this is interior point by the Theorem of open maps, so the interior of D forced to be mapped $\varphi(D)$.

Lemma. *D is a set with volume. and it satisfy all the previous assumptions, L is $L.T$ then $V(L(D)) = |Det(L)|\cdot V(D)$.*

Proof. we already show it when D is a box, now we want to prove it for D whch is not a box, if D has volume then we can find union of boxes which obtain and obtained in D i.e

$$\left\{\begin{array}{c} \underline{B}\subset D\subset\bar{B} \\ V(\underline{B})\subset V(D)\subset V(\bar{B}) \\ 0\le V(\bar{B})-V(\underline{B})<\epsilon \end{array}\right\}$$

Now we look at :

$$\left\{ \begin{array}{c} L(\underline{B}) \subset L(D) \subset L(\bar{B}) \\ |detL| \cdot V(\underline{B}) = V(L(\underline{B})) \subset V(L(D)) \subset V(L(\bar{B})) = |detL| \cdot V(B) \end{array} \right\}$$

Hence,

$$V(\underline{B}) \leq \frac{V(L(D))}{|detL|} \leq V(\bar{B})$$

$$V(\underline{B}) \subset V(D) \subset V(\bar{B})$$

$$0 \leq V(\bar{B}) - V(\underline{B}) < \epsilon$$

therefore, □

Lemma.

$$|\frac{V(L(D))}{|detL|} - V(D)| \leq V(\bar{B}) - V(\underline{B}) < \epsilon, \forall \epsilon > 0$$

Proof. So we conclude that:

$$\frac{V(L(D))}{|detL|} = V(D) \Rightarrow V(L(D)) = |detL| \cdot V(D)$$

□